BEYOND SMALL BALLS: A DETERMINANT INEQUALITY FOR FUNCTIONS OF MATRIX GAUSSIAN SERIES

SUVRIT SRA*

ABSTRACT. We prove a functional extension of Gaussian operator-norm small-ball inequalities. Let $X = \sum_{i=1}^{d} g_i B_i$, where $g \sim N(0, I_d)$, the matrices $B_i \in \mathbb{R}^{d\times d}$ are symmetric, and $\sum_i B_i^2 \preceq I_d$. For every even log-concave $f\colon \mathbb{R} \to [0,\infty)$, we obtain $\mathbb{E}\det f(X) \geq (\mathbb{E}f(\kappa G))^d$ at dilation $\kappa = 13/5$, where G is standard Gaussian. The proof reduces each function to one small-ball estimate, obtained from matrix-weighted Poincaré inequalities, and controls the rest by averages over the spectrum. We give an equivalent entropy formulation. An explicit counterexample shows that no universal dilation below $\sqrt{\pi/2}$ can work; we conjecture that this value is optimal.

hexagon:2610.00718v1

1. INTRODUCTION

When working with random vectors, we often need to know how likely their norms are to be small. Small-ball inequalities answer precisely such a question; they estimate the probability that a random vector lies in a small norm ball around the origin. While this topic is classical for Gaussian vectors (Li and Shao, 2001), our interest stems from matrix Gaussian series, where small-ball inequalities were instrumental in resolving the matrix Spencer conjecture (Akbas and Sra, 2026b).

Concretely, say $X = \sum_{i=1}^{d} g_i B_i$ is a real symmetric matrix Gaussian series with variance $V_X = \sum_i B_i^2 \preceq I_d$, then a small-ball inequality is simply $\mathbb{P}(\|X\| \leq C) \geq e^{-cd}$ for absolute constants C and c. We extend these small-ball inequalities from the operator-norm ball to functions.

Let $f\colon \mathbb{R} \to [0,\infty)$. Define the expectation $\mathbb{E}\det f(X) := \mathbb{E}\prod_j f(\lambda_j(X))$; it equals the small-ball probability $\mathbb{P}(\|X\| \leq C)$ when $f = \mathbf{1}_{[-C,C]}$, the *indicator* of the interval $[-C, C]$. More generally, we assume f is even and log-concave, and compare the expectation $\mathbb{E}\det f(X)$ with the "diagonal matrix" benchmark $(\mathbb{E}f(G))^d$, where G is a standard Gaussian. When the coefficient matrices B_i commute, this comparison succeeds: the eigenvalues of X are linear functions of the Gaussian coefficients, and Šidák's inequality (Šidák, 1967) immediately gives $\mathbb{E}\det f(X) \geq (\mathbb{E}f(G))^d$ (Theorem E.6).

Challenges. Without commutativity, the eigenvalues of X need not be Gaussian, and the comparison with the benchmark can fail (already for a 2×2 series built from two Pauli matrices). We therefore ask whether X satisfies a weaker *determinant inequality* against a *dilated Gaussian* κG, with κ independent of d and f. Because f peaks at zero, this inequality says that the spectrum of X is no more spread out than d independent $N(0,\kappa^2)$ variables, so we seek the smallest such κ.

To address this determinant inequality, we must first overcome a more basic obstruction: determinants couple all eigenvalues through products, while moment and trace estimates control only averages. Jensen's inequality can help us by turning bounds on the expected trace of $-\log f$ into bounds on the determinant, but this expectation is infinite whenever f has zeros. We therefore use one small-ball estimate to bound the probability that the spectrum lies in an interval where $-\log f$ is finite, an *averaged comparison* with the scalar Gaussian to control $-\log f$ there, and Gaussian correlation to combine the two. This strategy, with explicit constants, gives our main result.

2020 *Mathematics Subject Classification.* Primary 60B20, 60E15; Secondary 15B52, 52A40.

Key words and phrases. matrix Gaussian series, random matrices, log-concave functions, small-ball probabilities.

* TU Munich, Dept. of Mathematics, School of CIT; Garching, Germany. Email: s.sra@tum.de.

First public release: 19 October 2026, as v1 in the author's preprint repository, https://github.com/suvrit/preprints/tree/main/preprints/gsd; every version is kept there with its timestamp.

Theorem (informal; Theorem 2.1)**.** *Let X be as above. For every even log-concave $f\colon \mathbb{R} \to [0,\infty)$,*

$$\mathbb{E}\det f(X) \geq \left(\mathbb{E} f(\kappa G)\right)^d, \qquad \text{with } \kappa = 13/5.$$

The choice of κ in the theorem is (up to minor fractions) the currently best known value. Matrices with anticommuting coefficients show that $\kappa = 13/5$ cannot be lowered below $\kappa_* = \sqrt{\pi/2}$. This lower bound is approached only as the number of anticommuting coefficients grows; with two of them, the two-Pauli series above needs only $2/\sqrt{\pi}$ for all even log-concave functions and $\sqrt{2/\pi}$ for interval indicators. We conjecture that dilation κ_* is the best possible in general (Conjecture 4.1).

Main contributions of the paper.

(1) The determinant inequality at dilation $\kappa = 13/5$. Its indicator case gives a hereditary small-ball bound (Corollary 2.3) for $m \leq d$ contractions at radius of order $\sqrt{m\log(4d/m)}$, improving the radius $\sqrt{m}\,(1+\log(2d/m))^2$ of Akbas and Sra (2026b, Theorem 1.3).
(2) The comparison inequality $\mathbb{E}\operatorname{tr}\psi(|X|) \leq d\mathbb{E}\psi(\kappa_*|G|)$ for convex nondecreasing ψ with $\psi(0) = 0$ (Theorem 2.2). We also prove Conjecture 4.1 for some special cases in Theorems 4.3 and E.2.
(3) An entropy characterization of the determinant inequality, via the Gibbs variational principle (Theorem 4.2). It reduces to the averaged comparison at zero entropy, so the entropy quantifies the gap between averages and determinants.

1.1. **Related work.** For $d = 1$, the determinant inequality at dilation κ says that X is more peaked than κG in the sense of Birnbaum (1948). The Gaussian correlation inequality (Royen, 2014) enters the proof of Theorem 2.1. We prove the small-ball estimate with the matrix-weighted Poincaré criterion of Akbas and Sra (2026b, Lemma 2.3), a convex-domain form of the Brascamp–Lieb variance inequality (Brascamp and Lieb, 1976) for matrix weights (Cordero-Erausquin, 2019).

With the shifted partial-coloring theorem of Reis and Rothvoss (2023) (in the form of Dadush et al., 2022, Theorem 3.1), the hereditary bound of Corollary 2.3 produces signs with small matrix discrepancy, and hence another derivation of matrix Spencer (Akbas and Sra, 2026b). Gaussian-measure bounds for slices of the discrepancy body had earlier given matrix Spencer up to polylogarithmic rank (Bansal et al., 2023). For random signs in place of Gaussian coefficients, the Boolean small-ball inequality of Akbas and Sra (2026a) bounds determinant-weighted sign counts.

Theorem 4.2 parallels the entropic dual of forward–reverse Brascamp–Lieb (FRBL) inequalities (Liu et al., 2018, Theorem 1), with a comparison constraint on the average spectrum in place of prescribed marginals. The Gaussian saturation principle of Courtade and Liu (2021, Theorem 1.3) reduces linear FRBL inequalities to Gaussian functions. It does not extend to the eigenvalue maps here, since the two-Pauli series satisfies the determinant inequality for every $f(x) = e^{-bx^2}$ at dilation one but fails it for $f(x) = e^{-a|x|}$ with a large (Theorem E.8 and (E.8)).

2. The determinant inequality for functions

Let $B_1, \ldots, B_m \in \mathrm{Sym}_d(\mathbb{R})$, and define

$$X = \sum\nolimits_{i=1}^m g_i B_i, \qquad V_X := \sum\nolimits_i B_i^2, \qquad r_X := \dim \operatorname{span}\{B_i\},$$

where g_i are independent standard Gaussians. We call V_X the (matrix) *variance* and r_X the (coefficient) *rank* of X. Throughout G denotes a standard scalar Gaussian. For a real symmetric A, the matrix $f(A)$ is defined by spectral functional calculus, so that $\det f(A) = \prod_{j=1}^d f(\lambda_j(A))$. We say that X satisfies the *determinant inequality at dilation* κ if $\mathbb{E}\det f(X) \geq (\mathbb{E} f(\kappa G))^d$ for all even, log-concave $f\colon \mathbb{R} \to [0,\infty)$. Observe that smaller dilations mean stronger inequalities, since f is maximal at 0 and nonincreasing on $[0,\infty)$. We are now ready to state our main result.

Theorem 2.1. *Let X be a real symmetric matrix Gaussian series of order d with $V_X \preceq I_d$ and $r_X \leq d$. For every even log-concave $f\colon \mathbb{R} \to [0,\infty)$, we have $\mathbb{E}\det f(X) \geq \left(\mathbb{E} f(\kappa G)\right)^d$ with $\kappa = \frac{13}{5}$.*

For $f \not\equiv 0$, normalize $f(0) = 1$ without loss and define $\psi(t) := -\log f(t)$ for $t \geq 0$. Thus $\psi\colon [0,\infty) \to [0,\infty]$ is convex and nondecreasing, with $\psi(0) = 0$; we call such a function a *penalty*. Applying the determinant inequality to $x \mapsto f(x/\kappa)$ gives its equivalent *penalty form*

$$\mathcal{Z}_\psi(X;\kappa) := \mathbb{E}e^{-\operatorname{tr}\psi(|X|/\kappa)} \geq p_\psi^d, \qquad p_\psi := \mathbb{E}e^{-\psi(|G|)}. \tag{2.1}$$

With (2.1), our proof of Theorem 2.1 then rests on the following penalty based comparison:

Theorem 2.2 (Averaged comparison)**.** *If $V_X \preceq I_d$, with r_X arbitrary, then*

$$\mathbb{E}\operatorname{tr}\psi(|X|) \leq d\mathbb{E}\psi(\kappa_*|G|), \qquad \kappa_* := \sqrt{\pi/2}, \tag{2.2}$$

for every convex nondecreasing $\psi\colon [0,\infty) \to [0,\infty]$ with $\psi(0) = 0$. The constant κ_* is optimal.

To understand this special value of the constant κ_* consider $\psi(t) = t$, so that Theorem 2.2 claims $d^{-1}\mathbb{E}\operatorname{tr}|X| \leq \kappa_*\mathbb{E}|G| = 1$. It can be shown that there is an *anticommuting family* (i.e., a matrix Gaussian series with symmetric anticommuting coefficients) which fulfills Theorem 2.2 in the limit (Section 3.1). The same family bounds the dilation in Theorem 2.1 from below, because the determinant inequality also controls first moments. Indeed, for $f_\epsilon(x) = e^{-\epsilon|x|}$, both sides of the determinant inequality equal 1 at $\epsilon = 0$, so comparing their derivatives at $\epsilon = 0$ gives $\kappa \geq \kappa_*\, d^{-1}\mathbb{E}\operatorname{tr}|X|$, which tends to κ_* along the family. Thus, if κ_{univ} denotes the infimum of dilations valid for every matrix series X in Theorem 2.1 and every even log-concave function, then

$$\sqrt{\pi/2} \leq \kappa_{\mathrm{univ}} \leq 13/5. \tag{2.3}$$

We close the section with the consequence that motivated us: a proof of the hereditary bound needed by partial coloring for matrix Spencer discrepancy (Akbas and Sra, 2026b).

Corollary 2.3 (Hereditary small-ball bound)**.** *Let $A_1, \dots, A_n \in \mathrm{Sym}_d(\mathbb{R})$ be contractions, let $\varnothing \neq S \subseteq \{1,\dots,n\}$ have size m, put $D = \max\{d, m\}$, and let $0 < c \leq 1$. Then*

$$\mathbb{P}\Big(\Big\|\sum\nolimits_{i\in S} g_i A_i\Big\| \leq \tfrac{13}{5}\sqrt{2m\log(2D/(cm))}\Big) \geq e^{-cm}.$$

Proof. Pad each A_i by a zero block as needed, so that $Y = m^{-1/2}\sum_{i\in S} g_i A_i$ has order D, variance $V_Y \preceq I_D$ and $r_Y \leq m \leq D$. With $a = \sqrt{2\log(2D/(cm))}$, Theorem 2.1 for $f = \mathbf{1}_{[-13a/5,\,13a/5]}$ gives

$$\mathbb{P}(\|Y\| \leq \tfrac{13}{5}a) \geq (1-u)^D, \qquad u = \mathbb{P}(|G| > a) \leq e^{-a^2/2} = \tfrac{cm}{2D} \leq \tfrac{1}{2}.$$

Since $\log(1-u) \geq -2u$ for $0 \leq u \leq 1/2$, $(1-u)^D \geq e^{-2uD} \geq e^{-cm}$; also $\|\sum_{i\in S} g_i A_i\| = \sqrt{m}\,\|Y\|$. □

For $1 \leq m \leq d$ and fixed c, Corollary 2.3 improves the dependence of the radius on d/m from $(1+\log(2d/m))^2$ to $\sqrt{\log(2d/(cm))}$, compared with Akbas and Sra (2026b, Theorem 1.3), whose multiplicative constant $C(c)$ is unspecified; the corollary also covers $m > d$.

3. Proof of the determinant inequality

The proof of Theorem 2.1 rests on the averaged comparison, so we prove Theorem 2.2 first. The proofs use closed (lower semicontinuous) convex penalties, which by Lemma A.2 loses nothing.

3.1. The averaged comparison.

A convex function is an integral of hinges, so we first compare their expectations. Moment and tail estimates give the comparison; anticommuting coefficients then establish its optimality. Throughout, $V_X \preceq I_d$. We use the scalar Gaussian functions

$$c_0 = \sqrt{2/\pi}, \qquad q(r) = \mathbb{P}(|G| \leq r), \qquad T(r) = 1 - q(r),$$

$$H(r) = \mathbb{E}(|G| - r)_+ = c_0 e^{-r^2/2} - rT(r), \qquad M(r) = \frac{T(r)}{c_0 e^{-r^2/2}}.$$

Even moments. The bound $V_X \preceq I_d$ on the matrix variance controls every even moment:

$$d^{-1}\mathbb{E}\,\mathrm{tr}\,X^{2p} \le (2p-1)!! \qquad (p \ge 1). \tag{3.1}$$

For $p \ge 1$, Gaussian integration by parts gives

$$\mathbb{E}\,\mathrm{tr}\,X^{2p} = \sum_i \sum_{k=0}^{2p-2} \mathbb{E}\,\mathrm{tr}(B_i X^k B_i X^{2p-2-k}).$$

In a spectral basis of X, take absolute values of the scalar monomials and use the weighted arithmetic–geometric mean inequality

$$|x|^k |y|^{2p-2-k} \le \frac{k}{2p-2}|x|^{2p-2} + \frac{2p-2-k}{2p-2}|y|^{2p-2}$$

when $p > 1$. Summing the squared matrix entries of B_i and using $V_X \preceq I_d$ bounds the term of each k by $\mathbb{E}\,\mathrm{tr}\,X^{2p-2}$, so $\mathbb{E}\,\mathrm{tr}\,X^{2p} \le (2p-1)\mathbb{E}\,\mathrm{tr}\,X^{2p-2}$. The case $p = 1$ is $\mathrm{tr}\,V_X \le d$, and induction proves (3.1).

Comparison for hinge functions. Let U be the absolute value of a uniformly chosen eigenvalue of X, so that $\mathbb{E}U^{2p} = d^{-1}\mathbb{E}\,\mathrm{tr}\,X^{2p}$; then $\mathbb{E}U^2 \le 1$, $\mathbb{E}U^4 \le 3$, and $\mathbb{E}U^6 \le 15$. The even-moment bounds (3.1) give, for $t > 0$,

$$\mathbb{E}(U-t)_+ \le \min\left\{\frac{1}{4t}, \frac{81}{256t^3}, \frac{15625}{15552t^5}, \frac{2T(t)}{c_0}\right\}. \tag{3.2}$$

For $0 \le t \le 1/2$, the stronger bound $1 - t$ is available. These bounds imply

$$\mathbb{E}(U-t)_+ \le \kappa_* H(t/\kappa_*). \tag{3.3}$$

Proof of (3.2), (3.3), and Theorem 2.2. The bound $\mathbb{E}(U-t)_+ \le 1-t$ for $t \le 1/2$ follows from the pointwise inequality $(u-t)_+ \le u^2/2 + 1/2 - t$ for $u \ge 0$. For any $k > 1$,

$$(u-t)_+ \le \frac{(k-1)^{k-1}}{k^k t^{k-1}} u^k.$$

Take $k = 2, 4, 6$ and use (3.1). The even moments (3.1) also give $\mathbb{E}e^{sU} \le 2\mathbb{E}\cosh(sU) \le 2e^{s^2/2}$ for $s \ge 0$, and Markov's inequality with $s = t$ gives $\mathbb{P}(U > t) \le 2e^{-t^2/2}$, whose integral over (t, ∞) is the fourth term of (3.2). On $0 \le t \le 1/2$, $\kappa_* H(t/\kappa_*) \ge \kappa_* H(0) - t = 1 - t$.

On $[1/2, 16/5]$, the interval-arithmetic computation of Appendix C.4 partitions the interval into 1,350 pieces of width $1/500$. The upper bounds in (3.2) decrease, so the computation evaluates their minimum at the left endpoint of each piece and $\kappa_* H(t/\kappa_*)$ at the right endpoint. The smallest certified gap exceeds 0.00125. For $t \ge 16/5$, the Mills inequalities

$$T(t) \le c_0 e^{-t^2/2}/t, \qquad H(y) \ge c_0 e^{-y^2/2}/(y^2+3)$$

reduce the claim to

$$\frac{\kappa_* c_0 t}{2(t^2/\kappa_*^2 + 3)} e^{(1-\kappa_*^{-2})t^2/2} \ge 1.$$

This inequality holds strictly at $t = 16/5$, and its logarithmic derivative is positive thereafter: the rational factor contributes at least $-1/t$, while $(1 - \kappa_*^{-2})t > 1/t$. For the lower bound on H, the rational upper bound $M(y) \le \bar{M}(y) := (y^2+2)/(y^3+3y)$ follows by differentiation. Indeed, $\bar{M}' - y\bar{M} + 1 = -6/(y^2(y^2+3)^2) < 0$, and integration toward infinity compares $\bar{M}$ with M, which solves $M' = yM - 1$. The identity $H(y) = c_0 e^{-y^2/2}(1 - yM(y))$ finishes the argument; at $y = 0$ the bound is immediate. Hence (3.3) holds at every threshold.

A finite ψ is a mixture of hinges, $\psi(u) = \int (u-t)_+ \, d\nu(t)$ with $\nu \ge 0$, and $\mathbb{E}(\kappa_*|G| - t)_+ = \kappa_* H(t/\kappa_*)$. Integrating (3.3) against ν therefore proves Theorem 2.2. I □

Sharpness and the universal lower bound. Take real symmetric anticommuting involutions $C_1, \dots, C_j$ and $X = j^{-1/2} \sum_i g_i C_i$. Then $|X| = (\chi_j/\sqrt{j}) I_d$ and $V_X = I_d$, where $\chi_j = |g|$ is the Euclidean norm of $(g_1, \dots, g_j)$. Such matrices exist for every j: in dimension $d = 2^j$, put

$$C_i = Z^{\otimes(i-1)} \otimes H_2 \otimes I_2^{\otimes(j-i)}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Pauli relation $ZH_2 = -H_2 Z$ gives $C_i C_k = -C_k C_i$ for $i \neq k$, and $C_i^2 = I_d$. To realize $X = d^{-1/2} \sum_{i=1}^d g_i A_i$ with contractions A_i, choose j along powers of two and repeat each C_i exactly $2^j/j$ times. Gaussian aggregation gives the same matrix series in distribution. Since $\mathbb{E}\chi_j/\sqrt{j} \to 1$, these matrices complete the first-moment argument of Section 2 and prove the lower bound in (2.3).

3.2. **Localization and convex penalties.** Theorem 2.2 bounds an expected spectral sum, whereas (2.1) concerns the expectation of its exponential. Jensen's inequality connects the two, but the resulting bound is empty for penalties that take the value $+\infty$ and lossy otherwise. On the norm event $\{\|X\| \le R\}$, however, only the values of ψ on a bounded interval matter, and beyond that interval ψ may be replaced by a function of linear growth. The price of restricting to the norm event is its probability, measured per dimension by

$$N_X(R) = -d^{-1} \log \mathbb{P}(\|X\| \le R).$$

Fix $\kappa > 0$. Let ψ be finite and nonzero, and choose $r > 0$ with $\psi(r) = 1$; this radius sets the scale of the argument. For $\alpha \in (0,1)$, let $\widehat{\psi}$ agree with ψ up to αr and continue along its right supporting tangent thereafter. On the norm event $E = \{\|X\| \le \kappa\alpha r\}$ the trace penalties $\operatorname{tr}\psi(|X|/\kappa)$ and $\operatorname{tr}\widehat{\psi}(|X|/\kappa)$ agree, so $\mathcal{Z}_\psi(X;\kappa) \ge \mathbb{E}[\mathbf{1}_E e^{-\operatorname{tr}\widehat{\psi}(|X|/\kappa)}]$. The function $A \mapsto \operatorname{tr}\widehat{\psi}(|A|/\kappa)$ is convex and even, so the superlevel sets of its negative exponential are symmetric and convex, as is E. The Gaussian correlation inequality, integrated over these sets, gives

$$\mathcal{Z}_\psi(X;\kappa) \ge \mathbb{P}(E)\, \mathbb{E} e^{-\operatorname{tr}\widehat{\psi}(|X|/\kappa)}.$$

Jensen's inequality and Theorem 2.2, applied to the convex function $\widehat{\psi}$ of linear growth, give

$$\mathcal{Z}_\psi(X;\kappa)^{1/d} \ge \exp\{-N_X(\kappa\alpha r) - \mathbb{E}\widehat{\psi}(b_f|G|)\}, \qquad b_f = \kappa_*/\kappa. \tag{3.4}$$

It remains to compare the right side of (3.4) with p_ψ for every convex penalty. A convex penalty is a mixture of hinges, $\psi(t) = \int (t-h)_+ \, d\nu(h)$, as in Section 3.1; its *hinge measure* ν has the initial slope as an atom at zero. A determinant inequality for each hinge separately would not combine, because p_ψ is not a product over hinges, so we compare the two sides hinge by hinge against one measure. Since $\psi(r) = 1$, the measure $d\omega(h) = (r-h)\, d\nu(h)$ on $[0, r)$ is a probability measure. Define

$$\psi_{r,h}(t) = \frac{(t-h)_+}{r-h}, \qquad p_{r,h} = \mathbb{E} e^{-\psi_{r,h}(|G|)}, \qquad \psi_0 = \int_{h<r} \psi_{r,h}\, d\omega(h).$$

Hölder's inequality and $\psi_0 \le \psi$ give

$$\log p_\psi \le \log p_{\psi_0} \le \int_{h<r} \log p_{r,h}\, d\omega(h). \tag{3.5}$$

The expectation in (3.4) is an integral against the same measure ω, because the hinge measure of $\widehat{\psi}$ is ν restricted to $[0, \alpha r]$. Comparing the two integrands reduces the determinant inequality to scalar inequalities in r and h, stated in Lemma 3.1 below. This reduction is where the norm event matters: its cost $N_X(\kappa\alpha r)$ is a single term shared by all hinges, not one event per hinge.

We make the sufficient scalar inequalities explicit, in the notation q, T, H, M, c_0 of Section 3.1. Throughout, $V_X \preceq I_d$; this reduction needs no bound on the coefficient rank r_X.

With r as above, a supporting tangent at r shows that $p_\psi = \mathbb{E} e^{-\psi(|G|)}$ satisfies

$$p_\psi \le q(r), \qquad 1 - p_\psi \ge T(r). \tag{3.6}$$

Indeed, $e^{-\psi}$ is at most the negative exponential of the positive part of that tangent, a function with values in $[0,1]$ and Lebesgue integral r on $[0,\infty)$; its integral against the decreasing half-normal density is at most $q(r)$.

For $0 \le h < r$, the normalized hinge $\psi_{r,h}$ gives the closed form

$$p_{r,h} = q(h) + c_0 e^{-h^2/2} M\left(h + \frac{1}{r-h}\right).$$

Lemma 3.1 (Reduction to scalar hinge inequalities)**.** *Let $b_f = \kappa_*/\kappa$, and fix $\alpha \in (0,1)$ and an upper bound $n \ge N_X(\kappa\alpha r)$. Suppose*

$$q(r) \le e^{-n}, \tag{3.7}$$

$$p_{r,h} \exp\left(\frac{b_f H(h/b_f)}{r-h}\right) \le e^{-n} \qquad (0 \le h \le \alpha r). \tag{3.8}$$

Then the penalty form (2.1) holds for every finite closed penalty ψ with $\psi(r) = 1$.

Proof. The hinge representation above, with the probability measure ω, gives

$$\mathbb{E}\widehat{\psi}(b_f|G|) = \int_{h \le \alpha r} \frac{b_f H(h/b_f)}{r-h}\, d\omega(h).$$

Substituting in the localization bound (3.4) bounds $\mathcal{Z}_\psi(X;\kappa)^{1/d}$ from below. For $h > \alpha r$, the bound $p_{r,h} \le q(r)$ and (3.7) give $\log p_{r,h} \le -n$. For $h \le \alpha r$, use (3.8) instead. Integrating both inequalities against ω and applying the Hölder bound (3.5) proves the result. Finite measures and finite mixtures suffice by approximation; a finite penalty has bounded slope on every compact subinterval of $(0,\infty)$. □

3.3. **A small-ball bound from matrix-weighted Poincaré inequalities.** The scalar inequalities depend on the matrix only through $N_X(R)$, so what remains is a small-ball bound for the operator norm of a matrix Gaussian series. We obtain it by tilting: multiply the Gaussian measure by $Z^{-1}e^{-V(X/R)}$, where V is nonnegative and infinite outside the operator-norm unit ball, so that $Z \le \mathbb{P}(\|X\| \le R)$ and a lower bound on Z is an upper bound on $N_X(R)$. Scaling from zero to X/R writes $-\log Z$ as an integral of expectations of spectral quantities under the tilted measure, and matrix-weighted Poincaré estimates bound those expectations. The choice of V sets the order of the bound: a determinant weight gives bounds polynomial in $1/R$, while a logarithmic barrier that vanishes on an interior interval gives exponential decay. Large radii require exponential decay, since the benchmark $-\log \mathbb{P}(|G| \le r)$ itself decays like $e^{-r^2/2}$. We prove the determinant-weight bound here; Appendix B proves the barrier bounds.

The bounds on N_X rest on one external criterion, a matrix-weighted Poincaré inequality for Gaussian measures tilted by a determinant or resolvent weight. If p is a positive probability density on an open convex domain $\Omega \subset \mathbb{R}^m$, K is a positive definite matrix function on Ω, and $W = pK$ is smooth and integrable, define

$$\Theta_{ij}(W) = (\partial_i W)W^{-1}(\partial_j W) - \partial_{ij} W.$$

Its *normalized curvature* is the block matrix

$$p^{-1}[K^{-1/2}\Theta_{ij}(W)K^{-1/2}]_{ij}.$$

If $[\Theta_{ij}(W)] \succeq \gamma I_m \otimes W$ for $\gamma > 0$, then

$$\mathbb{E}F^{\mathsf{T}}KF - (\mathbb{E}KF)^{\mathsf{T}}(\mathbb{E}K)^{-1}(\mathbb{E}KF) \le \gamma^{-1}\mathbb{E}\sum_i (\partial_i F)^{\mathsf{T}} K (\partial_i F). \tag{3.9}$$

Expectations are under p, and F is any C^1 vector field with finite weighted square integral. The right side may be infinite. The scalar case $K \equiv 1$ is the Poincaré inequality for strongly log-concave densities, a consequence of the Brascamp–Lieb variance inequality (Brascamp and Lieb, 1976);

for matrix weights on $\mathbb{R}^m$, the criterion follows from the Brascamp–Lieb variance inequality of Cordero-Erausquin (2019, Theorem 4). The convex-domain form used here imposes no zero boundary value on F; Akbas and Sra (2026b, Lemma 2.3) prove it by Neumann estimates on compact convex subdomains and exhaustion, following the route to Brascamp–Lieb-type inequalities through Reilly's formula (Reilly, 1977; Kolesnikov and Milman, 2017). Approximation extends the criterion to the Lipschitz tests used below. Appendix B.7 verifies the limiting hypotheses.

The simplest barrier is $V(Y) = -\log\det(I - Y^2)$ for $\|Y\| < 1$. Its normalizer gives the first bound on N_X; Lemma 3.4 and the verification of the scalar inequalities (Proposition C.1) both use it. Retaining the trace of the matrix variance in the same proof gives Proposition 3.3. Write $\ell(z) = 2/(1 + \sqrt{1 - 4z})$ for $0 \le z < 1/4$.

Proposition 3.2. *If $V_X \preceq I_d$ and $R > 2$, set $\eta = R^{-2}$. Then*

$$N_X(R) \le n_0(R) := 2\log \ell(\eta) - \ell(\eta) + 1. \tag{3.10}$$

In particular,

$$N_X(R) \le 2/R^2 \quad (R \ge 3).$$

The proof of Proposition 3.2 controls the determinant weight through $\operatorname{tr} V_X$, which can be much smaller than $d\|V_X\|$, and keeping that trace gives a small-ball bound with no condition on the coefficient rank. For $R > 0$ and $\beta > 0$, let

$$W_{\beta,R}(A) = \begin{cases} \det(I - A^2/R^2)^\beta, & \|A\| < R, \\ 0, & \|A\| \ge R. \end{cases}$$

Proposition 3.3 (Gaussian determinant small balls)**.** *Let $X = \sum_i g_i B_i$ and $v = \|V_X\|$. For every $\beta > 0$ and every $R > 0$ with $R \ge 3\sqrt{v}$,*

$$\mathbb{E} W_{\beta,R}(X) \ge \exp(-2\max\{1,\beta\} \operatorname{tr} V_X / R^2). \tag{3.11}$$

There is no restriction on the coefficient rank. In particular, $\mathbb{P}(\|X\| < R) \ge \exp(-2\operatorname{tr} V_X/R^2)$ under the same radius condition.

The proof below controls the logarithmic derivative of the determinant expectation by $\operatorname{tr} V_X$, then obtains all powers β from the first power by Jensen's inequality and monotonicity. Proposition 3.3 is a Gaussian counterpart of the Boolean small-ball inequality of Akbas and Sra (2026a), which works at a different radius scale and whose proof bounds derivatives of a supported Gaussian density before passing to signs.

Proof of Propositions 3.2 and 3.3. First assume $V_X \preceq I_d$ and $R > 2$. For $0 \le s \le 1$, put $Y_s = sX/R$ and tilt the Gaussian measure by $\det(I - Y_s^2)\mathbf{1}_{\{\|Y_s\|<1\}}$; call the normalizer Z_s and the resulting probability measure μ_s, with density p_s. Let $K_s = (I - Y_s)^{-1}$ and $S_s = \mathbb{E}_{\mu_s} K_s$. The derivative of $\log Z_s$ is $-(2/s)\operatorname{tr}(S_s - I)$, as shown below, so the task is an upper bound on the mean resolvent S_s, and the weighted Poincaré inequality with weight K_s supplies one. With $D_i = sB_i/R$ and $E_i = K_s^{1/2} D_i K_s^{1/2}$, $W_s = p_s K_s$ has normalized curvature

$$\delta_{ij} I + \Gamma^-_{ij} I + \Gamma^+_{ij} I - E_j E_i,$$

where $\Gamma^-_{ij} = \operatorname{tr}[(I - Y_s)^{-1} D_i (I - Y_s)^{-1} D_j]$ and $\Gamma^+_{ij} = \operatorname{tr}[(I + Y_s)^{-1} D_i (I + Y_s)^{-1} D_j]$. The block inequality

$$[\operatorname{tr}(E_i E_j) I - E_j E_i]_{ij} \succeq 0 \tag{3.12}$$

proves curvature at least one. For block vectors u_i and $L_b = \sum_i u_i (E_i e_b)^{\mathsf{T}}$, the quadratic form of the left side of (3.12) is $\sum_b (\|L_b\|_F^2 - \operatorname{tr} L_b^2) = \frac{1}{2}\sum_b \|L_b - L_b^{\mathsf{T}}\|_F^2 \ge 0$. Integrability follows from the identity $\det(I - Y_s^2)(I - Y_s)^{-1} = \det(I + Y_s)\operatorname{adj}(I - Y_s)$ on $\{\|Y_s\| < 1\}$.

Test (3.9) with $F = (I - Y_s)z$ for a fixed vector z. Evenness gives

$$I - S_s^{-1} \preceq \sum_i D_i S_s D_i \preceq s^2\eta \|S_s\| I.$$

Writing $\lambda_s = \|S_s\|$, we obtain $\lambda_s - 1 \le s^2\eta\lambda_s^2$. Continuity from $\lambda_0 = 1$ excludes the large root, so $S_s \preceq \ell(s^2\eta)I$. The numerator defining S_s is bounded by a Gaussian-integrable function, and its moving boundary has Gaussian measure zero; these two facts justify continuity, including at $s = 0$.

To retain $\operatorname{tr} V_X$, conjugate the inequality $I - S_s^{-1} \preceq \sum_i D_i S_s D_i$ by $S_s^{1/2}$ and take traces:

$$\begin{aligned}\operatorname{tr}(S_s - I) &\le \sum_i \operatorname{tr}(S_s D_i S_s D_i) \\ &= \sum_i \|S_s^{1/2} D_i S_s^{1/2}\|_F^2 \le \|S_s\|^2 \sum_i \operatorname{tr} D_i^2 \le s^2\eta \operatorname{tr} V_X \, \ell(s^2\eta)^2.\end{aligned}$$

Furthermore, $S_s - I = \mathbb{E}_{\mu_s}[Y_s^2(I - Y_s^2)^{-1}]$ and

$$-\frac{d}{ds}\log Z_s = \frac{2}{s}\operatorname{tr}(S_s - I) \le 2s\eta \operatorname{tr} V_X \, \ell(s^2\eta)^2.$$

The determinant vanishes on the boundary, so differentiation creates no boundary term. All integrals may be taken on the coefficient support: removing a kernel preserves V_X, and for $s > 0$ the boundary lies in the zero set of the nonzero polynomial $\det(I - Y_s^2)$. Since ℓ satisfies $(\ell(u) - 1)/u = \ell(u)^2$, substituting $u = s^2\eta$ and then $u = (\ell - 1)/\ell^2$ gives

$$-\log Z_1 \le \operatorname{tr} V_X \int_0^\eta \ell(u)^2 \, du = \operatorname{tr} V_X \, n_0(R). \tag{3.13}$$

Since $\operatorname{tr} V_X \le d$ and $Z_1 \le \mathbb{P}(\|X\| \le R)$, (3.10) follows. For $u \le 1/9$, $\ell(u) \le 4/3$, so $n_0(R) < 2\eta = 2/R^2$ when $R \ge 3$.

For general $v = \|V_X\| > 0$, apply (3.13) to $X/\sqrt{v}$ at radius $R/\sqrt{v}$. This rescaling yields

$$\mathbb{E}W_{1,R}(X) \ge \exp\Big(-\frac{\operatorname{tr} V_X}{v}\, n_0(R/\sqrt{v})\Big) \qquad (R > 2\sqrt{v}).$$

The determinant weight $W_{1,R}$ takes values in $[0, 1]$. For $\beta \ge 1$, Jensen gives $\mathbb{E}W_{1,R}^\beta \ge (\mathbb{E}W_{1,R})^\beta$; for $0 < \beta \le 1$, one has $W_{1,R}^\beta \ge W_{1,R}$. The bound $n_0(R/\sqrt{v}) \le 2v/R^2$ for $R \ge 3\sqrt{v}$ proves (3.11). The probability bound uses $W_{1,R} \le \mathbf{1}_{\{\|X\| < R\}}$. If $v = 0$, all B_i vanish and both conclusions hold with equality. □

Removing the quadratic term from the barrier improves (3.10) to order R^{-3} (Theorem B.2), higher moments improve its leading coefficient (Proposition B.3), and a logarithmic barrier that vanishes on an interior interval gives the exponential bound (B.10) (Proposition B.5); Appendix B proves all three. Let $n(R)$ be the minimum of (3.10), (B.4), (B.8) for $3 \le P \le 8$, and (B.10) with $a = 77/100$, $b = 7/10$, omitting a formula whenever its stated domain or positivity condition fails. Thus $N_X(R) \le n(R)$ wherever $R > 2$.

3.4. **Small radii.** At small r the norm event $\{\|X\| \le \kappa\alpha r\}$ is itself a small ball, and the tilt estimates, which need $R > 2$, no longer apply. The proof instead rescales the coefficient vector by $g \mapsto sg$, with $s \le 1$, and retains the Gaussian density ratio. The Jacobian contributes $s^{r_X} \ge s^d$, which is where the hypothesis $r_X \le d$ on the coefficient rank enters. Lemma 3.4 gives the resulting determinant bound.

Let $R \ge 11/4$ and set

$$\lambda = \ell(R^{-2}), \qquad v_R = (R^2 + 2)^{-1}, \qquad s = \kappa r/R \le 1.$$

Here $n_0(R) = 2\log\lambda - \lambda + 1$ is the bound (3.10).

Lemma 3.4 (Localization by Gaussian rescaling)**.** *If $V_X \preceq I_d$ and $r_X \le d$, then every finite closed penalty ψ with $\psi(r) = 1$ satisfies*

$$\mathcal{Z}_\psi(X;\kappa)^{1/d} \ge s\exp\{(1-s^2)(3/2-\lambda) - n_0(R) - \sqrt{v_R} - \log(1-v_R)\}. \tag{3.14}$$

The positive term $(1-s^2)(3/2-\lambda)$ comes from the Radon–Nikodym factor of the rescaling.

Proof of Lemma 3.4. Remove the kernel of the coefficient map, so $m = r_X \le d$. Let $Y = X/R$ and

$$d\mu = Z_R^{-1}\det(I-Y^2)\mathbf{1}_{\{\|Y\|<1\}}d\gamma_m, \qquad V = -\log\det(I-Y^2).$$

Proposition 3.2 and its proof give $Z_R \ge e^{-dn_0(R)}$ and

$$A := \mathbb{E}_\mu \operatorname{tr}[Y^2(I-Y^2)^{-1}] \le d(\lambda-1).$$

Scalar Gaussian integration by parts gives the exact identity

$$\mathbb{E}_\mu|g|^2 = m - 2A. \tag{3.15}$$

If $Q = [\operatorname{tr} B_iB_j]$, the total potential of μ has Hessian at least $I_m + 2Q/R^2$. The Brascamp–Lieb variance inequality (Brascamp and Lieb, 1976), applied to linear functions, yields

$$\operatorname{Cov}_\mu(g) \preceq (I_m + 2Q/R^2)^{-1},$$

and evenness makes the mean zero. The eigenvalues of Q sum to at most d and there are at most d of them. Concavity of $u/(1+2u/R^2)$ gives

$$v := d^{-1}\mathbb{E}_\mu \operatorname{tr} Y^2 \le \frac{1}{R^2+2} = v_R.$$

Spectral Jensen and Cauchy–Schwarz also give

$$d^{-1}\mathbb{E}_\mu V \ge -\log(1-v), \qquad d^{-1}\mathbb{E}_\mu \operatorname{tr}|Y| \le \sqrt{v}.$$

With $s = \kappa r/R$, the change of variables $g \mapsto sg$ gives exactly

$$\mathcal{Z}_\psi(X;\kappa) = s^m\mathbb{E}_\gamma \exp\{(1-s^2)|g|^2/2 - \operatorname{tr}\psi(s|X|/\kappa)\}.$$

Restricting this integral to the support of μ, replacing $d\gamma$ by $Z_Re^V\,d\mu$, and applying Jensen's inequality bound it from below. Because $\psi(r) = 1$ and $\psi(0) = 0$, convexity gives $\psi(rt) \le t$ for $0 \le t \le 1$, so $\operatorname{tr}\psi(s|X|/\kappa) = \operatorname{tr}\psi(r|Y|) \le \operatorname{tr}|Y|$ on this support. Combining these facts with (3.15) gives

$$\begin{aligned} d^{-1}\log\mathcal{Z}_\psi &\ge \frac{m}{d}\left(\log s + \frac{1-s^2}{2}\right) - n_0(R) - (1-s^2)(\lambda-1) - \log(1-v) - \sqrt{v} \\ &\ge \log s + (1-s^2)(3/2-\lambda) - n_0(R) - \log(1-v_R) - \sqrt{v_R}. \end{aligned}$$

For the second line, $\log s + (1-s^2)/2 \le 0$ and $m/d \le 1$. Moreover, $\sqrt{v} + \log(1-v)$ increases for $0 \le v \le (\sqrt{2}-1)^2$, and v_R is in this range for $R \ge 11/4$. Lemma 3.4 follows. Its right side increases as a function of r when $s \le 1$: its logarithmic derivative is $r^{-1}[1-2(3/2-\lambda)s^2] > 0$. This monotonicity validates the interval comparisons in the proof of Proposition C.1. □

3.5. **Completion of the proof.** Together these estimates cover all $r > 0$: Lemma 3.4 handles small radii, Lemma 3.1 handles a compact interval, and an analytic tail estimate handles large radii. Proposition C.1 verifies the scalar inequalities by exact interval arithmetic on the compact interval and by monotonicity at both ends; Appendix C proves it, together with the large-radius tangent bound (C.2).

Proof of Theorem 2.1. Fix $\kappa = 13/5$. The zero function is immediate. Otherwise normalize $f(0) = 1$; by Lemma A.2 its penalty ψ may be taken closed, and it suffices to prove the penalty form (2.1). The zero penalty gives equality.

Let ψ be finite and nonzero. Lemmas 3.1 and 3.4 are stated at the radius where the penalty reaches one, so choose $r > 0$ with $\psi(r) = 1$; by (3.6), $p_\psi \le q(r)$ and $T(r) \le 1 - p_\psi$. For $r \le 24/25$, Lemma 3.4 and part (1) of Proposition C.1 give $\mathcal{Z}_\psi(X;\kappa)^{1/d} \ge q(r) \ge p_\psi$. For $24/25 \le r \le 16/5$, part (2), with $n = n(\kappa\alpha r) \ge N_X(\kappa\alpha r)$, supplies the hypotheses of Lemma 3.1, which gives (2.1). For $r \ge 16/5$, the tangent bound (C.2) and part (3) give $-d^{-1}\log \mathcal{Z}_\psi(X;\kappa) \le (9/50 + 77/100)(1-p_\psi) < 1 - p_\psi \le -\log p_\psi$.

An extended-valued closed penalty is the increasing limit of the finite penalties $\psi_k(t) = \inf_{0\le u\le t}\{\psi(u) + k(t-u)\}$, and bounded convergence on both sides of (2.1) completes the proof. □

4. The determinant inequality at the conjectured dilation

Theorem 2.1 leaves the gap (2.3) between the dilation $13/5$ and the lower bound κ_* forced by the anticommuting family. The localization proof of Theorem 2.1 cannot close this gap: its small-ball estimate decays too slowly to reach any dilation at most $\sqrt{2}$. The lower bound κ_* suggests the following universal statement; we call κ_* the *conjectured dilation*.

Conjecture 4.1. *For every integer $d \ge 1$, let $X = \sum_{i=1}^m g_i B_i \in \mathrm{Sym}_d(\mathbb{R})$, where the g_i are independent standard Gaussians and the B_i are real symmetric matrices satisfying $\sum_i B_i^2 \preceq I_d$ and $\dim \mathrm{span}\{B_1,\dots,B_m\} \le d$. Then, for every even log-concave $f\colon \mathbb{R} \to [0,\infty)$,*

$$\mathbb{E}\det f(X) \ge \left(\mathbb{E} f(\kappa_* G)\right)^d, \qquad \kappa_* = \sqrt{\pi/2}.$$

4.1. **An entropy route to the conjectured dilation.** The averaged comparison alone does not control a determinant. An exact characterization becomes possible when we allow a change of the coefficient law and bound its relative entropy. To state it, regard X as a function of its coefficients, $X(x) = \sum_i x_i B_i$ for $x \in \mathbb{R}^m$, so that $X = X(g)$ with $g \sim \gamma_m$ standard Gaussian on $\mathbb{R}^m$, and let γ_+ be the law of $|G|$. For probability measures ν, μ, write $D(\nu\|\mu) = \int \log(d\nu/d\mu)\, d\nu$ when $\nu \ll \mu$, and $D(\nu\|\mu) = +\infty$ otherwise. For a coefficient law ν on $\mathbb{R}^m$, define its average absolute spectral law by

$$\int h\, d\sigma_{\nu,\kappa} = \frac{1}{d}\int \mathrm{tr}\, h(|X(x)|/\kappa)\, d\nu(x).$$

For laws on $[0,\infty)$ with finite first moments, write $\sigma \preceq_{\mathrm{icx}} \eta$ for *increasing convex order*: $\int h\, d\sigma \le \int h\, d\eta$ for every finite convex nondecreasing h, allowing infinite integrals.

To enforce this order, minimize the relative entropy of the coefficient law:

$$\mathcal{I}_{X,\kappa}(\eta) := \inf_{\nu:\, \sigma_{\nu,\kappa} \preceq_{\mathrm{icx}} \eta} D(\nu\|\gamma_m). \tag{4.1}$$

Theorem 4.2 (Entropy characterization of the determinant inequality)**.** *Let $\kappa > 0$. A matrix Gaussian series X satisfies the determinant inequality at dilation κ if and only if*

$$\mathcal{I}_{X,\kappa}(\eta) \le d D(\eta\|\gamma_+) \qquad \text{whenever } D(\eta\|\gamma_+) < \infty. \tag{4.2}$$

For such η, the infimum in (4.1) is finite and attained. In particular, (4.2) holds at $\kappa = 13/5$ under the hypotheses of Theorem 2.1.

Proof that (4.2) implies the determinant inequality. The Gibbs variational identity connects entropy with the determinant:

$$-\log \int e^{-W}\, d\mu = \inf_{\xi}\Big\{ D(\xi\|\mu) + \int W\, d\xi \Big\} \qquad (W \ge 0). \tag{4.3}$$

Fix a finite closed convex penalty ψ and tilt the scalar law to $d\eta = p_\psi^{-1} e^{-\psi}\, d\gamma_+$. This law has finite entropy. By (4.2) and attainment, choose a coefficient law ν with $D(\nu\|\gamma_m) \le dD(\eta\|\gamma_+)$ and $\sigma_{\nu,\kappa} \preceq_{\mathrm{icx}} \eta$. Then

$$-\log \mathcal{Z}_\psi(X;\kappa) \le D(\nu\|\gamma_m) + d\int \psi\, d\sigma_{\nu,\kappa} \;\le\; d\Big(D(\eta\|\gamma_+) + \int \psi\, d\eta\Big) = -d\log p_\psi.$$

Increasing finite convex approximations and Lemma A.2 give all functions. Appendix D.3 proves attainment and the converse: convex separation enforces finite collections of hinge constraints, and entropy compactness passes to all hinges. □

At $\eta = \gamma_+$, the entropy bound (4.2) forces $\nu = \gamma_m$ and reduces to the averaged comparison $\sigma_{\gamma_m,\kappa} \preceq_{\mathrm{icx}} \gamma_+$. Theorem 2.2 establishes this case at $\kappa = \kappa_*$; the determinant inequality requires the entropy bound for every scalar law η. Conjecture 4.1 is therefore equivalent to (4.2) at $\kappa = \kappa_*$ for every matrix Gaussian series it covers.

4.2. **Two special cases.** The localization proof uses Jensen's inequality in (3.4). At $\kappa = \kappa_*$, even a linear penalty $\psi(t) = at$ leaves the strict gap $e^{-a\mathbb{E}|G|} < \mathbb{E}e^{-a|G|}$ for $a > 0$, so (3.4) cannot reach κ_* for this penalty, even if the norm event has probability one. The entropy route of Section 4.1, with a Gaussian trial law for the coefficients, avoids this loss for linear–quadratic penalties and reaches κ_*. The result depends on the ratio r_X/d of coefficient rank to order, through $\theta_* := 1/(\pi - 2) = 0.875969\ldots$.

Theorem 4.3 (Linear–quadratic penalties at the conjectured dilation)**.** *Suppose the matrix variance and coefficient rank satisfy* $\operatorname{tr} V_X \le d$ *and* $r_X \le \theta_* d$*; since* $\theta_* > 7/8$*, the condition* $r_X \le 7d/8$ *suffices. For every* $a, b \ge 0$*, the function* $f(x) = e^{-a|x|-bx^2}$ *satisfies*

$$\mathbb{E}e^{-a\operatorname{tr}|X|/\kappa_* - b\operatorname{tr}X^2/\kappa_*^2} \ge \big(\mathbb{E}e^{-a|G|-bG^2}\big)^d. \tag{4.4}$$

The constant κ_* *cannot be decreased uniformly over this class, even when* $b = 0$*.*

With the trial law $N(0, s^2 I)$ in (4.3), the penalty term is proportional to d and the entropy to r_X. Optimizing over s reduces the proof to a scalar comparison with $-\log \mathbb{E}e^{-a|G|}$; Appendix E.1 proves it and identifies θ_* as the largest value of r_X/d allowed by this estimate.

To treat every even log-concave function, we compare the trace penalty with a sum of scalar penalties of d Gaussian projections of the coefficients. This comparison keeps the product inside the expectation, where the Gaussian correlation inequality applies. An ellipsoidal bound $\|X(x)\| \le \sqrt{x^{\mathsf{T}} Q x}$ with $\operatorname{tr} Q \le 1$ on the operator norm makes it possible even when the absolute eigenvalues differ, and gives the determinant inequality at κ_* once d exceeds a threshold depending on Q (Theorem E.2). The isotropic case $Q = I_r/r$ covers all sufficiently large members of the anticommuting family that forces κ_* (Proposition E.4). Orthogonal sums and averages of orthogonal conjugates extend the determinant inequality to further matrix Gaussian series (Appendix E.5).

AI USAGE

The results in this paper were developed through extensive use of AI assistants: OpenAI's GPT Pro for the main line of argument, including Theorem 2.1, the verification of its scalar inequalities, and Theorem 4.3; and Anthropic's Claude for the results under an ellipsoidal norm bound in Appendix E.3, Proposition 3.3, Appendix D.3, and the writing, positioning, and literature search. None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs.

ACKNOWLEDGEMENTS

The author (SS) thanks the Alexander von Humboldt (AvH) Foundation for their generous support via a Humboldt Professorship in AI.

References

Emrullah Akbas and Suvrit Sra. Boolean small-ball inequalities for discrepancy theory, 2026a. URL https://arxiv.org/abs/2609.20785.

Emrullah Akbas and Suvrit Sra. A proof of the Matrix Spencer conjecture, 2026b. URL https://arxiv.org/abs/2608.28816.

Nikhil Bansal, Haotian Jiang, and Raghu Meka. Resolving matrix Spencer conjecture up to poly-logarithmic rank. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing*, pages 1814–1819. ACM, 2023. doi: 10.1145/3564246.3585103.

Z. W. Birnbaum. On random variables with comparable peakedness. *The Annals of Mathematical Statistics*, 19(1):76–81, 1948. doi: 10.1214/aoms/1177730293.

Herm Jan Brascamp and Elliott H. Lieb. On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *Journal of Functional Analysis*, 22(4):366–389, 1976. doi: 10.1016/0022-1236(76)90004-5.

Dario Cordero-Erausquin. On matrix-valued log-concavity and related Prékopa and Brascamp–Lieb inequalities. *Advances in Mathematics*, 351:96–116, 2019. doi: 10.1016/j.aim.2019.04.046.

Thomas A. Courtade and Jingbo Liu. Euclidean forward–reverse Brascamp–Lieb inequalities: Finiteness, structure, and extremals. *The Journal of Geometric Analysis*, 31:3300–3350, 2021. doi: 10.1007/s12220-020-00398-y.

Daniel Dadush, Haotian Jiang, and Victor Reis. A new framework for matrix discrepancy: Partial coloring bounds via mirror descent. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 649–658. ACM, 2022. doi: 10.1145/3519935.3519967.

Alexander V. Kolesnikov and Emanuel Milman. Brascamp–Lieb-type inequalities on weighted Riemannian manifolds with boundary. *The Journal of Geometric Analysis*, 27(2):1680–1702, 2017. doi: 10.1007/s12220-016-9736-5.

Wenbo V. Li and Qi-Man Shao. Gaussian processes: inequalities, small ball probabilities and applications. In *Stochastic Processes: Theory and Methods*, volume 19 of *Handbook of Statistics*, pages 533–597. North-Holland, 2001. doi: 10.1016/S0169-7161(01)19019-X.

Jingbo Liu, Thomas A. Courtade, Paul W. Cuff, and Sergio Verdú. A forward-reverse Brascamp–Lieb inequality: Entropic duality and Gaussian optimality. *Entropy*, 20(6):418, 2018. doi: 10.3390/e20060418.

Robert C. Reilly. Applications of the Hessian operator in a Riemannian manifold. *Indiana University Mathematics Journal*, 26(3):459–472, 1977. doi: 10.1512/iumj.1977.26.26036.

Victor Reis and Thomas Rothvoss. Vector balancing in Lebesgue spaces. *Random Structures & Algorithms*, 62(3):667–688, 2023. doi: 10.1002/rsa.21113.

Thomas Royen. A simple proof of the Gaussian correlation conjecture extended to some multivariate gamma distributions. *Far East Journal of Theoretical Statistics*, 48(2):139–145, 2014.

Zbyněk Šidák. Rectangular confidence regions for the means of multivariate normal distributions. *Journal of the American Statistical Association*, 62(318):626–633, 1967. doi: 10.1080/01621459.1967.10482935.

Appendix contents

Appendix A. Coefficient rank and support endpoints 13

Appendix B. Matrix-weighted Poincaré estimates for small-ball probabilities 13

Appendix C. The scalar inequalities at dilation 13/5 20

Appendix D. Spectral comparisons and entropy 22

Appendix E. The conjectured dilation 25

Appendix A. Coefficient rank and support endpoints

This appendix proves two facts about the hypotheses of Theorem 2.1: the hypothesis $r_X \le d$ on the coefficient rank is necessary, and the support endpoints of a function can be closed without changing either side of the determinant inequality.

A.1. **Coefficient rank and small-ball probabilities.** This necessity follows from small-ball probabilities. Remove the kernel of the coefficient map $x \mapsto X(x)$, so that it is injective on $\mathbb{R}^r$ and $r = r_X$.

Proposition A.1 (Small balls on the coefficient support)**.** *Suppose $x \mapsto X(x)$ is injective on $\mathbb{R}^r$ and $r \ge 1$. Define*

$$K = \{x : \|X(x)\| \le 1\}, \qquad v_K = (2\pi)^{-r/2}\operatorname{vol}_r(K), \qquad R_K = \max_{x\in K}|x|.$$

For every $t > 0$,

$$v_K t^r e^{-R_K^2 t^2/2} \le \mathbb{P}(\|X\| \le t) \le v_K t^r. \tag{A.1}$$

Consequently $\mathbb{P}(\|X\| \le t) \sim v_K t^r$ as $t \downarrow 0$. If X satisfies the determinant inequality at a finite dilation κ, then $r \le d$; when $r = d$, necessarily $v_K \kappa^d \ge (2/\pi)^{d/2}$.

Proof. Injectivity makes $\|X(\cdot)\|$ a norm, so K is a compact convex body with positive volume. Changing variables $x = ty$ gives

$$\mathbb{P}(\|X\| \le t) = (2\pi)^{-r/2} t^r \int_K e^{-t^2|y|^2/2}\, dy.$$

Bounding the integrand proves (A.1) and its asymptotic. The interval-indicator instance of the determinant inequality gives $\mathbb{P}(\|X\| \le \kappa t) \ge \mathbb{P}(|G| \le t)^d$. Since $\mathbb{P}(|G| \le t) \sim \sqrt{2/\pi}\, t$, comparison of powers and then coefficients proves the final assertions. □

A.2. **Support endpoints of log-concave functions.** The proofs use closed penalties (Section 3); the following lemma justifies that convention.

Lemma A.2 (Support-endpoint closure)**.** *Let f be a normalized even log-concave function. There is an upper-semicontinuous normalized even log-concave function $\bar f$ that agrees with f except possibly at the two endpoints of a bounded positive support. For every real symmetric matrix Gaussian series X and every $\kappa > 0$, the scalar and determinant expectations agree:*

$$\mathbb{E} f(G) = \mathbb{E}\bar f(G), \qquad \mathbb{E}\det f(X/\kappa) = \mathbb{E}\det \bar f(X/\kappa).$$

Proof. The positive support of f is a symmetric interval containing zero. If it is $\{0\}$, then f is already upper semicontinuous. Otherwise $\log f$ is finite and concave in the interior of its support, and hence continuous there. If the support has finite radius $a > 0$, extend f at $\pm a$ by its interior one-sided limit and keep it zero outside. The endpoint value of f is at most this limit, by log-concavity along the segment from zero. Extending $\log f$ by its one-sided limits preserves concavity, so the resulting $\bar f$ is even, log-concave, and upper semicontinuous. An unbounded symmetric support is all of $\mathbb{R}$, where no extension is needed.

The scalar Gaussian assigns zero mass to $\pm a$. For the matrix expectation, write X on its Gaussian coefficient support. Each polynomial $\det(X \mp \kappa a I_d)$ is nonzero at the zero coefficient vector and therefore vanishes on a Lebesgue-null set. Consequently X/κ almost surely has no eigenvalue equal to $\pm a$. The products defining the determinants then agree almost surely, proving both identities. □

Appendix B. Matrix-weighted Poincaré estimates for small-ball probabilities

This appendix proves the sharper bounds on the negative log probability $N_X(R)$ stated in Section 3.3. Throughout, $V_X \preceq I_d$, the coefficient rank r_X is arbitrary, and ℓ is the function of Section 3.3.

B.1. **Convex tilts.** The tilts used below are all even and log-concave, so the following comparison applies to them.

Lemma B.1 (Even convex observables under an even convex tilt)**.** *Let $d\mu = Z^{-1}e^{-V}d\gamma_m$, where $V\colon \mathbb{R}^m \to (-\infty,\infty]$ is even and convex, and $0 < Z < \infty$. If $F \geq 0$ is an even convex function, then*

$$\mathbb{E}_\mu F \leq \mathbb{E}_{\gamma_m} F.$$

Proof. For each real u, the set $C_u = \{F \leq u\}$ is symmetric and convex. Integrating the Gaussian correlation inequality between C_u and the symmetric convex superlevel sets of e^{-V} gives $\mathbb{E}_\gamma[\mathbf{1}_{C_u}e^{-V}] \geq \gamma(C_u)Z$. Thus $\mu(F > u) \leq \gamma(F > u)$, and integration of the tails proves the claim. Truncation handles an infinite expectation. Adding a constant to V does not affect the argument. □

For $Y = \sum_i g_i D_i$ with $\sum_i D_i^2 \preceq \xi I$, change the Gaussian density by a factor proportional to e^{-V}, where V is even and convex. Under the resulting measure μ, Lemma B.1, applied to $\operatorname{tr} Y^{2p}$, gives

$$\mathbb{E}_\mu \operatorname{tr} Y^{2p} \leq d(2p-1)!!\xi^p. \tag{B.1}$$

The barrier argument of Section B.6 uses the following tail bound:

$$\mathbb{E}_\mu \operatorname{tr} \mathbf{1}_{\{|Y|>u\}} \leq \frac{3}{2} d \exp\left(-\frac{u^2}{2\xi}\right) \quad \text{if } u/\sqrt{\xi} \geq 4. \tag{B.2}$$

Its proof uses the factorial estimate $(2p-1)!! \leq \sqrt{2}(2p/e)^p$: if $a_p = (2p-1)!!/(2p/e)^p$ and $t = (2p+1)^{-1}$, then

$$\log(a_{p+1}/a_p) = 1 - \frac{\operatorname{arctanh} t}{t} - \frac{1}{2}\log(1-t^2) = \sum\nolimits_{k\geq 1}\left(\frac{1}{2k} - \frac{1}{2k+1}\right)t^{2k} > 0.$$

Stirling's formula gives $a_p \to \sqrt{2}$, establishing the bound. For $x = u/\sqrt{\xi} \geq 4$, put $A = x^2/2 \geq 8$ and take the integer p nearest to A. Taylor's theorem for $v\log(v/A) - v + A$ gives

$$p\log(p/A) - p + A \leq \frac{1}{8(A-1/2)} \leq \frac{1}{60}.$$

Markov's inequality and (B.1) now yield $\sqrt{2}e^{1/60}de^{-x^2/2} < \frac{3}{2}de^{-x^2/2}$. Indeed, $e^{1/60} \leq 60/59$ and $2(60/59)^2 < 9/4$.

B.2. **Block Gram identities.** Besides the block inequality (3.12), the curvature computations below use two further block inequalities for symmetric matrices D_i and E_i. The calculation behind (3.12), followed by Frobenius Cauchy–Schwarz, gives

$$[D_jE_i + E_jD_i]_{ij} \preceq b[\operatorname{tr}(D_iD_j)]_{ij} \otimes I + b^{-1}[\operatorname{tr}(E_iE_j)]_{ij} \otimes I \quad (b > 0). \tag{B.3}$$

Also, if $\sum_i D_i^2 \preceq tI$, then $[D_jD_i]_{ij} \preceq tI$: use $\sum_{ij}\langle D_ju_i, D_iu_j\rangle \leq \sum_{ij}\|D_ju_i\|^2$.

B.3. **Removing the quadratic term from the barrier.** The bound of Proposition 3.2 is of order R^{-2}, the order of the quadratic part of $-\log\det(I - Y^2)$. Removing that part from the barrier, and bounding $\operatorname{tr} Y^2$ by the convex-tilt domination of Lemma B.1 instead, improves the order to R^{-3}.

Theorem B.2. *For $\sum_i B_i^2 \preceq I_d$, $R > \sqrt{5}$, define*

$$\eta = R^{-2}, \quad \lambda = \frac{2}{1 + \sqrt{1 - 4\eta/(1-\eta)}}, \quad D = \frac{\lambda^2(\lambda+4)}{(1-\eta)^2}.$$

Then

$$N_X(R) \leq n_1(R) := \frac{\sqrt{D}}{R^3} + \frac{D}{2R^4}. \tag{B.4}$$

The leading coefficient in (B.4) is $\sqrt{5}$ as $R \to \infty$.

Proof. Let $s(x) = -\log(1-x^2) - x^2$ for $|x| < 1$. For $b > 0$, put $\alpha = 1 + b^{-1}$ and use the tilt

$$V(Y) = \frac{b}{2}\operatorname{tr} Y^2 + \alpha \operatorname{tr} s(Y), \qquad d\mu_t = Z_t^{-1} e^{-V(Y_t)} \mathbf{1}_{\{\|Y_t\|<1\}} d\gamma_m, \quad Y_t = tX/R.$$

The scalar function s is convex and nonnegative, so μ_t is an even log-concave tilt of Gaussian measure. Lemma B.1, applied to the even convex function $\operatorname{tr} Y_t^2$, gives

$$\mathbb{E}_{\mu_t} \operatorname{tr} Y_t^2 \le d\eta t^2. \tag{B.5}$$

We next verify curvature for the resolvent weight $K = (I - Y_t)^{-1}$. Write $D_i = tB_i/R$, $E_i = K^{1/2} D_i K^{1/2} = D_i + \Delta_i$, and $\Gamma_D = [\operatorname{tr} D_i D_j]$, $\Gamma_\Delta = [\operatorname{tr} \Delta_i \Delta_j]$. If $\mathcal{H}$ is the coefficient Hessian of $\operatorname{tr} s(Y_t)$, then

$$\Gamma_\Delta \preceq \mathcal{H}. \tag{B.6}$$

To check (B.6), diagonalize Y_t and set $r_a = (1-x_a)^{-1}$, $p_a = (1+x_a)^{-1}$. The respective coefficients in these Gram forms are $(\sqrt{r_a r_b} - 1)^2$ and $r_a r_b + p_a p_b - 2$. The second dominates the first because $p_a p_b + 2\sqrt{r_a r_b} \ge 3$, by AM–GM and $p_a p_b r_a r_b \ge 1$.

By (3.12) and (B.3),

$$[E_j E_i] \preceq \eta t^2 I + b\Gamma_D \otimes I + (1 + b^{-1})\mathcal{H} \otimes I.$$

The last two terms are exactly the Hessian of V. Thus $p_t K$ has curvature at least $c_t = 1 - \eta t^2$. All matrix weights are integrable: $\alpha > 1$ and the remaining factor $e^{(\alpha - b/2)\operatorname{tr} Y_t^2}$ is bounded on the spectral domain.

The inverse-weight test and continuation argument from Proposition 3.2 give

$$\|S_t\| \le \lambda_t := \frac{2}{1 + \sqrt{1 - 4\zeta_t}}, \qquad S_t = \mathbb{E}_{\mu_t} K, \qquad \zeta_t = \frac{\eta t^2}{1 - \eta t^2} < \frac{1}{4}.$$

Let $M_t = \operatorname{tr}(S_t - I)$ and $Q_t = (I - Y_t^2)^{-1}$. We claim

$$\mathbb{E}_{\mu_t} \operatorname{tr}[Y_t^4 Q_t] \le \zeta_t(\lambda_t + 4) M_t \le dD\eta^2 t^4. \tag{B.7}$$

Apply (3.9) to the columns of $F = Y_t^2$ and sum. The weighted-mean term is $\operatorname{tr}[(S_t - I)S_t^{-1}(S_t - I)] \le (\lambda_t - 1)M_t/\lambda_t$. For the derivative term, evenness permits replacing K by Q_t. In an eigenbasis of Y_t, write $q_a = (1 - x_a^2)^{-1}$. The elementary inequality

$$(x_a + x_b)^2 (q_a + q_b) \le 4(x_a^2 q_a + x_b^2 q_b)$$

follows from $(x_a + x_b)^2 \le 2(x_a^2 + x_b^2)$ and $(x_a^2 - x_b^2)(q_a - q_b) \ge 0$. Symmetrizing the derivative energy over a, b therefore gives

$$\mathbb{E}_{\mu_t} \sum\nolimits_i \operatorname{tr}[(D_i Y_t + Y_t D_i) Q_t (D_i Y_t + Y_t D_i)] \le 4\eta t^2 M_t.$$

Since $(\lambda_t - 1)/\lambda_t = \zeta_t \lambda_t$, $M_t \le d\zeta_t \lambda_t^2$, and $\lambda_t \le \lambda$, the claim follows.

Finally, differentiating the partition function yields

$$-\frac{d}{dt} \log Z_t = \frac{1}{t} \mathbb{E}_{\mu_t} \{ b \operatorname{tr} Y_t^2 + 2\alpha \operatorname{tr}[Y_t^4 Q_t] \}.$$

Integrating (B.5) and (B.7) gives

$$-d^{-1} \log Z_1 \le \frac{b\eta}{2} + \frac{(1 + b^{-1}) D\eta^2}{2}.$$

Choose $b = \sqrt{D\eta}$ and use $Z_1 \le \mathbb{P}(\|X\| \le R)$. Hence (B.4) holds. □

B.4. **Bounds from higher moments.** Keeping more moments of the logarithmic barrier improves the leading coefficient in (B.4). For fixed $P \geq 4$, the resulting leading coefficient is $\sqrt{3}$.

Proposition B.3 (Higher-moment bound)**.** *For $R > \sqrt{5}$, set $\eta = R^{-2}$, $z = \eta/(1-\eta)$, and $\delta = \ell(z) - 1$. For an integer $P \geq 3$ with $d_P = 1 - \delta - P^2 z > 0$, define*

$$B_P(R) = \sum\nolimits_{j=2}^{P-2} \frac{(2j-1)!!}{j}\eta^j + \frac{(2P-3)!!}{(P-1)d_P}\eta^{P-1}.$$

The sum is empty for $P = 3$. Then

$$N_X(R) \leq n_P(R) := \sqrt{2\eta B_P(R)} + B_P(R). \tag{B.8}$$

The verification of the scalar inequalities (Proposition C.1) uses the best of (3.10), (B.4), and (B.8) with $3 \leq P \leq 8$.

Proof. Fix $\beta > 0$ and use the tilt of Theorem B.2 with $b = \beta$, $Y_t = tX/R$, variance $\xi_t = t^2/R^2$, curvature $1 - \xi_t$, and $z_t = \xi_t/(1-\xi_t)$. Put

$$K = (I - Y_t)^{-1}, \quad Q = (I - Y_t^2)^{-1}, \quad S = \mathbb{E}_\mu K = \mathbb{E}_\mu Q, \quad \delta_t = \ell(z_t) - 1,$$

and $J_p = \mathbb{E}_\mu \operatorname{tr} Y_t^{2p} Q$. Apply the weighted Poincaré inequality to the columns of $F = Y_t^p$.

Its mean term is at most $\|\mathbb{E}KY_t^p\|_F^2$ because $S \succeq I$. If p is even, $\mathbb{E}KY_t^p = \mathbb{E}QY_t^p$. Factoring the integrand as $(Y_t Q^{1/2})(Y_t^{p-1}Q^{1/2})^T$ and using $\mathbb{E}Y_t^2 Q = S - I \preceq \delta_t I$ bounds the squared Frobenius norm by $\delta_t J_{p-1}$. If p is odd, evenness gives $\mathbb{E}KY_t^p = \mathbb{E}QY_t^{p+1}$; use the factors $Y_t^2 Q^{1/2}$ and $Y_t^{p-1}Q^{1/2}$ instead. The operator bound $\mathbb{E}Y_t^4 Q \preceq S - I$ gives the same conclusion. These operator bounds are weighted Cauchy–Schwarz inequalities; no independence is used.

For the derivative term, put $D_p(x,y) = \sum_{k=0}^{p-1} x^k y^{p-1-k}$ and $q_x = (1-x^2)^{-1}$. The integral formula for the divided difference, followed by convexity of $|\cdot|^{2p-2}$, gives

$$|D_p(x,y)|^2 \leq \frac{p^2}{2}(|x|^{2p-2} + |y|^{2p-2}).$$

Monotonicity of q_x as a function of $|x|$ then implies

$$|D_p(x,y)|^2(q_x + q_y) \leq p^2(x^{2p-2}q_x + y^{2p-2}q_y).$$

Spectral summation and $\sum_i (tB_i/R)^2 \preceq \xi_t I$ bound the Dirichlet energy by $p^2\xi_t J_{p-1}$. Thus

$$J_p \leq (\delta_t + p^2 z_t) J_{p-1}.$$

All tests have finite weighted energy by Section B.7.

Write $M_j = \mathbb{E}_\mu \operatorname{tr} Y_t^{2j}$, so $J_j = M_j + J_{j+1}$. If $d_P = 1 - \delta_1 - P^2 z_1 > 0$, then uniformly in $t \leq 1$,

$$J_{P-1} \leq M_{P-1}/d_P, \qquad J_2 \leq \sum\nolimits_{j=2}^{P-2} M_j + M_{P-1}/d_P.$$

Use the tilted moment bound (B.1) to bound each M_j. The normalizer satisfies

$$-\frac{1}{d}\frac{d}{dt}\log Z_t = \frac{1}{dt}\mathbb{E}_\mu\{\beta \operatorname{tr} Y_t^2 + 2(1+\beta^{-1})\operatorname{tr} Y_t^4 Q\}.$$

Integrating each moment term yields $-d^{-1}\log Z_1 \leq \beta\eta/2 + (1+\beta^{-1})B_P(R)$. Since the tilt stays fixed during the integration, minimizing afterwards in $\beta > 0$ gives (B.8). □

B.5. **A logarithmic barrier with a flat interior.** The large-radius bound tilts Gaussian measure by a logarithmic barrier that vanishes on $[-a, a]$ and agrees with $-\log(1-x)$ up to a linear term near $x = 1$. Vanishing on the interior is the point: the derivative energy in Section B.6 is then supported on eigenvalues beyond $b < a$, whose expected number decays like a Gaussian tail. This subsection constructs the barrier and bounds the curvature of the resolvent weight it produces. Fix $a \in (0,1)$ and set $L = (1-a)^{-1}$. Define

$$h_a(x) = \begin{cases} 0, & x \le a, \\ -\log((1-x)/(1-a)) - L(x-a), & a < x < 1, \end{cases} \qquad v_a(x) = h_a(x) + h_a(-x).$$

On $(-1,1)$, h_a and v_a are convex and nonnegative, and

$$h_a'(x) = \big((1-x)^{-1} - L\big)_+.$$

The derivative is continuous. Smooth approximations, specified in Section B.7, justify the curvature computations at its change of formula.

Let $Y = \sum_i g_i D_i$ with $\sum_i D_i^2 \preceq \xi I$, and on $\|Y\| < 1$ put

$$K = (I - Y)^{-1}, \qquad E_i = K^{1/2} D_i K^{1/2}.$$

In a spectral basis of Y, write $r(x) = (1-x)^{-1}$ and define

$$u_a(x) = \sqrt{\min\{r(x), L\}}, \quad C_i = u_a(Y) D_i u_a(Y), \quad U_i = E_i - C_i.$$

Let $\Gamma_C = [\operatorname{tr} C_i C_j]$ and $\Gamma_U = [\operatorname{tr} U_i U_j]$.

Lemma B.4 (Curvature of the clipped resolvent)**.** *With $\mathcal{H}_a$ the coefficient Hessian of $\operatorname{tr} h_a(Y)$,*

$$\Gamma_U \preceq \mathcal{H}_a, \qquad \Gamma_C \preceq L^2 [\operatorname{tr} D_i D_j], \qquad \textstyle\sum_i C_i^2 \preceq L^2 \xi I.$$

Consequently, for every $\beta > 0$, the Gaussian tilt

$$V(Y) = \frac{\beta L^2}{2} \operatorname{tr} Y^2 + (1 + \beta^{-1}) \operatorname{tr} v_a(Y) \tag{B.9}$$

has resolvent matrix weight pK with curvature at least $1 - L^2\xi$ in (3.9).

Proof. In a spectral basis, the multiplier of D_{ab} in U_i is

$$u(x,y) = \sqrt{r(x) r(y)} - u_a(x) u_a(y).$$

It suffices to prove $u(x,y)^2 \le h_a'[x,y]$, with the divided difference interpreted as a derivative on the diagonal. If $x, y \le a$, the left side is zero. If $x, y > a$, then

$$u(x,y)^2 = (\sqrt{r(x)r(y)} - L)^2 \le r(x) r(y) = h_a'[x,y].$$

If $x > a \ge y$, division of the desired inequality by its positive right side leaves

$$r(y)(x-y) \frac{\sqrt{r(x)} - \sqrt{L}}{\sqrt{r(x)} + \sqrt{L}} \le 1.$$

Both factors on the left are in $[0,1]$: in particular $r(y)(x-y) = (x-y)/(1-y) \le 1$. This factor bound proves the first Gram inequality by the spectral Hessian formula. The second follows because every entry multiplier in C_i is at most L. Finally,

$$\textstyle\sum_i C_i^2 = u_a(Y) \sum_i D_i u_a(Y)^2 D_i u_a(Y) \preceq L u_a(Y) \Big(\sum_i D_i^2 \Big) u_a(Y) \preceq L^2 \xi I.$$

Expand $E_i = C_i + U_i$. The block inequalities (3.12) and (B.3), and the last block inequality in Section B.2 give

$$[E_j E_i]_{ij} \preceq L^2 \xi I + \beta \Gamma_C \otimes I + (1 + \beta^{-1}) \Gamma_U \otimes I \preceq L^2 \xi I + \beta L^2 [\operatorname{tr} D_i D_j] \otimes I + (1 + \beta^{-1}) \mathcal{H}_a \otimes I.$$

For the weight pK, the normalized block curvature is

$$[\delta_{ij} I + \partial_{ij} V\, I - E_j E_i]_{ij}.$$

The Hessian of (B.9) absorbs the last two terms in the preceding bound. The Hessian of $\operatorname{tr} h_a(-Y)$ is additional and nonnegative. The remaining curvature is at least $1 - L^2\xi$. □

B.6. **An exponential small-ball bound.** The preceding determinant estimates give polynomial decay in R. For the barrier of Section B.5, the derivative energy can instead be bounded by the expected number of eigenvalues beyond b. The Gaussian tail estimate then gives exponential decay. Fix $0 < b < a < 1$ and put $L = (1-a)^{-1}$ and $w = a - b$. For R with $R^2 > L^2 + 4$, set

$$z = \frac{1}{R^2 - L^2}, \quad \delta = \ell(z) - 1, \quad C_{a,b}(R) = \frac{3 + 2z/((1-a^2)w^2)}{1 - 3\delta/2 - 2z/(1-b)^2}.$$

The expression is used only when its denominator is positive.

Proposition B.5 (Exponential bound for the negative log probability)**.** *If $bR \geq 4$ and the preceding conditions hold, define*

$$A_R = \frac{L^2}{2R^2}, \qquad B_R = \frac{3C_{a,b}(R)}{b^2R^2} e^{-b^2R^2/2}.$$

Then

$$N_X(R) \leq 2\sqrt{A_R B_R} + B_R. \tag{B.10}$$

Behind Proposition B.5 is a bound on the derivative energy of a spectral ramp, obtained by treating the eigenvalues beyond a and those in the transition band (b, a) separately. Let $Y = \sum_i g_i D_i$ with $\sum_i D_i^2 \preceq \xi I$, let μ be the Gaussian tilt (B.9) of Lemma B.4, and let $F = \min\{I, (|Y| - bI)_+/w\}$ and $Q = (I - Y^2)^{-1}$. Then

$$\mathbb{E}_\mu \sum\nolimits_i \operatorname{tr}[(\partial_i F)(I-Y)^{-1}(\partial_i F)] \leq \frac{2\xi}{(1-b)^2}\mathbb{E}_\mu \operatorname{tr}(FQF) + \frac{2\xi}{(1-a^2)w^2}\mathbb{E}_\mu \operatorname{tr} \mathbf{1}_{\{|Y|>b\}}. \tag{B.11}$$

The coefficient of the first term depends on $1 - b$, not the much smaller transition width $a - b$.

Proof of (B.11) *and Proposition B.5.* Fix a, b, L, w as in Proposition B.5, and use the tilt (B.9) at $Y_t = tX/R$. Its curvature is $1 - L^2\xi_t$, where $\xi_t = t^2/R^2$. As in Proposition 3.2, the inverse-weight test and continuation from $S_0 = I$ give

$$I \preceq S_t = \mathbb{E}_{\mu_t}(I - Y_t)^{-1} = \mathbb{E}_{\mu_t} Q_t \preceq \ell(z_t) I, \quad z_t = \frac{\xi_t}{1 - L^2\xi_t}.$$

The condition $R^2 > L^2 + 4$ ensures $z_t < 1/4$. Let $s(x) = \min\{1, (|x| - b)_+/w\}$, $F = s(Y_t)$, and

$$\mathcal{A}_t = \mathbb{E}_{\mu_t} \operatorname{tr}(FQ_tF), \qquad P_t = \mathbb{E}_{\mu_t} \operatorname{tr} \mathbf{1}_{\{|Y_t|>b\}}, \qquad \delta_t = \ell(z_t) - 1.$$

The weighted mean term is at most $3P_t + (3/2)\delta_t \mathcal{A}_t$. Indeed, split $\mathbb{E}K_tF = \mathbb{E}F + \mathbb{E}(K_t - I)F$ and use $\|u+v\|^2 \leq 3\|u\|^2 + (3/2)\|v\|^2$. The first squared norm is at most P_t, and weighted Cauchy–Schwarz with the exact identity

$$\mathbb{E}(K_t - I)K_t^{-1}(K_t - I) = S_t - I$$

bounds the second by $\delta_t \mathcal{A}_t$.

To prove (B.11), symmetrize the derivative energy using evenness and work in a spectral basis. For an unordered pair of distinct indices its coefficient is $s[x,y]^2(q_x + q_y)$. If both indices have absolute eigenvalue at least a, this coefficient is zero. If precisely one index is high, put $v = |x| \geq a$. The explicit ramp gives $|s[x,y]| \leq (v-b)^{-1}$ and $q_y \leq q_x$. The resolvent term splits as

$$\frac{q_v}{(v-b)^2} = \frac{q_v}{(1-b)^2} + \frac{1 + v - 2b}{(1+v)(v-b)^2(1-b)^2}. \tag{B.12}$$

The second term decreases in v. At $v = a$, it is at most $M = ((1-a^2)w^2)^{-1}$, since its ratio to M equals $1 - w^2/(1-b)^2 \le 1$. Consequently the high-index contribution is at most

$$2\xi_t \left\{ (1-b)^{-2} \operatorname{tr}_{\{|Y_t| \ge a\}} Q_t + M \operatorname{tr} \mathbf{1}_{\{|Y_t| \ge a\}} \right\}.$$

For pairs, including diagonal terms, with both eigenvalues in $(-a, a)$, the resolvent is at most $(1-a^2)^{-1}$ and the ramp Lipschitz constant is $1/w$. Its divided difference vanishes unless at least one index lies in the transition band $b < |x| < a$. Their entire contribution is at most $2\xi_t M \operatorname{tr} \mathbf{1}_{\{b<|Y_t|<a\}}$. The two indicator terms partition $\{|Y_t| > b\}$, so their expected traces sum to P_t. Also, $\operatorname{tr}_{\{|Y_t|\ge a\}} Q_t \le \operatorname{tr}(FQ_tF)$. Hence (B.11) holds. To verify monotonicity of the remainder in (B.12), set $u = v - b$. Its logarithmic derivative is

$$\frac{2b}{(1-b+u)(1+b+u)} - \frac{2}{u} < 0.$$

Weighted Poincaré therefore yields

$$\left(1 - \frac{3}{2}\delta_t - \frac{2z_t}{(1-b)^2}\right) \mathcal{A}_t \le \left(3 + \frac{2z_t}{(1-a^2)w^2}\right) P_t.$$

The denominator stays positive by hypothesis; replacing z_t, δ_t by their values at one gives $\mathcal{A}_t \le C_{a,b}(R) P_t$. This bound holds for each $\beta > 0$ and uniformly in $t \le 1$.

By the tilted tail bound (B.2) and $bR/t \ge 4$, $P_t \le (3/2) d e^{-b^2R^2/(2t^2)}$. Also $\mathbb{E}_{\mu_t} \operatorname{tr} Y_t^2 \le d\xi_t$ and $x v_a'(x) \le 2(1-x^2)^{-1} \mathbf{1}_{\{|x|>a\}}$. Differentiation of the normalizer gives

$$-\frac{1}{d}\frac{d}{dt} \log Z_t \le \frac{\beta L^2 t}{R^2} + \frac{3(1+\beta^{-1}) C_{a,b}(R)}{t} e^{-b^2R^2/(2t^2)}.$$

For $c > 0$, $\int_0^1 e^{-c/t^2}\, dt/t \le e^{-c}/(2c)$. Integration hence gives $\beta A_R + (1+\beta^{-1}) B_R$. Optimize at $\beta = \sqrt{B_R/A_R}$ to prove (B.10). Section B.7 justifies the regularity and limiting steps. □

Specializing the parameters makes the large-radius decay explicit.

Theorem B.6 (Exponential small-ball bound)**.** *If $V_X \preceq I_d$, then for every $R \ge 83/10$,*

$$N_X(R) \le \frac{160}{R^2} e^{-5329R^2/40000} + \frac{507}{R^2} e^{-5329R^2/20000}. \tag{B.13}$$

There is no restriction on r_X.

For the unbounded range, take $a = 4/5$, $b = 73/100$: exact arithmetic gives $C_{a,b}(R) \le 90$ for $R \ge 83/10$, and then rounds the two coefficients to 160 and 507. For the compact range we use $a = 77/100$, $b = 7/10$ and retain the unrounded formula.

B.7. **Regularity and boundary limits.** For barriers, choose a smooth nonnegative probability density ρ_ϵ supported in $(a-\epsilon, a)$ and set

$$h_{a,\epsilon}(x) = \int h_z(x) \rho_\epsilon(z)\, dz.$$

For $z < a$, the function $h_z - h_a$ is convex. Thus the Hessian domination in Lemma B.4, using the clip threshold a and the same quadratic coefficient L^2, remains valid for the smoothed barrier. Its derivative is supported on $|x| > a - \epsilon$. Use the ramp with upper threshold $a - \epsilon$ and lower threshold b in the proof of Proposition B.5. The constants tend to the stated ones, and the denominator stays positive for sufficiently small ϵ. Passing to the limit proves the exact formula with $C_{a,b}(R)$; strict slack in its rounded specializations is retained.

Lipschitz spectral ramps are admissible by smooth approximation on compact convex subdomains and exhaustion. Their divided-difference formulas hold almost everywhere. The derived energy inequalities provide a finite integrable majorant. For each fixed R and positive tilt parameter, the barrier power is $1 + \beta^{-1} > 1$, while the weights $(I - Y)^{-1}$ and $(I - Y^2)^{-1}$ have only first-order

poles. Density times weight is bounded on the spectral domain, with Gaussian decay in coefficient kernel directions. The same observations justify continuity of the resolvent mean from $S_0 = I$ and differentiation of the normalizer on compact subintervals of $t > 0$ with no boundary contribution. Integrate first from $\tau > 0$ and then let $\tau \downarrow 0$. Bounded convergence gives $Z_\tau \to 1$.

For the determinant weight, the product with its first resolvent is the adjugate expression displayed above and is integrable even though it need not vanish on the boundary. The curvature criterion imposes no zero-boundary test condition. Scalar integration by parts below uses the density itself, which does vanish. Its logarithmic derivative has an integrable first-order pole, and $-\log\det(I - Y^2)$ is integrable under the same density. These facts justify the rescaling proof and all its expectations by exhaustion.

Appendix C. The scalar inequalities at dilation 13/5

This appendix proves the large-radius tangent bound (C.2) and Proposition C.1.

C.1. **The tangent bound at large radii.** We bound the trace penalty in (3.4) by $1 - p_\psi$ and its norm cost by $T(r)$. Let ψ be a finite closed penalty with $\psi(r) = 1$ and $r \geq 16/5$. Set $\alpha = 999/1000$, $\rho = (13/5)\alpha$, $b_f = \kappa_*/(13/5)$, $a_0(r) = 1 - H(r)/H(0)$, and $\delta_0 = 1 - e^{-1}$. For $0 \leq h \leq \alpha r$,

$$H(h) - H(r) \geq a_0(r) H(h/b_f). \tag{C.1}$$

The difference vanishes at zero and its derivative has the sign of $-1 + (a_0(r)/b_f)T(h/b_f)/T(h)$. The ratio $T(h/b_f)/T(h)$ decreases: its logarithmic derivative is $1/M(h) - 1/(b_f M(h/b_f)) < 0$, since M decreases and $b_f < 1$. Thus the minimum of the difference is at an endpoint. At $h = \alpha r$, put $u = \alpha r/b_f$ and use

$$H(\alpha r) - H(r) \geq (1 - \alpha) r T(r), \qquad H(u) \leq T(u)/u \leq T(r) e^{-(u^2 - r^2)/2}/u.$$

The sufficient inequality

$$e^{-((\alpha/b_f)^2 - 1) r^2/2} \leq (1 - \alpha)(\alpha/b_f) r^2$$

holds at $r = 16/5$ by the interval arithmetic of Section C.2, and persists by monotonicity. Hence (C.1) holds.

For the positive hinge measure ν of the tangent-truncated $\widehat{\psi}$,

$$\mathbb{E}\widehat{\psi}(b_f|G|) = \int b_f H(h/b_f)\, d\nu(h), \quad \mathbb{E}\min\{\psi(|G|), 1\} \geq \int (H(h) - H(r))\, d\nu(h).$$

Together with $1 - e^{-x} \geq \delta_0 \min\{x, 1\}$, this inequality gives $\mathbb{E}\widehat{\psi}(b_f|G|) \leq b_f(1 - p_\psi)/(\delta_0 a_0(r))$. Substituting this estimate in the localization bound (3.4) gives

$$-d^{-1} \log \mathcal{Z}_\psi(X; 13/5) \leq N_X(\rho r) + \frac{b_f}{\delta_0 a_0(r)}(1 - p_\psi). \tag{C.2}$$

To extend the norm bound to the unbounded range, divide the exponential small-ball bound (B.13), at $R = \rho r$, by $T_-(r) = c_0 r(1 + r^2)^{-1} e^{-r^2/2} \leq T(r)$. Each term is a positive constant times $(1 + r^2) r^{-3} e^{-cr^2}$ with $c > 0$, because $(73/100)^2 \rho^2 > 2$. These terms decrease. Therefore the single endpoint comparison at $r = 16/5$ covers the full unbounded range.

C.2. **Verification of the scalar inequalities.**

Proposition C.1 (Scalar inequalities at dilation 13/5)**.** *Fix $\kappa = 13/5$ and $b_f = \kappa_*/\kappa = 5\sqrt{\pi/2}/13$.*
(1) For $0 < r \leq 24/25$, the right-hand side of (3.14), for one of $R = 11/4$ or $R = 3$, is at least $q(r)$.
(2) For $24/25 \leq r \leq 16/5$, the piecewise constant tangent fraction α in Table 1 satisfies (3.7)–(3.8) with $n = n(\kappa\alpha r)$.

(3) For $r \geq 16/5$, let $\alpha = 999/1000$, $\rho = \kappa\alpha$, $a_0(r) = 1 - H(r)/H(0)$, and $\delta_0 = 1 - e^{-1}$. Then

$$N_X(\rho r) \leq \frac{9}{50} T(r), \qquad \frac{b_f}{\delta_0 a_0(r)} < \frac{77}{100},$$

and the tangent bound (C.2) holds.

Proof of Proposition C.1. Let $L_R(r)$ denote the right side of (3.14). For $R = 11/4$, $7/6 \leq \lambda \leq 6/5$, $v_R = 16/153$, and $\sqrt{v_R} \leq 1/3$. At $r = 1/2$, the exponent after extracting $\lambda^{-2}(1 - v_R)^{-1}$ is $1/2 - (26/55)^2(3/2 - \lambda) - \sqrt{v_R} \geq 1673/18150$. Using $e^x \geq 1 + x$ gives

$$\frac{L_{11/4}(1/2)}{1/2} \geq \frac{52}{55} \frac{1 + 1673/18150}{(36/25)(137/153)} = \frac{4380883}{5470410} > \frac{4}{5} > c_0.$$

Since $3/2 - \lambda > 0$, the ratio $L_{11/4}(r)/r$ decreases. Together with $q(r) \leq c_0 r$, this monotonicity proves the first assertion for $r \leq 1/2$.

On $[1/2, 24/25]$, the check $\max_{R \in \{11/4, 3\}} L_R(r_l) > q(r_u)$ on 920 intervals of width $1/2000$ completes the first assertion. For the middle range, 2,240 radius intervals and 8,762 terminal hinge rectangles exhaust the domain. Section C.4 specifies the outward-rounded integer arithmetic and transcendental remainders. For the unbounded range, Appendix C.1 reduces the comparisons to certified endpoints. □

Table 1 lists the tangent fractions used on the middle range.

Table 1: Complete compact partition for the normalized-hinge comparison.

Radius interval	Tangent fraction α	Terminal rectangles
$[24/25, 6/5]$	$9/10$	1366
$[6/5, 3/2]$	$93/100$	1162
$[3/2, 9/5]$	$97/100$	900
$[9/5, 21/10]$	$197/200$	900
$[21/10, 23/10]$	$199/200$	785
$[23/10, 5/2]$	$499/500$	800
$[5/2, 16/5]$	$999/1000$	2849

C.3. **Monotonicity behind the enclosures.** The interval arithmetic of Section C.2 bounds each quantity on a rectangle of radii and hinge positions by its value at a corner; the following monotonicity properties justify these corners. For the normalized hinges write $h = yr$, $0 \leq y < 1$. The scalar partition $p_{r,yr}$ increases in both r and y. As r increases, the penalty decreases pointwise. For y, fix r and put $s = r - h$. The function $e^{-(t-h)_+/s}$ has Lebesgue integral r; its cumulative integral for $t \geq h$ is

$$h + s(1 - e^{-(t-h)/s}).$$

Its derivative in h is $e^{-(t-h)/s}(t - h)/s \geq 0$. Thus its integral against any decreasing density, including the half-normal density, increases in h.

The hinge term $P(r, y) = b_f H(ry/b_f)/(r(1 - y))$ decreases in r. On any y-interval, its maximum is at an endpoint: under $z = (1 - y)^{-1}$,

$$P(r, y) = \mathbb{E}\big(1 + z(b_f|G|/r - 1)\big)_+,$$

a convex function of z. All available norm bounds decrease in R on their valid domains. For the moment bounds their denominators increase with R, and for the exponential bounds $C_{a,b}(R)$ decreases, so this assertion follows directly from their formulas.

Consequently, for a radius interval $[r_l, r_u]$ and hinge interval $[y_l, y_u]$, an upper bound on the left side of (3.8) is

$$p_{r_u,y_u r_u} \exp\{\max(P(r_l, y_l), P(r_l, y_u))\},$$

while $\exp(-n(\kappa\alpha r_l))$ is a lower bound for its target. The condition (3.7) is covered by $q(r_u)$. The computation starts with $[0, \alpha]$ and bisects this hinge interval until the strict enclosure passes. All middle-range radius intervals have width $1/1000$. The final partition is exhaustive, with a maximum hinge depth of six.

C.4. **Arithmetic and explicit remainders.** The computation uses only integer arithmetic. Every interval has outward-rounded endpoints on the fixed grid $10^{-55}\mathbb{Z}$. Products, quotients and square roots use integer floor, ceiling, and integer square roots. Machin's identity $\pi = 16\arctan(1/5) - 4\arctan(1/239)$ is enclosed by alternating Taylor remainders. Exponentials are reduced to arguments in $[0, 1]$, summed through degree 60 with a geometric upper remainder, then squared back; negative arguments use reciprocals. The logarithms used are in $[1, 2]$ and use the positive arctanh series with its geometric remainder.

Gaussian integrals up to argument eight use the integrated alternating series for $e^{-t^2/2}$ through degree index 200 and the next-term remainder. For Mills ratios above six, the integral representation

$$M(z) = \int_0^\infty e^{-zu-u^2/2}\, du$$

and the signed Taylor remainder for $e^{-u^2/2}$ give

$$\sum\nolimits_{j=0}^{17} \frac{(-1)^j(2j-1)!!}{z^{2j+1}} \le M(z) \le \sum\nolimits_{j=0}^{16} \frac{(-1)^j(2j-1)!!}{z^{2j+1}}.$$

These bounds are integrated Taylor remainder inequalities, not an assumption that the asymptotic series converges. All intermediate roundoff is outward and included in the final bounds.

The complete computation has 20 groups of checks: four for the sharp averaged comparison and moment prefactor, two for small penalties, seven for normalized hinges, and seven for the analytic unbounded tail. The middle range uses the determinant bound on 480 radius intervals, the cubic bound on 354, moment orders three through six on 986 in total, and the exponential bound on 420.

Appendix D. Spectral comparisons and entropy

The determinant inequality controls symmetric polynomials and Schatten norms through different scalar comparisons. We derive them and their limitations, then prove the entropy characterization.

D.1. **Symmetric polynomials.** For a symmetric polynomial P in d variables, write $P(M) = P(\lambda_1(M), \dots, \lambda_d(M))$ when $M \succeq 0$, and put $\mathbf{1} = (1, \dots, 1)$. To apply the determinant inequality, we bound $P(M)$ below by a power of $\det M$.

Proposition D.1 (Symmetric polynomial comparison)**.** *Suppose X satisfies the determinant inequality at dilation κ. Let P be a nonzero symmetric homogeneous polynomial of degree $m \ge 1$ with nonnegative coefficients. For every even log-concave function $f\colon \mathbb{R} \to [0, \infty)$,*

$$\frac{\mathbb{E}P(f(X/\kappa))}{P(\mathbf{1})} \ge \left(\mathbb{E}f(G)^{m/d}\right)^d. \tag{D.1}$$

If $m \ge d$, the right side is at least $(\mathbb{E}f(G))^m$.

Proof. Write $P(y) = \sum_\alpha c_\alpha y^\alpha$, where $c_\alpha \ge 0$ and $\sum_i \alpha_i = m$. Normalize the coefficients by $w_\alpha = c_\alpha/P(\mathbf{1})$. Symmetry gives $\sum_\alpha w_\alpha \alpha_i = m/d$ for every i. Weighted arithmetic–geometric mean therefore yields, for $y_i > 0$,

$$\frac{P(y)}{P(\mathbf{1})} = \sum\nolimits_\alpha w_\alpha y^\alpha \ge \prod_\alpha (y^\alpha)^{w_\alpha} = \left(\prod_{i=1}^d y_i\right)^{m/d}. \tag{D.2}$$

Continuity gives the same inequality when some coordinates vanish. Taking $y_i = f(\lambda_i(X)/\kappa)$ in (D.2) and applying the determinant inequality to the even log-concave function $f^{m/d}$ proves (D.1). For $m \geq d$, Jensen's inequality gives $\mathbb{E}f(G)^{m/d} \geq (\mathbb{E}f(G))^{m/d}$. □

The choice $P = e_k$ gives $\mathbb{E}e_k(f(X/\kappa))/\binom{d}{k} \geq (\mathbb{E}f(G)^{k/d})^d$ for $1 \leq k \leq d$. The proposition also includes the complete homogeneous symmetric polynomials and products of elementary symmetric polynomials. For a general P, the quantity $P(\mathbf{1})(\mathbb{E}f(G))^m = P(\mathbb{E}f(G), \dots, \mathbb{E}f(G))$ need not equal $\mathbb{E}P(f(G_1), \dots, f(G_d))$ for independent G_i: repeated powers in a monomial involve higher scalar moments.

For elementary symmetric polynomials, the independent scalar benchmark is $\binom{d}{k}(\mathbb{E}f(G))^k$. If the coefficients commute and $V_X \preceq I_d$, applying the commuting case (Theorem E.6) to each $k \times k$ diagonal subseries in a simultaneous eigenbasis and summing gives this benchmark at dilation one. For general coefficients it fails: the two-Pauli series of Theorem E.8 has both absolute eigenvalues equal to $\sqrt{g_1^2 + g_2^2}/\sqrt{2}$. Thus, for $f_a = \mathbf{1}_{[-a,a]}$ and every fixed finite $\kappa > 0$,

$$\frac{1}{2}\mathbb{E}e_1(f_a(X/\kappa)) = 1 - e^{-\kappa^2 a^2} \sim \kappa^2 a^2, \qquad \mathbb{E}f_a(G) \sim \sqrt{2/\pi}\, a.$$

The different powers of a rule out that benchmark as $a \downarrow 0$.

D.2. **Norms and their Laplace transforms.** Power penalties control the Schatten norms $\|A\|_{S_p} = (\operatorname{tr}|A|^p)^{1/p}$, for $1 \leq p < \infty$.

Proposition D.2 (Schatten norm comparison)**.** *If X satisfies the determinant inequality at dilation κ, then for $1 \leq p < \infty$ and $t \geq 0$,*

$$\mathbb{E}e^{-t\|X/\kappa\|_{S_p}^p} \geq \left(\mathbb{E}e^{-t|G|^p}\right)^d = \mathbb{E}e^{-t\|(G_1,\dots,G_d)\|_p^p}, \tag{D.3}$$

where the G_i are independent standard Gaussians. In particular, for $a > 0$,

$$\mathbb{P}(\|X\|_{S_p} \leq \kappa a) \geq \sup_{t>0} \frac{\left[(\mathbb{E}e^{-t|G|^p})^d - e^{-ta^p}\right]_+}{1 - e^{-ta^p}}.$$

Proof. Apply the determinant inequality to $f(x) = e^{-t|x|^p}$. For nonnegative T and $u, t > 0$,

$$\mathbb{E}e^{-tT} \leq \mathbb{P}(T \leq u) + e^{-tu}\mathbb{P}(T > u).$$

Apply this bound with $T = \|X/\kappa\|_{S_p}^p$ and $u = a^p$, use (D.3), and optimize over t. □

The case $p = 1$ concerns the nuclear norm; $p = 2$ concerns the Frobenius norm. Theorem E.7 gives the latter comparison at $\kappa = 1$ when $V_X \preceq I_d$ and $r_X \leq d$.

The Laplace-transform bound does not imply small-ball domination by the corresponding scalar norm at every radius. For example, $X = gI_d$ satisfies the determinant inequality at dilation one, but $\|X\|_{S_2} = \sqrt{d}\,|g|$. Gaussian tail asymptotics give

$$\lim_{a\to\infty} \frac{1}{a^2} \log \mathbb{P}(\|X/\kappa\|_{S_2} > a) = -\frac{\kappa^2}{2d}, \qquad \lim_{a\to\infty} \frac{1}{a^2} \log \mathbb{P}(\|(G_1, \dots, G_d)\|_2 > a) = -\frac{1}{2}.$$

Hence a comparison of these small-ball probabilities at every radius would require $\kappa \geq \sqrt{d}$. A dilation independent of d cannot give that comparison for all unitarily invariant norms.

A unitarily invariant norm N depends only on singular values and satisfies $N(A) \leq N(I_d)\|A\|$. Thus, if X satisfies the determinant inequality at dilation κ, then

$$\mathbb{P}(N(X) \leq \kappa a N(I_d)) \geq \mathbb{P}(|G| \leq a)^d, \qquad a > 0.$$

D.3. **Proof of the entropy characterization.** To complete Theorem 4.2, we construct a coefficient law that satisfies all hinge constraints at the required entropy cost. Compactness will enforce these constraints simultaneously and give attainment of the infimum.

Completion of the proof of Theorem 4.2. First, the Gibbs variational identity (4.3) follows by subtracting its left side from the expression in braces: when the integral is positive, the difference is the relative entropy to the normalized tilt $e^{-W}\mu$. The tilt attains the infimum, since $We^{-W} \le 1/e$. If the integral is zero, both sides are infinite.

We will use compactness of entropy sublevel sets. To see tightness, the entropy inequality gives

$$a \int |x|^2 \, d\nu \le D(\nu\|\gamma_m) + \log \int e^{a|x|^2} \, d\gamma_m \qquad (0 < a < 1/2).$$

This inequality follows from Jensen's inequality after an exponential change of measure, first for truncated squared norms and then by monotone convergence. Relative entropy is lower semicontinuous, so its sublevel sets are weakly compact. The same bound holds for η relative to the law γ_+ of $|G|$. In particular, finite Gaussian relative entropy ensures finite second moments, so the constraint $\sigma_{\nu,\kappa} \preceq_{\mathrm{icx}} \eta$ is well defined for all such laws.

For $D(\eta\|\gamma_+) < \infty$, absolute continuity and the moment bound give $\bar{t} = \int t \, d\eta(t) \in (0, \infty)$. The event $E = \{x : \|X(x)\| \le \kappa\bar{t}\}$ contains a neighborhood of the origin, so $\nu = \gamma_m(\cdot \mid E)$ has relative entropy $-\log \gamma_m(E) < \infty$. Its average absolute spectral law is supported on $[0, \bar{t}]$, so monotonicity and Jensen give $\int h \, d\sigma_{\nu,\kappa} \le h(\bar{t}) \le \int h \, d\eta$ for every finite convex nondecreasing h. Thus the infimum in (4.1) is finite. A minimizing sequence has a weakly convergent subsequence in an entropy sublevel set. Lower semicontinuity of the hinge integrals preserves feasibility, and lower semicontinuity of relative entropy gives attainment.

In the increasing convex order it suffices to test the hinges $h_t(u) = (u - t)_+$, $t \ge 0$. Assume the determinant inequality at dilation κ and fix η with $D(\eta\|\gamma_+) < \infty$. Put $C = dD(\eta\|\gamma_+)$. For a finite collection of hinges $h_{t_1}, \dots, h_{t_n}$, let

$$S_j(x) = \operatorname{tr} h_{t_j}(|X(x)|/\kappa), \qquad b_j = d \int h_{t_j} \, d\eta.$$

We first find one law with entropy at most C and $\int S_j \, d\nu \le b_j$ for every j. Consider the upward closed convex set of vectors $(u, v_1, \dots, v_n)$ for which some law ν satisfies $D(\nu\|\gamma_m) \le u$ and $\int S_j \, d\nu \le v_j$. This set is closed: a bounded entropy sequence is tight, relative entropy is lower semicontinuous, and each S_j is continuous and nonnegative, so its integral is weakly lower semicontinuous.

In the following separation argument, the laws ν have finite entropy. If $(C, b_1, \dots, b_n)$ lay outside this set, strict separation would give nonnegative coefficients $a_0, a_1, \dots, a_n$, not all zero, such that

$$\inf_\nu \left\{ a_0 D(\nu\|\gamma_m) + \textstyle\sum_j a_j \int S_j \, d\nu \right\} > a_0 C + \textstyle\sum_j a_j b_j.$$

For $a_0 > 0$, divide by a_0 and apply (4.3) and the determinant inequality with $\psi = \sum_j (a_j/a_0) h_{t_j}$. They bound the infimum by

$$-d \log \int e^{-\psi} \, d\gamma_+ \le dD(\eta\|\gamma_+) + d \int \psi \, d\eta,$$

contradicting separation. If $a_0 = 0$, replace it by a sufficiently small positive number. Nonnegativity of relative entropy preserves the strict gap, and the same argument applies.

On the compact entropy sublevel set $\{\nu : D(\nu\|\gamma_m) \le C\}$, the closed constraints $\int S_j \, d\nu \le b_j$ have the finite intersection property by the preceding argument. Hence one law satisfies every hinge constraint with rational $t \ge 0$. The inequality $|h_t - h_s| \le |t - s|$ extends the constraints to all $t \ge 0$, giving a feasible law of entropy at most C. The moment bound above ensures that both spectral and scalar first moments are finite.

□

For interval indicators, take η to be γ_+ conditioned on $[0,a]$, where $a>0$. Its entropy is $-\log q(a)$, and the hinge h_a forces every feasible ν to be supported on $E_a=\{x:\|X(x)\|\le \kappa a\}$. The least relative entropy among all laws supported on E_a is $-\log \gamma_m(E_a)$, attained by conditioning. Hence

$$-\log \gamma_m(E_a)\le \mathcal{I}_{X,\kappa}(\eta)\le -d\log q(a),$$

which recovers the small-ball bound. The support constraint alone may be weaker than the order constraint $\sigma_{\nu,\kappa}\preceq_{\mathrm{icx}}\eta$ on the average absolute spectral law.

Appendix E. The conjectured dilation

This appendix proves Theorem 4.3 and the conjectured dilation under an ellipsoidal norm bound (Theorem E.2), applies the latter to the anticommuting family at large order, and collects the exact cases that support Conjecture 4.1: commuting coefficients, quadratic penalties, and the two-Pauli matrix Gaussian series.

E.1. **Proof of Theorem 4.3.** A Gaussian change of covariance reduces the linear–quadratic comparison to the following scalar inequality.

For $s>0$, define

$$D(s)=\frac{s^2-1-2\log s}{2},\qquad h(a)=-\log \mathbb{E}e^{-a|G|},\qquad c_0=\sqrt{2/\pi}.$$

The number $D(s)$ is the relative entropy per coordinate of $N(0,s^2I)$ with respect to standard Gaussian measure.

Lemma E.1 (A scalar Gaussian variational comparison)**.** *For $0<\theta\le\theta_*=1/(\pi-2)$ and $a\ge 0$,*

$$H_\theta(a):=\inf_{s>0}\{\theta D(s)+ac_0 s\}\le h(a). \tag{E.1}$$

The constant θ_* is the largest value of θ for which this scalar inequality holds for every $a\ge 0$.

Section E.2 proves the lemma by comparing the derivatives of h and H_{θ_*} through a Riccati equation; the optimality of θ_* is a second-order condition at $a=0$.

Proof of Theorem 4.3. Remove the kernel of the coefficient map, and put $m=r_X$ and $\theta=m/d$. The case $X=0$ is immediate. Since $\operatorname{tr}V_X\le d$,

$$\mathbb{E}\operatorname{tr}|X|\le\sqrt{d\,\mathbb{E}\operatorname{tr}X^2}\le d,\qquad \mathbb{E}\operatorname{tr}X^2\le d.$$

Use $N(0,s^2I_m)$ as the trial law ξ in the Gibbs variational identity (4.3). As $\kappa_*^{-1}=c_0$, it gives

$$-d^{-1}\log\mathcal{Z}_{at+bt^2}(X;\kappa_*)\le\inf_{s>0}\{\theta D(s)+ac_0s+bc_0^2s^2\}.$$

Set $w=1+2bc_0^2/\theta$ and $v=1+2b$. Substituting $u=s\sqrt{w}$ shows that the right-hand side is

$$\frac{\theta}{2}\log w+H_\theta(a/\sqrt{w})\le\frac{\theta}{2}\log w+h(a/\sqrt{w}).$$

On the scalar side, Gaussian rescaling gives the exact identity

$$-\log\mathbb{E}e^{-a|G|-bG^2}=\frac{1}{2}\log v+h(a/\sqrt{v}). \tag{E.2}$$

Two elementary facts compare these expressions. First, $xh'(x)\le 1$ for $x\ge 0$: under the half-normal tilt by $e^{-x|G|}$, integration by parts gives $1=\mathbb{E}_xG^2+x\mathbb{E}_x|G|$. Therefore

$$h(cx)-h(x)\le\log c\qquad(c\ge 1).$$

Second, $t\log(1+z/t)$ increases with $t>0$ for $z\ge 0$. If $w\ge v$, monotonicity of h and $\theta\le 1$ imply

$$\theta\log w\le\log(1+2bc_0^2)\le\log v,\qquad h(a/\sqrt{w})\le h(a/\sqrt{v}).$$

If $w < v$, then

$$h(a/\sqrt{w}) - h(a/\sqrt{v}) \le \tfrac{1}{2}\log(v/w),$$

so the excess over (E.2) is at most $(\theta-1)\log w/2 \le 0$. Hence (4.4) holds.

For sharpness, use the anticommuting families from Section 3.1. They satisfy $m/d \to 0$ and $d^{-1}\mathbb{E}\,\mathrm{tr}\,|X| \to 1$. The linear penalties with strength tending to zero force $\kappa \ge 1/\mathbb{E}|G| = \kappa_*$. □

E.2. **The scalar variational comparison.** We prove Lemma E.1 by comparing derivatives.

Let $\sigma = c_0^2 = 2/\pi$, $\beta = 2(1-\sigma)$, and $\theta_* = \sigma/\beta = 1/(\pi-2)$. Since $D(s) \ge 0$, $H_\theta(a) \le H_{\theta_*}(a)$ whenever $\theta \le \theta_*$. It suffices to treat $\theta = \theta_*$.

Recall the Mills ratio M of Section 3.1 in integral form, and put $m = h'$:

$$M(a) = \int_0^\infty e^{-at-t^2/2}\,dt, \qquad m(a) = h'(a) = \frac{1}{M(a)} - a.$$

The measure proportional to $e^{-at-t^2/2}\,dt$ is the half-normal law tilted by $e^{-a|G|}$. Thus $m(a) > 0$, $m(0) = c_0$, and $m'(a) = -\operatorname{Var}_a(|G|) \le 0$. Integrating a derivative gives $M'(a) = aM(a) - 1$, so

$$m'(a) = am(a) + m(a)^2 - 1. \tag{E.3}$$

The objective defining H_{θ_*} is strictly convex in $s > 0$ and has a unique minimizer. By its first-order equation and the envelope identity, $\ell(a) := H'_{\theta_*}(a)$ is the positive root of

$$\ell(a)^2 + \beta a\ell(a) - \sigma = 0, \qquad \ell(0) = c_0, \qquad \ell'(a) = -\frac{\beta\ell(a)^2}{\ell(a)^2+\sigma}. \tag{E.4}$$

Since m solves the Riccati equation (E.3), we compare ℓ with m through that equation: the residual below measures the failure of ℓ to solve it, and its sign pattern, with an integrating factor, gives $m \ge \ell$. Put $w = \ell(a)^2/\sigma$. As a increases from zero to infinity, w decreases continuously from one to zero. The Riccati residual of ℓ is

$$\begin{aligned}\mathcal{R}(a) &:= \ell'(a) - a\ell(a) - \ell(a)^2 + 1 \\ &= \frac{(w-1)\{\sigma(2\sigma-1)w + 3\sigma - 2\}}{2(1-\sigma)(1+w)}.\end{aligned} \tag{E.5}$$

To obtain it, substitute $a\ell = (\sigma - \ell^2)/\beta$ into (E.4) and clear the positive denominator.

The elementary bounds $3 < \pi < 22/7$ imply $7/11 < \sigma < 2/3$. Therefore $\sigma(2\sigma-1) > 0$, $3\sigma - 2 < 0$, and $\sigma(2\sigma-1) + 3\sigma - 2 = 2(\sigma^2 + \sigma - 1) > 0$. There is exactly one $a_0 > 0$ at which the braces in (E.5) vanish. Consequently

$$\mathcal{R}(a) < 0 \quad (0 < a < a_0), \qquad \mathcal{R}(a) > 0 \quad (a > a_0).$$

Let $\Delta = m - \ell$ and define

$$J(a) = \exp\left\{-\int_0^a [t + m(t) + \ell(t)]\,dt\right\}.$$

Equations (E.3)–(E.4) give $(J\Delta)' = -J\mathcal{R}$ and $\Delta(0) = 0$. On $[0, a_0]$, integration from zero shows $J\Delta \ge 0$. On $[a_0, \infty)$, integrate from infinity instead:

$$J(a)\Delta(a) = \int_a^\infty J(t)\mathcal{R}(t)\,dt \ge 0.$$

The endpoint is justified because $0 < m, \ell \le c_0$ and $J(a) \le e^{-a^2/2}$, so $J(a)\Delta(a) \to 0$. Thus $h'(a) \ge H'_{\theta_*}(a)$ for every $a \ge 0$. Both functions vanish at zero, proving (E.1).

For the optimality of the scalar threshold, Taylor expansion at zero gives

$$h(a) = c_0 a - \frac{1-c_0^2}{2}a^2 + O(a^3), \qquad H_\theta(a) = c_0 a - \frac{c_0^2}{4\theta}a^2 + O(a^3).$$

If $H_\theta \le h$ for all small positive a, then $\theta \le c_0^2/[2(1-c_0^2)] = \theta_*$. This bound on θ is an obstruction to the scalar variational estimate with only its stated first-moment bound; it is not a counterexample to the determinant inequality at the conjectured dilation when $r_X = d$.

E.3. **All functions under an ellipsoidal norm bound.** An ellipsoidal bound on the operator norm lets us compare the trace penalty with scalar penalties of Gaussian projections even when the absolute eigenvalues differ.

Theorem E.2 (The conjectured dilation under an ellipsoidal bound)**.** *Let X be a matrix Gaussian series. Suppose that a positive definite matrix Q satisfies*

$$\|X(x)\| \le \sqrt{x^{\mathsf{T}} Q x} \quad (\textit{every } x), \qquad \operatorname{tr} Q \le 1. \tag{E.6}$$

For a standard Gaussian vector z of the same dimension as x, put

$$m_Q = \mathbb{E}\sqrt{z^{\mathsf{T}} Q z}, \qquad \delta_Q = \frac{\sqrt{\lambda_{\min}(Q)}}{\kappa_*}\left(\frac{1}{m_Q} - 1\right) > 0.$$

If $d > 16\delta_Q^{-2}$, then X satisfies the determinant inequality at dilation κ_*: for every even log-concave function f,

$$\mathbb{E}\det f(X) \ge (\mathbb{E} f(\kappa_* G))^d.$$

The ellipsoidal bound (E.6) already implies $V_X \preceq I_d$: since $B_i = X(e_i)$, one has $B_i^2 \preceq Q_{ii} I_d$, hence $V_X \preceq (\operatorname{tr} Q) I_d \preceq I_d$.

The proof starts by choosing unit vectors $u_1, \dots, u_d$ such that

$$\frac{1}{d}\sum\nolimits_{j=1}^{d} |\langle u_j, x\rangle| \ge \frac{\sqrt{x^{\mathsf{T}} Q x}}{\kappa_*} \qquad (\text{every } x). \tag{E.7}$$

The strict inequality $m_Q < 1$ provides the margin needed to approximate a spherical average by d projections. The following sampling estimate does so for any distribution of directions; it uses only that each direction has unit norm.

Lemma E.3. *Let ρ be a probability measure on the unit sphere in $\mathbb{R}^r$. For every integer $d \ge 1$, there are unit vectors $u_1, \dots, u_d$ such that*

$$\sup_{\|x\|_2 \le 1} \left| \frac{1}{d}\sum\nolimits_{j=1}^{d} |\langle u_j, x\rangle| - \int |\langle u, x\rangle| \, d\rho(u) \right| \le \frac{4}{\sqrt{d}}.$$

Proof. Let $U_1, \dots, U_d$ be independent with law ρ, and let $\varepsilon_1, \dots, \varepsilon_d$ be independent uniform signs. Write $h_\rho(x) = \int |\langle u, x\rangle| \, d\rho(u)$. Symmetrization and contraction give

$$\begin{aligned} &\mathbb{E} \sup_{\|x\|_2 \le 1} \left| \frac{1}{d}\sum\nolimits_j |\langle U_j, x\rangle| - h_\rho(x) \right| \\ &\qquad \le \frac{2}{d}\mathbb{E} \sup_{\|x\|_2 \le 1} \left| \sum\nolimits_j \varepsilon_j |\langle U_j, x\rangle| \right| \le \frac{4}{d}\mathbb{E} \left\| \sum\nolimits_j \varepsilon_j U_j \right\|_2 \le \frac{4}{\sqrt{d}}. \end{aligned}$$

The first inequality follows by introducing an independent copy of the U_j, applying Jensen's inequality, and then using the symmetry of each difference and the triangle inequality. To verify contraction, for any finite family (A_t, b_t) and any 1-Lipschitz function ϕ one has

$$\frac{1}{2}\sup_t (A_t + \phi(b_t)) + \frac{1}{2}\sup_t (A_t - \phi(b_t)) \le \frac{1}{2}\sup_t (A_t + b_t) + \frac{1}{2}\sup_t (A_t - b_t).$$

Combine the two suprema on the left into a supremum over (s, t) and use $\phi(b_s) - \phi(b_t) \le |b_s - b_t|$; interchanging s and t removes the absolute value on the right. Apply this inequality successively to the signs, with $\phi(t) = |t|$. The absolute supremum costs at most a factor of two, since the index set contains $x = 0$ and the two signs of the sum have the same law. Finite subsets dense in the unit

ball give the stated bound by continuity. Finally, $\mathbb{E}\|\sum_j \varepsilon_j U_j\|_2^2 = d$. Some realization has error at most its expectation, proving the claim. □

Proof of Theorem E.2. We first construct unit vectors satisfying (E.7). Let $v = Q^{1/2}z$ and define a probability law ρ on the unit sphere by

$$\int h(u)\,d\rho(u) = \frac{1}{m_Q}\mathbb{E}\left[\|v\|_2 h\Big(\frac{v}{\|v\|_2}\Big)\right].$$

Since $\mathbb{E}|G| = 1/\kappa_*$, this law satisfies

$$\int |\langle u, x\rangle|\,d\rho(u) = \frac{\sqrt{x^{\mathsf{T}}Qx}}{\kappa_* m_Q} \geq \frac{\sqrt{x^{\mathsf{T}}Qx}}{\kappa_*} + \delta_Q\|x\|_2.$$

Strict Jensen gives $m_Q < \sqrt{\operatorname{tr} Q} \leq 1$. Lemma E.3 approximates this average uniformly on the unit ball with error at most $4/\sqrt{d} < \delta_Q$, since $d > 16\delta_Q^{-2}$; homogeneity gives (E.7).

Now normalize $f(0) = 1$ and write $f(t) = e^{-\psi(|t|)}$; the zero function is immediate. The norm bound, (E.7), and convexity give

$$\operatorname{tr}\psi(|X|/\kappa_*) \leq d\psi\left(\frac{\sqrt{g^{\mathsf{T}}Qg}}{\kappa_*}\right) \leq \sum\nolimits_{j=1}^{d}\psi(|\langle u_j, g\rangle|).$$

Each scalar projection is standard Gaussian. Their interval events are symmetric and convex, so the Gaussian correlation inequality and layer cake give

$$\mathbb{E}\det f(X/\kappa_*) \geq \mathbb{E}\prod_{j=1}^{d} f(\langle u_j, g\rangle) \geq (\mathbb{E}f(G))^d.$$

The inequalities remain valid for extended-valued convex penalties; Lemma A.2 handles support-endpoint values. □

E.4. **Radial matrix Gaussian series and compressions.** The isotropic case of Theorem E.2 applies to the anticommuting family that forces the lower bound κ_*.

Proposition E.4 (Radial matrix Gaussian series)**.** *For an integer $r \geq 1$, put*

$$\delta_r = \frac{\Gamma(r/2)}{\sqrt{\pi}\,\Gamma((r+1)/2)} - \sqrt{\frac{2}{\pi r}} > 0.$$

Suppose a real symmetric matrix Gaussian series X of order d satisfies $|X| = (\chi_r/\sqrt{r})I_d$, where χ_r is the Euclidean norm of its standard Gaussian coefficient vector in $\mathbb{R}^r$. If $d > 16\delta_r^{-2}$, then the determinant inequality holds at κ_* for every even log-concave function. In particular, it holds for all sufficiently large members of the anticommuting family of order $d = 2^r$ in Section 3.1, and κ_* is optimal uniformly over those members.

Proof. The Gaussian law has full support, so continuity extends the identity everywhere. Apply Theorem E.2 with $Q = I_r/r$. The radial integral gives $\mathbb{E}\chi_r = \sqrt{2}\,\Gamma((r+1)/2)/\Gamma(r/2)$, hence

$$\delta_Q = \frac{1}{\kappa_*\mathbb{E}\chi_r} - \frac{1}{\kappa_*\sqrt{r}} = \delta_r = \frac{\sqrt{2/\pi}}{4r^{3/2}} + O(r^{-5/2}),$$

where the expansion follows from Stirling's formula. Thus $16\delta_r^{-2} = O(r^3) < 2^r$ for all sufficiently large r. The anticommuting lower bound in Section 3.1 proves uniform optimality of κ_*. □

Compressions give nonradial examples. Let $C_1, \dots, C_r$ be symmetric anticommuting involutions of order N, and let $T\colon \mathbb{R}^d \to \mathbb{R}^N$ be an isometry. The matrix Gaussian series with

$$X(x) = T^{\mathsf{T}}\left(\sum\nolimits_{i=1}^{r}(Q^{1/2}x)_i C_i\right)T$$

satisfies (E.6), because the square of the matrix in parentheses is $(x^{\mathsf{T}}Qx)I_N$. Compression need not preserve its equal absolute eigenvalues. Theorem E.2 therefore applies to every such compression whose order d exceeds $16\delta_Q^{-2}$.

E.5. **Orthogonal sums and averages of conjugates.** Fixed orthogonal blocks give another way to combine the proved cases. If matrix Gaussian series X_ℓ with common Gaussian coefficients each satisfy the determinant inequality at dilation κ, then so does their orthogonal sum $\bigoplus_\ell X_\ell$, even when the blocks are dependent: determinants factor over orthogonal blocks, each factor $\det f(X_\ell/\kappa)$ has symmetric convex superlevel sets in the Gaussian coefficients, and the Gaussian correlation inequality bounds the expectation of the product below by the product of the expectations. Averaging orthogonal conjugates then produces further matrix Gaussian series without requiring an ellipsoidal bound for their operator norm.

Proposition E.5 (Averages of orthogonal conjugates)**.** *If X satisfies the determinant inequality at dilation κ and*

$$\Phi(M) = \sum\nolimits_{j=1}^{N} p_j O_j M O_j^{\mathsf{T}}, \qquad p_j \ge 0, \quad \sum\nolimits_j p_j = 1,$$

where the O_j are fixed orthogonal matrices, then $\Phi(X)$ satisfies the determinant inequality at the same dilation. This operation preserves $V_X \preceq I_d$ and cannot increase coefficient rank.

Proof. Fix a normalized function f and its penalty ψ. Convexity and orthogonal invariance of $M \mapsto \operatorname{tr}\psi(|M|)$ give

$$\operatorname{tr}\psi(|\Phi(M)|/\kappa) \le \sum\nolimits_j p_j \operatorname{tr}\psi(|O_j M O_j^{\mathsf{T}}|/\kappa) = \operatorname{tr}\psi(|M|/\kappa).$$

Thus $\det f(\Phi(M)/\kappa) \ge \det f(M/\kappa)$ pointwise. For a symmetric B, writing $M_j = O_j B O_j^{\mathsf{T}}$ gives

$$\Phi(B^2) - \Phi(B)^2 = \sum\nolimits_j p_j (M_j - \Phi(B))^2 \succeq 0.$$

Summing over coefficients yields $V_{\Phi(X)} \preceq \Phi(V_X) \preceq I_d$ whenever $V_X \preceq I_d$. Linearity of Φ proves the rank assertion. □

E.6. **Commuting coefficients and special penalties.** The shape of the function and the coefficient geometry permit smaller dilations than 13/5.

Theorem E.6 (Commuting coefficients)**.** *Suppose the B_i commute and $V_X \preceq I_d$. Then $\mathbb{E}\det f(X) \ge (\mathbb{E}f(G))^d$ for every even log-concave function f. More generally, any even nonnegative Borel function that is nonincreasing on $[0,\infty)$ suffices. The constant one is optimal uniformly over matrix Gaussian series with commuting coefficients.*

Proof. Simultaneous diagonalization gives coordinates $Z_j = \langle b_j, g\rangle$ with $\|b_j\|_2 \le 1$. The preimages of the interval superlevel sets of f are symmetric and convex, including when $b_j = 0$. Gaussian correlation and layer cake give

$$\mathbb{E}\prod_{j=1}^{d} f(Z_j) \ge \prod_{j=1}^{d} \mathbb{E}f(Z_j) \ge (\mathbb{E}f(G))^d,$$

where the last step uses $Z_j \overset{d}{=} \|b_j\|_2 G$ and monotonicity. The diagonal series $\operatorname{diag}(g_1,\dots,g_d)$ with $f(t) = e^{-t^2}$ rules out any dilation below one. □

Theorem E.7 (Exact quadratic scale)**.** *Let $r_X \le d$ and put $\tau = d^{-1}\operatorname{tr} V_X$. If $\tau > 0$, then*

$$\mathbb{E}e^{-t\operatorname{tr} X^2/\kappa^2} \ge (\mathbb{E}e^{-tG^2})^d \quad \text{for every } t \ge 0$$

holds with $\kappa = \sqrt{\tau}$. This value is the smallest dilation for which the displayed inequality holds for every $t \ge 0$ for the given matrix Gaussian series. In particular, $V_X \preceq I_d$ permits $\kappa = 1$.

Proof. The Gram matrix $Q = [\operatorname{tr}(B_i B_j)]_{ij}$ has rank $r_X \le d$. Pad its nonzero eigenvalues with zeros to obtain $\lambda_1, \dots, \lambda_d \ge 0$, with $\sum_j \lambda_j = d\tau$. The Gaussian quadratic-form identity and arithmetic–geometric mean give

$$\mathbb{E}e^{-t \operatorname{tr} X^2/\kappa^2} = \prod_{j=1}^{d} (1 + 2t\lambda_j/\kappa^2)^{-1/2} \ge (1 + 2t\tau/\kappa^2)^{-d/2}.$$

At $\kappa^2 = \tau$ this lower bound is the required scalar benchmark. Differentiating the proposed inequality at $t = 0$ shows that any valid dilation must satisfy $\kappa^2 \ge \tau$. If $\tau = 0$, then $X = 0$ and the displayed inequality is immediate for every positive dilation. □

E.7. **The real two-Pauli matrix Gaussian series.** The two-Pauli example distinguishes the optimal dilation for even log-concave functions from the optimal dilation for interval indicators. Comparing a diamond or square with a disk of the same area gives the upper bounds.

Theorem E.8 (Optimal constants for the real two-Pauli matrix Gaussian series)**.** *Let*

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad X = \frac{g_1 Z + g_2 H}{\sqrt{2}}.$$

The smallest dilation valid simultaneously for all even log-concave functions is $2/\sqrt{\pi}$*. The smallest dilation for all interval indicators is* $\sqrt{2/\pi}$*.*

Proof. Since $ZH = -HZ$ and $Z^2 = H^2 = I_2$, we have $|X| = RI_2$, where $R = \chi_2/\sqrt{2}$ and $\chi_2 = (g_1^2 + g_2^2)^{1/2}$. For the upper bounds, the zero function is immediate; otherwise normalize $f(0) = 1$ and apply Lemma A.2 to obtain a closed penalty ψ with $\mathbb{E} \det f(X/\kappa) = \mathbb{E}e^{-2\psi(R/\kappa)}$.

Among planar sets of a fixed finite area, the centered disk has largest standard Gaussian measure. Indeed, if A and the centered disk B have equal area, the Gaussian density on $A \setminus B$ is at most its boundary value on B, whereas on $B \setminus A$ it is at least that value. The two differences have equal area, so $\gamma_2(A) \le \gamma_2(B)$.

To apply this comparison to all functions, put $S = (|g_1| + |g_2|)/2$ and $c_2 = 2/\sqrt{\pi}$. The diamond $\{|g_1| + |g_2| \le 2t\}$ has area $8t^2$, equal to that of the disk of radius $2\sqrt{2/\pi}\, t = \sqrt{2}\, c_2 t$. Thus $\mathbb{P}(R/c_2 \le t) \ge \mathbb{P}(S \le t)$ for every $t \ge 0$. Since $e^{-2\psi}$ is nonincreasing, this probability comparison and convexity of ψ give

$$\mathbb{E}e^{-2\psi(R/c_2)} \ge \mathbb{E}e^{-2\psi(S)} \ge \mathbb{E}e^{-\psi(|g_1|)-\psi(|g_2|)} = (\mathbb{E}e^{-\psi(|G|)})^2.$$

Extended-valued penalties are covered too: their exponentials are bounded and nonincreasing.

For interval indicators, the square $[-t, t]^2$ has area $4t^2$, equal to that of the disk of radius $2t/\sqrt{\pi}$. The disk comparison gives $\mathbb{P}(\chi_2 \le 2t/\sqrt{\pi}) \ge q(t)^2$, proving the bound at $\kappa = \sqrt{2/\pi}$.

For necessity, the density of R is $2re^{-r^2}$ on $r > 0$. For the linear penalty $\psi(t) = at$, differentiation at $a = 0$ forces only $\kappa \ge \mathbb{E}R/\mathbb{E}|G| = \pi/(2\sqrt{2})$. The stronger restriction comes from large a:

$$\mathbb{E}e^{-2aR/\kappa} \sim \frac{\kappa^2}{2a^2}, \qquad (\mathbb{E}e^{-a|G|})^2 \sim \frac{2}{\pi a^2} \quad (a \to \infty). \tag{E.8}$$

Thus $\kappa \ge 2/\sqrt{\pi}$. For an interval indicator of radius $r \downarrow 0$,

$$\mathbb{P}(R \le \kappa r) = 1 - e^{-\kappa^2 r^2} \sim \kappa^2 r^2, \qquad q(r)^2 \sim (2/\pi) r^2,$$

so $\kappa \ge \sqrt{2/\pi}$. Each asymptotic follows from rescaling and dominated convergence. □