MONOTONICITY OF TURÁN-RATIOS OF SYMMETRIC POLYNOMIALS

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ABSTRACT. We study the *Turán-ratio* $S_{\lambda,i}(x)^2/(S_{\lambda,i-1}(x)S_{\lambda,i+1}(x))$, where $S_{\lambda,i}(x) := s_\lambda(1^{[n-i]}, x^{[i]})$, and s_λ is the Schur polynomial in n variables (indexed by partition λ) with i arguments equal to x and $n-i$ of them equal to 1. Specifically, we prove that this Turán-ratio is strictly increasing for $x > 1$ and $1 \le i < n$ (unless all n padded parts of λ are equal, then it is identically one). For one-row shapes, our result answers a question of Ait-Haddou about complete homogeneous symmetric polynomials. For this case, we also offer a second proof through the zeros of Meixner polynomials that gives an explicit positive formula for the logarithmic derivative. We conclude with a Turán-ratio conjecture for Macdonald polynomials.

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1. INTRODUCTION

Let s_λ be the Schur polynomial in n variables indexed by a partition λ with at most n parts. Writing $a^{[k]}$ for k repetitions, i.e., the *multiplicity* of a scalar a, we set

$$S_{\lambda,i}(x) = s_\lambda(1^{[n-i]}, x^{[i]}), \quad 0 \le i \le n \quad \mathcal{T}_{\lambda,i}(x) := \frac{S_{\lambda,i}(x)^2}{S_{\lambda,i-1}(x)S_{\lambda,i+1}(x)}, \quad 1 \le i < n. \tag{1.1}$$

If $\mathcal{T}_{\lambda,i}$ is strictly increasing for $x > 1$, then $S_{\lambda,i}(x)^2 > S_{\lambda,i-1}(x)S_{\lambda,i+1}(x)$, an inequality of Turán type; we therefore call $\mathcal{T}_{\lambda,i}$ a *Turán-ratio*. Our main result is a proof of this monotonicity property.

Theorem 1.1 (Schur multiplicity comparison)**.** *Let $n \ge 2$ and let λ be a partition with at most n parts, padded with zeros to length n. If $\lambda_1 > \lambda_n$, then $\mathcal{T}_{\lambda,i}$ is strictly increasing on $(1,\infty)$ for every $1 \le i < n$. If $\lambda = (r^n)$, all these ratios equal one.*

The monotonicity question originates with a question asked by Ait-Haddou (2024), who asked about the one-row $\lambda = (\ell)$ version of (1.1). More specifically, in this case $s_{(\ell)} = h_\ell$, where h_ℓ is the complete homogeneous symmetric polynomial of degree ℓ:

$$h_\ell(y) = \sum_{1 \le j_1 \le j_2 \le \cdots \le j_\ell \le n} y_{j_1} y_{j_2} \cdots y_{j_\ell}, \qquad h_0 = 1. \tag{1.2}$$

For integers $\ell \ge 1$ and $n \ge 2$, we rewrite $H_i = S_{(\ell),i}$ and $\Psi_{i,n} = \mathcal{T}_{(\ell),i}$, so that (1.1) becomes

$$H_i(x) = h_\ell(1^{[n-i]}, x^{[i]}), \qquad \Psi_{i,n}(x) = \frac{H_i(x)^2}{H_{i-1}(x)H_{i+1}(x)}. \tag{1.3}$$

Ait-Haddou (2024) asked whether $\Psi_{i,n}$ is increasing on $(1,\infty)$ for $n \ge 4$ and $2 \le i \le n-2$, and reported a proof for $\Psi_{2,4}$. Taking $\lambda = (\ell)$ in Theorem 1.1 resolves Ait-Haddou's question.

Corollary 1.2. *For integers $\ell \ge 1$, $n \ge 2$, and $1 \le i < n$, $\Psi_{i,n}$ is strictly increasing on $(1,\infty)$.*

Section 2 proves Theorem 1.1 through a Toda identity for the confluent determinants of González-Serrano and Maximenko (2025) and a maximum principle that treats all multiplicities i at once. For one-row shapes, Section 3 gives an independent proof through the zeros of Meixner polynomials,

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which also yields a positive formula for the log-derivative of $\Psi_{i,n}$. Section 4 *conjectures* (with some evidence) that the same monotonicity holds for Macdonald polynomials $P_\lambda(\,\cdot\,;q,t)$ when the repeated arguments are replaced by an aligned split of the principal specialization $(1,t,\dots,t^{n-1})$.

2. Proof of Theorem 1.1: Turán-ratios for Schur polynomials

The ratio $\mathcal{T}_{\lambda,i}$ compares three Schur values that differ only in the multiplicity of x. By the confluent bialternant formula for Schur polynomials, these values are determinants from one family $(D_0,\dots,D_n)$, and adjacent members of such a family satisfy a *Toda identity*: a second-order differential identity in $t=\log x$ that links D_i to $D_{i\pm1}$. This identity provides the key structure that we exploit and turn into a maximum principle for the derivatives of $\log \mathcal{T}_{\lambda,i}(e^t)$.

2.1. The Toda identity.

First, we make a change of coordinates $t=\log x$. Rewriting (1.1),

$$f_i(t) := \log S_{\lambda,i}(e^t), \qquad u_i(t) := \log \mathcal{T}_{\lambda,i}(e^t) = 2f_i(t) - f_{i-1}(t) - f_{i+1}(t),$$

our goal is to prove $u_i'(t)>0$ on $(0,\infty)$, but we have no direct handle on u_i'. What the determinant structure of the $S_{\lambda,i}$ gives is a differential equation, and it is second order in f_i: Lemma 2.1 expresses f_i'' through u_i. Taking the second difference in i of both sides then gives u_i'' in terms of u_{i-1},u_i,u_{i+1}, eliminating f; differentiating once more then gives a linear system for the u_i', whose sign Section 2.2 controls from its boundary values. The structure of Schur polynomials is key to this lemma.

Lemma 2.1 (Toda)**.** *Let $n\ge 2$; let λ be a partition with at most n parts; for $t>0$ and $1\le i<n$,*

$$f_i''(t) = \frac{i(n-i)}{4\sinh^2(t/2)}\bigl(1-e^{-u_i(t)}\bigr). \tag{2.1}$$

Proof. ***Step 1***. We first express $S_{\lambda,i}$ as a determinant. Pad λ with zeros and set $a_j=\lambda_{n-j}+j$ for $0\le j<n$, so that $a_0<\cdots<a_{n-1}$. The *confluent bialternant formula* of González-Serrano and Maximenko (2025, Theorem 1.1 and (4.5)) evaluates a Schur polynomial at repeated arguments as a quotient of two determinants; for the alphabet $(1^{[n-i]},x^{[i]})$ it reads

$$S_{\lambda,i}(x) = \det G_\lambda / \det G_\varnothing,$$

where G_λ has rows indexed by j and columns $\frac{1}{r!}\partial_y^r y^{a_j}$, at $y=1$ for $0\le r<n-i$ and at $y=x$ for $0\le r<i$; and $G_\varnothing$ is the same matrix for $\lambda=\varnothing$, so that $\det G_\varnothing=(x-1)^{i(n-i)}$ (González-Serrano and Maximenko, 2025, (4.6)), with sign $+$ for the row order $a_0<\cdots<a_{n-1}$.

The derivatives in y are not that convenient. Step 2 needs a determinant whose entries are successive derivatives in $t=\log x$ of one function, and $\partial_t = y\partial_y$ acts on $y^{a_j}=e^{a_jt}$ by multiplication with a_j. We thus replace ∂_y^r by $(y\partial_y)^r$: with the row vector $d_j=(1,a_j,\dots,a_j^{n-i-1},e^{a_jt},a_je^{a_jt},\dots,a_j^{i-1}e^{a_jt})$, define $D_i(t):=\det[d_j]_{j=0}^{n-1}$, whose columns are $(y\partial_y)^r y^{a_j}$ at $y=1$ for $0\le r<n-i$ and at $y=x=e^t$ for $0\le r<i$. Since $(y\partial_y)^r = y^r\partial_y^r + \sum_{k<r} c_{r,k}\,y^k\partial_y^k$ with constants $c_{r,k}$, subtracting from each column of D_i suitable multiples of the lower-order columns of the same block leaves $y^r\partial_y^r y^{a_j}$, which is $r!\,y^r$ times the corresponding column of G_λ. The product of these scalars over both blocks gives

$$D_i(t) = C_i\,x^{i(i-1)/2}\det G_\lambda, \quad \text{where} \quad C_i=\Bigl(\prod\nolimits_{r=0}^{n-i-1} r!\Bigr)\Bigl(\prod\nolimits_{r=0}^{i-1} r!\Bigr),$$

and the confluent bialternant formula turns this into

$$D_i(t) = C_i e^{i(n-1)t/2}[2\sinh(t/2)]^{i(n-i)} S_{\lambda,i}(e^t). \tag{2.2}$$

For $i=0,n$, this formula follows from ordinary Vandermonde; in particular, $D_i(t)>0$ for $t>0$.

Step 2. Identity (2.1) relates f_i'' to $u_i=2f_i-f_{i-1}-f_{i+1}$, and by (2.2) each f_k is $\log D_k$ up to explicit terms; so (2.1) is a relation between D_i and its neighbors $D_{i\pm1}$. Hankel determinants of successive derivatives of one function carry such a relation: for $K_i=\det[g^{(r+s)}]_{r,s<i}$, the Desnanot–Jacobi

identity gives $K_iK_i'' - (K_i')^2 = K_{i-1}K_{i+1}$. We show that D_i is such a determinant up to a constant, and then read off (2.1). Introduce

$$g(t) = \sum_{j=0}^{n-1} \frac{e^{a_j t}}{\prod_{k\neq j}(a_j - a_k)}, \qquad K_i(t) = \det[g^{(r+s)}(t)]_{r,s=0}^{i-1}, \qquad K_0 = 1.$$

Cauchy–Binet expands K_i over subsets of i exponents, and expanding D_i by its last i columns gives the same terms, with

$$K_i(t) = (-1)^{i(i-1)/2}\frac{D_i(t)}{\Delta(a)}, \qquad \Delta(a) = \prod_{j<k}(a_k - a_j). \tag{2.3}$$

In the Desnanot–Jacobi identity the two bordered minors are K_i' and K_i'': differentiating K_i row by row, only the derivative of the last row survives, since differentiating any other row duplicates the next one; differentiating that minor column by column likewise leaves only the last column, so K_i'' is the minor with last row and last column shifted. By (2.3), $\tau_i = (-1)^{i(i-1)/2}K_i = D_i/\Delta(a) > 0$, and the alternating signs give

$$(\log \tau_i)'' = -\frac{\tau_{i-1}\tau_{i+1}}{\tau_i^2}, \qquad 1 \le i < n.$$

By (2.2)–(2.3) and $C_{i-1}C_{i+1}/C_i^2 = i(n-i)$, the right-hand side is $-i(n-i)e^{-u_i}/[4\sinh^2(t/2)]$, and the left-hand side is $f_i'' - i(n-i)/[4\sinh^2(t/2)]$. This proves (2.1). □

2.2. **A maximum principle for the derivatives.** The proof of Theorem 1.1 differentiates (2.1) once more and arrives at the system (2.4) for the derivatives u_i'. The next lemma shows that components of such a system that are nonnegative at both ends of $(0,\infty)$ are nonnegative throughout, and positive unless they all vanish. The coupling in (2.4) has the right sign for the sum of the negative parts of the components to satisfy a scalar differential inequality, and the ordinary maximum principle then applies to that sum.

Lemma 2.2 (Maximum principle)**.** *Let $n \ge 2$, let p and $b_1, \dots, b_{n-1} > 0$ be continuous on $(0,\infty)$, and let $v_1, \dots, v_{n-1} \in C^2(0,\infty)$ satisfy*

$$v_i'' + pv_i' = 2b_iv_i - b_{i-1}v_{i-1} - b_{i+1}v_{i+1}, \qquad 1 \le i < n, \tag{2.4}$$

where $b_0v_0 = b_nv_n = 0$. If each v_i has nonnegative limits at 0 and ∞, then $v_i \ge 0$ for every i. If moreover $v_i(t_0) = 0$ for some i and some $t_0 > 0$, then $v_k \equiv 0$ for every k.

Proof. We show that $w = \sum_{i=1}^{n-1}\max\{-v_i, 0\}$ vanishes identically. At a point where the nonzero signs are fixed let $I = \{i : v_i < 0\}$. Summing the equations over these indices gives

$$w'' + pw' = -\sum_{j=1}^{n-1} b_jv_j(2\mathbf{1}_I(j) - \mathbf{1}_I(j-1) - \mathbf{1}_I(j+1)) \ge 0.$$

Here $\mathbf{1}_I(0) = \mathbf{1}_I(n) = 0$. For $j \in I$, the factor in parentheses is nonnegative and $v_j < 0$; for $j \notin I$, the signs are reversed. The positive-part chain rule preserves this inequality distributionally: a crossing through zero contributes a nonnegative jump to the derivative of the negative part.

With $P' = p$, this inequality reads $(e^P w')' \ge 0$, so $w \le \max\{w(a), w(b)\}$ on every $[a,b] \subset (0,\infty)$. Since $w \to 0$ at 0 and at ∞, letting $a \to 0$ and $b \to \infty$ gives $w \equiv 0$.

Suppose $v_i(t_0) = 0$ with $t_0 > 0$. Since $v_i \ge 0$, we have $v_i'(t_0) = 0$ and $v_i''(t_0) \ge 0$, whereas (2.4) forces $v_{i\pm1}(t_0) = 0$. Repeating in i gives $v_k(t_0) = v_k'(t_0) = 0$ for every k, and uniqueness for the linear system gives $v_k \equiv 0$. □

Proof of Theorem 1.1. We first bring Lemma 2.1 into the form (2.4). Write

$$\alpha_i = i(n-i), \qquad c(t) = \frac{1}{4\sinh^2(t/2)}, \qquad b_i(t) = \alpha_i c(t)e^{-u_i(t)}, \qquad v_i(t) = u_i'(t),$$

and set $b_0 = b_n = 0$. Since $2\alpha_i - \alpha_{i-1} - \alpha_{i+1} = 2$, Lemma 2.1 gives

$$u_i'' = 2c - 2b_i + b_{i-1} + b_{i+1}.$$

For $p(t) = \coth(t/2)$, the derivatives are $c' = -pc$ and $b_i' = -(p + v_i)b_i$. Differentiating the last display therefore cancels the terms involving c and gives exactly (2.4).

Lemma 2.2 needs the boundary values of v_i. At $t = 0$ the arguments coincide, and symmetry and homogeneity give $f_i'(0) = i|\lambda|/n$, so $v_i(0) = 0$. At infinity, $\deg S_{\lambda,i} = \lambda_1 + \cdots + \lambda_i =: d_i$. To see this, order the alphabet so that the i copies of x come first. A semistandard tableau then has at most i entries from these copies in each column, and filling row j with j attains the bound. Since the leading coefficient is positive,

$$\lim_{t\to\infty} v_i(t) = 2d_i - d_{i-1} - d_{i+1} = \lambda_i - \lambda_{i+1} \geq 0.$$

Lemma 2.2 now gives $v_i \geq 0$ for every i. If $\lambda_1 > \lambda_n$, then $\lambda_i > \lambda_{i+1}$ for some i, so $v_i \not\equiv 0$, and the lemma gives $v_k > 0$ on $(0, \infty)$ for every k; that is, each $\mathcal{T}_{\lambda,k}$ is strictly increasing on $(1, \infty)$. For $\lambda = (r^n)$, the identity $s_\lambda(y) = (y_1 \cdots y_n)^r$ gives $\mathcal{T}_{\lambda,i} = 1$ directly. □

3. Turán-ratios monotonicity for h_ℓ through Meixner zeros

This section gives a second proof of Corollary 1.2, independent of Section 2, together with the positive formula (3.18) for $(\log \Psi_{i,n})'$. To factor the ratio, we express H_i as a polynomial evaluated at the negative multiplicity $-i$. The zeros of this polynomial are positive and move monotonically with x, and factoring $\Psi_{i,n}$ over them proves the corollary.

3.1. **The Meixner representation.** The multiplicity i enters $H_i(x) = h_\ell(1^{[n-i]}, x^{[i]})$ only through the number of arguments equal to x, and the point of this section is that the dependence on i is polynomial, of a classical kind: $H_i(x)$ is a Meixner polynomial evaluated at $-i$. For $x > 1$ put $c = 1/x \in (0, 1)$ and let

$$M_\ell(u; n, c) := {}_2F_1\left(-\ell, -u; n; 1 - \frac{1}{c}\right) = \sum_{r=0}^{\ell} \frac{(-\ell)_r (-u)_r}{(n)_r\, r!} \left(1 - \frac{1}{c}\right)^r \tag{3.1}$$

denote the Meixner polynomial of degree ℓ in the variable u, in the normalization of Koekoek et al. (2010, §9.10). Here $(a)_r = a(a+1)\cdots(a+r-1)$ is the rising factorial, with $(a)_0 = 1$.

Lemma 3.1 (Meixner representation)**.** *For integers $0 \leq i \leq n$ and $x > 1$,*

$$H_i(x) = \binom{n+\ell-1}{\ell} M_\ell(-i; n, c), \qquad c = \frac{1}{x}. \tag{3.2}$$

Proof. Extract the coefficient of t^ℓ from

$$\sum_{L\geq 0} h_L(1^{[n-i]}, x^{[i]}) t^L = (1-t)^{-(n-i)}(1-xt)^{-i} = (1-t)^{-n}\left(1 - \frac{(x-1)t}{1-t}\right)^{-i}.$$

Expanding the last factor and using $[t^{\ell-r}]\,(1-t)^{-n-r} = \binom{n+\ell-1}{\ell-r}$ gives

$$H_i(x) = \sum_{r=0}^{\ell} \frac{(i)_r}{r!} \binom{n+\ell-1}{\ell-r} (x-1)^r,$$

and $\binom{n+\ell-1}{\ell-r} = \binom{n+\ell-1}{\ell}\binom{\ell}{r}\frac{r!}{(n)_r}$ turns this sum into

$$\frac{H_i(x)}{\binom{n+\ell-1}{\ell}} = \sum_{r=0}^{\ell} \binom{\ell}{r} \frac{(i)_r}{(n)_r} (x-1)^r. \tag{3.3}$$

Finally, $\frac{(-\ell)_r}{r!}(1-x)^r = \binom{\ell}{r}(x-1)^r$ and $(-(-i))_r = (i)_r$. Because $1-c^{-1} = 1-x$, the right side of (3.3) is therefore $M_\ell(-i;n,c)$. □

3.2. **Orthogonality and positive zeros.** The proof of the corollary in Section 3.4 factors M_ℓ over its zeros, so (3.2) reduces the problem to the location of those zeros and to their dependence on c. Orthogonality places them on the positive axis; differentiating orthogonality in Section 3.3 then determines how they move with c.

Lemma 3.2 (Orthogonality and positive zeros)**.** *Let $c \in (0,1)$. The polynomials $M_r(\,\cdot\,;n,c)$, $r \ge 0$, are orthogonal for the positive discrete weight*

$$w_c(m) = \frac{(n)_m}{m!}\,c^m, \qquad m = 0,1,2,\dots, \tag{3.4}$$

and $M_\ell(\,\cdot\,;n,c)$ has ℓ simple zeros

$$0 < \zeta_1(c) < \cdots < \zeta_\ell(c). \tag{3.5}$$

Proof. We derive the orthogonality from the generating function, since the derivation also fixes the weight that Section 3.3 differentiates. The generating function in Koekoek et al. (2010, §9.10) is

$$\sum_{r=0}^{\infty} \frac{(n)_r}{r!} M_r(m;n,c)\,t^r = \left(1-\frac{t}{c}\right)^m (1-t)^{-m-n}. \tag{3.6}$$

To compare different degrees, multiply two copies of (3.6), at arguments t and s with $|t|, |s|$ small, and sum against w_c. Including the factor c^m from $w_c(m)$, the dependence on m is $(n)_m z^m/m!$, where

$$z = \frac{(c-t)(c-s)}{c(1-t)(1-s)}, \qquad 1-z = \frac{(1-c)(c-ts)}{c(1-t)(1-s)},$$

so $\sum_{m\ge0} \frac{(n)_m}{m!} z^m = (1-z)^{-n}$ yields

$$\sum_{m=0}^{\infty} w_c(m) \left[\sum_{r\ge0} \frac{(n)_r}{r!} M_r(m;n,c)\,t^r\right]\left[\sum_{q\ge0} \frac{(n)_q}{q!} M_q(m;n,c)\,s^q\right] = (1-c)^{-n}\left(1-\frac{ts}{c}\right)^{-n}. \tag{3.7}$$

The right side contains only equal powers $t^r s^r$, so comparing coefficients gives

$$\sum_{m=0}^{\infty} w_c(m)\, M_r(m;n,c)\, M_q(m;n,c) = 0 \qquad (r \ne q). \tag{3.8}$$

Since $\deg M_r = r$, the polynomial M_ℓ is orthogonal to every polynomial of degree $< \ell$, and the usual sign-change argument yields (3.5). Suppose M_ℓ changed sign at fewer than ℓ points of $(0,\infty)$. Multiplying it by the polynomial of degree $< \ell$ that vanishes at those points would give a product of one fixed sign on the support $\{0,1,2,\dots\}$, not identically zero there, contradicting (3.8). Hence M_ℓ has ℓ sign changes on $(0,\infty)$; these sign changes account for all of its zeros, which are therefore simple and positive. □

3.3. **Monotonicity of the Meixner zeros.** By Markov's monotonicity theorem, the zeros (3.5) move monotonically with c; Castillo et al. (2017, Example 4.5) treat the Meixner case explicitly. We include the quadrature argument because it gives the positive formula (3.14) for ζ_j', which Section 3.4 turns into (3.18).

Put $\langle f,g\rangle_c = \sum_{m\ge0} f(m)\,g(m)\,w_c(m)$, let $p_\ell(u,c)$ be the monic normalization of $M_\ell(u;n,c)$, and write $p_\ell' = \partial_u p_\ell$.

Lemma 3.3 (Motion of the zeros)**.** *Each zero $c \mapsto \zeta_j(c)$ is strictly increasing on $(0,1)$.*

Proof. Fix j and write $\zeta = \zeta_j(c)$. The coefficients of p_ℓ are rational in c, and the zeros are simple by Lemma 3.2, so the implicit function theorem applied to $p_\ell(\zeta_j(c), c) = 0$ shows that ζ_j is differentiable with

$$\partial_c p_\ell(\zeta, c) = -p'_\ell(\zeta, c)\,\zeta'_j(c). \tag{3.9}$$

To find the sign of $\zeta'_j(c)$, we evaluate one inner product in two ways: differentiating orthogonality in c produces a sum of squares, and Gaussian quadrature at the zeros isolates the single zero ζ_j. Put

$$q(u, c) = \frac{p_\ell(u, c)}{u - \zeta}, \tag{3.10}$$

a monic polynomial of degree $\ell - 1$, so that $\langle p_\ell, q\rangle_c = 0$.

First, differentiate this identity in c; the geometric decay of w_c, locally uniformly in c, justifies differentiating term by term. Since $\partial_c \log w_c(m) = m/c$, and since $\langle p_\ell, \partial_c q\rangle_c = 0$ because $\deg \partial_c q < \ell$,

$$\langle \partial_c p_\ell,\, q\rangle_c = -\Big\langle p_\ell\, q,\ \frac{m}{c}\Big\rangle_c. \tag{3.11}$$

Orthogonality also allows us to subtract ζ/c in the second argument on the right of (3.11). Substituting $p_\ell = (m - \zeta)q$ then gives

$$\langle \partial_c p_\ell,\, q\rangle_c = -\frac{1}{c}\sum_{m=0}^{\infty} (m - \zeta)^2\, q(m, c)^2\, w_c(m). \tag{3.12}$$

Second, we evaluate the same inner product by Gaussian quadrature at the nodes (3.5). Let L_k be the Lagrange cardinal polynomials of the nodes and put $\lambda_k = \langle L_k, 1\rangle_c$. Division by p_ℓ shows that $\langle f, 1\rangle_c = \sum_k \lambda_k f(\zeta_k)$ whenever $\deg f \le 2\ell - 1$. Because p_ℓ is monic, $\deg\big((\partial_c p_\ell)\, q\big) \le 2\ell - 2$. Moreover, $q(\zeta_k, c) = 0$ for $k \ne j$ and $q(\zeta, c) = p'_\ell(\zeta, c)$; hence

$$\langle \partial_c p_\ell,\, q\rangle_c = \lambda_j\, \partial_c p_\ell(\zeta, c)\, p'_\ell(\zeta, c). \tag{3.13}$$

Since $\deg L_j^2 = 2\ell - 2$, quadrature gives $\lambda_j = \langle L_j, L_j\rangle_c > 0$. Combining (3.9), (3.12), and (3.13),

$$\boxed{\zeta'_j(c) = \frac{\displaystyle\sum_{m=0}^{\infty} (m - \zeta_j)^2\, q(m, c)^2\, w_c(m)}{c\,\lambda_j\, p'_\ell(\zeta_j, c)^2} \;>\; 0.} \tag{3.14}$$

The numerator is positive because the support is infinite while $q(\,\cdot\,, c)$ has only $\ell - 1$ zeros. □

3.4. **Factorization of the ratio.** Factoring M_ℓ over its zeros places the multiplicity i in each factor separately, as $i + \zeta_j(c)$. The Turán-ratio then becomes a product over the zeros of factors that depend on one zero each, and Lemma 3.3 decides how each factor moves with x.

Second proof of Corollary 1.2. Fix $x > 1$ and $c = 1/x$. Since $M_\ell(0; n, c) = 1$, factoring over the zeros (3.5) gives

$$M_\ell(-i; n, c) = \prod_{j=1}^{\ell} \frac{i + \zeta_j(c)}{\zeta_j(c)}. \tag{3.15}$$

Inserting (3.2) into (1.3) cancels the binomial prefactors and the factors $\zeta_j(c)$, leaving

$$\Psi_{i,n}(x) = \prod_{j=1}^{\ell} \frac{\big(i + \zeta_j(c)\big)^2}{\big(i - 1 + \zeta_j(c)\big)\big(i + 1 + \zeta_j(c)\big)}, \qquad c = \frac{1}{x}. \tag{3.16}$$

Each factor of (3.16) is $e^{\phi_i(\zeta_j(c))}$, where for $i \ge 1$ and $z > 0$

$$\phi_i(z) = 2\log(i + z) - \log(i - 1 + z) - \log(i + 1 + z).$$

With $a = i + z > 1$,

$$\phi_i'(z) = \frac{2}{a} - \frac{1}{a-1} - \frac{1}{a+1} = -\frac{2}{a(a^2-1)} < 0. \tag{3.17}$$

Since $\log \Psi_{i,n}(x) = \sum_j \phi_i(\zeta_j(1/x))$ and $\frac{d}{dx}\zeta_j(1/x) = -x^{-2}\zeta_j'(1/x)$, formulas (3.14) and (3.17) give

$$\boxed{\frac{d}{dx}\log \Psi_{i,n}(x) = \frac{2}{x^2}\sum_{j=1}^{\ell} \frac{\zeta_j'(1/x)}{\big(i+\zeta_j(1/x)\big)\Big(\big(i+\zeta_j(1/x)\big)^2 - 1\Big)} \;>\; 0.} \tag{3.18}$$

□

Remark 3.4 (Why one row)**.** *The proof uses three properties of $i \mapsto H_i(x)$ at fixed ℓ, n, and x: it is a polynomial in i, it has only real zeros, all negative, and these zeros move monotonically in x. The first property holds for every shape. The branching rule gives*

$$S_{\lambda,i}(x) = \sum_{\mu \subseteq \lambda} x^{|\mu|}\, s_\mu(1^{[i]})\, s_{\lambda/\mu}(1^{[n-i]}),$$

and $s_\mu(1^{[m]}) = \prod_{(a,b)\in\mu}(m+b-a)/h(a,b)$ is a polynomial in m (Macdonald, 1995, Chapter I, §3, Example 4), with $h(a,b)$ the hook length of the cell in row a and column b. Hence $S_{\lambda,i}(x)$ is a polynomial in i of degree $|\lambda|$. The second property fails already for one column: for $\lambda = (1,1)$ and $n = 3$, the values $3, 5, 8, 12$ at $i = 0, 1, 2, 3$ give $S_{\lambda,i}(2) = \frac{1}{2}(i^2 + 3i + 6)$, with complex zeros. The reason is a mismatch of parameter ranges. By the generating function $\sum_k e_k(1^{[n-i]}, x^{[i]})z^k = (1+z)^{n-i}(1+xz)^i$, a column is a Krawtchouk polynomial in i, $e_k(1^{[n-i]}, x^{[i]}) = \binom{n}{k}K_k(i; p, n)$ with $p = 1/(1-x)$ (Koekoek et al., 2010, §9.11), and p lies in the orthogonality range $0 < p < 1$ only when $x < 0$. For a row, the Meixner range $0 < c < 1$ is exactly $x > 1$. The factorization over zeros, and with it the explicit formula (3.18), is therefore a one-row phenomenon; for other shapes Theorem 1.1 rests on the Toda identity alone.

4. A Turán-ratio conjecture for Macdonald polynomials

The identity $P_\lambda(z; q, q) = s_\lambda(z)$ suggests extending Theorem 1.1 to Macdonald polynomials. At repeated arguments, however, the extension fails already for a single row (calculations omitted for brevity). A principal specialization gives a different comparison with an exact connection to elementary symmetric functions.

Fix $0 < q, t < 1$ and $n \geq 2$, and use the P-normalization of Macdonald (1995, Chapter VI). Starting from $\mathbf{t}_n = (1, t, \ldots, t^{n-1})$, dilate its first j entries:

$$\mathsf{A}_j(x) = (x, xt, \ldots, xt^{j-1}, t^j, \ldots, t^{n-1}), \qquad 0 \leq j \leq n, \tag{4.1}$$

where empty blocks are omitted. For a partition λ with at most n parts, put

$$\widehat{B}_j(x) = \frac{P_\lambda(\mathsf{A}_j(x); q, t)}{P_\lambda(\mathbf{t}_n; q, t)}, \qquad \widehat{T}_i(x) = \frac{\widehat{B}_i(x)^2}{\widehat{B}_{i-1}(x)\widehat{B}_{i+1}(x)}. \tag{4.2}$$

Then $\widehat{B}_j(1) = 1$, $\widehat{B}_0(x) = 1$, and $\widehat{B}_n(x) = x^{|\lambda|}$. The common normalization cancels from $\widehat{T}_i$, and $\widehat{T}_i \to \mathcal{T}_{\lambda,i}$ as $q = t \uparrow 1$.

Conjecture 4.1 (Macdonald principal-specialization comparison)**.** *Let $0 < q, t < 1$, $n \geq 2$, and pad λ with zeros to length n. If $\lambda_1 > \lambda_n$, then*

$$\widehat{T}_i'(x) > 0 \qquad (x > 1,\ 1 \leq i < n).$$

If $\lambda = (r^n)$, with $r \geq 0$, all these ratios equal one.

Remark 4.2 (Complementation)**.** *For $M \geq \lambda_1$, let $\lambda^c = (M-\lambda_n, \ldots, M-\lambda_1)$. Then $P_{\lambda^c}(z) = (z_1 \cdots z_n)^M P_\lambda(z_1^{-1}, \ldots, z_n^{-1})$ (Macdonald, 1995, Chapter VI, (4.14)), and the inverted alphabet $\mathsf{A}_j(x)^{-1}$ is, after reversal, $x^{-1}t^{1-n}\mathsf{A}_{n-j}(x)$. By symmetry and homogeneity,*

$$P_{\lambda^c}(\mathsf{A}_j(x); q, t) = x^{jM-|\lambda|}\, t^{M\binom{n}{2}-(n-1)|\lambda|}\, P_\lambda(\mathsf{A}_{n-j}(x); q, t),$$

and the prefactor, whose exponent is affine in j, cancels from the second difference. Hence $\widehat{T}_{\lambda^c,i} = \widehat{T}_{\lambda,n-i}$ for $1 \leq i < n$. Conjecture 4.1 therefore needs proof for only one shape of each complementary pair, and the rectangles (r^n) are the fixed points. The same argument gives $\mathcal{T}_{\lambda^c,i} = \mathcal{T}_{\lambda,n-i}$.

Two results support the conjecture: $\widehat{T}_i(q^{-1}) > 1$ when $\lambda_1 > \lambda_n$ (Theorem 4.3), and monotonicity near $x = 1$, for $n = 2$, and at infinity for single rows (Proposition 4.4).

4.1. **Evaluation duality and the first q-step.** The alignment (4.1) lets Macdonald's evaluation duality turn $\widehat{B}_j(q^{-1})$ into normalized elementary symmetric functions, to which the t-Newton inequality applies.

Theorem 4.3 (Comparison at the first q-step)**.** *Under the hypotheses of Conjecture 4.1,*

$$\widehat{T}_i(q^{-1}) \geq 1 \qquad (1 \leq i < n).$$

Every inequality is strict unless $\lambda = (r^n)$.

Pad λ to length n and set $y_\lambda = (q^{-\lambda_1}, q^{-\lambda_2}t, \ldots, q^{-\lambda_n}t^{n-1})$. Macdonald's evaluation symmetry (Macdonald, 1995, Chapter VI, (6.6)) gives, for every integer $k \geq 0$,

$$\widehat{B}_j(q^{-k}) = \frac{P_{(k^j)}(y_\lambda; q, t)}{P_{(k^j)}(\mathbf{t}_n; q, t)}, \qquad 0 \leq j \leq n. \tag{4.3}$$

To check the parameters, apply that symmetry at (q^{-1}, t^{-1}). The identity $P_\rho(z; q^{-1}, t^{-1}) = P_\rho(z; q, t)$ (Macdonald, 1995, Chapter VI, (4.14)(iv)) restores the original parameters, and multiplying each evaluation alphabet by t^{n-1} changes its rth entry from $q^{-\mu_r}t^{-(n-r)}$ to $q^{-\mu_r}t^{r-1}$. Homogeneity cancels this scaling in each normalized evaluation. Taking $\mu = (k^j)$ proves (4.3).

Proof of Theorem 4.3. At $k = 1$, the identity $P_{(1^j)} = e_j$ reduces (4.3) to

$$\widehat{B}_j(q^{-1}) = \frac{e_j(y_\lambda)}{c_j}, \qquad c_j = e_j(\mathbf{t}_n) = t^{j(j-1)/2} \prod_{r=1}^{j} \frac{1-t^{n-j+r}}{1-t^r}. \tag{4.4}$$

For positive $z_1, \ldots, z_n$ satisfying $z_{r+1} \leq t z_r$, the t-Newton inequality states that

$$\left(\frac{e_i(z)}{c_i}\right)^2 \geq \frac{e_{i-1}(z)}{c_{i-1}} \frac{e_{i+1}(z)}{c_{i+1}} \qquad (1 \leq i < n). \tag{4.5}$$

Indeed, the polynomial $\prod_r (1 + z_r u)$ has t-separated negative zeros, so Lamprecht (2012, Theorem 27) gives (4.5). If z is not a scalar multiple of $\mathbf{t}_n$, the polynomial is nonextremal; Theorem 4(c),(d) of the same paper then makes every interior inequality strict. Now

$$\frac{(y_\lambda)_{r+1}}{(y_\lambda)_r} = t q^{\lambda_r - \lambda_{r+1}} \leq t,$$

with equality for every r if and only if $\lambda_1 = \lambda_n$. Substituting $z = y_\lambda$ into (4.5) proves the assertion. In the exceptional case, $P_{(r^n)}(z) = (z_1 \cdots z_n)^r$, so $\widehat{B}_j(x) = x^{rj}$ and $\widehat{T}_i(x) = 1$. □

Since $\widehat{T}_i(1) = 1$, Conjecture 4.1 predicts $\widehat{T}_i(q^{-k}) > 1$ for every $k \geq 1$ when $\lambda_1 > \lambda_n$. By (4.3), this says that $j \mapsto P_{(k^j)}(y_\lambda; q, t)/P_{(k^j)}(\mathbf{t}_n; q, t)$ is strictly log-concave on $0 \leq j \leq n$. Theorem 4.3 is the case $k = 1$; for $k \geq 2$ we have no proof.

4.2. **Monotonicity near $x = 1$, in two variables, and at infinity.**

Proposition 4.4 (Partial monotonicity)**.** *Let λ be padded to length n, with $\lambda_1 > \lambda_n$.*

(i) $(\log \widehat{T}_i)'(1) > 0$ *for* $1 \le i < n$.

(ii) *If $n = 2$, then $\widehat{T}_1$ is strictly increasing on $(1, \infty)$.*

(iii) *If $\lambda = (m)$, then for $2 \le i < n$, with $\gamma_i > 0$ given by (4.10),*

$$x\frac{d}{dx}\log \widehat{T}_i(x) = \frac{\gamma_i}{x} + O(x^{-2}) \qquad (x \to \infty),$$

and for $i = 1$ the left side tends to m. Hence every $\widehat{T}_i$ is strictly increasing for all sufficiently large x.

Proof. (i) Write $D_r = z_r\partial_{z_r}$. Differentiating the first j coordinates in (4.1) and taking the adjacent second difference gives

$$\frac{d}{dx}\log \widehat{T}_i(x)\Big|_{x=1} = (D_i - D_{i+1}) \log P_\lambda(\mathbf{t}_n; q, t) > 0. \tag{4.6}$$

To see the sign, the branching formula (Macdonald, 1995, Chapter VI, (6.24) and (7.14′)) gives nonnegative monomial coefficients. Pair monomials exchanged by coordinates $i, i+1$. For exponents $a > b$, their contribution to $(D_i - D_{i+1})P_\lambda(z)$ is a nonnegative multiple of

$$(a - b)(z_i^a z_{i+1}^b - z_i^b z_{i+1}^a),$$

which is positive when $z_i > z_{i+1}$. Since $\lambda_1 > \lambda_n$, the monomial orbit of λ contains such a pair with positive coefficient, and $t^{i-1} > t^i$ gives strictness.

(ii) Put $d = |\lambda|$ and $f(u) = P_\lambda(u, 1; q, t)$. The function $F(v) = e^{-dv/2}f(e^v)$ is a nonnegative combination of hyperbolic cosines, with a nonconstant term since $\lambda_1 > \lambda_2$. Homogeneity gives

$$\widehat{T}_1(x) = \frac{t^d}{P_\lambda(1, t; q, t)^2} F(\log(x/t))^2.$$

For $x > 1$ we have $\log(x/t) > 0$, so $\widehat{T}_1$ increases strictly.

(iii) For a single row, the Q-normalization makes the generating function particularly simple. Write $C_j^{(m)}(x) = Q_{(m)}(\mathsf{A}_j(x); q, t)$ and $(a; q)_k = \prod_{r=0}^{k-1}(1 - aq^r)$, with the analogous infinite product for $k = \infty$. Macdonald's one-row generating product (Macdonald, 1995, Chapter VI, (2.8)) telescopes along the two aligned blocks:

$$\sum_{m \ge 0} C_j^{(m)}(x) z^m = \frac{(xt^j z; q)_\infty}{(xz; q)_\infty} \frac{(t^n z; q)_\infty}{(t^j z; q)_\infty}. \tag{4.7}$$

The q-binomial theorem therefore gives

$$C_j^{(m)}(x) = \sum_{k=0}^{m} \frac{(t^j; q)_k}{(q; q)_k} \frac{(t^{n-j}; q)_{m-k}}{(q; q)_{m-k}} t^{j(m-k)} x^k. \tag{4.8}$$

For fixed $m \ge 1$, abbreviate $C_j = C_j^{(m)}$. The scalar relating $P_{(m)}$ and $Q_{(m)}$ cancels, so $\widehat{T}_i = C_i^2/(C_{i-1}C_{i+1})$.

For $j \ge 1$, the top two coefficients in (4.8) give

$$C_j(x) = \frac{(t^j; q)_m}{(q; q)_m} x^m \left(1 + \frac{r_j}{x} + O(x^{-2})\right), \qquad r_j = \frac{1 - q^m}{1 - q} \frac{t^j - t^n}{1 - q^{m-1}t^j}. \tag{4.9}$$

Consequently, for $2 \le i < n$,

$$x\frac{d}{dx}\log \widehat{T}_i(x) = \frac{\gamma_i}{x} + O(x^{-2}), \qquad \gamma_i = r_{i-1} + r_{i+1} - 2r_i.$$

Writing $a = t^i$, $b = t^n$, and $c = q^{m-1}$, subtraction yields

$$\gamma_i = \frac{1-q^m}{1-q}\,\frac{a(1-t)^2(1+ac)(1-bc)}{(t-ac)(1-ac)(1-act)} > 0. \tag{4.10}$$

Every denominator is positive because $ac \le t^i < t$. For $i = 1$, C_0 is constant and $x(d/dx)\log \widehat{T}_1(x) \to m > 0$ instead. □

We note in passing that Conjecture 4.1 also holds for $\lambda = (2)$ and every $n \ge 2$, hence by Remark 4.2 for $\lambda = (2^{n-1})$: there $P_{(2)}(\mathsf{A}_j(x); q, t)$ is an explicit quadratic in x, and monotonicity follows from the positivity of a polynomial of degree four in $x - 1$, whose verification we omit.

AI Usage Statement. The results in this paper were developed through extensive use of an AI assistant (OpenAI's GPT Pro). None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs.

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