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ASYMPTOTIC CLOSED FORMS FOR THE FALTINGS HEIGHT OF FERMAT JACOBIANS AND THE SPECTRUM OF L'/L-VALUES

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Abstract. For the Fermat tower $F_N\colon x^N+y^N=1$, $N=p^n$, we assemble the complete asymptotic machine for the stable Faltings height $h_F(J(F_N))$: the main-chain closed form $\hat{h}_n = \frac{\log p}{4}n - 6\log A - \frac{p\log p}{4(p^2-1)} + \Theta\big(\frac{n\log p}{p^n}\big)$; an exact spectral identity $h(\Phi) = \frac{\mu(\Phi)}{2} - \sum_\chi m(\Phi,\chi)\rho_\chi$ with $\rho_\chi = R_\chi + \log f_\chi$; an arithmetic channel for the deviation functional dev through the Bernoulli-mass $M_1(\Phi) = \sum_\chi m(\Phi,\chi)\log|B_{1,\bar\chi}|$ ($R^2 = 0.78$, residual std 1.90); a profile law $\mathbb{E}[M_{c'}] = 3^{c'-2}\Lambda(n)$ proved by a carry-counting identity; and the layerwise height closure mean $h^{\mathrm{Col}}(n) = \frac{3^{n-1}\log 3}{4}(n-n^*)$ with $n^* \to 5.69$, which explains the sign flip at $n=5$ and the Δ_9 explosion ($\hat{h}_9^{\mathrm{Fal}} = +7.24$, twenty-five times outside the corridor 0.25–0.30). All identities are certified at the exact-arithmetic level on layers $n \le 12$ (2 187 orbits and 729 odd characters at layer 7, statistical verdicts at layers 8–12). The limiting integral $\mathcal{I}_p = \int_0^1 w_p \log\Gamma$ receives a Kummer closed form in $\zeta(2)$ and $\zeta'(2)$ (certified to 10^{-17} at $p = 3,5,7,11$), and the profile law's lattice-point count is given a presentation-grade proof with closed form $4\cdot 3^{2n-2} - 3^{2n-c}$. The irreducible content of the unified wall is reduced to a bounded per-type functional and an explicit five-parameter residual kernel ($R^2 = 1.0000$ on the orbit level).

1. Setup and notation

Let p be an odd prime and $N = p^n$. The Fermat curve $F_N\colon x^N + y^N = 1$ has genus $g_N = \frac{(N-1)(N-2)}{2}$; its Jacobian decomposes up to isogeny into CM factors indexed by orbits of $G_n = (\mathbb{Z}/N\mathbb{Z})^\times$ on

$$Y_n = \big\{(r,s,t) \in [1,N)^3 :\ r+s+t \equiv 0 \pmod N\big\}. \tag{1.1}$$

For a triple $\tau = (r,s,t)$ write $d_0(\tau) = N/\gcd(N,r,s,t)$ and let $\mathcal{O}(\tau)$ be its G_n-orbit.

Theorem 1.1 (Orbit factorisation)**.** *$J(F_d) \sim \bigoplus_{\mathcal{O}} A_{\mathcal{O}}$ over all G_n-orbits $\mathcal{O} \subset Y_n$, and*

$$\dim A_{\mathcal{O}} = \frac{\varphi(d_0)}{2}, \tag{1.2}$$

with $A_{\mathcal{O}}$ of CM type by $\mathbb{Q}(\zeta_{d_0})$, primitive (not a Weil restriction).

The genus check $d=5$: the twelve triples split into three orbits of dimension 2, totalling $6 = g(F_5)$. The *deep kernel* is the structural shadow of this theorem: the orbit with $d_0 = 3$ is the elliptic curve $J(F_3)$ ($j=0$, conductor 27), recovered independently from the stable-fibre tables of [4].

For each CM type Φ of a factor A with character group G,

$$h_F(A) = \sum_{\Phi} m(\Phi,\chi)\Big[-\tfrac{1}{2}\tfrac{L'}{L}(0,\chi) - \tfrac{1}{4}\log d_\Phi\Big] - \tfrac{g}{2}\log 2\pi, \tag{1.3}$$

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where the coefficient

$$m(\Phi,\chi)=\frac{1}{|G|}\sum_{g}\bigl(|\Phi\cap g\Phi|-\tfrac{g}{2}\bigr)\,\bar\chi(g) \tag{1.4}$$

vanishes on even characters (one-line proof: $c\Phi=\Phi^c$ forces $h_{\mathcal O}(cg)=-h_{\mathcal O}(g)$). We use the Kronecker–Lerch form

$$\frac{L'}{L}(0,\chi)=-\log f_\chi-\frac{1}{B_{1,\chi}}\sum_{a=1}^{f_\chi}\chi(a)\log\Gamma\Bigl(\frac{a}{f_\chi}\Bigr), \tag{1.5}$$

and write

$$R_\chi=\frac{L'}{L}(0,\bar\chi)\ \ \text{(Kronecker–Lerch)},\qquad \rho_\chi:=R_\chi+\log f_\chi. \tag{1.6}$$

The anchor values are machine-exact throughout: $h_F(j{=}0)=-1.667691$ (Faltings normalisation, differing from the Colmez normalisation by exactly $\frac12\log 2\pi$), $R(\chi_2)=0.948199$, and $\rho_\chi=2.046811$ at the trivial conductor layer.

Conventions and certification levels. We distinguish: *theorem* (proof closed and machine-certified), *proposition* (proof closed, partial certificate), and *data-level* (exact integer/rational output, no fitting). Every numerical table below is exact arithmetic unless marked as a statistical verdict; inter-block spread is the only currency of error bars (heavy-tailed M_1 classes move c_1 by ±0.044 at rate $\sim 1.5\times10^{-4}$; nominal standard errors understate by a factor up to 30).

2. Orbit mass and the counting closed form

Define the *top-multiplicity*

$$M_{\mathrm{top}}(n)=\#\{\mathcal O\subset Y_n : d_0(\mathcal O)=N\}. \tag{2.1}$$

The programme established $M_{\mathrm{top}}(n)=4\cdot3^{n-1}-3$ for $p=3$ by layer-by-layer enumeration through $n=8$. We record the Burnside closed form, which removes the enumeration from the hypothesis.

Theorem 2.1. *For $p=3$ and all $n\ge1$,*

$$M_{\mathrm{top}}(n)=\frac{|Y_n|}{|G_n|}=4\cdot3^{n-1}-3. \tag{2.2}$$

Proof. $|Y_n|=(3^n-1)(3^n-2)-(3^{n-1}-1)(3^{n-1}-2)$ (remove triples with all coordinates divisible by 3) and $|G_n|=2\cdot3^{n-1}$. The action of G_n on Y_n is free [2]: for $g\neq1$, $v_3(g-1)\le n-1$, hence $\mathrm{Fix}(g)=\ker(\times g-\times1)\subseteq 3\mathbb Z/3^n\mathbb Z$ in each coordinate, and every fixed triple has all coordinates divisible by 3, which is excluded from Y_n by definition. Thus $\mathrm{Fix}(g)=\varnothing$ for $g\neq1$ and Burnside gives (2.2). □

Corollary 2.2 (Genus recursion)**.** *$g(F_{3^n})=M_{\mathrm{top}}(n)\,\varphi(3^n)/2+g(F_{3^{n-1}})$ for all $n\ge1$.*

This is machine-checked through $n=8$: $M_{\mathrm{top}}(7)=2913$, $g(F_{2187})=2\,388\,205$; and recovers $g(F_{729})=264\,628$ (correcting the misprint 265 720 in the compiled transcript).

Remark 2.3 (Exceptional orbits)**.** The enumeration must exclude triples with $t\equiv0 \pmod{3^n}$: they form exactly one pseudo-orbit ($970-1=969$ at $n=6$), and $\gcd(r,s,t)=1$ holds on Y_n, so deep-kernel orbits live in the extended triple set.

Remark 2.4 (Layer weight closed form)**.** The intrinsic mass per orbit admits a two-FFT closed form $m_{\mathcal{O}}(v) = |X_0(v) - X_1(v)|^2/|G_{d_0}|$, computed on the orbit layer G_{d_0} and then scattered (computing the FFT on G_{3^n} instead scales by 3^{n-c}—bug class "index arithmetic $\neq$ field arithmetic", caught by a brute-force comparison chain at six orbits with $|\Delta m| \le 10^{-14}$).

Proposition 2.5 (Intrinsic weights and the normalisation identity)**.** *Index the odd characters of conductor dividing 3^n as χ_j, $j \in \{1, 3, 5, \dots, 2\cdot 3^{n-1} - 1\}$, with $f_{\chi_j} = 3^{\,n - v_p(j)}$; write $v_\chi := v_p(j)$ (the p-adic valuation of the index, not the conductor exponent). The intrinsic orbit weights close as*

$$w_\chi^{\mathrm{intr}} = \sum_{\mathcal{O}} m^{\mathrm{intr}}(\Phi_{\mathcal{O}}, \chi) = c_{v_\chi}\,|B_{1,\chi}|^2, \qquad c_v = \frac{3p\,(p^{v+1} - 1)}{(p-1)^2}, \tag{2.3}$$

and the tower sum obeys the exact Bernoulli-moment identity

$$\sum_{\chi \text{ odd}} c_{v_\chi}\,|B_{1,\chi}|^2 = \frac{g_n}{2}. \tag{2.4}$$

Certification. Identity (2.4) is machine-exact at seven layers. At $p = 3$ both sides equal 14, 162.5, 1580; at $p = 5$ they equal 138 and 3813; at $p = 7$ they equal 564 and 29155.5. The three $p = 3$ values were re-derived independently on 2026-11-03 by an FFT pipeline along the primitive root (indices j, valuation $v_p(j)$, conductors $3^{\,n - v_p(j)}$), matching $g_n/2$ digit by digit; the $v_\chi = v_p(j)$ convention is the one for which the identity holds exactly. Identity (2.3) is the stabiliser-normalised orbit attribution ($|\mathrm{stab}_{(r,s,t)}| = \gcd(N, r, s, t)$), certified at four layers by full orbit enumeration to 4×10^{-16}. □

The engine behind both identities is the deep-kernel recursion

$$w^{(n,\mathrm{full})} = w^{(n,\mathrm{free})} + p\, w^{(n-1,\mathrm{full})}. \tag{2.5}$$

3. The main-chain closed form

Lemma 3.1 (Digit structure [**?**, §T.2])**.** *Let $d_j(a)$ denote the j-th p-adic digit of a. After Gauss multiplication, the layer weights are carried by*

$$D_1(a) = 2d_0(a) - p, \qquad D_c(a) = p^{c-1}\big(2d_{c-1}(a) - p + 1\big) \quad (c \ge 2), \tag{3.1}$$

whose digit mean $\sum_{d=0}^{p-1}(2d - p + 1) = 0$ annihilates the $\log 2\pi$ term identically ($S_0(c) = 0$). The algebraic part closes as

$$S_a(1) = \frac{p(p-1)(p-2)}{6}, \qquad S_a(c) = \frac{(p-1)^2(p+1)}{6}\,p^{3c-3} \quad (c \ge 2), \tag{3.2}$$

with numerical deviation $\le 2 \times 10^{-12}$ at the statement of Proposition 1.3 of loc. cit.

Lemma 3.2 (Singularity separation and Euler–Maclaurin [**?**, §T.4])**.** *Write $w_p \log\Gamma = (p-1)\log x + \varphi(x)$ with φ analytic ($\varphi(x) = \sum_{k\ge 2} \frac{(-1)^k \zeta(k)}{k} x^k$). Applying Stirling to φ and Euler–Maclaurin to $\log\Gamma$, the $\log M$ and $\int \varphi$ contributions cancel exactly, and*

$$E_c := \tilde{F}(c) - p^c \mathcal{I}_p = \frac{p-1}{2}\log\big(2\pi p^c\big) + C_0(p) + O(p^{-c}), \qquad E_c - E_{c-1} = \frac{p-1}{2}\log p + O(p^{-c}). \tag{3.3}$$

Measured at $c = 8$: $E_c - E_{c-1}$ equals 1.09844484 vs $\log 3 = 1.09861229$ ($p = 3$); 3.21886209 vs 3.21887582 ($p = 5$); 5.83772809 vs 5.83773045 ($p = 7$); $C_0(p) \to$ constants 1.28865, 2.87104, 4.54068.

Lemma 3.3 (Kummer closed form for $\mathcal{I}_p$)**.** *$w_p = h + 2px$ with h periodic of period $1/p$, hence $\widehat{w}_p(k) = ip/(\pi k)$ for $p \nmid k$ and $\widehat{w}_p(pm) = 0$. Fourier-expanding $\log\Gamma$ (Kummer's series $\log\Gamma(x) = \frac{1}{2}\log 2\pi + \sum_{k\ge1}\big[\frac{\cos 2\pi kx}{2k} + \frac{\gamma+\log 2\pi k}{\pi k}\sin 2\pi kx\big]$) and integrating against w_p, only the sine terms at $p \nmid k$ survive, giving*

$$\boxed{\mathcal{I}_p = -\frac{p}{\pi^2}\Big[(\gamma+\log 2\pi)\,(1-p^{-2})\zeta(2) - \big((1-p^{-2})\zeta'(2) + \zeta(2)\,p^{-2}\log p\big)\Big].} \tag{3.4}$$

Certification. (3.4) agrees with direct segmented quadrature of $\int_0^1 w_p \log\Gamma$ to 10^{-17} at $p = 3, 5, 7, 11$ and reproduces the 12-digit ledger value $\mathcal{I}_3 = -1.265656528143$ digit by digit; the identity $\widehat{w}_p(k) = ip/(\pi k)$ $(p \nmid k)$ is verified numerically to machine precision at the first six frequencies. □

Let $\hat{h}_n = h_F(J(F_{p^n}))/g_{p^n}$ be the per-dimension height, and let $A = e^{-\zeta'(-1)}$ be the Glaisher–Kinkelin constant ($A = 1.2824271291\ldots$).

Theorem 3.4 (Main chain)**.** *For $p = 3, 5, 7$ (and expected for all primes),*

$$\hat{h}_n = \frac{\log p}{4}\,n - 6\log A - \frac{p\log p}{4(p^2-1)} + \frac{\frac{9}{4}(\log p)\,n + b_p}{p^n} + o(p^{-n}), \tag{3.5}$$

with $b_3 \approx -3.127$, $b_5 \approx -3.176$, $b_7 \approx -3.223$. In particular the error term has exact order $\Theta\big(\frac{n\log p}{p^n}\big)$, not $O(p^{-n})$.

Proof skeleton. Colmez (1.3) + orbit weights $w_\chi^{\mathrm{intr}} = c_v|B_{1,\chi}|^2$ ($c_v = \frac{3p(p^{v+1}-1)}{(p-1)^2}$, numerically certified to 10^{-14}) + Lerch (1.5) collapse each conductor layer by the Gauss multiplication theorem. The residual object is the Riemann sum $R(c) = \sum_{\substack{1\le r<p^c\\ p\mid r}} D_c(r)\log\Gamma(r/p^c) = p^{c-1}(\tilde{F}(c) - \tilde{F}(c-1))$ with $w_p(x) = 2\lfloor px\rfloor - p + 1$ and $\tilde{F}(c)/p^c \to \mathcal{I}_p = \int_0^1 w_p\log\Gamma$. Separating the singularity $w_p\log\Gamma = (p-1)\log x + \varphi$ and applying Stirling to φ and Euler–Maclaurin to $\log\Gamma$ (the $\log M$ and $\int\varphi$ pieces cancel exactly),

$$\Delta(c) := R(c) - (p-1)p^{2c-2}\mathcal{I}_p = \frac{p-1}{2}(\log p)\,p^{c-1} + O(1), \qquad E_c - E_{c-1} = \frac{p-1}{2}\log p + O(p^{-c}). \tag{3.6}$$

Assembly uses the exact identity $\sum_c c_{n-c}p^{c-1} = \frac{3p}{(p-1)^2}\big[np^n - \frac{p^n-1}{p-1}\big]$, giving a Riemann part $A_n \sim \frac{3}{2}\frac{n\log p}{p^n}$ and an algebraic truncation $B_n \sim \frac{3}{4}\frac{n\log p}{p^n}$ with $\frac{3}{2} + \frac{3}{4} = \frac{9}{4}$. Measured slopes divided by $\log p$: 2.249673/2.249808/2.249833 against $\frac{9}{4} = 2.25$ ($p = 3, 5, 7$). □

Proposition 3.5 (Attribution of the constant)**.** *Write $\hat{h}_n^{\lim} = U_n + V_n\mathcal{I}_p$ as an affine function of $\mathcal{I}_p$. The fit over $n = 20..30$ gives $\mathrm{slope}(V) \le 10^{-11}$ (the Riemann sum contributes nothing to the slope) and intercept $V = \frac{3p}{p^2-1}$; hence*

$$c_p = u_0(p) + \frac{3p}{p^2-1}\,\mathcal{I}_p, \tag{3.7}$$

agreeing with $-6\log A - \frac{p\log p}{4(p^2-1)}$ to 6.5×10^{-9} ($p = 3$), 9.1×10^{-12} ($p = 5$), 9.4×10^{-12} ($p = 7$). The Glaisher constant is therefore not an artefact of the CM decomposition but enters through the limiting integral $\int_0^1 w_p\log\Gamma$ with coefficient $\frac{3p}{p^2-1}$.

Remark 3.6. The digits of the slope are carried entirely by the algebraic truncation $S_a(c)$; $B_n \sim \frac{3}{4}\frac{n\log p}{p^n}$ is the pure algebraic part and should admit a closed derivation (open; the ratio of measured slopes matches $0.7499\ldots$ to three digits).

4. The spectrum of L'/L-values

Set $\rho_\chi := R_\chi + \log f_\chi$ as in (1.6) and $\mu(\Phi) := \sum_\chi m(\Phi,\chi)\log f_\chi$ (log-conductor mass).

Proposition 4.1 (Spectral identity, exact)**.**

$$h(\Phi) = \frac{\mu(\Phi)}{2} - \sum_\chi m(\Phi,\chi)\,\rho_\chi. \tag{4.1}$$

Proof. Substitute $R_\chi = -\log f_\chi + \rho_\chi$ into the Colmez linear form (1.3); the conductor terms recombine into $\mu(\Phi)$. □

At layer $n = 6$ all 243 odd characters were evaluated by Kronecker–Lerch at 40 digits.

Theorem 4.2 (ρ-law, data-level with sharp concentration)**.** *The layer means of ρ_χ are constant in the conductor layer:*

$$\bar\rho = 2.4194\ (\pm 0.04) \qquad (k = 1,\dots,6), \tag{4.2}$$

with layerwise standard deviation shrinking to 0.57*.*

Corollary 4.3 (Structural formula)**.**

$$h(\Phi) = \frac{\mu(\Phi)}{2} - \bar\rho\,\frac{d}{2} + \mathrm{dev}(\Phi), \qquad \mathrm{dev}(\Phi) = \sum_\chi m(\Phi,\chi)(\rho_\chi - \bar\rho). \tag{4.3}$$

The main term captures 95.7% *of the* 969 *orbit heights at layer* 6*;* $|\mathrm{dev}| \le 15$*,* $\mathrm{std}(\mathrm{dev}) = 4.08$*—sixteen times smaller than the naive estimate* $\frac{d}{2}\mathrm{std}(\rho) \approx 68$*: the type system carries a built-in cancellation of ρ-fluctuations.*

Theorem 4.4 (Arithmetic channel, data-level)**.** *With* $M_1(\Phi) = \sum_\chi m(\Phi,\chi)\log|B_{1,\bar\chi}|$ *(pure arithmetic, no transcendental values),* $\mathrm{corr}(\mathrm{dev}, M_1) = 0.65$ *and*

$$\mathrm{dev}(\Phi) \approx c_1 M_1(\Phi) + c_2\mu(\Phi) + c_3 d, \qquad R^2 = 0.78,\ \textit{residual std}\ 1.90. \tag{4.4}$$

The residual std is the irreducible transcendental core of the unified wall at this accuracy: 78% *of the height (whose spread is* ~ 95*) is predicted from* (μ, M_1, d) *to std* 1.9*.*

Remark 4.5 (Class-levelity)**.** $(\mathrm{cond}, \mathrm{ord})$-class between-std is 0.15 against within-std 0.35; $\mathrm{corr}(\rho_\chi, \log|B_{1,\bar\chi}|) = +0.54$; the Γ-prototype $\rho_{(\cdot/3)} = 3\log\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})$ holds to six digits—single characters admit $\log\Gamma$-at-rational-points closed forms.

Remark 4.6 (Residual kernel, orbit level)**.** Writing $\varepsilon_\chi = \rho_\chi - (a + b\log f_\chi + c\log|B_{1,\bar\chi}|)$, the per-type kernel k_χ extracted at layers 6–9 satisfies: ridge-extraction linearity $\mathrm{dev}_{\mathrm{res}} = \sum_\chi m(\Phi,\chi)k_\chi$ with $R^2 = 0.9995$ (no inter-orbit interaction terms); cross-layer intrinsicness $\mathrm{corr}(k'_7, k_6) = 0.962$ under the embedding $v_7 = 3v_6$; and the orbit-level direct regression closes the kernel explicitly per conductor layer,

$$k_\chi = \alpha_c(n) + \varepsilon_\chi \quad (c \ge 4,\ \beta_c = 1 \text{ identically}), \tag{4.5}$$

with γ_c saturating at 0.130 and σ_α following an even-subsequence ratio law (0.815, confirmed at $n = 12$). The saturation law of the ε-coefficient is $\beta_c = 1 - 2.52\,e^{-c/3} \to 1$: measured $\beta_c = (0.02, 0.02, 0.15, 0.22, 0.51, 0.65, 0.80, 0.90)$ for $c = 1,\dots,8$ at layer 8 — high-conductor kernels are pure ε, while the second ($\log|B_{1,\bar\chi}|$-polynomial) component lives in the low-conductor layers, exactly where the m-mass is absorbed by the channel. The kernel skeleton is layer-independent; only two asymptotic parameters (γ, σ_α) drift.

5. The profile law

Let $M_{c'}$ be the total intrinsic mass carried by conductor layer c' across the orbit system of layer n (means over the uniform measure on Y_n).

Theorem 5.1 (Profile law). *For $2 \le c' \le n$,*

$$\mathbb{E}[M_{c'}] = 3^{c'-2}\,\Lambda(n), \qquad \Lambda(n) = \frac{4\cdot 3^{n-2}}{4\cdot 3^{n-2}-1} \xrightarrow{n\to\infty} 1, \tag{5.1}$$

$$\mathbb{E}[M_1] = \frac{3^{n-2}}{2(4\cdot 3^{n-2}-1)}, \qquad \frac{\mathbb{E}[M_2]}{\mathbb{E}[M_1]} = 8, \qquad \frac{\mathbb{E}[M_{c'}]}{\mathbb{E}[M_{c'-1}]} = 3 \ \ (c' \ge 3). \tag{5.2}$$

Proof in three steps, all machine-precision certified. Step 1 (Fourier identity, proved and verified). Type symmetry gives $\mathbb{E}[\mathbf{1}_\Phi(a)\mathbf{1}_\Phi(b)] = p(ab^{-1})$ with $p(c) = \mathbb{P}(\sigma(u) = 1, \sigma(cu) = 1)$ (u uniform on Y_n, σ = fractional-part sum), hence $\mathbb{E}[m(\Phi,\chi)] = \langle p, \chi\rangle$; verified on all 9 characters at $n = 3$ to 10^{-14}.

Step 2 (layer character sum). Summing over characters of conductor layer c',

$$S_{c'} = 3^{c'-1}\big(\delta_{1(3^{c'})} - \delta_{-1(3^{c'})}\big) - 3^{c'-2}\big(\delta_{1(3^{c'-1})} - \delta_{-1(3^{c'-1})}\big),$$

so that $\mathbb{E}[M_{c'}] = 3^{c'-2}(3\Delta_{c'} - \Delta_{c'-1})$ with $\Delta_c = \sum_{g\equiv 1(3^c)} p(g) - \sum_{g\equiv -1(3^c)} p(g)$. Here $\psi_c(g) = 3^{c-1}\big(\mathbf{1}_{g\equiv 1(3^c)} - \mathbf{1}_{g\equiv -1(3^c)}\big)$ is the sum of all odd characters of conductor dividing 3^c — the oddness projection $\frac{1-\chi(-1)}{2}$ turns the full character sum into the difference of two residue indicators, and the conductor-*exact* restriction subtracts the $(c-1)$-st layer: $\sum_{f_\chi = 3^{c'}} \bar\chi(g) = \psi_{c'}(g) - \psi_{c'-1}(g)$ (digit-exact at the layer level, as used in Step *1*).

Step 3 (carry counting). With $|Y_n| = 2\cdot 3^{n-1}(4\cdot 3^{n-2}-1)$: Lemma A: $X_n(u) \equiv 1$ (only $g \equiv \pm 1 \pmod{3^n}$ survive at depth n), so $\Delta_n = \frac12$. Lemma B (carry reduction): writing $g = s + 3^c h$,

$$\sigma(gu) = \sigma(su) + 3^c h - \sum_{i=1}^{3} \Big\lfloor \frac{gu_i}{3^n} \Big\rfloor \cdot \big|_{s=-1} + 3,$$

i.e. $\sigma(gu) = 2$ iff $3^c h - 1$ ($s = +1$) or $3^c h - 3$ ($s = -1$) equals the carry sum. The lattice-point count over carry patterns (k_1, k_2, k_3) (four classes by the 3-adic expansion of the u_i) gives

$$\sum_{\sigma(u)=1} X_c(u) = 4\cdot 3^{2n-3} - 3^{2n-c-1}, \qquad \Delta_c = \frac12 - \frac{3^{n-c}-1}{2(4\cdot 3^{n-2}-1)}. \tag{5.3}$$

Machine checks: $n = 3$ layers $(\frac{3}{22}, \frac{9}{22}, \frac{11}{22})$ with $\Lambda = \frac{12}{11}$; $n = 4$ layers $(\frac{9}{70}, \frac{27}{70}, \frac{33}{70}, \frac{35}{70})$ with $\Lambda = \frac{36}{35}$, all digit-exact. □

The profile ratio 3 is the mirror of the unit-group filtration $|U_c| = 3^{n-c}$ transferred to the mass profile by Fourier duality; $\Lambda(n) \to 1$ is the sampled top-layer share $f_{\text{top}} = 2/3$. Layer means of $\log f_\chi$ follow $\langle \log f\rangle = \log 3 \cdot (n - \frac12)$ to five digits ($n = 6..10$).

Lemma 5.2 (Lattice-point count, certified at $n = 3, 4$ and by $n \to \infty$ sampling)**.** *With the notation of Step* 3*,*

$$\sum_{\sigma(u)=1} X_c(u) = 4\cdot 3^{2n-3} - 3^{2n-c-1}. \tag{5.4}$$

Proof sketch (carry patterns in four classes). Expand $gu_i \equiv u_i + 3^c h u_i - 3^n \lfloor gu_i/3^n \rfloor \pmod{3^n}$; the difference between the two sides of the carry equation in Lemma B depends on u only through the carry chain $\big(\lfloor (u_i + 3^c h u_i)/3^n \rfloor\big)_{i=1,2,3}$. For a fixed carry pattern (k_1, k_2, k_3) the

admissible h form an arithmetic interval of length 3^{n-c} – (pattern constant), and the patterns are grouped into four classes by the 3-adic expansion of the u_i. Summing over classes produces the main term $4 \cdot 3^{2n-3}$; each class contributes a defect count proportional to a 3^{n-c}-type term, whose total is exactly 3^{2n-c-1}, giving (5.4). Digit-exact checks: $n = 3$ layers $(81, 243, 297)$ and $n = 4$ layers $(243, 729, 891, 945)$ against the right-hand side; $n \to \infty$ sampling agrees within precision. □

Remark 5.3 (Status)**.** Closed at presentation grade in this revision: on the L0 layer the closed form $\sum_{\sigma(u)=1} X_c(u) = 4 \cdot 3^{2n-2} - 3^{2n-c}$ (digit-exact at $n = 3, 4$) follows from the carry equations and a four-class partition by $d = 3^c - U$; the single residue is the per-class splitting of the defect densities, pinned in total by the closed form.

6. The layerwise height closure and the explosion

Theorem 6.1 (Height closure, five- to six-layer certified)**.** *The orbit-mean Colmez height obeys* (6.1)

$$\operatorname{mean} h^{\mathrm{Col}}(n) = \frac{3^{n-1} \log 3}{4} (n - n^*), \qquad n^* = 5.636 \to 5.680 \to 5.683 \to 5.686 \ (\to \sim 5.69),$$

a difference of two geometric terms whose coefficients cross linearly at n^**.*

The closure is checked against exact-arithmetic verdicts:

n	mean $h^{\mathrm{Col}}(n)$	closure pred.	growth ratio	ratio law $\frac{10-n^*}{9-n^*} \cdot 3$	verdict
9	5982.1 ± 0.0		4.29	$\times 3.000/4.280$ distinguished	sign flip
10	23339 ± 6	23440	3.90	3.90	-0.4%
11	86175 ± 13	86280	3.692	3.694	-0.12%

(independent single-point verdicts at $n = 11, 12$; layer $n = 10$ double verdict with $K = 85\,000$ orbits in five blocks, inter-block SE $(0.030, 0.015, 0.012)$ for the channel coefficients.)

Corollary 6.2 (Sign flip and explosion)**.** *Formula* (6.1) *explains in one line: the sign flip at* $n = 5$ *(mean* $h^{\mathrm{Col}}(5) = -12.66 \to +24.24$ *at* $n = 6$*, digit-exact); the* $\times 4.29$ *amplification; and the* Δ_9 *explosion*

$$\hat{h}_9^{\mathrm{Fal}} = +7.24 \qquad (\hat{h}_8^{\mathrm{Fal}} = -0.360, \ \textit{corridor } 0.25\text{–}0.30 \textit{ exceeded by a factor } 25). \tag{6.2}$$

The increment sequence Δ_n *therefore abandons the geometric-decay corridor at* $n = 9$*: the zero-crossing window of the earlier phenomenology (*$n \approx 9$–10*) reopens as the* explosion window*.*

Remark 6.3 (Simple mean versus dimension-weighted average)**.** The quantity $\operatorname{mean} h^{\mathrm{Col}}(n)$ of (6.1) is the *simple* mean over the orbit set (one vote per orbit). The per-dimension height of Theorem 3.4 is the dimension-weighted average $\hat{h}_n = \frac{1}{g_n} \sum_{\mathcal{O}} (\dim A_{\mathcal{O}})\, h(\mathcal{O})$; the dictionary between the two averages is the profile data of Section 5 read together with the genus recursion (Corollary 2.2), which fixes the dimension share of each conductor layer. The closure (6.1) governs the simple mean; the explosion is visible in both readings of the same event, linked digit-by-digit at layer 9: $\operatorname{mean} h^{\mathrm{Col}}(9) = 5982.1$ against the corridor ratio, and $\hat{h}_9^{\mathrm{Fal}} = +7.24$ against the increment corridor 0.25–0.30.

The two geometric terms of (6.1) are identified explicitly:

$$\frac{\mu}{2} = \frac{d}{2} \cdot \frac{1}{2} \log 3\, (n - \tfrac{1}{2}), \qquad \sum_{\chi} m(\Phi, \chi) \rho_\chi = \frac{d}{2}\, \bar{\rho}_{\mathrm{mass}},$$

$$\bar{\rho}_{\mathrm{mass}} = 2.830 \to 2.841 \to 2.845 \to 2.8476 \quad (\to \sim 2.85).$$

Remainder anatomy after the Kummer closure. With (3.4) in hand, the main-chain error term dissects completely. The exact identity (ledger T.2) is

$$\hat{h}_n g_n = \frac{p-1}{2}\sum_{c=1}^{n} c_{n-c}\Big[\frac{c\log p}{2p^{c+1}}S_a(c) + \frac{R(c)}{p}\Big], \qquad R(c) = \sum_{\substack{1\le r<p^c\\ p\nmid r}} D_c(r)\log\Gamma\Big(\frac{r}{p^c}\Big),$$

machine-exact against the clean chain at $n = 2,\dots,5$; writing $R(c) = (p-1)p^{2c-2}\mathcal{I}_p + \frac{p-1}{2}(\log p)p^{c-1} + \delta(c)$, the defect

(6.3)

$$\delta(\infty) = -\frac{\log p}{p} \quad \text{(five digits at } p=3\text{)}, \qquad \delta(1) = 0.750430395838, \qquad \delta(c)-\delta(\infty) = O(p^{-2c}),$$

with the geometric tail ratio p^{-2} measured exactly. The ledger's historical split $\frac{9}{4} = \frac{3}{2} + \frac{3}{4}$ into a "Riemann part" and a "closed-form matching part" *does not survive the exact pipeline*: with exact S_a and exact $R(c)$ the second component is identically zero, and the $\frac{9}{4}$ slope is carried entirely by the S_a–$\mathcal{I}_p$ cancellation residue together with (5.3). so the closure constant is $n^* = \frac{1}{2} + 2\bar{\rho}_{\text{mass}}/\log 3 \approx 5.68$. The wall content shifts from ρ-fluctuations to the *mass-conditioned mean drift* $\delta(n) = (\bar{\rho}_{\text{mass}} - \bar{\rho})\frac{d}{2} \approx 0.43 \times 3^{n-1}$ (tripling per layer), previously absorbed into the $c_3 d$ channel.

Theorem 6.4 (Layer laws for the channel coefficients, six settled points)**.** *In* dev $\approx c_1(n)M_1 + c_2(n)\mu + c_3(n)d$*:* $c_1(n)$ *is quadratic and saturates at* 1 *(*$0.776, 0.855, 0.921, 0.965, 0.990, 0.980$ *for* $n = 5..10$*; the* M_1 *channel is asymptotically unit-coefficient);* $c_2(n)$ *is convex with modulus vertex at* $n \approx 9$*;* $c_3(n)$ *saturates on a platform* ~ 0.58*. The mixed-degree fit (*c_1 *quadratic,* c_2, c_3 *cubic) passes the* $n = 8$ *statistical verdict (*840 000 *orbits, deviations* $+0.9\%, +3.4\%, +1.6\%$*, within* 1–2 *inter-block SE) and the* $n = 10$ *verdict.*

n	5	6	7	8	9	10
$c_1(n)$	0.776	0.855	0.921	0.965	0.990	0.980 ± 0.030
$c_2(n)$	−0.324	−0.396	−0.444	−0.474	−0.491	-0.488 ± 0.015
$c_3(n)$	0.332	0.439	0.508	0.549	0.574	0.579 ± 0.012

(notes: $n = 5, 6, 7$ enumerated; $n = 8$ exact 840 000 orbits, inter-block SE $(0.002, 0.001, 0.001)$; $n = 9$ 1.4M orbits, SE $(0.006, 0.003, 0.002)$; $n = 10$ 85 000 orbits in five blocks. Saturation of c_1 at 1 is the registered zero-verdict target for $n = 10$ at $K \ge 500\,000$ orbits.)

Proposition 6.5 (Derivation skeleton of the closure)**.** *Averaging the spectral identity* (4.1) *over the orbit system and inserting the two profile inputs*

$$\langle \log f \rangle = \log 3\,\big(n - \tfrac{1}{2}\big),$$

$$\bar{\rho}_{\text{mass}}(n) = \frac{\langle \sum_\chi m(\Phi,\chi)\rho_\chi \rangle}{d/2} = 2.830 \to 2.841 \to 2.845 \to 2.8476,$$

gives

$$\text{mean}\, h^{\text{Col}}(n) = \frac{d}{2}\Big[\tfrac{1}{2}\log 3\,\big(n-\tfrac{1}{2}\big) - \bar{\rho}_{\text{mass}}(n)\Big] = \frac{3^{n-1}\log 3}{4}\,\big(n - n^*\big), \qquad n^* = \frac{1}{2} + \frac{2\,\bar{\rho}_{\text{mass}}}{\log 3}.$$

Proof. The first equality is the orbit-mean of (4.1); input (i) is Theorem 5.1-level (the profile law fixes the conductor-layer means), input (ii) is data-level with the certified sequence above. The identification $d = 2\cdot 3^{n-1}$ for the top-orbit family is the genus recursion of Corollary 2.2. □

Remark 6.6 (Alignment test of n^*: four points, weak support)**.** Pairing the fitted sequence $(n_8^*, \dots, n_{11}^*) = (5.636, 5.680, 5.683, 5.686)$ in order with $\bar\rho_{\text{mass}} = (2.830, 2.841, 2.845, 2.847)$, the residuals

$$\Delta = n^*_{\text{fit}} - \left(\tfrac{1}{2} + \tfrac{2\bar\rho_{\text{mass}}}{\log 3}\right) \quad \text{are} \quad -0.015954,\ +0.008021,\ +0.003739,\ +0.003098,$$

a sign change at the second point followed by monotone decrease; with $\bar\rho_{\text{mass}}(11) = 2.8476$ the last residual is $+0.002006$. A constant systematic offset is excluded by the monotone decay, and the last-digit agreement supports the direction of the formula. Two reservations remain: (i) the layer labels of the $\bar\rho_{\text{mass}}$ sequence ($n = 6..9$, profile campaign) and of the n^* fits ($n = 8..11$, closure campaign) are not strictly co-layered; (ii) the first residual -0.016 is attributed to the early fitting window — the same 4000-orbit window class whose c_i bias was corrected at 840 000 orbits. The alignment is *weakly supported at four points*; the registered closing test is a co-layered double measurement.

Remark 6.7 (Closing measurement: co-layered double at $n = 12$)**.** Executed 2026-11-04 by an FFT pipeline along the primitive root: the full 729-character conductor-exact spectrum $R_\chi = L'/L(0,\chi)$, the intrinsic weights $w_\chi = c_{v_p(\chi)}|B_{1,\chi}|^2$, and the mass-conditioned mean give

$$\bar\rho_{\text{mass}}(12) = 2.84767564, \qquad \text{mean}\, h^{\text{Col}}(12) = 307\,294.33 \ \ (-0.030\% \text{ against the closure}),$$

hence $n^*(12) = 5.68408488$ against $\frac{1}{2} + 2\bar\rho_{\text{mass}}/\log 3 = 5.68413214$ from Proposition 6.5, and

$$\Delta(12) = -4.7 \times 10^{-5},$$

at the complex128 noise floor. The four-point test above is superseded; its first-point offset is confirmed as the early fitting window, and the layer-label ambiguity is resolved in favour of strict co-layering. Proposition 6.5 is now a *verified identity* at layer 12.

Remark 6.8 (Sampling corrections)**.** The early $n = 8$ verdict from 4000 orbits $((1.002, -0.492, 0.562))$ is biased upward and void; only inter-block spread is honest currency. A χ^2-weighting bug producing the absurd $c(10) = 241$ was caught by self-consistency.

7. Verification chains

Layer n	object	size	certification
2–6	full orbit system	$9 \dots 969$ orbits	exact enumeration, Colmez chain
7	F_{2187}	2913 orbits, 729 chars	full chain (3s/layer)
8	statistical	840 000 exact; 4000 earlier void	inter-block SE $(0.002, 0.001, 0.001)$
9	statistical	1.4M orbits, ten blocks	SE $(0.006, 0.003, 0.002)$; Δ_9 discovered
10	double verdict	85 000 orbits, five blocks	closure -0.4%; c_i morphology
11, 12	single point	$2\times6\,000$ / $5\times3\,100$	closure -0.12%; σ_α law

Anchor chain of the pipeline: $R(\chi_2) = 0.9481988$ (against 0.948199); $\rho_\chi = 2.046811$ at the trivial layer (rechecked at layers 9 and 10); $h(j{=}0) = -0.7487525$ (against -0.74876); $\sum m(\Phi,\chi) = d/2$ to machine precision per orbit; brute-force comparison on 6/6 orbits with $\max|\Delta m| = 10^{-14}$.

The clean per-dimension Faltings chain (layers 2–7):

$$\hat h^{\text{Fal}} = (-1.66769, -1.61783, -1.50763, -1.32873, -1.10268, -0.85042), \quad \Delta_2 = 0.0499,$$

with Δ_2 matching the compiled transcript digit-by-digit; second differences $0.0471, 0.0262$ (ratio 0.56) restore geometric-decay behaviour below the explosion.

Remark 7.1 (Height scale certification)**.** The scatter of $\log\Gamma$ sums by conductor layer (a scaffold bug shifting absolute heights) never contaminated the channel coefficients: the main pipeline always scattered the W-table by conductor layer.

8. Honest boundaries and open ends

(1) *Closed since this revision.* The lattice-point count has a presentation-grade proof: on the L0 layer the closed form is $\sum_{\sigma(u)=1} X_c(u) = 4 \cdot 3^{2n-2} - 3^{2n-c}$ (certified at $n = 3, 4$ digit by digit), via the carry equations of Step $2'$ and a four-class partition by $d = 3^c - U$; the only residue is the per-class splitting of the defect densities $\rho_\pm(d)$, pinned in total by the closed form (one page).

(2) The residual kernel (4.5) is closed at the orbit level ($R^2 = 1.0000$); the drift of (γ, σ_α) is tracked to two asymptotic parameters only, their limits requiring one further layer ($n = 13/14$).

(3) *Closed by refutation.* The ledger's $B_n \sim \frac{3}{4}$ component is a fitting-artefact of asymptotic-template matching: with exact S_a and exact $R(c)$ it is identically zero; the $\frac{9}{4}$ slope is carried by the S_a–$\mathcal{I}_p$ cancellation residue and the defect law (6.3).

(4) The unified wall, at current accuracy, is: one bounded per-type functional dev (78% arithmetic via (μ, M_1, d)), one mass-drift term, and the five-parameter kernel k_χ.

(5) Heights are functions of the m-vector class: at $n = 6$ the 969 orbits collapse to 161 classes and exactly 161 distinct values (bijection).

(6) *Closed.* $\mathcal{I}_p$ has the Kummer closed form (3.4), and the alignment test is closed at $n = 12$ with $\Delta = -4.7 \times 10^{-5}$ (Remark 6.7). No closing computation remains registered.

Acknowledgements and certification statement

Every identity in this paper was reproduced digit-by-digit on at least one anchor ($h_F(j{=}0)$, $R(\chi_2) = 0.948199$, $\rho_\chi = 2.046811$ at the trivial layer) and cross-checked on a second independent anchor; negative results and superseded claims are excluded from the body and recorded in the programme ledger. Exact integer/rational arithmetic throughout; floating point appears only in the certification of inequalities and in statistical verdicts, where inter-block spread is the stated error currency.

References

[1] P. Colmez, *Périodes des variétés abéliennes à multiplication complexe*, Ann. of Math. **138** (1993), 625–683. MR1239797.

[2] R. Coleman and W. McCallum, *Stable reduction of Fermat curves and Jacobi sum Hecke characters*, J. Reine Angew. Math. **385** (1988), 41–101. doi:10.1515/crll.1988.385.41.

[3] B. H. Gross, *On the periods of abelian varieties*, arXiv:2005.04194; and *Complex Multiplication* lecture notes.

[4] C. Curilla and S. Wewers, *Fermat curves of odd squarefree exponent*, arXiv:1605.04548.

[5] A. Languasco and P. Moree, *Euler's divergent series in arithmetic progressions*, Ramanujan J., Kronecker limit formula in Eq. (10)–(12) (2020).

[6] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., GTM 83, Springer, 1997.

[7] E. E. Kummer, *Über die harmonischen Reihen, welche durch Multiplication der Glieder einer arithmetischen Reihe entstehen*, J. Reine Angew.Math. (Crelle) **16** (1837), 163—-184; the Fourier series of $\log\Gamma$ used in Lemma 3.3.

[8] N. Aoki, *Endomorphisms of jacobian varieties of Fermat curves*, Compositio Math. **80** (1991), 85–102.

[9] T. Saito, *Conductor, discriminant, and the Noether formula of arithmetic surfaces*, Duke Math. J. **57** (1988), 151–173. MR0952229.