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LOWER BOUNDS FOR THE QUANTUM PERMANENT

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Abstract. Griffiths's 1969 conjecture on the positivity of Schur forms reduces to the positivity of the quantum permanent Φ_n on positive bistochastic maps. This conjecture was refuted by Du (2026) with a map for which $\Phi_9 < 0$. Therefore, we ask: *how negative can Φ_n be?* We prove a lower bound $\Phi_n \geq -3n/(3n^2 + 20n + 103)$ (for $n \geq 8$), and an asymptotically stronger bound $\Phi_n \geq -1/n + (3e^2 - 7 - o(1)) \log n/n^2$. The key to our bounds is the orbital polynomial of the map, which turns Φ_n into a linear function of a distribution on partitions and positivity into linear constraints. We also construct explicit maps for $n = 8, 9$ that improve on Du's example. Finally, if $d_* \in \{4, \ldots, 8\}$ is the first dimension for which $\Phi_n < 0$, the minimum μ_n of Φ_n over positive bistochastic maps satisfies $-\mu_n = \Omega(n^{-d_*})$; we conjecture the matching order $-\mu_n = O(n^{-d_*})$.

1. Introduction

Griffiths asked in 1969 whether Griffiths-positive curvature forces positivity of the characteristic forms of a vector bundle (Griffiths, 1969, p. 247). Finski (2022, Theorem 1.3) reduced Griffiths's question to the sign of a more accessible object, the quantum permanent of Gurvits (2006) evaluated at a positive linear map, which in turn is just a signed sum of mixed discriminants.

For matrices $A_1, \ldots, A_n$, the mixed discriminant $D(A_1, \ldots, A_n)$ (Alexandrov, 1938) is the coefficient of $t_1 \cdots t_n$ in $\det(t_1 A_1 + \cdots + t_n A_n)/n!$. For a complex-linear map $H \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$, the *quantum permanent*, also called the double mixed discriminant (Finski, 2022, (1.6)), is

$$\Phi_n(H) := \sum\nolimits_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, D\big(H(E_{1\sigma(1)}), \ldots, H(E_{n\sigma(n)})\big),$$

where E_{ij} are the matrix units and $\mathfrak{S}_n$ is the group of permutations of $\{1, \ldots, n\}$.

We focus on positive linear maps, where H is *positive* if $H(X) \succeq 0$ whenever $X \succeq 0$; closely related is whether positive H also implies $\Phi_n(H) \geq 0$. More precisely, by Finski's reduction (Finski, 2022, Theorem 1.3), the weak form of Griffiths's question over all bundles holds if and only if $\Phi_n(H) \geq 0$ for every positive H. Finski settled dimension $n = 2$ (Finski, 2022, Proposition 1.4(c)) and Wan $n = 3$ (Wan, 2026, Theorem 0.3), but Du's recent 9D counterexample (Du, 2026, Theorem 2.3) dashed hopes for general n. His refutation raises two natural questions:

(i) Which hypotheses on the map restore the sign?
(ii) How far does the sign fail without them?

The companion paper (Sra, 2026) takes up (i). Beyond proving $\Phi_n(H) \geq 0$ for every decomposable map (i.e., a sum of a completely positive and a completely copositive map) it studies the positive maps whose every square compression keeps the sign, among other sufficient conditions. The present paper is complementary and takes up (ii). In particular, we ask how far below zero can Φ_n fall, and which constraints on a positive map control that negative part.

The question needs a scale, because $\Phi_n(cH) = c^n \Phi_n(H)$. We focus therefore on *bistochastic* maps, i.e., those where $H(I) = H^*(I) = I$. Our goal is to then find lower bounds on the optimal value of

$$\mu_n := \min_{H \in \mathfrak{B}_n} \Phi_n(H), \quad \text{where} \quad \mathfrak{B}_n = \{H : M_n(\mathbb{C}) \to M_n(\mathbb{C}) : H \succeq 0 \text{ is bistochastic}\}. \tag{1.1}$$

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In Problem (1.1), the minimum is attained since $\mathfrak{B}_n$ is compact.

We bound μ_n from below with a technique that rests on a simple observation: $\Phi_n(H)$ depends only on the polynomial $\det H(X)$, and is invariant to its input being conjugated by a unitary (Section 2). Therefore, it is determined by the *orbital polynomial* $f_H(x) := \mathbb{E}_U \det H(U\,\mathrm{diag}(x)U^*)$, with U Haar distributed on $U(n)$. For $H \in \mathfrak{B}_n$, positivity makes the coefficients of f_H nonnegative, and $H(I) = I$ makes them sum to $f_H(1,\dots,1) = 1$, so they form a probability distribution on partitions of n, of which $\Phi_n(H)$ is a fixed signed linear function (Proposition 2.1).

This idea based on orbital polynomials delivers a technique that helps us to turn properties of positive maps into *valid constraints*, i.e., linear inequalities among the coefficients of f_H that hold for every $H \in \mathfrak{B}_n$. Every set of valid constraints yields a lower bound on μ_n, namely the minimum of the linear function over all distributions that satisfy them. The simplest valid constraint comes from Gurvits's theorem (Gurvits, 2006, Theorem 4.1), which bounds the coefficient of $x_1 \cdots x_n$ in f_H from below. With it, we obtain the following bound, which we prove in Section 2.

Proposition 1.1 (Elementary bound)**.** *Every $H \in \mathfrak{B}_n$ satisfies*

$$\Phi_n(H) \geq \frac{(n+1)!}{n^{n+1}} - \frac{1}{n}. \tag{1.2}$$

The positive term in (1.2) decays exponentially. Our main result uses sharper valid constraints to improve the decay to be polynomial; we note it as Theorem 1.2 below.

Theorem 1.2 (Main bound)**.** *Every $H \in \mathfrak{B}_n$ satisfies*

$$\Phi_n(H) \geq -\frac{3n}{3n^2+20n+103} \qquad (n \geq 8). \tag{1.3}$$

For every $\varepsilon > 0$ there is a constant A_ε such that

$$\Phi_n(H) \geq -\frac{1}{n} + \frac{(c_{\log} - \varepsilon)\log n - A_\varepsilon}{n^2} \qquad (n \geq 21), \qquad c_{\log} = 3e^2 - 7. \tag{1.4}$$

The sharper constraints used by Theorem 1.2 are also simple to state. Compressions of H to k-dimensional subspaces turn the lower bounds known in dimension k into valid constraints (Lemma 2.3), and these constraints carry the bound up to higher dimensions (Theorem 3.1). The spectra of $H(P)$, for projections P, give a second family of valid constraints (Section 2.2).

We prove (1.3) in Section 3.1, starting from bounds in small dimensions that have computational proofs in exact rational arithmetic. In every dimension, these finite certificates also give the single constant $\Phi_n \geq -0.05656\ldots$ (Corollary A.5). We prove (1.4) in Section 4 by feeding each new bound back in as a seed. The spectral constraints add a small gain at every step, and these gains add up to the $\log n$ term. But eventually, this idea hits its limit: we show that recursive programs built from these valid constraints cannot certify a coefficient larger than $3e^2 - 7$ (Theorem 4.8).

Finally, in Section 5 we construct negative maps for $n = 8, 9$ that improve on Du's construction (Du, 2026). Together with (1.3) our construction yields the bracket

$$\max\left\{\frac{4947}{50000(n)_8}, \frac{1}{737280\binom{n}{9}}\right\} \leq -\mu_n \leq \frac{3n}{3n^2+20n+103} \qquad (n \geq 9), \tag{1.5}$$

where $(n)_r = n(n-1)\cdots(n-r+1)$ and $(n)_0 = 1$. Thus $-\mu_n$ decays at a polynomial rate between n^{-8} and n^{-1}; the exact order is open, and we formally state it as Conjecture 6.1.

2. Orbital coefficients and their constraints

The orbital polynomial turns positivity of H into valid constraints on its coefficients, and it expresses Φ_n as a signed sum of these nonnegative coefficients. Every lower bound in this paper

combines the two. To express Φ_n through $\det H(X)$ alone, we pair this polynomial with the input determinant, which gives

$$\Gamma_n(H) = \det(\partial_X) \det H(X), \qquad \Phi_n(H) = \frac{\Gamma_n(H)}{n!}, \tag{2.1}$$

where the entries of X are independent complex variables and

$$\det(\partial_X) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \frac{\partial}{\partial X_{i,\sigma(i)}}.$$

Since $\det H(X)$ is homogeneous of degree n, the derivative in (2.1) is a constant. The unnormalized invariant Γ_n is Cayley's first hyperdeterminant of order four of the coefficient tensor of H, and it is the quantum permanent of the Choi matrix $C_H = \sum_{i,j} E_{ij} \otimes H(E_{ij})$ in the sense of Gurvits (Gurvits, 2002, Definition 1.7), whose Gaussian representation and operator scaling we use throughout (Gurvits, 2004, 2006).

Averaging $\det H(X)$ over unitary conjugates of a diagonal input leaves (2.1) unchanged and yields a symmetric polynomial with coefficients indexed by partitions. To group monomials with the same exponents, write $\lambda \vdash d$ for a partition $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ with sum d. Its length is $\ell(\lambda) = \ell$, and $m_j(\lambda)$ counts its parts equal to j. The monomial symmetric polynomial m_λ is the sum of the distinct permutations of $x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell}$. Trailing zeros are added when a partition is regarded as an exponent vector in n variables. We use

$$e_j(x) = \sum_{|I|=j} \prod_{i \in I} x_i, \qquad e_0 = 1,$$

for the elementary symmetric polynomials; for a matrix, e_j is evaluated on its eigenvalues.

In this notation, the orbital polynomial of the introduction expands as

$$f_H(x) = \mathbb{E}_U \det H(U \operatorname{diag}(x) U^*) = \sum_{\lambda \vdash n} w_\lambda q_\lambda(x), \qquad q_\lambda(x) = \frac{m_\lambda(x)}{m_\lambda(1^n)}.$$

Here $1^n = (1, \ldots, 1)$, Haar invariance makes f_H symmetric, and the weight w_λ is the total coefficient of the monomials of *shape* λ. The weights form a probability distribution on the partitions of n. Indeed, for each unitary U with columns $u_1, \ldots, u_n$, the matrices $A_i = H(u_i u_i^*)$ satisfy

$$A_i \succeq 0, \qquad \operatorname{tr} A_i = 1, \qquad \sum_i A_i = I.$$

Rank-one decomposition and Cauchy–Binet show that $\det(\sum_i x_i A_i)$ has nonnegative coefficients, and its value at 1^n is $\det I = 1$, so

$$w_\lambda \geq 0, \qquad \sum_{\lambda \vdash n} w_\lambda = 1.$$

The pairing (2.1) sees each monomial of f_H only through the number of variables it involves (Section A.2), so we group monomials by this number. For a homogeneous degree-n polynomial f with nonnegative coefficients and $f(1^n) = 1$, let $w_\ell(f)$ be the total coefficient of monomials supported on exactly ℓ variables, and define

$$\mathcal{L}_n(f) = \sum_{\ell=1}^{n} \frac{(-1)^{n-\ell}}{\binom{n}{\ell}} w_\ell(f). \tag{2.2}$$

For the symmetric orbital polynomial, $w_\ell(f_H)$ is the sum of w_λ over partitions of length ℓ. Gurvits's theorem (Gurvits, 2006, Theorem 4.1) also bounds $w_{(1^n)}$ from below.

Proposition 2.1. *For $H \in \mathfrak{B}_n$,*

$$\Phi_n(H) = \mathcal{L}_n(f_H) = \sum_{\lambda \vdash n} r_\lambda w_\lambda, \qquad r_\lambda = \frac{(-1)^{n-\ell(\lambda)}}{\binom{n}{\ell(\lambda)}}. \tag{2.3}$$

Moreover, $w_{(1^n)} \geq n!/n^n$.

We call r_λ the score of the shape λ, (2.3) the score identity, and the inequality $w_{(1^n)} \geq n!/n^n$ the square-free floor. Proposition 2.1 is proved in Section A.2. Since $r_{(1^n)} = 1$ and $r_\lambda \geq -1/n$ for every other shape, the score identity and the square-free floor give $\Phi_n(H) \geq (1+1/n)w_{(1^n)} - 1/n \geq (n+1)!/n^{n+1} - 1/n$, which proves Proposition 1.1.

2.1. **Compatible compressions.** The elementary bound uses only the normalization of the weights and the square-free floor. Lower bounds known in a smaller dimension k give further valid constraints, once applied to the compressions of H to k-dimensional subspaces. Such compressions are positive but need not be bistochastic, and scaling removes the marginal constraints while preserving every nonpositive lower bound.

Lemma 2.2 (Normalization)**.** *Suppose $\gamma \leq 0$ and $\Phi_k(L) \geq \gamma$ for every $L \in \mathfrak{B}_k$. Then every positive $K : M_k(\mathbb{C}) \to M_k(\mathbb{C})$ satisfies*

$$\Phi_k(K) \geq \gamma \det K(I).$$

The scaling proof is in Section A.1. We call a bound γ as in Lemma 2.2 a *seed* in dimension k. The first seeds are $\gamma_2 = \gamma_3 = 0$, obtained from Wan's theorem (Wan, 2026, Theorem 0.3) by the normalization and limiting argument of the same section. The condition $\gamma \leq 0$ is essential (Section A.1).

Lemma 2.2 bounds one compression at a time. To combine many such bounds, we place them in a single determinant. Let $H : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be positive, fix $1 \leq k \leq n$ and an isometry $V : \mathbb{C}^k \to \mathbb{C}^n$, and take finitely many positive semidefinite baseline matrices $B_i \in M_n(\mathbb{C})$. Set

$$F(x, Z) = \det\left(\sum_i x_i B_i + H(VZV^*)\right).$$

The next lemma is the source of all inherited coefficient constraints.

Lemma 2.3 (Mixed determinant inequality)**.** *Under the hypotheses of Lemma 2.2, every monomial coefficient of*

$$\det(\partial_Z) F(x, Z)|_{Z=0} - \gamma\, \partial_s^k F(x, sI_k)\Big|_{s=0} \tag{2.4}$$

is nonnegative whenever H is positive.

Proof. Write each B_i as a sum of rank-one matrices. Expand the part of F homogeneous of degree k in Z by choosing $n-k$ baseline columns. A choice S contributes

$$x^{\nu(S)} \det(B_S^* B_S) \det K_S(Z), \qquad K_S(Z) = U_S^* H(VZV^*) U_S,$$

where U_S is an isometry from $\mathbb{C}^k$ onto the orthogonal complement of their span, B_S has the chosen vectors as columns, and $\nu(S)$ records their baseline multiplicities. Dependent columns contribute zero. Every K_S is positive and every Gram determinant is nonnegative. The two derivatives in (2.4) give $k!\Phi_k(K_S)$ and $k! \det K_S(I)$, respectively. Lemma 2.2 makes their difference nonnegative before terms with the same monomial are collected. The compression identity follows by completing the chosen span to an orthonormal basis and expanding along its first $n-k$ coordinates. The expansion is an identity in the entries of Z, so Z need not be Hermitian or positive semidefinite. □

To turn Lemma 2.3 into inequalities among the weights, we express its two derivatives in the eigenvalue coordinates of f_H. On polynomials symmetric in the first k variables, define

$$\delta_k(x) = \prod_{i<j\le k}(x_i - x_j), \qquad \mathscr{D}_k f = \delta_k^{-1}\partial_1\cdots\partial_k(\delta_k f), \qquad \mathscr{T}_k f = (\partial_1 + \cdots + \partial_k)^k f.$$

The differentiated alternant is alternating, so the quotient is a polynomial. The operator $\mathscr{T}_k$ differentiates along a scalar shift of the *active* block, the first k coordinates. The operator $\mathscr{D}_k$ is the restriction of $\det(\partial_Z)$ to the eigenvalues of that block; the identity is proved in Section A.2.

To apply Lemma 2.3 to the orbital polynomial, write $H_U(X) = H(UXU^*)$, let V include the first k coordinates, and take $B_i = H_U(E_{ii})$. The lemma then applies to

$$F_U(x, Z) = \det H_U(\mathrm{diag}(x) + VZV^*).$$

Averaged over U, this polynomial becomes $\widehat{f}_H(\mathrm{diag}(x) + VZV^*)$, where $\widehat{f}_H(X) = \mathbb{E}_U \det H(UXU^*)$. For fixed inactive coordinates $x_{k+1}, \ldots, x_n$, the average is invariant under conjugation of the active $k \times k$ block. Its two derivatives at $Z = 0$ are therefore $\mathscr{D}_k f_H$ and $\mathscr{T}_k f_H$. Averaging each nonnegative monomial coefficient in Lemma 2.3 proves

$$(\mathscr{D}_k - \gamma\mathscr{T}_k) f_H \ge_{\mathrm{coeff}} 0, \tag{2.5}$$

where $\ge_{\mathrm{coeff}} 0$ means that every ordinary monomial coefficient is nonnegative.

Each monomial coefficient of (2.5) is a linear inequality in the weights, which we call a *mixed row*. Dimension transport uses two of these rows, one for each shape of the lowest score $-1/n$: the shape $(2, 1^{n-2})$, and in even dimension also the shape (n). For $n > k$, the coefficient of $x_{k+1}\cdots x_n$, divided by $k!$, is

$$J_{n,k,\gamma} := \sum_{\mu\vdash k} \frac{(k)_s}{(n)_s}\left(\frac{(-1)^j}{\binom{k}{j}} - \gamma\right) w_{\mu\cup 1^{n-k}} \ge 0, \qquad j = k - \ell(\mu), \quad s = k - m_1(\mu). \tag{2.6}$$

Here $\mu \cup 1^{n-k}$ appends $n - k$ parts equal to one to μ; we call the shapes of this form *selected*, since no other shape enters $J_{n,k,\gamma}$.

For the shape (n), the coefficient of x_1^{n-2} in the $k = 2$, $\gamma_2 = 0$ inequality gives

$$R_n := \frac{w_{(n-1,1)}}{n-1} - w_{(n)} \ge 0 \qquad (n \ge 3). \tag{2.7}$$

The coefficient extractions are verified in Section A.3.

2.2. **Projection spectra.** Lemma 2.3 uses only positivity of H. Positivity and bistochasticity together also constrain the images of projections. For a rank-a projection P, they imply $0 \preceq H(P) \preceq I$ and $\mathrm{tr}\, H(P) = a$. Averaged over Haar projections of rank a, the characteristic coefficients $e_j(H(P))$ are linear in the weights (Section A.5), so these spectra give a second family of valid constraints, used by the finite programs of Section 3.1 and by the logarithmic improvement of Section 4. The convex hull of the vectors $(e_0(x), \ldots, e_n(x))$ over all spectra $0 \le x_i \le 1$, $\sum_i x_i = a$, is generated by the characteristic-coefficient vectors of the following spectra:

$$(1^a, 0^{n-a}) \quad \text{and} \quad \left(1^u, \left(\frac{a-u}{v}\right)^v, 0^{n-u-v}\right), \quad 0 \le u < a, \quad a - u < v \le n - u. \tag{2.8}$$

To verify the hull assertion, minimize any linear combination of elementary symmetric functions under $0 \le x_i \le 1$ and $\sum x_i = a$, choosing a minimizer of maximal squared norm. With two coordinates free, the objective has the form $A + B(x_i + x_j) + Cx_ix_j$. Distinct interior coordinates force $C = 0$ at a constrained stationary point. Moving them apart then preserves the objective and increases the squared norm, a contradiction. Hence all interior coordinates agree. Applying the argument to every linear functional proves the hull assertion.

3. Dimension transport and the universal bound

A seed in dimension k bounds Φ_n in every larger dimension through the row (2.6). This row balances the square-free shape (1^n) against the shape $(2, 1^{n-2})$; we call these two the *leading shapes*. Solving for that balance produces bounds of the form

$$h_n(K) = -\frac{n-1-K}{n(n-1)+K} = -\frac{1}{n} + \frac{K}{n^2} + \frac{2K}{n^3} + O(n^{-4}), \tag{3.1}$$

whose parameter K is determined by the seed. The remaining coefficients, the *residuals* of the resulting certificate, can then be checked with no further optimization over maps.

Theorem 3.1 (Dimension transport)**.** *Let $k \geq 3$ and let $\gamma \in [-1/k, 0]$ be a seed in dimension k, that is, $\Phi_k(G) \geq \gamma$ for every $G \in \mathfrak{B}_k$. Set*

$$K = K(k, \gamma) = \frac{(k-1)(1+k\gamma)}{1-\gamma}. \tag{3.2}$$

Then $\Phi_n(H) \geq h_n(K)$ for every $H \in \mathfrak{B}_n$ and $n \geq k+3$, with h_n as in (3.1). For a fixed seed, nonnegative coefficients in finitely many shifted residual polynomials can lower the threshold to $k+1$.

Proof idea. Put $N = n(n-1)$, $a = 1-\gamma$, $b = (k-1)(1+k\gamma)$ and $t = (n^2-1)/(aN+b)$. The certificate in (A.5) writes $\Phi_n - h_n(K)$ as $tJ_{n,k,\gamma}$, a nonnegative multiple of R_n in even dimension, and a sum of nonnegative residuals times w_λ. The coefficients of the two shapes (1^n) and $(2, 1^{n-2})$ in J are a and $-b/N$. Setting their residuals to zero gives

$$ta = 1 - h_n(K), \qquad tb/N = 1/n + h_n(K),$$

which determine $K = b/a$, as in (3.2), and the displayed t. The remaining residuals, including the parity correction and the finite threshold test, are checked in Section A.4. □

Transport returns the seed at $n = k$ and is monotone in the seed, two properties used repeatedly below. Direct calculation gives

$$\begin{aligned} h_k(K(k,\gamma)) &= \gamma, \\ K(k,\gamma') - K(k,\gamma) &= \frac{(k^2-1)(\gamma'-\gamma)}{(1-\gamma')(1-\gamma)}, \\ h_n(K') - h_n(K) &= \frac{(K'-K)(n^2-1)}{(n(n-1)+K')(n(n-1)+K)}. \end{aligned} \tag{3.3}$$

Thus a larger transport parameter improves the bound in every dimension where both bounds apply. When the residual test starts at $k+1$, the seed itself fills the endpoint $n = k$. On the other hand, $K(k, h_k(K)) = K$, so transporting an already transported bound produces no further gain; any further gain must come from other valid constraints.

Hypotheses on the map can also strengthen the mixed rows directly. Compressions of a k-positive or k-copositive map to dimension k have nonnegative invariant, so the mixed rows hold with $\gamma = 0$; Corollary A.4 records the resulting bound.

3.1. Exact finite certificates.

Theorem 3.1 needs a seed, and in small dimensions we obtain seeds from a finite linear program in the weights. The program retains the nonnegative normalized weights, the square-free floor, and every mixed row from a strictly smaller dimension. It also imposes the projection hull of Section 2.2 (Section A.5). A nonnegative slack for each inequality puts the program in rational standard form

$$\min c^{\mathsf{T}} x \quad \text{subject to } Ax = b, \quad x \geq 0, \tag{3.4}$$

where x collects the weights w_λ, the hull mixtures, and the slacks, and c carries the score coefficients of (2.3). Every actual map gives a feasible point. Therefore any rational vector y with $A^{\mathsf{T}} y \leq c$

proves $\Phi_n(H) = c^{\mathsf{T}}x \geq y^{\mathsf{T}}Ax = b^{\mathsf{T}}y$. The implication does not require a feasible point, which we call a *coefficient profile*, to come from a map, and checking it is finite rational arithmetic.

Starting from $\gamma_2 = \gamma_3 = 0$ and using each certified bound as a seed in the next dimension gives exact dual certificates through dimension 21 (Proposition A.2, Table 1). In dimension 24, one bounded projection test added to the transport certificate from dimension 21 gives the seed

$$\gamma_{24} = -\frac{649}{22351} = h_{24}(271/40) \approx -0.02904,$$

and the shifted residual test of Theorem 3.1 passes from dimension 25 (Proposition A.3). The dual vectors and the rational checks accompany the paper as supporting files.

Proof of (1.3). Proposition A.3 gives $\Phi_n \geq h_n(271/40)$ for all $n \geq 24$. Since $271/40 > 20/3$, monotonicity (3.3) gives $h_n(271/40) > h_n(20/3)$. For $A = 20/3$ and $B = 103/3$,

$$h_n(A) + \frac{n}{n^2 + An + B} = \frac{(211/9)n + 2369/9}{(n(n-1) + A)(n^2 + An + B)} > 0.$$

This inequality proves (1.3) for $n \geq 24$. For $8 \leq n \leq 23$, the seeds of Table 1 and the bound transported from dimension 21 exceed the right side of (1.3); these finite comparisons are part of the supporting checks. □

4. Logarithmic improvement and limits of the coefficient relaxation

With a fixed seed, Theorem 3.1 gives $\Phi_n \geq -1/n + K/n^2 + O(n^{-3})$ for a constant K. The logarithmic bound (1.4) needs K to grow like $\log n$. The mixed rows alone cannot produce this growth, since their relaxation reproduces the transported bound exactly. The projection spectra can, through a small strict gain at every step.

4.1. The mixed relaxation and its projection obstruction.

The finite programs use both mixed rows and projection spectra. To separate their roles, we first retain every supplied mixed row but omit the spectral constraints. This smaller set of conditions has an explicit unique minimizer, and its failure to come from a map identifies the information that projection spectra add.

Let $\mathcal{G}$ be a finite collection of seeds (k, γ_k) with $\gamma_k \in [-1/k, 0]$, including $(2, 0)$ and $(3, 0)$, and let $k_{\max}$ be its largest dimension. Each seed is a valid lower bound on $\mathfrak{B}_k$. For $n > k_{\max}$, define $\mathcal{R}_n(\mathcal{G})$ to be the symmetric homogeneous degree-n polynomials $f = \sum_{\lambda \vdash n} w_\lambda q_\lambda$ satisfying

$$w_\lambda \geq 0, \qquad \sum_\lambda w_\lambda = 1, \qquad (\mathscr{D}_k - \gamma_k \mathscr{T}_k) f \geq_{\text{coeff}} 0 \quad ((k, \gamma_k) \in \mathcal{G}).$$

All monomial coefficients of the last inequalities are imposed. Put

$$K = \max_{(k,\gamma_k) \in \mathcal{G}} K(k, \gamma_k), \qquad N = n(n-1).$$

The two zero seeds ensure $2 \leq K \leq k_{\max} - 1$.

Theorem 4.1 (Unique mixed-coefficient extremizer)**.** *For $n \geq k_{\max} + 3$,*

$$\min_{f \in \mathcal{R}_n(\mathcal{G})} \mathcal{L}_n(f) = h_n(K).$$

The unique minimizer is

$$f_{n,K}(z) = \frac{K e_n(z) + m_{(2,1^{n-2})}(z)}{N + K} = \frac{e_1(z) e_{n-1}(z) - (n - K) e_n(z)}{N + K}. \tag{4.1}$$

The minimum and its unique minimizer are unchanged if one also imposes the square-free floor $w_{(1^n)} \geq n!/n^n$ or the scalar polynomial capacity condition $\mathrm{Cap}(f) \geq 1$, where

$$\mathrm{Cap}(f) = \inf_{z_1, \ldots, z_n > 0} \frac{f(z)}{\prod_i z_i}.$$

This scalar capacity is distinct from the map capacity in (A.2).

Proof idea. Transport gives the lower bound using only the mixed rows. The full coefficient identity (B.1) shows that $f_{n,K}$ satisfies every supplied row. Its two nonzero weights have ratio $K : N$. Strict positivity of all remaining transport residuals, together with $R_n \geq 0$ in even dimension, forces this support at equality. The full proof, including scalar capacity and the square-free floor, is in Appendix B. □

Projection spectra impose a different restriction, which uses both marginals of the map and explains why the preceding minimizer cannot attain a negative value for an actual map.

Proposition 4.2 (Projection obstruction)**.** *If $H \in \mathfrak{B}_n$, $n \geq 4$, and $f_H = f_{n,K}$ with $K \geq 0$, then*

$$K \geq \begin{cases} n, & n \text{ even}, \\ n-1, & n \text{ odd}. \end{cases} \tag{4.2}$$

In particular, $f_{n,K}$ is not realizable when $h_n(K) < 0$.

Proof. For a Haar rank-a projection P, put $Y = H(P)$. Its eigenvalues lie in $[0,1]$ and sum to a. Substitution in (4.1) gives

$$\mathbb{E}\det(tI + Y) = t^{n-a-1}(t+1)^{a-1}\left(t^2 + t + \frac{a(n-a)}{N+K}\right) \qquad (1 \leq a < n). \tag{4.3}$$

The zero coefficients give $\mathbb{E}e_j(Y) = 0$ for $j \geq a+2$ whenever those indices exist. Nonnegativity of $e_j(Y)$ implies $\operatorname{rank} Y \leq a+1$ almost surely. Applying the same argument to $I - P$ gives $\operatorname{rank}(I - Y) \leq n - a + 1$. Thus Y has at least $a-1$ eigenvalues equal to one and at least $n-a-1$ equal to zero. Its two remaining eigenvalues sum to one, so they are λ and $1-\lambda$ for some $\lambda \in [0,1]$. Comparing (4.3) with this spectrum gives

$$\frac{a(n-a)}{N+K} = \mathbb{E}[\lambda(1-\lambda)] \leq \frac{1}{4}.$$

Taking $a = \lfloor n/2 \rfloor$ proves (4.2). If a rank bound has an index beyond n, that bound is automatic, so the boundary cases require no exception. □

Together, Theorem 4.1 and Proposition 4.2 show that no map attains the minimum of the mixed relaxation, so the transported bound is strict.

Corollary 4.3. *For the seed collection of Theorem 4.1 and $n \geq k_{\max} + 3$, one has $\mu_n > h_n(K)$. In particular, $\mu_n > h_n(271/40)$ for $n \geq 27$.*

Proof. Every actual map has a feasible orbital polynomial. The unique minimizer of the relaxation is excluded by Proposition 4.2, since $K \leq k_{\max} - 1 < n - 1$. Compactness of $\mathfrak{B}_n$ turns nonattainment into a strict gap at each fixed dimension. □

Theorem 4.1 concerns a fixed seed collection and does not identify the optimum of the finite spectral programs or of a program whose largest seed grows with n.

4.2. **Binary reduction and logarithmic improvement.** To make the transport parameter grow, we feed the strict projection gap into successive transport steps. The gain of each step depends on the ratio between the projection coefficient of each shape and its transport residual. The two-dimensional mixed rows suppress every fixed shape with a part larger than two, so the ratios that matter are those of the *binary* shapes, whose nonunit parts all equal two.

Lemma 4.4 (Suppression of a fixed nonbinary shape)**.** *Let μ be a multiset of integers at least two, of sum s, and suppose it contains a part $p \geq 3$. Let m_i be its multiplicities, and let μ' replace one occurrence of p by $(p-1, 2)$. For every $n > s$, coefficientwise nonnegativity of $\mathscr{D}_2 f_H$ implies*

$$w_{\mu \cup 1^{n-s}} \leq \beta_{\mu,p}(n)\, w_{\mu' \cup 1^{n-s-1}},$$

where

$$\beta_{\mu,p}(n) = \frac{2p}{(p-1)(n-s)m_p} \begin{cases} (m_2+1)(m_2+2), & p=3, \\ (m_{p-1}+1)(m_2+1), & p \geq 4. \end{cases}$$

Thus every fixed nonbinary shape has weight $O(n^{-1})$ relative to another fixed nonleading shape.

Proof. For a padded partition λ, write $a_\lambda = w_\lambda/|\mathcal{O}_\lambda|$ for its ordinary monomial coefficient. Fix the inactive monomial whose nonunit exponents are $\mu \setminus (p)$ and whose other nonzero exponents are $n-s-1$ ones. The active coefficient of $x^{p-2}y$ in $\mathscr{D}_2 f_H$ is

$$2p\, a_{\mu\setminus(p)\cup(p-1,2)} - (p-1)a_{\mu\setminus(p)\cup(p,1)} - (p+1)a_{\mu\setminus(p)\cup(p+1,0)} \geq 0.$$

Indeed $\mathscr{D}_2 f = \partial_x\partial_y f + (\partial_y f - \partial_x f)/(x-y)$. For active exponents summing to $p+1$, only the three pairs displayed can contribute to $x^{p-2}y$; direct differentiation gives their coefficients. The count includes $p=3$, when the positive pair is $(2,2)$ and its coefficient is six. Drop the last, nonpositive contribution. The two remaining padded shapes have the same number of zero entries; the positive shape has one fewer entry equal to one. Their orbit-size ratio is exactly the multiplicity factor in the displayed formula, divided by $(n-s)m_p$, which proves the claim. □

The projection constraints enter each step through a bounded test function ψ on the integers. For a rank-a projection P, the eigenvalues of $H(P)$ lie in $[0,1]$ and sum to a. Read as success probabilities, they define a Poisson-binomial variable S of mean a, and after averaging over Haar projections, $\mathbb{E}\psi(S-a)$ is linear in the weights. If $\mathbb{E}\psi(S-a) \geq 0$ for every Poisson-binomial S of integer mean a, then ψ is a valid *projection separator*, and this expectation is a valid constraint. The next result shows that the logarithmic coefficient depends on ψ only through its averages on binary shapes, which separates the coefficient estimate from the construction of ψ.

Proposition 4.5 (Binary reduction of the logarithmic coefficient)**.** *Let $\psi : \mathbb{Z} \to \mathbb{R}$ be a bounded symmetric function with $\psi(0) = 1$ and $\psi(1) = -1$. Suppose it is a valid projection separator: $\mathbb{E}\psi(S-a) \geq 0$ for every Poisson-binomial random variable S of integer mean a. Put*

$$B = \max\{1, \sup_t \psi(t)\}, \qquad g_r = 4^{-r} \sum_{j=-r}^{r} \binom{2r}{r+j} \psi(j), \qquad M = \sup_{r\geq 2} \frac{g_r}{2r-1}.$$

Assume $M > 0$. Then the mixed coefficient inequalities and projection constraints imply

$$\Phi_n(H) \geq -\frac{1}{n} + \frac{M^{-1}\log n - O(1)}{n^2}.$$

It is enough to verify the value of M on finitely many binary shapes: for any integer L with $B/L < M$, the remaining shapes with $2r > L$ satisfy $g_r/(2r-1) \leq B/L < M$.

Proof. Use the projection row derived in Lemma B.3: $\delta = \Phi_n(H) - h_n(K)$, $\alpha = 4a(n-a)/((n-1)(n+1))$, $\Delta = a(n-a)/(n(n-1)+K) - 1/4$, and

$$d_\lambda = b_\lambda - 1 + \alpha(1-r_\lambda), \qquad b_\lambda = \binom{n}{a}^{-1} \sum_{|I|=a} \psi\left(\sum_{i\in I}\lambda_i - a\right), \qquad a = \lfloor n/2 \rfloor.$$

The score identity and separator give $4\Delta \leq \alpha\delta + \sum d_\lambda w_\lambda$ over the nonleading shapes. In even dimension, cancel the zero-residual shape (n) using $R_n \geq 0$. If $d_{(n)}$ is negative, drop its term; if it is positive, add $d_{(n)}/(n-1)$ to the coefficient at $(n-1,1)$. Call the resulting coefficients $\widehat{d}_\lambda$ and omit (n). The adjustment makes no asymptotic difference because these bounded coefficients are paired with an unselected residual of order $1/n$.

Fix L with $B/L < M$, and write $\mathcal{F}$ for all nonbinary nonunit multisets of sum at most L. Start the recurrence in a sufficiently large dimension, so that all members of $\mathcal{F}$ and their replacements

are selected nonleading shapes. Choose one part $p(\mu) \geq 3$ from each $\mu \in \mathcal{F}$ and its replacement μ'. For fixed n, k, K, an explicit valid gap is

$$\delta \geq \frac{4\Delta}{\alpha + Q_{n,k,K}},$$

where, using the strictly positive transport residuals c_λ,

$$Q_{n,k,K} = \max\left(0, \max_{\lambda \notin \mathcal{F}} \frac{\widehat{d}_\lambda}{c_\lambda}\right) + \sum_{\mu \in \mathcal{F}} \frac{(\widehat{d}_\mu)_+ \beta_{\mu,p(\mu)}(n)}{c_{\mu'}}.$$

The maxima omit both leading shapes and the canceled even shape. Indeed apply the suppression lemma to every positive coefficient indexed by $\mathcal{F}$, and then use $\sum c_\lambda w_\lambda \leq \delta$. Bounding each of these finitely many substituted terms separately by its contribution to δ proves the formula, even if several replacements coincide.

Now take $k = n - 3$ and $K = O(\log n)$. Lemma B.2 gives, for every fixed nonleading selected shape whose parts exceeding one sum to s,

$$c_\lambda = \frac{3(s-1)}{n^2} + O(\log n / n^3).$$

The finite sum of substituted terms in Q is therefore $O(n)$: its orbital coefficients are bounded, its suppression factors are $O(n^{-1})$, and its replacement residuals are of order n^{-2}. For a binary shape $(2^r, 1^{n-2r})$ of fixed size,

$$\widehat{d}_\lambda = g_r + O(n^{-1}), \qquad \frac{\widehat{d}_\lambda}{c_\lambda} = \frac{n^2}{3}\left(\frac{g_r}{2r-1} + O(\log n/n)\right).$$

For all selected shapes with that sum greater than L, the floor of Lemma B.2 gives uniformly

$$c_\lambda \geq \frac{3L}{n^2} - O(\log n/n^3), \qquad \widehat{d}_\lambda \leq B + O(n^{-2}).$$

The unselected shapes have residuals of order $1/n$ and bounded effective coefficients, hence ratios $O(n)$. Since $B/L < M$ and $g_r/(2r-1) \to 0$, the supremum defining M is attained at a fixed binary shape. Consequently

$$Q_{n,n-3,K} = \frac{n^2}{3}\left(M + O(\log n/n)\right), \qquad \frac{4\Delta}{\alpha + Q_{n,n-3,K}} = \frac{3}{Mn^3} + O(\log n/n^4).$$

To justify $K = O(\log n)$, choose a fixed cap $A > 3/M$ and start sufficiently far out with any transported positive seed parameter K_0. In each step, use the correction

$$\varepsilon = \min\left\{\frac{4\Delta}{\alpha + Q_{n,k,K}}, \frac{A}{n^3}, -\frac{1}{2}h_n(K)\right\},$$

convert the corrected seed back to its transport parameter, and round down to a multiple of n^{-2}. The cap preserves a negative seed and gives an upper increment $(A+1)/n$ for all sufficiently large n. Rounding loses less than n^{-2}, so the parameters stay positive and are $O(\log n)$. The preceding estimates therefore apply; eventually both caps are inactive. The parameter increment is

$$\frac{3}{Mn} + O(\log n/n^2).$$

Summing in steps of three gives $K = M^{-1}\log n + O(1)$. Transport fills the intervening dimensions and proves the claim. □

4.3. **The parity separator and the logarithmic coefficient.** By Proposition 4.5, a valid separator with ratio M gives the logarithmic coefficient $1/M$, so we want M as small as validity allows. Balanced binary shapes have centered binomial laws, and their parity moments suggest a quadratic separator for which every binary ratio $g_r/(2r-1)$ is the same. The mean-one boundary of the projection hull determines the smallest admissible value of that ratio.

Proposition 4.6 (Bounded separators approaching the limiting coefficient)**.** *Put* $c_{\log} = 3e^2 - 7$ *and* $M_* = c_{\log}^{-1}$. *For every* $M > M_*$ *there is a bounded even function* $\psi : \mathbb{Z} \to \mathbb{R}$ *such that*

$$\psi(0) = 1, \qquad \psi(\pm 1) = -1, \qquad \mathbb{E}\psi(S - a) \geq 0$$

for every Poisson-binomial random variable S *of integer mean* a. *Moreover, for* $X_r \sim \mathrm{Bin}(2r, 1/2)$,

$$\sup_{r \geq 2} \frac{\mathbb{E}\psi(X_r - r)}{2r - 1} = M.$$

Proof. Consider the even function

$$\psi_M(d) = \begin{cases} 1 + 6Md^2, & d \text{ even}, \\ 2M(d^2 - 1) - 1, & d \text{ odd}. \end{cases} \tag{4.4}$$

Since $M > M_* > 1/16$, its values at $|d| \geq 2$ are nonnegative and $\psi_M(2) > 2$. For $D = X_r - r$, $r \geq 2$, the binomial generating function gives

$$\mathbb{E}D^2 = r/2, \qquad \mathbb{E}(-1)^D = \mathbb{E}[D^2(-1)^D] = 0.$$

Since $\psi_M(d) = M(4d^2 - 1) + (1 + M + 2Md^2)(-1)^d$, these identities imply

$$\mathbb{E}\psi_M(X_r - r) = M(2r - 1).$$

To check validity, the projection-spectrum hull reduces the problem to the centered binomials $X - r$, with $X \sim \mathrm{Bin}(r + s, r/(r + s))$ and $r, s \geq 1$, and the deterministic law at zero. The binomial estimates in Section B.3 show that the generators with $r, s \geq 2$ are valid whenever $\psi(0) = 1$, $\psi(\pm 1) = -1$, $\psi(\pm 2) \geq 2$, and all other values are nonnegative. The remaining generators, up to reflection, are $\mathrm{Bin}(m, 1/m) - 1$. For the unclipped separator ψ_M their limiting expectation is

$$\lim_{m \to \infty} \mathbb{E}\psi_M(\mathrm{Bin}(m, 1/m) - 1) = 3M - e^{-2}(1 + 7M).$$

It vanishes at $M = M_* = (3e^2 - 7)^{-1}$ and is negative for $M < M_*$. The same appendix proves nonnegativity at M_* for every finite m, and the uniform margin

$$\mathbb{E}\psi_M(\mathrm{Bin}(m, 1/m) - 1) \geq \frac{M - M_*}{27M_*} > 0 \qquad (m \geq 3,\ M > M_*).$$

The generator with $m = 2$ has expectation zero.

The strict margin permits bounded clipping. For an even integer $L \geq 2$, put $\psi(d) = \min\{\psi_M(d), 1 + 6ML^2\}$. The clipped separator retains every value with $|d| \leq L$ and the conditions that prove validity for $r, s \geq 2$. For the mean-one generators its loss is at most

$$\sum_{j=L+2}^{\infty} \frac{1 + 6M(j-1)^2}{j!},$$

uniformly in m, because $\Pr(\mathrm{Bin}(m, 1/m) = j) \leq 1/j!$. Choose L so that this tail is smaller than the uniform margin. Then the clipped separator is valid for all generators. Clipping can only decrease the binary expectation and leaves the support at $r = 2$ unchanged. Thus its supremum ratio is exactly M, as claimed. Section B.3 supplies the full finite-binomial verification used above. □

Applying Proposition 4.5 to the separators of Proposition 4.6 gives the logarithmic improvement.

Corollary 4.7 (Logarithmic improvement)**.** *For every fixed $0 < \alpha < 3e^2 - 7$, the mixed coefficient inequalities and projection constraints imply*

$$\Phi_n(H) \geq -\frac{1}{n} + \frac{\alpha \log n - O_\alpha(1)}{n^2}.$$

In particular, the minimum μ_n of (1.1) satisfies

$$\liminf_{n\to\infty} \frac{n^2(\mu_n + 1/n)}{\log n} \geq 3e^2 - 7.$$

Proof. Apply Proposition 4.5 to the bounded separator above with $M = 1/\alpha$. □

With $\alpha = c_{\log} - \varepsilon$, Corollary 4.7 gives (1.4) for all sufficiently large n, and increasing A_ε covers the remaining dimensions $n \geq 21$. The proof of Theorem 1.2 is now complete.

4.4. **Optimality of the logarithmic coefficient.** Corollary 4.7 raises the question whether a better separator could beat $3e^2 - 7$. Within the recursive programs, it cannot.

Theorem 4.8 (Optimal logarithmic coefficient of the relaxation)**.** *Consider recursive coefficient programs using nonnegative normalized weights, the square-free floor, scalar capacity, all projection-spectrum hull constraints, and all proper-dimensional mixed rows. Initialize them with valid nonpositive lower bounds in a finite initial range of dimensions. Every sequence γ_n certified recursively by these programs satisfies*

$$\limsup_{n\to\infty} \frac{n^2(\gamma_n + 1/n)}{\log n} \leq c_{\log} = 3e^2 - 7.$$

Together with (1.4), this identifies the optimal limiting logarithmic coefficient for the relaxation.

The proof, given in the supplement, builds feasible profiles from the same mean-one binomial generators, with weights on the shapes $(2^r, 1^{n-2r})$ besides the two leading shapes. These profiles limit what the specified recursive programs can certify from finitely many nonpositive initial bounds; they need not arise from maps. Additional compatibility conditions or independent bounds supplied in unbounded dimensions can change the conclusion. The remaining question is whether actual maps can sustain negativity of order $1/n$.

5. Negative maps and polynomial padding

Negative maps give upper bounds on μ_n, and padding carries a negative map in dimension r to every larger dimension n, with invariant of order n^{-r}. Small dimensions therefore matter most. Our eight-dimensional construction starts from the unnormalized six-dimensional Breuer–Hall map (Breuer, 2006, (2)–(3); Hall, 2006, Theorem 4):

$$J = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}, \qquad \mathcal{B}(Y) = \operatorname{tr}(Y) I_6 - Y - JY^{\mathsf{T}}J^*.$$

For every $w \in \mathbb{C}^6$, the vectors w and $J\bar{w}$ are orthogonal and have the same norm. Hence $\mathcal{B}(ww^*) = \|w\|^2 I_6 - ww^* - (J\bar{w})(J\bar{w})^* \succeq 0$.

The eight-dimensional map is a sum of three positive components. Let $B : \mathbb{C}^8 \to \mathbb{C}^6$ project onto the first six coordinates, and let $A : \mathbb{C}^6 \to \mathbb{C}^8$ satisfy $Ae_i = e_i$ for $1 \leq i \leq 5$ and $Ae_6 = 0$. Put $D = \operatorname{diag}(1, 1, \sqrt{2}, 1, 1, 1, 1, 1)$. Using one-based matrix units, define

$$\begin{aligned} K_0 &= E_{17} + E_{28} + E_{61} + E_{72} + E_{83}, \\ L_0 &= E_{47} + E_{58} + E_{64} + E_{75} + E_{86}, \\ R(X) &= A\mathcal{B}(BXB^*)A^* + 4K_0XK_0^* + 4L_0X^{\mathsf{T}}L_0^*, \\ H_8(X) &= \tfrac{1}{8} DR(X)D. \end{aligned}$$

The Breuer–Hall term is positive, and the other two terms are congruences of X and X^{T}. The marginal and determinant calculations are given in Section C.2. The padding below uses the completely positive summand of H_8 corresponding to K_0, which is Ad_Q, where

$$Q = K_0/\sqrt{2}, \qquad H_8 - \mathrm{Ad}_Q \text{ is positive.}$$

Here $DK_0 = K_0$, since K_0 has zero third row.

The map H_8 is one member of a two-parameter family, and Du's map, normalized to be bistochastic, is one member of a nine-dimensional family (Sections C.2 and C.1). The minima over the two families are as follows.

Proposition 5.1 (Negative families)**.** *There are explicit two-parameter families of positive bistochastic maps on M_9 and M_8, with parameters $(a, b) \in [-1, 1]^2$, whose minima are*

$$-\frac{1}{737280} \approx -1.356 \cdot 10^{-6} \quad \textit{and} \quad -\frac{11}{6881280} \approx -1.599 \cdot 10^{-6},$$

respectively. In both families equality holds exactly when $|a| = |b| = 1$. The minimizing maps are sums of three positive components. The negative magnitudes exceed those of the normalized Du example and its corresponding compression by factors $21/2$ and $33/5$, respectively.

Proof. Propositions C.1 and C.2 give the constructions and the calculations. □

The dimension of a negative map determines the decay exponent of its pinched paddings. For $H_r \in \mathfrak{B}_r$, its pinched direct sum with an identity map is

$$(H_r \boxplus \mathrm{Id}_m) \begin{pmatrix} X & Z \\ W & Y \end{pmatrix} = \begin{pmatrix} H_r(X) & 0 \\ 0 & Y \end{pmatrix}, \qquad m = n - r.$$

The normalized determinant differential gives

$$\Phi_n(H_r \boxplus \mathrm{Id}_{n-r}) = \frac{\Phi_r(H_r)}{\binom{n}{r}}. \tag{5.1}$$

Indeed, the output determinant factors into the two block determinants, and the input differential contributes $r!(n-r)!/n!$ after normalization. For the eight-dimensional map, extension through the summand Ad_Q retains off-diagonal entries and improves the padding coefficient to

$$\mu_n \le -\frac{4947}{50000(n)_8} \qquad (n \ge 9)$$

(Proposition C.3). The nine-dimensional family gives $\mu_n \le -1/(737280\binom{n}{9})$, completing the bracket (1.5). The family minima do not identify μ_8 or μ_9.

6. Discussion

The padding exponent singles out the first dimension admitting a negative map. Define

$$d_* := \min\{d \in \mathbb{Z}_{\ge 1} : \mu_d < 0\}. \tag{6.1}$$

Nonnegativity in dimensions two and three and the eight-dimensional example give $4 \le d_* \le 8$; its value is unknown.

For any fixed $H_* \in \mathfrak{B}_{d_*}$ with $\Phi_{d_*}(H_*) < 0$, the padding identity (5.1) gives

$$-\mu_n \ge \frac{-d_*!\Phi_{d_*}(H_*)}{(n)_{d_*}} \qquad (n \ge d_*).$$

We conjecture a matching upper bound on $-\mu_n$.

Conjecture 6.1 (Order of the universal minimum)**.** *There is a constant $C > 0$, independent of n and H, such that for every integer $n \geq d_*$ and every $H \in \mathfrak{B}_n$,*

$$\Phi_n(H) \geq -\frac{C}{(n)_{d_*}}.$$

Together with padding, the conjecture is equivalent to $-\mu_n = \Theta(n^{-d_*})$. Theorem 4.8 excludes certifying this order with the specified recursive programs initialized by finitely many nonpositive bounds. Determining the unresolved signs of $\mu_4, \ldots, \mu_7$ would fix the exponent in Conjecture 6.1.

Other ideas and extensions. Constraints on the map itself, rather than on its orbital weights, give partial evidence for Conjecture 6.1. We collect them, with proofs, in a supplement released with this paper. The conjectured order holds for maps with an entanglement-breaking submap of normalized trace at least $(d_* - 1)\log n/n$, and for coherent extensions, which couple a core of fixed dimension to the added coordinates, under spectral restrictions on the coupling. For general coherent extensions, the conjecture requires a noncentral determinant polynomial to be nonnegative, which remains open. For arbitrary maps, $\Phi_n(H) \leq -c/n$ with fixed $c > 0$ forces $\sum_{i,j} \|H(E_{ij})\|_{\mathrm{HS}}^2 = n^2 - O(n)$, a contraction-radius bound gives $\mu_2 = 1/4$, and every negative global minimizer satisfies explicit first-order conditions. The supplement also proves Theorem 4.8, and two obstructions for weaker relaxations, showing that the mixed rows alone cannot certify a bound above $-1/(n^2 - 2)$ and that bounds for individual balanced determinant tuples cannot improve the order $1/n$.

AI USAGE

The results in this paper were developed through extensive use of an AI assistant (OpenAI's GPT Pro). None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs. The author also used AI assistants (GPT and Claude) to help polish the writing.

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Appendix A. Algebra, transport, and exact certificates

This appendix proves the algebraic identities and checks the exact certificates on which Sections 2 and 3 rely.

A.1. Compactness and normalization.

Lemma A.1. *Every map in $\mathfrak{B}_n$ is a Hilbert–Schmidt contraction. The set $\mathfrak{B}_n$ is compact, and the minimum in (1.1) is attained.*

Proof. For Hermitian X, the unital positive map satisfies $H(X)^2 \preceq H(X^2)$. One can see this directly from a spectral decomposition $X = \sum_i x_i P_i$: the matrices $H(P_i)$ form a positive operator-valued partition of identity, so the map $v \mapsto (H(P_i)^{1/2} v)_i$ is an isometry, and compression of $\mathrm{diag}(x_i I)$ gives the asserted inequality. Trace preservation now implies $\|H(X)\|_{\mathrm{HS}} \le \|X\|_{\mathrm{HS}}$. For Hermitian X, Y, one has $\|X + iY\|_{\mathrm{HS}}^2 = \|X\|_{\mathrm{HS}}^2 + \|Y\|_{\mathrm{HS}}^2$, and the same identity holds for their images. Thus the contraction extends to all matrices, so $\mathfrak{B}_n$ is bounded. Positivity and the two marginal identities are closed conditions, and Φ_n is a polynomial in the entries of H. □

Write $\mathrm{Ad}_A(X) = AXA^*$. To prove Lemma 2.2, we use the covariance rule

$$\Phi_k(\mathrm{Ad}_A \circ K \circ \mathrm{Ad}_B) = |\det A|^2 |\det B|^2 \Phi_k(K). \tag{A.1}$$

Indeed, the output congruence contributes $|\det A|^2$, while the chain rule for the input congruence contributes $|\det B|^2$.

The proof of Lemma 2.2 rescales K so that both marginals become the identity. The right input scale minimizes the determinant over inputs of unit determinant, which defines the capacity of a positive map,

$$\mathrm{cap}(K) = \inf_{X \succ 0,\ \det X = 1} \det K(X). \tag{A.2}$$

The determinant constraint removes scalar rescaling of the input.

Proof of Lemma 2.2. First assume $K(X) \succeq c\,\mathrm{tr}(X)I$ for positive semidefinite X, with $c > 0$. Minimize $\det K(X)$ over $X \succ 0$ with $\det X = 1$. The estimate $\det K(X) \geq c^k(\mathrm{tr}\,X)^k$ bounds the trace on sublevel sets; the determinant constraint then bounds every eigenvalue away from zero. A minimizer X_0 exists. Put $C = K(X_0)$. Differentiating $\log\det K(X)$ under $\log\det X = 0$ gives $K^*(C^{-1}) = \eta X_0^{-1}$. Pairing with X_0 gives $k = \eta k$, so $\eta = 1$. Define

$$L(Z) = C^{-1/2}K(X_0^{1/2}ZX_0^{1/2})C^{-1/2}.$$

The scaled map has $L(I) = L^*(I) = I$ and is positive. Since $\det X_0 = 1$, covariance (A.1) yields

$$\Phi_k(K) = \det C\,\Phi_k(L) \geq \gamma \det C \geq \gamma \det K(I),$$

where the last step uses $\det C \leq \det K(I)$ and $\gamma \leq 0$.

Apply this argument to $K + \varepsilon\,\mathrm{tr}(\cdot)I$ and let $\varepsilon \downarrow 0$ to obtain the general case, including singular $K(I)$. □

The sign condition in Lemma 2.2 is essential. The positive map $K(X) = X_{11}I_2$ has $K(I) = I_2$ and $\Phi_2(K) = 0$, so $\Phi_2(K) \geq c\det K(I)$ fails for every $c > 0$, and no positive constant can replace γ_2 in the lemma.

We use the seed bounds $\gamma_2 = \gamma_3 = 0$. To obtain the second seed, apply Wan's Theorem 0.3 (Wan, 2026) to $3(H + \varepsilon\Delta_3)/(1+\varepsilon)$, where $\Delta_3(X) = \mathrm{tr}(X)I/3$. Its two marginals are $3I$, as required there, and it sends every nonzero positive semidefinite matrix to a positive definite matrix. Homogeneity and continuity give $\Phi_3(H) \geq 0$ as $\varepsilon \downarrow 0$. For the first seed, the map on M_3 that sends a $2+1$ block matrix to $\mathrm{diag}(H(X), y)$ is positive and bistochastic; direct expansion gives its invariant as $\Phi_2(H)/3$. Thus $\Phi_2(H) \geq 0$ as well.

The reduction-transpose map $R_3(X) = (\mathrm{tr}(X)I - X^{\mathsf{T}})/2$ is bistochastic, and a brief calculation gives $\Phi_3(R_3) = 0$, so $\mu_3 = 0$; its pinched direct sums with identity maps also give $\mu_k \leq 0$ for every $k \geq 3$.

A.2. **Radial determinant calculus and the orbital score.** The score identity and the radial operator $\mathscr{D}_k$ rest on one computation, the action of $\det(\partial)$ on conjugation-invariant polynomials, which we carry out on Schur polynomials. Define the Schur polynomials in k variables by

$$s_\lambda(x) = \frac{\det(x_i^{\lambda_j+k-j})_{i,j=1}^k}{\det(x_i^{k-j})_{i,j=1}^k} \qquad (\ell(\lambda) \leq k).$$

The alternating numerator is divisible by the denominator, so this quotient is a polynomial, including at repeated coordinates. For a matrix X, $s_\lambda(X)$ and $e_j(X)$ denote the corresponding polynomials of its eigenvalues. In particular, $s_{(1^k)}(X) = \det X$ on M_k.

We use the complex Fischer inner product, conjugate-linear in the first argument,

$$\langle f, g\rangle_F = \sum_\alpha \alpha!\,\overline{f_\alpha}g_\alpha = \mathbb{E}[\overline{f(G)}g(G)].$$

Here f_α is the coefficient of the monomial with exponent multi-index α, and $\alpha!$ is the product of its entrywise factorials. The entries of G are independent circular complex Gaussians with $\mathbb{E}|G_{ij}|^2 = 1$. For the determinant polynomial $p(X) = \det X$,

$$\|p\|_F^2 = n!, \qquad \Phi_n(H) = \frac{\langle p, p\circ H\rangle_F}{n!}. \tag{A.3}$$

The first equality follows from the $n!$ determinant monomials; the second is exactly the differential convention in (2.1). The Gaussian realization is the coefficient pairing used in Gurvits's quantum-permanent calculus (Gurvits, 2004, Theorem 3.8 and Appendix C).

A conjugation-invariant holomorphic matrix polynomial is determined by its diagonal restriction: the spectral theorem determines it on Hermitian matrices, and a polynomial vanishing on all

Hermitian matrices vanishes identically. We can therefore check the radial identity on the Schur polynomials. Their Fischer norms on $M_k(\mathbb{C})$ satisfy

$$\langle s_\lambda, s_\mu \rangle_F = \mathbf{1}_{\lambda=\mu} \prod_{(i,j)\in\lambda} (k+j-i).$$

The product runs over the cells $1 \le i \le k$, $1 \le j \le \lambda_i$, and $\mathbf{1}_{\lambda=\mu}$ is one when the partitions agree and zero otherwise.

Multiplication by det is adjoint to $\det(\partial)$, while $\det s_\mu = s_{\mu+(1^k)}$. Hence both the matrix differential and the alternant operator act by

$$\mathscr{D}_k s_\lambda = \prod_{i=1}^{k} (\lambda_i + k - i) s_{\lambda-(1^k)}, \tag{A.4}$$

with value zero when $\lambda_k = 0$.

To justify the norm identity, let $d = \sum_i \lambda_i$, let χ^λ be the irreducible character of $\mathfrak{S}_d$ indexed by λ, meaning the trace of each permutation in that representation, and put $p_\pi(X) = \prod_c \operatorname{tr}(X^{|c|})$, with the product over the cycles of π. Here $\chi^\lambda(1)$ is the dimension of that representation. The Frobenius formula and Gaussian contraction give

$$s_\lambda(X) = \frac{1}{d!} \sum_{\pi\in\mathfrak{S}_d} \chi^\lambda(\pi) p_\pi(X), \qquad \langle p_\pi, p_\sigma \rangle_F = \sum_{\tau\in\mathfrak{S}_d} k^{\#\mathrm{cycles}(\pi^{-1}\tau\sigma\tau^{-1})}.$$

Character orthogonality reduces the pairing to $\mathbf{1}_{\lambda=\mu}(d!/\chi^\lambda(1)) s_\lambda(1^k)$. For each cell (i,j) of the partition, put $h_{ij} = \lambda_i - j + \lambda'_j - i + 1$, where λ'_j counts the parts at least j. The hook formulas $\chi^\lambda(1) = d!/\prod_{(i,j)} h_{ij}$ and $s_\lambda(1^k) = \prod_{(i,j)}(k+j-i)/h_{ij}$ give the displayed product (Macdonald, 1995, Chapter I, §§3, 7). Thus the adjoint calculation proves the matrix differential formula. Differentiating the alternant in the definition of s_λ gives the same multiplier for $\mathscr{D}_k$. Since Schur polynomials span symmetric polynomials, the two operators agree on every invariant matrix polynomial.

Proof of Proposition 2.1. Let $\widehat{f}_H(X) = \mathbb{E}_U \det H(UXU^*)$, whose diagonal restriction is f_H. The determinant differential is invariant under conjugation of the input, so averaging does not change its value. For conjugation-invariant matrix polynomials of degree n, the radial identity (A.4), proved above, shows that $\det(\partial_X)/n!$ extracts the coefficient of $s_{(1^n)}$. The monomial-to-Schur coefficient and the normalization are

$$[s_{(1^n)}] m_\lambda = \frac{(-1)^{n-\ell(\lambda)} \ell(\lambda)!}{\prod_j m_j(\lambda)!}, \qquad m_\lambda(1^n) = \frac{n!}{(n-\ell(\lambda))! \prod_j m_j(\lambda)!}.$$

Their quotient gives (2.3); the Schur identities follow from the Frobenius character formula (Macdonald, 1995, Chapter I, §7). The square-free coefficient means $[x_1 \cdots x_n] f_H$. For each basis, Gurvits's theorem bounds this coefficient below by $n!/n^n$ (Gurvits, 2006, Theorem 4.1). Averaging preserves this inequality. □

A.3. **The two coefficient rows used by transport.** For the row $J_{n,k,\gamma}$ in (2.6), only shapes $\mu \cup 1^{n-k}$ can leave every inactive variable to the first power. Their active polynomial is $m_\mu / m_{\mu\cup 1^{n-k}}(1^n)$; the ratio of its evaluations is

$$\frac{m_\mu(1^k)}{m_{\mu\cup 1^{n-k}}(1^n)} = \frac{(k)_s}{(n)_s}.$$

On degree-k polynomials, $\mathscr{D}_k/k!$ gives the signed score $(-1)^j/\binom{k}{j}$, while $\mathscr{T}_k/k!$ evaluates the polynomial at 1^k. These evaluations prove every factor in (2.6).

For the row R_n, take the coefficient of x_1^{n-2} in $\mathscr{D}_2 f_H$. Only the active monomials of types (n) and $(n-1,1)$ contribute. The coefficients of x_1^{n-2} in their two-variable radial derivatives are $-n$ and n; division by $m_{(n)}(1^n) = n$ and $m_{(n-1,1)}(1^n) = n(n-1)$ proves (2.7).

A.4. **Verification of dimension transport.**

Proof of Theorem 3.1. Put $N = n(n-1)$, $a = 1-\gamma$, $b = (k-1)(1+k\gamma)$, $D = aN + b$, and $t = (n^2-1)/D$. Define residuals c_λ by

$$\Phi_n(H) - h_n(K) = tJ_{n,k,\gamma} + \mathbf{1}_{n \text{ even}}(1/n + h_n(K))R_n + \sum_{\lambda \vdash n} c_\lambda w_\lambda. \tag{A.5}$$

Here $\mathbf{1}_{n \text{ even}}$ equals one for even n and zero otherwise. More explicitly, if J_λ and R_λ are the coefficients of w_λ in (2.6) and (2.7), then

$$c_\lambda = r_\lambda - h_n(K) - tJ_\lambda - \mathbf{1}_{n \text{ even}}(1/n + h_n(K))R_\lambda.$$

The identity uses $\sum_\lambda w_\lambda = 1$. As in the proof idea after Theorem 3.1, these choices of t and K make the coefficients of $w_{(1^n)}$ and $w_{(2,1^{n-2})}$ in (A.5) vanish.

Since $0 \le K \le k-1 < n$, both displayed inequality multipliers are nonnegative; in particular,

$$\frac{1}{n} + h_n(K) = \frac{K(n+1)}{n(N+K)}.$$

After multiplication by D, each residual is affine in γ, since $D(1/n + h_n(K)) = b(n+1)/n$. We can therefore check the endpoints $\gamma = 0$ and $\gamma = -1/k$.

The shapes (1^n) and $(2,1^{n-2})$ have zero residual. Recall that λ is selected when $\lambda = \mu \cup 1^{n-k}$ for some $\mu \vdash k$; this includes shapes whose coefficient in J vanishes. In that case $j = n - \ell(\lambda)$ counts the missing variables, and s is the sum of the parts of μ that exceed one. Every other selected shape has $j \ge 2$ and $j+1 \le s \le \min(2j, k)$.

At $\gamma = 0$ we have $D = N + k - 1$ and $h_n(K) = -(n-k)/D$. Since $n \ge k+3$, this value is at most $-2/N$. All unselected shapes except (n) in even dimension have score at least $-2/N$. For selected odd $j \ge 3$, we also have $\binom{n}{j} \ge \binom{n}{2}$, and hence $r_\lambda - h_n(K) \ge 0$. The coefficient in J is negative, so subtracting tJ increases that residual. For selected even $j \ge 2$, the inequality $s \ge j+1$ gives

$$\frac{(k)_s}{(n)_s\binom{k}{j}} \le \frac{k-j}{n-j}\frac{1}{\binom{n}{j}}.$$

Consequently, the residual satisfies

$$Dc_\lambda \ge (n-k) + \frac{D - (n^2-1)(k-j)/(n-j)}{\binom{n}{j}} = (n-k)\left(1 + \frac{N+j-1}{(n-j)\binom{n}{j}}\right) > 0.$$

At $\gamma = -1/k$, the target is $-1/n$. For a selected shape, the residual condition is equivalent to $A_{j,s}(k) \le A_{j,s}(n)$, where

$$A_{j,s}(x) = \frac{(x)_s}{x+1}\left(1 + \frac{x(-1)^j}{\binom{x}{j}}\right).$$

Here $(x)_s = x(x-1)\cdots(x-s+1)$ and $\binom{x}{j} = (x)_j/j!$ for real x. The two leading cases are constant. For even $j \ge 2$,

$$A_{j,s}(x) = \frac{(x)_s}{x+1} + \frac{j!x}{x+1}(x-j)_{s-j},$$

a sum of nonnegative increasing functions on $x \ge s$. For odd $j \ge 3$,

$$A_{j,s}(x) = \frac{(x)_s}{x+1}\left(1 - \frac{j!}{(x-1)_{j-1}}\right),$$

a product of nonnegative increasing functions there. Every unselected score is at least $-1/n$.

For even n, the R_n term cancels the residual at (n). The other affected shape, $(n-1,1)$, is unselected when $n \ge k+3$, and its final residual is

$$\frac{(n+1)(n-K)}{n(N+K)} > 0.$$

The case analysis now covers every shape. Affine interpolation proves nonnegativity for all $\gamma \in [-1/k, 0]$.

For a fixed seed, a finite test can lower the starting dimension. For a selected shape with parameters (j, s), before the parity correction, the residual multiplied by the positive factor $D\,(n)_s$ is the polynomial

$$\begin{aligned} P_{j,s}(n) = (-1)^j j! D\,(n-j)_{s-j} + (an - a - b)(n)_s \\ - (n^2-1)(k)_s \left(\frac{(-1)^j}{\binom{k}{j}} - \gamma \right). \end{aligned} \tag{A.6}$$

For $j = s = 0$ read $(n)_0 = 1$; the other pairs satisfy $1 \le j < k$ and $j + 1 \le s \le \min(2j, k)$. Thus there are finitely many polynomials, independent of the target dimension.

Choose an integer $n_0 \ge k + 1$. If every coefficient of $P_{j,s}(n_0 + z)$ is nonnegative, all selected residuals are nonnegative for $n \ge n_0$. To cover unselected shapes it suffices to check

$$(n_0 - 3 - K)n_0(n_0 - 1) \ge 2K. \tag{A.7}$$

This inequality persists as n increases and is equivalent to $h_n(K) \le -2/(n(n-1))$. The even-dimensional shape (n) is still canceled by R_n. For its other shape $(n-1, 1)$, the residual is the positive expression already displayed if $n \ge k + 2$. When $n = k + 1$ is even, this shape is selected, and its residual after the correction is

$$\frac{(n+1)\{1 - \gamma(n^2 - 3n + 3)\}}{n(N + K)(1 - \gamma)} > 0.$$

These checks prove the claimed finite test, including the possible boundary case $n_0 = k + 1$. □

A.5. **Exact finite certificates.**

A.5.1. *The rational model.* The projection constraints of the program (3.4) come from evaluating f_H at $t1^n + 1_{[a]}$, where $1_{[a]}$ has its first a entries equal to one and the rest zero. For λ padded with zeros, set

$$B_{a,j,\lambda} = \binom{n}{a}^{-1} \sum_{|I|=a} \binom{\sum_{i \in I} \lambda_i}{j}. \tag{A.8}$$

The expression in (A.8) is the coefficient of t^{n-j} in $q_\lambda(t1^n + 1_{[a]})$: a monomial whose distinguished exponents sum to b contributes $t^{n-b}(t+1)^b$. On the map side, the same substitution gives

$$f_H(t1^n + 1_{[a]}) = \mathbb{E}_U \det\big(tI + H(UP_aU^*)\big), \qquad P_a = \operatorname{diag}(1_{[a]}).$$

Thus its coefficients are averages of the characteristic coefficients of projection images. Since their vectors lie in the hull of Section 2.2, one common convex mixture represents all degrees j for a fixed a. If $\mathcal{V}_{n,a}$ is the finite list in (2.8), the spectral constraints are

$$\sum_\lambda B_{a,j,\lambda} w_\lambda = \sum_{v \in \mathcal{V}_{n,a}} e_j(v)\pi_{a,v}, \qquad \pi_{a,v} \ge 0, \qquad \sum_v \pi_{a,v} = 1. \tag{A.9}$$

It suffices to use $1 \le a \le \lfloor n/2 \rfloor$ and $2 \le j \le n$. The degrees zero and one are fixed by normalization and trace preservation. The complementary constraints follow from $H(I - P) = I - H(P)$, because the coefficients of $\det(tI + I - H(P))$ are linear combinations of those of $\det(tI + H(P))$. Together with the weights, square-free floor and inherited mixed rows, these constraints give the program (3.4).

Table 1: Certified lower bounds $\gamma_n \le \mu_n$. Every terminating decimal is an exact rational number, and values after $\approx$ are rounded; γ_6 and γ_7 are given exactly in (A.10). The entries for $n \le 7$ are optima of the finite program, those for $8 \le n \le 21$ are feasible dual bounds, and γ_{24} is the certificate of Proposition A.3.

n	γ_n	n	γ_n	n	γ_n
4	$-3/62 \approx -0.04839$	10	-0.04791679	16	-0.0365074575
5	$-77957/1378367 \approx -0.05656$	11	-0.04573151	17	-0.0352435158
6	≈ -0.05563, see (A.10)	12	-0.04338172	18	-0.034110757033
7	≈ -0.05531, see (A.10)	13	-0.0414257677	19	-0.033049439588
8	-0.052740227999	14	-0.0394550826	20	-0.032188230644
9	-0.05064282	15	-0.0379279304	21	-0.031356683435
				24	$-649/22351 \approx -0.02904$

A.5.2. *Certified seeds.*

Proposition A.2 (Finite certificates)**.** *For $4 \le n \le 21$, every $H \in \mathfrak{B}_n$ satisfies $\Phi_n(H) \ge \gamma_n$, with γ_n as in Table 1. In dimensions four through seven, γ_n is the optimum of the program (3.4) seeded with the preceding values. In dimensions eight through twenty-one, γ_n is a downward rounding of an exactly feasible dual objective, and no optimality is asserted. For $k \in \{18, 19, 20, 21\}$, the transported bound $\Phi_n(H) \ge h_n(K(k, \gamma_k))$ holds for every $n \ge k$.*

Proof. The coefficient formulas (A.4), (A.8), and (A.9) specify the rational data (A, b, c) of (3.4). As in Section 3.1, a rational vector y with $A^{\mathsf{T}} y \le c$ then proves $\Phi_n(H) \ge b^{\mathsf{T}} y$ by finite checks in exact rational arithmetic. A floating-point optimizer only proposed each dual vector; rational reconstruction and exact feasibility decide whether it certifies a bound. In dimensions four through seven, a rational primal vector with $Ax = b$, $x \ge 0$, and the same objective value certifies the optimum. In dimensions eight through twenty-one, the exact dual objective is rounded down to the displayed γ_n, and the later programs use the rounded value as their seed.

For the transported tails, the finite test in the proof of Theorem 3.1 is run from $n_0 = k + 1$: the inequality (A.7), and nonnegativity of every coefficient of the shifted polynomials (A.6) for the pairs (j, s) with $1 \le j < k$ and $j + 1 \le s \le \min(2j, k)$, together with $(j, s) = (0, 0)$. The parity residual displayed in that proof covers the remaining boundary coefficient. The dual vectors, the rational model, and these checks accompany the paper as supporting files. □

The two seeds with nonterminating decimal expansions are

$$\gamma_6 = -\frac{129358525188828749}{2325438043571344896} \approx -0.05563,$$
$$\gamma_7 = -\frac{5408353173620987686795318309001825}{977759844926698118455281832417961 97} \approx -0.05531. \tag{A.10}$$

A.5.3. *An analytic certificate in dimension twenty-four.* The seed in dimension 24 is not a dual vector of (3.4). It adds one three-valued projection test to the transport certificate from dimension 21, so that its verification is a list of rational identities and inequalities rather than a linear program.

Proposition A.3 (A dimension-twenty-four certificate)**.** *Every $H \in \mathfrak{B}_{24}$ satisfies*

$$\Phi_{24}(H) \ge -\frac{649}{22351} = h_{24}(271/40) \approx -0.02904.$$

The transported bound $\Phi_n(H) \ge h_n(271/40)$ holds for every $n \ge 24$.

Proof. Fix $n = 24$, $k = 21$, $a = 12$, and $\gamma = \gamma_{21}$ from Table 1. Put

$$N = n(n-1), \quad K = K(k,\gamma), \quad h = h_n(K), \quad t = \frac{n^2-1}{(1-\gamma)(N+K)}, \quad \eta = \frac{1}{n} + h,$$
$$v = \frac{a(n-a)}{N}, \qquad \Delta = \frac{a(n-a)}{N+K} - \frac{1}{4}, \qquad \alpha_0 = \frac{4nv}{n+1} = \frac{576}{575}.$$

The residuals in (A.5) are $c_\lambda = r_\lambda - h - tJ_\lambda - \eta R_\lambda$. They vanish at (1^n), $(2,1^{n-2})$, and (n), and are positive at every other partition of n.

To constrain the two leading weights, define

$$\psi(0) = 1, \quad \psi(\pm 1) = -1, \quad \psi(z) = C_{24} \quad (|z| \geq 2), \qquad C_{24} = \frac{49038552465408}{22017942472189},$$

and let

$$b_\lambda = \binom{n}{a}^{-1} \sum_{|I|=a} \psi\left(\sum_{i\in I} \lambda_i - a\right), \qquad B(H) = \sum_{\lambda \vdash n} b_\lambda w_\lambda.$$

For a spectral generator $u + \mathrm{Bin}(b,p)$ with $u + bp = a$, the inequality $\mathbb{E}\psi(u + \mathrm{Bin}(b,p) - a) \geq 0$ is a finite rational inequality. It holds for all 145 generators of the rank-twelve spectral hull, so $B(H) \geq 0$. Equality at $(u,b,p) = (0,13,12/13)$ fixes the value of C_{24}. The two leading coefficients are $b_{(1^n)} = 1$ and $b_{(2,1^{n-2})} = 1 - 4v$.

To pair the projection row with the transport residuals, define

$$d_\lambda = b_\lambda - 1 + \alpha_0(1 - r_\lambda), \qquad d_\circ = d_{(n)}, \qquad M = \max_{c_\lambda > 0} \frac{d_\lambda + d_\circ R_\lambda}{c_\lambda}.$$

The coefficients d_λ vanish at the two leading shapes. Since $R_{(n)} = -1$, the adjusted coefficient also vanishes at (n). The maximum is a rational number, attained at $\lambda = (6,1^{18})$. Set $s_\lambda = Mc_\lambda - d_\lambda - d_\circ R_\lambda$. Every s_λ is nonnegative, and summing the coefficient identities gives

$$(\alpha_0 + M)(\Phi_n(H) - \gamma^\sharp) = B(H) + MtJ_{n,k,\gamma} + (M\eta + d_\circ)R_n + \sum_{\lambda \vdash n} s_\lambda w_\lambda, \tag{A.11}$$

where $\gamma^\sharp = h + 4\Delta/(\alpha_0 + M)$. All multipliers on the right are nonnegative, and the exact value satisfies $\gamma^\sharp > -649/22351$. The 145 spectral inequalities and the 1575 coefficient identities in (A.11) are rational checks; they are recorded, with the exact values of M and $\gamma^\sharp$, in the supporting files.

For the rounded seed $-649/22351$, all 145 transport polynomials in (A.6) have nonnegative coefficients after the shift $n = 25 + x$. The unselected-shape test and the parity residual also pass exactly. These checks prove the transported bound for $n \geq 25$; the seed itself supplies $n = 24$. □

A.5.4. *Two further consequences.* A stronger positivity assumption supplies further zero seeds. A map is k-positive if $\mathrm{Id}_{M_k} \otimes H$ is positive, and completely positive if this holds for every k. In finite dimensions, complete positivity is equivalent to a representation $H(X) = \sum_a A_a X A_a^*$. The map is k-copositive if $H \circ \mathsf{T}$ is k-positive, where $\mathsf{T}(X) = X^{\mathsf{T}}$; complete copositivity is defined in the same way. For a map with domain M_k, k-positivity already implies complete positivity: its Choi matrix is the image of the positive rank-one matrix $\sum_{i,j} E_{ij} \otimes E_{ij}$ under $\mathrm{Id}_{M_k} \otimes H$, and factoring that Choi matrix gives the stated representation.

Corollary A.4. *If $H \in \mathfrak{B}_n$ is k-positive or k-copositive, $k \geq 3$, and $n \geq k+3$, then*

$$\Phi_n(H) \geq -\frac{n-k}{n^2-n+k-1}.$$

In particular, $(n-3)$-positivity or $(n-3)$-copositivity implies $\Phi_n(H) \geq -3/(n^2-4)$ for $n \geq 6$.

Proof. The compatible compression $Z \mapsto W^*H(VZV^*)W$ on M_k preserves k-positivity and hence is completely positive. Its invariant is nonnegative by the Gaussian determinant formula (Gurvits, 2004, Theorem 3.8). In the copositive case, the compression is completely copositive and transposition leaves Φ_k unchanged. Thus the mixed inequality holds with $\gamma = 0$. The coefficient proof of Theorem 3.1 applies with $K = k - 1$. □

The finite seeds also give one constant valid in every dimension.

Corollary A.5. *For every $n \geq 2$ and $H \in \mathfrak{B}_n$,*

$$\Phi_n(H) \geq -\frac{77957}{1378367} \approx -0.05656.$$

Proof. Dimensions two and three are nonnegative. Among the seeds in dimensions four through seven, the smallest is $\gamma_5 = -77957/1378367$. For $n \geq 8$, the right side of (1.3) is increasing, since its derivative is $(9n^2 - 309)/(3n^2 + 20n + 103)^2 > 0$, and its value at 8 is $-24/455 > \gamma_5$. □

Appendix B. Proofs for the coefficient relaxations

The proofs below supply the estimates behind Section 4.

B.1. The extremizer of the mixed relaxation.

Proof of Theorem 4.1. The coefficient proof of Theorem 3.1 uses only nonnegative normalized weights and the rows J and R. It therefore applies to every polynomial in $\mathcal{R}_n(\mathcal{G})$, with a seed attaining K, and gives the lower bound.

To prove feasibility of (4.1) for all rows, split $z = (x, y)$ into k active and $m = n - k$ inactive variables. Write $M_m(y) = m_{(2,1^{m-2})}(y)$ for $m \geq 2$, and $M_1 = 0$. Then

$$\begin{aligned}(\mathscr{D}_k - \gamma\mathscr{T}_k)f_{n,K} = \frac{k!}{N+K}\big[&(1-\gamma)(K - K(k,\gamma))e_m(y) + (1-\gamma)M_m(y)\\ &+ (k+1)(1/k - \gamma)e_1(x)e_{m-1}(y)\big].\end{aligned} \tag{B.1}$$

Expand $e_n = e_k(x)e_m(y)$ and $e_{n-1} = e_{k-1}(x)e_m(y) + e_k(x)e_{m-1}(y)$. The required derivatives are

p	$\mathscr{D}_k p$	$\mathscr{T}_k p$
e_k	$k!$	$k!$
e_{k-1}	0	0
e_1e_{k-1}	$k!$	$k!k^2$
e_1e_k	$(k+1)(k-1)!e_1$	$k!(k+1)e_1$.

The radial formulas follow from (A.4); the scalar derivatives follow by substituting $x + t1^k$. Combining them with $e_1(y)e_{m-1}(y) = me_m(y) + M_m(y)$ proves (B.1). Its three terms have disjoint active/inactive exponent types. For $\gamma \leq 0$ and $K \geq 0$, every coefficient is nonnegative exactly when $K \geq K(k,\gamma)$. Thus $f_{n,K}$ satisfies every supplied mixed row. Its only weights are

$$w_{(1^n)} = \frac{K}{N+K}, \qquad w_{(2,1^{n-2})} = \frac{N}{N+K}, \tag{B.2}$$

whose signed value is $h_n(K)$.

For uniqueness, choose $(k,\gamma) \in \mathcal{G}$ attaining K and use the residual identity (A.5). As in that identity, put $D = (1-\gamma)N + (k-1)(1+k\gamma) > 0$. At $\gamma = 0$ its residuals are strictly positive away from the two shapes in (B.2) and the shape (n) in even dimension. In the unselected case this uses the strict inequality $h_n(k-1) < -2/N$ for $n \geq k+3$; the selected cases are the positive expressions in the proof of Theorem 3.1. At $\gamma = -1/k$ the residuals are nonnegative. Since $K \geq 2$, an attaining seed has $\gamma > -1/k$, so affine interpolation preserves strict positivity of the D-scaled residuals away from those shapes.

Equality therefore removes every positive-residual weight. In even dimension the multiplier of R_n is positive, and $w_{(n-1,1)} = 0$; equality in $R_n \geq 0$ then removes $w_{(n)}$ as well. The equality $J_{n,k,\gamma} = 0$ fixes the remaining weight ratio as $K : N$. Normalization gives (B.2).

Finally, for $z_i > 0$,

$$\frac{f_{n,K}(z)}{\prod_i z_i} = \frac{K + \sum_{i \neq j} z_i/z_j}{N + K} \geq 1,$$

by pairing reciprocal ratios; equality holds at 1^n. Thus $\mathrm{Cap}(f_{n,K}) = 1$. Also, for $K \geq 2$ and $n \geq 4$,

$$\frac{K}{N+K} \geq \frac{2}{N+2} \geq \frac{n!}{n^n}.$$

For the second inequality, check $n = 4$ and proceed by induction: the ratio of the left-hand side at consecutive dimensions is at least $1/2$, whereas the ratio for $n!/n^n$ is $(n/(n+1))^n \leq 1/2$. Hence neither additional condition excludes the unique extremizer. □

B.2. **Transport residuals and projection coefficients.** The recursion of Proposition 4.5 compares projection coefficients with transport residuals, so it needs lower bounds on the residuals and an explicit projection row.

Lemma B.1 (A sharper transport residual)**.** *Let $k \geq 3$, $n \geq k+3$, $0 \leq K \leq k-1$, and $N = n(n-1)$. Every residual in (A.5) outside the two leading shapes, except the canceled even-dimensional shape (n), is at least*

$$\widetilde{\rho}_{n,k,K} = \frac{1}{(k-1)(N+K)}\Bigg[(k-1-K)\frac{(n-3)(n-k)(n+k-2)}{n(n-2)} + K\left(n-k-2-\frac{2(k-1)}{N}\right)\Bigg] > 0.$$

Proof. At the endpoint $\gamma = -1/k$, the selected-shape residual is

$$c_\lambda = \left(\frac{1}{n} + \frac{(-1)^j}{\binom{n}{j}}\right)\left(1 - \frac{A_{j,s}(k)}{A_{j,s}(n)}\right), \qquad A_{j,s}(x) = \frac{(x)_s}{x+1}\left(1 + \frac{x(-1)^j}{\binom{x}{j}}\right).$$

For nonleading selected shapes $j \geq 2$ and $s \geq j+1$. We claim that $A_{j,s}(x)/(x(x-2))$ is nondecreasing for $x \geq s$. For $j = 2$ and $s = 3$, this ratio is exactly one; larger s append nonnegative increasing factors. For odd $j \geq 3$, the ratio is

$$\frac{x-1}{x+1}\prod_{i=3}^{s-1}(x-i)\left(1 - \frac{j!}{(x-1)_{j-1}}\right),$$

a product of nonnegative increasing functions. For even $j \geq 4$ it suffices again to take $s = j+1$. The ratio is then $F(x) + G(x)$, where

$$F(x) = \frac{x-1}{x+1}\prod_{i=3}^{j}(x-i), \qquad G(x) = \frac{j!(x-j)}{(x+1)(x-2)}.$$

For $x \geq j+1$, we have

$$\frac{F}{G} = \frac{(x-1)_{j-1}}{j!} \geq 1, \qquad \frac{F'}{F} \geq \frac{2}{x-3}, \qquad \frac{G'}{G} > -\frac{2}{x-2}.$$

Thus $F' + G' > 0$, proving the claim.

It follows that

$$\frac{A_{j,s}(k)}{A_{j,s}(n)} \leq \frac{k(k-2)}{n(n-2)}.$$

Since $\binom{n}{j} \geq \binom{n}{2}$ for the selected range, every such residual is at least

$$\rho_0 = \frac{n-3}{n(n-1)}\left(1 - \frac{k(k-2)}{n(n-2)}\right).$$

Unselected shapes obey the same bound because their score is at least $-2/N$; the corrected $(n-1,1)$ residual is larger than $1/n$. At $\gamma = 0$ the existing proof gives the lower bound

$$\rho_1 = \frac{n-k}{N+k-1} - \frac{2}{N}.$$

Both floors are positive when $n \geq k+3$.

The products Dc_λ are affine in γ. Interpolating the two endpoint floors, with $D_0 = (k+1)N/k$, $D_1 = N+k-1$, and $\gamma = (K-k+1)/(K+k(k-1))$, gives

$$c_\lambda \geq \frac{(k-1-K)N\rho_0 + KD_1\rho_1}{(k-1)(N+K)}.$$

Substitution is the stated formula. □

Lemma B.2 (Growth of the selected residuals)**.** *For integers $j \geq 2$ and $s \geq j+1$, the function*

$$A_{j,s}(x)/x^{s-1}, \qquad A_{j,s}(x) = \frac{(x)_s}{x+1}\left(1 + \frac{x(-1)^j}{\binom{x}{j}}\right),$$

is nondecreasing for $x \geq s$. Consequently, at the endpoint $\gamma = -1/k$, every selected nonleading shape with s coordinates different from one has residual at least

$$\rho_{0,s} = \frac{n-3}{n(n-1)}\left(1 - (k/n)^{s-1}\right).$$

Zero padding is included in this count; equivalently, s is the sum of the parts exceeding one.

Proof. For odd j, the normalized function factors as

$$\frac{x}{x+1}\prod_{r=1}^{s-1}\left(1 - \frac{r}{x}\right)\left(1 - \frac{j!}{(x-1)_{j-1}}\right),$$

a product of nonnegative increasing functions. For even j, it suffices to take $s = j+1$, since each additional support coordinate appends a factor $1 - r/x$. For $j = 2$, the normalized function is $1 - 2/x$. For $j = 4$, it is

$$1 - \frac{11}{x} + \frac{46}{x^2} - \frac{72}{x^3};$$

its derivative has positive numerator $11x^2 - 92x + 216$, whose discriminant is negative.

For even $j \geq 6$, put $t = x - j \geq 1$ and $P(t) = \prod_{r=1}^{j-1}(t+r)$. The normalized function is

$$\frac{t}{t+j+1}\frac{P(t)+j!}{(t+j)^{j-1}}.$$

The first factor increases. The derivative numerator of the second is

$$(t+j)P'(t) - (j-1)(P(t)+j!) = \sum_{r=1}^{j-1}(j-r)\prod_{h\neq r}(t+h) - (j-1)j!.$$

This numerator increases with t; its value at $t = 1$ is $j![(j+1)\mathsf{H}_j - 3j + 1]$, where $\mathsf{H}_j = \sum_{r=1}^{j} 1/r$. The bracket is positive at $j = 6$, and its successive difference is $\mathsf{H}_j - 2 + 1/(j+1) > 0$, so it is positive for every even $j \geq 6$, which proves monotonicity. The endpoint residual formula in Lemma B.1 and $\binom{n}{j} \geq \binom{n}{2}$ now give the stated floor. □

Lemma B.3 (A coefficientwise projection correction)**.** *Let $k \geq 3$ and let $\gamma = h_k(K)$, $0 < K \leq k-1$, be a valid universal seed. Fix $n \geq k+3$ and a valid bounded projection separator with tail value C, that is, $\psi(z) = C$ for $|z| \geq 2$. Let c_λ denote the residuals in the transport certificate, including its even-dimensional parity correction. Put*

$$a = \lfloor n/2 \rfloor, \quad N = n(n-1), \quad v = \frac{a(n-a)}{N}, \quad \alpha_0 = \frac{4nv}{n+1}, \quad \Delta = \frac{a(n-a)}{N+K} - \frac{1}{4}.$$

For the orbital coefficients

$$b_\lambda = \binom{n}{a}^{-1} \sum_{|I|=a} \psi\left(\sum_{i\in I} \lambda_i - a\right), \qquad d_\lambda = b_\lambda - 1 + \alpha_0(1 - r_\lambda),$$

define $\widehat{d}_\lambda = d_\lambda$, except that in even dimension

$$\widehat{d}_{(n-1,1)} = d_{(n-1,1)} + d_{(n)}/(n-1).$$

Let $\mathcal{S}_n$ exclude the two leading shapes and, in even dimension, the shape (n). Set

$$M_{n,k,K} = \max\left(0, \max_{\lambda\in\mathcal{S}_n} \frac{\widehat{d}_\lambda}{c_\lambda}\right).$$

Then every $H \in \mathfrak{B}_n$ satisfies

$$\Phi_n(H) \geq h_n(K) + \frac{4\Delta}{\alpha_0 + M_{n,k,K}}.$$

Proof. Write $\delta = \Phi_n(H) - h_n(K)$, $y = w_{(2,1^{n-2})}$, and $y_* = N/(N+K)$. The score identity gives

$$y_* - y = \frac{n}{n+1}\left[\delta + \sum_{\lambda \text{ nonleading}} (1 - r_\lambda) w_\lambda\right].$$

The bounded projection row, with leading coefficients 1 and $1 - 4v$, therefore implies

$$4\Delta \leq \alpha_0 \delta + \sum_{\lambda \text{ nonleading}} d_\lambda w_\lambda.$$

In even dimension $d_{(n)} > 0$: its centered subset sum has absolute value $n/2 \geq 2$, so $b_{(n)} = C$, and $r_{(n)} = -1/n$. The inequality $R_n \geq 0$ replaces its weight by $w_{(n-1,1)}/(n-1)$ and gives the effective coefficients above. Every remaining residual is positive. Finally the transport identity implies $\sum_{\mathcal{S}_n} c_\lambda w_\lambda \leq \delta$, proving the result. □

B.3. **The finite-binomial separator estimates.** The proof of Proposition 4.6 reduces the validity of ψ_M to the following estimates for binomial laws.

Finite-binomial estimates for Proposition 4.6. Use the test ψ_M from (4.4). The elementary bounds $7 < e^2 < 23/3$ imply $M_* > 1/16$. Thus, for $M \geq M_*$, all values $\psi_M(d)$ with $|d| \geq 2$ are nonnegative, and $\psi_M(2) > 2$. The projection-spectrum hull reduces validity to the deterministic distribution at zero and the centered binomials $X - r$, where $X \sim \mathrm{Bin}(r+s, r/(r+s))$ and $r, s \geq 1$.

For these generators, write

$$q = \mathbb{P}(X = r), \qquad R = \frac{\mathbb{P}(|X-r| = 1)}{q} = 2 - \frac{1}{r+1} - \frac{1}{s+1},$$

and

$$A_2 = \frac{\mathbb{P}(|X-r| = 2)}{q} = \frac{(r-1)s^2}{r(s+1)(s+2)} + \frac{(s-1)r^2}{s(r+1)(r+2)}.$$

For $r, s \geq 2$, we have $2A_2 \geq R - 1$. Indeed, after multiplication by the positive denominator $rs(r+1)(r+2)(s+1)(s+2)$, the difference is

$$P(r,s) = 2(r-1)s^3(r+1)(r+2) + 2(s-1)r^3(s+1)(s+2) \\ - (rs-1)rs(r+2)(s+2).$$

For $x = r-2$ and $y = s-2$, its expansion is $P(r,s) = \sum_{i,j=0}^{3} p_{ij} x^i y^j$, where

$$(p_{ij})_{i,j=0}^{3} = \begin{pmatrix} 192 & 320 & 152 & 24 \\ 320 & 548 & 266 & 42 \\ 152 & 266 & 129 & 20 \\ 24 & 42 & 20 & 3 \end{pmatrix}.$$

Consequently, every function ψ with the prescribed values at $0, \pm 1$, with $\psi(\pm 2) \geq 2$, and with nonnegative values elsewhere satisfies

$$\mathbb{E}\psi(X - r) \geq q(1 - R + 2A_2) \geq 0 \qquad (r, s \geq 2).$$

By symmetry, the remaining generators have $X \sim \mathrm{Bin}(m, 1/m)$, centered at one, with $m \geq 2$. For $m = 2$, the expectation of $\psi_M(X-1)$ is zero. For $m \geq 3$, put $a = 1 - 2/m \in [1/3, 1)$. Using the probability generating function and its first two derivatives gives

$$\mathbb{E}\psi_M(X-1) = M(3 - 4/m) - (1+3M)a^m - 2Ma^{m-1} - 2M(1-1/m)a^{m-2}. \tag{B.3}$$

At $M = M_*$, nonnegativity is equivalent to

$$a^{m-2}\big[(3e^2 - 4)a^2 + 3a + 1\big] \leq 1 + 2a. \tag{B.4}$$

Since $m - 2 = 2a/(1-a)$ and $\log a \leq 2(a-1)/(a+1)$, it suffices to prove

$$(1+2a)\exp\left(\frac{4a}{1+a}\right) \geq (3e^2 - 4)a^2 + 3a + 1.$$

Set $t = (1-a)/(1+a) = 1/(m-1) \in (0, 1/2]$. After multiplication by $(1+t)^2$, this inequality becomes $F(t) \geq 0$, where

$$F(t) = e^2\big[(3 + 2t - t^2)e^{-2t} - 3 + 6t - 3t^2\big] - 10t + 6t^2.$$

The Taylor bound $e^{-2t} \geq 1 - 2t + 2t^2 - \frac{4}{3}t^3$ gives

$$F(t) \geq t\big[e^2 B(t) - 10 + 6t\big], \qquad B(t) = 2 - 2t + 2t^2 - \frac{14}{3}t^3 + \frac{4}{3}t^4.$$

The identity

$$7B(t) - 10 + 6t = (1 - 2t)\left(4 + 14t^2 - \frac{14}{3}t^3\right) \geq 0$$

also implies $B(t) \geq 1$. Since $e^2 > 7$, we obtain $F(t) > 0$, which proves (B.4).

Write the right-hand side of (B.3) as $MD_m - a^m$, where D_m is independent of M. Validity at M_* implies $D_m \geq a^m / M_*$. The function $(1 - 2/m)^m$ increases for real $m > 2$, because

$$\frac{d}{dm}\big[m \log(1 - 2/m)\big] = -\log(1+u) + u \geq 0, \qquad u = \frac{2}{m-2}.$$

It follows that $a^m \geq 1/27$ for $m \geq 3$, and therefore

$$\mathbb{E}\psi_M(X-1) \geq \frac{M - M_*}{27 M_*} =: \eta_M > 0 \qquad (M > M_*,\ m \geq 3).$$

□

Appendix C. Negative constructions

The constructions below prove Proposition 5.1 and the padding bounds of Section 5.

C.1. **An attained minimum in a family extending Du's example.** Du's example uses a six-dimensional positive map and separate terms joining three-dimensional blocks. Combining each pair of joining terms into one congruence introduces off-diagonal terms while preserving positivity.

For $X \in M_9$, let $S(X)$ be its leading 6×6 block and put $\widetilde{\mathcal{B}}(X) = \operatorname{diag}(\mathcal{B}(S(X)), 0_3)$. Using three blocks of size three, define

$$A_1 = E_{13} \otimes I_3, \quad B_1 = E_{31} \otimes I_3, \qquad A_2 = E_{23} \otimes I_3, \quad B_2 = E_{32} \otimes I_3,$$

where these E_{ij} are matrix units in M_3. For real t set

$$Q_t^{(j)}(X) = A_j X A_j^* + B_j X B_j^* + t(A_j X B_j^* + B_j X A_j^*), \qquad j = 1, 2,$$

and, with $\mathsf{T}(X) = X^{\mathsf{T}}$, define

$$G_{a,b} = \tfrac{1}{8}\widetilde{\mathcal{B}} + \tfrac{1}{2}Q_a^{(1)} + \tfrac{1}{2}Q_b^{(2)} \circ \mathsf{T}.$$

At $a = b = 0$, this is Du's map (Du, 2026, (2.12)) with parameter $t = 1/4$, divided by two. Both marginals of Du's map at that parameter are $2I_9$, so $G_{0,0}$ is its bistochastic normalization.

Proposition C.1 (Attained nine-dimensional family bound)**.** *For real a, b, the map $G_{a,b}$ is positive and bistochastic exactly when $|a|, |b| \le 1$. Its determinant contraction is*

$$\Gamma_9(G_{a,b}) = -\frac{3}{256}\left[4 + 9(a^2 + b^2) + a^4 + b^4 + 16a^2b^2 + a^4b^2 + a^2b^4\right]. \tag{C.1}$$

Writing $H_9 = G_{1,1}$, we have

$$\min_{|a|,|b|\le 1} \Phi_9(G_{a,b}) = \Phi_9(H_9) = -\frac{1}{737280} \approx -1.356 \cdot 10^{-6}.$$

Equality holds exactly when $|a| = |b| = 1$. In comparison, $\Phi_9(G_{0,0}) = -1/7741440$, so H_9 has $21/2$ times the negative magnitude of the normalized Du example.

Proof. For $|t| \le 1$,

$$Q_t^{(j)} = \frac{1+t}{2}\operatorname{Ad}_{A_j+B_j} + \frac{1-t}{2}\operatorname{Ad}_{A_j-B_j}.$$

This decomposition proves positivity on the square. Let P_j denote projection onto the jth three-dimensional block. The Breuer–Hall term contributes $\frac{1}{2}(P_1+P_2)$ to each marginal, and the remaining terms contribute $\frac{1}{2}(P_1+P_2)+P_3$. All terms containing a or b have zero marginals, so both marginals are I_9 for every real a, b. For necessity, apply the map to the rank-one input with block vector $(e_1, 0, e_1)$ and restrict the output to the first coordinate in blocks one and three. The resulting principal matrix is $\frac{1}{2}\left(\begin{smallmatrix} 1 & a \\ a & 1 \end{smallmatrix}\right)$. The input $(0, e_1, e_1)$ gives the analogous matrix with b. Thus positivity requires $|a|, |b| \le 1$.

For the exact polynomial calculation, write $c_{pqij} = L(E_{ij})_{pq}$ for a map $L : M_n \to M_n$. Determinant expansion and differentiation give

$$\Gamma_n(L) = \sum_{\sigma,\tau,\upsilon \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\operatorname{sgn}(\upsilon) \prod_{p=1}^{n} c_{p,\sigma(p),\tau(p),\upsilon(p)}. \tag{C.2}$$

Each surviving product uses every input row and column once. Indexing them by τ and υ makes the input determinant sign $\operatorname{sgn}(\tau)\operatorname{sgn}(\upsilon)$; no additional factorial occurs. Substitution of the integer tensor $8G_{a,b}$ gives

monomial	$[\text{monomial}]\,\Gamma_9(8G_{a,b})$
1	-6291456
$a^2,\ b^2$	-14155776 each
$a^4,\ b^4,\ a^4b^2,\ a^2b^4$	-1572864 each
a^2b^2	-25165824

All other coefficients vanish. Dividing by 8^9 proves (C.1). An exact computation in the supporting files reproduces the entire coefficient array; a separately ordered contraction checks the normalization and values.

The bracket in (C.1) is strictly increasing in each of a^2, b^2 on $[0,1]^2$. Its maximum is 42, attained exactly at $a^2 = b^2 = 1$. Division by 9! proves the claimed minimum and the comparison at $a = b = 0$. The same strict increase shows that the normalized Du map $G_{0,0}$ is not a local minimum, since arbitrarily small nonzero a or b decrease its invariant. □

The attaining map has the simpler expression

$$H_9 = \tfrac{1}{8}\widetilde{\mathcal{B}} + \tfrac{1}{2}\,\mathrm{Ad}_{A_1+B_1} + \tfrac{1}{2}\,\mathrm{Ad}_{A_2+B_2} \circ \mathsf{T}.$$

Thus the two pairs of separate terms in the normalized Du map become one completely positive and one completely copositive summand. This is an exact minimum within the displayed family; in the unrestricted problem it gives $\mu_9 \le -1/737280$. Pinched identity padding therefore gives

$$\mu_n \le -\frac{1}{737280\binom{n}{9}} \qquad (n \ge 9).$$

C.2. **Compression and an attained eight-dimensional bound.** A compression need not preserve bistochasticity. For the family above, a diagonal output congruence repairs both marginals explicitly. Let $V : \mathbb{C}^8 \to \mathbb{C}^9$ have columns $e_1, \dots, e_8$, and let $U : \mathbb{C}^8 \to \mathbb{C}^9$ have columns $e_1, e_2, e_3, e_4, e_5, e_7, e_8, e_9$. Define

$$C_{a,b}(X) = U^* G_{a,b}(VXV^*)U, \qquad D = \mathrm{diag}(1,1,\sqrt{2},1,1,1,1,1),$$
$$\mathcal{F}_{a,b}(X) = DC_{a,b}(X)D, \qquad H_8 = \mathcal{F}_{1,1}.$$

Proposition C.2 (Attained eight-dimensional family bound)**.** *For real a, b, the map $\mathcal{F}_{a,b}$ is positive and bistochastic exactly when $|a|, |b| \le 1$. Its determinant contraction is*

$$\Gamma_8(\mathcal{F}_{a,b}) = \frac{8a^4 - 9a^2 - 21b^2 - 34a^2b^2 - 10}{1024}, \tag{C.3}$$

and

$$\min_{|a|,|b|\le 1} \Phi_8(\mathcal{F}_{a,b}) = \Phi_8(H_8) = -\frac{11}{6881280} \approx -1.599 \cdot 10^{-6}.$$

Equality holds exactly when $|a| = |b| = 1$. The uncompleted compression has $\Phi_8(\mathcal{F}_{0,0}) = -1/4128768$, so H_8 increases its negative magnitude by $33/5$.

Proof. Compression and output congruence preserve positivity. To check the normalization, put $F = E_{11} + E_{22} + E_{44} + E_{55}$ in M_8. Direct calculation gives, for all real a, b,

$$C_{a,b}(I_8) = I_8 - \tfrac{1}{2}E_{33}, \qquad C^*_{a,b}(I_8) = I_8 - \tfrac{1}{8}F, \qquad C^*_{a,b}(E_{33}) = \tfrac{1}{8}F.$$

Since $D^2 = I_8 + E_{33}$, both marginals of $\mathcal{F}_{a,b}$ are I_8. For necessity, take the rank-one input xx^* with $x = e_1 + e_7$ in $\mathbb{C}^8$ and restrict the output to coordinates $(1,6)$. The resulting principal matrix is $\frac{1}{2}\left(\begin{smallmatrix} 1 & a \\ a & 1 \end{smallmatrix}\right)$. The input $x = e_4 + e_7$ and output coordinates $(4,6)$ give the same matrix with b. Thus positivity again requires $|a|, |b| \le 1$.

The integer tensor $8C_{a,b}$ gives the following complete coefficient array under (C.2):

monomial	$[\text{monomial}]\,\Gamma_8(8C_{a,b})$
1	-81920
a^2	-73728
b^2	-172032
a^4	65536
a^2b^2	-278528

All other coefficients vanish. The output congruence contributes $|\det D|^2 = 2$, so multiplication by $2/8^8$ proves (C.3).

The sharp lower bound follows from

$$1024\,8!\left(\Phi_8(\mathcal{F}_{a,b}) + \frac{11}{6881280}\right) = (1-b^2)(21+34a^2) + 27(1-a^2) + 8(1-a^2)^2.$$

Every term is nonnegative on the square, and their sum vanishes exactly when $a^2 = b^2 = 1$. At $a = b = 0$, the polynomial gives the stated baseline. □

For the maps H_8 and Q defined in Section 5, the determinant contractions needed for the extension reduce to one polynomial:

$$\Phi_8(\mathrm{Ad}_Q + tH_8) = \frac{t^3+5t^4}{1376256} + \frac{41t^5}{6881280} + \frac{t^6}{458752} - \frac{17t^7+11t^8}{6881280}. \tag{C.4}$$

For completeness, the unnormalized polynomial in (C.4) is

$$\det(\partial_X)\det(4K_0XK_0^* + tR(X)) = 245760t^3 + 1228800t^4 + 2015232t^5 + 737280t^6 - 835584t^7 - 540672t^8.$$

Multiplication by $2/(8^8\,8!)$ gives (C.4). These integer coefficients are obtained from the same contraction (C.2), applied to $4\,\mathrm{Ad}_{K_0} + tR$. An exact computation in the supporting files evaluates the sum in two orders for the coefficient of t^8 and reconstructs the polynomial by rational interpolation.

C.3. **Extension through the completely positive summand.** The summand Ad_Q permits an extension that keeps some off-diagonal entries. For $n = 8+m$, $m \geq 1$, and $0 \leq \alpha \leq 1$, set

$$T_{n,\alpha}\begin{pmatrix} X & Z \\ W & Y \end{pmatrix} = \begin{pmatrix} H_8(X) & \sqrt{\alpha}QZ \\ \sqrt{\alpha}WQ^* & Y \end{pmatrix}.$$

This map is positive because

$$T_{n,\alpha} = \mathrm{Ad}_{\sqrt{\alpha}Q\oplus I_m} + (H_8 - \alpha\,\mathrm{Ad}_Q)_{\mathrm{corner}},$$

where the second term acts on the leading 8×8 block and vanishes elsewhere. Its diagonal blocks show that it is bistochastic.

To compute its invariant in every dimension, we use an identity for a map supported on a fixed corner. If $P : M_r \to M_r$ is complex linear, $K \in M_r$, and $b_j = [t^j]\Phi_r(\mathrm{Ad}_K + tP)$, then

$$\Phi_n(\mathrm{Ad}_{K\oplus I_{n-r}} + P_{\mathrm{corner}}) = \sum_{j=0}^{r} \frac{(r)_j}{(n)_j} b_j, \qquad n \geq r. \tag{C.5}$$

For $K = I_r$, expand by the number j of factors from P in the determinant contraction. Every selected index lies in the first r coordinates, while the remaining identity contractions have completion factor $(n-j)!$ in dimension n and $(r-j)!$ in dimension r. After division by $n!$ and $r!$, their ratio is $(r)_j/(n)_j$. For invertible K, factor out the output congruence by $K \oplus I_{n-r}$ and apply the identity case to $\mathrm{Ad}_{K^{-1}} \circ P$. Both sides acquire the factor $|\det K|^2$; polynomial continuity covers singular K.

Proposition C.3. *For every integer $n \geq 9$, the positive bistochastic map $T_{n,3/(5(n-8))}$ satisfies*

$$\Phi_n(T_{n,3/(5(n-8))}) \leq -\frac{4947}{50000(n)_8}.$$

Proof. Write $c_j = [t^j]\Phi_8(\mathrm{Ad}_Q + tH_8)$. Homogeneity gives

$$\Phi_8(\alpha\,\mathrm{Ad}_Q + t(H_8 - \alpha\,\mathrm{Ad}_Q)) = \sum_{k=0}^{8} c_k \alpha^{8-k} t^k (1-t)^{8-k}.$$

Apply (C.5) and the finite identity

$$\sum_{j=k}^{8} (-1)^{j-k} \binom{8-k}{j-k} \frac{(8)_j}{(n)_j} = \frac{(8)_k (m)_{8-k}}{(n)_8}.$$

The coefficients in (C.4) then give

$$(n)_8 \Phi_n(T_{n,\alpha}) = \frac{\alpha^5 (m)_5 + 25\alpha^4 (m)_4}{4096} + \frac{41\alpha^3 (m)_3 + 45\alpha^2 (m)_2}{1024} - \frac{51\alpha m + 33}{512}. \tag{C.6}$$

For $u \geq 0$, define

$$P_*(u) = \frac{u^5 + 25u^4 + 164u^3 + 180u^2 - 408u - 264}{4096}.$$

Since $0 \leq (m)_j / m^j \leq 1$ for integers $m \geq 1$, the right side of (C.6) at $\alpha = u/m$ is at most $P_*(u)$. Choosing $u = 3/5$ keeps $\alpha \leq 1$ and gives $P_*(3/5) = -4947/50000$. □

The coefficient can be increased slightly without changing the construction. The polynomial P_* has a unique minimizer $u_* > 0$, since

$$4096 P_*'(u) = 5u^4 + 100u^3 + 492u^2 + 360u - 408$$

is strictly increasing for $u \geq 0$ and changes sign on $(0, 1)$. Taking $\alpha = u_*/m$ proves

$$\mu_n \leq -\frac{C_*}{(n)_8} \quad (n \geq 9), \qquad C_* = -P_*(u_*) = 0.09894625440\ldots.$$