SHARP ORDER-FOUR SIGN REGULARITY OF A JACOBI THETA KERNEL

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ABSTRACT. We prove that the Hankel kernel $\Phi(\sqrt{x+y})$, where Φ is the Jacobi theta kernel whose cosine transform is the Riemann Ξ-function, is strictly reverse-regular through order four. The order is sharp: an interval enclosure of the fifth Hankel–Wronskian at $t = 10^{-2}$ shows failure at order five. The third-order condition is Csordas's conjecture that $s'^2 - ss''$ is strictly log-concave for $s(t) = \Phi(\sqrt{t})$. The proof reduces the kernel minors to Wronskian signs and establishes the signs by exact polynomial estimates for $t \geq 1$ and interval arithmetic on $[0, 1]$. The fourth-order sign also yields three positive iterates of $s \mapsto s'^2 - ss''$, the triple Turán inequalities for the normalized Taylor coefficients of Ξ, and quartic integral inequalities for Ξ.

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1. INTRODUCTION

The theta kernel

$$\Phi(x) = \sum_{n=1}^{\infty} \pi n^2 (2\pi n^2 e^{4x} - 3) \exp(5x - \pi n^2 e^{4x}) \tag{1.1}$$

is an even, rapidly decreasing, positive kernel connected with the Riemann ξ-function. In the notation of Csordas (2015), it appears in the cosine transform representation

$$H(z) = \int_0^{\infty} \Phi(t) \cos(zt)\, dt.$$

With the standard normalization

$$\xi(z) = \tfrac{1}{2} z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z), \qquad \Xi(z) = \xi(\tfrac{1}{2} + iz),$$

the exact relation is $H(z) = \Xi(z/2)/8$. Positivity, Turán inequalities, generalized Laguerre inequalities, and sign regularity connect properties of Φ to the zero distribution of Ξ; see (Csordas et al., 1986; Csordas and Dimitrov, 2000; Coffey and Csordas, 2013; Csordas, 2015; Nuttall, 2013; Karlin, 1968).

Csordas's Open Problem 4.14 (Csordas, 2015), which restates Problem 3.3 of Csordas and Dimitrov (2000), concerns the square-root composition

$$s(t) = \Phi(\sqrt{t}), \qquad t > 0,$$

and the Turán expression

$$f(t) = s'(t)^2 - s(t)s''(t).$$

The problem asks whether

$$(\log f)''(t) < 0 \qquad (t > 0). \tag{1.2}$$

The positivity $f(t) > 0$ follows from the strict logarithmic concavity of $s(t)$, proved by Csordas and Varga (1988, Theorem 2.1); see also (Csordas, 2015, Theorem 4.2(b)). Thus (1.2) asserts that the first Turán expression attached to s is itself strictly log-concave. Csordas and Dimitrov (2000) showed that (1.2) implies the double Turán inequalities.

Planat and Solé (2026) have recently proved (1.2), and with it the third-order sign-regularity condition for the kernel $\Phi(\sqrt{x+y})$, by two computer-assisted arguments that share a local interval certificate near $t = 0$: one continues with theta-tail estimates, and the other uses Jacobi's differential

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equation for θ_3. The argument in this paper was developed independently[1] and is also computer-assisted: exact polynomial estimates handle $t \geq 1$, and interval arithmetic handles $0 \leq t \leq 1$.

Inequality (1.2) is not an isolated condition. Section 2 shows that it is equivalent to the sign of a 3×3 Hankel determinant of derivatives of s, which is precisely the third infinitesimal sign-regularity condition for the Hankel kernel

$$K(x, y) = s(x + y) = \Phi(\sqrt{x + y}).$$

The equivalence raises the question of how far the sign regularity of K extends. For the related kernel $\Phi(x + y)$, Nuttall (2013) proved the third condition and reported numerical evidence that the fourth holds and the fifth fails. We prove that, for K, the fourth-order condition holds and the fifth fails. The third-order condition yields the double Turán inequalities discussed by Csordas and Dimitrov (2000), as Planat and Solé (2026) note, and the fourth-order condition also gives the triple Turán inequalities and certain quartic Fourier inequalities.

1.1. **Main results.** For a smooth function g, define the Hankel–Wronskians

$$W_m[g](t) = \det(g^{(i+j-2)}(t))_{i,j=1}^{m}. \tag{1.3}$$

Let

$$\varepsilon_m = (-1)^{m(m-1)/2}.$$

The order-m infinitesimal reverse-regularity sign is

$$\varepsilon_m W_m[g](t) > 0.$$

A kernel K on $[0, \infty)^2$ is *strictly reverse-regular through order q* if

$$\varepsilon_m \det[K(x_i, y_j)]_{i,j=1}^{m} > 0 \quad (1 \leq m \leq q)$$

for all $0 \leq x_1 < \cdots < x_m$ and $0 \leq y_1 < \cdots < y_m$. Dropping strictness defines RR_q. For a Hankel kernel $s(x + y)$, Lemma 2.2 shows that the Wronskian signs imply the signs of these ordered minors.

Theorem 1.1 (Finite reverse regularity and sharp failure)**.** *Let $s(t) = \Phi(\sqrt{t})$. Under the interval-arithmetic trust model in Assumption 1.5,*

$$\varepsilon_m W_m[s](t) > 0 \qquad (t \geq 0,\ m = 1, 2, 3, 4), \tag{1.4}$$

where the value at $t = 0$ is interpreted by analytic continuation of s. In particular,

$$W_1 > 0, \qquad W_2 < 0, \qquad W_3 < 0, \qquad W_4 > 0.$$

Furthermore, the fifth condition fails: at $t = 10^{-2}$ one has

$$W_5[s](10^{-2}) < 0, \tag{1.5}$$

whereas $\varepsilon_5 = +1$. Consequently the Hankel kernel $K(x, y) = \Phi(\sqrt{x + y})$ is strictly reverse-regular through order four, but is not RR_5.

The $m = 3$ case is exactly Csordas's Open Problem 4.14. By Lemma 2.1,

$$(\log f)''(t) = \frac{s(t)W_3[s](t)}{f(t)^2}.$$

Since $s > 0$ and $f > 0$, the inequality $(\log f)'' < 0$ is equivalent to $W_3[s] < 0$. Through the Desnanot–Jacobi identity, the signs for $m \leq 4$ also make the first three iterates of $g \mapsto g'^2 - gg''$ positive at s (Corollary 4.1).

[1] Our reconstructed chronology of this AI-assisted work records an explicit global proof claim for Csordas's conjecture on 25 February 2026. The first reported successful full run of a corrected compact verifier dates to 6 March 2026. A further verifier bug was repaired on 1 May, but the available record does not establish a completed run of that repair. On 23 May, compact certification was explicitly reported successful for the stronger RR_4 theorem; the failure of RR_5 was also obtained that day. These results were not publicly reported at the time. We record these milestones to document the paper's development and make no claim of priority.

Theorem 1.1 has two consequences for the Riemann Ξ-function itself. The first concerns its Taylor coefficients. Let

$$H(x) = \int_0^\infty \Phi(t)\cos(xt)\,dt = \sum_{k=0}^{\infty}(-1)^k \frac{\beta_k}{(2k)!}x^{2k}, \qquad \gamma_k = \frac{k!\beta_k}{(2k)!}. \tag{1.6}$$

Then the infinite Hankel matrix $(\gamma_{i+j})_{i,j\geq 0}$ inherits strict reverse regularity through order four. The same transfer works for real indices: for $a \geq 0$, set

$$\beta_a = \int_0^\infty \Phi(r) r^{2a}\,dr, \qquad \gamma_a = \frac{\sqrt{\pi}\,\beta_a}{4^a\Gamma(a+1/2)}.$$

At integer indices, γ_a agrees with (1.6).

Theorem 1.2 (Coefficient reverse regularity)**.** *Under Assumption 1.5, for $m = 1, 2, 3, 4$, and for all strictly increasing nonnegative real sequences*

$$r_0 < \cdots < r_{m-1}, \qquad c_0 < \cdots < c_{m-1},$$

one has

$$\varepsilon_m \det(\gamma_{r_i+c_j})_{i,j=0}^{m-1} > 0. \tag{1.7}$$

In particular, the consecutive Hankel minors satisfy

$$\varepsilon_m \det(\gamma_{n+i+j})_{i,j=0}^{m-1} > 0 \qquad (n \geq 0,\ m \leq 4).$$

At integer indices, determinant condensation gives the first three iterates of the Turán operator.

Corollary 1.3 (Turán, double Turán, and triple Turán)**.** *Define*

$$T_k = \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \quad (k \geq 1),$$
$$E_k = T_k^2 - T_{k-1}T_{k+1} \quad (k \geq 2),$$

and

$$Q_k = E_k^2 - E_{k-1}E_{k+1} \quad (k \geq 3).$$

Under Assumption 1.5,

$$T_k > 0 \quad (k \geq 1), \qquad E_k > 0 \quad (k \geq 2), \qquad Q_k > 0 \quad (k \geq 3).$$

The second consequence is a Fourier-side family of multilinear integral inequalities for the Riemann Ξ-function. Define

$$\Xi_\Phi(\lambda) = 2\int_0^\infty \Phi(r)\cos(\lambda r)\,dr. \tag{1.8}$$

Thus $\Xi_\Phi(\lambda) = \Xi(\lambda/2)/4 = 2H(\lambda)$. Fourier inversion gives

$$\Phi(r) = \frac{1}{\pi}\int_0^\infty \Xi_\Phi(\lambda)\cos(r\lambda)\,d\lambda.$$

For $r > 0$, put

$$\Theta_j(r,\lambda) = \left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^j \cos(r\lambda). \tag{1.9}$$

Theorem 1.4 (Fourier-side determinant inequalities)**.** *Under Assumption 1.5, for every $r > 0$ and every $m = 1, 2, 3, 4$,*

$$\varepsilon_m \int_{(0,\infty)^m} \left(\prod_{\ell=1}^m \Xi_\Phi(\lambda_\ell)\right) \det(\Theta_{i+j-2}(r,\lambda_j))_{i,j=1}^m\,d\lambda_1\cdots d\lambda_m > 0. \tag{1.10}$$

For $m = 5$, the left-hand side is negative at $r = 1/10$.

The $m = 3$ case is the Fourier-inverted form of Csordas's conjecture; the $m = 4$ case is a quartic constraint on Ξ_Φ.

1.2. **Strategy and computer assistance.** By Lemma 2.2, Theorem 1.1 reduces to the signs of the Wronskians $W_m[s]$. For $m = 3, 4$ the proof has an analytic tail component and a compact computer-assisted component. For $t \geq 1$ the first term of the theta series dominates, and the signs of W_3 and W_4 reduce to exact polynomial and integer inequalities. For $0 \leq t \leq 1$ we verify the signs in interval arithmetic, and a single interval enclosure at $t = 10^{-2}$ gives the failure at order five. The coefficient, differential, and Fourier consequences inherit the same arithmetic assumption.

Assumption 1.5 (Interval-arithmetic trust model)**.** *The computer-assisted statements use exact Python integer arithmetic, exact symbolic polynomial operations in SymPy, and the outward interval operations of* `mpmath.iv`*. We assume these implementations are correct.*

All error bounds and sign tests run in interval arithmetic with 90 decimal digits, with integer factorials and rational partition endpoints; the overlap of the t- and u-ranges is checked exactly.
Limitations. The signs on the compact range rest on Assumption 1.5; an independent implementation in another interval package would provide an additional check. The sharp order-four boundary concerns the specific Hankel kernel $\Phi(\sqrt{x+y})$. The order-five failure rules out reverse regularity of all orders for that kernel, but not higher positivity orders after a different transform: the coefficient integration in Section 4.2 need not preserve a failure of sign, so the fifth-order witness does not prove failure of the fifth coefficient Hankel condition or of later Turán iterates. Finally, the coefficient and Fourier inequalities are not pointwise Laguerre inequalities for Ξ; in particular, they do not by themselves prove $\Xi'^2 - \Xi\Xi'' \geq 0$ on $\mathbb{R}$. For orders two and three, the consecutive coefficient inequalities are necessary conditions for the Riemann hypothesis (Csordas and Dimitrov, 2000); for order four this is not known, since Karlin's conjecture fails in general.

1.3. **Related work.** The theta kernel Φ and its Fourier transform have long been used to formulate necessary conditions for the Riemann hypothesis. Csordas et al. (1986) proved Pólya's Turán inequalities for the Taylor coefficients of ξ; their key step is the strict log-concavity of $\int_t^\infty \Phi(\sqrt{u})\,du$, that is, reverse regularity of order two for the corresponding Hankel kernel. For the kernel $\Phi(x+y)$, Nuttall (2013) worked through the Wronskians $\det[\Phi^{(i+j-2)}(t)]$ and proposed a cumulant-based route past the obstruction at order five. Whether the regularity orders of $\Phi(x+y)$ and $\Phi(\sqrt{x+y})$ are related is not addressed here. Csordas (2015) lists further inequalities and open problems for Φ, and the general theory of totally positive and sign-regular kernels is due to Karlin (1968).

2. From logarithmic concavity to Hankel–Wronskians

Two reductions turn Open Problem 4.14 and its higher-order analogues into sign conditions on Hankel–Wronskians. Lemma 2.1 identifies (1.2) with the sign of $W_3[s]$, and Lemma 2.2 shows that Wronskian signs give the signs of all ordered kernel minors.

Let s be a C^4 function and put

$$f = s'^2 - ss''.$$

Define

$$W_3[s](t) = \det \begin{pmatrix} s & s' & s'' \\ s' & s'' & s^{(3)} \\ s'' & s^{(3)} & s^{(4)} \end{pmatrix}(t).$$

Lemma 2.1. *For every C^4 function s,*

$$ff'' - (f')^2 = sW_3[s]. \tag{2.1}$$

Thus, wherever $s > 0$ and $f > 0$,

$$(\log f)'' = \frac{sW_3[s]}{f^2}. \tag{2.2}$$

Proof. Differentiating $f = s'^2 - ss''$ gives

$$f' = s's'' - ss^{(3)}, \qquad f'' = (s'')^2 - ss^{(4)}.$$

Expanding $ff'' - (f')^2$ gives

$$s\{ss''s^{(4)} - s(s^{(3)})^2 - s'^2 s^{(4)} + 2s's''s^{(3)} - (s'')^3\},$$

which is $sW_3[s]$. Dividing by f^2 proves (2.2). □

For $s(t) = \Phi(\sqrt{t})$, the Csordas–Varga theorem gives $s > 0$ and $f > 0$ on $(0, \infty)$. Hence Open Problem 4.14 is equivalent to

$$W_3[s](t) < 0 \qquad (t > 0). \tag{2.3}$$

The second reduction passes from Wronskians to the ordered minors in the definition of reverse regularity. We include the classical Chebyshev-system argument (Karlin, 1968), with the derivative signs appropriate to reverse regularity, because Section 3.4 also needs its converse.

Lemma 2.2 (Wronskians and ordered minors)**.** *Let $s \in C^{2q-2}([0, \infty))$. If $\varepsilon_j W_j[s](t) > 0$ for $t \geq 0$ and $1 \leq j \leq q$, then $s(x+y)$ is strictly reverse-regular through order q. Conversely, reverse regularity through order q implies $\varepsilon_j W_j[s](t) \geq 0$ for $t > 0$.*

Proof. We first recall the evaluation criterion for a Chebyshev system. If the initial Wronskians of $f_1, \dots, f_k$ are positive on an interval, then $\det[f_j(x_i)]_{i,j=1}^k > 0$ at increasing nodes. For completeness, divide the ith row by $f_1(x_i) > 0$ and put $g_j = (f_{j+1}/f_1)'$. Their initial Wronskians satisfy

$$W(g_1, \dots, g_j) = \frac{W(f_1, \dots, f_{j+1})}{f_1^{j+1}} > 0.$$

Subtract consecutive normalized rows, from bottom to top, and expand in the first column. The remaining determinant is the integral of $\det[g_j(t_i)]$ over $\prod_{i=1}^{k-1}[x_i, x_{i+1}]$. Induction makes its integrand positive in the interior, proving the criterion, including endpoints.

Fix $z \leq 0$ and apply this criterion in $x \geq 0$ to $(-1)^j s^{(j)}(x - z)$, $0 \leq j < k$. Their initial Wronskians are $\varepsilon_j W_j[s](x - z) > 0$. For fixed increasing x_i, the resulting evaluation determinants are the transposes of the initial Wronskians in z of $s(x_i - z)$. A second application gives $\det[s(x_i - z_j)] > 0$ for increasing $z_j \leq 0$. Set $z_j = -y_{k+1-j}$ and reverse the columns; their reversal has sign ε_k.

For the converse, coalesce increasing x- and y-nodes around positive points whose sum is t. Divide the determinant by the two positive Vandermonde products and pass to the limit. The limit is $W_j[s](t)/\prod_{i=0}^{j-1}(i!)^2$, proving the necessary sign. □

3. Proof of Theorem 1.1

By Lemma 2.2, Theorem 1.1 reduces to the signs of $W_m[s]$. To read these signs off the theta series, we first write each $W_m[s]$ as a positive factor times a polynomial in normalized theta sums; the tail, compact, and order-five arguments all work with this polynomial.

3.1. Derivative polynomials and normalized determinants.

Set

$$t = u^2, \qquad y = \pi e^{4u}.$$

Define polynomials q_k by

$$q_0(y) = 2y - 3, \qquad q_{k+1}(y) = 4yq_k'(y) + (5 - 4y)q_k(y). \tag{3.1}$$

Then

$$\Phi^{(k)}(u) = \pi e^{5u-y} U_k(u), \tag{3.2}$$

where

$$U_k(u) = \sum_{n=1}^{\infty} n^2 q_k(n^2 y) e^{-(n^2-1)y}. \tag{3.3}$$

Induction proves this identity: differentiating a summand applies the operator $4y\partial_y + (5-4y)$ to the polynomial factor.

The substitution $t = u^2$ introduces powers of u^{-1}, so we remove them before evaluating the determinants. Define integers $a_{r,k}$ by $a_{0,0} = 1$, zero outside $0 \le k \le r$, and

$$a_{r+1,k} = (k-2r)a_{r,k} + a_{r,k-1}.$$

Repeated application of $(2u)^{-1}\partial_u$ gives

$$\frac{s^{(r)}(u^2)}{\pi e^{5u-y}} = \frac{R_r(U;u)}{2^r u^{2r}}, \qquad R_r(U;u) = \sum_{k=0}^{r} a_{r,k} u^k U_k.$$

For $r \ge 1$, $a_{r,0} = 0$, so R_r is divisible by u. Every term in $\det[R_{i+j}]_{i,j=0}^{m-1}$ contains at least $m-1$ such factors. Thus

$$P_m(U;u) = u^{1-m} \det[R_{i+j}(U;u)]_{i,j=0}^{m-1}$$

is a polynomial, homogeneous of degree m in U, and

$$W_m[s](u^2) = \frac{(\pi e^{5u-y})^m}{2^{m(m-1)} u^{(m-1)(2m-1)}} P_m(U;u), \qquad u > 0. \tag{3.4}$$

In particular, the denominators for $m = 3, 4$ are $64u^{10}$ and $4096u^{21}$. The recurrence determines every polynomial using exact integer arithmetic. Near $u = 0$ we instead use the Taylor series in t, which avoids division by small powers of u.

The compact verification of W_3 in Section 3.3 works with the cubic polynomial explicitly:

$$\begin{aligned} N(u) =& 6U_0U_1^2 - 12uU_0U_1U_2 - u^3U_0U_1U_4 + 6u^2U_0U_2^2 + u^4U_0U_2U_4 - u^4U_0U_3^2 \\ &+ 10uU_1^3 - 6u^2U_1^2U_2 + 4u^3U_1^2U_3 - u^4U_1^2U_4 - 3u^3U_1U_2^2 + 2u^4U_1U_2U_3 - u^4U_2^3, \end{aligned} \tag{3.5}$$

and

$$W_3[s](u^2) = \frac{(\pi e^{5u-y})^3}{64u^{10}} N(u).$$

3.2. **Analytic tail estimates.** For $t \ge 1$ the first theta summand dominates every U_k, and exact estimates fix the signs of W_3 and W_4.

Proposition 3.1 (Tail signs)**.** *For $t \ge 1$,*

$$W_3[s](t) < 0, \qquad W_4[s](t) > 0.$$

Proof. For $u \ge 1$, put $y = \pi e^{4u} > 159$ and write $U_k = A_k + \eta_k$, where $A_k = q_k(y)$ is the $n = 1$ contribution. We bound $P_m(A;u)$ away from zero, then show that the $n \ge 2$ tail cannot change its sign.

The first summand. For $m = 3$, write $P_3(A;u) = \sum_{j=0}^{6} a_j(u) y^j$. The leading coefficient is $a_6(u) = -1024u(4u-1)^3 \le -27648u^4$. The absolute coefficient sums of $a_0, \dots, a_5$ are

$$(S_0, \dots, S_5) = (10800, 268200, 1450800, 3197280, 2223936, 776192).$$

Each a_j has degree at most four. Since $\sum_{j=0}^{5} S_j/159^{6-j} < 5000$, it follows that

$$P_3(A;u) < -22648u^4y^6 \le -10000u^4y^6.$$

For $m = 4$, write $P_4(A;u) = \sum_{j=0}^{10} b_j(u) y^j$. Here $b_{10}(u) = 786432u^3(4u-1)^6 \ge 573308928u^9$. For $j = 0, \dots, 9$, the constants

$$(C_0, \dots, C_9) = (1, 1, 3, 9, 4, 66, 60, 53, 23, 6)$$

satisfy $|b_j(u)| \le C_j b_{10}(u)$: substituting $u = v + 1$ makes every coefficient of $C_j b_{10} \pm b_j$ nonnegative. The rational inequality $\sum_{j=0}^{9} C_j/159^{10-j} < 1/25$ therefore gives

$$P_4(A;u) \ge \frac{24}{25}\,573308928u^9y^{10} > 500000000u^9y^{10}.$$

The recurrences of Section 3.1 generate the coefficients in these finite checks, which we verify in exact integer arithmetic.

The remaining summands. Let P_* be the largest absolute coefficient sum among the required q_k. Since $y \ge 159$,

$$|A_k| \le P_* y^d =: M, \qquad |\eta_k| \le P_* y^d 2^{2d+3} e^{-3y} =: \delta,$$

where $(d, P_*) = (5, 151269)$ for $m = 3$ and $(d, P_*) = (7, 51450613)$ for $m = 4$. Indeed, $|q_k(n^2y)| \le P_*(n^2y)^d$, and the ratio of successive terms $n^{2d+2}e^{-(n^2-1)y}$ is at most $(3/2)^{2d+2}e^{-5y} < 1/2$ for $n \ge 2$. Summing from $n = 2$ gives the displayed bound; also $\delta < M$.

The absolute coefficient sums of P_3 and P_4, after bounding their powers of u by u^4 and u^9, are 54 and 17348. A degree-m monomial changes by at most $((M+\delta)^m - M^m) \le (2^m - 1)M^{m-1}\delta$. Using the constant $648 \ge 7 \cdot 54$ for the cubic bound gives

$$|P_3(U;u) - P_3(A;u)| \le 648 \cdot 2^{13} P_*^3 u^4 y^{15} e^{-3y},$$
$$|P_4(U;u) - P_4(A;u)| \le 15 \cdot 17348 \cdot 2^{17} P_*^4 u^9 y^{28} e^{-3y}.$$

Both comparisons reduce to integer inequalities, because y^9e^{-3y} and $y^{18}e^{-3y}$ decrease for $y \ge 159$ and $e^{-477} < 2^{-477}$:

$$648 \cdot 2^{13}(151269)^3 159^9 < 5000 \cdot 2^{477},$$
$$15 \cdot 17348 \cdot 2^{17}(51450613)^4 159^{18} < 50000000 \cdot 2^{477}.$$

Consequently $P_3(U;u) < -5000u^4y^6$ and $P_4(U;u) > 450000000u^9y^{10}$. The prefactors in (3.4) are positive, which proves the proposition. □

3.3. **Compact verification.** On $0 \le t \le 1$ no theta summand dominates, and we verify the signs in interval arithmetic: with Taylor polynomials in t near zero, and with the theta series in u elsewhere.

Proposition 3.2 (Compact signs)**.** *Under Assumption 1.5,*

$$W_3[s](t) < 0, \qquad W_4[s](t) > 0, \qquad 0 \le t \le 1.$$

Proof. ***Small*** t**.** The evenness and analyticity of Φ at zero give

$$s(t) = \sum_{j \ge 0} a_j t^j, \qquad a_j = \frac{\Phi^{(2j)}(0)}{(2j)!}.$$

For $0 < \rho < \pi/8$, the real part of e^{4u} is at least $e^{-4\rho}\cos(4\rho) > 0$ on $|u| \le \rho$, so the theta series gives the explicit Cauchy bound

$$|\Phi(u)| \le M_\rho := \sum_{n \ge 1} \pi n^2 (2\pi n^2 e^{4\rho} + 3) e^{5\rho - \pi n^2 e^{-4\rho}\cos(4\rho)}.$$

Thus $|a_j| \le M_\rho \rho^{-2j}$. If $p_M(t) = \sum_{j=0}^{M} a_j t^j$ and $q = t_0/\rho^2 < 1$, then, for $0 \le t \le t_0$,

$$|s^{(r)}(t) - p_M^{(r)}(t)| \le M_\rho t_0^{-r} \frac{(M+1)^r q^{M+1}}{1 - ((M+2)/(M+1))^r q}, \tag{3.6}$$

provided the denominator is positive. To prove (3.6), bound the falling factorial by j^r and sum the resulting geometric majorant. The coefficients a_j and M_ρ are themselves enclosed by theta series with geometric tail bounds.

For W_3, we use $(t_0, \rho, M) = (1/100, 3/10, 30)$ and 500 equal rational subintervals. For W_4, we use $(t_0, \rho, M) = (3/200, 19/50, 50)$ and 750 subintervals. We expand $\det[p_M^{(i+j)}]$ as a polynomial before

evaluating it on each interval. If its entries have modulus at most B, and every derivative error is at most δ, the determinant error is at most $m!m(B+\delta)^{m-1}\delta$. The determinant bound follows by telescoping the m factors in each of the $m!$ determinant terms.

Away from zero. For the cubic, we use the mean-value bound

$$N(u) \le N(c) + h \sup_{v\in[c-h,c+h]} |N'(v)|$$

on a subdivision of $99/1000 \le u \le 1$ into 28175 rational intervals. The derivative is obtained from $U_k' = U_{k+1} - (5-4y)U_k$. Each U_k is enclosed by its first four theta summands and the geometric tail bound of Section 3.4 with $N = 4$. For the quartic, substitute the first four theta summands into P_4, producing a finite function $F_4(u)$. On each interval $[c-h, c+h]$, use

$$F_4(u) \ge F_4(c) - \sum_{j=1}^{6} \frac{|F_4^{(j)}(c)|h^j}{j!} - \frac{h^7}{7!} \sup_{v\in[c-h,c+h]} |F_4^{(7)}(v)|.$$

An additional bound $4C(B+\delta)^3\delta$ controls the omitted $n \ge 5$ summands; here B bounds the four-summand values of the U_k and δ their tails on the interval, and C is the sum of the absolute coefficient bounds of P_4 on the interval. There are 71 rational subintervals covering $153/1250 \le u \le 1$, with widths at most $1/200$ below $1/5$, $1/100$ up to $1/2$, and $1/50$ thereafter. The strict rational inequalities

$$(99/1000)^2 < 1/100, \qquad (153/1250)^2 < 3/200$$

ensure overlap with the small-t regions.

Evaluated in interval arithmetic, these bounds give a positive signed lower bound on every subinterval. □

3.4. **Failure at order five.** The fifth-order witness is not needed for Open Problem 4.14; it shows that order four in Theorem 1.1 is sharp.

Proposition 3.3 (Failure of RR_5)**.** *Under Assumption 1.5, the kernel $K(x,y) = \Phi(\sqrt{x+y})$ is not RR_5.*

Proof. For $m = 5$, $\varepsilon_5 = +1$. If K were RR_5, coalescing five increasing nodes in each variable would force $W_5[s](t) \ge 0$ for every $t > 0$ (Lemma 2.2). We show that this sign fails at $t = 10^{-2}$. Set $u = 1/10$ and $y = \pi e^{4u}$, and define the positively scaled determinant

$$\widetilde{W}_5(u) = \frac{W_5[s](u^2)}{(\pi e^{5u-y})^5}.$$

Interval arithmetic gives

$$-1.106 \cdot 10^{29} < \widetilde{W}_5(1/10) < -1.105 \cdot 10^{29}. \tag{3.7}$$

The tail bound used for each U_k, $0 \le k \le 8$, is

$$\left|\sum_{n>N} n^2 q_k(n^2 y) e^{-(n^2-1)y}\right| \le S_k y^{k+1} \frac{(N+1)^{2k+4} e^{-((N+1)^2-1)y}}{1 - ((N+2)/(N+1))^{2k+4} e^{-(2N+3)y}},$$

where S_k is the absolute coefficient sum of q_k and $N = 12$. The denominator is positive. Its geometric ratio bounds all subsequent ratios because both factors decrease with n. We insert these enclosures into the exact chain-rule recurrence and evaluate all 5! signed determinant terms in interval arithmetic. Thus (3.7) encloses both series truncation and arithmetic error, and $W_5[s](10^{-2}) < 0$ contradicts the necessary sign. □

Proof of Theorem 1.1. Propositions 3.1 and 3.2 prove the $m = 3, 4$ signs. Positivity gives $m = 1$, and the strict log-concavity of s (Csordas and Varga, 1988, Theorem 2.1) gives $m = 2$ for $t > 0$. At zero, an interval enclosure of the theta series gives $-38.050 < W_2[s](0) < -38.049$. Lemma 2.2

now proves the strict kernel-minor assertion. Finally, (3.7) gives (1.5), and Proposition 3.3 gives the failure of RR_5. □

4. Consequences of the order-four signs

Theorem 1.1 gives strict signs for $W_1[s], \dots, W_4[s]$. We transfer them in three directions: to iterates of $g \mapsto g'^2 - gg''$ by the Desnanot–Jacobi identity, to the Taylor coefficients of Ξ by a moment integral, and to integral inequalities for Ξ by Fourier inversion.

4.1. **A differential hierarchy.** The fourth-order condition has a differential consequence for s itself. Write $W_0[s] = 1$ and $\tau_m = \varepsilon_m W_m[s]$. The Desnanot–Jacobi identity for consecutive derivative determinants gives

$$W_m W_m'' - (W_m')^2 = W_{m-1} W_{m+1}. \tag{4.1}$$

Indeed, differentiating W_m leaves only the term in which its last row is differentiated; all earlier terms have equal adjacent rows. Differentiating this last-row determinant by columns gives W_m''. Apply Desnanot–Jacobi to the $(m+1) \times (m+1)$ derivative matrix, deleting its last two rows and columns. The four border minors are W_m, W_m', W_m', W_m'', and the central minor is W_{m-1}. Since $\varepsilon_{m-1}\varepsilon_{m+1} = -1$, wherever the τ_j are positive,

$$(\log \tau_m)'' = -\frac{\tau_{m-1}\tau_{m+1}}{\tau_m^2}. \tag{4.2}$$

Corollary 4.1 (Differential concavity hierarchy)**.** *Under Assumption 1.5, the functions s, $s'^2 - ss'' = -W_2[s]$, and $-W_3[s]$ are strictly log-concave on $[0,\infty)$. If $\mathcal{L}(g) = g'^2 - gg''$, then*

$$\mathcal{L}(s) = \tau_2 > 0, \qquad \mathcal{L}^2(s) = s\tau_3 > 0, \qquad \mathcal{L}^3(s) = \tau_2(\tau_3^2 + s^2\tau_4) > 0. \tag{4.3}$$

In contrast, $(\log W_4[s])''(10^{-2}) > 0$.

Proof. Theorem 1.1 and (4.2) give the three concavity assertions. Applying $\mathcal{L}(g) = -g^2(\log g)''$ first to τ_2 and then to $s\tau_3$ gives (4.3). Finally, $(\log W_4)'' = W_3 W_5 / W_4^2 > 0$ at the certified witness. □

4.2. **Coefficient inequalities.** The numbers β_a are moments of s, so integrating the kernel $s(x+y)$ against powers in each variable transfers its minors to the matrix $(\gamma_{r_i+c_j})$. The transfer applies the continuous Cauchy–Binet (Andréief) formula twice and handles the power determinants as Chebyshev systems.

Proof of Theorem 1.2. Since

$$\beta_k = \int_0^\infty \Phi(r) r^{2k}\, dr = \frac{1}{2}\int_0^\infty s(u) u^{k-1/2}\, du,$$

consider, for increasing nonnegative real sequences r_i, c_j, the matrix

$$I_{ij} = \int_0^\infty \int_0^\infty x^{r_i - 3/4} y^{c_j - 3/4} s(x+y)\, dx\, dy.$$

All integrals converge: the exponents at zero exceed -1, and s decays faster than any power at infinity. Applying the continuous Cauchy–Binet formula twice gives

$$\det I = \int_{x_1 < \cdots < x_m} \int_{y_1 < \cdots < y_m} \det[x_j^{r_i - 3/4}] \det[s(x_i + y_j)] \det[y_i^{c_j - 3/4}]\, d\boldsymbol{y}\, d\boldsymbol{x},$$

with all variables positive. The power determinants are positive: their initial Wronskians have the sign of the Vandermonde in the increasing exponents, so the evaluation criterion applies. By Theorem 1.1, the middle determinant has sign ε_m for $m \le 4$. Hence $\varepsilon_m \det I > 0$, with strictness on the whole integration domain. Changing variables $w = x + y$, $v = x/(x+y)$, gives

$$I_{ij} = B\left(r_i + \frac{1}{4}, c_j + \frac{1}{4}\right) \int_0^\infty s(w) w^{r_i + c_j - 1/2}\, dw = 2B\left(r_i + \frac{1}{4}, c_j + \frac{1}{4}\right) \beta_{r_i + c_j}.$$

Using

$$\gamma_a = \frac{\sqrt{\pi}}{4^a \Gamma(a+1/2)} \beta_a,$$

we may write

$$I_{ij} = \frac{2}{\sqrt{\pi}} \big[4^{r_i}\Gamma(r_i+1/4)\big]\big[4^{c_j}\Gamma(c_j+1/4)\big]\gamma_{r_i+c_j}.$$

All displayed row and column factors are positive. Hence $\det(I_{ij})$ and $\det(\gamma_{r_i+c_j})$ have the same sign, which proves (1.7). □

Proof of Corollary 1.3. Let

$$\Delta_m(n) = \det(\gamma_{n+i+j})_{i,j=0}^{m-1}.$$

Theorem 1.2 gives $\Delta_2(n) < 0$, $\Delta_3(n) < 0$, and $\Delta_4(n) > 0$. Since

$$\Delta_2(n) = \gamma_n\gamma_{n+2} - \gamma_{n+1}^2 = -T_{n+1},$$

we get $T_k > 0$. Desnanot–Jacobi condensation gives

$$\Delta_3(n)\gamma_{n+2} = \Delta_2(n)\Delta_2(n+2) - \Delta_2(n+1)^2 = -E_{n+2}.$$

Since $\Delta_3(n) < 0$ and $\gamma_{n+2} > 0$, the identity gives $E_k > 0$. A second condensation identity gives

$$\Delta_4(n)\Delta_2(n+2) = \Delta_3(n)\Delta_3(n+2) - \Delta_3(n+1)^2.$$

Because $\Delta_4(n) > 0$ and $\Delta_2(n+2) < 0$, the sequence $D_n = -\Delta_3(n) > 0$ is strictly log-concave:

$$D_{n+1}^2 > D_n D_{n+2}.$$

Since $E_{n+2} = \gamma_{n+2}D_n$, the same identities give the more explicit factorization

$$Q_k = T_k(D_{k-2}^2 + \gamma_{k-1}\gamma_{k+1}\Delta_4(k-3)) > 0, \qquad k \geq 3. \tag{4.4}$$

□

4.3. **Fourier-side inequalities.** Fourier inversion writes each derivative of s as an integral of Ξ_Φ against an explicit kernel, so each Wronskian sign becomes a multilinear integral inequality for Ξ_Φ. The theta kernel and all its derivatives decay rapidly on the real line, so Φ and Ξ_Φ are Schwartz functions. For fixed $r > 0$, each $\Theta_j(r,\lambda)$ has at most polynomial growth in λ. Thus differentiation under the Fourier integral and the applications of Fubini below are justified by absolute convergence. Fourier inversion gives

$$s(t) = \Phi(\sqrt{t}) = \frac{1}{\pi}\int_0^\infty \Xi_\Phi(\lambda)\cos(\lambda\sqrt{t})\,d\lambda.$$

Writing $r = \sqrt{t}$, we have

$$s^{(j)}(t) = \frac{1}{\pi}\int_0^\infty \Xi_\Phi(\lambda)\Theta_j(r,\lambda)\,d\lambda,$$

where Θ_j is defined by (1.9). Multilinearity of the determinant in its columns gives

$$W_m[s](r^2) = \frac{1}{\pi^m}\int_{(0,\infty)^m}\left(\prod_{\ell=1}^m \Xi_\Phi(\lambda_\ell)\right)\det(\Theta_{i+j-2}(r,\lambda_j))_{i,j=1}^m\,d\lambda_1\cdots d\lambda_m.$$

Multiplying by ε_m and using Theorem 1.1 gives (1.10) for $m \leq 4$. The failure for $m = 5$ follows from (1.5), completing the proof of Theorem 1.4.

The first four kernels are

$$\Theta_0(r,\lambda) = \cos(r\lambda),$$

$$\Theta_1(r,\lambda) = -\frac{\lambda}{2r}\sin(r\lambda),$$

$$\Theta_2(r,\lambda) = \frac{\lambda}{4r^3}\sin(r\lambda) - \frac{\lambda^2}{4r^2}\cos(r\lambda),$$

$$\Theta_3(r,\lambda)=\frac{r^2\lambda^3\sin(r\lambda)+3r\lambda^2\cos(r\lambda)-3\lambda\sin(r\lambda)}{8r^5}.$$

For $m=2$, the inequality is the quadratic integral constraint

$$\int_0^\infty\int_0^\infty \Xi_\Phi(\lambda)\Xi_\Phi(\mu)\mathcal{K}_2(r;\lambda,\mu)\,d\lambda\,d\mu>0,$$

where symmetrizing in λ,μ gives

$$\begin{aligned}\mathcal{K}_2(r;\lambda,\mu)=&\frac{\lambda\mu}{4r^2}\sin(r\lambda)\sin(r\mu)+\frac{\lambda^2+\mu^2}{8r^2}\cos(r\lambda)\cos(r\mu)\\&-\frac{\lambda\sin(r\lambda)\cos(r\mu)+\mu\cos(r\lambda)\sin(r\mu)}{8r^3}.\end{aligned}$$

The $m=3$ and $m=4$ cases are the corresponding cubic and quartic determinant inequalities.

AI USAGE

The results in this paper were developed with an AI assistant (OpenAI's GPT Pro). None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs.

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