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THE q-FOULKES PROBLEM I: A FUSION INEQUALITY FOR MACDONALD PLETHYSMS

SUVRIT SRA*

ABSTRACT. We prove that the plethysm $H_m[\widetilde{H}_\mu]$ of a Macdonald polynomial by the Hall–Littlewood polynomial H_m lies above the Macdonald polynomial $\widetilde{H}_{m\mu}$ obtained by fusing m copies of $\widetilde{H}_\mu$. More specifically, the quotient $(H_m[\widetilde{H}_\mu] - \widetilde{H}_{m\mu})/(1-q^m)$ is Schur positive with coefficients in $\mathbb{N}[q,t]$, and it is the character of an explicit module that we call the *fusion defect*. For one-row shapes the inequality places both q-Foulkes plethysms $H_b[H_a]$ and $H_a[H_b]$ above the same polynomial H_{ab}, and comparing their two defects we obtain a distorted version of Bergeron's q-analog of Foulkes' conjecture, in which the subtracted plethysm is shifted by $q^{a(a-1)\lfloor b/2\rfloor}$.

1. INTRODUCTION

We study two simple modifications of the q-analog of Foulkes' conjecture proposed by Bergeron (2017). Recall that Foulkes's conjecture (1950) states that, for $a \le b$, the difference of plethysms $h_b[h_a] - h_a[h_b]$, of complete homogeneous symmetric functions is Schur positive, i.e., it expands with nonnegative integer coefficients in the Schur functions s_λ, $\lambda \vdash ab$. This positivity is known for $a \le 6$ (Thrall, 1942; Dent and Siemons, 2000; McKay, 2008; Cheung et al., 2017; OpenAI, 2026) and for b large with respect to a (Brion, 1993), but it remains open in general.

Bergeron's q-analog replaces h_n by the (modified) Hall–Littlewood polynomial

$$H_n(X;q) := (q;q)_n h_n\left[\tfrac{X}{1-q}\right], \quad \text{where} \quad (q;q)_n := \prod_{j=1}^{n}(1-q^j), \tag{1.1}$$

and his analog of Foulkes' conjecture is the following:

Conjecture 1.1 (Bergeron, 2017)**.** *For all integers $a \le b$, the divided difference*

$$F_{a,b} := \frac{H_b[H_a] - H_a[H_b]}{1-q} \tag{1.2}$$

is Schur positive with coefficients in $\mathbb{N}[q]$.

Since $H_n(X;1) = h_1^n$, both plethysms in (1.2) equal h_1^{ab} at $q=1$, so the numerator of $F_{a,b}$ is divisible by $1-q$; since $H_n(X;0) = h_n$, the conjecture at $q=0$ reduces to Foulkes' conjecture. Bergeron proved that Conjecture 1.1 holds for $q=1$ (Bergeron, 2017, Section 3).

Conjecture 1.1 compares degrees. To see this, fix a partition $\lambda \vdash ab$, and let d_λ be the coefficient of s_λ in h_1^{ab}. Since both plethysms equal h_1^{ab} at $q=1$, the coefficient of s_λ in $H_b[H_a]$ is a polynomial $q^{\alpha_1} + \cdots + q^{\alpha_{d_\lambda}}$ with $\alpha_1 \le \cdots \le \alpha_{d_\lambda}$; in $H_a[H_b]$ it is $q^{\beta_1} + \cdots + q^{\beta_{d_\lambda}}$ with $\beta_1 \le \cdots \le \beta_{d_\lambda}$. Dividing by $1-q$ takes partial sums, so the coefficient of $q^k s_\lambda$ in $F_{a,b}$ equals the difference of cardinalities $|\{i : \alpha_i \le k\}| - |\{i : \beta_i \le k\}|$. Thus Conjecture 1.1 asks that $|\{i : \alpha_i \le k\}| \ge |\{i : \beta_i \le k\}|$ for every k, which, since both lists are sorted, says exactly that $\alpha_i \le \beta_i$ for every λ and i.

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* TU Munich, Dept. of Mathematics, School of CIT; Garching, Germany. Email: s.sra@tum.de.

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Our first modification compares each of the two plethysms in (1.2) with the polynomial obtained by fusing its copies of the inner factor. The second modification shifts the subtracted plethysm by q^s, which amounts to asking only for $\alpha_i \le \beta_i + s$ ($s = 0$ is Bergeron's conjecture).

For the first modification, let $\widetilde{H}_\mu(X;q,t)$ be the modified Macdonald polynomial of a partition μ, the bigraded character of its Garsia–Haiman module, so that $\widetilde{H}_{(n)}(X;q,t) = H_n(X;q)$ (Section 1.1). Placing m copies of a current-algebra module with character $\widetilde{H}_\mu$ at distinct points of a line and letting the points collide gives the fusion product of Feigin and Loktev (1999), named after the fusion of fields in conformal field theory. Its character is the fused polynomial $\widetilde{H}_{m\mu}$, the Macdonald polynomial of μ with every row multiplied by m (Khoroshkin and Makedonskyi, 2026). We show that it lies below the plethysm $H_m[\widetilde{H}_\mu]$, and call the difference the *fusion defect*.

Theorem 1.2 (Fusion defect)**.** *For every partition μ and $m \in \mathbb{N}$, with $m\mu = (m\mu_1, m\mu_2, \ldots)$,*

$$\frac{H_m[\widetilde{H}_\mu] - \widetilde{H}_{m\mu}}{1-q^m} \succeq_s 0 \qquad \text{over } \mathbb{N}[q,t],$$

and the quotient is the bigraded character of an explicit module; $\succeq_s 0$ denotes Schur positivity.

The proof builds both characters on the tensor product of the m copies. Fusion lets the current algebra act on it, and the plethysm arises when every endomorphism of a single copy is allowed to act as well (Kedem, 2004). More operators reach more vectors in each degree, and the extra vectors, degree by degree, span a module whose character is the fusion defect divided by $1-q$ (Section 2).

For the one-row shapes $\mu = (a)$, $m = b$ and $\mu = (b)$, $m = a$, Theorem 1.2 places both $H_b[H_a]$ and $H_a[H_b]$ above H_{ab}, and Conjecture 1.1 says that the fusion defect of $H_b[H_a]$ is at least that of $H_a[H_b]$. To compare the two plethysms with each other, we turn the inequality for $H_a[H_b]$ around, using a symmetry of the modified Macdonald polynomials that reverses degrees (Haiman, 2001). This costs a shift of degrees, which we determine exactly (Theorem 2.1). Against H_{ab} the shift is quadratic in b, but it becomes linear in b if H_a and H_b are graded by separate variables q and t, so that both plethysms lie above the Macdonald polynomial of the rectangle (a^b) (Theorem 3.2). Specializing $t = q^c$ afterwards, and writing $H_n(z) = H_n(X;z)$, we obtain the following shifted form of Conjecture 1.1, which is our second main result.

Theorem 1.3 (q-distorted Foulkes comparison)**.** *For all integers $a, b \ge 2$ and $c \ge 1$,*

$$\frac{H_b(q^c)[H_a(q)] - q^{a(a-1)\lfloor b/2\rfloor} H_a(q)[H_b(q^c)]}{1-q} \succeq_s 0 \qquad \text{over } \mathbb{N}[q].$$

In particular, $(H_b[H_a] - q^{a(a-1)\lfloor b/2\rfloor} H_a[H_b])/(1-q) \succeq_s 0$.

Bergeron's rectangular version (Bergeron, 2017, Conjecture 4) replaces the inner H_a and H_b by the modified Macdonald polynomials $\widetilde{H}_{(a^k)}$ and $\widetilde{H}_{(b^k)}$ of rectangles with k rows. Applying Theorem 1.3 to the column character $H_k(X;t)$ proves it up to a shift linear in b for fixed a and k (Theorem 3.5). The fusion defects of two inner shapes comparable in dominance are ordered once m is large, after the specialization $t = q^c$ (Theorem B.1 and Corollary B.3); the proof pulls back along $x \mapsto x^m$ the family of Guo et al. (2026) that joins the Garsia–Haiman modules of two shapes differing by one box. Conjecture 1.1 itself would follow from a graded embedding of the fusion defect module of $H_a[H_b]$ into that of $H_b[H_a]$, whose existence remains open (Section 4).

The companion papers show that negative coefficients of $F_{a,b}$ are rare under Plancherel sampling (Sra, 2026a), and prove that $h_b[H_a] - H_a[h_b]$ is Schur positive for $b \ge (a-1)^2$, together with positivity of families of Schur coefficients of $F_{a,b}$ (Sra, 2026b). Appendix C polarizes each Schur-positive quotient proved here into a positive semidefinite kernel on tuples of vectors.

1.1. **Conventions.** We work over $\mathbb{C}$, and permuting tensor factors introduces no signs. Plethysm acts on every grading variable through the Adams operations $\psi_r(p_j) = p_{rj}$, $\psi_r(q) = q^r$, and $\psi_r(t) = t^r$, and the involution ω sends s_λ to $s_{\lambda'}$. The coinvariant algebra $R_m = \mathbb{C}[z_1, \dots, z_m]/\langle \mathbb{C}[z_1, \dots, z_m]_+^{S_m} \rangle$, graded by $\deg z_i = 1$, has graded Frobenius character H_m, and $H_0 = 1$. We write $H_m[f; z] = H_m(z)[f]$ for the plethysm with outer coinvariant grading z; grading variables in f keep their meanings, and $H_m[f] = H_m[f; q]$. For a partition μ and an integer n, set

$$n(\mu) = \sum_i (i-1)\mu_i, \qquad c_n = \binom{n}{2}, \qquad \varepsilon_n = n \bmod 2, \qquad [n]_q = \frac{1-q^n}{1-q},$$

so that $[0]_q = 0$, and write $\begin{bmatrix} n \\ k \end{bmatrix}_q = (q;q)_n/((q;q)_k(q;q)_{n-k})$ for $0 \le k \le n$. For $\rho = 1^{m_1}2^{m_2}\cdots$, put $z_\rho = \prod_j j^{m_j} m_j!$; d_λ is the dimension of the Specht module S^λ, and $\succeq$ is the dominance order.

Our normalization of the modified Macdonald polynomial is

$$\widetilde{H}_\mu[X; q, t] = t^{n(\mu)} J_\mu \left[\frac{X}{1-t^{-1}}; q, t^{-1} \right],$$

where J_μ is the integral Macdonald form. It is the normalization of Bergeron et al. (1999) and Haiman (2001); see also Macdonald (1995, Chapter VI, (8.1') and (8.3)). By Haiman's $n!$ theorem, for $\mu \vdash n$, $\widetilde{H}_\mu$ is the bigraded Frobenius character of the Garsia–Haiman module of μ (Haiman, 2001): the span of all partial derivatives of the determinant $\det[x_k^j y_k^i]$, whose rows are indexed by $k = 1, \dots, n$ and whose columns by the cells (i, j) of μ, with zero-based row index i and column index j; here q counts the degree in $x_1, \dots, x_n$ and t the degree in $y_1, \dots, y_n$. For $\mu = (n)$ the determinant is the Vandermonde determinant in $x_1, \dots, x_n$, whose derivatives span a copy of the coinvariant algebra, so $\widetilde{H}_{(n)}(X; q, t) = H_n(X; q)$; similarly, $\widetilde{H}_{(1^n)}(X; q, t) = H_n(X; t)$.

2. Fusion defects and their reversal

Fusion places each plethysm above a fused Macdonald polynomial of the same degree, and sign duality reverses that order after a shift. These two comparisons are the inputs to every shifted comparison in Section 3.

2.1. **Two fusion filtrations.** To realize the fusion defect as a module, we filter one tensor power of a module with character $\widetilde{H}_\mu$ in two ways. The first filtration comes from the current algebra $\mathfrak{gl}(V)[x, y] = \mathfrak{gl}(V) \otimes \mathbb{C}[x, y]$, spanned by the currents $Y x^r y^\ell$ with $Y \in \mathfrak{gl}(V)$; the second comes from all endomorphisms of the module. The first filtration is contained in the second, and the quotient has the fusion defect as its character. Placing the evaluation points at the mth roots of unity refines the inclusion by degree modulo m, which gives the denominator $1 - q^m$.

The fusion product of cyclic modules over a current algebra, evaluated at distinct points $c_1, \dots, c_m$, is the associated graded of their tensor product for the filtration by current degree, where the variable x acts on the ith factor as $x + c_i$ (Feigin and Loktev, 1999). Khoroshkin and Makedonskyi (2026, Corollary 6.16 and Theorem 6.19) construct cyclic bigraded $\mathfrak{gl}(V)[x, y]$-modules $U_\mu(V)$ with character $\widetilde{H}_\mu(X; q, t)$, where q and t count the x- and y-degrees; the construction needs $\dim V \ge 3N + 1$, where N is the total polynomial degree of the module constructed. Their theorem identifies fusion in y with the union of rows, the partition whose rows are those of all factors. Exchanging x and y and conjugating diagrams (Khoroshkin and Makedonskyi, 2026, Lemma 8.2) turns this into fusion in x: the fusion in x of m copies of U_μ has character $\widetilde{H}_{m\mu}$, where $m\mu = (m\mu_1, m\mu_2, \dots)$. These fusion products do not depend on the evaluation points, and Haiman's $n!$ theorem is an input to the identification of their characters.

Kedem's fusion theorem (Kedem, 2004, Theorem 4.3) identifies the m-fold fusion product of the defining $\mathfrak{gl}(W)$-module with $(R_m \otimes W^{\otimes m})^{S_m}$, whose graded character is $H_m[W; q]$. The identification

respects any grading of W, so that grading survives in the character. For the denominator $1-q^m$, we use the action of rotating the tensor factors on fusion at roots of unity, described by Westbury (2016, Proposition 8.1), on both filtrations at once and compare its eigenspaces. The precise form of Theorem 1.2 names the two quotient modules that the proof constructs.

Theorem 1.2 (precise form)**.** *For every partition μ and integer $m \geq 1$,*

$$\mathcal{D}_{\mu,m}(X;q,t) := \frac{H_m[\widetilde{H}_\mu(X;q,t);q] - \widetilde{H}_{m\mu}(X;q,t)}{1-q} \in \mathbb{N}[q,t]\{s_\lambda\}. \tag{2.1}$$

For $\dim V \geq 3m|\mu|+1$, $\mathcal{D}_{\mu,m}$ is the character of the finite-dimensional module $\mathcal{K}_{\mu,m}(V)$ in (2.3). The quotient with denominator $1-q^m$ is also a character:

$$\mathcal{E}_{\mu,m}(X;q,t) := \frac{H_m[\widetilde{H}_\mu(X;q,t);q] - \widetilde{H}_{m\mu}(X;q,t)}{1-q^m} \in \mathbb{N}[q,t]\{s_\lambda\}, \qquad \mathcal{D}_{\mu,m} = [m]_q \mathcal{E}_{\mu,m}. \tag{2.2}$$

The module $\mathcal{L}_{\mu,m}(V)$ in (2.4) realizes this character.

Proof. For the empty partition, take $U_\varnothing = \mathbb{C}$ in degree $(0,0)$ with trivial current action. Then $H_m[1;q] = 1$, both filtrations below are constant, and the defect is zero. Suppose $|\mu| \geq 1$. Set $U = U_\mu(V)$, choose its cyclic vector v in bidegree $(0,0)$, and write ρ for the current-algebra action. On $T = U^{\otimes m}$, evaluate the ith factor at $c_i = \zeta^i$ in the x variable, where ζ is a primitive mth root of unity and indices are taken modulo m. The first filtration is the fusion filtration by x-degree,

$$A_s = U(\mathfrak{gl}(V)[x,y])_{\deg_x \leq s} v^{\otimes m}.$$

It preserves the total y-degree, and the fusion realization of Khoroshkin and Makedonskyi (2026) gives $\operatorname{ch}_{q,t} \operatorname{gr}_A T = \widetilde{H}_{m\mu}$.

For the second filtration, regard U as the defining representation of $\mathfrak{gl}(U)$, and let G_j be Kedem's fusion filtration of T at the same evaluation points. Each G_j is $\mathrm{GL}(U)$-invariant, so it decomposes by the internal x- and y-degrees of T. Writing T_d for the part of T of internal x-degree d, set

$$B_s = \bigoplus_{d \geq 0} (G_{s-d} \cap T_d), \qquad G_j = 0 \ (j < 0).$$

The index s adds the fusion degree and the internal x-degree, and Kedem's theorem, with the internal grading kept, gives $\operatorname{ch}_{q,t} \operatorname{gr}_B T = H_m[\widetilde{H}_\mu]$.

We claim $A_s \subseteq B_s$. A current $Yx^r y^\ell$ acts on T as

$$\sum_{i=1}^{m} \sum_{h=0}^{r} \binom{r}{h} c_i^{r-h} \rho(Yx^h y^\ell)^{(i)},$$

and we write $\rho_{\mathrm{sh}}(Yx^r y^\ell)$ for this operator. For fixed h, the sum over i is a multiple of the fusion current $\sum_i c_i^{r-h} E^{(i)}$ of $E = \rho(Yx^h y^\ell) \in \operatorname{End}(U)$. This current has fusion degree $r-h$, and E raises the internal x-degree by h, so the term maps $G_j \cap T_d$ into $G_{j+r-h} \cap T_{d+h}$, and hence B_s into B_{s+r}. Since $v^{\otimes m} \in B_0$, every word in currents of total x-degree at most s maps $v^{\otimes m}$ into B_s, which proves $A_s \subseteq B_s$.

The quotients of the two filtrations form the module

$$\mathcal{K}_{\mu,m}(V) = \bigoplus_{s \geq 0} B_s / A_s, \tag{2.3}$$

in which the summand B_s/A_s has q-degree s and keeps its y-degree. Both filtrations eventually equal T, so $\mathcal{K}_{\mu,m}(V)$ is finite-dimensional. Taking differences of successive filtration spaces gives

$$\operatorname{ch} \operatorname{gr}_B T - \operatorname{ch} \operatorname{gr}_A T = (1-q) \sum_{s \geq 0} q^s \operatorname{ch}_t(B_s/A_s).$$

Since $\dim V \geq 3m|\mu|+1$, the fusion realization applies and $GL(V)$-characters detect every Schur coefficient of degree $m|\mu|$, so this identity of characters proves (2.1).

To keep track of degrees modulo m, let R rotate the tensor factors backward, so $RE^{(i)}R^{-1} = E^{(i-1)}$ for $E \in \mathrm{End}(U)$, let Z act on T_d by ζ^d, and set $\Gamma = ZR$. The operators R and Z commute, so $\Gamma^m = 1$, and Γ commutes with $GL(V)$ and preserves the y-degree. In $\rho_{\mathrm{sh}}(Yx^ry^\ell)$, conjugation by R multiplies the term of index h by ζ^{r-h}, and Z multiplies it by ζ^h, so

$$\Gamma\rho_{\mathrm{sh}}(Yx^ry^\ell)\Gamma^{-1} = \zeta^r\rho_{\mathrm{sh}}(Yx^ry^\ell).$$

Since Γ fixes $v^{\otimes m}$, it acts on A_s/A_{s-1} as ζ^s. For the second filtration,

$$R\Big(\sum_i c_i^r E^{(i)}\Big)R^{-1} = \zeta^r \sum_i c_i^r E^{(i)},$$

so R acts on G_j/G_{j-1} as ζ^j, and Γ acts on its part of internal x-degree d as ζ^{j+d}. By the definition of B_s, Γ therefore acts on B_s/B_{s-1} as ζ^s.

Write $T^{[s]} = \ker(\Gamma - \zeta^s)$ and set filtration terms of negative index equal to zero. On $T^{[s]}$, either filtration can grow only at indices congruent to s modulo m, so

$$A_{s-1} \cap T^{[s]} = A_{s-m} \cap T^{[s]}, \qquad B_{s-1} \cap T^{[s]} = B_{s-m} \cap T^{[s]}.$$

The inclusions $A_s \subseteq B_s$ define a finite bigraded module

$$\mathcal{L}_{\mu,m}(V) = \bigoplus_{s\geq 0} \frac{B_s \cap T^{[s]}}{A_s \cap T^{[s]}}, \tag{2.4}$$

with degree q^s on the indicated summand and its original y-degree. Taking differences at successive indices in each residue class gives

$$\mathrm{ch}_{q,t}\,\mathrm{gr}_B T - \mathrm{ch}_{q,t}\,\mathrm{gr}_A T = (1-q^m)\,\mathrm{ch}_{q,t}\,\mathcal{L}_{\mu,m}(V).$$

Both filtrations exhaust T, so the sum is finite, which proves (2.2). □

Coefficientwise, Theorem 1.2 orders cumulative multiplicities. For a Schur label $\nu \vdash m|\mu|$ and integers $j,k \geq 0$, expanding $1/(1-q)$ in (2.1) gives

$$\sum_{i=0}^{k} [q^i t^j s_\nu](H_m[\widetilde{H}_\mu;q] - \widetilde{H}_{m\mu}) = [q^k t^j s_\nu]\mathcal{D}_{\mu,m} \geq 0.$$

Summed over all q-degrees, the two multiplicities agree, because both filtrations exhaust T; equivalently, $\widetilde{H}_{m\mu}(X;1,t) = \widetilde{H}_\mu(X;1,t)^m$.

The refinement (2.2) orders these multiplicities within each residue class. For $0 \leq r < m$ and $j,k \geq 0$,

$$\sum_{\ell=0}^{k} [q^{r+m\ell}t^j s_\nu](H_m[\widetilde{H}_\mu;q] - \widetilde{H}_{m\mu}) = [q^{r+mk}t^j s_\nu]\mathcal{E}_{\mu,m} \geq 0.$$

The totals in every residue class agree once k exceeds the support.

The cumulative order is weaker than an inequality in each degree. For $\mu = (2)$ and $m = 2$,

$$[s_{(1^4)}](H_2[H_2] - H_4) = q^2 - q^6, \qquad [s_{(1^4)}]\mathcal{D}_{(2),2} = q^2[4]_q,$$

so the difference has the negative coefficient $-q^6$ while the defect is positive. The module of $H_2[H_2]$ is induced from the wreath product $S_2 \wr S_2$, the stabilizer in S_4 of a splitting of four points into two pairs. The sign character of S_4 restricts to the sign characters of the two inner factors and to the

trivial character of the outer one, so the sign constituent of $H_2[H_2]$ lies in degree two, while that of H_4 lies in degree six. The refined quotient has sign coefficient $[s_{(1^4)}]\mathcal{E}_{(2),2} = q^2 + q^4$.

2.2. **The least shift for reversing fusion.** Theorem 1.2 places the plethysm $H_m[\widetilde{H}_\mu]$ above the fused polynomial $\widetilde{H}_{m\mu}$. Comparing two plethysms also needs the opposite order. Sign duality supplies it: reversing both gradings turns the fusion comparison around at the cost of a degree shift. The least shift is decided by the sign constituent, which occurs once in each character. In the example of Section 2.1, it lies in degree 6 of H_4 and in degree 2 of $H_2[H_2]$, so the reversed comparison of H_4 with $q^s H_2[H_2]$ needs $s \geq 4$; by the theorem below, $s = 4$ suffices.

Theorem 2.1 (Sharp reverse fusion)**.** *Let $\mu \vdash n \geq 1$ and $m \geq 1$. Set*

$$\Delta_{\mu,m} = c_m \left(\sum_i \mu_i^2 - \varepsilon_n \right).$$

For every integer $s \geq 0$,

$$\frac{\widetilde{H}_{m\mu}(X;q,t) - q^s H_m[\widetilde{H}_\mu(X;q,t);q]}{1-q} \succeq_s 0 \quad \Longleftrightarrow \quad s \geq \Delta_{\mu,m}. \tag{2.5}$$

Positivity is coefficientwise over $\mathbb{N}[q,t]$. With the denominator $1-q^m$,

$$\frac{\widetilde{H}_{m\mu}(X;q,t) - q^s H_m[\widetilde{H}_\mu(X;q,t);q]}{1-q^m} \in \mathbb{N}[q,t]\{s_\lambda\} \quad \Longleftrightarrow \quad s \geq \Delta_{\mu,m} \text{ and } m \mid s. \tag{2.6}$$

Proof. For $m = 1$, both quotients are $[s]_q \widetilde{H}_\mu$ and $\Delta_{\mu,1} = 0$. For $\mu = (1)$ the two numerator characters coincide with H_m, and the assertions follow from the geometric sum $(1-q^s)/(1-q^m)$ and its trivial Schur coefficient. Suppose $m, n \geq 2$, and abbreviate

$$A = \widetilde{H}_\mu, \qquad T = H_m[A;q], \qquad R = \widetilde{H}_{m\mu}, \qquad u = n(\mu'), \quad v = n(\mu).$$

Stretching μ horizontally to $m\mu$ gives

$$U := n((m\mu)') = m^2 u + c_m n, \qquad n(m\mu) = mv.$$

Sign self-duality of the Garsia–Haiman module (Haiman, 2001, Section 3.7, (45), and Section 3.9), with the sign twist explicit in (Guo et al., 2026, (2.3)), gives

$$q^u t^v \omega A(q^{-1}, t^{-1}) = A(q,t), \qquad q^U t^{mv} \omega R(q^{-1}, t^{-1}) = R(q,t).$$

Since these characters have nonnegative degrees and coefficients, A has bidegrees bounded by (u,v), R by (U, mv), and T by $(mu + c_m, mv)$. The bound for R dominates the one for T, because

$$U - mu - c_m = c_m(2u + n - 1) = c_m \left(\sum_i \mu_i^2 - 1 \right) \geq 0.$$

Theorem 1.2 makes $E = (T - R)/(1 - q^m)$ a polynomial. Its numerator has q-degree at most U, so the bidegrees of a nonzero E are bounded by $(U - m, mv)$, where $U \geq m$ because $m, n \geq 2$.

To reverse the degrees of T, we use coinvariant reciprocity and the plethystic parity rule,

$$q^{c_j} H_j(X; q^{-1}) = \omega H_j(X;q), \qquad \omega(f[g]) = (\omega^{\deg g} f)[\omega g], \tag{2.7}$$

where ω acts only on the symmetric-function alphabet. The second identity follows on power sums from $\omega p_j = (-1)^{j-1} p_j$, because Adams operations fix an integer sign. Applied to T, the two identities give

$$\begin{aligned} \omega T(q^{-1}, t^{-1}) &= q^{-mu} t^{-mv} (\omega^n H_m[-; q^{-1}])[A] \\ &= q^{-mu - c_m} t^{-mv} \overline{T}, \qquad \overline{T} = (\omega^{n+1} H_m[-;q])[A]. \end{aligned}$$

Reversing the degrees of E within its bounding box preserves positivity, so, with $d_0 = c_m(\sum_i \mu_i^2 - 1)$,

$$q^{U-m}t^{mv}\omega E(q^{-1},t^{-1}) = \frac{R - q^{d_0}\overline{T}}{1-q^m} \succeq_s 0. \tag{2.8}$$

For odd n, $\overline{T} = T$ and $d_0 = \Delta_{\mu,m}$, so (2.8) is (2.6) at $s = \Delta_{\mu,m}$.

For even n, $\overline{T} = (\omega H_m)[A]$, and Lemma A.1 supplies the positive character

$$J_m^{(m)} := \frac{\omega H_m - q^{c_m} H_m}{1-q^m} \succeq_s 0.$$

Adding $q^{d_0} J_m^{(m)}[A]$ to (2.8) proves (2.6) at $s = d_0 + c_m = \Delta_{\mu,m}$. In either parity m divides $\Delta_{\mu,m}$, because $m \mid c_m$ for odd m, and for even m the integer $\sum_i \mu_i^2 - \varepsilon_n$ is even. For a larger multiple s of m, adding $q^{\Delta_{\mu,m}}[(s-\Delta_{\mu,m})/m]_{q^m} T$ proves (2.6). Multiplying by $[m]_q$ gives (2.5) at the threshold, and adding $q^{\Delta_{\mu,m}}[s-\Delta_{\mu,m}]_q T$ gives it for every larger integer s.

For necessity, we compute sign coefficients. The ungraded Garsia–Haiman module is the regular S_n-module, so its sign multiplicity is one, and sign self-duality pairs its constant trivial constituent with $q^u t^v \operatorname{sgn}$. The sign coefficient of A is therefore $q^u t^v$. The sign representation of S_{mn} restricts to the sign representation in each inner S_n factor and to sgn^n on the outer S_m, so

$$[s_{(1^{mn})}]R = q^U t^{mv}, \qquad [s_{(1^{mn})}]T = q^{mu+\varepsilon_n c_m} t^{mv}.$$

If $L = s + mu + \varepsilon_n c_m < U$, the sign coefficient of the quotient in (2.5) is $-q^L[U-L]_q t^{mv}$, which is nonzero and negative. Positivity therefore requires $L \geq U$, which is $s \geq \Delta_{\mu,m}$. Positivity in (2.6) implies (2.5) after multiplication by $[m]_q$, so the same bound is necessary there. Finally, T and R both have trivial Schur coefficient 1, since each is a graded regular representation whose trivial constituent lies in degree zero (for T by wreath restriction). The trivial coefficient of the quotient in (2.6) is therefore $(1-q^s)/(1-q^m)$, a polynomial exactly when $m \mid s$. □

Since $\widetilde{H}_{\mu'}(X;q,t) = \widetilde{H}_\mu(X;t,q)$, conjugating μ and exchanging q,t turn Theorem 2.1 into a comparison in the variable t. Let $\mu^{\sqcup m} = (m\mu')'$ be the partition that repeats each row of μ exactly m times. The least shift in

$$\frac{\widetilde{H}_{\mu^{\sqcup m}}(X;q,t) - t^s H_m[\widetilde{H}_\mu(X;q,t);t]}{1-t}$$

is $c_m(\sum_i(\mu_i')^2 - \varepsilon_n)$. With the denominator $1-t^m$, positivity holds exactly for the shifts of at least this size that are divisible by m.

3. Shifted Foulkes comparisons

Adding a forward fusion comparison to a reversed one cancels the fused Macdonald polynomial and leaves a shifted comparison of two plethysms. This section proves Theorem 1.3 in this way and extends it to rectangular inner characters.

3.1. **A q-distorted Foulkes comparison.** To combine the forward comparison for $H_b[H_a]$ with the reversed comparison for $H_a[H_b]$, we record the coinvariant grading of H_a by q and that of H_b by t, in both plethysms:

$$\mathcal{P} = H_b[H_a[X;q];t], \quad \mathcal{Q} = H_a[H_b[X;t];q], \quad \mathcal{R} = \widetilde{H}_{(a^b)}(X;q,t). \tag{3.1}$$

Both plethysms fuse to $\mathcal{R}$, one by fusion in y and the other by fusion in x; Macdonald transposition, $\widetilde{H}_{(a^b)}(q,t) = \widetilde{H}_{(b^a)}(t,q)$, exchanges the two cases. Theorem 1.2 therefore places each plethysm above $\mathcal{R}$.

Proposition 3.1 (Two rectangular filtration quotients)**.** *For every $a, b \geq 2$,*

$$U = \frac{\mathcal{P} - \mathcal{R}}{1-t} \succeq_s 0, \qquad V = \frac{\mathcal{Q} - \mathcal{R}}{1-q} \succeq_s 0 \tag{3.2}$$

over $\mathbb{N}[q,t]$. As in Theorem 1.2, the denominators refine to $1-t^b$ and $1-q^a$:

$$\frac{\mathcal{P} - \mathcal{R}}{1-t^b} \succeq_s 0, \qquad \frac{\mathcal{Q} - \mathcal{R}}{1-q^a} \succeq_s 0.$$

Proof. For U, fuse b copies of the one-row module with character $H_a[X;q]$ in the variable y. The currents give $\mathcal{R}$, and all endomorphisms give $\mathcal{P}$. A current Yx^hy^r acts as $\sum_i c_i^r \rho(Yx^h)^{(i)}$, which is the fusion current of the endomorphism $\rho(Yx^h)$ in the same y-degree r. The filtration from the currents therefore lies inside the one from all endomorphisms, and both keep the q-grading. The quotient argument of Theorem 1.2 gives $U \succeq_s 0$, and its cyclic part gives the denominator $1-t^b$. Transposing the rectangle and exchanging q and t gives the same statements for V, with the denominator $1-q^a$. □

To place the rectangle above a shifted $\mathcal{Q}$, the second comparison must be reversed. For the rectangle, the least shift of Theorem 2.1 is linear in b.

Theorem 3.2 (Sharp reverse fusion for a rectangle)**.** *For integers $a, b \geq 2$, put $\varepsilon_b = b \bmod 2$ and*

$$\delta_{a,b} = (b - \varepsilon_b)c_a = a(a-1)\lfloor b/2 \rfloor.$$

For an integer $s \geq 0$,

$$\frac{\mathcal{R} - q^s\mathcal{Q}}{1-q} \succeq_s 0 \quad \Longleftrightarrow \quad s \geq \delta_{a,b}. \tag{3.3}$$

Positivity in (3.3) is over $\mathbb{N}[q,t]$.

Proof. Apply Theorem 2.1 to $\mu = (1^b)$ and $m = a$. Then $\widetilde{H}_\mu = H_b(X;t)$, $m\mu = (a^b)$, and $\Delta_{\mu,a} = c_a(b - \varepsilon_b)$. □

The explicit specialization of the reversal identity is recorded in Appendix A.1.

We combine Proposition 3.1 and Theorem 3.2 while the two gradings are still separate. With $G = (\mathcal{R} - q^{\delta_{a,b}}\mathcal{Q})/(1-q)$,

$$\mathcal{P} - q^{\delta_{a,b}}\mathcal{Q} = (1-t)U + (1-q)G, \qquad U, G \succeq_s 0. \tag{3.4}$$

Specializing $t = q^c$ in (3.4) proves Theorem 1.3.

Theorem 1.3. *For all integers $a, b \geq 2$ and $c \geq 1$, with $H_n(z) = H_n(X;z)$ and $\delta_{a,b} = a(a-1)\lfloor b/2 \rfloor$,*

$$\frac{H_b(q^c)[H_a(q)] - q^{\delta_{a,b}} H_a(q)[H_b(q^c)]}{1-q} \succeq_s 0 \qquad \text{over } \mathbb{N}[q].$$

Proof. Substitute $t = q^c$ in (3.4), so that $\mathcal{P}$ and $\mathcal{Q}$ become the two plethysms above, and divide by $1-q$. The resulting quotient is $[c]_q U(q, q^c) + G(q, q^c)$, and both summands are positive. □

The identity (3.4) also allows both gradings to be rescaled. For positive integers r, s, substituting $q = z^r$ and $t = z^s$ and dividing by $1-z$ turns its right side into $[s]_z U(z^r, z^s) + [r]_z G(z^r, z^s)$, which is Schur positive. This is a comparison in the degree that gives q weight r and t weight s.

3.2. **Rectangular modified-Macdonald comparisons.** Bergeron's rectangular conjecture replaces the inner H_a by $\widetilde{H}_{(a^k)}(X;q,t)$, whose two parameters both grade the inner module, while the outer grading is again q. No variable is left for the outer grading, as in Section 3.1, so we compare the rectangles through one-row plethysms instead.

For $a,b\geq 2$ and $k\geq 1$, set

$$\mathcal{F}^{(k)}_{a,b}=\frac{H_b[\widetilde{H}_{(a^k)}]-H_a[\widetilde{H}_{(b^k)}]}{1-q}.$$

Both plethysms fuse to $\widetilde{H}_{((ab)^k)}$, which cancels in the difference:

$$\begin{aligned}\mathcal{F}^{(k)}_{a,b}&=\mathcal{D}_{(a^k),b}-\mathcal{D}_{(b^k),a}\\&=[b]_q\mathcal{E}_{(a^k),b}-[a]_q\mathcal{E}_{(b^k),a}.\end{aligned}\tag{3.5}$$

At $k=1$ the fused polynomial is H_{ab}, and the two defects are

$$\mathcal{D}_{(a),b}=\frac{H_b[H_a]-H_{ab}}{1-q},\qquad \mathcal{D}_{(b),a}=\frac{H_a[H_b]-H_{ab}}{1-q}.$$

Both are positive by Theorem 1.2, and Conjecture 1.1 says that the first is at least the second, coefficient by coefficient. For every k, the rectangular conjecture says the same of $\mathcal{D}_{(a^k),b}$ and $\mathcal{D}_{(b^k),a}$. For integers c,d with $cd=ab$, which Bergeron's conjecture also allows, the same cancellation gives

$$\frac{H_c[\widetilde{H}_{(d^k)}]-H_a[\widetilde{H}_{(b^k)}]}{1-q}=\mathcal{D}_{(d^k),c}-\mathcal{D}_{(b^k),a}.\tag{3.6}$$

As a baseline, comparing q-supports alone gives a shift that works in every t-degree separately.

Proposition 3.3 (A rectangular support bound)**.** *Let $a,b\geq 2$ and $k\geq 1$, and put*

$$P^{(k)}_{a,b}=H_b[\widetilde{H}_{(a^k)};q],\qquad Q^{(k)}_{a,b}=H_a[\widetilde{H}_{(b^k)};q],\qquad M_{a,b,k}=bkc_a+c_b.$$

Then the shift $M_{a,b,k}$ suffices:

$$\frac{P^{(k)}_{a,b}-q^{M_{a,b,k}}Q^{(k)}_{a,b}}{1-q}\in\mathbb{N}[q,t]\{s_\lambda\}.$$

More generally, if $ab=cd$, the same assertion with $P^{(k)}_{a,b}$ replaced by $H_c[\widetilde{H}_{(d^k)};q]$ holds with exponent ckc_d+c_c.

At $q=1$, fusion identifies both plethysms with $H_k(X;t)^{ab}$, so their total multiplicities agree separately in every Schur label and t-degree. Separating their q-supports gives the bound; the proof is in Appendix A.2.

To beat the support bound, we pass the one-row comparisons of Section 3.1 through the outer plethysm $H_m[-;q^c]$, which preserves cumulative order in each t-degree.

Lemma 3.4 (Preservation of cumulative order)**.** *Let $f,g\in\mathbb{N}[q,t]\{s_\lambda\}$ be homogeneous of the same positive symmetric-function degree. Suppose $(f-g)/(1-q)\in\mathbb{N}[q,t]\{s_\lambda\}$. For all positive integers m,c,*

$$\frac{H_m[f;q^c]-H_m[g;q^c]}{1-q}\in\mathbb{N}[q,t]\{s_\lambda\}.\tag{3.7}$$

Proof. Choose a vector space V large enough to detect all Schur types in degree $m\deg f$, and realize f,g as characters of bigraded polynomial $GL(V)$-modules M,N. Fix an irreducible type and a t-degree. The hypothesis says that the cumulative q-multiplicities in M dominate those in N, and that their totals agree. List the degrees of the multiplicity basis vectors in increasing order as $\alpha_1\leq\cdots\leq\alpha_r$ for M and $\beta_1\leq\cdots\leq\beta_r$ for N. The cumulative inequalities are equivalent to $\alpha_i\leq\beta_i$

for every i. Matching these vectors gives a $GL(V)$-equivariant bijection $\phi\colon N \to M$ that preserves t-degree and does not increase q-degree.

Since $\phi^{\otimes m}$ commutes with permutations of the factors, we may tensor it with the identity of R_m, graded with degrees multiplied by c, and take S_m-invariants. This gives a bijection

$$(R_m \otimes N^{\otimes m})^{S_m} \longrightarrow (R_m \otimes M^{\otimes m})^{S_m}$$

that preserves t-degree and does not increase total q-degree. Its source and target have characters $H_m[g;q^c]$ and $H_m[f;q^c]$, so the filtration inclusions give (3.7), and the equality of total characters makes the quotient a polynomial. □

With Lemma 3.4, the rectangular comparison reduces to the one-row one. The column character $H_k(X;t) = \widetilde{H}_{(1^k)}$ fuses to both rectangles: a copies give (a^k), and b copies give (b^k). Each rectangular plethysm can therefore be compared with a plethysm of one-row plethysms of $H_k(X;t)$, and Theorem 1.3 compares those. The three shifts add up as follows.

Theorem 3.5 (A distorted rectangular comparison)**.** *Let $a, b \geq 2$ and $k, c \geq 1$ be integers. Put*

$$S_{a,b,k} = c_a\big(b(k+1-\varepsilon_k) - \varepsilon_b\big).$$

Then the shift $S_{a,b,k}$ suffices:

$$\frac{H_b[\widetilde{H}_{(a^k)}(X;q,t);q^c] - q^{S_{a,b,k}} H_a[\widetilde{H}_{(b^k)}(X;q^c,t);q]}{1-q} \in \mathbb{N}[q,t]\{s_\lambda\}. \tag{3.8}$$

In particular, at $c=1$ the theorem compares $P^{(k)}_{a,b}$ and $q^{S_{a,b,k}}Q^{(k)}_{a,b}$ with both Macdonald parameters retained. At $k=1$, it is Theorem 1.3.

Proof. The plethysms of the column character $A = H_k(X;t)$ link the two rectangles: after a shift, $\widetilde{H}_{(a^k)}$ lies above $H_a[A;q]$, and $H_b[A]$ lies above $\widetilde{H}_{(b^k)}$, so Theorem 1.3 for the alphabet A closes the chain. For the first step, set $d_a = c_a(k-\varepsilon_k)$; Theorem 2.1, applied to the column (1^k), gives

$$\frac{\widetilde{H}_{(a^k)} - q^{d_a} H_a[A;q]}{1-q} \succeq_s 0.$$

Lemma 3.4 therefore gives the first of the following three positive quotients, where

$$X_0 = H_b[H_a[A;q];q^c], \qquad Y_0 = H_a[H_b[A;q^c];q],$$

and P_c, Q_c denote the first and second plethysms in (3.8) before shifting:

$$E_1 = \frac{P_c - q^{bd_a} X_0}{1-q} \succeq_s 0,$$

$$E_2 = \frac{X_0 - q^{\delta_{a,b}} Y_0}{1-q} \succeq_s 0,$$

$$E_3 = \frac{Y_0 - Q_c}{1-q} \succeq_s 0.$$

The inner shift becomes q^{bd_a} because the outer plethysm has degree b. For E_2, substitute the positive alphabet A into Theorem 1.3; the coefficients of A depend only on t, so the substitution keeps the q-grading and the denominator $1-q$. For E_3, forward fusion gives

$$\frac{H_b[A;q^c] - \widetilde{H}_{(b^k)}(X;q^c,t)}{1-q} = [c]_q \mathcal{D}_{(1^k),b}(X;q^c,t) \succeq_s 0,$$

and Lemma 3.4 applies to the outer $H_a[-;q]$. Finally, $S_{a,b,k} = bd_a + \delta_{a,b}$ and

$$\frac{P_c - q^{S_{a,b,k}} Q_c}{1-q} = E_1 + q^{bd_a} E_2 + q^{S_{a,b,k}} E_3 \succeq_s 0.$$

□

At $c = 1$, Proposition 3.3 and Theorem 3.5 both apply, so the smaller of $S_{a,b,k}$ and $M_{a,b,k}$ is a valid shift. Appendix A compares the two bounds, including the parity cases and the larger shift obtained by direct reverse fusion.

4. Open problems

Conjecture 1.1 and its rectangular version would follow from a graded embedding of one fusion defect module into the other. By (3.5), the question is whether there is an embedding

$$\mathcal{K}_{(b^k),a}(V) \hookrightarrow \mathcal{K}_{(a^k),b}(V), \qquad 2 \le a \le b, \quad k \ge 1,$$

that preserves both gradings and the $GL(V)$-action, for $\dim V \ge 3abk + 1$, where the constructions of Section 2.1 apply; their stable characters do not depend on $\dim V$. For c, d with $cd = ab$ and $a \le c, d \le b$, the same question compares $\mathcal{K}_{(b^k),a}$ with $\mathcal{K}_{(d^k),c}$.

The least shift is also open. Reversing one fusion comparison costs exactly the shift of Theorem 2.1, while Theorem 3.5 compares the two plethysms with a shift linear in b, which beats the support bound of Proposition 3.3 for odd k, and for even k once b is large (Appendix A).

An intermediate shape does not help. For a non-row shape $\rho \vdash a$, the defect of ρ with b fused copies has a smaller coefficient of the standard character $s_{(ab-1,1)}$ and a smaller total dimension than the defect of $H_a[H_b]$ (Proposition B.5), so it cannot lie between the two defects.

AI usage

The results in this paper were developed through extensive use of AI assistants in sustained mathematical discussion, including failed attempts, counterexamples, audits, and repairs. AI assistants also helped with exact computations, proof checks, and exposition. The external representation-theoretic results are cited at their points of use.

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Appendix A. Shift bounds and exact checks

This appendix proves the cyclic duality lemma and the support bound used in Sections 2 and 3, and compares the admissible shifts with one another and with exact computations.

A.1. **Duality and its rectangular specialization.** For even n, the proof of Theorem 2.1 corrects the outer twist by the character $J_m^{(m)}$, which must be positive with the denominator $1-q^m$ to keep the comparison in each residue class. Its positivity comes from the rotation of tensor factors in the proof of Theorem 1.2.

Lemma A.1. *For every $m \geq 1$,*

$$J_m^{(m)} = \frac{\omega H_m(X;q) - q^{c_m} H_m(X;q)}{1-q^m} \in \mathbb{N}[q]\{s_\lambda\}.$$

Proof. On Kedem's fusion filtration of $W^{\otimes m}$, the backward rotation R acts on G_j/G_{j-1} as ζ^j, by the proof of Theorem 1.2, so $H_m(X;\zeta^k)$ is the trace of R^k on $W^{\otimes m}$, as a character of $GL(W)$. Since R^k and R^{-k} are conjugate permutations, $H_m(X;\zeta^k) = H_m(X;\zeta^{-k})$ for every integer k, and coinvariant reciprocity (2.7) then gives

$$\omega H_m(X;\zeta^k) = \zeta^{kc_m} H_m(X;\zeta^k).$$

With $\dim W \geq m$, so that all Schur labels are detected, the numerator of $J_m^{(m)}$ vanishes at every mth root of unity and is divisible by $1-q^m$.

For each Schur label, the first numerator term is supported in $[0, c_m]$, and the second in $[c_m, 2c_m]$. By divisibility, their total multiplicities agree in every residue class modulo m. Before degree c_m, each residue-wise cumulative difference is nonnegative because only the first term contributes. At and after c_m, the first term has reached the shared total, while the second has contributed at most that total. Every coefficient of the quotient is therefore a nonnegative integer.

With the ordinary denominator, writing $\omega H_m = \sum_{\nu,j} a_{\nu,j} q^j s_\nu$ and using reciprocity gives

$$[m]_q J_m^{(m)} = \sum_{\nu,j} a_{\nu,j} q^j [2(c_m - j)]_q s_\nu, \qquad a_{\nu,j} \geq 0, \quad 0 \leq j \leq c_m.$$

□

For the rectangle of Section 3.1, with $a, b \geq 2$, the characters (3.1), the quotient V of (3.2), and $\overline{\mathcal{Q}} = (\omega^{b+1} H_a[-;q])[H_b(X;t)]$, the reversal identity (2.8) reads

$$q^{bc_a-1} t^{ac_b} \omega V(q^{-1}, t^{-1}) = \frac{\mathcal{R} - q^{(b-1)c_a} \overline{\mathcal{Q}}}{1-q} \succeq_s 0.$$

The parity correction and the sign obstruction are those of the proof of Theorem 2.1.

A.2. **Support separation in each grading.** The support bound uses only that the two plethysms agree at $q=1$ and have bounded q-degree.

Proof of Proposition 3.3. Theorem 1.2, at $q=1$ with inner partition (1^k), gives $\widetilde{H}_{(a^k)}(X;1,t) = H_k(X;t)^a$. Since $H_b[-;1]$ is the ordinary bth power,

$$P_{a,b}^{(k)}(X;1,t) = Q_{a,b}^{(k)}(X;1,t) = H_k(X;t)^{ab}. \tag{A.1}$$

The q-degree of $\widetilde{H}_{(a^k)}$ is at most kc_a, so $\deg_q P_{a,b}^{(k)} \leq M_{a,b,k}$. Fix a Schur label and a t-degree. Below degree $M_{a,b,k}$ the shifted source $q^{M_{a,b,k}} Q_{a,b}^{(k)}$ contributes nothing, and from degree $M_{a,b,k}$ on the target has reached its total multiplicity, which equals that of the source by (A.1). Every cumulative difference is therefore nonnegative, and the equal totals make the quotient a polynomial. For $cd = ab$ the same argument uses $\deg_q H_c[\widetilde{H}_{(d^k)};q] \leq ckc_d + c_c$. □

A.3. **Comparing admissible shifts.** A shift is admissible when the corresponding shifted quotient is Schur positive. Evaluated at $c=1$ and $0 \leq q < 1$, Theorem 1.3 compares real Schur coefficients,

$$H_b[H_a](q) \succeq_s q^{\delta_{a,b}} H_a[H_b](q),$$

so, for any $0 < \eta < 1$,

$$q \geq (1-\eta)^{1/\delta_{a,b}} \quad \Longrightarrow \quad H_b[H_a](q) \succeq_s (1-\eta) H_a[H_b](q), \qquad 0 \leq q < 1.$$

At $k=1$, Proposition 3.3 gives the shift $M = bc_a + c_b$, quadratic in b for fixed a, while Theorem 1.3 gives $\delta_{a,b} = a(a-1)\lfloor b/2 \rfloor$, linear in b; by Theorem 3.2, $\delta_{a,b}$ is the least shift for the comparison of $\mathcal{R}$ with $\mathcal{Q}$.

For $c=1$, both $S_{a,b,k}$ and $M_{a,b,k}$ are admissible, and their difference is

$$M_{a,b,k} - S_{a,b,k} = \begin{cases} c_b + \varepsilon_b c_a, & k \text{ odd}, \\ c_b - c_a(b - \varepsilon_b), & k \text{ even}. \end{cases}$$

For odd k, $S_{a,b,k}$ is always the smaller; for even k, it is smaller exactly when $c_b > c_a(b-\varepsilon_b)$. For general c, the support argument gives the exponent $bkc_a + cc_b$.

Reversing fusion directly is costlier. Theorem 2.1 with $\mu = (b^k)$ and $m = a$ gives the least shift $\Delta_{(b^k),a} = c_a(kb^2 - \varepsilon_{bk})$ for the fused character $\widetilde{H}_{((ab)^k)}$ to dominate the shifted $Q^{(k)}_{a,b}$, and adding the forward defect of $P^{(k)}_{a,b}$ turns this into a full comparison, but with an exponent larger than $M_{a,b,k}$:

$$\Delta_{(b^k),a} - M_{a,b,k} = c_b(2kc_a - 1) - \varepsilon_{bk}c_a > 0.$$

Theorem 3.5 avoids this cost by passing through one-row plethysms of the column character, where the shift is the smaller $\delta_{a,b}$.

Since $\widetilde{H}_{\mu'}(X;q,t) = \widetilde{H}_{\mu}(X;t,q)$, Theorem 3.5 also compares the conjugate rectangles, with the same shift: for $a, b \geq 2$ and $k, c \geq 1$,

$$\frac{H_b[\widetilde{H}_{(k^a)}(X;q,t);t^c] - t^{S_{a,b,k}} H_a[\widetilde{H}_{(k^b)}(X;q,t^c);t]}{1-t} \in \mathbb{N}[q,t]\{s_\lambda\},$$

and at $c = 1$ the exponent $\min\{S_{a,b,k}, M_{a,b,k}\}$ is also admissible.

A.4. **Exact checks.** Exact Schur expansions over $\mathbb{Q}[q,t]$ check forward and reverse fusion for 25 pairs (μ, m) with $|\mu| \leq 4$, $m \leq 3$, and $m|\mu| \leq 8$, including $m = 1$ and both parities of $|\mu|$. At every positive threshold $\Delta_{\mu,m}$, decreasing the shift by one produces a negative coefficient, as the sign obstruction requires.

Six rectangular comparisons through total degree twelve check both the shift $S_{a,b,k}$ of Theorem 3.5 and the support bound, including $c = 2$, the case $a = b$, and both orders of a, b, with Adams operations acting on both grading variables. Representative parameters are

(a,b,k,c)	$S_{a,b,k}$	$bkc_a + cc_b$
$(2,3,1,2)$	2	9
$(2,3,2,1)$	8	9
$(2,2,3,2)$	6	8

Another 36 cases check the denominator $1 - q^m$ over $\mathbb{Q}[q,t]$, for forward positivity and for reverse positivity at the sharp shift: all nontrivial pairs with $m|\mu| \leq 8$, all partitions of size at most four at $m = 1$, and four cases with empty shape. A bivariate example is

$$\mathcal{E}_{(1,1),2} = s_{(2,2)} + t\, s_{(2,1,1)} + t^2 s_{(1^4)}.$$

The residue condition in (2.6) is a polynomiality requirement; increasing the sharp shift by one when $m > 1$ violates it, as the trivial coefficient already shows.

Appendix B. The stretched Butler curve and defect comparisons

To compare fusion defects with different inner shapes, we deform one Garsia–Haiman module into another. Haiman's proof of the $n!$ theorem realizes the Garsia–Haiman module of μ as the fiber, at the monomial ideal of μ (spanned by the monomials $x^j y^i$ with $(i,j) \notin \mu$), of a vector bundle on the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$ of n points in the plane, the Procesi bundle (Haiman, 2001). When two shapes differ by moving one box, Guo et al. (2026, Lemmas 2.1–2.2 and Theorem 2.3) join their monomial ideals by a family of ideals over a projective line. A vector bundle on a projective line splits into line bundles, and for the restricted Procesi bundle the splitting writes both Macdonald polynomials in terms of the same positive characters.

Let $\lambda, \mu \vdash n$ differ by moving one box up and to the right,

$$\lambda = \eta \cup \{(i, j+d)\}, \qquad \mu = \eta \cup \{(i+e, j)\}, \qquad d, e > 0, \tag{B.1}$$

where a cell (i,j) has zero-based row index i, counted by t, and column index j, counted by q. For example, $\lambda = (2)$ and $\mu = (1,1)$ differ by such a move, with $\eta = (1)$ and $d = e = 1$. The union of λ and μ is a partition of $n+1$, and their intersection is η.

Along the projective line joining the monomial ideals of λ and μ, the tautological bundle, whose fiber at an ideal is the quotient ring by that ideal, has a determinant of degree one, which confines the

splitting degrees of the Procesi bundle to 0 and 1. At $m = 1$, the theorem below writes $\widetilde{H}_\lambda = I + q^d J$ and $\widetilde{H}_\mu = I + t^e J$ with $I, J \succeq_s 0$; in the example, $\widetilde{H}_{(2)} = s_{(2)} + qs_{(1,1)}$ and $\widetilde{H}_{(1,1)} = s_{(2)} + ts_{(1,1)}$, so $I = s_{(2)}$ and $J = s_{(1,1)}$. The positivity of I is Butler's conjecture of 1994, proved by Guo et al. (2026). Pulling the family back along $x \mapsto x^m$ stretches λ and μ to $m\lambda$ and $m\mu$, and the same splitting then has $m+1$ pieces, with $\mathcal{J}^{(m)}_{\lambda,\mu}$ the m-fold version of J.

Theorem B.1 (Positive stretched divided difference)**.** *For every $m \geq 1$, there are characters $C_0, \ldots, C_m \in \mathbb{N}[q,t]\{s_\nu\}$ such that*

$$\widetilde{H}_{m\lambda} = \sum_{l=0}^{m} q^{mdl} C_l, \qquad \widetilde{H}_{m\mu} = \sum_{l=0}^{m} t^{el} C_l. \tag{B.2}$$

Consequently

$$\mathcal{J}^{(m)}_{\lambda,\mu} := \frac{\widetilde{H}_{m\lambda} - \widetilde{H}_{m\mu}}{q^{md} - t^e} = \sum_{l=1}^{m} C_l \sum_{r=0}^{l-1} q^{md(l-1-r)} t^{er} \succeq_s 0. \tag{B.3}$$

B.1. **The geometric construction.** The characters C_l of Theorem B.1 are the quotients of the splitting filtration of an equivariant vector bundle on $\mathbb{P}^1$; we construct the bundle and record its weights at both endpoints.

By Haiman's theory (Haiman, 2001), the Procesi bundle $\mathcal{P}_n$ on $\mathrm{Hilb}^n(\mathbb{C}^2)$ is globally generated, has fiber character $\widetilde{H}_\mu$ at the monomial ideal of μ, and, with $\mathcal{B}_n$ the tautological bundle, satisfies (Guo et al., 2026, (2.3))

$$\mathcal{P}_n \cong \mathcal{P}_n^\vee \otimes \det \mathcal{B}_n \otimes \mathrm{sgn}\,.$$

A vector bundle on $\mathbb{P}^1$ has a canonical filtration by its splitting degrees, which is equivariant because it is intrinsic.

Proof of Theorem B.1. Put $\nu = \lambda \cup \mu$ and use coordinates z, y on the original affine plane. For a partition ρ, let I_ρ denote the monomial ideal whose standard monomials are indexed by ρ. In $\mathbb{C}[z,y]/I_\nu$, let

$$\alpha = z^{j+d} y^i, \qquad \beta = z^j y^{i+e}, \qquad W = \mathrm{span}\{\alpha, \beta\}.$$

Both classes are corners of ν, so z and y annihilate them, and for a line $\ell \subset W$ the ideal $I_\ell = I_\nu + (\ell)$ has colength n. These ideals form a flat family $f\colon \mathbb{P}(W) \to \mathrm{Hilb}^n(\mathbb{C}^2)$, with endpoints I_λ at $\ell = \mathbb{C}\beta$ and I_μ at $\ell = \mathbb{C}\alpha$, and with tautological sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(W)}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(W)} \otimes (\mathbb{C}[z,y]/I_\nu) \longrightarrow f^*\mathcal{B}_n \longrightarrow 0.$$

By Guo et al. (2026, Lemma 2.1), the sequence gives $\deg \det f^*\mathcal{B}_n = 1$.

To stretch, pull the family back under $(z,y) = (x^m, y)$. The extension $\mathbb{C}[z,y] \subset \mathbb{C}[x,y]$ is free with basis $1, x, \ldots, x^{m-1}$, so the family stays flat, has length mn, and has endpoints $m\lambda$ and $m\mu$. With z of weight q^m, the quotient line bundle $(\mathcal{O} \otimes W)/\mathcal{O}(-1)$ has endpoint weights $q^{m(j+d)}t^i$ and $q^{mj}t^{i+e}$, and twisting by the character $q^{-mj}t^{-i}$ gives a degree-one line bundle $\mathcal{L}$ with endpoint weights q^{md} and t^e, the weights in (B.2).

Pulling $\mathcal{P}_{mn}$ back gives a vector bundle E on $\mathbb{P}^1$ whose fibers at the endpoints have the two Macdonald characters in (B.2). The tautological bundle of the pulled-back family is a direct sum of m copies of the original one, twisted by characters, so its determinant has degree m. If $\mathcal{O}(k)$ occurs in the splitting of E, self-duality makes $\mathcal{O}(m-k)$ occur as well, and global generation gives $k, m-k \geq 0$, so every splitting degree lies in $[0,m]$.

The canonical filtration of E by splitting degree is equivariant for the torus and for S_{mn}, and its degree-l quotient is $\mathcal{L}^l \otimes C_l$ for a representation C_l whose torus character is a priori a Laurent polynomial. Taking fibers proves (B.2) as identities of Laurent characters. A weight $q^\sigma t^\tau$ of C_l contributes $q^{\sigma+mdl}t^\tau$ at one endpoint and $q^\sigma t^{\tau+el}$ at the other; neither endpoint has negative degrees,

and all multiplicities are nonnegative, so $\sigma, \tau \geq 0$. Each C_l is therefore a positive polynomial character, and subtraction proves (B.3). □

The quotient $\mathcal{J}^{(m)}_{\lambda,\mu}$ is the character of the module $H^0(\mathbb{P}^1, E \otimes \mathcal{L}^{-1})$: after the twist every splitting degree is at least -1, so H^1 vanishes, and the character of $H^0(\mathcal{L}^{l-1})$ is the geometric sum in (B.3).

B.2. **Defect comparisons under a box move.** A fusion defect compares a plethysm with a fused polynomial, so across a box move both differences are needed. The fused side is Theorem B.1; at $m = 1$, with $I = C_0$ and $J = C_1$, it reads

$$\widetilde{H}_\lambda = I + q^d J, \qquad \widetilde{H}_\mu = I + t^e J, \qquad I, J \succeq_s 0.$$

For the plethysms, the one-row Cauchy identity $H_m[I + zJ] = \sum_r \begin{bmatrix} m \\ r \end{bmatrix}_q z^r H_{m-r}[I] H_r[J]$, at $z = q^d$ and at $z = t^e$, gives a second positive quotient, in which every term is a graded character:

$$\begin{aligned} \Xi_{\lambda,\mu,m} &:= \frac{H_m[\widetilde{H}_\lambda] - H_m[\widetilde{H}_\mu]}{q^d - t^e} \\ &= \sum_{r=1}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_q H_{m-r}[I] H_r[J] \sum_{l=0}^{r-1} q^{d(r-1-l)} t^{el} \succeq_s 0. \end{aligned}$$

The difference of the two defects combines $\Xi_{\lambda,\mu,m}$ and $\mathcal{J}^{(m)}_{\lambda,\mu}$, with coefficients that are positive once $t = q^c$ for suitable c.

Theorem B.2 (Fusion defects under a box move)**.** *For the move* (B.1)*, let* $c \in \mathbb{Z}_{>0}$ *satisfy* $d \leq ce \leq md$*. Then*

$$\begin{aligned} &\mathcal{D}_{\lambda,m}(q, q^c) - \mathcal{D}_{\mu,m}(q, q^c) \\ &\quad = q^d [ce - d]_q \, \Xi_{\lambda,\mu,m}(q, q^c) + q^{ce} [md - ce]_q \, \mathcal{J}^{(m)}_{\lambda,\mu}(q, q^c) \succeq_s 0. \end{aligned}$$

Proof. Before specializing,

$$(1 - q)(\mathcal{D}_{\lambda,m} - \mathcal{D}_{\mu,m}) = (q^d - t^e)\Xi_{\lambda,\mu,m} - (q^{md} - t^e)\mathcal{J}^{(m)}_{\lambda,\mu}.$$

Setting $t = q^c$ and dividing each monomial difference by $1 - q$ gives the stated formula, whose q-integers $[ce - d]_q$ and $[md - ce]_q$ are nonnegative because $d \leq ce \leq md$. □

Dominance is generated by box moves, so applying Theorem B.2 along a chain orders the defects of all comparable shapes once m is large.

Corollary B.3 (Uniform dominance comparison of defects)**.** *If* $\lambda \succeq \mu$ *are partitions of* $n \geq 2$ *and* $m \geq (n-1)^2$*, then*

$$\mathcal{D}_{\lambda,m}(q, q^{n-1}) - \mathcal{D}_{\mu,m}(q, q^{n-1}) \succeq_s 0.$$

In particular, for $\rho \vdash 6$ *and* $b \geq 25$*,*

$$\mathcal{D}_{(6),b}(q) - \mathcal{D}_{\rho,b}(q, q^5) \succeq_s 0.$$

Proof. Each box move (B.1) in a chain from λ to μ has $1 \leq d, e \leq n - 1$, so $c = n - 1$ gives $d \leq ce \leq (n-1)^2 \leq m \leq md$. Theorem B.2 therefore applies to every move, and adding the comparisons along the chain proves the corollary. □

B.3. **Why a non-row intermediate cannot bridge the two defects.** The unshifted comparison of $\mathcal{D}_{(a),b}$ with $\mathcal{D}_{(b),a}$ could pass through the defect $\mathcal{D}_{\rho,b}$ of a non-row partition $\rho \vdash a$: Corollary B.3 compares $\mathcal{D}_{(a),b}$ with $\mathcal{D}_{\rho,b}$ for the same $m = b$, so it would remain to show $\mathcal{D}_{\rho,b} \succeq_s \mathcal{D}_{(b),a}$. Proposition B.5 shows that this comparison fails, already for total dimensions. Both obstructions use two invariants of a defect, its coefficient of the standard character and its total dimension, which we compute first.

Lemma B.4 (Standard multiplicity and total dimension)**.** *Let $\mu \vdash n \geq 1$, $m \geq 2$, and $N = mn$, and define*

$$C_{r,m}(q) = \frac{[r]_q[m]_q - [rm]_q}{1-q}.$$

Then the coefficient of the standard character $s_{(N-1,1)}$ in the defect is

$$[s_{(N-1,1)}]\mathcal{D}_{\mu,m} = \sum_{i\geq 0} t^i C_{\mu_{i+1},m}(q). \tag{B.4}$$

For $r \geq 2$,

$$C_{r,m} = [r]_q \sum_{j=1}^{m-1} q^j [j(r-1)]_q$$

has degree $rm-2$ and leading coefficient one; $C_{1,m} = 0$. Writing $\operatorname{Dim} f = \sum_{\lambda\vdash N} d_\lambda [s_\lambda] f(1,1)$, we also have

$$\operatorname{Dim} \mathcal{D}_{\mu,m} = \frac{N!}{4}\, m(m-1)\left(\sum_i \mu_i^2 - 1\right). \tag{B.5}$$

Proof. With $B_\mu(q,t) = \sum_{i\geq 0} t^i [\mu_{i+1}]_q$, the modified Macdonald evaluation formula (Macdonald, 1995; Haiman, 2001), in zero-based cell coordinates, reads

$$\widetilde{H}_\mu[1-u;q,t] = \prod_{(i,j)\in\mu} (1-uq^j t^i) = 1 - uB_\mu(q,t) + O(u^2).$$

To expand the plethysm, we use the coinvariant Cauchy identity, which follows from (1.1),

$$H_m[1+Y;q] = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q H_j[Y;q].$$

Adams operations send u to u^ℓ, so an alphabet $Y = O(u)$ gives $H_j[Y;q] = O(u^j)$, and substituting the Macdonald expansion gives

$$H_m[\widetilde{H}_\mu;q][1-u] = 1 - u[m]_q B_\mu(q,t) + O(u^2).$$

Schur coefficients are read off from such evaluations, since each homogeneous f of degree N satisfies

$$f[1-u] = (1-u)\sum_{r=0}^{N-1} (-u)^r [s_{(N-r,1^r)}] f;$$

by Jacobi–Trudi, non-hook Schur functions vanish at $1-u$, and a hook evaluates to $(1-u)(-u)^r$. Both numerator characters in (2.1) have trivial coefficient one, so comparing the coefficients of u gives (B.4). Finally, $[rm]_q = [r]_q \sum_{j=0}^{m-1} q^{rj}$ gives the positive formula for $C_{r,m}$ and its leading term.

For the dimension, both numerator characters in (2.1) have total S_N-dimension $N!$, so the divided difference at $q=1$ is $N!$ times the mean x-degree of the fused character minus that of the plethysm. Sign reciprocity gives the inner Garsia–Haiman module mean x-degree $n(\mu')/2$, and the wreath construction adds the degrees of m inner factors and an outer coinvariant factor, so the mean degrees of the plethysm and of the fused character are, respectively,

$$\frac{mn(\mu') + c_m}{2}, \qquad \frac{n((m\mu)')}{2}.$$

Since $n((m\mu)') - mn(\mu') = c_m \sum_i \mu_i^2$, the difference of the two means gives (B.5). □

For a non-row inner shape, both invariants fall short of those of $\mathcal{D}_{(b),a}$.

Proposition B.5 (Obstruction to a non-row intermediate)**.** *Let $a \geq 2$, $b \geq a$, and $\rho \vdash a$ with $\rho \neq (a)$. Then*

$$[q^{ab-2} s_{(ab-1,1)}](\mathcal{D}_{\rho,b}(q, q^{a-1}) - \mathcal{D}_{(b),a}(q)) = -1. \tag{B.6}$$

Moreover, independently of the specialization $t = q^c$,

$$\operatorname{Dim} \mathcal{D}_{\rho,b} - \operatorname{Dim} \mathcal{D}_{(b),a} \leq -\frac{(ab)!}{4}(a-1)(b-1)(a+b) < 0.$$

Equality in the displayed bound occurs for $\rho = (a-1, 1)$.

Proof. By (B.4), a nonzero summand of the standard coefficient of $\mathcal{D}_{\rho,b}(q, q^{a-1})$ has $r = \rho_{i+1} \geq 2$ and degree $br - 2 + (a-1)i$. If $i = 0$, then $r < a$ because $\rho \neq (a)$; if $i \geq 1$, the rows above give $a - r \geq i$, hence

$$ab - br - (a-1)i \geq (b - a + 1)i > 0.$$

Every such degree is therefore below $ab - 2$, while the standard coefficient of $\mathcal{D}_{(b),a}$ is $C_{b,a}$, of degree $ab - 2$ with leading coefficient one, which proves (B.6).

For every non-row partition ρ of a, $\sum_i \rho_i^2 - 1 \leq (a-1)^2$, with equality for $(a-1, 1)$, and (B.5) applied to both defects gives

$$\begin{aligned} \operatorname{Dim} \mathcal{D}_{\rho,b} - \operatorname{Dim} \mathcal{D}_{(b),a} &\leq \frac{(ab)!}{4}(b(b-1)(a-1)^2 - a(a-1)(b^2-1)) \\ &= -\frac{(ab)!}{4}(a-1)(b-1)(a+b). \end{aligned}$$

Evaluation at $(q, t) = (1, 1)$ does not see the specialization $t = q^c$. □

The standard coefficient alone also gives an obstruction independent of the specialization, since

$$C_{r,m}(1) = m(m-1)\binom{r}{2}, \qquad [s_{(N-1,1)}]\mathcal{D}_{\mu,m}(1,1) = m(m-1)n(\mu').$$

For a non-row $\rho \vdash a$, this value is strictly smaller than the corresponding total for $\mathcal{D}_{(b),a}$. In contrast, $C_{a,b} = C_{b,a}$, so the standard coefficient of the full $F_{a,b}$ vanishes identically, while

$$\operatorname{Dim} F_{a,b} = \frac{(ab)!}{4}(a-1)(b-1)(b-a) \geq 0.$$

Thus $\mathcal{D}_{(a),b}$ has sufficient total dimension, but every non-row intermediate is too small. Constant convex combinations of such intermediates have the same obstruction.

Appendix C. Matrix polarization and positive kernels

Polarization turns each positive character of this paper into a kernel on tuples of vectors that is positive semidefinite coefficient by coefficient, which is stronger than nonnegativity of the corresponding matrix function on Gram matrices.

For a permutation $\sigma \in S_n$ of cycle type ρ, the graded trace of σ on the coinvariant algebra R_n is

$$\kappa_n(\sigma; q) = \frac{(q;q)_n}{\prod_{j \in \rho}(1 - q^j)},$$

because quotienting the polynomial ring by invariant generators of degrees $1, \ldots, n$ turns the graded trace $1/\det(I - q\sigma)$ into $(q;q)_n/\det(I - q\sigma)$. Hence $H_n = \sum_{\rho \vdash n} \kappa_n(\rho; q) p_\rho / z_\rho$.

C.1. **Polarization and positive kernels.** For a homogeneous symmetric function f of degree n and an $n \times n$ matrix A, with ordinary commuting entries, put

$$\mathfrak{M}_n(f)(A) = [x_1 \cdots x_n] f(XA), \qquad X = \operatorname{diag}(x_1, \ldots, x_n),$$

where $f(XA)$ is evaluated through $p_r(XA) = \operatorname{tr}((XA)^r)$, so that no diagonalization is needed. If $f = \sum_{\mu \vdash n} a_\mu p_\mu$, define the central function $\chi_f(\sigma) = z_{\mu(\sigma)} a_{\mu(\sigma)}$. Then

$$\mathfrak{M}_n(f)(A) = \sum_{\sigma \in S_n} \chi_f(\sigma) \prod_i a_{i,\sigma(i)},$$

and in particular $\mathfrak{M}_n(s_\lambda)$ is the immanant $\operatorname{Imm}_\lambda(A) = \sum_\sigma \chi^\lambda(\sigma) \prod_i a_{i,\sigma(i)}$ of the irreducible character χ^λ of S_n.

To see the formula, expand each trace $p_r(XA) = \operatorname{tr}((XA)^r)$. A squarefree term in $p_\rho(XA)$ is a permutation with cycle type ρ; choosing the starting point of each cycle and ordering cycles of equal length gives exactly z_ρ copies of its matrix monomial. Hence $\mathfrak{M}_n(p_\rho/z_\rho)$ is the sum of those monomials.

Since $H_n = \sum_\mu \kappa_n(\mu; q) p_\mu / z_\mu$, the formula gives $\mathfrak{M}_n(H_n)(A; q) = \sum_\sigma \kappa_n(\sigma; q) \prod_i a_{i,\sigma(i)}$. In particular,

$$\mathfrak{M}_n(H_n)(A; 0) = \operatorname{per} A, \qquad \mathfrak{M}_n(H_n)(A; 1) = n! \prod_i a_{ii}.$$

For tuples $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{C}^d$, the cross-Gram kernel of f is

$$K_f(\mathbf{u}, \mathbf{v}) = \mathfrak{M}_n(f)([\langle u_i, v_j \rangle]_{i,j=1}^n).$$

The kernel K_f detects the Schur coefficients of f.

Proposition C.1 (Schur coefficients are kernel eigenvalues)**.** *Suppose $f = \sum_{\lambda \vdash n} c_\lambda s_\lambda$ has real coefficients. The kernel K_f on $(\mathbb{C}^d)^n$ is positive semidefinite if and only if $c_\lambda \geq 0$ for every λ with $\ell(\lambda) \leq d$. For $d \geq n$, this condition is equivalent to Schur positivity of f. The same equivalence holds coefficient by coefficient for polynomial parameters.*

Proof. Let $T_f = \sum_\sigma \chi_f(\sigma) U_\sigma$, where U_σ permutes the factors of $(\mathbb{C}^d)^{\otimes n}$. For the pure tensors associated with $\mathbf{u}, \mathbf{v}$, the kernel is $K_f(\mathbf{u}, \mathbf{v}) = \langle u_1 \otimes \cdots \otimes u_n, T_f(v_1 \otimes \cdots \otimes v_n) \rangle$. Pure tensors span the tensor space, so positivity of the kernel is equivalent to $T_f \succeq 0$. Character orthogonality gives T_f the eigenvalue $(n!/d_\lambda) c_\lambda$ on the S_n-isotypic component of type λ, and by Schur–Weyl duality exactly the types with $\ell(\lambda) \leq d$ occur in $(\mathbb{C}^d)^{\otimes n}$. Applying the argument to each coefficient of the parameters gives the last assertion. □

By Proposition C.1, Bergeron's conjecture says exactly that the kernel $K_{F_{a,b}}$ is positive semidefinite coefficient by coefficient. Nonnegativity of $\mathfrak{M}_{ab}(F_{a,b})(A)$ on positive semidefinite A is weaker, because it tests $T_{F_{a,b}}$ only on pure tensors.

C.2. **Kernels from the positive characters.** By Proposition C.1, each positive character of this paper defines a kernel that is positive semidefinite coefficient by coefficient. In degree $m|\mu|$, the defects $\mathcal{E}_{\mu,m}$ and $\mathcal{D}_{\mu,m}$ give such kernels in q, t, and the shifted quotient of Theorem 1.3 gives one in q for all $a, b \geq 2$ and $c \geq 1$. The same holds, in their stated parameter ranges, for the reversed quotients of Theorem 2.1, the rectangular quotients of Proposition 3.3 and Theorem 3.5, the quotient $\mathcal{J}^{(m)}_{\lambda,\mu}$ of Theorem B.1, and the dominance differences of Corollary B.3. All these kernels act on arbitrary tuples of vectors; consequences of the unshifted results on restricted tuples are in the companion paper (Sra, 2026b).

C.3. **Rectangular plethysms.** Bergeron's rectangular conjecture replaces the inner one-row character by a modified Macdonald polynomial. By Haiman's positivity theorem and Proposition C.1, the polarizations $\mathfrak{M}_n(\widetilde{H}_\mu)(A;q,t)$ define positive semidefinite cross-Gram kernels for $q,t \geq 0$; for example,

$$\mathfrak{M}_3(\widetilde{H}_{(2,1)})(A;q,t) = \operatorname{per} A + (q+t)\operatorname{Imm}_{(2,1)}(A) + qt \det A.$$

For $a \leq c, d \leq b$, $ab = cd$, and $k \geq 1$, Bergeron (2017, Conjecture 4) asserts Schur positivity of (3.6), and by Proposition C.1 this is equivalent to the positivity of the cross-Gram kernel of the difference in (3.6), with the parameters read in the same way. Sra (2026b) realizes these rectangular plethysms through wreath products.