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A PROOF OF PETZ'S CONJECTURE ON THE SCALAR CURVATURE OF THE KUBO–MORI METRIC

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Abstract. Petz (1994) conjectured that the scalar curvature of the Kubo–Mori metric on invertible density matrices increases as the state becomes more mixed. Dittmann (2000) reduced the conjecture to the concavity of an explicit symmetric function h on $(0,\infty)^3$. We prove that h is strictly concave, and hence Petz's conjecture. The proof writes $h = -1/(8M)$ with M positive and homogeneous of degree one, and its central idea is to show that $t \mapsto 1/M(X+tY)$ is a Stieltjes function for all $X, Y > 0$; Cauchy–Schwarz and homogeneity then give concavity of h on $(0,\infty)^3$.

1. Introduction

The Kubo–Mori metric, also called the Bogoliubov–Kubo–Mori metric, is one of the monotone Riemannian metrics on the manifold of invertible $n \times n$ density matrices (Petz, 1994, 1996). For two such matrices ω_1 and ω_2, call ω_1 *more mixed* than ω_2 if, for each k, the sum of the k largest eigenvalues of ω_1 is at most the corresponding sum for ω_2. Petz (1994) conjectured that the scalar curvature of the Kubo–Mori metric increases under this order: $\mathrm{Scal}(\omega_1) \geq \mathrm{Scal}(\omega_2)$ whenever ω_1 is more mixed than ω_2 (Dittmann, 2000, Section VI).

Dittmann (2000) reduced this conjecture to the concavity of one explicit function. His curvature calculation leads to a symmetric function of three positive variables, written here in terms of the Morozova–Chentsov kernel of the Kubo–Mori metric. This kernel is the reciprocal of the logarithmic mean: for $x, y > 0$,

$$L(x,y) = \frac{x-y}{\log x - \log y} = \int_0^1 x^{1-s} y^s \, \mathrm{d}s,$$

with the usual continuous value $L(x,x) = x$, and

$$c(x,y) := \frac{1}{L(x,y)} = \frac{\log x - \log y}{x-y}, \qquad c(x,x) = \frac{1}{x}. \tag{1.1}$$

For pairwise distinct $x, y, z > 0$, define

$$\begin{aligned} h(x,y,z) := {} & \frac{(y-z)\,c(x,y)c(x,z) - (x-z)\,c(x,y)c(y,z) + (x-y)\,c(x,z)c(y,z)}{6(x-y)(x-z)(y-z)\,c(x,y)c(x,z)c(y,z)} \\ & + \frac{-xy(x-y)\,c(x,y) + xz(x-z)\,c(x,z) - yz(y-z)\,c(y,z)}{3xyz(x-y)(x-z)(y-z)\,c(x,y)c(x,z)c(y,z)}. \end{aligned} \tag{1.2}$$

All diagonal and partial-diagonal values are understood by continuous extension. Dittmann stated the concavity of this function as a labelled *Assertion*, writing that he had no formal proof of it but that it was confirmed by several million numerical tests (Dittmann, 2000, Eq. (61) and the discussion following it). He proved that the Assertion implies Petz's conjecture (Dittmann, 2000, Theorem 3).

Key words and phrases. Kubo–Mori metric, scalar curvature, logarithmic mean, strict concavity, Stieltjes function, Morozova–Chentsov kernel.

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The Assertion has remained unproved since 2000, and Petz's conjecture with it. Two later works approach the same conjecture. Andai (2003a) regroups the summands of the scalar curvature and proves monotonicity for some of them, leaving the remainder supported only by numerics; Gibilisco and Isola (2005) conjecture that the Wigner–Yanase–Dyson metrics near the Kubo–Mori metric have monotone scalar curvature, which would imply Petz's conjecture (Gibilisco and Isola, 2005, Conjecture 8.1). Neither closes the gap, and neither establishes the Assertion. Not every monotone metric has monotone scalar curvature: Andai (2003b) exhibits monotone metrics for which it fails.

Our main result is the following.

Theorem 1.1 (Strict concavity of Dittmann's function)**.** *The function h defined by (1.2) extends real-analytically to $(0,\infty)^3$ and is strictly concave there. That is, for all distinct $X, Y \in (0,\infty)^3$ and every $0 < \lambda < 1$,*

$$h(\lambda X + (1-\lambda)Y) > \lambda h(X) + (1-\lambda)h(Y).$$

Theorem 1.1 proves Dittmann's Assertion, so his Theorem 3 gives the following.

Corollary 1.2 (Petz's conjecture)**.** *Let ω_1 and ω_2 be invertible $n \times n$ density matrices. If ω_1 is more mixed than ω_2, then the scalar curvature of the Kubo–Mori metric satisfies* $\mathrm{Scal}(\omega_1) \geq \mathrm{Scal}(\omega_2)$.

Corollary 1.2 rests on Dittmann's Theorem 3, whose proof we use as published and do not re-derive. Our own work is the proof of Theorem 1.1. It passes through a positive, symmetric function M on $(0,\infty)^3$, homogeneous of degree one, defined in (2.13) below, such that

$$h = -\frac{1}{8M}. \tag{1.3}$$

Set

$$m(a,b) := M(a,b,1). \tag{1.4}$$

By homogeneity $M(x,y,z) = z\, m(x/z, y/z)$, so M is the perspective of m, and concavity of M is equivalent to concavity of m. The two-variable form is the statement that a Hessian computation would target, and we record it separately.

Theorem 1.3 (Strict concavity in two variables)**.** *The function m in (1.4) is strictly concave on $(0,\infty)^2$.*

The proof has three by-products that do not depend on Petz's conjecture. The normal form of Section 2 writes Dittmann's two-line expression (1.2) as $h = -1/(8M)$ with M positive, symmetric, homogeneous of degree one, and normalized by $M(x,x,x) = x$. Theorem 2.10 shows that $1/M$ is a Stieltjes function along every line with positive base point and positive direction, so M is operator monotone along such lines (Remark 2.12), which is more than the concavity the proof uses. Lemma 2.9 shows that a function homogeneous of degree one that is concave along lines with positive direction is concave on the whole cone, which is why positive directions suffice.

1.1. **Idea of the proof.** By (1.3) and homogeneity, concavity of h follows from concavity of m, so the direct route is to show that the 2×2 Hessian of m is negative semidefinite at every point of $(0,\infty)^2$. Its entries are second derivatives of a rational expression in logarithmic divided differences. The route we take replaces this second-order condition by a sign condition that involves no derivatives. For $X, Y > 0$, consider the restriction $t \mapsto F(X+tY) = 1/M(X+tY)$ of F to a line with positive base point and positive direction. The resolvent integrals behind the normal form continue this function analytically to the complex plane cut along a compact interval of the negative axis, and we show that it is a Stieltjes function, the integral of $1/(t-s)$ against a finite positive measure on that interval (Theorem 2.10). The Stieltjes property gives concavity for free: a single Cauchy–Schwarz inequality shows that the reciprocal of a Stieltjes function is concave, so M is concave along each of these lines, and the same property makes each of these restrictions of M operator monotone (Remark 2.12). Degree-one homogeneity then shows that positive directions already control every

chord of the cone: concavity along lines with positive direction gives superadditivity of M, hence concavity, and excluding point masses gives the strict statements.

The Stieltjes property is where the work lies. We continue $t \mapsto F(X + tY)$ analytically to the plane cut along a compact interval of the negative axis, and a keyhole contour represents it through its boundary values on the cut. The representing measure is positive when the upper boundary value has nonpositive imaginary part. This exchange is the point of the detour: concavity, a second-order condition at every interior point, becomes a sign condition on the boundary values alone. On the cut, the three coordinates of $X + sY$ are real with mixed signs, and symmetry, homogeneity and Schwarz reflection reduce them to $(-1, p, q)$ with $p, q > 0$. There the logarithms are explicit, since the logarithm of a negative number is $\log|x| \pm \mathrm{i}\pi$, and the sign condition becomes the positivity of the function $N(p, q)$ of Lemma 3.2, which we prove in logarithmic coordinates by elementary estimates for hyperbolic functions. No computer calculation is part of the proof, and the only numerical facts it uses are $\pi > 3$, $\pi^2 < 10$ and $e > 8/3$.

The argument uses two properties of the logarithmic mean: its resolvent integral, which gives h the nonsingular form of Section 2, and the explicit values of the principal logarithm on the negative axis, which turn the sign condition of Section 3.2 into an explicit inequality. It is therefore specific to the Kubo–Mori metric, and it does not say which other monotone metrics have monotone scalar curvature; the conjecture of Gibilisco and Isola (2005), which places the Kubo–Mori metric at the end of a one-parameter family of such metrics, is not addressed here.

Section 2 derives the normal form (1.3) and reduces both theorems to the Stieltjes property, Section 3 proves it, and the elementary estimates behind the boundary sign are in Appendix A.

2. The normal form and the reduction to lines

This section writes h as $-1/(8M)$ for an explicit positive function M, removing the apparent singularities of (1.2), and then reduces the concavity of M to a Stieltjes property of $1/M$ along lines, Theorem 2.10.

2.1. Resolvent integrals.

For $x, y, z > 0$, set

$$T(x,y,z) := \int_0^\infty \frac{\mathrm{d}t}{(t+x)(t+y)(t+z)}, \qquad U(x,y,z) := \int_0^\infty \frac{t\ \mathrm{d}t}{(t+x)(t+y)(t+z)}. \tag{2.1}$$

We shall usually abbreviate

$$A := c(x,y), \qquad B := c(x,z), \qquad C := c(y,z). \tag{2.2}$$

Lemma 2.1. *For $x, y, z > 0$,*

$$c(x,y) = \int_0^\infty \frac{\mathrm{d}t}{(t+x)(t+y)}, \tag{2.3}$$

and

$$A = U + zT, \qquad B = U + yT, \qquad C = U + xT. \tag{2.4}$$

Proof. The identity (2.3) follows from partial fractions. Then

$$A = \int_0^\infty \frac{t+z}{(t+x)(t+y)(t+z)}\,\mathrm{d}t = U + zT,$$

and the other two identities are cyclic variants. □

2.2. **Exact reduction of h.** Define

$$Q(x,y,z) := \frac{2U(x,y,z)}{xyz} - T(x,y,z)^2. \tag{2.5}$$

Proposition 2.2 (Nonsingular normal form)**.** *For all $x,y,z>0$,*

$$h(x,y,z) = -\frac{Q(x,y,z)}{6ABC}. \tag{2.6}$$

Consequently, h has a real-analytic extension through every partial diagonal.

Proof. It suffices first to work with pairwise distinct variables. In the abbreviations (2.2), the first fraction in (1.2) reads

$$\frac{(y-z)AB-(x-z)AC+(x-y)BC}{6(x-y)(x-z)(y-z)ABC}. \tag{2.7}$$

By (2.4),

$$B = A+(y-z)T, \qquad C = A+(x-z)T.$$

A direct expansion gives

$$(y-z)AB-(x-z)AC+(x-y)BC = (x-y)(x-z)(y-z)T^2. \tag{2.8}$$

Hence (2.7) equals $T^2/(6ABC)$.

The numerator of the second fraction is

$$-xy(x-y)A + xz(x-z)B - yz(y-z)C.$$

Substituting (2.4) and simplifying yields

$$-xy(x-y)A + xz(x-z)B - yz(y-z)C = -(x-y)(x-z)(y-z)U. \tag{2.9}$$

Thus the second fraction equals $-U/(3xyzABC)$, proving (2.6). The integral expressions in Lemma 2.1 are real-analytic on the positive cone, including coincident variables, and $ABC>0$; therefore the extension is analytic. □

2.3. **Positivity of Q.** Let

$$\Delta_2 := \{\lambda\in[0,\infty)^3 : \lambda_1+\lambda_2+\lambda_3 = 1\}$$

with ordinary two-dimensional Lebesgue measure, and write

$$s_\lambda := \lambda_1 x+\lambda_2 y+\lambda_3 z, \qquad I_k := \int_{\Delta_2} s_\lambda^{-k}\,\mathrm{d}\lambda.$$

Lemma 2.3 (Simplex representation)**.** *One has*

$$U = I_1, \qquad T = I_2, \qquad I_3 = \frac{1}{2xyz}. \tag{2.10}$$

Consequently,

$$Q = 4I_1I_3 - I_2^2 > 0. \tag{2.11}$$

Proof. The elementary Feynman-parameter identity

$$\frac{1}{abc} = 2\int_{\Delta_2}\frac{\mathrm{d}\lambda}{(\lambda_1 a+\lambda_2 b+\lambda_3 c)^3} \tag{2.12}$$

holds for $a,b,c>0$. Applying it to $a=t+x, b=t+y, c=t+z$, followed by integration in t, gives

$$T = 2\int_{\Delta_2}\int_0^\infty (t+s_\lambda)^{-3}\,\mathrm{d}t\,\mathrm{d}\lambda = I_2,$$

whereas

$$U = 2\int_{\Delta_2}\int_0^\infty t(t+s_\lambda)^{-3}\,\mathrm{d}t\,\mathrm{d}\lambda = I_1.$$

Putting $t = 0$ in (2.12) gives $1/(xyz) = 2I_3$. Finally, Cauchy–Schwarz gives

$$I_2^2 = \left(\int_{\Delta_2} s_\lambda^{-1/2} s_\lambda^{-3/2} \, \mathrm{d}\lambda \right)^2 \le I_1 I_3.$$

Thus $Q = 4I_1 I_3 - I_2^2 \ge 3I_1 I_3 > 0$; equivalently one may use $Q \ge 3I_2^2 > 0$. □

Remark 2.4. *Formula (2.12) can be verified directly from*

$$\frac{1}{abc} = \int_{[0,\infty)^3} e^{-au-bv-cw} \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w$$

by the change of variables $(u, v, w) = r(\lambda_1, \lambda_2, \lambda_3)$.

We now introduce the functions that drive the proof.

Definition 2.5. Set

$$M(x,y,z) := \frac{3ABC}{4Q}, \qquad F(x,y,z) := \frac{1}{M(x,y,z)} = \frac{4Q}{3ABC}. \tag{2.13}$$

Proposition 2.6. *The functions M and F are positive and symmetric on $(0,\infty)^3$. They are homogeneous of degrees 1 and -1, respectively, and*

$$h = -\frac{1}{8M} = -\frac{F}{8}. \tag{2.14}$$

Moreover,

$$M(x,x,x) = x, \qquad h(x,x,x) = -\frac{1}{8x}. \tag{2.15}$$

Proof. Positivity follows from Lemma 2.3. Symmetry follows either from (1.2) or from the integral expressions. Under the scaling $(x,y,z) \mapsto (rx, ry, rz)$, one has

$$A, B, C \mapsto r^{-1}A, r^{-1}B, r^{-1}C, \quad U \mapsto r^{-1}U, \quad T \mapsto r^{-2}T, \quad Q \mapsto r^{-4}Q,$$

which proves the homogeneity statements. Equation (2.14) follows from (2.6). At the diagonal,

$$A = B = C = \frac{1}{x}, \qquad U = \frac{1}{2x}, \qquad T = \frac{1}{2x^2}, \qquad Q = \frac{3}{4x^4},$$

which gives (2.15). □

2.4. **Reduction to a Stieltjes property along lines.** We now reduce Theorems 1.1 and 1.3 to one statement about the restrictions of F to lines, Theorem 2.10. Two elementary lemmas carry the reduction: the reciprocal of a Stieltjes function is concave (Lemma 2.8), and a function homogeneous of degree one that is concave along every line with positive direction is concave (Lemma 2.9). Both are proved where they are stated. Theorem 2.10 itself is proved in Section 3.

Definition 2.7. A function $f : (0,\infty) \to (0,\infty)$ will be called a compactly supported Stieltjes function if

$$f(t) = \int_I \frac{\mathrm{d}\mu(s)}{t - s} \tag{2.16}$$

for a finite positive measure μ supported on a compact interval $I \subset (-\infty, 0)$.

Schilling et al. (2012) is the general reference for Stieltjes and Bernstein functions.

Lemma 2.8 (Reciprocal Stieltjes concavity)**.** *Let f have the representation (2.16), and set $g = 1/f$. Then $g'' \le 0$ on $(0,\infty)$. If μ is not a point mass, then $g'' < 0$.*

Proof. Differentiating under the integral sign gives

$$-f'(t) = \int_I (t-s)^{-2}\,\mathrm{d}\mu(s), \qquad \frac{1}{2}f''(t) = \int_I (t-s)^{-3}\,\mathrm{d}\mu(s).$$

Therefore

$$g''(t) = \frac{2f'(t)^2 - f(t)f''(t)}{f(t)^3}. \tag{2.17}$$

Cauchy–Schwarz yields

$$\left(\int_I (t-s)^{-2}\,\mathrm{d}\mu(s)\right)^2 \le \left(\int_I (t-s)^{-1}\,\mathrm{d}\mu(s)\right)\left(\int_I (t-s)^{-3}\,\mathrm{d}\mu(s)\right),$$

which is exactly $2f'^2 \le ff''$. Equality holds only when $(t-s)^{-1/2}$ and $(t-s)^{-3/2}$ are proportional μ-almost everywhere, that is, only when s is constant on the support of μ. □

The next elementary lemma explains why it is enough to control lines with a positive direction.

Lemma 2.9 (Concavity from positive directions)**.** *Let $G : (0,\infty)^n \to (0,\infty)$ be continuous and homogeneous of degree one. Suppose that, for every $X, Y > 0$, the map*

$$t \longmapsto G(X + tY), \qquad t \ge 0,$$

is concave. Then G is concave on $(0,\infty)^n$.

If, in addition, G is continuously differentiable and every such line restriction is strictly concave whenever X and Y are not proportional, then

$$G(X+Y) > G(X) + G(Y) \tag{2.18}$$

for nonproportional $X, Y > 0$, and equality holds in superadditivity exactly on common rays.

Proof. Fix $X, Y > 0$ and write $g(t) = G(X + tY)$. Concavity gives, for $t > 1$,

$$g(1) - g(0) \ge \frac{g(t) - g(1)}{t-1}.$$

By homogeneity and continuity,

$$\lim_{t\to\infty} \frac{g(t) - g(1)}{t-1} = G(Y).$$

Hence

$$G(X+Y) \ge G(X) + G(Y). \tag{2.19}$$

Homogeneity and (2.19) imply, for $0 < \lambda < 1$,

$$G(\lambda X + (1-\lambda)Y) \ge \lambda G(X) + (1-\lambda)G(Y),$$

so G is concave.

For strictness, assume X, Y are not proportional. Since g is differentiable and strictly concave, its derivative is strictly decreasing. Homogeneity gives

$$g(t) = tG\left(Y + \frac{X}{t}\right), \qquad g'(t) = G\left(Y + \frac{X}{t}\right) - \frac{1}{t}\,\mathrm{d}G_{Y+X/t}(X),$$

so $g'(t) \to G(Y)$ as $t \to \infty$. Thus $g'(t) > G(Y)$ for finite t, and

$$G(X+Y) - G(X) = \int_0^1 g'(t)\,\mathrm{d}t > G(Y).$$

If X and Y are proportional, equality in (2.19) follows from homogeneity. □

The analytic heart of the paper is the following theorem.

Theorem 2.10 (Stieltjes property along lines)**.** *For every $X, Y \in (0,\infty)^3$, there is a finite positive measure $\mu_{X,Y}$, supported on a compact interval in $(-\infty, 0)$, such that*

$$F(X+tY) = \int \frac{\mathrm{d}\mu_{X,Y}(s)}{t-s}, \qquad t \geq 0. \tag{2.20}$$

If X and Y are not proportional, the measure is not a point mass.

Assuming this theorem, the main concavity statements are immediate.

Proposition 2.11. *Theorem 2.10 implies Theorems 1.3 and 1.1.*

Proof. By Lemma 2.8,

$$t \longmapsto M(X+tY) = \frac{1}{F(X+tY)}$$

is concave, and is strictly concave unless X, Y are proportional. By Proposition 2.2 and (2.14), $M = -1/(8h)$ is real-analytic, so Lemma 2.9 applies and makes M globally concave and strictly superadditive off common rays.

For distinct $(a,b), (a',b') \in (0,\infty)^2$, set

$$X = \lambda(a,b,1), \qquad Y = (1-\lambda)(a',b',1).$$

These vectors are nonproportional. Strict superadditivity gives

$$m(\lambda a + (1-\lambda)a', \lambda b + (1-\lambda)b') > \lambda m(a,b) + (1-\lambda)m(a',b'),$$

proving Theorem 1.3.

Finally, $h = \phi \circ M$, where $\phi(s) = -1/(8s)$ is increasing and strictly concave. If two points are not proportional, strictness already comes from M. On a common ray, M is affine but takes distinct endpoint values, so strictness comes from ϕ. This proves Theorem 1.1. □

Remark 2.12. *Theorem 2.10 gives more than concavity. Fix $X, Y > 0$ and let $\beta := \min_j x_j/y_j$, so that $X + tY > 0$ exactly for $t > -\beta$. The proof in Section 3.3 places $\mu = \mu_{X,Y}$ in the interval I of (3.2), whose right endpoint is $-\beta$, and represents the analytic continuation $\mathcal{F}$ of $t \mapsto F(X+tY)$ to $\mathbb{C} \setminus I$, defined in (3.3), as $\mathcal{F}(\zeta) = \int (\zeta - s)^{-1}\,\mathrm{d}\mu(s)$ (3.28). For $\operatorname{Im}\zeta > 0$,*

$$\operatorname{Im}\mathcal{F}(\zeta) = -\operatorname{Im}\zeta \int \frac{\mathrm{d}\mu(s)}{|\zeta - s|^2} < 0,$$

so $1/\mathcal{F}$ is analytic in the upper half-plane with positive imaginary part. It is also analytic across $(-\beta, \infty)$, where it is real and equals $M(X+tY)$. By Löwner's theorem (Bhatia, 1997, Chapter V), $t \mapsto M(X+tY)$ is therefore operator monotone on $(-\beta, \infty)$. Operator monotone functions are also the functions that classify monotone metrics (Petz, 1996). Here they appear in a different role, along lines in the space of eigenvalue triples. The proof of Theorem 1.1 uses only the concavity.

3. The Stieltjes property along lines

This section proves Theorem 2.10. The theorem asks for a Stieltjes representation rather than for concavity directly because the representation turns a second-order condition into a boundary sign. Concavity of M along a line is a condition on second derivatives, and the second derivatives of $F = 4Q/(3ABC)$ are unwieldy. If instead the continuation $\mathcal{F}$ of $t \mapsto F(X+tY)$, defined in (3.3), is analytic off a compact interval I of the negative axis, decays at infinity, and has nonpositive imaginary part on the upper edge of I, then Cauchy's formula on a keyhole contour writes it as the integral of $1/(\zeta - s)$ against the positive measure $-\pi^{-1}\operatorname{Im}\mathcal{F}(s + \mathrm{i}0)\,\mathrm{d}s$, and Lemma 2.8 converts this representation into concavity along the line. The analytic continuation comes from the resolvent integrals of Section 2, and on the cut the logarithms in A, B, C take explicit values, so the sign of the boundary value is a question about one explicit real function.

Fix $X, Y > 0$. The proof has three steps, one per subsection, in this order.

(1) *Continuation.* The function $t \mapsto F(X + tY)$ continues analytically, as the function $\mathcal{F}$ of (3.3), to the complement of a compact interval I of the negative real axis, and decays like $F(Y)/\zeta$ at infinity (Lemma 3.1).
(2) *Boundary sign.* Away from the branch points, its upper boundary value on I has negative imaginary part. Symmetry, homogeneity and Schwarz reflection reduce this to boundary triples $(-1, p, q)$, and there to the positivity of an explicit function of two real variables (Section 3.2); the elementary estimates that prove this positivity are in Appendix A.
(3) *Reconstruction.* A keyhole contour recovers the function from its boundary values; the representing measure has density $-\pi^{-1} \operatorname{Im} \mathcal{F}(s + \mathrm{i}0)$, which is positive by step (2). Coincident branch points follow by approximation (Section 3.3).

3.1. **Analytic continuation along a line.** To prove Theorem 2.10 we continue $t \mapsto F(X + tY)$ from $t \geq 0$ to the complex plane cut along an interval of the negative axis. Fix

$$X = (x_1, x_2, x_3), \qquad Y = (y_1, y_2, y_3) \in (0, \infty)^3,$$

and define

$$Z_j(\zeta) := x_j + \zeta y_j, \qquad \alpha_j := \frac{x_j}{y_j} > 0. \tag{3.1}$$

Let

$$I := [-\max_j \alpha_j, -\min_j \alpha_j] \subset (-\infty, 0), \tag{3.2}$$

and set

$$\mathcal{F}(\zeta) := F(Z_1(\zeta), Z_2(\zeta), Z_3(\zeta)). \tag{3.3}$$

The integral formulas admit analytic continuation to the slit plane. For complex arguments, use

$$c(z_i, z_j) = \int_0^1 \frac{\mathrm{d}\lambda}{(1-\lambda)z_i + \lambda z_j}, \tag{3.4}$$

which agrees with the divided difference of the principal logarithm wherever the latter expression is directly available. Similarly,

$$U(z_1, z_2, z_3) = \int_{\Delta_2} \frac{\mathrm{d}\lambda}{\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3}, \tag{3.5}$$

$$T(z_1, z_2, z_3) = \int_{\Delta_2} \frac{\mathrm{d}\lambda}{(\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3)^2}. \tag{3.6}$$

Lemma 3.1 (Slit-plane analyticity)**.** *The function $\mathcal{F}$ is analytic on $\mathbb{C} \setminus I$, satisfies*

$$\mathcal{F}(\overline{\zeta}) = \overline{\mathcal{F}(\zeta)}, \tag{3.7}$$

and has the expansion

$$\mathcal{F}(\zeta) = \frac{F(Y)}{\zeta} + O(|\zeta|^{-2}) \qquad (\zeta \to \infty). \tag{3.8}$$

Proof. Every possible zero of a denominator in (3.4)–(3.6) has the form

$$-\frac{\sum_j \lambda_j x_j}{\sum_j \lambda_j y_j},$$

which lies in I. Hence A, B, C, U, T are analytic on $\mathbb{C} \setminus I$. The factor $1/(Z_1 Z_2 Z_3)$ in Q is analytic there as well, because its poles $-\alpha_j$ lie in I. Therefore Q is analytic on $\mathbb{C} \setminus I$.

We must also check that the factors $c(Z_i, Z_j)$ do not vanish off the cut. If $\operatorname{Im}\zeta > 0$, then every denominator in (3.4) has positive imaginary part, so its reciprocal has strictly negative imaginary part. Thus

$$\operatorname{Im} c(Z_i(\zeta), Z_j(\zeta)) < 0.$$

The lower-half-plane assertion follows by conjugation. On either real component of $\mathbb{R} \setminus I$, all affine denominators have the same nonzero sign, so the integral is again nonzero. This proves analyticity of $\mathcal{F}$. Equation (3.7) is immediate.

Finally, degree -1 homogeneity gives

$$\mathcal{F}(\zeta) = \frac{1}{\zeta} F\left(Y + \frac{X}{\zeta}\right),$$

and smoothness of F near Y yields (3.8). □

The remaining task is to prove

$$\operatorname{Im}\mathcal{F}(s + \mathrm{i}0) \le 0, \qquad s \in I. \tag{3.9}$$

Section 3.2 establishes this sign (Lemma 3.6).

3.2. **The boundary sign.** We now prove the sign condition (3.9). At a non-branch point of the cut, the real limiting triple $(Z_1(s), Z_2(s), Z_3(s))$ has mixed signs. The case with two negative coordinates reduces to the case with one by reflection (Lemma 3.6). By permutation symmetry and positive homogeneity, the case with exactly one negative coordinate reduces to

$$(-1, p, q), \qquad p > q > 0. \tag{3.10}$$

The case $p = q$ will be obtained by a limit. We write $F_+(Z)$ and $F_-(Z)$ for the upper and lower boundary values of F at a real triple Z with mixed signs, so that $\mathcal{F}(s \pm \mathrm{i}0) = F_\pm(Z_1(s), Z_2(s), Z_3(s))$.

Put

$$u := \log p, \qquad v := \log q, \qquad \delta := u - v > 0, \qquad d := p - q > 0. \tag{3.11}$$

For the upper boundary value, the principal logarithm gives, in the abbreviations (2.2),

$$A = c(-1, p) = \frac{u - \mathrm{i}\pi}{1 + p}, \qquad B = c(-1, q) = \frac{v - \mathrm{i}\pi}{1 + q}, \qquad C = c(p, q) = \frac{\delta}{d} > 0. \tag{3.12}$$

The relations (2.4) become

$$T = \frac{B - A}{d}, \qquad U = \frac{pA - qB}{d}. \tag{3.13}$$

Since $xyz = -pq$,

$$\frac{Q}{AB} = -\frac{2}{pqd}\left(\frac{p}{B} - \frac{q}{A}\right) - \frac{1}{d^2}\left(\frac{B}{A} + \frac{A}{B} - 2\right). \tag{3.14}$$

The required sign is encoded by one real expression.

Lemma 3.2 (Boundary numerator)**.** *Define*

$$\begin{aligned} N(p, q) := {} & (1 + q)^2(u^2 + \pi^2)\left(\delta + \frac{2d(1 + p)}{q}\right) \\ & - (1 + p)^2(v^2 + \pi^2)\left(\delta + \frac{2d(1 + q)}{p}\right). \end{aligned} \tag{3.15}$$

Then

$$\frac{1}{\pi}\operatorname{Im}\frac{Q}{AB} = -\frac{N(p, q)}{(u^2 + \pi^2)(v^2 + \pi^2)(1 + p)(1 + q)d^2}. \tag{3.16}$$

Consequently,

$$N(p, q) > 0 \quad \Longrightarrow \quad \operatorname{Im} F_+(-1, p, q) < 0. \tag{3.17}$$

Proof. From (3.12),

$$\operatorname{Im}\frac{1}{A}=\frac{\pi(1+p)}{u^2+\pi^2},\qquad \operatorname{Im}\frac{1}{B}=\frac{\pi(1+q)}{v^2+\pi^2},$$

while

$$\operatorname{Im}\frac{B}{A}=-\frac{\pi\delta(1+p)}{(1+q)(u^2+\pi^2)},\qquad \operatorname{Im}\frac{A}{B}=\frac{\pi\delta(1+q)}{(1+p)(v^2+\pi^2)}.$$

Substitution in (3.14) gives (3.16). Since

$$F_+(-1,p,q)=\frac{4}{3C}\frac{Q}{AB}$$

and $C>0$, the sign implication follows. □

We now convert N into hyperbolic coordinates. Write

$$p=we^{\rho},\qquad q=we^{-\rho},\qquad w=e^{\sigma},\qquad \rho>0. \tag{3.18}$$

Then

$$u=\sigma+\rho,\qquad v=\sigma-\rho.$$

Set

$$S:=\sinh\rho,\qquad \gamma:=\cosh\rho,\qquad E:=1+2w\gamma+w^2, \tag{3.19}$$

$$\mathcal{D}:=ES-\rho w(1+w\gamma), \tag{3.20}$$

and

$$\mathcal{B}:=\rho(1+2w\gamma+w^2\cosh(2\rho))+2SE(w+\gamma). \tag{3.21}$$

Lemma 3.3 (Hyperbolic normal form)**.** *With the notation above,*

$$\boxed{N=8\left[S\mathcal{D}(\sigma^2+\rho^2+\pi^2)+\rho\sigma\mathcal{B}\right].} \tag{3.22}$$

Proof. Insert (3.18) into (3.15), use $d=2wS$ and $\delta=2\rho$, and collect the coefficients of σ^2,σ, and $\rho^2+\pi^2$. The resulting identity is (3.22). No inequality is used in this step. □

The proof of the boundary sign is therefore reduced to the following real-variable proposition.

Proposition 3.4 (Core inequality)**.** *For every $\rho>0$ and every $\sigma\in\mathbb{R}$, the right-hand side of (3.22) is strictly positive.*

Proposition 3.4 is proved in Appendix A. The proof splits according to the sign of σ, because the second term in (3.22) changes sign with σ. For $\sigma\le 0$ the factor $\mathcal{D}$ is positive, and dividing N by $8S\mathcal{D}$ leaves a quadratic in $\tau=-\sigma$ whose linear coefficient is bounded by its value at $w=0$ plus $16w$. For $\sigma>0$ only the case $\mathcal{D}<0$ needs an argument; that case forces τ beyond the threshold $\rho-\log\rho$, where a lower bound for N that increases in τ is already positive. The case $p=q$, that is $\rho\downarrow 0$, is decided by the limit of $N/(8\rho^2)$, computed in Appendix A.3. Together these give the following.

Theorem 3.5 (Boundary sign with one negative coordinate)**.** *For every $p,q>0$, the upper boundary value satisfies*

$$\operatorname{Im}F_+(-1,p,q)<0. \tag{3.23}$$

Lemma 3.6. *For every interior point $s\in I$ that is not a branch point,*

$$\operatorname{Im}\mathcal{F}(s+\mathrm{i}0)<0. \tag{3.24}$$

Proof. If exactly one limiting coordinate is negative, positive scaling and permutation symmetry reduce the boundary value to Theorem 3.5.

Suppose instead that two coordinates are negative, and write $Z := (Z_1(s), Z_2(s), Z_3(s))$. Multiplication of the upper boundary triple by -1 produces the lower boundary value at $-Z$, which has exactly one negative coordinate. The integral definitions (3.4)–(3.6) show directly that the analytic continuation of F remains homogeneous of degree -1, even for the scalar -1. Therefore

$$F_+(Z) = -F_-(-Z).$$

By Schwarz reflection, $F_-(-Z) = \overline{F_+(-Z)}$. Theorem 3.5 then gives $\operatorname{Im} F_+(Z) < 0$. □

3.3. **The Stieltjes representation.** We now return to the restriction $\mathcal{F}$ of F to the line $X + \zeta Y$, defined in (3.3), and complete the proof of Theorem 2.10.

Behavior at branch points. Assume first that the three numbers α_j in (3.1) are distinct. Near a branch point $\zeta_0 = -\alpha_j$, precisely one coordinate, say Z_1, tends to zero while the other two tend to nonzero limits. If $\xi = Z_1(\zeta)$, then

$$c(\xi, r) = -\frac{\operatorname{Log} \xi}{r} + O(1) \qquad (\xi \to 0), \tag{3.25}$$

for every fixed $r \neq 0$ in a slit neighborhood. Consequently,

$$\mathcal{F}(\zeta) = O\left(\frac{1}{(\zeta - \zeta_0)\operatorname{Log}^2(\zeta - \zeta_0)}\right). \tag{3.26}$$

Indeed, the two c-factors involving ξ each grow logarithmically, the third tends to a nonzero limit, $T = O(\operatorname{Log} \xi)$, and U stays bounded; the term $2U/(Z_1 Z_2 Z_3)$ is therefore $O(1/\xi)$ and dominates the possible logarithmic-square growth of T^2. In particular, the integral of $\mathcal{F}$ over a circle of radius ε around ζ_0 is

$$O\left(\frac{1}{|\log \varepsilon|^2}\right) \longrightarrow 0. \tag{3.27}$$

Keyhole reconstruction.

Proposition 3.7 (Cauchy representation for distinct branch points)**.** *If the α_j are pairwise distinct, then*

$$\mathcal{F}(\zeta) = \int_I \frac{\mathrm{d}\mu(s)}{\zeta - s}, \qquad \zeta \in \mathbb{C} \setminus I, \tag{3.28}$$

where

$$\mathrm{d}\mu(s) = -\frac{1}{\pi} \operatorname{Im} \mathcal{F}(s + \mathrm{i}0)\, \mathrm{d}s. \tag{3.29}$$

The measure is finite, positive, and has total mass

$$\mu(I) = F(Y). \tag{3.30}$$

Proof. Apply Cauchy's formula to a large circle with a thin keyhole removed around I, and remove small circles around the three branch points. The large circle contributes zero in the limit by (3.8); the small circles contribute zero by (3.27). The two lips of the cut give

$$\mathcal{F}(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_I \frac{\mathcal{F}(s + \mathrm{i}0) - \mathcal{F}(s - \mathrm{i}0)}{s - \zeta}\, \mathrm{d}s.$$

Using Schwarz reflection and rearranging yields (3.28)–(3.29). Positivity follows from Lemma 3.6. Finally, comparing the coefficient of $1/\zeta$ at infinity in (3.28) with (3.8) gives (3.30). □

Coincident branch points. If two or three α_j coincide, perturb X, Y slightly so that the ratios become distinct while remaining positive. The resulting measures are supported in a common compact negative interval and have masses tending to $F(Y)$, hence uniformly bounded. By weak compactness, a subsequence converges to a finite positive measure μ. The corresponding Stieltjes transforms converge locally uniformly off the interval, while the analytic functions $F(X^{(n)} + \zeta Y^{(n)})$ converge locally uniformly to $F(X + \zeta Y)$. Thus (3.28) passes to the limit. Uniqueness of the Stieltjes transform makes the limiting measure independent of the subsequence.

Point masses. If all three ratios coincide, then $X = \kappa Y$ for some $\kappa > 0$, and homogeneity gives the explicit point-mass representation

$$F(X + tY) = \frac{F(Y)}{t + \kappa}. \tag{3.31}$$

If the ratios are not all equal, the cut contains a nondegenerate mixed-sign interval. On every compact subinterval avoiding the branch points, the Stieltjes inversion formula identifies the absolutely continuous part as

$$\frac{\mathrm{d}\mu}{\mathrm{d}s}(s) = -\frac{1}{\pi} \operatorname{Im} \mathcal{F}(s + \mathrm{i}0).$$

By Lemma 3.6, this density is strictly positive there; therefore the representing measure is not a point mass.

Conclusion. This completes the proof of Theorem 2.10, and hence, by Proposition 2.11, the proofs of Theorems 1.3 and 1.1.

AI USAGE

The results in this paper were developed with an AI assistant (OpenAI's GPT Pro). None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs.

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Appendix A. Proof of the core inequality

This appendix proves Proposition 3.4 and treats the case $p = q$. We split according to the sign of σ. The two halves require different estimates because the second term in (3.22) changes sign.

A.1. **The half-plane $\sigma \le 0$.** The case $\sigma = 0$ is immediate: then $w = 1$, and

$$\mathcal{D} = (\gamma + 1)(2S - \rho) > 0. \tag{A.1}$$

Assume henceforth

$$\sigma = -\tau, \qquad \tau > 0, \qquad w = e^{-\tau} \in (0, 1).$$

Lemma A.1. *For $0 \le w \le 1$, one has $\mathcal{D} > 0$.*

Proof. Expanding (3.20) as a quadratic in w gives

$$\mathcal{D} = S + (2\gamma S - \rho)w + (S - \rho\gamma)w^2. \tag{A.2}$$

Since $\rho\gamma - S > 0$, this quadratic is concave. Its endpoint values are

$$\mathcal{D}(0) = S > 0, \qquad \mathcal{D}(1) = (\gamma + 1)(2S - \rho) > 0.$$

A concave function lies above its endpoint chord, proving the claim. □

Define

$$R(\rho, w) := \frac{\rho\mathcal{B}}{S\mathcal{D}}, \qquad R_0(\rho) := R(\rho, 0) = 2\rho \coth \rho + \rho^2 \operatorname{csch}^2 \rho. \tag{A.3}$$

Then (3.22) becomes

$$\frac{N}{8S\mathcal{D}} = \tau^2 + \rho^2 + \pi^2 - \tau R(\rho, w). \tag{A.4}$$

Lemma A.2 (Comparison with the value at $w = 0$)**.** *For $\rho > 0$ and $0 \le w \le 1$,*

$$R(\rho, w) \le R_0(\rho) + 16w. \tag{A.5}$$

Proof. Set

$$\Delta := S\big[(R_0 + 16w)S\mathcal{D} - \rho\mathcal{B}\big]. \tag{A.6}$$

It is enough to prove $\Delta \ge 0$. Put

$$\theta := \frac{\rho}{S} \in (0, 1).$$

A direct expansion in the Bernstein basis on $[0, 1]$ gives

$$\Delta = w\left[\beta_0(1 - w)^2 + 2\beta_1 w(1 - w) + \beta_2 w^2\right], \tag{A.7}$$

where

$$\frac{\beta_0}{S^3} = 16 - 2\theta - 2\gamma\theta^2 - \theta^3, \tag{A.8}$$

$$\begin{aligned}\frac{2\beta_1}{S^3} &= 32(\gamma + 1) - 4\gamma\theta - 4\gamma\theta^2 - \gamma\theta^3 \\ &\quad - 4\rho^2 - 20\theta - 2\theta^2 - 2\theta^3,\end{aligned} \tag{A.9}$$

and

$$\frac{\beta_2}{S^3} = (\gamma + 1)(32 - 20\theta - 2\theta^2 - \theta^3) - 4\rho^2. \tag{A.10}$$

We use only

$$\rho^2 \le 2(\gamma-1), \qquad \gamma\theta^2 = \frac{\gamma\rho^2}{S^2} \le \frac{2\gamma}{\gamma+1} \le 2. \tag{A.11}$$

The first inequality is the quadratic Taylor lower bound for $\cosh\rho$. Equations (A.8) and (A.11) give

$$\beta_0 \ge 9S^3.$$

For β_1, use $\theta \le 1$, $\gamma\theta^3 \le \gamma\theta^2 \le 2$, and $4\rho^2 \le 8(\gamma-1)$; then

$$2\beta_1 \ge (20\gamma+6)S^3 > 0.$$

Finally,

$$32 - 20\theta - 2\theta^2 - \theta^3 \ge 9,$$

so

$$\beta_2 \ge S^3\{9(\gamma+1) - 8(\gamma-1)\} = (\gamma+17)S^3 > 0.$$

All Bernstein coefficients in (A.7) are positive; therefore $\Delta \ge 0$, proving (A.5). □

Lemma A.3 (A bound for R_0)**.** *For every $\rho > 0$,*

$$R_0(\rho)^2 \le 4\rho^2 + 9. \tag{A.12}$$

Proof. Let $\theta = \rho/S$, as in the proof of Lemma A.2. Since $\theta\gamma = \sqrt{\rho^2+\theta^2}$,

$$R_0 = 2\sqrt{\rho^2+\theta^2} + \theta^2. \tag{A.13}$$

The estimate $\sinh\rho \ge \rho + \rho^3/6$ implies

$$\theta^2 \le \frac{3}{\rho^2+3}. \tag{A.14}$$

Indeed, $(1+\rho^2/6)^{-2} \le 3/(\rho^2+3)$. The right-hand side of (A.13) is increasing in θ^2. It is therefore enough, with

$$t = \rho^2, \qquad \nu = \frac{3}{t+3},$$

to prove

$$2\sqrt{t+\nu} + \nu \le \sqrt{4t+9}. \tag{A.15}$$

The quantities on the right after each rearrangement below are nonnegative. Squaring twice reduces (A.15) to

$$(9+\nu^2-4\nu)^2 - 4\nu^2(4t+9) \ge 0.$$

After substituting $\nu = 3/(t+3)$, the left-hand side is

$$\frac{9t(t+2)(9t^2+50t+72)}{(t+3)^4} \ge 0. \tag{A.16}$$

This proves (A.12). □

Completion for $\sigma < 0$. By Lemma A.2,

$$\begin{aligned}\tau^2+\rho^2+\pi^2-\tau R &\ge \tau^2+\rho^2+\pi^2-\tau R_0 - 16\tau e^{-\tau} \\ &= \left(\tau - \frac{R_0}{2}\right)^2 + \rho^2 + \pi^2 - \frac{R_0^2}{4} - 16\tau e^{-\tau}.\end{aligned}$$

Using Lemma A.3 and $\tau e^{-\tau} \le 1/e$,

$$\tau^2+\rho^2+\pi^2-\tau R \ge \pi^2 - \frac{9}{4} - \frac{16}{e} > 0. \tag{A.17}$$

For example, $\pi > 3$ and $e > 8/3$ make the final lower bound exceed $3/4$. Equations (A.4) and Lemma A.1 now give $N > 0$. □

A.2. **The half-plane $\sigma > 0$.** Write

$$\sigma = \tau > 0, \qquad w = e^{\tau} > 1.$$

Set

$$a_\rho := \rho\gamma - S > 0, \qquad d_\rho := 2\gamma S - \rho > 0. \tag{A.18}$$

Then

$$\mathcal{D} = S + d_\rho w - a_\rho w^2. \tag{A.19}$$

Also,

$$\mathcal{B} = b_0 + b_1 w + b_2 w^2 + 2S w^3, \tag{A.20}$$

where

$$b_0 = \rho + 2\gamma S > 0, \tag{A.21}$$

$$b_1 = 4\gamma^2 S + 2\gamma\rho + 2S > 0, \tag{A.22}$$

and

$$b_2 = \rho\cosh(2\rho) + 6\gamma S > 0. \tag{A.23}$$

If $\mathcal{D} \geq 0$, both terms in (3.22) are positive and there is nothing to prove. Assume from now on that

$$\mathcal{D} < 0. \tag{A.24}$$

Discarding the positive b_0, b_1 terms in $\mathcal{B}$, and the positive $S + d_\rho w$ part of $\mathcal{D}$, gives

$$\frac{N}{8} > w^2 P_\rho(\tau), \tag{A.25}$$

where

$$P_\rho(t) := \rho t(b_2 + 2Se^t) - Sa_\rho(t^2 + \rho^2 + \pi^2). \tag{A.26}$$

We shall show $P_\rho(\tau) > 0$.

Lemma A.4 (The negative-$\mathcal{D}$ threshold)**.** *Assumption* (A.24) *implies*

$$\tau > t_* := \rho - \log\rho. \tag{A.27}$$

Moreover, $t_* \geq 1$.

Proof. From (A.19) and $\mathcal{D} < 0$,

$$a_\rho w^2 > S + d_\rho w > d_\rho w,$$

so $w > d_\rho/a_\rho$. We claim

$$\frac{d_\rho}{a_\rho} \geq \frac{e^\rho}{\rho}. \tag{A.28}$$

Indeed,

$$\rho d_\rho - e^\rho a_\rho = \frac{1}{2}\left[e^{2\rho} - 1 - \rho(1 + e^{-2\rho}) - 2\rho^2\right] \geq 0, \tag{A.29}$$

where we used $e^{2\rho} \geq 1 + 2\rho + 2\rho^2$ and $1 + e^{-2\rho} \leq 2$. Thus

$$e^\tau = w > \frac{e^\rho}{\rho},$$

which is (A.27). Finally, $\rho - \log\rho \geq 1$ because this function has its global minimum at $\rho = 1$. □

Lemma A.5 (Monotonicity of P_ρ)**.** *For every $\rho > 0$, the function $P_\rho(t)$ is strictly increasing on $(0, \infty)$.*

Proof. First,

$$b_2 > 2Sa_\rho. \tag{A.30}$$

Indeed, direct simplification gives

$$b_2 - 2Sa_\rho = 2e^{2\rho} - 1 + (\rho - 1)e^{-2\rho} > 0. \tag{A.31}$$

For $\rho \geq 1$ positivity is immediate. If $0 < \rho < 1$, then $2e^{2\rho} - 1 > 1$ and $(\rho - 1)e^{-2\rho} > \rho - 1$, so the sum exceeds ρ.

Differentiating (A.26),

$$P'_\rho(t) = \rho b_2 + 2\rho S e^t(t + 1) - 2Sa_\rho t. \tag{A.32}$$

If $t \leq \rho$, (A.30) gives

$$P'_\rho(t) > 2Sa_\rho(\rho - t) + 2\rho S e^t(t + 1) > 0.$$

If $t > \rho$, then

$$a_\rho = \rho\gamma - S < \rho e^\rho < \rho e^t,$$

so

$$\rho e^t(t + 1) - a_\rho t = t(\rho e^t - a_\rho) + \rho e^t > 0.$$

Equation (A.32) is again positive. □

We need a somewhat stronger lower bound for b_2.

Lemma A.6. *For every $\rho > 0$,*

$$b_2 > \left(2 + \frac{8}{\rho}\right) Sa_\rho. \tag{A.33}$$

Proof. An exact calculation gives

$$\rho\left[b_2 - \left(2 + \frac{8}{\rho}\right) Sa_\rho\right] = (\rho^2 + \rho)e^{-2\rho} - \rho + 4(\cosh(2\rho) - 1). \tag{A.34}$$

If $0 < \rho \leq 3$, use $e^{-2\rho} \geq 1 - 2\rho$ and $\cosh(2\rho) - 1 \geq 2\rho^2$. The right-hand side of (A.34) is then at least

$$7\rho^2 - 2\rho^3 = \rho^2(7 - 2\rho) > 0.$$

If $\rho \geq 3$, discard the first positive term and use

$$4(\cosh(2\rho) - 1) - \rho \geq 8\rho^2 - \rho > 0.$$

□

Lemma A.7 (Positivity at the threshold)**.** *For $t_* = \rho - \log\rho$, one has*

$$P_\rho(t_*) > 0. \tag{A.35}$$

Proof. Since $e^{t_*} = e^\rho/\rho$, Lemma A.6 and (A.26) give

$$\frac{P_\rho(t_*)}{Sa_\rho} > 2\rho t_* + 8t_* + \frac{2t_* e^\rho}{a_\rho} - t_*^2 - \rho^2 - \pi^2$$

$$= 8t_* + \frac{2t_* e^\rho}{a_\rho} - (\log\rho)^2 - \pi^2. \tag{A.36}$$

Furthermore,

$$a_\rho < \frac{\rho e^\rho}{2}, \tag{A.37}$$

because

$$\rho e^\rho - 2a_\rho = e^\rho - (\rho + 1)e^{-\rho} > 0.$$

Thus

$$\frac{P_\rho(t_*)}{Sa_\rho} > \Psi(\rho), \tag{A.38}$$

where

$$\Psi(\rho) := 8(\rho - \log\rho) + \frac{4(\rho - \log\rho)}{\rho} - (\log\rho)^2 - \pi^2. \tag{A.39}$$

We prove $\Psi(\rho) > 0$ in two ranges. If $0 < \rho \le 1$, write $\rho = e^{-s}$, $s \ge 0$. Then

$$\Psi(\rho) = 8e^{-s} + 8s + 4 + 4se^s - s^2 - \pi^2.$$

Its derivative with respect to s is

$$8(1 - e^{-s}) + 4e^s(1 + s) - 2s > 0.$$

Therefore

$$\Psi(\rho) \ge \Psi(1) = 12 - \pi^2 > 0. \tag{A.40}$$

If $\rho \ge 1$, write $\rho = e^s$, $s \ge 0$. Then

$$\Psi(\rho) = 8e^s - 8s + 4 - 4se^{-s} - s^2 - \pi^2.$$

Using

$$e^s \ge 1 + s + \frac{s^2}{2} + \frac{s^3}{6}, \qquad e^{-s} \le 1,$$

we obtain

$$\begin{aligned}\Psi(\rho) &\ge 12 - 4s + 3s^2 + \frac{4}{3}s^3 - \pi^2 \\ &= \frac{131}{12} + \frac{(2s-1)^2(4s+13)}{12} - \pi^2 > 0,\end{aligned} \tag{A.41}$$

where the last inequality uses $\pi^2 < 10$. This proves (A.35). □

Completion for $\sigma > 0$. Under $\mathcal{D} < 0$, Lemma A.4 gives $\tau > t_*$. By Lemmas A.5 and A.7,

$$P_\rho(\tau) > P_\rho(t_*) > 0.$$

Equation (A.25) implies $N > 0$. Together with the immediate case $\mathcal{D} \ge 0$, this completes the proof for $\sigma > 0$. □

A.3. **The case** $p = q$**.** The preceding argument assumed $\rho > 0$, equivalently $p > q$. At $p = q = w$, note that $d = 2w\sinh\rho \sim 2w\rho$. Hence (3.16), after canceling the common quadratic factor, shows that the continuous boundary sign is determined by the exact limit

$$\lim_{\rho\downarrow 0} \frac{N}{8\rho^2} = (1 + w)\left[\sigma^2 + \pi^2 + \sigma(1 + w)(3 + 2w)\right]. \tag{A.42}$$

For $\sigma \ge 0$, this is plainly positive. If $\sigma = -\tau < 0$, then $w = e^{-\tau}$, and the bracket is

$$\tau^2 + \pi^2 - \tau(1 + e^{-\tau})(3 + 2e^{-\tau}).$$

Since

$$(1 + e^{-\tau})(3 + 2e^{-\tau}) = 3 + 5e^{-\tau} + 2e^{-2\tau},$$

we get

$$\begin{aligned}&\tau^2 + \pi^2 - \tau(1 + e^{-\tau})(3 + 2e^{-\tau}) \\ &\qquad \ge \left(\tau - \frac{3}{2}\right)^2 + \pi^2 - \frac{9}{4} - \frac{7}{e} > 0.\end{aligned}$$

Thus the upper boundary sign remains strict when the two positive coordinates coincide.

This proves Proposition 3.4 and the case $p = q$ of Theorem 3.5.