A closed-form bound for linear forms in three p-adic logarithms of rational numbers

David Niedbala Giraudin
ORCID 0009-0009-1526-1178

23 October 2026; English version, 30 October 2026

Abstract

We deduce from the p-adic kit for three logarithms of [4] a fully explicit bound in closed form. If $\gamma_1, \gamma_2, \gamma_3$ are multiplicatively independent rationals, congruent to 1 modulo p (modulo 4 if $p = 2$), of heights at most $a_j \log p$ with $a_j \geq 1$, and if $|b_j| \leq B$, then

$$v_p\big(\gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_3} - 1\big) \leq a_1 a_2 a_3\, \mathcal{B}^2 \max\Big\{27\,711,\ C_2(p) \max_j a_j\Big\}, \qquad \mathcal{B} = \max\Big\{\frac{\log B}{\log p}, 10\Big\},$$

with $C_2(p) = 11\,107\,\frac{p}{p-1}\big(1 + \frac{0.31}{\log p}\big)^2$, as soon as $\gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_3} \neq 1$. By a change of basis in the multiplicative group, the result applies to arbitrary rationals prime to p. On the examples of [4], the bound is 42 to 956 times smaller than that of Yu's explicit theorem, and stays within a factor between 1.2 and 2.5 of the numerical optimisation of the parameters of the kit.

hexagon:2610.00708v2

1 Statements

For a non-zero rational $\gamma = u/w$ written in lowest terms, put $h(\gamma) = \log \max\{|u|, |w|\}$. We call *principal unit* a rational γ prime to p with $\gamma \equiv 1 \pmod p$ if p is odd, and $\gamma \equiv 1 \pmod 4$ if $p = 2$.

Theorem 1. *Let p be a prime, $\gamma_1, \gamma_2, \gamma_3$ multiplicatively independent principal units, and h_1, h_2, h_3 real numbers with $h_j \geq \max\{h(\gamma_j), \log p\}$. Put $a_j = h_j/\log p$ (≥ 1). Let $b \in \mathbb{Z}^3$, $B \geq \max\{1, |b_1|, |b_2|, |b_3|\}$ and $\mathcal{B} = \max\{\log B/\log p, 10\}$. If $\Lambda = \gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_3} - 1 \neq 0$, then*

$$v_p(\Lambda) \leq a_1 a_2 a_3\, \mathcal{B}^2 \max\big\{C_1,\ C_2(p) \max\{a_1, a_2, a_3\}\big\},$$

with $C_1 = 27\,711$ and $C_2(p) = 11\,107\,\frac{p}{p-1}\big(1 + \frac{0.31}{\log p}\big)^2$.

Numerically, $C_2(2) \leq 46\,528$, $C_2(3) \leq 27\,390$, $C_2(5) \leq 19\,748$, $C_2(7) \leq 17\,416$, $C_2(101) \leq 12\,776$, and $C_2(p) \to 11\,107$. The term C_1 comes from the main case, the term $C_2(p) \max a_j$ from the degenerate cases.

For arbitrary rationals, one reduces to the theorem by a change of basis. Let $\alpha_1, \alpha_2, \alpha_3$ be non-zero rationals prime to p, multiplicatively independent. For p odd, let ζ be a primitive $(p-1)$-th root of unity in $\mathbb{Z}_p$ and n_i the Teichmüller index of α_i ($\alpha_i \equiv \zeta^{n_i} \pmod p$); put $g = p - 1$. For $p = 2$, put $g = 2$, $n_i = 0$ if $\alpha_i \equiv 1 \pmod 4$ and $n_i = 1$ otherwise. Let

$$\Lambda_0 = \{x \in \mathbb{Z}^3 : n_1 x_1 + n_2 x_2 + n_3 x_3 \equiv 0 \pmod g\},$$

e_1, e_2, e_3 a basis of Λ_0, E the matrix with rows e_j, $\gamma_j = \alpha_1^{e_{j1}}\alpha_2^{e_{j2}}\alpha_3^{e_{j3}}$ and $c_j = \sum_k |(E^{-1})_{kj}|$.

Corollary 2. *With these notations, let $h_j \geq \max\{h(\gamma_j), \log p\}$, $a_j = h_j/\log p$, $b \in \mathbb{Z}^3$ with $|b_i| \leq B$ and $\alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3} \neq 1$. Put $\mathcal{B}' = \max\{\log(B \max_j c_j)/\log p, 10\}$. Then*

$$v_p\big(\alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3} - 1\big) \leq a_1 a_2 a_3\, \mathcal{B}'^2 \max\Big\{C_1,\ C_2(p) \max_j a_j\Big\}.$$

Since $\Lambda = \alpha_3^{b_3}\big(\alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{-b_3} - 1\big)$, the corollary also applies to $\Lambda = \alpha_1^{b_1}\alpha_2^{b_2} - \alpha_3^{b_3}$.

Remark 3. The bound is in $(\log B)^2$, whereas that of Yu [5] is in $\log B$; like the kit, it is therefore meant for the reduction of bounds. In the example $p = 7$, $\alpha = (2, 3, 5)$ of Section 6, it remains better than the explicit form of Yu given in [1] as long as $\log B < 1.77 \cdot 10^4$, that is $B < 10^{7711}$. It contains no factor p: for principal units, none appears in Corollary 2 of [2] either. For arbitrary rationals, it moves into the heights. For instance, if the numerators and denominators of the α_i are pairwise coprime, then $h(\alpha^e) \geq \frac{1}{2}\sum_i |e_i| h(\alpha_i)$ for $e \in \mathbb{Z}^3$, and Hadamard's inequality gives $h_1 h_2 h_3 \geq \frac{[\mathbb{Z}^3:\Lambda_0]}{8} h(\alpha_1)h(\alpha_2)h(\alpha_3)$; without this coprimality hypothesis, this lower bound may fail.

Remark 4. The constants are dictated by the case $\mathcal{B} = 10$. For large $\mathcal{B}$, if ℓ is allowed to depend on $\mathcal{B}$ (instead of being fixed at 7.2), the same construction gives, in the main case, a constant tending to $520\,\ell_\infty^2 \approx 1.39 \cdot 10^4$, where $\ell_\infty = 2/(1 - \theta_\infty)$ and θ_∞ is the limit of $\theta(L)$ defined in Section 3.

2 Results used

The kit with $g = 1$

The following result is Theorem 11 of [4], that is, its Theorem 2 with $g = 1$; the exceptional cases (2.14) and (2.15) are those of Mignotte and Voutier [3]. Let $\gamma_1, \gamma_2, \gamma_3$ be multiplicatively independent principal units, $h_j \geq h(\gamma_j)$, and $\Lambda' = \gamma_1^{\beta_1}\gamma_2^{\beta_2} - \gamma_3^{\beta_3} \neq 0$ with $\beta_j \geq 1$, $\gcd(\beta_1, \beta_2, \beta_3) = 1$, $p \nmid \beta_3$ and $\beta_j \leq B$. Write $d_1 = \gcd(\beta_1, \beta_3)$, $d_2 = \gcd(\beta_2, \beta_3)$,

$$F_K = \Big(\prod_{k=1}^{K-1} (k!)^{K-k}\Big)^{-12/(K^3-K)}, \qquad b = \frac{(R-1)\beta_3 + (T-1)\beta_1}{2d_1} \cdot \frac{(S-1)\beta_3 + (T-1)\beta_2}{2d_2} \cdot F_K,$$

$E(K, L) = (2K + 1)(L - 1)/3$, $\lambda_L = 2\lfloor L^2/4 \rfloor / L$ and $N = K(K + 1)L/2$.

Theorem 5 ([4])**.** *Let $K \geq 2$, $L \geq 2$, $R_i, S_i, T_i \geq 1$ $(i = 1, 2, 3)$ be integers, $R = R_1 + R_2 + R_3 + 1$, $S = S_1 + S_2 + S_3 + 1$, $T = T_1 + T_2 + T_3 + 1$, $\chi > 0$, $V_1 = (R_1 + 1)(S_1 + 1)(T_1 + 1)$ and $\mathcal{M} = \max\{R_1 + S_1 + 1, S_1 + T_1 + 1, R_1 + T_1 + 1, \chi\sqrt{V_1}\}$. Assume*

$$\begin{aligned} &\text{(Z1)} \quad V_1 > K\mathcal{M} \quad \text{and} \quad V_1 > L, \\ &\text{(Z2)} \quad (R_2 + 1)(S_2 + 1)(T_2 + 1) > \max\{2KL, K^2\}, \\ &\text{(Z3)} \quad (R_3 + 1)(S_3 + 1)(T_3 + 1) > 3K^2 L, \end{aligned}$$

$$E(K, L)\log p > \log N + \frac{2(K-1)}{3}\log b + \lambda_L\big((R-1)h_1 + (S-1)h_2 + (T-1)h_3\big). \tag{1}$$

Then $v_p(\Lambda') \leq KL - 2$, except in one of the following two cases:

(2.14) $\beta_1 \leq \max\{R_1, R_2\}$, $\beta_2 \leq \max\{S_1, S_2\}$, $\beta_3 \leq \max\{T_1, T_2\}$; then $v_p(\Lambda') \leq (\log 2 + \beta_1 h_1 + \beta_2 h_2 + \beta_3 h_3)/\log p$;

(2.15) there are coprime integers $(u_1, u_2, u_3) \neq 0$ with $u_1\beta_1 + u_2\beta_2 + u_3\beta_3 = 0$ and $|u_i| \leq U_i$, where

$$U_1 = \frac{(S_1 + 1)(T_1 + 1)}{\mathcal{M} - \max\{S_1, T_1\}}, \quad U_2 = \frac{(R_1 + 1)(T_1 + 1)}{\mathcal{M} - \max\{R_1, T_1\}}, \quad U_3 = \frac{(R_1 + 1)(S_1 + 1)}{\mathcal{M} - \max\{R_1, S_1\}}.$$

Two logarithms

Theorem 6 ([2, Corollaire 2])**.** *Let γ, γ' be multiplicatively independent principal units, $c, c' \geq 1$ integers, $H \geq \max\{h(\gamma), \log p\}$, $H' \geq \max\{h(\gamma'), \log p\}$ and $b^\star = c/H' + c'/H$. Then*

$$v_p\big(\gamma^c - \gamma'^{c'}\big) \leq \frac{24p}{(p-1)(\log p)^4}\Big(\max\big\{\log b^\star + \log\log p + 0.4,\ 10\log p,\ 10\big\}\Big)^2 HH'.$$

For $p = 2$, the integer t of [2] equals 1 and the hypothesis $\gamma, \gamma' \in U_v^1$ is satisfied.

A lemma on factorials

Lemma 7. *For every integer $K \ge 2$, $\log F_K \le -2\log K + \frac{11}{3} + \frac{4\log K}{K}$.*

Proof. We have $\sum_{k=1}^{K-1}(K-k)\log k! = \sum_{j=1}^{K} w_j \log j$ with $w_j = \frac{(K-j)(K-j+1)}{2}$. Let $w(x) = \frac{(K-x)(K-x-1)}{2}$. For $2 \le j \le K-1$ and $x \in [j-1, j]$, we have $0 \le w(x) \le w_j$ and $0 \le \log x \le \log j$; on $[0,1]$, $w \ge 0$ and $\log x \le 0$; on $[K-1, K]$, $w \le 0$ and $\log x > 0$, while $w_1 \log 1 = w_K \log K = 0$. Hence

$$\sum_{j=1}^{K} w_j \log j \;\ge\; \int_0^K w(x)\log x\,dx = \frac{K^3}{6}\Big(\log K - \frac{11}{6}\Big) - \frac{K^2}{4}\Big(\log K - \frac{3}{2}\Big),$$

using $\int_0^1 (1-t)^n \log t\,dt = -H_{n+1}/(n+1)$, where H_{n+1} is the harmonic number. Thus $-\log F_K \ge G_K := \big(2K^3\log K - \frac{11}{3}K^3 - 3K^2\log K + \frac{9}{2}K^2\big)/(K^3-K)$, and

$$G_K - \Big(2\log K - \frac{11}{3} - \frac{4\log K}{K}\Big) = \frac{K^2\log K + \frac{9}{2}K^2 + 2K\log K - \frac{11}{3}K - 4\log K}{K^3 - K} > 0$$

for $K \ge 2$. □

3 Choice of the parameters

We fix $m = 520$, $\kappa = 0.65$, $\ell = 7.2$. Let $a_1, a_2, a_3 \ge 1$, $A = a_1a_2a_3$, $P = \log p$, $\mathcal{B} \ge 10$, and put

$$L = \lceil \ell\mathcal{B} \rceil, \quad K = \lfloor mLA \rfloor, \quad \chi = \kappa\sqrt{L}, \quad c_1 = (m\kappa)^{2/3}L, \quad c_2 = (mL)^{2/3}, \quad c_3 = (3m^2)^{1/3}L,$$

$$R_i = \lfloor c_i a_2 a_3 \rfloor, \quad S_i = \lfloor c_i a_1 a_3 \rfloor, \quad T_i = \lfloor c_i a_1 a_2 \rfloor \qquad (i = 1, 2, 3).$$

We have $L \ge L_0 = 72$ and $K \ge K_0 = \lfloor mL_0 \rfloor = 37\,440$. Put $\sigma = c_1 + c_2 + c_3 = \tau(L)\,mL$, where

$$\tau(L) = \kappa^{2/3}m^{-1/3} + (mL)^{-1/3} + (3/m)^{1/3}, \qquad \theta(L,K) = \frac{9}{4}\tau(L)\Big(1 + \frac{1}{K}\Big).$$

These quantities decrease in L and K; we have $\tau(L_0) \le 0.30256$ and $\theta(L,K) \le \theta_0 = \theta(L_0, K_0) \le 0.68078$. Finally $R_i + 1 > c_i a_2 a_3$, so that

$$(R_i+1)(S_i+1)(T_i+1) > c_i^3 A^2, \qquad R_i \le c_i a_2 a_3,\ S_i \le c_i a_1 a_3,\ T_i \le c_i a_1 a_2. \tag{2}$$

Lemma 8. *We have $K \ge 2$, $L \ge 2$, $R_i, S_i, T_i \ge 1$, and conditions (Z1), (Z2), (Z3) hold.*

Proof. The c_i are ≥ 1 and $a_j \ge 1$. Since $K \le mLA$ and $A \ge 1$, (2) gives $\sqrt{V_1} > c_1^{3/2}A = \kappa\sqrt{L}\cdot mLA \ge \chi K$, hence $V_1 > K\chi\sqrt{V_1}$. Next $R_1 + S_1 + 1 \le (2c_1+1)A \le 3c_1A$, and $V_1 > c_1^3A^2 \ge 3mLc_1A^2 \ge K(R_1+S_1+1)$ because $c_1^2 = (m\kappa)^{4/3}L^2 \ge 3mL$, which is equivalent to $m^{1/3}\kappa^{4/3}L \ge 3$ (the left-hand side is at least 326); likewise for S_1+T_1+1 and R_1+T_1+1. Hence $V_1 > K\mathcal{M}$, and $V_1 > c_1^3 \ge L$. For (Z2), $c_2^3A^2 = m^2L^2A^2 \ge K^2$ and $m^2L^2A^2 \ge 2mL^2A^2 \ge 2KL$. For (Z3), $c_3^3A^2 = 3m^2L^3A^2 \ge 3K^2L$. □

Lemma 9. *Let $h_j = a_jP$ and $\beta_j \le B$ with $\log B \le \mathcal{B}P$. Criterion (1) holds.*

Proof. Height term. By (2), $R - 1 = R_1 + R_2 + R_3 \le \sigma a_2a_3$, whence $(R-1)h_1 \le \sigma AP$, and likewise for S, T. Since $\lambda_L \le L/2$ and $A \le (K+1)/(mL)$,

$$\lambda_L\big((R-1)h_1 + (S-1)h_2 + (T-1)h_3\big) \le \frac{3}{2}L\sigma AP \le \frac{3}{2}\tau(L)L(K+1)P.$$

Term $\log b$. We have $(R-1)\beta_3 + (T-1)\beta_1 \le (R+T-2)B \le \sigma(a_2a_3 + a_1a_2)B \le 2\sigma AB$, and likewise for the second factor; with Lemma 7, $\log b \le 2\log B + 2\log(\sigma A/K) + \frac{11}{3} + \frac{4\log K}{K}$. Since $K \ge mLA(1 - \frac{1}{mL})$, we have $\sigma A/K \le \tau(L)/(1-\frac{1}{mL})$, hence $\log b \le 2\log B + \beta$ with

$$\beta = 2\log\tau(L) - 2\log\Big(1 - \frac{1}{mL}\Big) + \frac{11}{3} + \frac{4\log K}{K} \le \beta_0 \le 1.27688,$$

β_0 being the value at (L_0, K_0) (each term decreases). Thus $\frac{2(K-1)}{3}\log b \le \frac{2K}{3}(2\log B + \beta_0)$, since $\beta_0 > 0$.

Term $\log N$. $\log N \le \log K + \log(K+1) + \log L \le 3\log(K+1)$.

Conclusion. Since $E(K,L) \ge \frac{2K}{3}(L-1)$, it suffices, after division by $\frac{2K}{3}P$, that

$$L\big(1-\theta(L,K)\big) > 1 + \frac{2\log B}{P} + \frac{1}{P}\Big(\beta_0 + \frac{9\log(K+1)}{2K}\Big).$$

We have $L(1-\theta(L,K)) \ge \ell\mathcal{B}(1-\theta_0)$, $2\log B/P \le 2\mathcal{B}$, $P \ge \log 2$ and $\frac{9\log(K+1)}{2K} \le \frac{9\log(K_0+1)}{2K_0}$. It therefore suffices that $\mathcal{B}\big(\ell(1-\theta_0)-2\big) > 1 + \big(\beta_0 + \frac{9\log(K_0+1)}{2K_0}\big)/\log 2$, that is $0.2983\,\mathcal{B} > 2.8440$, which holds for $\mathcal{B} \ge 10$. □

Lemma 10. *We have* $\mathcal{M} > \max\{R_1, S_1, T_1\}$ *and* $U_j \le \mu\eta\, a_j$ *for* $j = 1, 2, 3$, *where*

$$\mu = m^{1/3}\kappa^{-2/3} \le 10.7167, \qquad \eta = \frac{(1+1/c_1(L_0))^2}{1 - 1/(\kappa^{4/3}m^{1/3}L_0)} \le 1.00366.$$

Proof. Put $x_1 = c_1a_2a_3$, $x_2 = c_1a_1a_3$, $x_3 = c_1a_1a_2$, so that $R_1 \le x_1 < R_1 + 1$, etc. We have $\mathcal{M} \ge \chi\sqrt{V_1} > \chi c_1^{3/2}A$ and $\max\{S_1, T_1\} \le c_1a_1\max\{a_2, a_3\} \le c_1A$, hence $\mathcal{M} - \max\{S_1, T_1\} > c_1A(\chi\sqrt{c_1} - 1) > 0$. Next $(S_1+1)(T_1+1) \le x_2x_3(1+1/c_1)^2 = c_1^2a_1A(1+1/c_1)^2$. Hence

$$U_1 \le \frac{c_1a_1(1+1/c_1)^2}{\chi\sqrt{c_1} - 1} = \frac{\sqrt{c_1}}{\chi}\, a_1\, \frac{(1+1/c_1)^2}{1 - 1/(\chi\sqrt{c_1})},$$

with $\sqrt{c_1}/\chi = \mu$, $\chi\sqrt{c_1} = \kappa^{4/3}m^{1/3}L \ge \kappa^{4/3}m^{1/3}L_0$ and $c_1 \ge c_1(L_0)$. Likewise for U_2, U_3. □

4 Proof of Theorem 1

Reduction. Let $\delta = \gcd(b_1, b_2, b_3)$ and $y = \gamma_1^{b_1/\delta}\gamma_2^{b_2/\delta}\gamma_3^{b_3/\delta}$. It is a principal unit, $y \ne 1$, and $v_p(\Lambda) = v_p(y^\delta - 1) = v_p(y-1) + v_p(\delta)$ (for $p = 2$, one uses $y \equiv 1 \pmod 4$). We have $v_p(\delta) \le \log B/P \le \mathcal{B} \le A\mathcal{B}^2/10$. Since b/δ is primitive, one of its coordinates, say j_0, is prime to p. Replacing some γ_j by γ_j^{-1} (which does not change $h(\gamma_j)$) and multiplying by a unit, $v_p(y-1)$ is the valuation of $\gamma_i^{\beta_i}\gamma_{i'}^{\beta_{i'}} - \gamma_{j_0}^{\beta_{j_0}}$, with $\{i, i', j_0\} = \{1,2,3\}$, $\beta_j = |b_j|/\delta \le B$ and $p \nmid \beta_{j_0}$. We renumber so that $j_0 = 3$; the statements being symmetric in a_1, a_2, a_3, this does not change the bound.

One or two zero exponents. If two of the β_j vanish, $v_p(y-1) = v_p(\gamma_j^\beta - 1) = v_p(\gamma_j - 1) + v_p(\beta) \le (h_j + \log 2 + \log B)/P \le a_j + 1 + \mathcal{B}$. If only one vanishes, Theorem 6, applied to the two units γ_j, γ_k whose exponents do not vanish, with $H = h_j$ and $H' = h_k$, gives the bound of case (2.15) below with 24 instead of $96\mu^2\eta^2$ and $a_ja_k \le A$ instead of $A\max a_j$. In both cases we stay below the stated bound.

Three non-zero exponents. We apply Theorem 5 with the parameters of Section 3 and $h_j = a_jP$; its hypotheses hold (Lemmas 8 and 9). Outside cases (2.14), (2.15), $v_p(y-1) \le KL - 2 \le mL^2A$, and $L \le \ell\mathcal{B} + 1 \le (\ell + \frac{1}{10})\mathcal{B}$, whence

$$v_p(\Lambda) \le \Big(m\big(\ell + \tfrac{1}{10}\big)^2 + \tfrac{1}{10}\Big)A\mathcal{B}^2 \le 27\,711\,A\mathcal{B}^2.$$

Case (2.14). Since $c_2 \le c_1$ (as $L \ge 3$), $\beta_1h_1 + \beta_2h_2 + \beta_3h_3 \le 3c_1AP$, and $v_p(y-1) \le 1 + 3c_1A \le 1 + 3 \cdot 48.53 \cdot 7.3\, A\mathcal{B} \le 107\, A\mathcal{B}^2$.

Case (2.15). Put $x = \gamma_3^{\beta_3}\gamma_1^{-\beta_1}\gamma_2^{-\beta_2}$, so that $v_p(y-1) = v_p(x-1)$. We have $x \ne \pm 1$, since $\Lambda' \ne 0$ and x is a product of powers of the γ_j, which are multiplicatively independent. For every integer $u \ne 0$, therefore, $x^u \ne 1$ and $v_p(x-1) \le v_p(x^u - 1)$. If $u_3 \ne 0$, then $|u_3| \ge 1$, hence $U_3 \ge 1$, and

$$x^{u_3} = \gamma_1'^{-\beta_1}\gamma_2'^{-\beta_2}, \qquad \gamma_1' = \gamma_1^{u_3}\gamma_3^{u_1}, \quad \gamma_2' = \gamma_2^{u_3}\gamma_3^{u_2},$$

where γ_1', γ_2' are multiplicatively independent principal units with $h(\gamma_1') \le H_1 = U_3h_1 + U_1h_3$ and $h(\gamma_2') \le H_2 = U_3h_2 + U_2h_3$; we have $H_i \ge P$. If $u_3 = 0$, then $u_1u_2 \neq 0$ and $x^{u_1} = (\gamma_1^{u_2}\gamma_2^{-u_1})^{\beta_2}(\gamma_3^{u_1})^{\beta_3}$, with heights $H_1 = U_2h_1 + U_1h_2$ and $H_2 = U_1h_3$, both $\ge P$. In both cases, up to a unit and after possibly inverting γ_1' or γ_2', we are reduced to $\gamma^c - \gamma'^{c'}$ with $c, c' \le B$. By Lemma 10, $H_1H_2 \le 4\mu^2\eta^2 A \max_j a_j P^2$. In Theorem 6, $b^\star \le 2B/P$, hence $\log b^\star + \log\log p + 0.4 \le \log B + 1.1 \le \mathcal{B}P + 1.1$; moreover $10\log p \le \mathcal{B}P$ and $10 \le \mathcal{B}P + 3.1$. Since $\mathcal{B} \ge 10$, the maximum is at most $\mathcal{B}P + 0.31\mathcal{B} = \mathcal{B}P(1 + 0.31/P)$. Hence

$$v_p(y-1) \le \frac{24p}{(p-1)P^4}\mathcal{B}^2P^2\Big(1+\frac{0.31}{P}\Big)^2 \cdot 4\mu^2\eta^2 A \max_j a_j P^2 = 96\mu^2\eta^2\frac{p}{p-1}\Big(1+\frac{0.31}{\log p}\Big)^2 A \max_j a_j\, \mathcal{B}^2.$$

With $96\mu^2\eta^2 \le 11\,106$ and $v_p(\delta) \le A\mathcal{B}^2/10$, we obtain $v_p(\Lambda) \le C_2(p)A\max_j a_j\, \mathcal{B}^2$.

In all cases, $v_p(\Lambda) \le A\mathcal{B}^2 \max\{C_1, C_2(p)\max_j a_j\}$. □

5 Proof of Corollary 2

If $v_p(\alpha^b - 1) = 0$, there is nothing to prove; for $p = 2$, neither if $v_2(\alpha^b - 1) = 1$. Otherwise $\alpha^b \equiv 1 \pmod p$ (modulo 4 if $p = 2$), so the Teichmüller component of α^b, equal to $\zeta^{n\cdot b}$ (to $(-1)^{n\cdot b}$ if $p = 2$), is 1, and $b \in \Lambda_0$. The γ_j have trivial Teichmüller component; they are principal units, multiplicatively independent since E is invertible. Write $b = \sum_j b_j' e_j$ with $b' = bE^{-1} \in \mathbb{Z}^3$, whence $|b_j'| \le c_jB$ and $\alpha^b = \gamma^{b'}$. Apply Theorem 1 to $\gamma^{b'} - 1$ with $B' = B\max_j c_j$; we have $B' \ge 1$, since $b' \neq 0$ (because $\alpha^b \neq 1$), so $1 \le \max_j |b_j'| \le B'$. □

6 Examples

The table takes up the examples of [4, Section 7], with $\alpha = (2, 3, 5)$ if $p = 7$ and $\alpha = (2, 3, 7)$ otherwise, and bases of Λ_0 with the same heights and the same c_j. The column "kit" is the numerical optimisation of the parameters of [4]; the column "Yu" is the explicit form of [1], applied to the same α_i.

For instance, for $p = 7$ and $\alpha = (2, 3, 5)$, the basis $(-3, 0, 0)$, $(-1, 2, 0)$, $(-1, 1, -1)$ of Λ_0 gives $\gamma = (1/8, 9/2, 3/10)$, $c = (2/3, 1, 1)$, $a = (1.069; 1.129; 1.183)$, and for $B = 10^{30}$, $\mathcal{B}' = 35.5$, whence $v_7 \le 4.99 \cdot 10^7$.

The bounds in the table are rounded up; the ratio is their integer part.

p	B	heights h_j	Corollary 2	kit [4]	Yu	Yu / Cor. 2
5	10^{10}	1.79 1.95 2.08	$1.01 \cdot 10^7$	$4.62 \cdot 10^6$	$3.08 \cdot 10^9$	306
5	10^{30}	1.79 1.95 2.08	$8.94 \cdot 10^7$	$3.98 \cdot 10^7$	$9.24 \cdot 10^9$	103
7	10^{10}	2.08 2.20 2.30	$5.54 \cdot 10^6$	$2.68 \cdot 10^6$	$4.27 \cdot 10^9$	771
7	10^{30}	2.08 2.20 2.30	$4.99 \cdot 10^7$	$2.11 \cdot 10^7$	$1.28 \cdot 10^{10}$	257
11	10^{10}	2.48 2.64 3.30	$4.35 \cdot 10^6$	$2.10 \cdot 10^6$	$4.16 \cdot 10^9$	956
11	10^{30}	2.48 2.64 3.30	$3.61 \cdot 10^7$	$1.60 \cdot 10^7$	$1.25 \cdot 10^{10}$	345
13	10^{10}	2.64 2.77 3.30	$3.97 \cdot 10^6$	$1.75 \cdot 10^6$	$6.37 \cdot 10^8$	160
13	10^{30}	2.64 2.77 3.30	$2.90 \cdot 10^7$	$1.24 \cdot 10^7$	$1.91 \cdot 10^9$	66
101	10^{10}	4.68 4.85 8.88	$5.69 \cdot 10^6$	$2.33 \cdot 10^6$	$2.43 \cdot 10^8$	42
101	10^{30}	4.68 4.85 8.88	$1.25 \cdot 10^7$	$9.98 \cdot 10^6$	$7.29 \cdot 10^8$	58

Table 1: Upper bounds for $v_p(\alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3} - 1)$ for $|b_i| \le B$.

Verification

The ancillary file `VERIF_FERME3P.py` (plain Python, no argument, a few seconds; messages in French) checks Lemma 7 for $2 \le K \le 3000$ and two large values; checks the numerical values

quoted ($\tau(L_0)$, θ_0, β_0, μ, η, $96\mu^2\eta^2$, C_1, $C_2(p)$); and checks directly, on a grid of 27 440 points (p up to 10^9+7, a_j up to 77 in all orders, $\log B/\log p$ up to 10^5), conditions (Z1)–(Z3), criterion (1), the inequality $\mathcal{M} > \max\{R_1, S_1, T_1\}$, the bounds for the U_j and those of all branches. It recomputes the ten rows of the table (bases, indices, congruences $\gamma_j \equiv 1 \pmod p$, c_j, bounds, ratios).

Acknowledgements. Technical assistance: Claude (Anthropic).

References

[1] A. Alvarado, A. Koutsianas, B. Malmskog, C. Rasmussen, C. Vincent, M. West, A robust implementation for solving the S-unit equation and several applications, arXiv:1903.00977.

[2] Y. Bugeaud, M. Laurent, Minoration effective de la distance p-adique entre puissances de nombres algébriques, *J. Number Theory* **61** (1996), 311–342.

[3] M. Mignotte, P. Voutier, A kit for linear forms in three logarithms, *Math. Comp.* **93** (2024), 1903–1951; arXiv:2205.08899.

[4] D. Niedbala Giraudin, A kit for linear forms in three p-adic logarithms, Hexagon preprint hexagon:2610.00706, 2026.

[5] K. Yu, p-adic logarithmic forms and group varieties III, *Forum Math.* **19** (2007), 187–280.