Claw-free perfect graphs are chromatic-choosable

Suchismita Mishra[1], Narayanan N[2], and Umesh Shankar[2]

[1]National Institute of Informatics, Tokyo
[2]Department of Mathematics, Indian Institute of Technology Madras, Chennai, India

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Abstract

We prove that every claw-free perfect graph G satisfies $\mathrm{ch}(G) = \chi(G) = \omega(G)$. More strongly, if S is a clique of simplicial vertices of G and $k \geq \omega(G)$, every proper precoloring of S extends to a proper coloring of G from arbitrary lists of size k on $V(G) \setminus S$. The proof has three coloring steps: a mixed-list theorem for cobipartite graphs, a precoloring-extension theorem for elementary graphs built on Galvin's kernel method and a homogeneous-pair exchange argument, and a polynomial method (Combinatorial Nullstellensatz) argument for graphs of interval type in which a Vandermonde factor is transferred across each incomplete pair. The three coloring steps are assembled along the structure theorems of Chvátal–Sbihi, Maffray–Reed and Chudnovsky–Seymour, using double clique cutsets for composition. As a consequence, every perfect line graph is chromatic-choosable, recovering the theorem of Peterson and Woodall. Our precoloring extension theorem also resolves a conjecture posed by Chudnovsky and Seymour.

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1 Introduction

A central theme in list colouring is to determine when restrictions on the available colours leave the chromatic number unchanged. A *list assignment* L on a graph G assigns to each vertex v a set $L(v)$ of admissible colours; an *L-colouring* is a proper colouring f with $f(v) \in L(v)$ for all $v \in V(G)$. The *choice number* $\mathrm{ch}(G)$ is the least k for which an L-colouring exists whenever $|L(v)| \geq k$ at every vertex. Thus $\chi(G) \leq \mathrm{ch}(G)$, and G is *chromatic-choosable* when equality holds. List colouring was introduced by Vizing [17] and independently by Erdős, Rubin and Taylor [7]. Even bipartite graphs need not be chromatic-choosable, so perfectness alone does not force $\mathrm{ch}(G) = \chi(G)$.

The List Colouring Conjecture asserts that every line graph is chromatic-choosable. Galvin [8] proved it for line graphs of bipartite multigraphs, and Peterson and Woodall [15, 16] established it for perfect line graphs. Since line graphs are claw-free, these results belong to the broader conjecture of Gravier and Maffray [10] that *every claw-free graph is chromatic-choosable*. Recall that a graph is claw-free if it contains no induced $K_{1,3}$, and perfect if each of its induced subgraphs H satisfies $\chi(H) = \omega(H)$.

The conjecture has been studied intensively within claw-free perfect graphs. Gravier and Maffray proved the result for 3-chromatic claw-free perfect graphs [11], extending their earlier work on 3-colourable elementary graphs [9]. Gravier, Maffray and Pastor [12] subsequently established it whenever $\omega(G) \leq 4$. More recently, Narayanan, Vasudevan and Somasundaram [14]

removed the clique-number restriction for the entire class of elementary graphs. We settle the problem for all claw-free perfect graphs.

Theorem 1.1. *Every claw-free perfect graph G satisfies*

$$\mathrm{ch}(G) = \chi(G) = \omega(G).$$

The difficulty in passing from basic structural classes to arbitrary claw-free perfect graphs was identified by Chudnovsky and Seymour [5]. Their structure theorem uses compositions along cliques, but list colourings on different pieces cannot in general be reconciled by permuting colour names. Motivated by this obstacle, they formulated a precolouring-extension conjecture for certain *simplicial cliques*.

Conjecture 1 (Chudnovsky–Seymour [5, Conjecture 4.2])**.** *Let G be a perfect claw-free graph and let K be a simplicial clique in G. Let $\{L_v\}_{v\in V(G)}$ be a set of lists of colours with $|L_v| \geq \chi(G)$ for all $v \in V(G)$. Let the vertices of K be coloured in $|K|$ different colours, each with a colour from its list. Then this colouring of K can be extended to a proper list-colouring of G with the lists $\{L_v\}_{v\in V(G)}$.*

We establish a stronger extension principle. A vertex is *simplicial* if its neighbourhood is a clique.

Theorem 1.2 (Precolouring extension)**.** *Let G be a claw-free perfect graph, let $k \geq \omega(G)$ be an integer, and let $S \subseteq V(G)$ be a clique of simplicial vertices, possibly empty. If ϕ is any proper colouring of $G[S]$ and L is any list assignment on $V(G) \setminus S$ such that $|L(v)| \geq k$ for every $v \notin S$, then G has a proper colouring f satisfying*

$$f|_S = \phi \qquad \text{and} \qquad f(v) \in L(v) \quad (v \in V(G) \setminus S).$$

Taking $S = \varnothing$ and $k = \omega(G)$ yields Theorem 1.1. Our precoloring extension theorem also resolves the above conjecture of Chudnovsky and Seymour. Their simplicial cliques are cliques of simplicial vertices with the additional requirement that $G - (S \cup N_G(S))$ be connected. Our theorem does not require this connectivity condition. Moreover, the conjecture requires the prescribed colours to belong to the original lists, whereas here they are arbitrary and no lists on S are needed.

The structural backbone for these results is the decomposition theorem of Chvátal and Sbihi [4]. A *clique cutset* of a connected graph is a clique whose deletion disconnects the graph; a connected graph without a clique cutset is an *atom*.

Theorem 1.3 (Chvátal–Sbihi [4])**.** *Every claw-free perfect atom is elementary or peculiar.*

Elementary graphs are those whose edges admit a two-colouring in which the edges of every induced three-vertex path have different colours. Maffray and Reed [13] characterized them as augmentations of line graphs of bipartite multigraphs. Peculiar graphs have an explicit clique-based description, and their chromatic-choosability was established by Gravier and Maffray [10]. Thus both kinds of atoms are known to be chromatic-choosable. This fact does not by itself prove Theorem 1.1, however: clique cutsets introduce compatibility constraints on lists, while unrestricted clique sums need not preserve claw-freeness [6]. The extension theorem is designed to meet precisely these difficulties.

1.1 Proof strategy

Our argument makes the structural decomposition compatible with lists by alternating between precolouring extension and a *mixed-list* property. The mixed-list property allows lists of size $k - r$ on an attachable clique Q and lists of size k elsewhere, where $r \geq 0$ and $|Q| + r \leq k$. If S

is a precoloured clique of simplicial vertices, then $Q = N_G(S)$ is attachable in $G - S$; deleting the colours of S from the lists on Q gives precisely such a mixed-list problem. This provides the link between local colouring arguments and extension across separators.

To prove the main result, three ingredients are needed. First, we prove an asymmetric mixed-list theorem for cobipartite graphs. A saturation argument reduces the problem to a canonical graph, where a nonzero coefficient of its graph polynomial establishes the required choosability by the Combinatorial Nullstellensatz of Alon [1]. Second, we establish an extension theorem for elementary graphs. Galvin's kernel method handles the underlying bipartite line graphs, while an extremal argument and a Hall-type deficiency lemma control homogeneous pairs arising from incomplete augmentations. These steps reduce a potential counterexample to the line-graph case. Third, we treat graphs of interval type by organizing their vertices into ordered clique fibres. The polynomial argument eliminates these fibres successively, transferring and then recovering Vandermonde factors across incomplete pairs. A refinement of the fibre representation permits chosen simplicial vertices to be retained throughout the elimination.

For the final step we use the structural refinement of Chudnovsky and Seymour [5]. Call a clique cutset *rich* if each side of the separation contains two vertices that are not true twins. Their theorem states that a claw-free graph with a rich clique cutset is either of interval type or has a *double clique cutset*: a clique cutset that splits into two cliques C_1, C_2 such that one side sees only C_1 and the other only C_2 (precise definitions are in Section 7.1). The extension property composes across double clique cutsets.

2 Preliminaries

2.1 Notation and conventions

We allow the empty graph and set its clique number, chromatic number and choice number equal to zero. The empty set is a clique. Clique parts in a cobipartition may be empty. All graph parameters and list sizes used as bounds are integers. Empty products equal 1, and $0! = 1$. In counterexamples to (P_k), we may assume without loss of generality that every unprescribed list has size exactly k.

For $A \subseteq V(G)$, let $G[A]$ denote the subgraph induced by A, and write $G - A = G[V(G) \setminus A]$. For a vertex v, let $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$ denote its open and closed neighborhoods. For a set A, its *external neighborhood* is

$$N_G(A) = \left(\bigcup_{a \in A} N_G(a) \right) \setminus A.$$

We omit the subscript when the graph is clear. Two adjacent vertices are *true twins* if their closed neighborhoods are equal. Disjoint vertex sets form a *complete pair* if every edge between them is present, an *anticomplete pair* if none is present, and an *incomplete pair* otherwise.

A graph is *cobipartite* if its vertex set can be partitioned into two cliques, and *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. The *line graph* of a graph G, denoted by $\mathcal{L}(G)$, has vertex set $E(G)$, with two distinct vertices adjacent if their corresponding edges have a common endpoint. We write $\mathrm{st}_G(w)$ for the set of edges incident with a vertex w of G. Multigraphs occur only as roots of line graphs; they may have parallel edges but no loops. For a list assignment L and a set T in its domain, put $L(T) = \bigcup_{v \in T} L(v)$. For a coloring f, let $f(T) = \{f(v) : v \in T\}$. A proper coloring of a clique is injective, so Hall's theorem gives

$$K \text{ is } L\text{-colorable} \quad \Longleftrightarrow \quad |L(T)| \geq |T| \quad \text{for every } T \subseteq K, \tag{2.1}$$

whenever K is a clique. The same inequalities characterize an injective choice of colors on any vertex set, whether or not it is a clique.

See Table 1 in the appendix for the recurring notation. Symbols introduced inside an individual proof have the contextual meaning specified there.

2.2 The two extension properties

Definition 2.1. A clique K of a graph G is *attachable* if $N_G(v) \setminus K$ is a clique for every $v \in K$. By convention, the empty clique is attachable.

We use repeatedly the following property which immediately follows from the definition of attachable clique.

If $x \in K$ has a neighbour $y \notin K$ then every neighbour of x that is not adjacent to y lies in$K$. (2.2)

Definition 2.2. Let $k \geq \omega(G)$ be an integer.

- A (P_k)-*instance* is a tuple (G, S, ϕ, L), where S is a clique of simplicial vertices of G, possibly empty, ϕ is a proper coloring of $G[S]$, and L is a list assignment on $V(G) \setminus S$ with $|L(v)| \geq k$ for every $v \notin S$. The instance is *colorable* if G has a proper coloring extending ϕ and using a color from $L(v)$ at every $v \notin S$. We say that G satisfies (P_k) if every (P_k)-instance on G is colorable.
- An (M_k)-*instance* is a tuple (G, K, r, L), where $r \geq 0$ is an integer, K is an attachable clique of G with $|K| + r \leq k$, and L is a list assignment on $V(G)$ satisfying

$$|L(v)| \geq \begin{cases} k - r, & v \in K, \\ k, & v \notin K. \end{cases}$$

  The instance is *colorable* if G has a proper L-coloring. We say that G satisfies (M_k) if every (M_k)-instance on G is colorable.

The next lemma explains both the definition of attachability and the form of the reduced list bound in (M_k).

Lemma 2.3 (Simplicial cliques)**.** *Let K be a clique of simplicial vertices in a graph G, and put $Q = N_G(K)$. Then the vertices of K are pairwise true twins and $K \cup Q$ is a clique. Moreover, if G is claw-free, then:*

(a) *The clique Q is attachable in $G - K$.*

(b) *Given a (P_k)-instance (G, K, ϕ, L), define lists on $G - K$ by*

$$L'(v) := \begin{cases} L(v) \setminus \phi(K), & v \in Q, \\ L(v), & v \notin K \cup Q. \end{cases}$$

Then $(G - K, Q, |K|, L')$ is an (M_k)-instance, and it is colorable if and only if (G, K, ϕ, L) is colorable.

Proof. The assertions are immediate when $K = \varnothing$, so assume that $K \neq \varnothing$. If $u, v \in K$ are distinct, simpliciality of u implies that every neighbor of u other than v is adjacent to v. Thus $N_G[u] \subseteq N_G[v]$, and the reverse inclusion follows symmetrically. Consequently all vertices of K have closed neighborhood $K \cup Q$, which is a clique.

Suppose that G is claw-free. If a vertex $q \in Q$ had nonadjacent neighbors x, y outside $K \cup Q$, then, for any $u \in K$, the vertices q, u, x, y would induce a claw centered at q. Thus Q is attachable in $G - K$, proving (a).

For (b), the clique $K \cup Q$ gives $|K| + |Q| \le \omega(G) \le k$. At most $|K|$ colors are removed from each list on Q, so

$$|L'(v)| \ge \begin{cases} k - |K|, & v \in Q, \\ k, & v \in V(G) \setminus (K \cup Q). \end{cases}$$

Together with (a), this makes $(G - K, Q, |K|, L')$ an (M_k)-instance. Every proper coloring of G extending ϕ and respecting L outside K restricts to an L'-coloring of $G - K$, since Q is complete to K. Conversely, an L'-coloring of $G - K$ combines with ϕ to color G, because $Q = N_G(K)$ and no vertex of Q uses a color in $\phi(K)$. □

Lemma 2.4 (Minimum counterexamples)**.** *Fix a nonnegative integer k and a class $\mathcal{C}$ closed under induced subgraphs. Among all uncolorable (P_k)-instances (G, S, ϕ, L) with $G \in \mathcal{C}$ and $\omega(G) \le k$, choose one with $|V(G)|$ minimum. Then G is connected, every simplicial vertex of G belongs to S, and every proper induced subgraph of G satisfies (P_k).*

Proof. Minimality is taken over all permissible choices of S, ϕ, L. Consequently, every proper induced subgraph satisfies (P_k) for every permissible instance on that subgraph. In particular, this is not a statement restricted to the inherited lists.

The empty graph is colorable. If G were disconnected, each component would be a proper induced subgraph. A nonempty clique S lies in one component; in that component use its prescribed coloring, and in every other component use the empty prescribed clique. The resulting colorings combine to color G, a contradiction.

Suppose that a simplicial vertex v lies outside S. In $G - v$ the vertices of S remain simplicial, so minimality gives a coloring extending ϕ. The set $N_G[v]$ is a clique, whence $|N_G(v)| \le \omega(G) - 1 \le k - 1$. At least one of the k or more colors in $L(v)$ is absent from its neighborhood. Giving that color to v completes a coloring of G, again a contradiction. □

2.3 Minimum order and constrained edge maximality

Every minimum-counterexample argument fixes the integer k first. Minimum order is taken over *all* uncolorable (P_k)-instances in the specified induced-subgraph-closed class $\mathcal{C}$ with clique number at most k. Thus it is available for every permissible list assignment on a smaller graph, including one obtained by changing lists. Lemma 2.4 records its immediate consequences.

Some arguments add a second extremal condition: with the vertex set, S, ϕ and the lists fixed (lists trimmed to size exactly k as in Section 2.1), the number of edges is maximized among counterexamples in $\mathcal{C}$ with clique number at most k in which S is still a clique of simplicial vertices. Such a choice exists because there are finitely many graphs on a fixed vertex set, adding edges cannot create a coloring, and the instance still has minimum order. Whenever an edge is added in a proof, membership in $\mathcal{C}$, the clique bound and simpliciality of S are checked before edge maximality is invoked.

Theorem 3.3 uses a different, explicitly stated saturation: the vertex set, lists, and designated attachable clique are fixed, and only cobipartite supergraphs preserving attachability and the clique bound are considered. No minimum-order assumption is used there. The interval-type and final assembly arguments need minimum order only.

All extremal choices are local to the proof in which they are made. An extremal property is used only under the hypotheses specified for that choice. In particular, when an earlier theorem is applied later, only its stated conclusion is used; the counterexample selected in the later proof need not satisfy the extremal conditions employed in proving the earlier theorem.

2.4 Cobipartite graphs and matching saturation

Let H be a cobipartite graph with clique partition $V(H) = X \dot\cup Y$. Its *nonedge graph* F is the graph on the same vertex set whose edges are the nonadjacent X–Y pairs of H. Thus $F = \overline{H}$

is bipartite. A set is a clique of H if and only if it is independent in F. By Kőnig's theorem, the minimum size of a vertex cover of F equals its matching number $\nu(F)$; therefore

$$\omega(H) = |X| + |Y| - \nu(F). \tag{2.3}$$

Lemma 2.5 (Saturation)**.** *Let M be a maximum matching of F, and put $H^+ = K_{X\cup Y} - M$. Then $H \subseteq H^+$ and $\omega(H^+) = \omega(H)$. In particular, every L-coloring of H^+ is an L-coloring of H.*

Proof. Only nonedges of H are added, and the nonedge graph of H^+ is the matching M. Its matching number is $|M| = \nu(F)$, so (2.3) gives the equality of clique numbers. A coloring proper on a supergraph is proper on the original graph. □

Pairing the ends of M and giving every remaining vertex its own color also shows that $\chi(H) = \omega(H)$. The argument applies to each induced subgraph, so cobipartite graphs are perfect. They are claw-free because they have no independent set of size three.

Saturation preserves the clique number, but it need not preserve a designated attachable clique. The proof of Theorem 3.3 will therefore add edges only when attachability is preserved as well.

2.5 A coefficient criterion for list coloring

Fix a linear order on $V(G)$. The *graph polynomial* is

$$P_G(\mathbf{x}) = \prod_{\substack{uv\in E(G)\\ u<v}} (x_v - x_u) \in \mathbb{R}[x_v : v \in V(G)].$$

For nonnegative integers $d(v)$, write $\mathbf{x}^d = \prod_v x_v^{d(v)}$ and let $[\mathbf{x}^d]P$ denote its coefficient in P. The following interpolation identity yields the form of the Combinatorial Nullstellensatz [1] used below.

Lemma 2.6 (Coefficient formula)**.** *Let $P \in \mathbb{R}[x_1, \dots, x_n]$, and let $d_1, \dots, d_n$ be nonnegative integers with $\deg P \le \sum_i d_i$. For each i, let $T_i \subseteq \mathbb{R}$ have $d_i + 1$ elements and put $\psi_i(x) = \prod_{a\in T_i}(x - a)$. Then*

$$[x_1^{d_1} \cdots x_n^{d_n}]P = \sum_{(a_1,\dots,a_n)\in T_1\times\cdots\times T_n} \frac{P(a_1, \dots, a_n)}{\prod_{i=1}^n \psi_i'(a_i)}. \tag{2.4}$$

In particular, a nonzero coefficient implies that P is nonzero at some point of this Cartesian product.

Proof. For a set T of $d+1$ distinct real numbers, Lagrange interpolation gives

$$\sum_{a\in T} \frac{a^j}{\prod_{b\in T\setminus\{a\}}(a-b)} = \begin{cases} 0, & 0 \le j < d, \\ 1, & j = d. \end{cases}$$

All denominators are nonzero. Apply the product of these linear functionals to each monomial $x_1^{e_1} \cdots x_n^{e_n}$ of P. If $e_i < d_i$ for some i, the result is zero. If $e_i > d_i$ for some i, the total-degree bound forces $e_j < d_j$ for another j, so the result is again zero. The only surviving monomial has $e_i = d_i$ for all i, and its multiplier is 1. Linearity proves the identity. For $n = 0$, it is the identity for a constant polynomial, with the usual empty-product convention. □

All colors appearing in a finite list-coloring instance, including prescribed colors, can be identified injectively with real numbers. We make this identification whenever graph polynomials are used.

Corollary 2.7. *Let $Z \subseteq V(G)$, and let ϕ be a* proper coloring of $G[Z]$. *Define the polynomial in the variables indexed by $V(G) \setminus Z$ by*

$$P_{G,\phi} = \prod_{\substack{uv \in E(G-Z) \\ u<v}} (x_v - x_u) \prod_{\substack{vz \in E(G) \\ v \notin Z,\ z \in Z}} (x_v - \phi(z)).$$

Suppose that nonnegative integers $d(v)$, $v \notin Z$, satisfy

$$\sum_{v \notin Z} d(v) = \deg P_{G,\phi} \qquad \textit{and} \qquad \left[\prod_{v \notin Z} x_v^{d(v)} \right] P_{G,\phi} \neq 0.$$

Then ϕ extends to a proper coloring of G from every list assignment on $V(G) \setminus Z$ with $|L(v)| \geq d(v) + 1$.

Proof. For each $v \notin Z$, choose a subset of $L(v)$ of size $d(v)+1$ and use it as the interpolation set in Lemma 2.6. At a point where $P_{G,\phi}$ is nonzero, adjacent uncolored vertices receive different colors, and each uncolored vertex avoids the colors on its neighbors in Z. Edges within Z are already properly colored by hypothesis. The resulting coloring therefore extends ϕ and respects every list. If $Z = V(G)$, the polynomial is 1 and the conclusion simply says that the already proper coloring requires no extension. □

3 Cobipartite graphs: the mixed-list theorem

We prove that in a cobipartite graph an attachable clique C can be colored from lists of size $|C|$, provided that all remaining lists have size at least the clique number. The short lists need not have any relation to the other lists. The proof has two parts: an explicit coefficient calculation for a canonical graph, and a saturation argument reducing a hypothetical counterexample to that graph.

3.1 Asymmetric lists

We first derive an asymmetric list-coloring lemma from [14, Corollary 2.6(b)]. Besides handling several boundary cases, it will be used again in the elementary-graph argument.

Lemma 3.1 (Asymmetric cobipartite lists)**.** *Let H admit a clique partition $V(H) = X \dot\cup Y$, where $|X| = a$ and $|Y| = b$. If a list assignment L satisfies*

$$|L(x)| \geq \omega(H) \ \textit{ for all } x \in X \qquad \textit{and} \qquad |L(y)| \geq b \ \textit{ for all } y \in Y$$

then H is L-colorable.

Proof. The assertion is immediate if either part is empty. Otherwise, let M be a maximum matching in the bipartite graph of missing X–Y pairs. By Lemma 2.5, adding all missing pairs outside M yields a supergraph

$$H^+ = K_{X \cup Y} - M$$

with $\omega(H^+) = \omega(H) = q$. The lists have size at least $q \geq a$ on X and at least b on Y. Corollary 2.6 (b) of [14] therefore gives an L-coloring of H^+, which also colors H. □

3.2 The canonical graph and its coefficient

Let $m, t \geq 1$ and $u \geq 0$ be integers, and put $q_0 = m + t$ and $q = q_0 + u$. Start with pairwise disjoint sets

$$C = \{c_1, \ldots, c_m\}, \qquad Z = \{z_1, \ldots, z_m\}, \qquad U = \{v_1, \ldots, v_u\},$$
$$X = \{x_1, \ldots, x_t\}, \qquad Y = \{y_1, \ldots, y_t\},$$

and a partition $C = C^+ \dot\cup C^-$. Either of the two parts may be empty. Now, define $J = J(m, t, u; C^+, C^-)$ by specifying that its nonedges are exactly

$$\begin{aligned} &c_i z_i \quad (1 \leq i \leq m), \qquad x_j y_j \quad (1 \leq j \leq t), \\ &c y_j \quad (c \in C^+,\ 1 \leq j \leq t), \qquad c x_j \quad (c \in C^-,\ 1 \leq j \leq t). \end{aligned} \tag{3.1}$$

This means that every vertex of U is universal in J. Figure 1 illustrates the nonedge graph.

Now, the complement of $J - U$ has bipartition

$$C^+ \cup X \cup \{z_i : c_i \in C^-\}, \qquad C^- \cup Y \cup \{z_i : c_i \in C^+\}.$$

Both parts have size q_0. The vertices of U are isolated in the nonedge graph and can be assigned to either part, so J is cobipartite. Its nonedge graph has a matching of size q_0, consisting of the pairs $c_i z_i$ and $x_j y_j$. No larger matching is possible, since there are only $2q_0$ nonisolated vertices. Thus (2.3) gives

$$\omega(J) = 2q_0 + u - q_0 = q.$$

The clique C is attachable: outside C, the neighbors of c_i are $Z \setminus \{z_i\}$, all of U, and all of X or all of Y according as $c_i \in C^+$ or $c_i \in C^-$. These sets together form a clique.

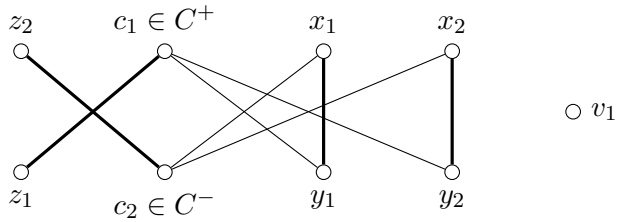


Figure 1: The nonedge graph of $J(2, 2, 1; \{c_1\}, \{c_2\})$. Thick edges are the matching pairs $c_i z_i$ and $x_j y_j$; thin edges are the additional pairs in (3.1). The isolated vertex v_1 is universal in J. Two distinct vertices are adjacent in J exactly when they are not joined in this drawing.

The exponents below deliberately give every vertex of C the smallest bound compatible with coloring the clique C. The other exponents are chosen so that an interpolation grid forces colors to occur on matched nonedges. This makes the coefficient calculation explicit.

Order the vertices as

$$c_1, z_1, \ldots, c_m, z_m, x_1, y_1, \ldots, x_t, y_t, v_1, \ldots, v_u,$$

and define P_J using this order.

Lemma 3.2 (Canonical coefficient)**.** *Define nonnegative integers $d(w)$ by*

$$\begin{aligned} d(c_i) &= m - 1, & d(z_i) &= q - 1, \\ d(x_j) &= d(y_j) = q - 1, & d(v_\ell) &= m + \ell - 1. \end{aligned}$$

Then $\sum_w d(w) = |E(J)|$ and

$$[\mathbf{x}^d] P_J = (-1)^{m(t+u)} m!\, t! \neq 0. \tag{3.2}$$

Consequently, J is colorable from arbitrary lists having at least m colors on C, at least q on $Z \cup X \cup Y$, and at least $m + \ell$ at v_ℓ for $1 \leq \ell \leq u$.

Proof. We first verify the degree condition, then identify every nonzero term in the coefficient formula and compute its value.

Degree. Put $J_0 = J - U$. This graph has $2q_0$ vertices and $q_0 + mt$ nonedges. Hence

$$\begin{aligned} |E(J_0)| &= \binom{2q_0}{2} - q_0 - mt \\ &= m(m-1) + (m+2t)(q_0 - 1), \end{aligned}$$

which is the exponent sum when $u = 0$. Adjoining U adds $2q_0u + \binom{u}{2}$ edges. The exponent sum increases by

$$u(m+2t) + \sum_{\ell=1}^{u}(m+\ell-1) = 2q_0u + \binom{u}{2}.$$

Thus the exponent sum equals $|E(J)|$ for every $u \geq 0$.

Interpolation sets. In Lemma 2.6, use the sets

$$\begin{aligned} T(c_i) &= \{1, \ldots, m\}, \\ T(z_i) = T(x_j) = T(y_j) &= \{1, \ldots, q\}, \\ T(v_\ell) &= \{1, \ldots, m\} \cup \{q_0 + 1, \ldots, q_0 + \ell\}. \end{aligned}$$

Each has $d(w) + 1$ elements. Let a be a point of this grid with $P_J(a) \neq 0$, viewed as a proper coloring of J. The clique C uses all colors $1, \ldots, m$. Since v_1 is universal, its only available color is $h_1 = q_0 + 1$. Inductively, v_ℓ must use $h_\ell = q_0 + \ell$: all its other available colors occur on C or on $v_1, \ldots, v_{\ell-1}$.

Every vertex of J_0 is adjacent to all of U, so J_0 uses only the colors $1, \ldots, q_0$. Its two clique parts both have q_0 vertices. Each color therefore occurs once in each part, and its two occurrences form a nonedge of J_0. The only nonneighbor of z_i in J_0 is c_i, so $a(z_i) = a(c_i)$. These pairs use the colors $1, \ldots, m$. The remaining colors $m+1, \ldots, q_0$ must occur on the pairs x_j, y_j. Conversely, all such assignments are proper. Thus there are exactly $m!t!$ nonzero grid points: independently permute the first m colors among the c_i, z_i pairs and the last t colors among the x_j, y_j pairs.

The value of a summand for $u = 0$. Write $A_0 = \{1, \ldots, m\}$ and $B_0 = \{m+1, \ldots, q_0\}$, and put

$$V_A = \prod_{\substack{a<a' \\ a,a' \in A_0}} (a' - a), \qquad V_B = \prod_{\substack{b<b' \\ b,b' \in B_0}} (b' - b),$$

$$\Gamma = \prod_{a \in A_0} \prod_{b \in B_0} (b - a).$$

Start with the complete graph on $V(J_0)$ minus the q_0 equal-colored pairs. Two such pairs of distinct colors contribute the fourth power of their color difference: their vertices form consecutive two-vertex blocks in our order. To obtain J_0, remove one further edge between each C-pair and each X, Y-pair. Its factor is always $b - a$, where $a \in A_0$ and $b \in B_0$, since the C-pair precedes the X, Y-pair. It follows that

$$P_{J_0}(a) = V_A^4 V_B^4 \Gamma^3. \tag{3.3}$$

In particular, this numerator is independent of both permutations and of the partition C^+, C^-.

Let $\varphi_s(x) = \prod_{j=1}^{s}(x - j)$. The denominator in the coefficient formula is

$$D_0 = \prod_{a \in A_0} \varphi'_m(a)\varphi'_{q_0}(a) \prod_{b \in B_0} \varphi'_{q_0}(b)^2.$$

For $a \in A_0$ and $b \in B_0$,

$$\varphi'_{q_0}(a) = \varphi'_m(a) \prod_{b \in B_0} (a - b),$$
$$\varphi'_{q_0}(b) = \prod_{a \in A_0} (b - a) \prod_{b' \in B_0 \setminus \{b\}} (b - b').$$

The products of the squared within-set derivative factors are V_A^4 and V_B^4. The cross factors contribute $(-1)^{mt}\Gamma$ from the first line and Γ^2 from the second. Consequently,

$$D_0 = (-1)^{mt} V_A^4 V_B^4 \Gamma^3. \tag{3.4}$$

Every nonzero summand therefore equals $(-1)^{mt}$.

Adjoining universal vertices. Suppose $u \geq 1$, and compare the calculation with the one for $u - 1$. Write $h = h_u = q_0 + u$. Every interpolation set on $Z \cup X \cup Y$ gains the element h; the sets on C and on the earlier universal vertices are unchanged. The numerator gains the factor

$$\prod_{a \in A_0} (h - a)^2 \prod_{b \in B_0} (h - b)^2 \prod_{\ell < u} (h - h_\ell).$$

The denominator gains

$$\left(\prod_{a \in A_0} (a - h) \prod_{b \in B_0} (b - h)^2 \right) \left(\prod_{a \in A_0} (h - a) \prod_{\ell < u} (h - h_\ell) \right).$$

The second parenthesis is the derivative factor for the new vertex v_u. Taking the quotient shows that every summand is multiplied by

$$\prod_{a \in A_0} \frac{h - a}{a - h} = (-1)^m.$$

After all u universal vertices have been adjoined, each of the $m!t!$ nonzero summands is $(-1)^{m(t+u)}$. Summing proves (3.2). The list-coloring conclusion follows from Corollary 2.7 with $Z = \varnothing$. □

3.3 The mixed-list theorem

Theorem 3.3 (Mixed lists on cobipartite graphs)**.** *Let H be a cobipartite graph, let C be an attachable clique of H, and let $k \geq \omega(H)$ be an integer. Then H is colorable from every list assignment satisfying*

$$|L(c)| \geq |C| \quad (c \in C), \qquad |L(v)| \geq k \quad (v \notin C).$$

Proof. If $C = \varnothing$, take any clique partition X, Y of H and apply Lemma 3.1; the lists have size at least $k \geq \omega(H) \geq |Y|$. This includes the empty graph. If $C = V(H)$, Hall's theorem colors the clique C from its lists.

Suppose, for a contradiction, that the theorem fails for an instance (H, C, L, k) not covered by these cases. Put $A = V(H) \setminus C$ and $m = |C| \geq 1$. Among the cobipartite supergraphs of H on the same vertex set, choose one with as many edges as possible subject to

$$\omega(H) \leq k \qquad \text{and} \qquad C \text{ remaining attachable.}$$

Such a supergraph exists, since the original graph is admissible. It is still not L-colorable, since a coloring of it would color the original graph. Keep the notation H for this supergraph, and let $F = \overline{H}$ be its bipartite nonedge graph. Choose a maximum matching M of F minimizing

$$t = |M \cap E(F[A])|.$$

For each $c \in C$, the set $N_F(c)$ is contained in A and is independent in F. Attachability of C is equivalent to the following *cover condition*:

$$N_F(c) \text{ meets every edge of } F[A] \qquad (c \in C). \tag{3.5}$$

Indeed, its complement in A is $N_H(c) \setminus C$, which must be a clique of H.

Deleting edges of F corresponds to adding edges of H and preserves bipartiteness. If M is retained, the matching number cannot decrease, and it cannot increase under edge deletion. Thus (2.3) shows that the clique number is unchanged. Our choice of H consequently forbids any deletion of an edge of F that preserves both M and the cover condition.

Claim 1. *The graph $F[A]$ consists of precisely the t edges of M with both ends in A.*

Proof. Delete from $F[A]$ every edge not in M. The sets $N_F(c)$ are unchanged, so the cover condition survives, and M survives as well. Edge maximality of H implies that no edge was deleted. □

Write these t edges as x_jy_j, $1 \le j \le t$.

Claim 2. *For every $c \in C$ and j, exactly one of cx_j, cy_j belongs to $E(F)$. Apart from these covering edges, the only C–A edges of F are the edges of M. Every C–A edge of M ends at a vertex isolated in $F[A]$.*

Proof. The cover condition gives at least one of cx_j, cy_j. Both cannot be present, since together with x_jy_j they would form a triangle in the bipartite graph F. Keep all these covering edges and all edges of M, and delete any other C–A edge. This preserves the cover condition and M, so maximality permits no such edge. Finally, an endpoint in A of a C–A matching edge cannot also be an endpoint of one of the matching edges x_jy_j. The last assertion follows from Claim 1. □

Claim 3. *We have $t \ge 1$, and every vertex $c \in C$ is matched by M to a distinct vertex $z_c \in A$ isolated in $F[A]$.*

Proof. If $t = 0$, Claim 1 says that A is a clique of H. Lemma 3.1, applied with first part A and second part C, gives an L-coloring, a contradiction. Hence $t \ge 1$.

Suppose that some $c \in C$ is unmatched. By Claim 2, it has an edge to one end of x_jy_j for any j. Replace x_jy_j in M by this edge. The new matching has the same size as M, but has one fewer edge with both ends in A, contradicting the choice of M. Thus every c is matched. Its partner lies in A because C is independent in F, and is isolated in $F[A]$ by Claim 2. □

Claim 4. *The endpoints of the edges x_jy_j can be labeled, and C can be partitioned into $C^+ \dot\cup C^-$, so that in F every vertex of C^+ is adjacent to all y_j and every vertex of C^- is adjacent to all x_j.*

Proof. Fix a bipartition (R_0, R_1) of F, and label the endpoints so that $x_j \in R_0$ and $y_j \in R_1$. Put $C^+ = C \cap R_0$ and $C^- = C \cap R_1$. For each c and j, Claim 2 supplies exactly one edge, and bipartiteness forces its other endpoint to lie in the opposite part. This is the claimed pattern. □

Label $C = \{c_1, \dots, c_m\}$ and write $z_i = z_{c_i}$. Every remaining vertex of A, outside the sets $\{z_i\}$ and $\{x_j, y_j\}$, is isolated in F by Claims 1 and 2; let these vertices be $U = \{v_1, \dots, v_u\}$. The preceding claims show that the nonedges of H are exactly those in (3.1). Thus $H = J(m, t, u; C^+, C^-)$ and

$$q = \omega(H) = m + t + u \le k.$$

The lists on C have at least m colors, those on $Z \cup X \cup Y$ have at least $k \geq q$, and those on v_ℓ have at least $k \geq q \geq m + \ell$. Hence Lemma 3.2 gives an L-coloring of H, the required contradiction. □

Corollary 3.4. *Every cobipartite graph H satisfies* (M_k) *for every integer $k \geq \omega(H)$.*

Proof. In an (M_k)-instance (H, Q, r, L), the bound $|Q| + r \leq k$ gives $|L(q)| \geq k - r \geq |Q|$ on Q. All other lists have size at least k. Apply Theorem 3.3 with $C = Q$. This also covers $Q = \varnothing$ and $r = 0$. □

4 Line graphs of bipartite multigraphs

We prove the mixed-list property (M_k) for line graphs of bipartite multigraphs. Galvin's kernel argument first gives a stronger list bound on a full star. A classification of attachable cliques then reduces the general case to this full-star statement.

Throughout this section, B is a bipartite multigraph with bipartition (P, P'), and $G = \mathcal{L}(B)$. Write $\mathrm{st}_B(w)$ for the set of edges incident with w, counting parallel edges separately. Every nonempty clique of $\mathcal{L}(B)$ is contained in a star: two nonparallel incident edges force a common endpoint for all edges of the clique, since B has no triangle; a family of parallel edges already has a common endpoint. Consequently $\omega(\mathcal{L}(B)) = \Delta(B)$, with both parameters taken to be zero when $E(B) = \varnothing$. Write $E(w, u)$ for the parallel edges joining w and u. We call u *pendant relative to* w when w is its only distinct neighbor; this condition does not exclude parallel edges.

4.1 Kernels and Galvin's orientation

An *orientation* of a graph G replaces each edge by one or both of its arcs. A *kernel* of a digraph is an independent set K such that every vertex outside K has an arc into K; a digraph is *kernel-perfect* if every induced subdigraph has a kernel. Outdegrees $d^+(v)$ count distinct out-neighbors. We begin with the kernel lemma of Bondy, Boppana and Siegel (see Alon and Tarsi [2, Remark 2.4], or Bondy and Murty [3, Theorem 14.19]).

Lemma 4.1 (Kernel lemma)**.** *Let D be a kernel-perfect orientation of G. If $|L(v)| \geq d_D^+(v) + 1$ for every v, then G is L-colorable.*

The following lemma is essentially a retelling of [8, Theorem 4.1].

Lemma 4.2 (Galvin's orientation)**.** *Let $c : E(B) \to \{1, \ldots, k\}$ be a proper edge coloring. Orient $\mathcal{L}(B)$ by directing, at each vertex of P, from smaller to larger colors, and at each vertex of P', from larger to smaller colors (parallel edges get both arcs). The result is a kernel-perfect orientation of $\mathcal{L}(B)$ with $d^+(e) \leq k - 1$ for every edge e. If $w \in P$ and $\mathrm{st}(w)$ receives exactly the colors $1, \ldots, q$, then $d^+(e) \leq q - 1$ for $e \in \mathrm{st}(w)$.*

Kőnig's edge-coloring theorem provides a proper k-edge-coloring whenever $k \geq \Delta(B)$, and a permutation of the colors makes any prescribed star of size q receive $\{1, \ldots, q\}$.

Lemma 4.3 (Full stars)**.** *Let $k \geq \Delta(B)$, and let either $Q = \varnothing$ or $Q = \mathrm{st}_B(w)$ for some $w \in V(B)$. Let $r \geq 0$ satisfy $|Q| + r \leq k$. If $|L(e)| \geq k - r$ on Q and $|L(e)| \geq k$ elsewhere, then $\mathcal{L}(B)$ is L-colorable.*

Proof. If $Q = \varnothing$, use any proper k-edge-coloring of B and apply Lemmas 4.2 and 4.1; this is Galvin's ordinary list-edge-coloring theorem. Otherwise exchange the bipartition classes if necessary so that $w \in P$, and permute the colors of a proper k-edge-coloring so that $\mathrm{st}_B(w)$ receives $1, \ldots, |Q|$. Lemma 4.2 gives outdegree at most $|Q| - 1 \leq k - r - 1$ on Q and at most $k - 1$ elsewhere. An application of the kernel lemma finishes the proof. □

4.2 Attachable cliques of a bipartite line graph

Lemma 4.4 (Classification)**.** *Let $Q \neq \varnothing$ be an attachable clique of $\mathcal{L}(B)$. Then $Q \subseteq \mathrm{st}(w)$ for some vertex w, and for every edge $wu \in Q$,*

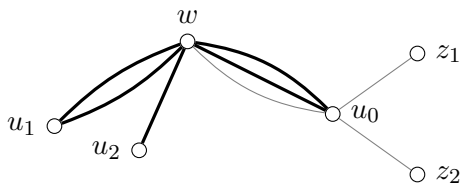
$$\mathrm{st}(w) \setminus \big(Q \cup E(w,u)\big) = \varnothing \quad \textit{or} \quad u \textit{ is pendant relative to } w. \tag{4.1}$$

Consequently at least one of the following holds:

(i) $Q = \mathrm{st}(w)$*;*

(ii) *every far end u of an edge $wu \in Q$ is pendant relative to w;*

(iii) *w has exactly one nonpendant neighbor u_0, and $Q = (\mathrm{st}(w) \setminus E(w,u_0)) \cup Q_0$ with $\varnothing \neq Q_0 \subsetneq E(w,u_0)$ (Figure 2).*



Thick edges form Q; u_1, u_2 are pendant relative to w.
The bundle $E(w,u_0)$ has three edges, two in Q_0.
$N_0 = \mathrm{st}(u_0) \setminus E(w,u_0) = \{u_0z_1, u_0z_2\}$.

Figure 2: An attachable clique in $\mathcal{L}(B)$, drawn in the root B. Any edge at u_0 other than wu_0 is nonadjacent (in $\mathcal{L}(B)$) to the edges wu_1, wu_2, so (4.1) forces everything outside $E(w,u_0)$ into Q.

Proof. Pairwise incident edges of a triangle-free multigraph share a vertex; this gives w. Let $wu \in Q$, let $wt \in \mathrm{st}(w) \setminus Q$ with $t \neq u$, and let ut' be an edge with $t' \neq w$. Bipartiteness puts t and t' in different parts, so wt and ut' are disjoint edges; they are neighbors of wu in $\mathcal{L}(B)$, nonadjacent, and outside Q (as $ut' \notin \mathrm{st}(w)$). This contradicts attachability and proves (4.1).

Assume $Q \neq \mathrm{st}(w)$ and some far end u_0 of an edge of Q is nonpendant. Then (4.1) for wu_0 gives $\mathrm{st}(w) \setminus E(w,u_0) \subseteq Q$, so $Q_0 = Q \cap E(w,u_0)$ is nonempty, and it is proper because $Q \neq \mathrm{st}(w)$. Any other neighbor u of w has an edge in Q (all of $\mathrm{st}(w) \setminus E(w,u_0)$ lies in Q), and $E(w,u_0) \setminus Q_0 \neq \varnothing$ lies outside $Q \cup E(w,u)$, so (4.1) makes u pendant. If no such u_0 exists, (ii) holds. □

4.3 The mixed-list lemma for bipartite line graphs

Lemma 4.5 (Mixed lists on bipartite line graphs)**.** *Let $G = \mathcal{L}(B)$ with B bipartite, and $k \geq \omega(G)$. Then G satisfies* (M_k).

Proof. Let (G, Q, r, L) be an (M_k)-instance. Note $\Delta(B) = \omega(G) \leq k$. If $Q = \varnothing$ or $r = 0$, all lists have at least k colors and Lemma 4.3 with its explicit empty-set alternative applies. Otherwise use Lemma 4.4.

Case (i). This is Lemma 4.3.

Case (ii). Color the clique Q injectively from its lists, which have at least $k - r \geq |Q|$ colors. Since the far ends of Q are pendant, the vertices of $G - Q$ adjacent to Q are exactly $Q' = \mathrm{st}(w) \setminus Q$. Delete the colors used on Q from the lists on Q'; these lists keep at least $k - |Q|$ colors. In $B - Q$ (delete the edges of Q) the set Q' is the full star at w, and $|Q'| + |Q| = \deg_B(w) \leq k$, so Lemma 4.3 colors $\mathcal{L}(B - Q) = G - Q$ compatibly.

Case (iii). Put $E_0 = E(w, u_0)$ and $N_0 = \mathrm{st}(u_0) \setminus E_0 \neq \varnothing$. First color the whole clique $\mathrm{st}(w) = Q \cup (E_0 \setminus Q_0)$ from L. Hall's condition (2.1) holds: a nonempty set $T \subseteq Q$ has $|L(T)| \geq k - r \geq |Q| \geq |T|$, and a set T meeting $E_0 \setminus Q_0$ has $|L(T)| \geq k \geq \deg_B(w) \geq |T|$. Let f be this coloring. The remaining graph is $G - \mathrm{st}(w) = \mathcal{L}(B - w)$. All neighbors of w other than u_0 are pendant, so the only vertices of $\mathcal{L}(B - w)$ adjacent to $\mathrm{st}(w)$ are those of N_0, which are adjacent exactly to E_0. Delete $f(E_0)$ from the lists on N_0: they keep at least $k - |E_0|$ colors, N_0 is the full star at u_0 in $B - w$, and $|N_0| + |E_0| = \deg_B(u_0) \leq k$. Lemma 4.3 colors $\mathcal{L}(B - w)$, and together with f this is an L-coloring of G. □

5 Elementary graphs

We now prove (P_k) for elementary graphs. In this section, we strengthen the chromatic-choosability theorem of Narayanan, Vasudevan and Somasundaram [14] to a precoloring extension form. We use their Deficiency Lemma directly. We then prove what we call the homogeneous-pair lemma.

5.1 Representation and auxiliary results

An edge is *flat* if it lies in no triangle. *Augmenting* a flat edge xy means replacing x and y by disjoint nonempty cliques X and Y, joined by at least one X–Y edge, with X complete to $N(x) \setminus \{y\}$ and Y complete to $N(y) \setminus \{x\}$. The pair (X, Y) is the *augment*, and X, Y are its *shores*. Several flat edges forming a matching may be augmented simultaneously [13]. Adjacencies between different augments are inherited from the original vertices.

Theorem 5.1 (Maffray–Reed [13])**.** *A graph is elementary if and only if it is obtained from the line graph of a bipartite multigraph by augmenting pairwise vertex-disjoint flat edges. Elementary graphs are claw-free and perfect, and the class is closed under induced subgraphs.*

We call this a *representation*, and its bipartite multigraph the *root*. By [14, Lemma 3.3], we may normalize the root so that each augmented pair consists of edges pw, wq with $p \neq q$ and $\deg(w) = 2$: an isolated pair of parallel edges is replaced by a two-edge path.

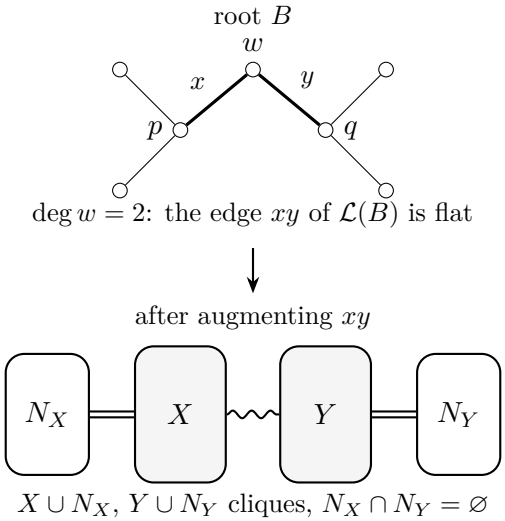


Figure 3: Augmenting a flat edge. Double lines denote complete pairs, the wavy line a specified X–Y adjacency with at least one cross edge. The graph on $X \cup Y$ is connected; the bipartite graph of cross edges need not be connected.

Lemma 5.2 (Augments)**.** *Fix a normalized representation of an elementary graph G. For an augment (X, Y), put $N_X = N(X) \setminus Y$ and $N_Y = N(Y) \setminus X$.*

(a) *Every vertex outside $X \cup Y$ is complete or anticomplete to each of X and Y; $X \cup N_X$ and $Y \cup N_Y$ are cliques; and $N_X \cap N_Y = \varnothing$, so no outside vertex sees both X and Y.*

(b) *Adding edges between X and Y preserves elementarity.*

(c) *Any collection of complete augments can be absorbed into the root, leaving the remaining augments unchanged. Moreover, if G' is an induced subgraph in which the surviving sides of every augment form a complete pair, then G' is a bipartite line graph; surviving sides may be empty.*

Proof. Part (a) is [14, Lemma 3.3], and (b) follows from the representation. For (c), use the construction of [14, Lemma 3.8]: replace pw, wq by parallel edges of the required sizes, allowing size zero for a deleted side. Delete root edges corresponding to deleted unaugmented vertices. Note that, this process can be done partially as well. That is, replace only the complete augments to be absorbed. The center of another designated flat pair cannot be p or q as its degree would be two, so its support would contain one of pw, wq, contrary to disjointness of the supports. Thus the other designated pairs remain flat and keep their original augments. This proves the partial-absorption assertion as well.

□

We need the deficiency lemma proved in [14], which we restate here adjusting the notation.

Lemma 5.3 ([14, Lemma 2.5])**.** *Let $H = K_{a+b} - M$, where the vertex set is $X \dot\cup Y$ with $|X| = a$, $|Y| = b$, and M is a matching of μ pairs xy with $x \in X$, $y \in Y$. Let X_0, Y_0 be the sets of matched vertices and $X_* = X \setminus X_0$, $Y_* = Y \setminus Y_0$. Let R be lists with $|R(x)| \geq a$ on X and $|R(y)| \geq b$ on Y. If H is not R-colorable, there is a set $T \subseteq X_* \cup Y_*$ meeting both X_* and Y_* with*

$$|R(T)| \leq \mu + |T| - 1. \tag{5.1}$$

In particular, H is R-colorable if either unmatched part is empty.

5.2 A homogeneous-pair obstruction

Fix $k \geq 0$ and a hereditary class $\mathcal{C}$ of claw-free perfect graphs. An *extremal counterexample* (G, S, ϕ, L) is chosen as follows:

(i) minimize $|V(G)|$ over all uncolorable (P_k)-instances with $G \in \mathcal{C}$ and $\omega(G) \leq k$;

(ii) keeping $V(G)$, S, ϕ and L fixed, maximize $|E(G)|$ among all uncolorable instances on this vertex set with $G \in \mathcal{C}$, $\omega(G) \leq k$, and S a clique of simplicial vertices.

By Lemma 2.4? every proper induced subgraph of G satisfies (P_k).

Lemma 5.4 (Homogeneous-pair lemma)**.** *Let (G, S, ϕ, L) be an extremal counterexample chosen as above. There are no disjoint nonempty cliques $A_1, A_2 \subseteq V(G) \setminus S$ such that:*

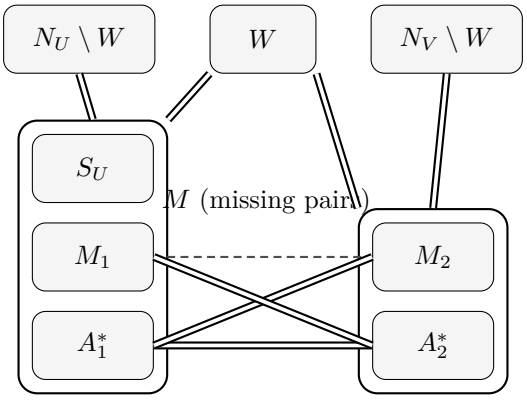
(i) *every vertex outside $A_1 \cup A_2$ is complete or anticomplete to each A_i;*

(ii) *writing $Q_i = N(A_i) \setminus (A_1 \cup A_2)$, each $A_i \cup Q_i$ is a clique;*

(iii) *$G[A_1 \cup A_2]$ is not complete, and $G + uv \in \mathcal{C}$ for every missing pair $u \in A_1$, $v \in A_2$.*

Proof. We adapt the list-copying argument of [14, Proposition 3.7]. Put $a_i = |A_i|$ and $W = Q_1 \cap Q_2$; then $|Q_i| \leq k - a_i$.

Saturation. Let F be the bipartite graph of missing A_1–A_2 pairs, and let M be a maximum matching of F, of size $\mu \geq 1$. If $uv \in E(F) \setminus M$, adding uv leaves the clique number of



$Q_1 = N_U \cup S_U$, $Q_2 = N_V$; $A_i \cup Q_i$ are cliques.
M_i are the matched vertices and A_i^* the unmatched vertices.
$K^* = M_1 \cup A_1^* \cup A_2^* \cup W$ is a clique of $G - M_2$.

Figure 4: The homogeneous pair of Lemma 5.4 after saturation. Double lines are complete pairs; S_U is anticomplete to V; the only missing pairs between A_1 and A_2 form the matching M.

$G[A_1 \cup A_2]$ unchanged by Lemma 2.5. Every new clique lies in $A_1 \cup A_2 \cup W$. Since W is a clique complete to $A_1 \cup A_2$, a maximum clique of $G[A_1 \cup A_2]$ together with W is a clique of G, so $\omega(G[A_1 \cup A_2]) + |W| \le \omega(G) \le k$; hence $\omega(G + uv) \le k$.

The addition preserves simpliciality and the prescribed coloring of S, since its ends lie outside S. Together with (iii), this contradicts edge maximality. Hence $F = M$.

Let M_i be the matched vertices of A_i, and put $A_i^* = A_i \setminus M_i$, $\alpha_i = |A_i^*|$, so $a_i = \mu + \alpha_i$. Both A_i^* are nonempty: otherwise color $G - (A_1 \cup A_2)$ by minimality, extending ϕ, and remove the colors on Q_i from lists on A_i. The remaining lists have size at least a_i, and Lemma 5.3 completes the coloring.

Copying a list. Set

$$m_i = \min_{\varnothing \neq T \subseteq A_i^*} (|L(T)| - |T|),$$

and interchange the indexed data so that $m_1 + \alpha_1 \le m_2 + \alpha_2$. Choose $\varnothing \neq Z \subseteq A_1^*$ attaining m_1, and $z \in Z$. On $G' = G - M_2$, replace every list on M_1 by $L(z)$. Minimum order gives a coloring f extending ϕ: S is retained and all unprescribed lists still have size k. Put $F_i = f(Q_i)$ and $R(v) = L(v) \setminus F_i$ for $v \in A_i$. Then $|R(v)| \ge a_i$. An R-coloring of $G[A_1 \cup A_2]$ would combine with f outside the pair to color G. Thus Lemma 5.3 gives $T \subseteq A_1^* \cup A_2^*$ meeting both parts with $|R(T)| \le \mu + |T| - 1$.

Write $t_i = |T \cap A_i^*|$. Since $L(T \cap A_2^*) \subseteq R(T) \cup F_2$,

$$m_2 + t_2 \le \mu + t_1 + t_2 - 1 + k - a_2, \qquad t_1 \ge m_2 + \alpha_2 - k + 1.$$

But $|L(Z)| = m_1 + |Z| \ge k$, so $T \cap Z = \varnothing$ would give

$$t_1 \le \alpha_1 - |Z| \le m_1 + \alpha_1 - k \le m_2 + \alpha_2 - k,$$

a contradiction. Put $I = T \cap Z \neq \varnothing$ and $T' = T \cup Z$. The choice of Z gives $|L(Z) \setminus L(I)| \le |Z \setminus I|$. All lists on Z lose the same set F_1, whence

$$|R(T')| \le |R(T)| + |Z \setminus T| \le \mu + |T'| - 1.$$

On the other hand, $M_1 \cup T'$ is a clique of G'. Its vertices in M_1 receive distinct colors from $L(z) \setminus F_1 = R(z) \subseteq R(T')$, and those in T' receive colors from their unchanged residual lists. Thus f assigns $\mu + |T'|$ distinct colors from $R(T')$, a contradiction. □

5.3 Precoloring extension

Theorem 5.5 (Elementary precoloring extension)**.** *Every elementary graph G satisfies* (P_k) *for every $k \geq \omega(G)$.*

Proof. Fix k and suppose the assertion fails. Choose an extremal counterexample in the elementary class. By Lemma 2.4, G is connected. Set $Q = N(S)$, with $Q = \varnothing$ when $S = \varnothing$. Lemma 2.3 reduces the extension problem to an (M_k)-instance $(G - S, Q, |S|, L')$. In particular, G is not cobipartite, by Corollary 3.4.

Fix a normalized representation. The clique S meets at most one side of each augment (X, Y): vertices of S are true twins, whereas a vertex of N_X sees X and misses Y, and conversely for N_Y. If S met both sides, therefore, $N_X = N_Y = \varnothing$; connectedness would give $G = G[X \cup Y]$, a contradiction. Orient each augment so that $S \cap Y = \varnothing$, and put $S_X = S \cap X$.

Suppose that $G[(X \setminus S_X) \cup Y]$ is not complete. If $S_X = \varnothing$ or $N_X \neq \varnothing$, then S_X is anticomplete to Y: a neighbor in Y and a vertex of N_X would be nonadjacent neighbors of a simplicial vertex in S_X. In particular, $X \setminus S_X \neq \varnothing$, because the augment has at least one $X - -Y$ edge and none of these edges leaves S_X. The pair $(A_1, A_2) = (X \setminus S_X, Y)$ now contradicts Lemma 5.4, with $Q_1 = N_X \cup S_X$ and $Q_2 = N_Y$; edge additions are allowed by Lemma 5.2(b).

It remains that $S_X \neq \varnothing$ and $N_X = \varnothing$. Let $Y_S = N(s) \cap Y$ for $s \in S_X$; this is independent of s because the vertices of S are twins. Simpliciality makes Y_S complete to X. Hence every missing pair of $G[(X \setminus S_X) \cup Y]$ joins $A_1 = X \setminus S_X$ to $A_2 = Y \setminus Y_S$, and both sets are nonempty because at least one such pair exists.

These are homogeneous cliques with

$$Q_1 = S_X \cup Y_S, \qquad Q_2 = Y_S \cup N_Y.$$

Both $A_i \cup Q_i$ are cliques, and their missing edges lie within the original augment. Lemma 5.4 again gives a contradiction.

Consequently every augment becomes complete after deleting S. By Lemma 5.2(c), $G - S$ is a bipartite line graph. Lemma 4.5 colors the reduced (M_k)-instance, and restoring ϕ gives the required extension. □

Remark 5.6. The attachment result in Section 7.4 will establish when adjoining a prescribed simplicial clique preserves elementarity; this closure property is not asserted here.

6 Graphs of interval type

The interval-type argument uses the ordered clique structure of the graphs of interval type introduced by Chudnovsky–Seymour [5]. After saturating each incomplete pair, we extract a coefficient one fiber at a time. Processing the first end of an incomplete pair transfers a Vandermonde factor to its other end; processing that end replaces the factor by a nonzero factorial. This local transfer is the reason crossing incomplete pairs can be handled.

For variables or constants $\zeta_1, \dots, \zeta_h$, write

$$\mathrm{V}(\zeta_1, \dots, \zeta_h) = \prod_{i<j} (\zeta_j - \zeta_i) = \det\left[\zeta_i^{j-1}\right]_{i,j=1}^{h}, \qquad \mathrm{V}(\varnothing) = 1.$$

We use vertex names also for their variables.

6.1 Fibers, windows and saturation

Definition 6.1. A graph G is *of interval type* if its vertex set has an ordered partition into nonempty cliques $C_1, \ldots, C_m$, called *fibers*, such that:

(I1) if $u \in C_i$, $v \in C_j$, $w \in C_\ell$ with $i < j < \ell$ and $uw \in E(G)$, then $uv, vw \in E(G)$;

(I2) each fiber forms an incomplete pair with at most one other fiber, its *incomplete partner*.

Here an *incomplete pair* is neither complete nor anticomplete. We allow $m = 0$ for the empty graph. Intersecting the fibers with an induced subgraph and discarding empty intersections preserves (I1) and (I2); a former incomplete pair may become complete or anticomplete. Reversing the order also preserves both conditions. Thus the class is hereditary, and we may process a representation from either end.

Lemma 6.2 (Windows)**.** *Let $(U, V) = (C_i, C_j)$, $i < j$, be an incomplete pair and define $W = C_{i+1} \cup \cdots \cup C_{j-1}$ to be its* window*.*

(a) *W is a clique complete to $U \cup V$.*

(b) *No fiber before U has a neighbor in V and no fiber after V has a neighbor in U. Hence every clique meeting U and V lies in $U \cup W \cup V$, and the neighbors of U in later fibers lie in $W \cup V$.*

(c) *If two fibers A, B are such that A, B is a complete pair, then every fiber in between together with A, B forms a clique.*

Proof. (a) First, we show that every edge between W and $U \cup V$ exists in G. Since U, V forms an incomplete pair, there exists $u \in U, v \in V$ such that $uv \in E(G)$. This forces u, v to be adjacent to every vertex in W, by (I1). Since U, V are the unique incomplete partners of each other, every other fiber is a complete or anti-complete pair for U, V, by (I2). The existence of one edge rules out anti-completeness and so, every fiber in W forms a complete pair with U, V. By (I1), W forms a clique and since every C_r in W is a complete pair of U, V, W is a complete pair of $U \cup V$.

(b) If a fiber D before U had a neighbor in V, then (D, V) would be complete, since U is the unique incomplete partner of V. An edge from D to the endpoint in V of a missing U–V pair would then force that missing edge by (I1), a contradiction. Reversing the order proves the assertion for fibers after V. The remaining assertions follow immediately.

(c) WLOG, assume A appears before $B = C_r$ such that A, B is a complete pair. So C_{r-1} forms a complete pair with A, by (I1). Recursively, each fiber between A and C_r is a complete pair with both A and C_r and this proves the assertion. □

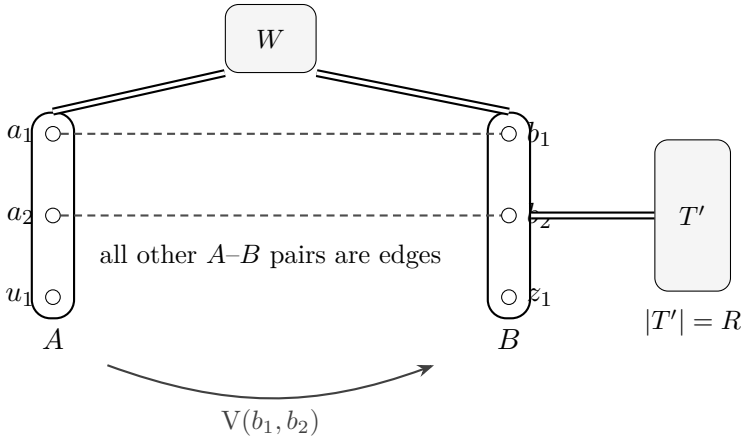
Lemma 6.3 (Matching saturation)**.** *Let G be of interval type. Adding, for each incomplete pair (U, V), all missing U–V pairs outside a maximum matching M_{UV} of the missing pairs yields a graph of interval type with the same ordered fibers and the same clique number. In the resulting graph, the missing pairs between the ends of each incomplete pair form a nonempty matching. If a vertex set S meets no endpoint fiber of an incomplete pair, no edge at S is added.*

Proof. Let G' be the graph after adding the edges mentioned in the statement of the lemma. It is clear that there are no new complete or anti-complete pairs. Suppose $i < j < k$ and let $uw \in E(G') \setminus E(G)$ with $u \in C_i, w \in C_k$. This implies that C_i, C_k is an incomplete pair and we may apply Lemma 6.2(a) to get that $uv, vw \in E(G)$ for any $v \in C_j$. This gives us Condition (I1) showing that G' is interval type.

Let $u \in C_i, v \in C_j$ with $uv \notin E(G)$. We claim that adding uv does not produce a larger clique if the edge addition does not decrease the size of the maximum matching on the complement of $G[C_i \cup C_j]$. If such a larger clique K is formed, it necessarily contains u and v. From Lemma

6.2(b), it is clear that the common neighbors of u, v are a subset of $C_i \cup W \cup C_j$ where W is the window of C_i, C_j and so, $K \subseteq C_i \cup W \cup C_j$. Since W is complete to $C_i \cup C_j$ (by Lemma 6.2), the largest clique K is entirely controlled by the clique number of $G[C_i \cup C_j]$. This contradicts Lemma 2.5. Therefore, adding the edges one by one shows that the clique number does not change. The rest of the assertions are straightforward. □

A graph of interval type is *saturated* if the missing edges of every incomplete pair form a matching; we then say the pair (U, V) has *matched* vertices $a_1, \dots, a_\mu \in U$ and $b_1, \dots, b_\mu \in V$, with $a_i b_i \notin E(G)$, and *unmatched* vertices otherwise.



sending at A ($n = |W| + |B|$, $\mu = 2$, $\alpha = 1$): $d(a_1) = d(a_2) = n + \alpha - 1$, $d(u_1) = n$
receiving at B ($\mu = 2$, $\beta = 1$): $d(b_1) = d(b_2) = R + \mu + \beta - 1 = R + 2$, $d(z_1) = R$

Figure 5: Eliminating the incomplete pair (A, B) with window W. Dashed: the matched nonedges a_1b_1, a_2b_2. The window is complete to $A \cup B$; T' is the set of neighbors of B in later fibers. Eliminating A leaves the factor $\mathrm{V}(b_1, b_2)$, which is absorbed when B is eliminated.

Consider a saturated pair (A, B) with matched vertices $a_i \in A$, $b_i \in B$ $(1 \le i \le \mu)$ and unmatched vertices $u_1, \dots, u_\alpha \in A$ (Figure 5). Set $T = W \cup B$, where W is the window of (A, B), and put $n = |T|$. Every neighbor of A in a later fiber belongs to T. The only nonadjacent pairs between A and T are a_ib_i; in particular, T includes b_i even if it has no neighbor in A. Define

$$P(x) = \prod_{\tau \in T} (x - \tau).$$

Up to sign, the factors of the graph polynomial that involve A and vertices of $A \cup T$ are

$$F_A = \mathrm{V}(a_1, \dots, a_\mu, u_1, \dots, u_\alpha) \prod_{i=1}^{\mu} \frac{P(a_i)}{a_i - b_i} \prod_{r=1}^{\alpha} P(u_r), \tag{6.1}$$

the quotients being polynomials since $b_i \in T$.

Lemma 6.4 (Sending identity)**.** *Let $d(a_i) = n + \alpha - 1$ and $d(u_r) = n + r - 1$. Then*

$$\Big[\prod_i a_i^{d(a_i)} \prod_r u_r^{d(u_r)}\Big] F_A = \pm \mathrm{V}(b_1, \dots, b_\mu),$$

the coefficient being taken in the variables of A. All these exponents are at most $\omega(G[A \cup T]) - 1$. The identity holds verbatim if some elements of $T \setminus \{b_1, \dots, b_\mu\}$ are constants.

Proof. Write $m = \mu + \alpha$ and $P(x) = x^n + c_1 x^{n-1} + \cdots + c_n$ with $c_0 = 1$. Each term of the Vandermonde determinant assigns a permutation of $0, \dots, m-1$ to the variables. Since

$Q_i(x) = P(x)/(x - b_i)$ has degree $n-1$, a term contributing to the desired coefficient must assign Vandermonde exponent at least α to each a_i. Thus the matched variables receive precisely the exponents $\alpha, \dots, \alpha+\mu-1$, and the unmatched variables receive $0, \dots, \alpha-1$.

Similarly, since P has degree n, the exponent assigned to u_r must be at least $r-1$. Working backwards from $r = \alpha$ forces this exponent to be exactly $r-1$. Therefore each factor $P(u_r)$ contributes only its leading term u_r^n.

Let Q be the $\mu \times \mu$ matrix with rows indexed by $i = 1, \dots, \mu$ and columns indexed by $j = 0, \dots, \mu-1$, where $Q_{i,j} = [x^{n-1-j}]Q_i(x)$. For each $\sigma \in S_\mu$, the corresponding Vandermonde term assigns exponent $\alpha + \sigma(i) - 1$ to a_i and $Q_i(a_i)$ must contribute exponent $n - \sigma(i)$, whose coefficient is $Q_{i,\sigma(i)-1}$. Up to a common sign independent of σ, the contribution is therefore $\operatorname{sgn}(\sigma) \prod_{i=1}^{\mu} Q_{i,\sigma(i)-1}$. Summing over all permutations, the desired coefficient is, up to sign,

$$\sum_{\sigma \in S_\mu} \operatorname{sgn}(\sigma) \prod_{i=1}^{\mu} Q_{i,\sigma(i)-1} = \det Q.$$

Polynomial division gives $Q_{i,j} = \sum_{\ell=0}^{j} c_\ell b_i^{j-\ell} = b_i^j + c_1 b_i^{j-1} + \cdots + c_j$. Hence $Q = V_B C$, where V_B is the Vandermonde matrix with entries $(V_B)_{i,k} = b_i^k \qquad (1 \le i \le \mu,\ 0 \le k \le \mu-1)$, and

$$C_{k,j} = \begin{cases} c_{j-k}, & k \le j, \\ 0, & k > j, \end{cases} \qquad 0 \le k, j \le \mu - 1.$$

Since C is upper triangular with diagonal entries $c_0 = 1$, we obtain $\det Q = \det V_B$.

Finally, the saturated graph on $A \cup T$ is complete except for the μ disjoint missing pairs $a_i b_i$. Its clique number is therefore

$$|A| + n - \mu = n + \alpha.$$

Every prescribed exponent is at most $n + \alpha - 1$, with equality attained. Thus the largest prescribed exponent is one less than the clique number of this block. □

For the first nonconstant transfer, take $\mu = 2$, $\alpha = 0$ and $T = \{b_1, b_2\}$. The identity reads $[a_1 a_2]\,(a_2 - a_1)(a_1 - b_2)(a_2 - b_1) = b_2 - b_1$. The output involves only the matched vertices of B and nothing in the window; this is what makes crossing incomplete pairs independent.

Now let B have matched vertices $b_1, \dots, b_\mu$ carrying the factor $\mathrm{V}(b_1, \dots, b_\mu)$, unmatched vertices $z_1, \dots, z_\beta$, and let T' be the set of neighbors of B in later fibers, with $R = |T'|$; by Lemma 6.2(c), $B \cup T'$ is a clique. The factors involving B, including the received one, are up to sign

$$F_B = \mathrm{V}(b_1, \dots, b_\mu, z_1, \dots, z_\beta)\, \mathrm{V}(b_1, \dots, b_\mu) \prod_{v \in B} \prod_{\tau \in T'} (v - \tau). \tag{6.2}$$

Lemma 6.5 (Receiving identity)**.** *Let $d(b_i) = R + \mu + \beta - 1$ and $d(z_r) = R + r - 1$. Then*

$$\Big[\prod_i b_i^{d(b_i)} \prod_r z_r^{d(z_r)}\Big] F_B = \pm\, \mu!\,,$$

and all these exponents are at most $\omega(G[B \cup T']) - 1$. The case $\mu = 0$ (no received factor) is included, and elements of T' may be constants.

Proof. Set $m = \mu + \beta$. The exponent sum $\mu(R + \mu + \beta - 1) + \beta R + \binom{\beta}{2}$ equals the degree of F_B in the variables of B, so only the top homogeneous part contributes. Therefore each factor $v - \tau$ contributes v, whether τ is a variable or a constant. Removing $\prod_{v \in B} v^R$ leaves

$$\mathrm{V}(b)^2\, \mathrm{V}(z) \prod_{i,r} (z_r - b_i),$$

and we need the coefficient of $\prod_i b_i^{\mu+\beta-1} \prod_r z_r^{r-1}$. $\mathrm{V}(z)$ contributes monomials of total degree $\binom{\beta}{2}$, which coincides with the total degree of $\prod_r z_r^{r-1}$. Therefore, the cross difference factors should only contribute $-b_i$. In the remaining $[\prod_i b_i^{\mu-1}]\mathrm{V}(b)^2$, expanding both determinants, once a permutation σ of exponents is chosen in the first, the second must use $\mu - 1 - \sigma(i)$ at b_i, i.e. the reversal of σ; the $\mu!$ terms all have the same sign $(-1)^{\binom{\mu}{2}}$. Hence the coefficient is $\pm\mu!$. Every exponent is nonnegative and at most $R + |B| - 1$. Since $B \cup T'$ is a clique, this upper bound is exactly $\omega(G[B \cup T']) - 1$. For $\mu = 0$ the matched block and its Vandermonde factor are empty, and the coefficient is $0! = 1$. □

6.2 The elimination lemma

Let G be saturated of interval type with fibers $C_1, \dots, C_m$, and let $p \in \{1, \dots, m\}$. The *elimination order toward* C_p is defined to be the sequence

$$C_1,\ C_2,\ \dots,\ C_{p-1},\ C_m,\ C_{m-1},\ \dots,\ C_{p+1};$$

the *full elimination order* is defined to be $C_1, \dots, C_m$.

Lemma 6.6 (Elimination)**.** *Let G be saturated of interval type with fibers $C_1, \dots, C_m$ and let $\omega = \omega(G)$. Let either $Z = \varnothing$, or $Z = C_p$ for a fiber C_p that has no incomplete partner and is straddled by no incomplete pair (no pair (C_i, C_j) with $i < p < j$). Let $\phi : Z \to \mathbb{R}$ be any map, and form $P_{G,\phi}$ by the product formula in Corollary 2.7, omitting edges internal to Z. Properness of ϕ is not needed for the following algebraic identity; it will be required when the identity is used to extend a coloring. Then there are integers $0 \le d(v) \le \omega - 1$, $v \notin Z$, with $\sum_{v \notin Z} d(v) = \deg P_{G,\phi}$ and*

$$\Big[\prod_{v \notin Z} x_v^{d(v)}\Big] P_{G,\phi} = \pm \prod_{(U,V)} \mu(U,V)!\ \neq 0,$$

the product over all incomplete pairs, $\mu(U,V)$ being the size of the matching of missing edges.

Proof. Our strategy will be as follows. We will compute $[\prod_{v \notin Z} x_v^{d(v)}] P_{G,\phi}$ fiber by fiber. We will move through the fibers outside Z in some order $A_1, \dots, A_q$, where q is the number of such fibers. At each step, we will extract the prescribed powers of the variables in the current fiber, obtaining a polynomial in the remaining variables. Iterated extraction will give the required coefficient.

We will change the order of processing based on whether $Z = \varnothing$ or $Z = C_p$. When $Z = \varnothing$, we will process the fibers in the order $C_1, \dots, C_m$. When $Z = C_p$, we will process them in the order

$$C_1, \dots, C_{p-1}, C_m, \dots, C_{p+1},$$

leaving Z unprocessed. We will start from $P^{(0)} = P_{G,\phi}$. Processing a fiber A at step k will mean choosing exponents $d(v)$, $v \in A$, and passing from $P^{(k-1)}$ to

$$P^{(k)} = \Big[\prod_{v \in A} x_v^{d(v)}\Big] P^{(k-1)}.$$

An *unprocessed* fiber is a fiber that has not yet been processed. In particular, Z remains unprocessed throughout. We will orient each incomplete pair (A, B) according to the processing order, with A processed before B, and fix all corresponding vertex labelings independently of ϕ.

We will see that

$$P^{(k)} = \pm c_k \times \prod_{e=uv} (x_u - x_v) \times \prod_{(A,B)} V(b_1, \dots, b_\mu), \tag{6.3}$$

where x_z is interpreted as $\phi(z)$ for $z \in Z$, e ranges over the edges with both ends unprocessed and not both in Z, and (A, B) ranges over the incomplete pairs with A processed and B unprocessed. Here c_k is the product of $\mu!$ over the incomplete pairs whose two fibers have both been processed. Initially, the invariant holds with $c_0 = 1$.

Let $\mathcal{U}_A$ be the set of unprocessed fibers other than the current fiber $A = C_i$; this includes Z when Z is nonempty. If $Z = \varnothing$, or if $Z = C_p$ and $i < p$, then

$$\mathcal{U}_A = \{C_{i+1}, \ldots, C_m\}.$$

If $Z = C_p$ and $i > p$, then

$$\mathcal{U}_A = \{C_p, \ldots, C_{i-1}\}.$$

Thus all other unprocessed fibers lie on one side of A. There are two possible scenarios.

1. If the incomplete partner B of A is unprocessed, let T be the union of B and all fibers strictly between A and B. These fibers are all unprocessed. By Lemma 6.2(a)–(b), applied in the appropriate direction, T is a clique containing every unprocessed neighbor of A. The only missing A–T pairs are $a_i b_i$. Moreover, $Z \cap T = \varnothing$, since Z is neither an endpoint nor an interior fiber of an incomplete pair.

2. Otherwise, let T' be the set of unprocessed neighbors of A. Since A has no incomplete partner on the unprocessed side, Lemma 6.2(c), applied in the appropriate direction, shows that $A \cup T'$ is a clique, possibly containing Z.

Suppose that the invariant holds before processing A. The factors of $P^{(k-1)}$ involving the variables of A are exactly the edge factors inside A, the edge factors from A to unprocessed fibers, and a received Vandermonde factor if the incomplete partner of A has already been processed. By condition (I2), at most one received factor involves A, and none is present in Case (1). All other factors are independent of the variables of A. At this step, all edge factors between A and processed fibers have disappeared from the remaining polynomial.

In Case (1), the edge factors at A form F_A of (6.1), and we use the exponents of Lemma 6.4. Their extraction leaves the factor $\pm V(b_1, \ldots, b_\mu)$, which will be processed when we process B.

In Case (2), the edge factors at the current fiber A, together with its received factor if one is present, form the receiving polynomial F_B of (6.2), after relabeling. We take $\mu = 0$ when nothing is received and use the exponents of Lemma 6.5. The coefficient is $\pm\mu!$, which is absorbed into c_k. In both cases, the invariant (6.3) is maintained, and the exponents are at most $\omega - 1$ by the clique-number bounds in the sending and receiving lemmas. In a receiving step, vertices of Z are still counted in the clique $A \cup T'$, even though their colors occur as constants.

Since Z is never an endpoint of an incomplete pair, both fibers of every incomplete pair are processed. When all fibers outside Z have been processed, no edge factor and no pending Vandermonde factor remains. Therefore,

$$P^{(\text{final})} = \pm \prod_{(U,V)} \mu(U, V)!.$$

The exponents chosen above depend only on the graph, the elimination order and the fixed vertex labelings, and not on ϕ. Specialize the coefficient identity just proved to the zero map $\phi_0 : Z \to \mathbb{R}$. The polynomial P_{G,ϕ_0} is homogeneous, and its degree equals that of $P_{G,\phi}$, since both are products of the same number of nonconstant linear factors. The coefficient of $\prod_{v \notin Z} x_v^{d(v)}$ in P_{G,ϕ_0} is still nonzero. Homogeneity therefore gives

$$\sum_{v \notin Z} d(v) = \deg P_{G,\phi_0} = \deg P_{G,\phi},$$

as required. □

Corollary 6.7. *Every graph G of interval type is $\omega(G)$-choosable.*

Proof. The empty graph is immediate. Otherwise saturate the incomplete pairs without changing the clique number. Lemma 6.6, with $Z = \varnothing$, gives a nonzero coefficient with $d(v) + 1 \leq \omega(G)$ at every vertex. Corollary 2.7 colors the saturated graph from any lists of size at least $\omega(G)$. The same coloring is proper in G. □

6.3 Placing the prescribed clique in a central fiber

Ordinary choosability uses full elimination. To preserve a prescribed coloring, we instead eliminate toward its clique. The following two lemmas put that clique in one fiber and ensure that no Vandermonde transfer passes through it. This is where simpliciality enters the interval argument.

Lemma 6.8 (Protected refinement)**.** *Let G be of interval type, with fixed* representation *$C_1, \ldots, C_m$. Let S be a clique consisting of simplicial vertices of G.*

Then this representation can be refined *by replacing each C_i with an ordered partition of C_i into one or more nonempty cliques, placed consecutively in the position originally occupied by C_i, so that the resulting ordered partition still satisfies* (I1) *and* (I2) *and no new fiber meeting S has an incomplete partner. Equivalently, every new fiber meeting S forms a complete or anticomplete pair with every other new fiber. Only the ordered partition is changed while the vertices and edges of G remain unchanged.*

Proof. We will simultaneously refine all incomplete pairs having an endpoint that meets S.

Consider such a pair (U, V), with U before V, and suppose first that $T = S \cap U \neq \varnothing$. By Lemma 2.3, the vertices of T have the same neighbors in V. Put

$$U' = U \setminus T, \qquad V_0 = N(T) \cap V, \qquad V_1 = V \setminus V_0.$$

Since a vertex of T is simplicial and adjacent to every vertex of $U' \cup V_0$, the set V_0 is complete to U. Also, T is anticomplete to V_1. Moreover, $S \cap V = \varnothing$. Otherwise adjacent simplicial vertices in U and V would be true twins whose common closed neighborhood contains $U \cup V$, making (U, V) complete.

We will replace U by the consecutive fibers T, U' and V by the consecutive fibers V_0, V_1, omitting empty parts. If S meets V instead of U, we will use the reversed construction.

These replacements are compatible. Indeed, by (I2), distinct incomplete pairs have no endpoint fiber in common, so each original fiber is split at most once. Furthermore, every complete or anticomplete relation between old fibers is inherited by all their parts. Within the replacement of (U, V), the only possibly incomplete pair is (U', V_1), whose two fibers are disjoint from S. Thus (I2) is preserved, and no new fiber meeting S has an incomplete partner.

If we check (I1), then we would be done. Triples whose vertices belonged to three distinct old fibers are covered by the original condition. Otherwise, two vertices belong to one split old fiber. If the third vertex belongs to an old fiber other than its incomplete partner, that fiber is complete or anticomplete to the split fiber, so the condition follows.

The remaining triples lie within a refined incomplete pair. For the construction above, the parts occur in the relative order T, U', V_0, V_1. Since T is anticomplete to V_1, the only triples of distinct parts whose outside vertices can be adjacent have types (T, U', V_0) or (U', V_0, V_1). In both cases, the required edges exist because V_0 is complete to U and each old fiber is a clique. The reversed construction satisfies the condition by the same argument with the order reversed.

Hence the new ordered partition is a representation of interval type, and every fiber meeting S has no incomplete partner. □

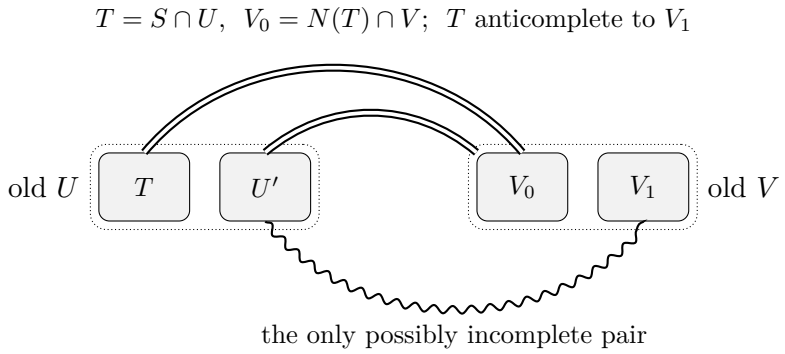


Figure 6: Refining an incomplete pair (U, V) that meets S in T. Double lines are complete pairs. The window between U and V (not drawn) is complete to all four new fibers.

Lemma 6.9 (Central fiber)**.** *Let G be of interval type and let $S \neq \varnothing$ be a clique of simplicial vertices containing every simplicial vertex of G. Then G has an interval representation in which S is a single fiber C_p that has no incomplete partner and is straddled by no incomplete pair.*

Proof. Consider the refinement produced by Lemma 6.8. Any fiber C that meets S has no incomplete partner, so all its vertices have the same closed neighborhood. This common closed neighborhood is a clique, since C contains a simplicial vertex from S. Therefore every vertex of C is simplicial, and hence $C \subseteq S$. Thus S is a union of fibers.

Let C_a and C_b be the first and last fibers contained in S. We will show that these fibers form a consecutive block. Let C be a fiber strictly between C_a and C_b that is not already contained in S, and let D be any fiber contained in S. The pair (C, D) cannot be incomplete, since D has no incomplete partner. If D lies before C, an edge between D and C_b spans C. If D lies after C, an edge between C_a and D spans C. In either case, (I1) rules out anticompleteness. Therefore C is complete to every fiber contained in S.

By Lemma 2.3, the vertices of S have a common closed neighborhood K, which is a clique. The completeness just established gives

$$C_a \cup \cdots \cup C_b \subseteq K.$$

Now let $x \in C_i$ with $a < i < b$. We will show that $N[x] \subseteq K$. Suppose that x has a neighbor $y \notin K$. Then y lies outside $C_a \cup \cdots \cup C_b$. If y lies in a fiber before C_a, the edge yx and (I1) imply that y is adjacent to a vertex of C_a. If y lies in a fiber after C_b, the same condition implies that y is adjacent to a vertex of C_b. In either case, $y \in K$, a contradiction. Thus $N[x] \subseteq K$, so x is simplicial and therefore belongs to S. Consequently, every intervening fiber is contained in S, and

$$S = C_a \cup \cdots \cup C_b.$$

We will merge this consecutive block into a single fiber. Condition (I1) is preserved, since any three vertices in distinct new fibers belonged to distinct old fibers in the same order. Moreover, every old fiber contained in S has no incomplete partner, and all vertices of S have the same closed neighborhood. Hence the merged fiber is complete or anticomplete to every other fiber and has no incomplete partner. All other fiber pairs are unchanged, so (I2) is also preserved.

Finally, suppose that an incomplete pair (U, V) straddles the merged fiber S. By Lemma 6.2(a), S is complete to $U \cup V$. Choose nonadjacent vertices $u \in U$ and $v \in V$. Then every vertex of S has the two nonadjacent vertices u, v in its neighborhood, contradicting simpliciality. After relabeling the fibers, S is the required fiber C_p. □

6.4 The interval precoloring theorem

Theorem 6.10 (Interval precoloring extension)**.** *Every graph G of interval type satisfies* (P_k) *for every $k \geq \omega(G)$.*

Proof. Fix k and suppose the assertion fails. Minimize $|V(G)|$ over all counterexamples (G, S, ϕ, L) in the hereditary interval-type class with $\omega(G) \leq k$. By Lemma 2.4, G is connected and every simplicial vertex of G lies in S. If $S = \varnothing$, Corollary 6.7 colors G; so $S \neq \varnothing$, and if $V(G) = S$ there is nothing to do.

By Lemma 6.9 we may take $S = C_p$ with no incomplete partner and no straddling pair. Saturate by Lemma 6.3: no edge at S is added, ω is unchanged, S is still a clique of simplicial vertices, and a coloring of the supergraph extending ϕ is one of G. Represent the finitely many colors by distinct real numbers. Lemma 6.6, applied to the saturated graph with $Z = S$, gives a nonzero coefficient whose exponents satisfy $0 \leq d(v) \leq \omega(G) - 1 \leq k - 1$ and have the required total degree. The prescribed coloring ϕ is proper on S by the definition of a (P_k)-instance. Thus every hypothesis of Corollary 2.7 holds, and it extends ϕ from the lists of size at least $k \geq d(v) + 1$. This contradicts the choice of G. □

7 Structural reductions

The coloring lemmas supply the required pieces of the proof. We first show that (P_k) is preserved under double clique sums. A minimum counterexample consequently has no rich clique cutset, and deleting its prescribed simplicial clique leaves an atom. The final two reductions determine how this clique can attach to the two types of atoms.

7.1 Rich and double clique cutsets

A clique cutset K of a connected graph G is *rich* if $V(G) \setminus K$ can be partitioned into nonempty anticomplete sets A, B each of which contains two vertices that are not true twins in G (the adjacent clones of [5]).

A *double clique cutset* of a graph G is a clique cutset $D = C_1 \dot\cup C_2$ for which

$$V(G) = V_1 \dot\cup C_1 \dot\cup C_2 \dot\cup V_2, \qquad V_1, V_2 \neq \varnothing,$$

with V_1 anticomplete to $C_2 \cup V_2$ and V_2 anticomplete to C_1. In this case G is the *double clique-sum* of $G[V_1 \cup D]$ and $G[V_2 \cup D]$ (Figure 7).

The next lemma shows that (P_k) is preserved by double clique sums.

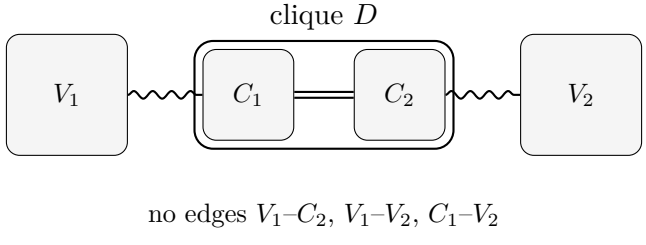


Figure 7: A double clique cutset. Wavy lines denote arbitrary adjacency.

Lemma 7.1 (Double clique sums)**.** *Let G be the double clique-sum of $G_1 = G[V_1 \cup D]$ and $G_2 = G[V_2 \cup D]$, where $D = C_1 \dot\cup C_2$, and let $k \geq \omega(G)$. If both G_1 and G_2 satisfy* (P_k)*, then G satisfies* (P_k)*.*

Proof. Let (G, S, ϕ, L) be a (P_k)-instance. Since S is a clique and V_1 is anticomplete to V_2, we may assume by symmetry that $S \cap V_2 = \varnothing$. Thus $S \subseteq V(G_1)$, and its vertices remain simplicial in G_1. Apply (P_k) in G_1 to obtain a coloring f_1 extending ϕ and respecting the lists outside S.

In G_2, the set

$$T = C_1 \cup (S \cap C_2)$$

is a clique of simplicial vertices: every vertex of C_1 has all its neighbors in the clique D, and vertices of $S \cap C_2$ remain simplicial. Precolor T by $f_1|_T$. This coloring agrees with ϕ on $S \cap C_2$, and every vertex of $G_2 - T$ has its original list of at least k colors. Apply (P_k) in G_2 to obtain a coloring f_2 extending $f_1|_T$ and respecting these lists.

Define $f = f_1$ on $V_1 \cup C_1$ and $f = f_2$ on $V_2 \cup C_2$. The only edges between these two parts are the C_1–C_2 edges; f_2 is proper on them and agrees with f_1 on C_1. Thus f is proper, respects the lists outside S, and extends ϕ. □

We recall the following structural result of Chudnovsky and Seymour.

Lemma 7.2 (Chudnovsky–Seymour [5, Theorem 1.4])**.** *A connected claw-free graph with a rich clique cutset is of interval type or admits a double clique cutset.*

Consequently, by Lemmas 7.2 and 7.1 and Theorem 6.10, a minimum-order counterexample to (P_k) in the class of claw-free perfect graphs has no rich clique cutset.

7.2 Atoms after removing the prescribed clique

A noncomplete atom has no simplicial vertex: a nonuniversal simplicial vertex v has the clique cutset $N(v)$, and a universal simplicial vertex makes the graph complete.

Lemma 7.3. *Let G be a connected graph with no rich clique cutset, and let S be a clique of simplicial vertices containing every simplicial vertex of G. Then $G - S$ is either empty or an atom.*

Proof. Put $H = G - S$ and $Q = N_G(S)$, and suppose $H \neq \varnothing$. If $S = \varnothing$, then $H = G$ is connected. Otherwise, every component of H contains a vertex of Q, since G is connected. As Q is a clique by Lemma 2.3, H is connected in this case as well.

Suppose that D is a clique cutset of H, and partition $V(H) \setminus D$ into nonempty anticomplete sets V_1, V_2. Since Q is a clique, we may assume that $Q \setminus D \subseteq V_1$. Then $V_1 \cup S$ and V_2 are nonempty anticomplete sets partitioning $V(G) \setminus D$.

We show that each side contains two vertices that are not true twins in G. If V_2 contained no such pair, it would be a clique of pairwise true twins (or a singleton). Each vertex of V_2 has neighbors only in $V_2 \cup D$; its neighbors in D are complete to V_2, by the twin property. Since D is a clique, every vertex of V_2 would be simplicial in G, contrary to $V_2 \cap S = \varnothing$. If $S = \varnothing$, the same argument applies to V_1. If $S \neq \varnothing$, choose $s \in S$ and $v \in V_1$. They cannot be true twins, since that would make v simplicial as well, contrary to $v \notin S$.

Thus D is a rich clique cutset of G, a contradiction. Hence H has no clique cutset and is an atom. □

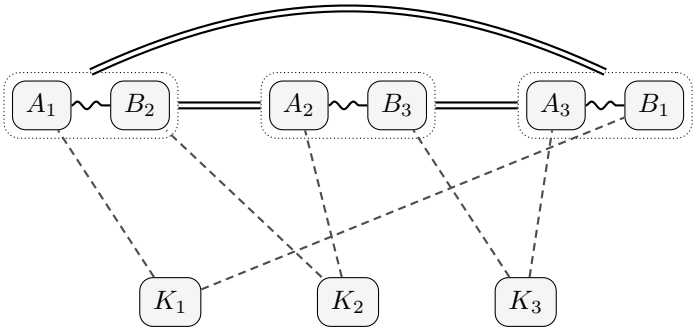
7.3 Nonempty attachments to peculiar graphs

Definition 7.4 (Peculiar graphs [4])**.** A graph G is called *peculiar* if its vertex set admits a partition into nine pairwise disjoint cliques $(A_1, B_1, A_2, B_2, A_3, B_3, K_1, K_2, K_3)$ satisfying the following conditions, with indices taken modulo 3:

1. The six cliques A_i, B_i are nonempty. We allow the K_i to be empty; the standard nondegenerate definition requires all nine cliques to be nonempty.

2. Each pair (A_i, B_{i+1}) contains a nonadjacent pair, and the three sets $A_1 \cup B_2$, $A_2 \cup B_3$, $A_3 \cup B_1$ are pairwise complete to one another.
3. The cliques K_1, K_2, K_3 are pairwise anticomplete.
4. For each i, the clique K_i is anticomplete to $A_i \cup B_i$ and complete to $A_j \cup B_j$ for every $j \neq i$.

See Figure 8. We call the A_i, B_i the *core cliques* and the K_i the *non-core cliques*.



Dashed lines show the anticomplete pairs K_i–A_i and K_i–B_i.
Every other K_i–core pair is complete.

Figure 8: A peculiar graph. Each dotted box is a cobipartite piece with at least one nonadjacent pair; its wavy line represents a noncomplete adjacency, possibly anticomplete. Distinct pieces are complete to each other, and the K_i are pairwise anticomplete.

Lemma 7.5. *A peculiar graph is either cobipartite or has no nonempty attachable clique.*

Proof. If a non-core clique is empty, say $K_3 = \varnothing$, then

$$K_1 \cup A_2 \cup B_2 \cup A_3, \qquad K_2 \cup A_1 \cup B_1 \cup B_3$$

are cliques partitioning the vertex set. Thus the graph is cobipartite. Assume that all three K_i are nonempty, and suppose that Q is a nonempty attachable clique.

By (2.2), two distinct nonadjacent neighbors of q cannot both lie outside Q.

Suppose first that Q meets a core clique. The cyclic shift of indices and the map $A_i \mapsto B_{-i}$, $B_i \mapsto A_{-i}$, $K_i \mapsto K_{-i}$ preserve the definition and act transitively on the core cliques. Thus we may assume that $a \in A_1 \cap Q$. Since a is complete to the nonempty anticomplete cliques K_2, K_3, attachability implies that one of them lies entirely in Q; the other is disjoint from Q, since Q is a clique. Also $K_1 \cap Q = \varnothing$.

If $K_2 \subseteq Q$, a vertex of K_3 is an outside neighbor of a nonadjacent to every vertex of $A_3 \cup B_3$. These vertices are all adjacent to a, so (2.2) gives $A_3 \cup B_3 \subseteq Q$. For $b_3 \in B_3$, an outside neighbor in K_1 similarly forces $B_1 \subseteq Q$. Then Q contains $A_3 \cup B_1$, which has a nonadjacent pair, a contradiction.

If $K_3 \subseteq Q$, outside neighbors in K_2 first force $A_2 \subseteq Q$. For $a_2 \in A_2$, outside neighbors in K_1 force $A_1 \cup B_1 \subseteq Q$. For $b_1 \in B_1$, outside neighbors in K_2 then force $B_2 \subseteq Q$. Now Q contains the noncomplete pair (A_1, B_2), again a contradiction.

Finally, if Q meets no core clique, then $Q \subseteq K_i$ for some i. A vertex of Q is complete to $A_{i+1} \cup B_{i+2}$, which contains a nonadjacent pair outside Q. This contradicts attachability. □

7.4 Attaching a simplicial vertex to an elementary atom

In this subsection we show that adjoining a prescribed simplicial clique to a non-cobipartite elementary atom gives an elementary graph (Corollary 7.8). By Lemma 7.6 it suffices to add a single vertex s whose neighbourhood is an attachable clique Q of the atom H. The remaining vertices of the clique are then added as true twins of s. We work directly with the defining edge 2-colouring rather than with the root. First we show that Q is either an entire *hub clique* C_u (the clique formed by the vertices coming from the root edges at a root vertex u), or meets both shores of a single augment in a set $X_0 \cup Y_0$ of a restricted form. In either case, every edge from Q to the rest of H receives the same colour. Giving each new edge sq the other colour then makes every new induced path s–q–x bichromatic.

Lemma 7.6. *Adding a true twin of a vertex to an elementary graph gives an elementary graph.*

Proof. Start with an elementary two-coloring of the edges of the old graph. When v is added as a true twin of u, color vx as ux for every old neighbor x of u, and color uv arbitrarily. The edge uv lies in no induced three-vertex path, and every other induced path $x\,v\,y$ or $x\,y\,v$ corresponds to $x\,u\,y$ or $x\,y\,u$ in the old graph. □

Lemma 7.7. *Let H be an elementary atom that is not cobipartite, and let Q be an attachable clique of H. Adding a vertex s with $N(s) = Q$ gives an elementary graph.*

Proof: If $Q = \varnothing$, the new vertex is isolated and the assertion is immediate. Assume $Q \neq \varnothing$. By the Maffray–Reed structure theorem, fix a normalized representation of H with bipartite root B, and absorb every complete augment by Lemma 5.2(c). Thus every remaining augment is noncomplete. Since H is a noncomplete atom, it has no simplicial vertex.

For each augment (X, Y), both exterior neighborhoods N_X, N_Y are nonempty: otherwise deleting the opposite shore would separate a shore from a vertex outside $X \cup Y$, which exists because H is not cobipartite. Every vertex of either shore has a neighbor in the other, since otherwise it would be simplicial.

Every nonisolated root vertex has at least two distinct neighbors. Indeed, if u had only neighbor v, a root edge $e = uv$ would be simplicial in $\mathcal{L}(B)$. If e were unaugmented, its neighbors would still form a clique after augmentation: no flat pair can lie entirely among them, since e would complete a triangle. If e belonged to a designated flat pair ef, simpliciality and flatness would make f its only neighbor, giving an augment shore with empty exterior neighborhood. Both possibilities are excluded.

We use the elementary 2-edge-coloring arising from this representation.

Fix a bipartition (A, B') of the root multigraph. If two nonparallel root edges meet at u, color their line-graph edge according to the part containing u. Edges joining vertices represented by parallel root edges may be colored arbitrarily: those vertices are true twins and their common edge lies in no induced three-vertex path. For an augment replacing a flat edge corresponding to two root edges uc, cv where $d(c) = 2$, let its two shores be X and Y. Since u and v belong to the same side of the bipartition, say with colour α, colour all edges inside X and inside Y, and all edges joining either shore to its exterior attachment clique, with colour α; colour the edges between X and Y with the other colour. This coloring is elementary: an induced three-vertex path that uses an edge within a shore or across an augment has one shore-side edge and one cross-shore edge; every other such path projects to an induced three-vertex path in the original line graph, whose two edges receive different colors.

For a nonisolated root vertex u that is not the center of a designated augmented pair, let C_u denote the corresponding *hub clique*: the vertices arising from root edges incident with u, with an augmented endpoint replaced by its shore. Its incidence blocks are indexed by the distinct neighbors of u: a parallel bundle forms one block, and an augmented root edge contributes its entire shore. The set C_u is a clique, since no designated flat pair has both root edges incident with u.

We first record the form of an attachable clique Q.

Claim. Either

1. $Q = C_u$ for some hub clique C_u, or
2. Q meets both shores X, Y of a single augment and

$$Q = X_0 \cup Y_0, \qquad \varnothing \neq X_0 \subseteq X, \quad \varnothing \neq Y_0 \subseteq Y,$$

where X_0 is complete to Y_0, X_0 is anticomplete to $Y \setminus Y_0$, and Y_0 is anticomplete to $X \setminus X_0$.

Suppose first that Q is contained in a hub clique C_u. The root degree observation gives at least two incidence blocks. Every vertex of C_u has a neighbor outside C_u, since otherwise it would be simplicial. If q belongs to the block indexed by a neighbor v of u, an outside neighbor of q arises through v, or in the opposite shore of its augment. Such a neighbor is nonadjacent to every other block of C_u, by bipartiteness and Lemma 5.2(a).

If $Q \neq C_u$, choose $q \in Q$ and $r \in C_u \setminus Q$ from different incidence blocks; this is possible because there are at least two blocks and both Q and $C_u \setminus Q$ are nonempty. Choose a neighbor $z \notin C_u$ of q through the block containing q. By the augmentation geometry, r is not adjacent to z. Hence

$$r, z \in N_H(q) \setminus Q$$

are nonadjacent, contradicting the assumption that Q is attachable. Thus $Q = C_u$.

Now suppose that Q is not contained in a hub clique. The root edges whose corresponding vertices or shores meet Q are pairwise incident, hence share a common endpoint, since B is bipartite. This endpoint must be the center of a designated augmented pair; otherwise Q would lie in its hub. Therefore Q meets both shores X, Y of that augment (a single shore lies in its exterior hub). No outside vertex sees both shores, by Lemma 5.2(a), so

$$Q \subseteq X \cup Y.$$

Put

$$X_0 = Q \cap X, \qquad Y_0 = Q \cap Y.$$

Both sets are nonempty, and, since Q is a clique, X_0 is complete to Y_0.

Let C_u and C_v be the two exterior hub cliques of this augment. Since H is a non-cobipartite atom, each of these hub cliques has another incidence block. Choose

$$a \in C_u \setminus X, \qquad b \in C_v \setminus Y.$$

If some $x \in X_0$ had a neighbour $y \in Y \setminus Y_0$, then

$$a, y \in N_H(x) \setminus Q,$$

whereas a and y are nonadjacent. This contradicts attachability. Hence X_0 is anticomplete to $Y \setminus Y_0$. By symmetry, Y_0 is anticomplete to $X \setminus X_0$. This proves the claim.

We now add the new vertex s with $N(s) = Q$ and extend the elementary edge-colouring.

Suppose first that $Q = C_u$. Let α be the colour corresponding to the root vertex u. Every edge

$$qx, \qquad q \in Q, \quad x \in N_H(q) \setminus Q,$$

has the colour opposite to α: such an edge leaves the hub C_u through the other endpoint of the corresponding root edge, or through the opposite shore of an augment. Colour every edge sq, $q \in Q$, with colour α.

Suppose next that

$$Q = X_0 \cup Y_0$$

is as in the second case of the claim. Let the corresponding flat edge come from the root edges uc and cv, and let α be the common colour associated with u and v. By the claim, a vertex of Q has no cross-shore neighbour outside Q. Hence every edge joining a vertex of Q to a vertex of $H - Q$ is a hub-side edge and therefore has colour α. Colour every edge sq, $q \in Q$, with the other colour.

In either case, the only new induced paths on three vertices are of the form

$$s - q - x, \qquad q \in Q, \quad x \in N_H(q) \setminus Q.$$

Indeed, no path $q - s - q'$ with $q, q' \in Q$ is induced, since Q is a clique. By construction, the two edges of every new induced P_3 have different colours. All induced P_3's contained in H were already properly bichromatic.

Thus the extended graph admits an elementary edge-colouring, and hence the graph obtained from H by adding s with $N(s) = Q$ is elementary. □

Corollary 7.8. *Let G be connected and claw-free, let $S \neq \varnothing$ be a clique of simplicial vertices, and suppose $G - S$ is an elementary atom that is not cobipartite. Then G is elementary.*

Proof. By Lemma 2.3, $Q = N(S)$ is an attachable clique of $G - S$ and every vertex of S has neighborhood $Q \cup (S \setminus \{s\})$. Add one vertex of S by Lemma 7.7 and the others as true twins by Lemma 7.6. □

8 Proof of the main theorem

Gravier and Maffray [10, Corollary 2, p. 89] proved that every peculiar graph H satisfies $\mathrm{ch}(H) = \chi(H)$. Their cliques Q_i correspond to the cliques denoted here by K_i. In our application, H is an induced subgraph of the perfect graph G, so

$$\mathrm{ch}(H) = \chi(H) = \omega(H) \leq \omega(G) \leq k.$$

Hence H is colorable from any lists of size at least k. In the proof below, we invoke this result only in the branch $S = \varnothing$, where no colors are prescribed, and only when K_1, K_2, K_3 are all nonempty. If some K_i is empty, then H is cobipartite and Theorem 3.3 applies instead.

Proof of Theorem 1.2. Fix k and suppose the theorem fails. Claw-free perfect graphs form an induced-subgraph-closed class, so Lemma 2.4 provides a counterexample (G, S, ϕ, L) of minimum order in which G is connected, every simplicial vertex lies in S, and every proper induced subgraph satisfies (P_k).

G has no double clique cutset, since its pieces would be proper induced subgraphs and Lemma 7.1 would color G; and G is not of interval type by Theorem 6.10. Hence, by Lemma 7.2, G has no rich clique cutset.

If $V(G) = S$ there is nothing to prove. Otherwise $H = G - S$ is a nonempty atom by Lemma 7.3, hence elementary or peculiar by Theorem 1.3.

Empty prescribed clique. If $S = \varnothing$, then $G = H$ is colored by Theorem 5.5 if elementary, by Theorem 3.3 (with $C = \varnothing$) if cobipartite, and by Gravier–Maffray [10] if peculiar with all K_i nonempty. So $S \neq \varnothing$.

Nonempty prescribed clique. Since $H \neq \varnothing$ and G is connected, $Q = N(S)$ is nonempty. Lemma 2.3 makes it an attachable clique of H, and deleting $\phi(S)$ from the lists on Q gives an (M_k)-instance $(H, Q, |S|, L')$ whose colorings extend ϕ.

- If H is cobipartite, Corollary 3.4 colors this instance.
- If H is peculiar and not cobipartite, all K_i are nonempty. Lemma 7.5 excludes the *nonempty* attachable clique Q.

- If H is elementary and not cobipartite, Corollary 7.8 shows that G itself is elementary, and Theorem 5.5 extends ϕ.

In every case ϕ extends, contradicting the choice of G. □

Proof of Theorem 1.1. Theorem 1.2 with $S = \varnothing$ and $k = \omega(G)$ gives $\mathrm{ch}(G) \le \omega(G)$, and $\omega(G) \le \chi(G) \le \mathrm{ch}(G)$ always. □

9 Consequences

Corollary 9.1. *Every claw-free perfect graph G satisfies* (M_k) *for every $k \ge \omega(G)$.*

Proof. Let (G, Q, r, L) be an (M_k)-instance; if $r = 0$ apply Theorem 1.2 with $S = \varnothing$. For $r > 0$, form G^+ by adding a clique S of r new vertices, complete to Q and anticomplete to $V(G) \setminus Q$.

We check the hypotheses needed for (P_k). Each new vertex is simplicial. For $q \in Q$, its neighborhood in G^+ is covered by the two cliques $(Q \setminus \{q\}) \cup S$ and $N_G(q) \setminus Q$. At every other old vertex the neighborhood is unchanged. Hence G^+ is claw-free.

To check perfection, consider any induced subgraph J of G^+. The graph $J - S$ is perfect and can be colored with at most $\omega(J)$ colors. Insert the vertices of $J \cap S$ one at a time. Each is simplicial in J and has at most $\omega(J) - 1$ neighbors, so an available color exists. Thus $\chi(J) \le \omega(J)$, proving that G^+ is perfect. Finally, every clique meeting S lies in $Q \cup S$, and the mixed-list hypothesis gives $|Q| + r \le k$. Consequently $\omega(G^+) \le k$.

Choose r distinct colors absent from all original lists, precolor S with them, and add them to every list on Q. The enlarged lists on Q have size at least $(k - r) + r = k$, and all other old lists already have at least k colors. Theorem 1.2 extends this precoloring; Q cannot use the fresh colors, being complete to S, so the restriction to G is an L-coloring. □

The following corollary recovers the theorem of Peterson and Woodall [15, 16].

Corollary 9.2. *Every perfect line graph is chromatic-choosable. Equivalently, if the line graph of a loopless multigraph B is perfect, then* $\mathrm{ch}'(B) = \chi'(B)$.

Proof. Line graphs are claw-free: the neighbors of the vertex uv of $\mathcal{L}(B)$ are covered by the two cliques of edges at u and at v. Apply Theorem 1.1; proper vertex colorings of $\mathcal{L}(B)$ are proper edge colorings of B, with the same correspondence for lists. □

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A Notation table

Table 1: Notation and scope. The references in the right column give the formal definitions of specialized objects.

Notation	Meaning and scope
Graphs, lists, and extension properties	
$G[A]$, $G - A$	Induced subgraphs on A and $V(G) \setminus A$.
$N_G(v)$, $N_G[v]$	Open and closed neighborhoods.
$N_G(A)$	External neighborhood of A, excluding A itself.
$\omega(G), \chi(G), \mathrm{ch}(G)$	Clique number, chromatic number, and choice number.
$L(v)$, $L(T)$	The list at v and $\bigcup_{v\in T} L(v)$.
$f(T)$, ϕ	Colors used on T; a prescribed proper coloring.
S, Q, r, k	A clique of simplicial vertices; an attachable clique; a nonnegative list deficit; a fixed integer list bound.
(P_k), (M_k)	Precoloring extension and mixed-list properties (Definition 2.2); in (M_k), $\lvert Q\rvert + r \le k$.
$\mathcal{C}$	A class closed under induced subgraphs; fixed before choosing a minimum counterexample.
Nonedges and polynomial coefficients	
F, $\nu(F)$	The bipartite nonedge graph of a cobipartite graph and its maximum matching size (Sections 2–3).
$K_W - M$	The complete graph on W with matching pairs M deleted.
P_G, $[\mathbf{x}^d]P$	Graph polynomial and coefficient of the indicated monomial.
Z, $P_{G,\phi}$	Fixed-color set and polynomial with its internal edges omitted (Corollary 2.7); ϕ is proper when the polynomial is used for coloring extension.
$\mathrm{V}(\zeta_1, \dots, \zeta_h)$	Vandermonde product $\prod_{i<j}(\zeta_j - \zeta_i)$, equal to 1 for $h \le 1$.
m, t, u; q_0, q	Canonical-graph parameters: $m, t \ge 1$, $u \ge 0$, $q_0 = m + t$, $q = m + t + u$ (Section 3).
Line graphs and elementary graphs	
B, $\mathcal{L}(B)$	A loopless bipartite root multigraph and its line graph.
$\mathrm{st}_B(w)$, $E(w, u)$	Root edges at w; the parallel bundle joining w, u.
$\Delta(B)$, $d_D^+(v)$	Maximum root degree, counting multiplicities; number of distinct out-neighbors in a orientation D.
(X, Y), N_X, N_Y	An augment and its external neighborhoods (Lemma 5.2).
A_i, Q_i, M_i, A_i^*	Unprecolored side, its external neighborhood, matched vertices, and unmatched vertices (Lemma 5.4).
a_i, α_i, μ	Sizes $\lvert A_i\rvert, \lvert A_i^*\rvert, \lvert M\rvert$ in that lemma.
m_i, $R(v)$	Minimum list-union excess on A_i^*; residual list at v in that lemma.
Interval representations and the structural reduction	
$C_1, \dots, C_m$	Ordered clique fibers of an interval representation (Definition 6.1).
W, μ	Window of an incomplete pair; size of its matching of nonedges (Section 6).
R, X_e, F	Root multigraph, clique represented by root edge e, and selected root edges (Lemma 7.7).
A_i, B_i, K_i	The nine cliques of a peculiar representation, with indices modulo 3 (Section 7.3).