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THE q-FOULKES PROBLEM II: SCHUR POSITIVITY WITH HIGH PROBABILITY

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ABSTRACT. Bergeron's q-analog of Foulkes' conjecture asserts that $(H_b[H_a] - H_a[H_b])/(1-q)$ is Schur positive for $a \le b$, where H_n is the (modified) Hall–Littlewood polynomial. We view this conjecture probabilistically: on each irreducible representation of S_{ab}, the two plethysms induce degree distributions, and positivity means that the first is stochastically smaller than the second. We prove that, for a random (Plancherel) irreducible representation and a degree sampled from its graded multiplicity, the corresponding coefficient is negative with probability at most $C\exp(-c(ab)^{1/3})$, uniformly in $a \le b$. Consequently, the q-Foulkes conjecture holds after deleting a fraction at most $C\exp(-c(ab)^{1/3})$ of each of the two graded modules. Additionally, we show that exact conditional means and variances yield complete positivity for explicit width and height families.

1. INTRODUCTION

We study the q-analog of Foulkes' conjecture proposed by Bergeron (2017) from a probabilistic point of view. Recall that Foulkes's conjecture (1950) states that, for $a \le b$, the difference of plethysms $h_b[h_a] - h_a[h_b]$ of complete homogeneous symmetric functions is Schur positive, i.e., it expands with nonnegative integer coefficients in the Schur functions s_λ, $\lambda \vdash ab$. This positivity is known for $a \le 6$ (Thrall, 1942; Dent and Siemons, 2000; McKay, 2008; Cheung et al., 2017; OpenAI, 2026) and for b large with respect to a (Brion, 1993), but it remains open in general.

Bergeron's q-analog replaces h_n by the (modified) Hall–Littlewood polynomial

$$H_n(X;q) := (q;q)_n h_n\left[\tfrac{X}{1-q}\right], \quad \text{where} \quad (q;q)_n := \textstyle\prod_{j=1}^{n}(1-q^j),$$

and his analog of Foulkes' conjecture is the following:

Conjecture 1.1 (Bergeron, 2017). *For all integers $a \le b$, the divided difference*

$$F_{a,b} := \frac{H_b[H_a] - H_a[H_b]}{1-q}$$

is Schur positive with coefficients in $\mathbb{N}[q]$.

Since $H_n(X;0) = h_n$, at $q = 0$ the conjecture reduces to Foulkes' conjecture. Bergeron proved the conjecture at $q = 1$ (Bergeron, 2017, Section 3).

Our starting point is a probabilistic view on Conjecture 1.1. The plethysms $H_b[H_a]$ and $H_a[H_b]$ are the graded characters of two S_N-modules $M_{a,b}$ and $M_{b,a}$, where $N = ab$, and both are regular once the grading is forgotten (Lemma 2.1). Each irreducible S^λ therefore occurs $d_\lambda = \dim S^\lambda$ times in each module, and the degree of a uniformly chosen copy defines a random variable T_λ for $M_{a,b}$, the target, and S_λ for $M_{b,a}$, the source. Dividing by $1-q$ takes cumulative sums, so $[q^k s_\lambda]F_{a,b}$ equals $d_\lambda(\mathbb{P}(T_\lambda \le k) - \mathbb{P}(S_\lambda \le k))$ (Proposition 2.2), and Conjecture 1.1 says that T_λ is stochastically smaller than S_λ for every $\lambda \vdash N$. The Plancherel-averaged degree laws admit an exact monotone coupling (Lemma 3.5), but a small absolute error can reverse this order where the averaged gap is small. We control the character corrections relative to positive averaged gaps and tails.

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Theorem 1.2 (Probabilistic q-Foulkes)**.** *There are absolute constants $C, c > 0$ with the following property. For integers $1 \le a \le b$, put $N = ab$, sample $\Lambda \vdash N$ from Plancherel measure, and, conditionally on Λ, let K have the law of T_Λ or of S_Λ according to a fair coin. Then*

$$\mathbb{P}([q^K s_\Lambda]F_{a,b} < 0) \le C\exp(-cN^{1/3}).$$

Theorem 1.2 says that Conjecture 1.1 holds with high probability, in that the probability of a q-Foulkes violation, a negative coefficient at a random shape and degree, is at most $Ce^{-cN^{1/3}}$. Concretely, suppose we weight each pair of a shape λ and a degree k by the dimension of the λ-isotypic part of the two modules in degree k. Then the pairs at which the conjectured inequality fails carry a fraction at most $Ce^{-cN^{1/3}}$ of the total weight. Moreover, outside a set of Plancherel mass at most $Ce^{-cN^{1/3}/2}$, the failing degrees carry at most $e^{-cN^{1/3}/2}$ of the shape's graded mass (Corollary 3.1). When a and b are close, positivity holds throughout a central degree window.

Corollary (informal Cor. 3.7)**.** *There are absolute constants $\kappa, C, c > 0$ such that, if $2 \le a < b$, $N = ab$, and $b - a \le \kappa N^{1/6}$, then a Plancherel-random shape Λ satisfies, with probability at least $1 - Ce^{-cN^{1/3}}$, that all coefficients of $[s_\Lambda]F_{a,b}$ in a deterministic central degree interval are nonnegative, and that the interval carries all but $Ce^{-cN^{1/3}}$ of the mass of T_Λ and of S_Λ.*

Since the conjecture asks for a degree-nonincreasing equivariant bijection $M_{b,a} \to M_{a,b}$, Theorem 1.2 has an exact consequence too. Match each copy of an irreducible in $M_{b,a}$ to a copy of the same irreducible in $M_{a,b}$ of no larger degree, and delete the copies that cannot be matched. Then, the deleted copies form a fraction at most $Ce^{-cN^{1/3}}$ of the dimension of each module, and the submodules that remain satisfy the conjectured Schur positivity (Theorem 3.4). We note that this fraction is the *smallest possible*, and it is exponentially small in N once $b \ge (1+\delta)a$ for a fixed $\delta > 0$ (Corollary 3.6). Because target degrees never exceed a fixed endpoint, both the failure probability in Theorem 1.2 and the deleted fraction are at most e^{-cN} when $b \ge 6a \ge 18$, and at most $\exp[-(1/3+o(1))N\log N]$ when $b \asymp a^2$ and $a \to \infty$ (Corollary 3.9). Both extend to Bergeron's rectangular conjecture, where the inner factor H_a becomes the modified Macdonald polynomial of a rectangle, with a second grading that the matching preserves (Theorem 5.1, Corollary 5.2).

Theorem 1.2 and its consequences concern typical shapes. Our second main result proves the conjecture outright, at every degree, for explicit infinite families of shapes. Such families exist because a width bound separates the source support from the target mean by order b^2, while both conditional standard deviations are $O(b^{3/2})$. The exact conditional means and variances then order the two cumulative functions at every degree (see Theorem 4.5).

Theorem 1.3 (Positivity at bounded width)**.** *Let $a \ge 2$, and let $1 \le w \le 2a-1$, or $w = 2a$ if a is even. There is an explicit $B(a,w)$ such that $[s_\lambda]F_{a,b} \in \mathbb{N}[q]$ for every $b \ge B(a,w)$ and every partition $\lambda \vdash ab$ with $\lambda_1 \le w$.*

The same criterion gives positivity under a height bound, even for widths growing linearly in b (Corollary 4.6). For fixed $a \ge 2$, let $\lambda_b \vdash ab$ have normalized transposition characters satisfying $\chi^{\lambda_b}(2,1^{ab-2})/d_{\lambda_b} \to x < 1$. The moments certify a positive interval of length $\Theta(b^2)$ reaching within $O(b)$ of both conditional means (Corollary 4.8). In the macroscopic interior, the coefficients are asymptotic to their upper bound d_{λ_b} (Theorem 4.7). Positivity at every degree of a typical shape remains open, since the rate $e^{-cN^{1/3}}$ does not rule out individual negative coefficients (Section 4.4).

1.1. **Conventions.** Throughout, $a, b \ge 2$ are integers and $N = ab$ unless stated otherwise, $c_n = \binom{n}{2}$, and $[n]_q = (1-q^n)/(1-q)$. For a partition λ, λ' is its conjugate, $\ell(\lambda)$ its length, $n(\lambda) = \sum_i (i-1)\lambda_i$, and its width, height, and depth are λ_1, $\ell(\lambda)$, and $|\lambda| - \lambda_1$. For $\rho = 1^{m_1}2^{m_2}\cdots$, put $z_\rho = \prod_r r^{m_r} m_r!$.

For $\lambda \vdash N$, S^λ is the irreducible S_N-module over $\mathbb{C}$, χ^λ its character, and $d_\lambda = \dim S^\lambda$. Plethysm acts on every grading variable through the Adams operations $p_j \mapsto p_{rj}$, $q \mapsto q^r$, and $t \mapsto t^r$.

2. Regular modules and conditional degree laws

This section turns Conjecture 1.1 into a comparison of two probability laws on each shape, which is the form every later result uses. The plethysms $P_{a,b} = H_b[H_a]$ and $Q_{a,b} = H_a[H_b]$ define the degree laws T_λ and S_λ of Theorem 1.2 because their underlying modules are regular.

2.1. **The graded character of the coinvariant algebra.** The coinvariant algebra is where H_n comes from, and its regularity is what makes the degree laws exist. The coinvariant algebra R_n is the quotient of the polynomial ring by invariant generators of degrees $1, \dots, n$, so the graded trace of $\sigma \in S_n$ on R_n is $(q;q)_n / \det(I - q\sigma)$. If μ is the cycle type of σ, this trace is

$$\kappa_n(\sigma; q) := \frac{(q;q)_n}{\prod_{r \in \mu}(1 - q^r)}. \tag{2.1}$$

Thus κ_n is the graded character of R_n, and at $q = 1$ it is the regular character. Expand $h_n = \sum_{\mu \vdash n} p_\mu / z_\mu$. The plethystic alphabet $1/(1-q)$ replaces p_r by $p_r/(1-q^r)$, so $H_n = \sum_{\mu \vdash n} \kappa_n(\mu; q) p_\mu / z_\mu$ is the graded Frobenius character of R_n.

2.2. **Wreath induction.** Wreath induction realizes plethysm (Macdonald, 1995, Chapter I, Appendix A, (6.2)), and its graded form produces the modules behind $H_b[H_a]$. Partition $[N]$ into b blocks of size a, with a fixed labeling (i, j), $1 \le i \le b$, $1 \le j \le a$, and let $W_{a,b} = S_a \wr S_b$, of order $(a!)^b b!$, preserve this partition. Write an element as $(\pi_1, \dots, \pi_b; \sigma)$, acting by $(i, j) \mapsto (\sigma(i), \pi_i(j))$. For an outer cycle $C = (i_1, \dots, i_r)$, define the monodromy $\pi_C = \pi_{i_r} \cdots \pi_{i_1}$, and put

$$\Theta_{a,b}(\boldsymbol{\pi}, \sigma; q) = \kappa_b(\sigma; q) \prod_{C \in \mathrm{Cyc}(\sigma)} \kappa_a(\pi_C; q^{|C|}). \tag{2.2}$$

Lemma 2.1 (Graded wreath induction)**.** *The function $\Theta_{a,b}$ is the graded character of $R_a^{\otimes b} \otimes R_b$, with the natural wreath action, permuting tensor factors without signs, and*

$$\mathrm{ch}\, \mathrm{Ind}_{W_{a,b}}^{S_N}(R_a^{\otimes b} \otimes R_b) = H_b[H_a].$$

Proof. The graded trace on an outer r-cycle is $\kappa_a(\pi_C; q^r)$: a homogeneous vector of inner degree d is carried around r tensor factors and contributes q^{rd}. Multiplication by the trace on R_b proves (2.2). The cycle-index formula for induction gives

$$\frac{1}{|W_{a,b}|} \sum_{w \in W_{a,b}} \Theta_{a,b}(w; q) p_{\mu(w)} = H_b[H_a].$$

Indeed, an outer r-cycle multiplies every cycle length of π_C by r, exactly the Adams operation $p_j \mapsto p_{rj}$ together with $q \mapsto q^r$. □

Put

$$M_{a,b} = \mathrm{Ind}_{W_{a,b}}^{S_N}(R_a^{\otimes b} \otimes R_b). \tag{2.3}$$

By Lemma 2.1, $\mathrm{ch}_q M_{a,b} = P_{a,b}$. Ungraded, $R_a^{\otimes b} \otimes R_b$ is regular for $W_{a,b}$: a nonidentity outer permutation has zero trace on R_b, while for the identity outer permutation the inner traces vanish unless every inner permutation is the identity. Induction preserves regularity, so $M_{a,b}$ and $M_{b,a}$ are ungraded copies of $\mathbb{C}[S_N]$.

Regularity turns the coefficient polynomials into probability laws. For $\lambda \vdash N$, write

$$u_\lambda(q) = [s_\lambda] P_{a,b}, \qquad v_\lambda(q) = [s_\lambda] Q_{a,b}.$$

These polynomials have nonnegative integer coefficients, and $u_\lambda(1) = v_\lambda(1) = d_\lambda$. Define integer-valued random variables by

$$\mathbb{E}q^{T_\lambda} = u_\lambda(q)/d_\lambda, \qquad \mathbb{E}q^{S_\lambda} = v_\lambda(q)/d_\lambda. \tag{2.4}$$

Proposition 2.2 (The exact conditional comparison)**.** *For every integer $k \geq 0$,*

$$\frac{[q^k s_\lambda]F_{a,b}}{d_\lambda} = \mathbb{P}(T_\lambda \leq k) - \mathbb{P}(S_\lambda \leq k). \tag{2.5}$$

Thus $F_{a,b}$ is Schur positive with coefficients in $\mathbb{N}[q]$ if and only if T_λ is stochastically smaller than S_λ for every $\lambda \vdash N$.

Proof. Dividing by $1-q$ takes cumulative sums of coefficients, and the resulting series terminates because both totals equal d_λ. □

Proposition 2.3 (Equivalent filtered formulations)**.** *For fixed a, b, the following are equivalent:*
(1) $F_{a,b}$ is Schur positive with coefficients in $\mathbb{N}[q]$;
(2) there is a degree-nonincreasing S_N-equivariant linear bijection $M_{b,a} \to M_{a,b}$;
(3) there is a degree-preserving S_N-equivariant $\mathbb{C}[z]$-linear injection

$$M_{b,a} \otimes \mathbb{C}[z] \longrightarrow M_{a,b} \otimes \mathbb{C}[z], \qquad \deg z = 1,$$

with finite-dimensional cokernel of graded Frobenius character $F_{a,b}$.

Proof. Fix an irreducible type and list its multiplicity degrees in increasing order as $s_1 \leq \cdots \leq s_{d_\lambda}$ in the source and $t_1 \leq \cdots \leq t_{d_\lambda}$ in the target. The cumulative inequalities are equivalent to $t_i \leq s_i$ for every i. Matching these copies gives (2). Over $\mathbb{C}[z]$, send the source generator to $z^{s_i - t_i}$ times the target generator; the cokernel of this map has Hilbert polynomial $(q^{t_i} - q^{s_i})/(1-q)$. Conversely, either map gives the cumulative inequalities. The maps depend on the chosen matching. □

3. Positivity with high probability

To measure how often the conjecture fails, we weight each coefficient $c_{\lambda,k} = [s_\lambda q^k]F_{a,b}$ by the dimension of its irreducible type and by its graded multiplicity in the two modules. The weights define the probability measure

$$\boxed{\mathsf{G}_{a,b}(\lambda, k) = \frac{d_\lambda^2}{2N!}\big(\mathbb{P}(T_\lambda = k) + \mathbb{P}(S_\lambda = k)\big).} \tag{3.1}$$

Its shape marginal is $\mathrm{Pl}_N(\lambda) = d_\lambda^2/N!$. Throughout, Λ is a random shape with law Pl_N and λ is a fixed shape, so T_Λ and S_Λ are the conditional degree laws at Λ. Conditional on λ, it chooses either module with equal probability and samples the degree of a uniformly chosen element of a homogeneous basis of the multiplicity space. We call this law the *graded Plancherel mixture*. Theorem 1.2 becomes, for all integers $1 \leq a \leq b$,

$$\mathbb{P}_{(\Lambda,K)\sim\mathsf{G}_{a,b}}(c_{\Lambda,K} < 0) \leq C\exp(-cN^{1/3}).$$

For $a = 1$ or $a = b$ the divided difference is zero, so all nontrivial estimates below assume $2 \leq a < b$.

For each shape, put

$$V_\lambda = \tfrac{1}{2}\big(\mathbb{P}(c_{\lambda,T_\lambda} < 0) + \mathbb{P}(c_{\lambda,S_\lambda} < 0)\big).$$

Thus V_λ is the fraction of the graded mass of λ on which the coefficient inequality fails, and Theorem 1.2 is the assertion $\mathbb{E}_{\mathrm{Pl}_N} V_\Lambda \leq Ce^{-cN^{1/3}}$. Markov's inequality turns this averaged bound into a statement about most shapes.

Corollary 3.1 (Violations carry little graded mass on most shapes)**.** *With the constants of Theorem 1.2,*

$$\mathbb{P}_{\mathrm{Pl}_N}\left(V_\Lambda > e^{-cN^{1/3}/2}\right) \leq Ce^{-cN^{1/3}/2}.$$

Proof. Apply Markov's inequality to V_Λ. □

A shape's first negative cutoff always carries source mass.

Proposition 3.2 (The sampling law detects every violation)**.** *The following are equivalent:*

$$\mathsf{G}_{a,b}(c<0)=0; \qquad V_\lambda = 0 \text{ for every } \lambda; \qquad c_{\lambda,k}\geq 0 \text{ for every } \lambda, k.$$

Proof. If a shape has a negative coefficient, take its first negative cutoff k_0, with $c_{\lambda,-1}=0$. Since $c_{\lambda,k_0}-c_{\lambda,k_0-1}=d_\lambda(\mathbb{P}(T_\lambda=k_0)-\mathbb{P}(S_\lambda=k_0))<0$, we have $\mathbb{P}(S_\lambda=k_0)>0$. Hence the negative coefficient has positive mass under (3.1). □

3.1. **Positivity after a minimal deletion.** Proposition 2.3 turns the conjecture into a degree-nonincreasing bijection $M_{b,a}\to M_{a,b}$. We seek the smallest deletion that repairs all degree cutoffs at once. Write $F_{T,\lambda}$ and $F_{S,\lambda}$ for the cumulative distribution functions of T_λ and S_λ in (2.5). The *cumulative defect* of λ,

$$D_\lambda=\sup_k(F_{S,\lambda}(k)-F_{T,\lambda}(k))_+,$$

measures the obstruction to matching each source copy to a target copy of no larger degree, and the following elementary lemma shows that the obstruction is exact.

Proposition 3.3 (Optimal ordered matching)**.** *Let $x_1,\dots,x_f$ and $y_1,\dots,y_f$ be two lists of integers. The largest partial matching with $x_i\leq y_j$ has size*

$$f-\max_k\big(\#\{j:y_j\leq k\}-\#\{i:x_i\leq k\}\big)_+.$$

Proof. An initial segment of the y-list with $y_j\leq k$ can only be matched to $x_i\leq k$, so its excess is an unavoidable deficiency. Conversely, the neighbor sets of the y-vertices, namely $\{i:x_i\leq y_j\}$, are nested. For any subset of y-vertices its Hall deficiency is bounded by the deficiency of the full initial segment ending at its largest degree. The maximum initial-segment deficiency is therefore the full Hall deficiency, and Hall's theorem, or greedy matching in increasing order, attains it. □

Theorem 3.4 (Optimal filtered matching)**.** *For integers $1\leq a\leq b$, put $N=ab$ and $\Delta_{a,b}=\mathbb{E}_{\mathrm{Pl}_N}D_\Lambda$. There are graded submodules $M^\circ_{a,b}\subseteq M_{a,b}$ and $M^\circ_{b,a}\subseteq M_{b,a}$ of the same codimension, with an S_N-equivariant isomorphism $M^\circ_{b,a}\longrightarrow M^\circ_{a,b}$ preserving the increasing degree filtration, such that*

$$\frac{\operatorname{codim} M^\circ_{a,b}}{N!}=\frac{\operatorname{codim} M^\circ_{b,a}}{N!}=\Delta_{a,b}\leq Ce^{-cN^{1/3}},$$

$$\frac{\operatorname{ch}_q M^\circ_{a,b}-\operatorname{ch}_q M^\circ_{b,a}}{1-q}\in\sum_{\lambda\vdash N}\mathbb{N}[q]s_\lambda. \tag{3.2}$$

The normalized codimension is optimal. Equivalently,

$$\max_\phi\frac{\operatorname{rank}\phi}{N!}=1-\Delta_{a,b}, \tag{3.3}$$

where $\phi\colon M_{b,a}\to M_{a,b}$ ranges over equivariant filtration-preserving linear maps. The minimum probability of $T_\Lambda>S_\Lambda$ in a coupling of the two conditional degree laws that preserves Λ is also exactly $\Delta_{a,b}$.

Proof. For each λ, list the d_λ homogeneous multiplicity degrees of the two modules. The unmatched number in Proposition 3.3 is the integer $\max_k(-c_{\lambda,k})_+=d_\lambda D_\lambda$. Delete this many copies of S^λ on each side, and match the rest. The common dimension loss is $\sum_\lambda d_\lambda^2 D_\lambda=N!\Delta_{a,b}$. Every matched pair $x\leq y$ contributes

$$s_\lambda\frac{q^x-q^y}{1-q}=s_\lambda(q^x+\cdots+q^{y-1}),$$

with the contribution zero when $x = y$; summing over matched pairs proves (3.2). The same counting of initial segments bounds the nullity of any filtered equivariant map from below by $\sum_\lambda d_\lambda^2 D_\lambda$, proving optimality and the rank identity (3.3).

A coupling of two uniform degree lists is a doubly stochastic matrix. The linear matching cost attains its minimum at a permutation matrix. Greedy ordered matching gives failure probability exactly D_λ; the same counting of initial segments gives the reverse inequality. Average over λ. Finally, $D_\lambda \le 2V_\lambda$ (Lemma A.1) and Theorem 1.2 give the rate. □

3.2. **The averaged degree laws.** The proof of Theorem 1.2 measures each conditional law against its Plancherel average, so we first show that the averages are explicit and already ordered. They are sums of independent uniform coordinates, which can be coupled monotonically. Let U_j be uniform on $\{0, \dots, j\}$ and define $\mathsf{M}_n = \sum_{j=1}^{n-1} U_j$, with all summands independent. For $M_{u,v}$, the Plancherel-averaged generating function is its graded dimension divided by $N!$:

$$g_{u,v}(q) = \left(\frac{[u]_q!}{u!}\right)^v \frac{[v]_q!}{v!}. \tag{3.4}$$

For $P_{a,b}$ and $Q_{a,b}$ these laws are those of

$$X = \sum_{i=1}^{b} \mathsf{M}_a^{(i)} + \mathsf{M}_b, \qquad Y = \sum_{i=1}^{a} \mathsf{M}_b^{(i)} + \mathsf{M}_a. \tag{3.5}$$

Repeated terms denote independent copies. Write $p_X(k) = \mathbb{P}(X = k)$ and $p_Y(k) = \mathbb{P}(Y = k)$, and put $\mu_X = \mathbb{E}X$ and $\mu_Y = \mathbb{E}Y$. Their mean difference is

$$d = \mu_Y - \mu_X = \frac{(a-1)(b-1)(b-a)}{4}.$$

Also

$$\mathbb{E}\mathsf{M}_n = \frac{n(n-1)}{4}, \qquad \operatorname{Var} \mathsf{M}_n = \frac{n(n-1)(2n+5)}{72}.$$

Lemma 3.5 (Positive coupling and separation)**.** *There is a coupling of X, Y with $X \le Y$ almost surely. In this coupling, if $a < b$,*

$$\mathbb{P}(Y - X < d/2) \le e^{-c_0(a-1)(b-1)} \tag{3.6}$$

for an absolute $c_0 > 0$. There is also a deterministic real cutoff t such that

$$\mathbb{P}(X > t) + \mathbb{P}(Y \le t) \le 2 \exp\left(-\frac{N(b-a)^2}{256 b^2}\right). \tag{3.7}$$

Proof. Remove the common summands $\mathsf{M}_a + \mathsf{M}_b$. There remain $m_0 = (a-1)(b-1)$ uniform coordinates on each side. For $1 \le i \le m_0$, put

$$A_i = \left\lceil \frac{i}{b-1} \right\rceil, \qquad B_i = \left\lceil \frac{i}{a-1} \right\rceil.$$

With independent uniform $\xi_i \in [0,1)$, couple the remaining coordinates as $\lfloor (A_i+1)\xi_i \rfloor$ and $\lfloor (B_i+1)\xi_i \rfloor$. Their laws are the required multiset of uniform coordinates, and $A_i \le B_i$. The total expected displacement is $m_0(b-a)/4 = d$. Each coordinate displacement is between zero and $b-a$. Hoeffding's inequality Hoeffding (1963) gives (3.6), for example with $c_0 = 1/32$.

For the last assertion, let Q_X, Q_Y be the sums of squared uniform widths in (3.5). Then

$$Q_X = b\frac{a(a-1)(2a-1)}{6} + \frac{b(b-1)(2b-1)}{6},$$

and Q_Y is obtained by interchanging a, b. Both are at most $Nb^2/2$, and $d \ge N(b-a)/16$. At $t = \mu_X + d\sqrt{Q_X}/(\sqrt{Q_X} + \sqrt{Q_Y})$, Hoeffding bounds the sum of the two tails by $2\exp\{-2d^2/(\sqrt{Q_X} + \sqrt{Q_Y})^2\}$. The displayed estimates imply (3.7). □

The cutoff of Lemma 3.5 bounds the optimal loss $\Delta_{a,b}$ and the failure probability when the degree is sampled from the target.

Corollary 3.6 (Separated parameters)**.** *For $2 \le a < b$ and $N = ab$,*

$$\max\{\Delta_{a,b}, \mathbb{P}(c_{\Lambda,T_\Lambda} < 0)\} \le 2\exp\left(-\frac{N(b-a)^2}{256b^2}\right),$$

where Λ has Plancherel law and T_Λ is sampled conditionally on Λ. If $b \ge (1+\delta)a$ for a fixed $\delta > 0$, both quantities are at most $2\exp[-\delta^2 N/(256(1+\delta)^2)]$.

Proof. For every real t and every shape,

$$D_\lambda,\ \mathbb{P}(c_{\lambda,T_\lambda} < 0) \le 1 - F_{T,\lambda}(t) + F_{S,\lambda}(t).$$

For the defect, cutoffs at most t have defect at most $F_{S,\lambda}(t)$, and larger cutoffs have defect at most $1 - F_{T,\lambda}(t)$. The bound on the sign when the degree is sampled from the target is (A.1). Average and use the cutoff of Lemma 3.5. The last assertion follows from $(b-a)/b \ge \delta/(1+\delta)$. □

Writing $G(k) = \mathbb{P}(X \le k) - \mathbb{P}(Y \le k)$, the same coupling gives

$$\mathbb{E}_{\mathrm{Pl}_N}\frac{c_{\Lambda,k}}{d_\Lambda} = G(k) = \mathbb{P}(X \le k < Y) \ge 0.$$

Since $E \otimes \mathbb{C}[S_N] \cong (\dim E)\,\mathbb{C}[S_N]$ for every representation E, this averaged inequality is equivalent to the comparison of Conjecture 1.1 after tensoring both modules with the regular representation in degree zero. For individual shapes, the proof of Theorem 1.2 bounds the character corrections relative to $G(k)$, even where $G(k)$ is small.

3.3. **Proof of Theorem 1.2.** When the mean gap d is large compared with the standard deviations, a cutoff separates the averaged laws, and one conditional source tail suffices. This separation bound has constant order when $b - a = 1$. For close means, we use the positive cumulative gap $G(k)$ and control the character corrections relative to it on a window of moderate deviations. The character estimates are proved in Appendix A.

The conditional cumulative and tail probabilities expand in normalized characters (Lemma A.5). For a reduced cycle type ν whose parts are at least two, let $s = |\nu|$ be the number of moved points and $r = s - \ell(\nu)$ its index. Retain the identity and the classes with $1 \le r < m$, for which $s \le 2r < 2m$. These classes are few, and their corrections are explicit finite differences of the averaged law. The residual $\mathcal{R}_m$ sums the contributions of the remaining classes, which are controlled only in mean square. For $H \subseteq S_N$, let $\gamma_m(H)$ be the maximum of $|C \cap H|/|C|$ over conjugacy classes C of index at least m. Lemma A.4 bounds these ratios, and (A.7) bounds the mean square of the residual by $\gamma_m(H)$ times a normalized trace.

Put

$$z = \varepsilon N^{1/6}, \qquad p = \lceil z^2 \rceil, \qquad m = \lceil \Gamma z^2 \rceil,$$

where the absolute constants ε and Γ are chosen at the end of the proof. We work with sufficiently large N and absorb the remaining finite range into the absolute prefactor C of Theorem 1.2.

3.3.1. *One conditional source tail.* For the source $Q_{a,b}$, let $u = b \ge v = a$ and $\sigma = b\sqrt{N}$. Take the window multiplier $L = 1$ in Lemma A.7, and put

$$k_+ = \lceil \mu_Y + \sigma z \rceil, \qquad \delta_B = \mathbb{P}(Y > k_+).$$

Lemma A.7 gives constants $B, c_1 > 0$ with

$$e^{-Bz^2} \le \delta_B \le Ce^{-c_1 z^2}, \qquad \delta_B \ge c\frac{\sigma}{1+z}p_Y(k_+). \tag{3.8}$$

After fixing Γ, choose ε sufficiently small as below. Proposition A.9 bounds the sum of the retained tail corrections by $\delta_B/4$ outside an event of probability $4 \cdot 16^{-p}$.

Lemma A.4, with $m = \lceil \Gamma z^2 \rceil$, gives

$$\gamma_m(W_{b,a}) \le \exp(-\Gamma(\log 2)z^2 + o(1)) + Ce^{-c_2\sqrt{N}}.$$

For the tail projector, use $\mathbb{E}|\mathcal{R}_m|^2 \le \gamma_m$. Chebyshev and (3.8) show that its residual exceeds $\delta_B/4$ with probability at most $16\gamma_m e^{2Bz^2}$. Choosing Γ so that $\Gamma \log 2 > 2B + 10$ gives

$$\mathbb{P}_{\mathrm{Pl}_N}\left(1 - F_{S,\Lambda}(k_+) < \delta_B/2\right) \le Ce^{-c_3 z^2}. \tag{3.9}$$

3.3.2. *Separated means.* Suppose $d \ge \Omega\sigma z$, where Ω is a sufficiently large absolute constant chosen after B. The optimized cutoff from Lemma 3.5 gives $t \in (\mu_X, \mu_Y)$ such that

$$\mathbb{P}(X > t) + \mathbb{P}(Y \le t) \le Ce^{-(B+10)z^2} \tag{3.10}$$

after increasing Ω. Markov's inequality and $\delta_B \ge e^{-Bz^2}$ imply

$$\mathbb{P}_{\mathrm{Pl}_N}\left(1 - F_{T,\Lambda}(t) > \delta_B/4\right) \le Ce^{-10z^2}.$$

On this good event and the one in (3.9), monotonicity gives, simultaneously for all integers $t < k \le k_+$,

$$\frac{c_{\lambda,k}}{d_\lambda} = (1 - F_{S,\lambda}(k)) - (1 - F_{T,\lambda}(k)) \ge \delta_B/4 > 0.$$

When the coin selects the source, a violation can therefore occur only below t, above k_+, or on an exceptional shape, with total probability at most Ce^{-cz^2}. When it selects the target, averaging the distribution-free bound (A.1) and using (3.10) gives the same conclusion, and so does the mixture.

3.3.3. *Close means.* Suppose instead $d < \Omega b\sqrt{N}z$. Since $d \ge N(b-a)/16$, we have $(b-a)/b \le 16\Omega z/\sqrt{N} = o(1)$. Consequently $a, b \asymp \sqrt{N}$ and $b - a \le C_\Omega z$, and the variances of X and Y are both comparable to N^2. Take the integer window

$$I_N = [\mu_X - L_0 N z,\ \mu_Y + L_0 N z] \cap \mathbb{Z}$$

with a fixed $L_0 \ge 1$. The two averaged degree laws have mass at most Ce^{-cz^2} outside I_N.

To compare corrections with $G(k)$, we bound $G(k)$ below relative to the point masses throughout this window. On this window, both X and Y satisfy the estimates of Appendix A.2, with multiplier depending only on Ω, L_0. Take an integer $\ell_0 \asymp N/(1+z)$, small enough that $\ell_0 \le d/2$; such an ℓ_0 exists because $b - a \ge 1$ gives $d \ge cN$. In the monotone coupling of Lemma 3.5,

$$G(k) = \mathbb{P}(X \le k < Y) \ge \mathbb{P}(k - \ell_0 < X \le k) - \mathbb{P}(Y - X < \ell_0),$$

and also

$$G(k) \ge \mathbb{P}(k < Y \le k + \ell_0) - \mathbb{P}(Y - X < \ell_0).$$

Lemma A.7 compares the masses within these short intervals, while (3.6) bounds the subtracted term by e^{-cN}. The latter is negligible relative to the moderate probabilities. We obtain, uniformly for $k \in I_N$,

$$G(k) \ge c_{\Omega,L_0}\frac{N}{1+z}(p_X(k) + p_Y(k)) \ge e^{-B_{\Omega,L_0}z^2}.$$

Apply Proposition A.9 to both wreath subgroups with reference function $G(k)$. On the common character event, the sum of their retained absolute corrections is less than $G(k)/2$ for every $k \in I_N$, if ε is sufficiently small. For both groups, $u, v \asymp \sqrt{N}$, so the stronger part of Lemma A.4 gives

$$\gamma_m \le C\exp(-c\Gamma z^2 \log N) + Ce^{-c\sqrt{N}}.$$

The full degree support has at most $\binom{N}{2} + 1 \le N^2$ cutoffs. Use (A.7), Chebyshev, and a union bound over them. The probability that the two residuals together exceed $\frac{1}{2}\min_{k\in I_N} G(k)$ is at most

$$CN^2 e^{2B_{\Omega,L_0}z^2}\left(e^{-c\Gamma z^2 \log N} + e^{-c\sqrt{N}}\right) \le Ce^{-cz^2}.$$

Thus all coefficient inequalities hold simultaneously on I_N except for shapes in an event of probability Ce^{-cz^2}. The graded mixture leaves I_N with probability Ce^{-cz^2}, completing this regime.

The constants are fixed in Appendix A.2.1. With $z^2 = \varepsilon^2 N^{1/3}$, both regimes fail with probability at most Ce^{-cz^2}, which proves Theorem 1.2.

The argument for close means compares all cutoffs in its window on the same set of shapes.

Corollary 3.7 (Simultaneous positivity on a central window)**.** *There are absolute constants $A, \kappa, C, c > 0$ such that, whenever $2 \le a < b$, $N = ab$, and $b - a \le \kappa N^{1/6}$, the interval*

$$I_{a,b} = [\mu_X - AN^{7/6},\ \mu_Y + AN^{7/6}] \cap \mathbb{Z}$$

has the following property. The shapes $\lambda \vdash N$ that satisfy both

$$c_{\lambda,k} \ge 0 \quad (k \in I_{a,b}), \qquad \mathbb{P}(T_\lambda \notin I_{a,b}) + \mathbb{P}(S_\lambda \notin I_{a,b}) \le Ce^{-cN^{1/3}}$$

have Plancherel probability at least $1 - Ce^{-cN^{1/3}}$.

Proof. Use the fixed constants ε, Ω, L_0 from the proof of Theorem 1.2, and take $\kappa = \Omega\varepsilon$ and $A = L_0\varepsilon$. Since $b \ge \sqrt{N}$ and $d \le N(b-a)/4$, the hypothesis places the parameters in the case of close means. The proof there establishes all coefficient inequalities on $I_{a,b}$ outside an event of probability $Ce^{-cN^{1/3}}$. It also gives $\mathbb{P}(X \notin I_{a,b}) + \mathbb{P}(Y \notin I_{a,b}) \le Ce^{-cN^{1/3}}$. This sum is the expectation of the conditional tail sum in the statement. Markov's inequality at threshold $e^{-cN^{1/3}/2}$ and a union bound prove the result after adjusting C, c. □

3.4. **Faster rates when b is much larger than a.** Beyond the target support, its cumulative distribution is one and a negative coefficient is impossible. The source mass below this endpoint therefore bounds failures; its Plancherel average is a tail of Y in (3.5).

Theorem 3.8 (A support bound for signs and filtered loss)**.** *For $1 \le a \le b$, put $N = ab$ and*

$$D_+ = bc_a + c_b, \qquad \eta_{a,b} = \mathbb{P}(Y < D_+), \qquad R_\lambda = F_{S,\lambda}(D_+ - 1).$$

Then every $\lambda \vdash N$ satisfies

$$\max(V_\lambda, D_\lambda) \le R_\lambda, \qquad \mathbb{E}_{\mathrm{Pl}_N} R_\Lambda = \eta_{a,b}. \tag{3.11}$$

In particular,

$$\max\{\mathsf{G}_{a,b}(c < 0), \Delta_{a,b}\} \le \min\{Ce^{-cN^{1/3}}, \eta_{a,b}\}, \tag{3.12}$$

$$\mathbb{P}_{\mathrm{Pl}_N}(\max(V_\Lambda, D_\Lambda) > u) \le \eta_{a,b}/u \qquad (u > 0). \tag{3.13}$$

There is an explicit filtered comparison with normalized dimension loss exactly $\eta_{a,b}$ from each module; the optimal loss $\Delta_{a,b}$ of Theorem 3.4 may be smaller.

Hoeffding's inequality and integer-tuple counting bound this scalar tail. The proofs and boundary cases are in Appendix A.3.

Corollary 3.9 (Exponential and entropy rates)**.** *For $a \ge 3$ and $b \ge 6a$,*

$$\max\{\mathsf{G}_{a,b}(c < 0), \Delta_{a,b}\} \le \eta_{a,b} \le e^{-8N/2187}.$$

For $b \ge a \ge 3$,

$$\eta_{a,b} \le \min\left\{1, \left[e^2\left(\frac{a}{b} + \frac{1}{a}\right)\right]^{N-1}\right\}. \tag{3.14}$$

Consequently, if $m_* := \min(a, b/a) \to \infty$, then

$$\log \eta_{a,b} \le -(N-1)\log m_* + O(N).$$

In particular, for $b \asymp a^2$ with $a \to \infty$,

$$\max\{\mathsf{G}_{a,b}(c < 0), \Delta_{a,b}\} \le \exp\big[-(1/3 + o(1))N \log N\big]. \tag{3.15}$$

4. Positivity on explicit shape families

Theorem 1.2 is a statement about typical shapes. This section proves the conjecture outright on explicit families, using the exact conditional means and variances. The means of T_λ and S_λ are explicit character values, and for fixed a their variances are $O(b^3)$ (Theorem B.4). Whenever a shape family separates each mean from the opposite support endpoint by order b^2, a one-sided Chebyshev bound therefore compares the cumulative functions at every degree. Width and height families have this separation; for other families the same moments locate positive intervals. Write $r_\lambda(\rho) = \chi^\lambda(\rho 1^{N-|\rho|})/d_\lambda$ when the specified nontrivial cycle type fits in S_N. In particular, $x_\lambda = r_\lambda(2)$ and $r_j^\lambda = r_\lambda(2^j)$. We omit the index λ when it is fixed.

Proposition 4.1 (Exact conditional means)**.** *For every $\lambda \vdash ab$,*

$$\mu_T := \mathbb{E}T_\lambda = \frac{bc_a}{2}(1 - x_\lambda) + \frac{c_b}{2}(1 - r_a^\lambda), \tag{4.1}$$

$$\mu_S := \mathbb{E}S_\lambda = \frac{ac_b}{2}(1 - x_\lambda) + \frac{c_a}{2}(1 - r_b^\lambda). \tag{4.2}$$

Also

$$x_\lambda = \frac{n(\lambda') - n(\lambda)}{c_N}. \tag{4.3}$$

The proof is in Appendix B.2.

Write σ_T^2 and σ_S^2 for the variances of T_λ and S_λ. They have explicit character expressions (Theorem B.2), and the uniform estimate (B.8) gives $\sigma_T\sigma_S \le 4a^3b^3$ for $b \ge a \ge 2$.

4.1. Support and moment criteria.

Support bounds settle thresholds outside the overlap of the two degree distributions. For a partition λ, define

$$\eta_m(\lambda) = \sum_{i \ge 1} \left\lfloor \frac{i-1}{m} \right\rfloor \lambda_i. \tag{4.4}$$

Recall $D_+ = bc_a + c_b$.

Theorem 4.2 (Support bounds)**.** *For every $\lambda \vdash ab$,*

$$\eta_b(\lambda) \le T_\lambda \le D_+ - \eta_b(\lambda'),$$
$$\eta_a(\lambda) \le S_\lambda \le ac_b + c_a - \eta_a(\lambda').$$

The proof is in Appendix C.1.

Fusion and a one-variable expansion sharpen the upper bounds when the first row is long.

Corollary 4.3 (Upper support from the shape)**.** *For $\lambda \vdash N = ab$ of depth $r = N - \lambda_1$,*

$$T_\lambda, S_\lambda \le \min\{c_N - n(\lambda'),\ r(a+b-2)\}.$$

If $r \le a \le b$, the second bound can be decreased to $r(a+b-2) - c_r$. At every integer cutoff at or above any of these common upper bounds, $[q^k s_\lambda]F_{a,b} = 0$.

The proof is in Appendix C.1. For $\lambda = (N-r, \nu)$, $c_N - n(\lambda') = Nr - r(r+1)/2 - n(\nu')$. At fixed a, r, the depth bound confines nonzero coefficients to degrees of order b.

Set $L_\lambda = \eta_a(\lambda)$ and

$$U_\lambda = \min\{D_+ - \eta_b(\lambda'),\ c_N - n(\lambda'),\ r(a+b-2)\}, \qquad r = N - \lambda_1.$$

If $L_\lambda \ge U_\lambda$, every coefficient of $[s_\lambda]F_{a,b}$ is nonnegative by support separation. Otherwise only the integer thresholds $L_\lambda \le k < U_\lambda$ need comparison.

4.1.1. *A one-sided probability test.* Within the support overlap, the product of the two margins from the means controls the cumulative comparison.

Lemma 4.4 (A Cantelli comparison)**.** *Let T, S have means μ_T, μ_S and standard deviations σ_T, σ_S. If $\mu_T < k < \mu_S$, then*

$$\mathbb{P}(T \le k) - \mathbb{P}(S \le k) \ge \frac{(k-\mu_T)^2(\mu_S-k)^2 - \sigma_T^2\sigma_S^2}{(\sigma_T^2 + (k-\mu_T)^2)(\sigma_S^2 + (\mu_S-k)^2)}.$$

For integer-valued T, S and integer k, $k-\mu_T$ in this formula may be replaced by $k+1-\mu_T$, provided both displayed margins are positive.

Proof. For a centered variable ξ with variance σ^2 and $s > 0$, Markov's inequality applied to $(\xi + \sigma^2/s)^2$ on $\{\xi \ge s\}$ gives $\mathbb{P}(\xi \ge s) \le \sigma^2/(\sigma^2 + s^2)$. Apply this inequality to the upper tail of T and the lower tail of S. Subtracting both bounds from one gives the stated expression. For integer-valued T, $\{T > k\} = \{T \ge k+1\}$, giving the stated lattice improvement. □

The exact variances in Theorem B.2 give a rational criterion. For an integer k with $k+1 > \mu_T$ and $k < \mu_S$,

$$\big((k+1-\mu_T)(\mu_S-k)\big)^2 \ge \sigma_T^2\sigma_S^2 \quad \Longrightarrow \quad [q^k s_\lambda]F_{a,b} \ge 0. \tag{4.5}$$

A strict inequality gives a strictly positive coefficient; Example C.7 certifies one coefficient inside a support overlap in this way. The concavity of $(k-\mu_T)(\mu_S-k)$ reduces the comparison at all degrees to the two endpoints of the support overlap.

Theorem 4.5 (Complete support–moment comparison)**.** *Suppose $S \ge L$, $T \le U$, and $\mu_T < L \le U < \mu_S$. If*

$$\min\{(L-\mu_T)(\mu_S-L),\ (U-\mu_T)(\mu_S-U)\} \ge \sigma_T\sigma_S, \tag{4.6}$$

then $\mathbb{P}(T \le k) \ge \mathbb{P}(S \le k)$ for every real threshold k. Applied to the laws (2.4), it proves $[s_\lambda]F_{a,b} \in \mathbb{N}[q]$.

Proof. Outside $[L, U]$, the support bounds give the comparison. Inside it, $(k-\mu_T)(\mu_S-k)$ is a concave quadratic, so its minimum occurs at an endpoint. Apply Lemma 4.4. □

4.2. **Width and height families.** A width bound separates the source support from the target mean by order b^2. The conditional standard deviations are $O(b^{3/2})$, so the support–moment criterion eventually applies at every degree.

Theorem 1.3 (precise form)**.** *Fix integers $a \ge 2$ and $w \ge 1$. Put*

$$g = \frac{2a/w - 1 + (-1/w)^a}{4}, \qquad h = \frac{a(1+1/w) - 2}{4}.$$

If $g, h > 0$, define

$$B(a,w) = \left\lceil \max\left\{ a, 4, \frac{2(c_a + 2a + w + 1)}{g}, \frac{2(c_a + a + w + 1)}{h}, \frac{16a^3}{gh} \right\} \right\rceil. \tag{4.7}$$

For every $b \ge B(a,w)$ and every $\lambda \vdash ab$ of width at most w, $[s_\lambda]F_{a,b} \in \mathbb{N}[q]$. In particular, $g, h > 0$ for $w \le 2a-1$, and for $w \le 2a$ if a is even, so the conclusion holds in these cases.

The proof is in Appendix C.2.

For $a = 6$, sharper variance bounds give the smaller thresholds of Theorem C.3.

Height bounds give the same separation for families whose first row can grow linearly in b.

Corollary 4.6 (Complete positivity at large height)**.** *Fix $a \geq 2$ and a real $h \leq a$. Suppose*

$$h > \max\{1, \sqrt{a/2}\} \quad \text{if } a \text{ is even}, \qquad h > \sqrt{a} \quad \text{if } a \text{ is odd}.$$

For all sufficiently large b, depending only on a, h, every $\lambda \vdash ab$ with $\ell(\lambda) \geq hb$ satisfies $[s_\lambda]F_{a,b} \in \mathbb{N}[q]$.

The proof is in Appendix C.2. For $a = 6$, Theorem C.3 gives the explicit range $b \geq 10240$ and $\ell(\lambda) \geq \lceil 7b/4 \rceil$.

4.3. **Positive intervals on general shapes.** Overlapping supports can still leave a middle interval where the target cumulative probability is near one and the source cumulative probability is near zero.

Theorem 4.7 (A conditional macroscopic interval)**.** *Fix $a \geq 2$ and let $\lambda_b \vdash ab$ have $x_{\lambda_b} \to x < 1$. If*

$$\frac{1 - x^a}{4} < u < \frac{a(1-x)}{4}, \qquad k_b/b^2 \to u,$$

then

$$\frac{[q^{k_b} s_{\lambda_b}]F_{a,b}}{d_{\lambda_b}} \longrightarrow 1.$$

In particular these coefficients are positive for all sufficiently large b.

Proof. Equations (4.1)–(4.2) and (B.6) give $\mu_T/b^2 \to (1 - x^a)/4$ and $\mu_S/b^2 \to a(1-x)/4$. The variance bound divided by b^4 tends to zero. Hence the first cumulative probability in (2.5) tends to one and the second to zero. The interval is nonempty because $1 - x^a < a(1-x)$ for $-1 \leq x < 1$. □

For $a = 6$, Corollary C.4 gives explicit degree windows and a lower bound $d_\lambda/2$ on the coefficients.

The Cantelli criterion gives strict positivity closer to the means: a margin of order b at one end suffices because the other margin has order b^2, while $\sigma_T \sigma_S = O(b^3)$.

Corollary 4.8 (Positive intervals near the conditional means)**.** *Fix $a \geq 2$ and let $\lambda_b \vdash ab$ satisfy $x_{\lambda_b} \to x < 1$. Put*

$$G_a(x) = \frac{a(1-x) - 1 + x^a}{4}, \qquad K_a = \sqrt{\frac{7a(9a+5)}{162}}.$$

For every $c > K_a/G_a(x)$ and all sufficiently large b,

$$[q^k s_{\lambda_b}]F_{a,b} > 0 \quad \text{whenever} \quad \lceil \mu_T + cb \rceil \leq k \leq \lfloor \mu_S - cb \rfloor.$$

Proof. The mean calculation in Theorem 4.7 gives $(\mu_S - \mu_T)/b^2 \to G_a(x) > 0$. In (B.7), $I(a,b) = O_a(b)$, $O(a,b)/b^3 \to (9a+5)/18$, and $I(b,a)/b^3 \to 7a/9$; the source outer term is bounded by $c_a/2$. Hence

$$\limsup_{b \to \infty} \frac{\sigma_T \sigma_S}{b^3} \leq K_a.$$

At each endpoint of the asserted interval, $(k + 1 - \mu_T)(\mu_S - k)/b^3 \to cG_a(x) > K_a$. By concavity, the margin product is minimized at an endpoint. The strict form of (4.5) proves the claim. □

For hooks of growing depth, the exact variances of Theorem B.2 replace K_a by the limit of $\sigma_T \sigma_S/b^3$ and so sharpen the constant c (Theorem C.5); for $a = 6$ and the hooks $(5b, 1^b)$ the margin $b/5$ suffices for all sufficiently large b (Corollary C.6).

4.4. **The whole-shape question.** Theorems 1.2 and 1.3 approach the conjecture from two sides, typical shapes at sampled degrees and every degree on special families. This subsection records what each side says about an arbitrary shape at every degree. By integrality, a negative coefficient of $[s_\lambda]F_{a,b}$ forces $V_\lambda \geq 1/(2d_\lambda)$. The dimension bound $d_\lambda \leq \sqrt{N!}$ is too large to convert the rate $e^{-cN^{1/3}}$ into a vanishing bound on the Plancherel probability that a shape has a negative coefficient.

Theorem 1.2 weights each shape by d_λ^2. If instead every shape and degree is counted once, the support bounds and the exact means still make negative coefficients rare, because every degree between the target endpoint D_+ and the top of the source support carries a strictly positive coefficient.

Corollary 4.9 (Unweighted count of negative coefficients)**.** *For integers $2 \leq a < b$, let B and P be the numbers of negative and of strictly positive coefficients $[q^k s_\lambda]F_{a,b}$, over all $\lambda \vdash ab$ and all $k \geq 0$. Then $B + P > 0$ and*

$$\frac{B}{B+P} \leq \min\Big\{1,\ \frac{2}{a} + \frac{2(a-1)}{b-1}\Big\}.$$

Proof. Let $p(N)$ be the number of partitions of N. By Theorem 4.2, $T_\lambda \leq D_+$, so (2.5) gives $[q^k s_\lambda]F_{a,b} = d_\lambda(1 - \mathbb{P}(S_\lambda \leq k)) \geq 0$ for $k \geq D_+$. Hence every negative coefficient has $k < D_+$, and $B \leq p(N)D_+$. Let m_λ be the largest value of S_λ. For $D_+ \leq k < m_\lambda$ the same identity gives a strictly positive coefficient, and $m_\lambda \geq \mu_{S,\lambda}$, so

$$P \geq \sum_{\lambda \vdash N} (m_\lambda - D_+)_+ \geq \sum_{\lambda \vdash N} \mu_{S,\lambda} - p(N)D_+.$$

By (4.3), $x_{\lambda'} = -x_\lambda$, so $\sum_\lambda x_\lambda = 0$, and $r_b^\lambda \leq 1$ since it is a normalized character value. Summing (4.2) over λ therefore gives $\sum_\lambda \mu_{S,\lambda} \geq p(N)ac_b/2$. If $ac_b/2 \leq D_+$, only the bound 1 is claimed. Otherwise $P \geq p(N)(ac_b/2 - D_+) > 0$, and since $B \leq p(N)D_+$ and $B/(B+P)$ increases with B,

$$\frac{B}{B+P} \leq \frac{D_+}{D_+ + (ac_b/2 - D_+)} = \frac{2D_+}{ac_b} = \frac{2}{a} + \frac{2(a-1)}{b-1}.$$

Finally, summing (2.5) over k and over λ with weights d_λ gives $\sum_\lambda d_\lambda(\mu_{S,\lambda} - \mu_{T,\lambda})$, which equals $N!(a-1)(b-1)(b-a)/4$ by (4.1), (4.2), and $\sum_\lambda d_\lambda \chi^\lambda(g) = 0$ for $g \neq 1$. This is positive for $a < b$, so some coefficient is nonzero and $B + P > 0$. □

For $b = a^2$ the bound is $2/a + 2/(a+1) < 4N^{-1/3}$, so the negative coefficients are a vanishing fraction of the nonzero ones as $a \to \infty$ and $b/a \to \infty$. The positive coefficients counted here lie above the target support, so the sign pattern below D_+ is untouched.

For general shapes, the moment criteria certify the intervals of Corollary 4.8, and the thresholds inside the support overlap remain open.

5. The rectangular conjecture

Both arguments of Section 3 extend to Bergeron's rectangular conjecture (Bergeron, 2017, Conjecture 4), which replaces the inner factor H_a by the modified Macdonald polynomial of a rectangle and so carries a second grading. An equivariant bijection preserving this grading requires equal multiplicities in each shape and t-degree after forgetting the q-degree. The rectangular plethysms agree at $q = 1$, so the matching argument of Section 3 applies within each such fiber, and the averaged degree laws bound the dimension loss.

Our normalization of the modified Macdonald polynomial is

$$\widetilde{H}_\mu[X; q, t] = t^{n(\mu)} J_\mu \left[\frac{X}{1 - t^{-1}}; q, t^{-1}\right],$$

where J_μ is the integral Macdonald form. It is the normalization of Bergeron et al. (1999) and Haiman (2001); see also Macdonald (1995, Chapter VI, (8.1') and (8.3)). In particular $\widetilde{H}_{(n)}(X;q,t) = H_n(X;q)$.

Let $r \geq 1$, $2 \leq a < b$, and now $N = rab$. Set

$$\mathcal{T}^{(r)} = H_b[\widetilde{H}_{(a^r)}(X;q,t)], \qquad \mathcal{S}^{(r)} = H_a[\widetilde{H}_{(b^r)}(X;q,t)], \tag{5.1}$$

$$\mathcal{F}^{(r)}_{a,b} = \frac{\mathcal{T}^{(r)} - \mathcal{S}^{(r)}}{1-q} = \sum_{\lambda,k,j} c^{(r)}_{\lambda,k,j} s_\lambda q^k t^j. \tag{5.2}$$

Schur positivity of $\mathcal{F}^{(r)}_{a,b}$ is the rectangular conjecture for this pair of rectangles. By Haiman's $n!$ theorem (Haiman, 2001), $\widetilde{H}_\mu$ is the bigraded character of a module that is regular when the grading is forgotten, so both numerators have nonnegative multiplicities. The specializations (Bergeron, 2017, §2, equations (13) and (16)) give

$$\widetilde{H}_{(a^r)}(X;1,t) = H_r(X;t)^a, \qquad \widetilde{H}_{(a^r)}(X;q,1) = H_a(X;q)^r. \tag{5.3}$$

Thus $\mathcal{T}^{(r)}(1,t) = \mathcal{S}^{(r)}(1,t) = H_r(X;t)^{ab}$.

For each shape and t-degree, the multiplicities in (5.1) therefore have the same total over all q-degrees,

$$m_{\lambda,j} = [s_\lambda t^j] H_r(X;t)^{ab}, \qquad \sum_j m_{\lambda,j} = d_\lambda.$$

Define

$$w(\lambda,j) = \frac{d_\lambda m_{\lambda,j}}{N!}, \qquad D^{(r)}_{\lambda,j} = \frac{\sup_k(-c^{(r)}_{\lambda,k,j})_+}{m_{\lambda,j}}, \qquad \Delta^{(r)}_{a,b} = \mathbb{E}_w D^{(r)}_{\Lambda,J}.$$

Fibers with $m_{\lambda,j} = 0$ have probability zero, and we set their defect to zero. The shape marginal of w is Pl_N.

Theorem 5.1 (A comparison keeping the second grading)**.** *The two bigraded modules with characters* (5.1) *have bigraded submodules of common normalized codimension* $\Delta^{(r)}_{a,b}$*, with an equivariant isomorphism that preserves t-degree and the increasing q-filtration, and the divided difference of their characters is Schur positive over* $\mathbb{N}[q,t]$*. This codimension is optimal, and so is the failure probability of the corresponding optimal coupling that preserves the shape and the t-degree. Moreover,*

$$\boxed{\Delta^{(r)}_{a,b} \leq 2\exp\left(-\frac{N(b-a)^2}{256b^2}\right).} \tag{5.4}$$

At fixed a, b, r, $\Delta^{(r)}_{a,b} = 0$ is equivalent to every coefficient of (5.2) *being nonnegative.*

Proof. Apply Proposition 3.3 separately within each (λ, j) fiber. The unmatched multiplicity is $m_{\lambda,j} D^{(r)}_{\lambda,j}$, an integer. Matching a pair of q-degrees $x \leq y$ contributes $s_\lambda t^j (q^x + \cdots + q^{y-1})$. Proposition 3.3 gives optimality.

By (5.3), the q-degree laws averaged over w are those of

$$X_r = \sum_{i=1}^{rb} \mathsf{M}_a^{(i)} + \mathsf{M}_b, \qquad Y_r = \sum_{i=1}^{ra} \mathsf{M}_b^{(i)} + \mathsf{M}_a.$$

Their mean separation and sums of squared uniform widths are

$$d_r = \frac{(b-a)(N-a-b+1)}{4},$$
$$Q_{A,r} = rb\frac{a(a-1)(2a-1)}{6} + \frac{b(b-1)(2b-1)}{6},$$
$$Q_{B,r} = ra\frac{b(b-1)(2b-1)}{6} + \frac{a(a-1)(2a-1)}{6}.$$

For any cutoff x_0, the conditional defect is at most the conditional X_r upper tail plus the conditional Y_r lower tail at x_0. Averaging gives

$$\Delta_{a,b}^{(r)} \le \mathbb{P}(X_r > x_0) + \mathbb{P}(Y_r \le x_0).$$

The optimized Hoeffding cutoff yields the sharper formula

$$\Delta_{a,b}^{(r)} \le 2\exp\left\{-\frac{2d_r^2}{(\sqrt{Q_{A,r}}+\sqrt{Q_{B,r}})^2}\right\}.$$

Since $N-a-b+1 \ge N/4$ and $Q_{A,r}, Q_{B,r} \le Nb^2/2$, we obtain (5.4). □

The exponent in (5.4) is $N(b-a)^2/b^2 = r(a/b)(b-a)^2$; it grows when $b-a$ or r does.

The target support also controls sampled signs over the full q-degree range. Sample (Λ, J) with law w, and then sample a q-degree K from its target or source multiplicity with equal probabilities. This law is the bigraded Plancherel mixture.

Corollary 5.2 (A support bound for bigraded signs)**.** *Put $D_r = rbc_a + c_b$, $\eta_r = \mathbb{P}(Y_r < D_r)$, and $M_r = ra(b-1) + a - 1$. For the bigraded Plancherel mixture,*

$$\max\{\mathbb{P}(c_{\Lambda,K,J}^{(r)} < 0), \Delta_{a,b}^{(r)}\} \le \eta_r \le \min\left\{1, \frac{\binom{D_r+M_r-1}{M_r}}{(b!)^{ra}a!}\right\}.$$

For $a \ge 3$,

$$\eta_r \le \min\left\{1, \left[e^2\left(\frac{a}{b}+\frac{1}{ra}\right)\right]^{M_r}\right\}.$$

If $b \ge 6a \ge 18$, then $\eta_r \le e^{-8N/2187}$ for every $r \ge 1$. If $b \asymp ra^2$, $ra \to \infty$, and $a \ge 3$, then

$$\log \eta_r \le -N\log(ra) + O(N).$$

In particular, at fixed $a \ge 3$ and $b \asymp ra^2$ the last bound gives $\eta_r \le \exp[-(1/2+o(1))N\log N]$ as $r \to \infty$.

The proof is in Appendix A.3.

For rectangles with r fixed and $b-a$ bounded, a decaying bound would need estimates after conditioning on the t-degree, which neither specialization supplies.

AI USAGE

The results in this paper were developed through extensive use of AI assistants in sustained mathematical discussion, including failed attempts, counterexamples, audits, and repairs. AI assistants also helped with exact computations, proof checks, and exposition. The external representation-theoretic results are cited at their points of use.

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Appendix A. Character and support estimates for probabilistic positivity

The proof of Theorem 1.2 in Section 3.3 needs the character corrections to be small relative to the averaged cumulative gap, and that gap can itself be small. An absolute error bound would erase it, so every estimate in this appendix keeps the degree density as a factor when summing the character corrections. The support tails of Section 3.4 are proved at the end.

Lemma A.1 (Three distribution-free facts)**.** *Let P, Q be finitely supported integer probability laws, and put $g = F_P - F_Q$, $E = \{k : g(k) < 0\}$, and $D = \sup_k(-g(k))_+$. For any real t,*

$$P(E) \le 1 - F_P(t) + F_Q(t), \tag{A.1}$$

$$D \le Q(E). \tag{A.2}$$

For any real $L \le R$, if $g(k) < 0$ for an integer $k \in [L, R]$, then

$$(1 - F_P(L)) + F_Q(R) > 1. \tag{A.3}$$

Proof. The part of $P(E)$ above t is at most $1 - F_P(t)$. On the remaining part, $F_P(K) < F_Q(K) \le F_Q(t)$. For $K \sim P$, $P(F_P(K) < x) \le x$, which proves (A.1).

For (A.2), suppose $D > 0$, and take k_* with $g(k_*) = -D$. Let $j < k_*$ be the last cutoff where $g(j) \ge 0$, allowing the formal initial cutoff before the support. Every $j < \ell \le k_*$ belongs to E, and

$$D \le g(j) - g(k_*) = \sum_{j<\ell\le k_*} (Q\{\ell\} - P\{\ell\}) \le Q(E).$$

Finally, throughout $[L, R]$, $g(k) \ge F_P(L) - F_Q(R)$, which proves (A.3). □

In particular, (A.2) implies $D_\lambda \le 2V_\lambda$, so the bounds on D_λ in Section 3 follow from those on V_λ.

A.1. **Growing character moments and a uniform spectral remainder.** The cutoff grows with N, so estimates for fixed conjugacy classes do not suffice. Plancherel moments control the retained classes, while wreath-subgroup intersection bounds control the residual $\mathcal{R}_m$ of Section 3.3.

A reduced cycle type $\nu = (\nu_1, \dots, \nu_\ell)$ has parts at least two. Write

$$s = |\nu|, \qquad r = \sum_i (\nu_i - 1) = s - \ell, \qquad z_\nu = \prod_{j\ge 2} j^{m_j} m_j!,$$

where m_j is the number of parts equal to j; we call r the index of ν. Thus $r + 1 \le s \le 2r$, and the class C_ν of S_N of type $\nu 1^{N-s}$ has size $|C_\nu| = (N)_s/z_\nu$, where $(N)_s = N(N-1)\cdots(N-s+1)$. The normalized character is $\rho_\nu(\lambda) = \chi^\lambda(\nu, 1^{N-s})/d_\lambda$. There are at most 2^r reduced types of index r, because subtracting one from each part identifies them with partitions of r.

Lemma A.2 (A character-moment bound with growing parameters)**.** *For integers $p \ge 1$ with $s \le N/2$ and $ps \le N$,*

$$\boxed{\|\rho_\nu\|_{L^{2p}(\mathrm{Pl}_N)} \le (7\sqrt{ps})^s N^{-s/2}.}$$

If these conditions hold for all $r < m$, then, except on an event of probability at most $4 \cdot 16^{-p}$, simultaneously for all such types,

$$|\rho_\nu(\Lambda)| \le (14\sqrt{ps})^s N^{-s/2}. \tag{A.4}$$

Proof. The Plancherel even moment is the coefficient of the identity in the $2p$-th power of the class sum, divided by $|C_\nu|^{2p}$. Every label in an identity-product tuple occurs in at least two supports. If only one factor moved a label, preceding factors would fix it and later factors could not return its image to that label, by bijectivity. The union of supports therefore has size at most $M = ps$.

On a fixed set of j labels there are $(j)_s/z_\nu$ permutations of reduced type ν. Dropping the identity-product constraint after restricting the union, and cancelling the centralizer factors, gives

$$\mathbb{E}\rho_\nu^{2p} \le \frac{\sum_{j\le M} \binom{N}{j} j^{2M}}{(N)_s^{2p}}.$$

For $j \le M \le N$, set $h = M - j$. The elementary factorial bound gives

$$\frac{N^j j^{2M}}{j!} \le e^M N^j j^{2M-j} \le e^M N^M M^M (M/N)^h \le e^M N^M M^M.$$

There are at most $M+1$ summands. Taking $2p$-th roots, using $(N)_s \geq (N/2)^s$ and $(M+1)^{1/(2p)} \leq 2^s$, yields $(4\sqrt{e}\sqrt{ps})^s N^{-s/2}$, which is less than the stated bound.

Markov's inequality gives failure probability at most 2^{-2ps} for an individual type in (A.4). Summing by index and using $s \geq r+1$ bounds the total by $\sum_{r\geq 1} 2^r 2^{-2p(r+1)} \leq 4 \cdot 16^{-p}$. This probability bound is independent of the cutoff m. □

Lemma A.3 (A centralizer bound)**.** *There is an absolute constant C_0 such that*

$$z_\nu \geq s^\ell e^{-C_0 r}, \qquad \frac{s^{s/2}}{z_\nu} \leq C_0^r r^{r-s/2}.$$

Proof. The number $\ell!/\prod_j m_j!$ counts orderings of the positive parts $\nu_i - 1$ and is at most the number of compositions of r, hence at most 2^r. Thus $z_\nu \geq \ell!/2^r$. Since $\ell \log(s/\ell) = \ell \log(1 + r/\ell) \leq r$ and $\ell \leq r$, $\ell!/2^r \geq (\ell/e)^\ell 2^{-r} \geq s^\ell e^{-(2+\log 2)r}$. Since $s/2 - \ell = r - s/2 \geq 0$ and $s \leq 2r$, the second bound follows after adjusting the absolute constant. □

The index $\mathrm{ind}(g)$ of a permutation g is the sum of its nontrivial cycle lengths minus the number of those cycles, so a permutation of reduced type ν has index r. For $H = W_{u,v} \subset S_N$, put

$$\gamma_m(H) = \max_{\mathrm{ind}(g)\geq m} \frac{|g^{S_N} \cap H|}{|g^{S_N}|}.$$

Lemma A.4 (A fixed-point-ratio bound for wreath products)**.** *If $u \geq v \geq 2$, $N = uv$, and $m \geq 1$ is an integer with $2m < N$, then*

$$\gamma_m(H) \leq \left(\frac{u-1}{N-2m+1}\right)^m + 4u^2 2^{-u}. \tag{A.5}$$

For $m \leq N/4$, the first term is at most $v^{-m}\exp(Cm^2/N)$. The same bounds hold when $v \leq 2u$, with an absolute adjustment of the coefficient of $u^2 2^{-u}$. If $r < u$, then additionally

$$\frac{|C_\nu \cap H|}{|C_\nu|} \leq v^{-r}\exp(Cr^2/N) \qquad (2r \leq N/2). \tag{A.6}$$

Proof. The class intersection ratio is the probability that a fixed permutation preserves a uniformly random partition into v blocks of size u. We may order the blocks without changing the probability.

If every block is fixed individually, choose m edges in a spanning forest of the permutation's cycles. Expose roots, then new child vertices in turn. At most $2m$ labels are exposed. Given the blocks of exposed labels, each required child joins its parent's block with probability at most $(u-1)/(N-2m+1)$. The probability that every block is fixed is at most the first term of (A.5).

If the induced permutation of blocks is nonidentity and $v \geq 3$, then $gB_i = B_j$ for some $i \neq j$. Conditional on B_i, B_j is a uniform u-subset of its complement. The probability is either zero or $\binom{N-u}{u}^{-1}$. A union bound gives $v(v-1)/\binom{N-u}{u} \leq v^2 2^{-u}$, since $N - u \geq 2u$ and $\binom{2u}{u} \geq 2^u$. For $v = 2$, swapping the blocks requires every permutation cycle to have even length. There are at most 2^u alternating two-colorings among $\binom{2u}{u}$ balanced colorings. This probability is at most $(2u+1)2^{-u}$. For $v \leq 2u$ these bounds are still $O(u^2 2^{-u})$.

Finally, a nonidentity block permutation moves at least $2u$ labels and has index at least u. A type of index $r < u$ can therefore occur only in the base subgroup. Apply the same forest argument to all its r edges. The exponential forms follow from $(1 - (2m-1)/N)^{-m} \leq \exp(Cm^2/N)$. □

Lemma A.5 (Spectral projections and the residual)**.** *The conditional degree laws of $M_{u,v}$ are the normalized traces of orthogonal degree projections $P_k \in \mathbb{C}[W_{u,v}]$ on each S_N-irreducible. For any $y \in \mathbb{C}[W_{u,v}]$, discard from its normalized character expansion every class of index below m, including the identity, and denote the remaining function of λ by $\mathcal{R}_m(y;\lambda)$. Then*

$$\mathbb{E}_{\mathrm{Pl}_N} |\mathcal{R}_m(y;\Lambda)|^2 \leq \gamma_m(W_{u,v})\, \tau(y^* y), \tag{A.7}$$

where τ is normalized regular trace. In particular the last factor is at most one for a cumulative or tail degree projection.

Proof. Transport the orthogonal graded decomposition of the regular $W_{u,v}$-module $R_u^{\otimes v} \otimes R_v$ to its group algebra by an equivariant unitary isomorphism. Its projections commute with the left action and hence belong to the right group algebra. Inducing to S_N gives the stated trace interpretation on multiplicity spaces.

Write $y = \sum_{g \in H} t_g g$. Column orthogonality gives

$$\mathbb{E}_{\mathrm{Pl}_N} \rho_C(\Lambda) \overline{\rho_D(\Lambda)} = \frac{\mathbf{1}_{\{C=D\}}}{|C|}.$$

Here the classes of S_N are inverse-closed. Cauchy–Schwarz on each class therefore bounds the residual second moment by

$$\sum_{\mathrm{ind}(C) \geq m} \frac{|\sum_{g \in C \cap H} t_g|^2}{|C|} \leq \gamma_m(H) \sum_{g \in H} |t_g|^2 = \gamma_m(H) \tau(y^* y).$$

The argument uses sums over global conjugacy classes and does not require the degree projector to be central. □

For a fixed cutoff k, let $\phi_\nu(k)$ be the coefficient of $\rho_\nu(\lambda)$ in the normalized character expansion of the cumulative projection. Let Z have generating function (3.4) and mass p_Z.

Lemma A.6 (Exact cycle factors)**.** *Suppose $r < u$ and set $t_\nu = |C_\nu \cap W_{u,v}|$. There is a probability law W_ν with*

$$Z \stackrel{d}{=} W_\nu + \sum_{i=1}^{\ell} U_{\nu_i - 1}, \tag{A.8}$$

all summands on the right independent, provided $s \leq u$. For the cumulative projection,

$$\phi_\nu(k) = \frac{t_\nu}{\prod_i \nu_i} \nabla^{r-1} p_{W_\nu}(k), \qquad \nabla f(k) = f(k) - f(k-1). \tag{A.9}$$

The corresponding tail coefficient is $-\phi_\nu(k)$.

Proof. Such a permutation belongs to the base subgroup. Dividing its graded trace by the identity trace in (2.1) gives $(1-q)^r / \prod_i [\nu_i]_q$. Summing the group-algebra coefficients over a global class equals the sum of the graded traces over its intersection with the subgroup, divided by the subgroup order. Thus its mass coefficient has generating function

$$t_\nu g_{u,v}(q) \frac{(1-q)^r}{\prod_i [\nu_i]_q}.$$

We justify the positive factorization, including repeated cycles. Sort $\nu_1 \geq \cdots \geq \nu_\ell$. At stage i, select an unused multiple L_i from $\nu_i, 2\nu_i, \ldots, i\nu_i$. At most $i-1$ choices were used, and $i\nu_i \leq s$. The selected lengths are distinct and at most $s \leq u$, so all their uniform factors are available in a single inner block. If $L_i = d_i \nu_i$, factor $U_{L_i - 1} \stackrel{d}{=} U_{\nu_i - 1} + \nu_i U_{d_i - 1}$. These factorizations prove (A.8); the remaining dilated uniform factors are positive convolution kernels. As $\mathbb{E} q^{U_{\nu_i - 1}} = [\nu_i]_q / \nu_i$, the mass coefficient is $t_\nu \nabla^r p_{W_\nu} / \prod_i \nu_i$. Cumulative summation proves (A.9). The total of a nonidentity class coefficient is zero, giving the tail statement. □

A.2. **Relative smoothing on a growing moderate window.** The retained class corrections are finite differences of the averaged law (Lemma A.6). To compare them with the density p_Z on a moderate-deviation window, we tilt Z to the window and use the log-concavity of the tilted uniform factors.

Assume $v \le 2u$, $u, v \ge 2$, $N = uv$, and put $\sigma = u\sqrt{N}$. Then $\operatorname{Var} Z \asymp \sigma^2$, every underlying uniform has width at most $2u$, and $\sigma \ge N/\sqrt{2}$. The comparison constants may depend on a fixed window multiplier L, but not on the block sizes.

Lemma A.7 (Tilted point probabilities and tails)**.** *Let $z \ge 1$ with $z = o(\sqrt{N})$. For integers satisfying $|k - \mathbb{E}Z| \le L\sigma z$, exponential tilting of Z to mean k, with parameter θ_k, gives*

$$|\theta_k| \le C_L z/\sigma, \qquad \operatorname{Var}_{\theta_k} Z \asymp \sigma^2, \tag{A.10}$$

$$c_L \sigma^{-1} e^{-I(k)} \le p_Z(k) \le C_L \sigma^{-1} e^{-I(k)}, \qquad I(k) \le C_L z^2, \tag{A.11}$$

where I is the Legendre transform of $\log \mathbb{E}e^{\theta Z}$, and $|I'(x)| \le C_L z/\sigma$ in this window. Probabilities at points within distance $c\sigma/(1+z)$ of each other are consequently comparable, after enlarging the window multiplier by a fixed amount.

If $k_+ = \lceil \mathbb{E}Z + L\sigma z \rceil$ and $z^2/\log N \longrightarrow \infty$, then for some $B_L, c_L > 0$,

$$e^{-B_L z^2} \le \mathbb{P}(Z > k_+) \le C_L e^{-c_L z^2}, \qquad \mathbb{P}(Z > k_+) \ge c_L \frac{\sigma}{1+z} p_Z(k_+). \tag{A.12}$$

Proof. Put $\psi(\theta) = \log \mathbb{E}e^{\theta Z}$. A uniform coordinate of width $w \le 2u$ has tilted point probabilities between $e^{-w|\theta|}$ and $e^{w|\theta|}$ times its original probabilities. The identity $\operatorname{Var} U = \frac{1}{2}\mathbb{E}(U - U')^2$, with U' an independent copy, therefore bounds its tilted variance between $e^{-2w|\theta|} \operatorname{Var} U$ and $e^{2w|\theta|} \operatorname{Var} U$. For fixed A, the condition $z = o(\sqrt{N})$ gives $2uAz/\sigma = o(1)$, so $c\sigma^2 \le \psi''(\theta) \le C\sigma^2$ uniformly on $|\theta| \le Az/\sigma$. Choose $A = A_L$ so that $cA > L$. Integrating this lower bound shows that ψ' at the two endpoints brackets every k in the stated window. Its strict monotonicity gives the mean-setting tilt and (A.10). Since $I'(x) = \theta_x$, integration from $\mathbb{E}Z$ proves the bounds on I.

Each tilted uniform is discrete log-concave, and convolution preserves this property (Bobkov et al., 2021, Corollary 4.3). At an integer mean k, any such law p satisfies

$$p(k) \ge \{2\pi e(\operatorname{Var}(p) + 1/12)\}^{-1/2}. \tag{A.13}$$

Indeed, the piecewise linear interpolation of $\log p$ is concave, so Jensen gives $\log p(k) \ge -H(p)$. Adding an independent uniform variable on $[-1/2, 1/2]$ gives differential entropy $H(p)$ and variance $\operatorname{Var}(p) + 1/12$; the Gaussian entropy upper bound proves (A.13). The upper bound $\max p \le 2/\sqrt{1 + 4\operatorname{Var}(p)}$ is the discrete log-concave maximal-atom bound (Bobkov et al., 2021, Theorem 1.1). Undoing the tilt proves (A.11).

The derivative bound on I proves comparison over intervals of length $c\sigma/(1+z)$. Sum over an interval immediately to the right of k_+ to obtain the last inequality in (A.12) and the lower exponential bound. Polynomial factors are absorbed using $\log N = o(z^2)$. The upper bound is Hoeffding's inequality, since the sum of squared uniform widths is $O(\sigma^2)$. □

Lemma A.8 (Growing finite differences, relative to the density)**.** *Suppose $r < m$, $2m \le u$, $m = o(N)$, $m^3 = o(\sigma^2)$, and $zm/\sigma = o(1)$. For W_ν in (A.8), uniformly on the window of Lemma A.7,*

$$\frac{|\nabla^{r-1} p_{W_\nu}(k)|}{p_Z(k)} \le C_L \sqrt{r} \left(\frac{C_L(\sqrt{r} + z)}{\sigma} \right)^{r-1}. \tag{A.14}$$

The constant C_L depends only on L.

Proof. In the extraction of Lemma A.6, each altered factor has length at most $s \le 2r$, and there are at most r of them. Discarding those factors loses $O(r^3)$ variance. The untouched ordinary

uniform factors retain total variance $V \asymp \sigma^2$, also under the mean-setting tilt of Z. Their largest individual variance A is $O(u^2)$, so $rA/V = O(r/N) = o(1)$. For sufficiently large N, $A \le V/(2r)$. Accumulate factors until their variance first reaches $V/(2r)$, and repeat r times. Each group has variance between $V/(2r)$ and V/r; after fewer than r groups, at least V/r remains available. Unused factors form a positive convolution kernel.

Every tilted group law is log-concave and has maximal mass at most $C\sqrt{r}/\sigma$. For a unimodal probability mass function p, $\|\nabla p\|_1 = 2\max p$. Distribute finite differences among distinct groups and retain one group in supremum norm. Positive convolution by the leftover factors cannot increase either norm. Thus, for $0 \le j \le r-1$, the tilted W_ν law satisfies $\|\nabla^j p_{W_\nu,\theta}\|_\infty \le (C\sqrt{r}/\sigma)^{j+1}$. This step uses only positivity of the remaining dilated factors, not log-concavity.

Write $M_W(\theta) = \mathbb{E}e^{\theta W_\nu}$ and $M_Z(\theta) = \mathbb{E}e^{\theta Z}$, and let $Sf(k) = f(k-1)$. Undoing the tilt at $\theta = \theta_k$ gives

$$\frac{\nabla^{r-1} p_{W_\nu}(k)}{p_Z(k)} = \frac{M_W(\theta)}{M_Z(\theta)} \frac{(I - e^\theta S)^{r-1} p_{W_\nu,\theta}(k)}{p_{Z,\theta}(k)}.$$

The independent deleted summand in (A.8) lies in $[0, r]$, so $M_W/M_Z \le e^{|\theta| r} = O(1)$. Expand $I - e^\theta S = (I - S) + (1 - e^\theta)S$ and apply the preceding norm bounds. Since $p_{Z,\theta}(k) \ge c/\sigma$ by (A.13), the quotient is at most

$$C\sqrt{r}\left(\frac{C\sqrt{r}}{\sigma} + C|\theta|\right)^{r-1}.$$

The bound $|\theta| \le C_L z/\sigma$ proves (A.14). □

Proposition A.9 (Summing the retained classes at the endpoint scale)**.** *Set*

$$z = \varepsilon N^{1/6}, \qquad p = \lceil z^2 \rceil, \qquad m = \lceil \Gamma z^2 \rceil,$$

with fixed $\Gamma \ge 1$ *and fixed* $\varepsilon > 0$. *Suppose a nonnegative reference function* $J(k)$ *satisfies* $J(k) \ge c_L \sigma p_Z(k)/(1+z)$ *on a moderate window of multiplier* L. *On the character event* (A.4)*, the sum of the absolute class corrections of indices* $1 \le r < m$*, relative to* $J(k)$*, is at most*

$$\sum_{r=1}^{m-1} \left(C_{\Gamma,L}\varepsilon^3\right)^r. \tag{A.15}$$

The bound holds uniformly on the window for all sufficiently large N*, and the exceptional shape probability is at most* $4 \cdot 16^{-p}$.

Proof. For these scales, $ps = O(N^{2/3}) < N$, $s \le 2m < N/2$, and $2m < u$ eventually. All hypotheses of Lemma A.8 hold: $m^3 = O(N) = o(\sigma^2)$ and $zm/\sigma = O(N^{-1/2})$.

Combine (A.9), (A.6), (A.4), and (A.14). Keeping the centralizer in $|C_\nu| = (N)_s/z_\nu \le N^s/z_\nu$ gives the relative bound

$$C_L^r N^{s/2-3r/2} p^{s/2} \frac{s^{s/2}}{z_\nu} (\sqrt{r} + z)^r.$$

We used $v\sigma = N^{3/2}$; the factor $\sqrt{r}(1+z)/(\sqrt{r}+z)$ is at most $2\sqrt{r}$ and is absorbed in C_L^r. The factor $\exp(Cr^2/N)$ is uniformly bounded. Set $h = r - s/2 \ge 0$. By Lemma A.3, the bound becomes

$$\left(\frac{C_L p(\sqrt{r}+z)}{\sqrt{N}}\right)^r \left(\frac{r}{Np}\right)^h.$$

Since $r \le m = O(\Gamma p)$, the last factor is at most one eventually. Also $p = O(z^2)$ and $\sqrt{r} + z = O_\Gamma(z)$, so the first base is at most $C_{\Gamma,L} z^3/\sqrt{N} = C_{\Gamma,L}\varepsilon^3$. Absorb the at most 2^r types of index r into the same constant and sum. Lemma A.2 gives the exceptional probability. □

A.2.1. *Order of the constants.* The proof of Theorem 1.2 in Section 3.3 fixes its constants in this order: the endpoint multiplier $L = 1$ and the constant B in (3.8); then Ω for (3.10), with L_0 and the window constants for close means; then Γ with $\Gamma \log 2 > 2B + 10$; finally an absolute $\varepsilon > 0$ so small that each geometric series in (A.15), for both uses, has the required sum, which is possible because the constants in these series depend only on Γ, Ω, L_0. The close residual has an extra $\log N$ in its exponent and absorbs every fixed B_{Ω,L_0}.

With these fixed choices, all necessary growth conditions hold:

$$ps = O(N^{2/3}) < N, \qquad m = O(N^{1/3}) < u/2, \qquad m^2/N = o(1),$$
$$m^3 = O(N) = o(\sigma^2), \qquad zm/\sigma = o(1).$$

Both regimes have failure at most Ce^{-cz^2}, and $z^2 = \varepsilon^2 N^{1/3}$. Absorb the fixed factor ε^2 into c to obtain Theorem 1.2.

A.3. **Support tails.** The tails in Theorem 3.8 and Corollary 5.2 are tails of sums of independent uniform coordinates, so Hoeffding's inequality and tuple counting suffice.

Proof of Theorem 3.8. The tensor description of $M_{a,b}$ gives $T_\lambda \le D_+$ for every λ. Hence the negative-cutoff set $E_\lambda = \{k : F_{T,\lambda}(k) < F_{S,\lambda}(k)\}$ lies below D_+. If it is nonempty, its largest element k_* satisfies

$$\mathbb{P}(T_\lambda \in E_\lambda) \le F_{T,\lambda}(k_*) < F_{S,\lambda}(k_*) \le R_\lambda.$$

Also $\mathbb{P}(S_\lambda \in E_\lambda) \le R_\lambda$, and the cumulative defect at any negative cutoff is at most R_λ. These bounds prove (3.11); the case $E_\lambda = \emptyset$ is immediate. Averaging, Markov's inequality, and the two uniform bounds in Theorems 1.2 and 3.4 give (3.12)–(3.13).

For the explicit comparison, delete every source copy of S^λ whose degree is below D_+, and delete the same number of arbitrary homogeneous target copies. The remaining source degrees are at least D_+, and the target degrees at most D_+. Matching within each λ therefore preserves the filtration. The common dimension loss divided by $N!$ is $\mathbb{E}R_\Lambda = \eta_{a,b}$. □

The generating function (3.4) gives

$$\eta_{a,b} = \frac{\sum_{j=0}^{D_+-1} [q^j]([b]_q!)^a [a]_q!}{(b!)^a a!}.$$

An empty sum is zero. For $N \ge 2$, define

$$\delta = \mathbb{E}Y - D_+ = \frac{(a-2)b(b-1) - (2b-1)a(a-1)}{4},$$
$$Q_Y = \frac{ab(b-1)(2b-1) + a(a-1)(2a-1)}{6}.$$

Here Q_Y sums the squared widths of the independent uniform variables forming Y.

Proposition A.10 (Explicit scalar bounds)**.** *For $1 \le a \le b$ and $N = ab \ge 2$,*

$$\eta_{a,b} \le \exp\left(-\frac{2(\delta+1)_+^2}{Q_Y}\right), \tag{A.16}$$

$$\eta_{a,b} \le \min\left\{1, \frac{\binom{D_++N-2}{N-1}}{(b!)^a a!}\right\}. \tag{A.17}$$

Proof. The strict cutoff is $Y \le D_+ - 1$. If $\delta + 1 > 0$, Hoeffding's inequality Hoeffding (1963) gives (A.16); otherwise the bound is one. There are exactly $a(b-1) + (a-1) = N - 1$ nontrivial uniform coordinates. Each point in their product sample space has probability $1/((b!)^a a!)$. Dropping the

upper coordinate bounds leaves $\binom{D_+ + N-2}{N-1}$ nonnegative integer tuples whose sum is at most $D_+ - 1$, proving (A.17). □

Proof of Corollary 3.9. For $a \le b$, the inequality $(a-1)(2a-1) \le 2a^2 \le 3b^2 - b$ gives $Q_Y \le Nb^2/3$. For $a \ge 3$, the normalized gap is

$$\frac{\delta}{Nb} = \frac{1}{4}\left(1 - \frac{2}{a} - \frac{2a - 1 - 2/a}{b} + \frac{a-1}{b^2}\right).$$

Its derivative in $b \ge a$ is positive. At $b = 6a$ it equals $1/6 - 65/(144a) + 11/(144a^2)$, which increases for $a \ge 3$ and takes the value $2/81$ at $a = 3$. Thus $\delta \ge 2Nb/81$ in the stated range, and (A.16) gives $2\delta^2/Q_Y \ge 8N/2187$.

For the entropy estimate put $n = D_+ + N - 2$ and $k = N - 1$. The bounds $\binom{n}{k} \le (en/k)^k$ and $j! \ge (j/e)^j$ imply

$$\frac{\binom{n}{k}}{(b!)^a a!} \le \left(\frac{e^2 n}{bk}\right)^k \frac{e^{a+1}}{ba^a} \le \left(\frac{e^2 n}{bk}\right)^k,$$

where the last step uses $b \ge a \ge 3 > e$. Also

$$\frac{D_+ + N - 2}{b(N-1)} \le \frac{2D_+}{bN} + \frac{1}{b} = \frac{a}{b} + \frac{1}{a} - \frac{1}{ab}.$$

For the first inequality, clearing the positive denominators leaves $D_+(N-2) + N \ge 0$. The last two displays prove (3.14). Since $a/b + 1/a \le 2/m_*$, $\log \eta_{a,b} \le -(N-1)\log m_* + (N-1)\log(2e^2)$. If $b \asymp a^2$, then $N \asymp a^3$ and $\log m_* = (1/3)\log N + O(1)$, proving (3.15). □

The restriction $a \ge 3$ is necessary: for $a = 2$, $\delta = 1/2 - b$ and (A.16) gives no decay, while for $a = 3$, $\delta = (b^2 - 13b + 6)/4$ and the strict-cutoff Hoeffding bound is less than one precisely for integers $b \ge 13$.

Nonnegative multiplicities make the support bound at $t = 1$ valid in each t-degree.

Proof of Corollary 5.2. By (5.3), $\mathcal{T}^{(r)}(X; q, 1) = H_b[H_a(X; q)^r]$ has q-degree at most $D_r = rbc_a + c_b$. Since all bigraded Schur multiplicities are nonnegative, each target (λ, j) fiber has the same support bound. Apply the proof of Theorem 3.8 in that fiber: both its conditional sign-failure mass and its cumulative defect are bounded by its source probability of degree below D_r. Averaging with w gives the bound by η_r.

The independent-uniform representation of Y_r has M_r nontrivial coordinates and $(b!)^{ra}a!$ equiprobable outcomes. Dropping the upper coordinate bounds leaves $\binom{D_r + M_r - 1}{M_r}$ nonnegative integer tuples with sum at most $D_r - 1$, proving the counting bound. Write $M = M_r$. The binomial and factorial bounds give

$$\frac{\binom{D_r + M - 1}{M}}{(b!)^{ra}a!} \le \left[\frac{e^2(D_r + M)}{bM}\right]^M \frac{e^{ra+1}}{b^{ra-a+1}a^a}.$$

For $a \ge 3$, the last factor is at most $(e/a)^{ra+1} \le 1$. Also $M \ge N/2$ and $D_r = N(a-1)/2 + b(b-1)/2 \le (a-1)M + bM/(ra)$. Thus $(D_r + M)/(bM) \le a/b + 1/(ra)$, proving the entropy bound.

For the exponential bound, let $\delta_r = \mathbb{E}Y_r - D_r$. Directly,

$$\frac{\delta_r}{Nb} = \frac{b - 2a + 1}{4b} - \frac{2b(b-1) - a(a-1)}{4rab^2}.$$

This expression increases with $r \ge 1$. The $r = 1$ calculation in Corollary 3.9 therefore gives $\delta_r \ge 2Nb/81$ when $b \ge 6a \ge 18$. The squared-width sum satisfies $Q_{B,r} \le rQ_{B,1} \le Nb^2/3$. Hoeffding's inequality at the strict cutoff $D_r - 1$ yields

$$\eta_r \le \exp\left(-\frac{2(\delta_r + 1)^2}{Q_{B,r}}\right) \le e^{-8N/2187}.$$

Finally, if $b \asymp ra^2$ and $ra \to \infty$, then $a/b + 1/(ra) \asymp 1/(ra)$ and $(N - M_r)\log(ra) = O(N)$. The entropy bound gives $\log \eta_r \le -N \log(ra) + O(N)$. At fixed a, $N = rab \asymp r^2a^3$, so $\log(ra) = \frac{1}{2}\log N + O(1)$. □

Appendix B. Spectral realization and conditional moments

Section 4 uses the exact conditional means and variances of T_λ and S_λ, and its width and height families need variances of order b^3 uniformly in the shape. Expanding the plethysms gives neither. Instead, a Hermitian element of the group algebra realizes each grading on the irreducible spaces, and conjugation averages turn its moments into character values, which can be bounded uniformly.

B.1. **Operators on irreducible spaces.** The fake-degree polynomial of S^ν in R_n is $f^\nu(q) = \sum_{T \in \mathrm{SYT}(\nu)} q^{\mathrm{maj}(T)}$. In a unitary Wedderburn decomposition of $\mathbb{C}[S_n]$, choose a Hermitian element $Z_n \in \mathbb{C}[S_n]$ whose matrix on S^ν is diagonal in the Young basis with entries $\mathrm{maj}(T)$. It exists because $\mathbb{C}[S_n] \cong \bigoplus_\nu \mathrm{End}(S^\nu)$. Right multiplication by Z_n on the regular module has graded Frobenius character H_n, and $0 \le Z_n \le c_n I$ in every unitary representation.

Embed identical copies of Z_a in the b blocks, and embed Z_b in the subgroup permuting whole blocks. Put

$$A_{a,b} = K_{a,b} + L_{a,b}, \qquad K_{a,b} = \sum_{j=1}^{b} Z_a^{(j)}, \quad L_{a,b} = Z_b^{\mathrm{blocks}}. \tag{B.1}$$

The inner sum is invariant under block permutations, so $K_{a,b}$ and $L_{a,b}$ commute.

Proposition B.1 (Spectral realization)**.** *For every $\lambda \vdash N$,*

$$u_\lambda(q) = \mathrm{Tr}_{S^\lambda}\, q^{A_{a,b}}, \qquad v_\lambda(q) = \mathrm{Tr}_{S^\lambda}\, q^{A_{b,a}}. \tag{B.2}$$

Thus T_λ, S_λ are normalized eigenvalue distributions of explicit Hermitian elements on the same irreducible space.

Proof. Use the same graded left-module isomorphism $R_a \cong \mathbb{C}[S_a]$ in every factor, and $R_b \cong \mathbb{C}[S_b]$. The resulting tensor identification with $\mathbb{C}[W_{a,b}]$ respects outer permutations. Right multiplication by the sum of the inner Z_a's acts independently of the outer coordinate, because that sum is invariant under its permutations. Right multiplication by the outer Z_b changes only the outer coordinate. Hence their sum gives the product grading in (2.3). Induction identifies it with right multiplication by $A_{a,b}$ on $\mathbb{C}[S_N]$. In the bimodule decomposition of the regular representation, the left S^λ-multiplicity space is its dual. The right action on that dual has the same eigenvalue multiset as the chosen Hermitian matrix, proving (B.2). □

B.2. **Classical fixed-shape moments and conditional means.** Conjugation averaging reduces the conditional means to the mean and variance of the major index on a random tableau, which are classical. Let $\mathrm{ct}(\nu) = \sum_{(i,j)\in\nu}(j - i)$ be the content sum of $\nu \vdash n$, put $v_n = n(n-1)(2n+5)/72$, and let $\mathbb{E}_\nu$ and Var_ν refer to a uniformly random standard Young tableau of shape ν. The hook formula gives

$$\mathbb{E}_\nu\, \mathrm{maj} = \frac{c_n - \mathrm{ct}(\nu)}{2}, \qquad \mathrm{Var}_\nu\, \mathrm{maj} = \frac{1}{12}\left(\sum_{j=1}^{n} j^2 - \sum_{c\in\nu} h_c^2\right) \le v_n. \tag{B.3}$$

These moment identities are due to Adin–Roichman (Adin and Roichman, 2001, Propositions 3.1–3.2). They follow by taking first and second cumulant derivatives of $q^{n(\nu)}[n]_q!/\prod_{c\in\nu}[h_c]_q$ at $q = 1$. The last bound uses $\sum_c h_c^2 \ge n$. Averaging Z_n over conjugation by S_n (twirling) therefore gives

$$\mathsf{E}_n Z_n = \frac{1}{2}\left(c_n I - \sum_{i<j}(ij)\right). \tag{B.4}$$

The normalized squared trace of $Z_n - \mathsf{E}_n Z_n$ on each irreducible is at most v_n.

Proof of Proposition 4.1. Twirling preserves normalized traces, so apply (B.4) to the inner and outer groups. An inner transposition has cycle type 2, and a transposition of two blocks has cycle type 2^a. The content eigenvalue of the transposition class sum gives (4.3). □

B.3. **Exact inner, outer, and mixed moments.** To compute the variance of $A_{n,m} = K_{n,m} + L_{n,m}$ on S^λ, we must retain the conditional covariance of the inner and outer parts. Conjugation averages that preserve a block swap reduce this covariance to the two cycle types of its product with an inner transposition. Write $\tau_\lambda = d_\lambda^{-1} \operatorname{Tr}_{S^\lambda}$, and $\operatorname{Var}_\lambda$ for the variance with respect to τ_λ.

Theorem B.2 (Exact conditional variances)**.** *Let $n, m \geq 2$, $N = nm$, and fix $\lambda \vdash N$. Write $r = r_\lambda$, take c_j from the conventions and v_j from (B.3), and put*

$$\alpha_j = \frac{4}{3}\binom{j}{3}, \qquad \beta_j = \frac{3}{2}\binom{j}{4}, \qquad x = r(2), \quad y = r(2^n).$$

Let $K = K_{n,m}$ and $L = L_{n,m}$ be as in (B.1), with identical copies of Z_n on the m blocks, and put $V_K = \operatorname{Var}_\lambda K$, $V_L = \operatorname{Var}_\lambda L$, and $C_{KL} = \tau_\lambda(KL) - \tau_\lambda(K)\tau_\lambda(L)$. Then

$$\begin{aligned} V_K &= m v_n + m\alpha_n r(3) + \left(m\beta_n + \frac{m(m-1)c_n^2}{4}\right) r(2^2) - \frac{m^2 c_n^2}{4} x^2, \\ V_L &= v_m + \alpha_m r(3^n) + \beta_m r(2^{2n}) - \frac{c_m^2}{4} y^2, \\ C_{KL} &= \frac{c_n c_m}{4}\Big((m-2) r(2^{n+1}) + 2r(4, 2^{n-2}) - mxy\Big), \\ \sigma_{n,m}^2(\lambda) &:= \operatorname{Var}_\lambda A_{n,m} = V_K + V_L + 2C_{KL}. \end{aligned}$$

A term whose combinatorial coefficient is zero is omitted. In particular, there is no need to evaluate $r(3^n)$ when $m = 2$, or $r(2^{2n})$ when $m < 4$. All other displayed cycle types fit in S_N. The formulas are independent of the choices of Young bases used to define the Z_j.

Proof of Theorem B.2. Write E_j for conjugation averaging over S_j. In the central element $\mathsf{E}_j Z_j^2$, the graded regular character gives the coefficient of a permutation g as

$$\frac{1}{j!}\,(q\partial_q)^2 \kappa_j(g; q)\Big|_{q=1}.$$

Indeed, the regular-module realization of Z_j gives

$$\kappa_j(g; q) = \sum_{h \in S_j} [h^{-1} g^{-1} h] q^{Z_j} = j! [g^{-1}] \mathsf{E}_j q^{Z_j}.$$

Differentiating twice proves the coefficient identity. The inverse has the same cycle type, and the conjugation average depends only on the major-index eigenvalue multiset on each irreducible, not on its basis.

Put $Q_j(q) = [j]_q!$. Since

$$\kappa_j(\rho; q) = Q_j(q) \frac{(1-q)^{j-\ell(\rho)}}{\prod_{s \in \rho} [s]_q},$$

a permutation with $j - \ell(\rho) > 2$ contributes zero to this derivative. For cycle types 1^j, $2, 1^{j-2}$, $3, 1^{j-3}$, and $2^2, 1^{j-4}$, the second Euler derivatives over $j!$ are $c_j^2/4 + v_j$, $-c_j/2$, $2/3$, and $1/2$, respectively.

For the identity, $Q_j/j!$ is the probability generating function of a sum of independent uniforms on $\{0, \dots, i-1\}$, $1 \leq i \leq j$. Their means sum to $c_j/2$ and their variances sum to v_j. For a transposition, differentiate $Q_j(q)(1-q)/(1+q)$. For the last two classes differentiate $Q_j(q)(1-q)^2/[3]_q$ and $Q_j(q)(1-q)^2/[2]_q^2$.

Thus, with C_ρ denoting the sum of all permutations of the indicated nontrivial cycle type in S_j,

$$\mathsf{E}_j Z_j = \frac{1}{2}(c_j I - C_2),$$

$$\mathsf{E}_j Z_j^2 = \left(\frac{c_j^2}{4} + v_j\right) I - \frac{c_j}{2} C_2 + \frac{2}{3} C_3 + \frac{1}{2} C_{2^2}.$$

There are c_j transpositions, $2\binom{j}{3}$ three-cycles, and $3\binom{j}{4}$ double transpositions. Taking normalized traces gives

$$\tau(Z_n^2) = \frac{c_n^2}{4} + v_n - \frac{c_n^2}{2} x + \alpha_n r(3) + \beta_n r(2^2).$$

Independent averaging over distinct inner blocks i, j, whose supports are disjoint, gives

$$\tau(Z_n^{(i)} Z_n^{(j)}) = \frac{c_n^2}{4}(1 - 2x + r(2^2)).$$

Summing the m diagonal terms and $m(m-1)$ ordered off-diagonal terms, then subtracting $\tau(K)^2 = m^2 c_n^2 (1-x)^2/4$, proves the formula for V_K. Applying the same one-block second-moment identity to the outer S_m multiplies each cycle length by n, proving the formula for V_L.

For the covariance, K is invariant under the block permutation group. Consequently $\tau(KL) = \tau(K\mathsf{E}_m L)$. Let B_{uv} exchange blocks u, v. To average $Z_n^{(i)}$ while preserving B_{uv}, use arbitrary conjugation in block i if $i \notin \{u, v\}$, and simultaneous identical conjugation in blocks u, v otherwise. An inner transposition multiplied by B_{uv} has cycle type

$$2^{n+1} \quad (i \notin \{u, v\}), \qquad (4, 2^{n-2}) \quad (i \in \{u, v\}).$$

It follows that

$$\tau(KB_{uv}) = \frac{mc_n}{2} y - \frac{c_n}{2}((m-2) r(2^{n+1}) + 2r(4, 2^{n-2})).$$

Use $\mathsf{E}_m L = (c_m I - \sum_{u<v} B_{uv})/2$, sum over the c_m block swaps, and subtract $\tau(K)\tau(L) = mc_n c_m (1-x)(1-y)/4$ to obtain C_{KL}. All reductions used conjugation averages and cycle types, establishing the stated independence from the bases.

□

B.4. **Uniform bounds.** For fixed a, the operator ranges give only $O(b^4)$ variance bounds. Twirling and approximate factorization of normalized characters on disjoint supports reduce this to $O(b^3)$, uniformly over all shapes, as needed in Section 4.

Lemma B.3 (Disjoint-support factorization)**.** *Suppose g, h move u, v points and their disjoint union fits in S_N. Then, for every irreducible normalized character,*

$$|r(g \sqcup h) - r(g) r(h)| \le \frac{2uv}{N}. \tag{B.5}$$

Consequently, for $j \le N/2$,

$$|r_j - x^j| \le \frac{4j(j-1)}{N}. \tag{B.6}$$

Proof. Average h over all conjugates. By Schur's lemma, its average matrix is $r(h)I$. A random conjugate has support meeting that of g with probability at most uv/N, by the union bound. Off that event the product has the fixed disjoint-union class. Normalized characters have absolute value at most one, proving (B.5). Apply the estimate successively to $j-1$ disjoint transpositions and one further transposition. The errors sum to $\sum_{h=1}^{j-1} 8h/N = 4j(j-1)/N$. □

For inner size n and outer count m, define

$$I(n,m) = m\left(v_n + \frac{c_n(2n-3)}{4} + \frac{2c_n^2}{n}\right),$$
$$O(n,m) = v_m + \frac{c_m(2m-3)}{4} + \frac{2nc_m^2}{m}.$$

Theorem B.4 (A conditional variance bound without a shape cutoff)**.** *For $n, m \geq 2$, in every irreducible representation of S_{nm},*

$$\operatorname{Var} A_{n,m} \leq \left(\sqrt{I(n,m)} + \min\left\{\sqrt{O(n,m)}, \frac{c_m}{2}\right\}\right)^2. \tag{B.7}$$

In particular, for $b \geq a \geq 2$,

$$\operatorname{Var} T_\lambda, \operatorname{Var} S_\lambda \leq 4a^3b^3. \tag{B.8}$$

Proof of Theorem B.4. Work in a fixed irreducible representation with its normalized trace. Twirling is orthogonal projection for the Hilbert–Schmidt inner product.

Independent twirling over the m inner groups sends $K = \sum_{j=1}^m Z_n^{(j)}$ to $K_0 = \frac{1}{2}(mc_nI - C_E)$, where C_E is the sum of transpositions in the m disjoint complete graphs on n vertices. Write $Z_n^{(j)} = \bar{Z}_j + D_j$, where $\bar{Z}_j$ is the twirl of $Z_n^{(j)}$ and the twirl of D_j is zero. Independent twirling makes all cross terms D_iD_j, $i \neq j$, vanish in trace. Equation (B.3) bounds the remaining squared traces by mv_n. Orthogonality to the twirled part gives $\operatorname{Var} K \leq mv_n + \frac{1}{4}\operatorname{Var} C_E$. There are $e = mc_n$ edges, $o = mn(n-1)(n-2)$ ordered pairs of distinct edges sharing one endpoint, and $d = e(e-1) - o$ disjoint ordered pairs. Consequently

$$\operatorname{Var} C_E = e(1-x^2) + o(r(3) - x^2) + d(r(2^2) - x^2).$$

The first two parentheses are at most one. The third is at most $8/(nm)$ by Lemma B.3. Thus

$$\operatorname{Var} C_E \leq e + o + \frac{8e^2}{nm} = mc_n(2n-3) + \frac{8mc_n^2}{n}, \qquad \text{so} \qquad \operatorname{Var} K \leq I(n,m).$$

For the outer operator $L = Z_m^{\text{blocks}}$, twirl over the block group. Its noncentral squared trace is at most v_m. Its central part is $(c_mI - C_B)/2$, where C_B sums all swaps of two blocks. There are $e = c_m$ swaps and $o = m(m-1)(m-2)$ ordered overlapping pairs. Disjoint swaps move $2n$ points each, so their normalized-character covariance is at most $8n^2/(nm) = 8n/m$. Therefore

$$\operatorname{Var} L \leq v_m + \frac{1}{4}\left\{c_m(2m-3) + \frac{8nc_m^2}{m}\right\} = O(n,m).$$

If $m < 4$ there are no disjoint pairs, and the same upper bound follows by retaining the positive extra term. The range $0 \leq L \leq c_mI$ also gives $\operatorname{Var} L \leq c_m^2/4$. The triangle inequality for the centered Hilbert–Schmidt norm proves (B.7).

For the coarse estimate, $v_n \leq n^3/24$ and $c_n \leq n^2/2$ imply $I(n,m) \leq mn^3$ and $O(n,m) \leq nm^3$. Hence $\operatorname{Var} A_{n,m} \leq 2mn^3 + 2nm^3$. Applying this bound to $(n,m) = (a,b)$ and (b,a) gives (B.8). □

Appendix C. Support calculations and explicit shape families

The thresholds and intervals stated in Section 4 come with explicit constants, and those constants come from bounding the two margins in the support–moment criterion by hand on each shape family, which is what this appendix does.

C.1. **Support bounds.** The lower bounds and the first upper bounds come from the plethystic alphabet and coinvariant reciprocity; fusion and a one-letter expansion then give the sharper upper bounds of Corollary 4.3.

Proof of Theorem 4.2. Consider $Q_{a,b} = H_a[H_b]$. Since the outer coinvariant degrees are nonnegative, no constituent occurs earlier than in H_b^a: indeed, $H_a[H_b] = \sum_\nu f^\nu(q) s_\nu[H_b]$, whereas H_b^a contains every $s_\nu[H_b]$ with positive multiplicity in degree shift zero. Here $f^\nu(q)$ is the fake-degree polynomial of Section B.1.

Replacing H_b by $h_b[X/(1-q)] = H_b/(q;q)_b$ introduces only additional nonnegative shifts. The product $h_b[X/(1-q)]^a$ is a multidegree piece of $\mathrm{Sym}(V \otimes \mathbb{C}^a \otimes \mathbb{C}[z])$. By the Cauchy identity, the earliest possible $S^\lambda V$ occurs at the minimum degree of $s_\lambda[1^a/(1-q)]$. The ordered alphabet has a letters in degree zero, a in degree one, and so on. Column strictness forces an entry in row i to be at least the i-th letter. Filling each row with that letter attains $\eta_a(\lambda)$, proving the lower source bound. Interchange a, b for the target.

Coinvariant reciprocity is $q^{c_n} H_n(q^{-1}) = \omega H_n$, where ω is the involution sending s_λ to $s_{\lambda'}$. For a homogeneous inner character g of symmetric-function degree a, the power-sum signs give $(\omega f)[\omega g] = \omega[(\omega^{a+1} f)[g]]$. Applying these identities to both layers gives

$$q^{D_+} P_{a,b}(q^{-1}) = \omega \left[(\omega^{a+1} H_b)[H_a] \right].$$

The outer character $\omega^{a+1} H_b$ still has nonnegative degrees, so the lower-bound argument applied to its λ'-coefficient gives the target upper bound; interchanging a, b gives the other one. □

Proof of Corollary 4.3. The fusion theorem (Sra, 2026, Theorem 1.2) makes both $(P_{a,b} - H_N)/(1-q)$ and $(Q_{a,b} - H_N)/(1-q)$ Schur positive. Thus the cumulative functions of T_λ, S_λ dominate that of the normalized fake-degree polynomial $[s_\lambda] H_N / d_\lambda$. By the q-hook formula, or coinvariant reciprocity, its largest degree is $c_N - n(\lambda')$, which also bounds both conditional supports.

For the depth bound, put $H_0 = 1$; the one-letter expansion is

$$H_n[x + Y] = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q x^{n-j} H_j[Y].$$

It follows by expanding $h_n[(x+Y)/(1-q)]$ and using $h_i[x/(1-q)] = x^i/(q;q)_i$. The coefficient of Y-degree j has q-degree at most $j(n-j) + c_j \le j(n-1)$. Write $H_a[x+Y] = x^a + Z$. Every term of Z has positive Y-degree, and its part of Y-degree j has q-degree at most $j(a-1)$. Expanding the outer factor gives

$$H_b[x^a + Z] = \sum_{m=0}^{b} \begin{bmatrix} b \\ m \end{bmatrix}_q x^{a(b-m)} H_m[Z].$$

Only $m \le r$ contributes to Y-degree r. In $H_m[Z]$, the outer grading contributes at most c_m, and the inner grading at most $r(a-1)$: every Adams operation multiplies the inner q-degree and Y-degree by the same integer. Including the Gaussian factor bounds the total q-degree by $m(b-m) + c_m + r(a-1) \le r(a+b-2)$. If $\lambda = (N-r, \nu)$, then $s_\lambda[x+Y]$ contains $x^{N-r} s_\nu[Y]$ with coefficient one. Schur positivity and branching therefore transfer this bound to $[s_\lambda] P_{a,b}$. Interchanging a, b gives it for $Q_{a,b}$. When $r \le a \le b$, maximizing $mb - m(m+1)/2$ over $m \le r$ gives $r(a+b-2) - c_r$ in both orientations. Once both cumulative functions reach one their difference is zero, proving the cutoff assertion. □

Lemma C.1 (Shape estimates)**.** *Let $\lambda \vdash N$.*
(1) If $\lambda_1 \le w$, then

$$\eta_m(\lambda) \ge \frac{N^2}{2mw} - \frac{N}{2}, \qquad x_\lambda \le \frac{w - N/w}{N-1}.$$

(2) If $\ell(\lambda) \geq l$, then

$$\eta_m(\lambda) \geq \sum_{i=0}^{l-1} \lfloor i/m \rfloor, \qquad x_\lambda \leq 1 - \frac{2(l-1)}{N-1}.$$

(3) For all λ,

$$\frac{n(\lambda)}{m} - \frac{(m-1)N}{m} \leq \eta_m(\lambda) \leq \frac{n(\lambda)}{m}.$$

Proof. Under a width cap, each level of m consecutive rows, which share one weight in (4.4), holds at most mw boxes. Filling the levels in increasing order minimizes $\eta_m(\lambda)$ under these capacities, so $\eta_m(\lambda) \geq N^2/(2mw) - N/2$. Indeed, if $N = mwr + s$ with $0 \leq s < mw$, the exact minimum under the capacities is $mw\, r(r-1)/2 + rs$, which exceeds that bound by $s(mw-s)/(2mw)$. Also $\sum_i \lambda_i^2 \leq wN$ and $\sum_j (\lambda'_j)^2 \geq N^2/w$; substitute in (4.3). The height lower bound follows by placing one box in each row. Among partitions of height at least l, the hook $(N-l+1, 1^{l-1})$ maximizes the content sum, because moving a box to a higher row increases that sum and the hook dominates all such partitions. Its transposition ratio is $1 - 2(l-1)/(N-1)$. The last pair of inequalities follows from $(i-1)/m - (m-1)/m \leq \lfloor (i-1)/m \rfloor \leq (i-1)/m$. □

C.2. **Width and height thresholds.** Both families satisfy the support–moment criterion (4.6) once b is large; the two proofs differ only in how they bound the margins.

Proof of Theorem 1.3. Let $\gamma = N^{-2} \sum_j (\lambda'_j)^2 \in [1/w, 1]$. The content formula gives

$$0 \leq x_\lambda + \gamma = \frac{2n(\lambda') + N(1-\gamma)}{N(N-1)} \leq \frac{w}{N-1}. \tag{C.1}$$

Since $|x|, \gamma \leq 1$, Lemma B.3 implies $|r_a - (-\gamma)^a| \leq (4(a-1) + 2w)/b$. Define

$$t(\gamma) = \frac{1 - (-\gamma)^a}{4}, \qquad s(\gamma) = \frac{a(1+\gamma)}{4}.$$

The exact means give

$$\mu_T \leq t(\gamma) b^2 + (c_a + a - 1 + w/2) b,$$
$$\mu_S \geq s(\gamma) b^2 - \frac{a+w}{2} b.$$

The second inequality uses $r_b \leq 1$ and (C.1). Since $2n(\lambda) = N^2\gamma - N$, Lemma C.1 gives

$$L_\lambda \geq \frac{a\gamma}{2} b^2 - (a - \tfrac{1}{2}) b, \qquad U_\lambda \leq \frac{1}{2} b^2 + (c_a - \tfrac{1}{2}) b.$$

Therefore

$$L_\lambda - \mu_T \geq \frac{2a\gamma - 1 + (-\gamma)^a}{4} b^2 - (c_a + 2a + w + 1) b,$$
$$\mu_S - U_\lambda \geq \frac{a(1+\gamma) - 2}{4} b^2 - (c_a + a + w + 1) b.$$

The leading-coefficient functions are increasing on $[0,1]$. Under (4.7), the two gaps are at least $gb^2/2$ and $hb^2/2$. If $L_\lambda \geq U_\lambda$, use support separation. Otherwise each endpoint product in (4.6) is at least $ghb^4/4$, and Theorem B.4 bounds $\sigma_T \sigma_S$ by $4a^3b^3$, so the last requirement in (4.7) proves the result.

For odd a and $w \leq 2a - 1$, $2a/w - 1 - w^{-a} \geq w^{-1} - w^{-a} > 0$ when $w > 1$; at $w = 1$ it is $2a - 2 > 0$. For even a, the boundary $w = 2a$ has $4g = (2a)^{-a} > 0$. The second coefficient h is positive in all these cases. □

Lemma C.1 bounds the source support and mean in terms of height, without restricting width.

Proof of Corollary 4.6. Put $l = \lceil hb \rceil$ and suppose $\ell(\lambda) \geq l$. Lemma C.1 gives

$$L_\lambda \geq \frac{l(l-1)}{2a} - \frac{(a-1)l}{a} = \frac{h^2}{2a}b^2 - O_{a,h}(b).$$

If a is even, (B.6) gives $r_a \geq x_\lambda^a - 4(a-1)/b \geq -4(a-1)/b$. The target mean therefore satisfies $\mu_T \leq b^2/4 + O_a(b)$. For odd a, the bound $r_a \geq -1$ instead gives $\mu_T \leq b^2/2 + O_a(b)$. For the source mean, $r_b \leq 1$ and the height estimate for x_λ give

$$\mu_S \geq \frac{ab(b-1)(l-1)}{2(ab-1)} \geq \frac{h}{2}b^2 - O_{a,h}(b).$$

Also $U_\lambda \leq D_+ = b^2/2 + O_a(b)$. Thus $\mu_S - U_\lambda \geq \frac{h-1}{2}b^2 - O_{a,h}(b)$, while the leading coefficient of $L_\lambda - \mu_T$ is at least $h^2/(2a) - 1/4$ for even a and $h^2/(2a) - 1/2$ for odd a. Both gaps are positive of order b^2 under the stated hypotheses. If $L_\lambda \geq U_\lambda$, use support separation. Otherwise each endpoint product in (4.6) is at least $(L_\lambda - \mu_T)(\mu_S - U_\lambda)$, of order b^4 uniformly over the shapes. Since $\sigma_T\sigma_S \leq 4a^3b^3$, Theorem 4.5 applies for all sufficiently large b. □

The same argument can use the column sizes directly. For fixed a, let $\lambda_b \vdash ab$ and $\gamma_b = (ab)^{-2}\sum_j (\lambda_b')_j^2$. The shape and mean bounds give

$$\frac{L_{\lambda_b} - \mu_T}{b^2} \geq \frac{2a\gamma_b - 1 + x_{\lambda_b}^a}{4} - O_a(b^{-1}), \qquad \frac{\mu_S - U_{\lambda_b}}{b^2} \geq \frac{a(1 - x_{\lambda_b}) - 2}{4} - O_a(b^{-1}).$$

Consequently, positive lower limits of both displayed leading terms suffice for complete positivity, without a width or height cutoff.

C.3. **Sixth-power families.** At $a = 6$ the uniform variance bound of Theorem B.4 is too crude for the thresholds in Theorem C.3, so we first sharpen it.

Corollary C.2 (Variance bounds for the sixth power)**.** *For $a = 6$, the sharper bounds*

$$\sqrt{\operatorname{Var} T_\lambda} \leq B_T(b) := \sqrt{\frac{695}{6}}b + \sqrt{\frac{59}{18}}b^3, \tag{C.2}$$

$$\sqrt{\operatorname{Var} S_\lambda} \leq B_S(b) := \sqrt{\frac{14}{3}b^3} + \frac{15}{2} \tag{C.3}$$

hold for every $\lambda \vdash 6b$. For $b \geq 32$, each variance is at most $5b^3$.

Proof. For $n = 6$, direct simplification gives $I(6,b) = \frac{695}{6}b$, $O(6,b) \leq \frac{59}{18}b^3$, and $I(b,6) \leq (14/3)b^3$. Use the range bound $c_6/2 = 15/2$ for the source outer term. Then (B.7) gives (C.2)–(C.3). For $b \geq 32$, $B_T(b)/b^{3/2} < 11/32 + 11/6 = 209/96 < \sqrt{5}$, while $B_S(b)/b^{3/2} < 13/6 + 1/24 = 53/24 < \sqrt{5}$. The inequality $15/(2b^{3/2}) < 1/24$ follows from $32^3 > 180^2$. □

For $a = 6$, define $M_b = (b^2 + 79b - 20)/4$ and $U_b = (b^2 + 29b)/2$. The exact means and (B.6) imply $\mu_T \leq M_b$ and $T_\lambda \leq U_b$. Indeed $r_6 \geq x^6 - 20/b \geq -20/b$ and $1 - x \leq 2$. With Corollary C.2, these bounds give smaller thresholds than (4.7).

Theorem C.3 (The explicit ranges)**.** *For $\lambda \vdash 6b$, each condition below implies $[s_\lambda]F_{6,b} \in \mathbb{N}[q]$:*

shape condition	*sufficient range*
$\lambda_1 \leq 6$	$b \geq 120$
$\lambda_1 \leq 11$	$b \geq 1200$
$\ell(\lambda) \geq 2b + 1$	$b \geq 350$
$\ell(\lambda) \geq \lceil 7b/4 \rceil$	$b \geq 10240.$

Proof. For width at most w, Lemma C.1 gives $S_\lambda \ge L_b := 3b^2/w - 3b$. When $6b \ge w^2$, the same lemma gives $x \le 0$, so $\mu_S \ge m_b := 3b(b-1)/2$. For an overlapping support interval, the endpoint products in Theorem 4.5 are at least $(L_b - M_b)(m_b - L_b)$ and $(U_b - M_b)(m_b - U_b)$.

If $w = 6$ and $b \ge 120$, then $L_b - M_b = (b^2 - 91b + 20)/4 \ge 29b/4$ and $m_b - L_b = b^2 + 3b/2$, whose product exceeds $5b^3$. Also $U_b - M_b \ge b^2/8$ and $m_b - U_b \ge b^2/2$, so the other product is at least $b^4/16 > 5b^3$.

If $w = 11$ and $b \ge 1200$, then $L_b - M_b = (b^2 - 1001b + 220)/44 \ge 199b/44$ and $m_b - L_b \ge 27b^2/22$, whose product is at least $(5373/968)b^3 > 5b^3$. The same upper-endpoint estimate applies. The support–moment theorem and (C.2)–(C.3) prove both width assertions.

Now suppose $\ell(\lambda) \ge 2b + 1$. Then $S_\lambda \ge L_b := (4b^2 - 8b - 5)/12$ by summing $\lfloor i/6 \rfloor \ge (i-5)/6$ for $0 \le i \le 2b$. Also $x \le 1 - 4b/(6b-1) < 1/3$, hence $\mu_S \ge m_b := b(b-1)$. For $b \ge 350$, $L_b - M_b = (b^2 - 245b + 55)/12 \ge 35b/4$ and $m_b - L_b \ge 3b^2/5$, whose product is at least $(21/4)b^3 > 5b^3$. At the other endpoint, $U_b - M_b \ge b^2/8$ and $m_b - U_b \ge b^2/4$. Their product is at least $b^4/32 > 5b^3$. Theorem 4.5 applies again, and disjoint relaxed supports give the result directly.

Finally suppose $\ell(\lambda) \ge \lceil 7b/4 \rceil$. The height estimate in Lemma C.1 and $\mu_T \le M_b \le b^2/4 + 20b$ give $L_\lambda - \mu_T \ge b^2/192 - 1037b/48 \ge b^2/384$ for $b \ge 8296$. The source mean is at least $(b-1)(7b/4 - 1)/2$, so $\mu_S - U_\lambda \ge 3b^2/8 - 127b/8 \ge 3b^2/16$ for $b \ge 85$. If the supports overlap, both endpoint products are at least $b^4/2048 \ge 5b^3$ for $b \ge 10240$. Corollary C.2 and Theorem 4.5 prove the last assertion; disjoint supports again give it directly. □

For $a = 6$, Chebyshev's inequality and the exact means give an effective form of Theorem 4.7.

Corollary C.4 (A uniform sixth-power interval)**.** *Let $0 < \varepsilon \le 1/2$, $b \ge 1280\varepsilon^{-4}$, and $\lambda \vdash 6b$ satisfy $\lambda_1 \le (1-\varepsilon)6b$. Every integer k with $\mu_T + \varepsilon^2 b^2/8 \le k \le \mu_T + \varepsilon^2 b^2/4$ satisfies $[q^k s_\lambda]F_{6,b} \ge d_\lambda/2$.*

Proof. The elementary estimate $x \le (\lambda_1 - 1)/(N-1)$ implies $1 - x \ge \varepsilon$. From the exact means and $|r_6 - x^6| \le 20/b$, $\mu_S - \mu_T \ge \frac{3}{4}b(b-1)(1-x)^2 - 20b \ge \frac{1}{2}\varepsilon^2 b^2$ in the stated range. Here $x^6 - 6x + 5 = (1-x)^2[3 + (1+x)^2(x^2+2)]$. Chebyshev's inequality in (2.5), with the variance bound $8b^3$, gives a total error at most $640/(\varepsilon^4 b) \le 1/2$, which proves the claim. □

C.4. **Hooks of growing depth.** Hook characters have the closed form below, so for hooks the exact variances of Theorem B.2 can be evaluated and sharpen the constant in the order-b margins of Corollary 4.8. The exterior powers of the standard representation give the hook characters:

$$\sum_{r=0}^{N-1} \chi^{(N-r,1^r)}(g) z^r = \frac{\prod_{C \in \mathrm{Cyc}(g)} (1 - (-z)^{|C|})}{1+z}. \tag{C.4}$$

Theorem C.5 (Hook intervals from exact variances)**.** *Fix $a \ge 2$, let $r_b/(ab) \to \rho \in (0,1)$, and set $\lambda_b = (ab - r_b, 1^{r_b})$. Write*

$$x = 1 - 2\rho, \qquad z = \rho(1-\rho), \qquad G = \frac{a(1-x) - 1 + x^a}{4} > 0,$$

$$C_a(\rho) = \frac{1}{36} + \frac{2}{9}(1-3z)^a - \frac{x^{2a}}{4} - \frac{azx^{2a-2}}{4}.$$

Then the exact conditional variances satisfy

$$\frac{\operatorname{Var} T_{\lambda_b}}{b^3} \longrightarrow C_a(\rho), \qquad \frac{\operatorname{Var} S_{\lambda_b}}{b^3} \longrightarrow \frac{az}{12}.$$

Consequently, for every constant $c > \sqrt{C_a(\rho)az/12}/G$, the degree $k_b = \lceil \mu_T(\lambda_b) + cb \rceil$ has $[q^{k_b} s_{\lambda_b}]F_{a,b} > 0$ for all sufficiently large b. The same conclusion holds for every integer degree between this lower endpoint and $\lfloor \mu_S(\lambda_b) - cb \rfloor$.

Proof. Put $M = ab - 1$, $\rho_M = r_b/M$, $x_M = 1 - 2\rho_M$, and $z_M = \rho_M(1 - \rho_M)$. The hook character at j disjoint transpositions is

$$r(2^j) = \sum_{h=0}^{j} (-2)^h \binom{j}{h} \frac{(r_b)_{\underline{h}}}{(M)_{\underline{h}}} = x_M^j - \frac{2j(j-1) z_M x_M^{j-2}}{M} + O(M^{-2})$$

for fixed $j \geq 2$; for $j = 1$ it is exactly x_M. The error is uniform when ρ_M stays in a compact subinterval of $(0, 1)$. Expand the falling-factorial quotients to first order in $1/M$ and differentiate $(1 - 2\rho_M)^j$ twice. The hook determinant formula also gives $r(3^a) \to (1 - 3z)^a$, by selecting the finitely many nontrivial eigenvalues of the a three-cycles. The expansion of $r(2^j)$ at $j = a$ and $j = 2a$ gives

$$r(2^{2a}) - r(2^a)^2 = -\frac{4a^2 z_M x_M^{2a-2}}{M} + O(M^{-2}).$$

In the exact outer variance, use $\beta_b - c_b^2/4 = -b(b-1)(2b-3)/8$ and group the last two terms as

$$V_L = v_b + \alpha_b r(3^a) + (\beta_b - c_b^2/4) r(2^{2a}) + \frac{c_b^2}{4}(r(2^{2a}) - r(2^a)^2).$$

Dividing by b^3 gives $C_a(\rho)$. For the target, $V_K = O_a(b)$ by the formula for V_K and $r(2^2) - x_M^2 = O(b^{-1})$. Cauchy–Schwarz gives $C_{KL} = O_a(b^2)$, so these two terms do not affect the limit.

For the source, the formula for V_K can be regrouped as

$$V_K = a v_b + a\alpha_b r(3) + a(\beta_b - c_b^2/4) r(2^2) + \frac{a^2 c_b^2}{4}(r(2^2) - x_M^2).$$

Here $r(3) \to 1 - 3z$ and the last difference is exactly $-4z_M/(M-1)$. The coefficient of b^3 is

$$a\left(\frac{1}{36} + \frac{2}{9}(1 - 3z) - \frac{1}{4}x^2\right) - \frac{az}{4} = \frac{az}{12}.$$

The source outer operator has range $[0, c_a]$, hence bounded variance; its covariance with the inner sum is $O_a(b^{3/2})$, proving the second variance limit. Proposition 4.1 gives $(\mu_S - \mu_T)/b^2 \to G$. At the stated two endpoints the lattice margin product divided by b^3 tends to cG, whereas the exact standard-deviation product divided by b^3 tends to $\sqrt{C_a(\rho) az/12}$. The strict inequality in the hypothesis proves eventual coefficient positivity.

Concavity of the lattice margin product extends the endpoint estimates to the full integer interval. □

Corollary C.6. *For all sufficiently large b, every integer k satisfying*

$$\lceil \mu_T + b/5 \rceil \leq k \leq \lfloor \mu_S - b/5 \rfloor$$

has $[q^k s_{(5b,1^b)}] F_{6,b} > 0$.

Proof. At $a = 6$ and $\rho = 1/6$, the constants in Theorem C.5 are $C_6(1/6) = 33733345/1088391168$, $a\rho(1-\rho)/12 = 5/72$, and $G = 793/2916$. The choice $c = 1/5$ satisfies the strict hypothesis because $1/25 - C_6(1/6)(5/72)/G^2 = 1578804259/144886809600 > 0$. □

Exact variances can certify coefficients when the supports overlap.

Example C.7. *For $b = 1200$, $\lambda = (6000, 1^{1200})$, and $k = 331423$, (C.4) and Theorem B.2 give the rational bounds*

$$\mu_T < 331217.24, \qquad \mu_S > 719507.43, \qquad \sigma_{6,b}^2 < 53592619, \qquad \sigma_{6,6}^2 < 119979996.$$

These bounds imply the strict inequality in (4.5); hence $[q^{331423} s_{(6000,1^{1200})}] F_{6,1200} > 0$. The support overlap is $[119600, 725400)$, so support separation alone does not settle this coefficient.