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HEIGHT AND LIMIT LAWS FOR RIORDAN LATTICE PATHS OF FIXED AREA

ILMAR GAHRAMANOV[1,2]

[1]Department of Physics, Boğaziçi University, 34342 Bebek, Istanbul, Türkiye
[2]Department of Mathematics, Khazar University, 41 Mehseti St., AZ1096 Baku, Azerbaijan

E-mail: ilmar.gahramanov@bogazici.edu.tr

Abstract. We study the maximum height H_n of a uniformly chosen Riordan path of area n. Riordan paths are Motzkin paths without horizontal steps on the axis. Using the continued fraction for the area generating function, we obtain an exact product for the height tail of a primitive component. This gives the approximation $\mathbb{P}(H_n \leq k) = \exp(-\Lambda n q_0^{(k+1)^2}) + o(1)$, uniformly over nonnegative integer thresholds k, with explicit constants q_0 and Λ. The height is concentrated on two explicit adjacent integers of order $\sqrt{\log n}$, and the probabilities of the two heights have nontrivial limits along suitable subsequences. We also obtain exponential counting asymptotics, a signed partition recurrence and a joint Gaussian law for the numbers of returns, horizontal steps and up steps.

1. Introduction

Area-weighted lattice paths appear in models of directed polymers and vesicles [1, 2, 3], in exclusion statistics [4, 5, 6], and in lattice-path formulations of knot invariants [7, 8, 9, 10, 11, 12, 13, 14]. Motzkin paths also label conserved quantities in certain integrable Hamiltonian systems [15, 16, 17].

Most of the probabilistic results in this area concern paths of fixed length, or polygons of fixed perimeter, where the area is a random variable with an Airy-type limit law [18]. Here we fix the area and choose a path uniformly at random. Recall that a *Motzkin path* of length n runs from $(0,0)$ to $(n,0)$ with steps $U=(1,1)$, $D=(1,-1)$ and $H=(1,0)$ and never passes below the x-axis. A *Riordan path* is a Motzkin path with no horizontal steps on the axis, and a *Dyck path* has no horizontal steps at all. A horizontal step on the axis does not change the area, so the Riordan condition is exactly what is needed for the number r_n of paths of area n to be finite. One has $r_n = 1, 2, 4, 9, 20, 45, 101, \ldots$ for $n = 1, 2, \ldots$.

A Riordan path of large area is a long sequence of small *primitive* components, i.e. of pieces between consecutive returns to the axis. The largest of their heights is the height H_n of the full path. We show (Theorem 5.3) that

$$\sup_{k\in\mathbb{Z}_{\geq 0}} \left|\mathbb{P}(H_n \leq k) - \exp\left(-\Lambda n\, q_0^{(k+1)^2}\right)\right| \longrightarrow 0 \qquad (n\to\infty), \tag{1.1}$$

with an explicit constant $\Lambda = 3.9586573854\ldots$, and that

$$\mathbb{P}\big(H_n \in \{m_n - 1, m_n\}\big) \longrightarrow 1, \qquad m_n = \left\lfloor \sqrt{\frac{\log(\Lambda n)}{\log(1/q_0)}} + \frac{1}{2} \right\rfloor$$

(Corollary 5.4). The smallest path of height $k+1$ is the pyramid $U^{k+1}D^{k+1}$ of area $(k+1)^2$, so a component of height larger than k has probability of order $q_0^{(k+1)^2}$, and the height is determined by the balance between this probability and the number of components, which is of order n. The proof

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is based on the exact product formula (5.3). Corollary 5.4 also describes how the probability splits between the two heights near a transition.

The number of paths satisfies

$$r_n = Cq_0^{-n}\big(1 + O(\theta^n)\big), \qquad q_0 = 0.4441165\ldots, \quad C = 0.3431582\ldots,$$

with $0 < \theta < 1$. The numbers of returns, horizontal steps and up steps satisfy a joint Gaussian limit law with explicit means and covariance (Corollary 5.2).

The area generating function of primitive Riordan paths is a specialization of the staircase-polygon generating function. Its functional equation, linearisation and ratio of q-series go back to Prellberg and Brak [2], Owczarek and Prellberg [19, 20, 21] and Polychronakos [3]. We recall these results in Section 3 in a uniform notation and give the precise dictionary. Gaussian limit laws at fixed area are known for staircase polygons [18, 22] and for locally restricted compositions [23, 24]. The latter include the positive height sequences of primitive Riordan paths, for which the largest part has order $\sqrt{\log n}$. The concentration of maxima of discrete random variables on two adjacent integers is a known phenomenon as well [25, 26, 27]. What we add is the exact tail formula with its constant, the approximation (1.1), which is uniform in k under the fixed-area conditioning, and the explicit identification of the two heights. As by-products we obtain the constants of the Gaussian laws and a linear recurrence for r_n whose coefficients are signed counts of partitions into distinct parts (Theorem 4.3).

The rest of the paper is organized as follows. In Section 2 we fix the notation. In Section 3 we recall the functional equation, the continued fraction and the closed form of the generating function, and compare them with the literature. In Section 4 we derive the asymptotics of r_n and the partition recurrence. In Section 5 we study the limit laws of the additive statistics and of the height. The proof of the height theorem is given in Appendix C, and explicit formulas for the area polynomials are collected in Appendices A and B.[1]

2. Riordan paths and area polynomials

In this section we fix the notation and state the problem. We assign the weight x to each up step and the weight y to each horizontal step. A step from height h to height h' contributes $(h + h')/2$ to the area, i.e. an up or down step between heights h and $h + 1$ contributes $h + \frac{1}{2}$, while a horizontal step at height h contributes h. The total area is an integer, since the numbers of up and down steps are equal. Every allowed step contributes at least $\frac{1}{2}$, so a Riordan path of area n has length at most $2n$, and in particular there are only finitely many such paths. Summing the contributions of the two endpoints of each step one also finds

$$\operatorname{area}(P) = \sum_{h \geq 1} h\, t_h, \tag{2.1}$$

where t_h is the number of interior vertices of P at height h. Thus the area is the sum of the heights of the interior vertices.[2] For $n \geq 1$ we define

$$S_n(x, y) = \sum_{\operatorname{area}(P) = n} x^{\#U(P)}\, y^{\#H(P)}, \qquad S_0 = 1, \tag{2.2}$$

where the sum runs over Riordan paths of area n, and $\#U(P)$ and $\#H(P)$ stand for the numbers of up and horizontal steps of P. A path with i up steps and j horizontal steps has weight $x^i y^j$ and

[1] Finite checks compare the polynomial recurrence and profile formula with direct enumeration through area 16, and the partition recurrence with step-by-step counts through area 1500. The Gaussian constants were evaluated independently and compared with exact moments at area 300; the height probabilities in Table 1 were also recomputed from exact counts. These checks complement the proofs below.

[2] This is the reason for excluding horizontal steps on the axis: a Motzkin path may contain arbitrarily many of them without changing the area.

length $2i+j$. One can easily compute the first polynomials,

$$S_1 = x, \qquad S_2 = x^2 + xy, \qquad S_3 = x^3 + 2x^2y + xy^2, \qquad S_4 = x^4 + 3x^3y + 3x^2y^2 + xy^3 + x^2,$$
$$S_5 = x^5 + 4x^4y + 6x^3y^2 + 4x^2y^3 + xy^4 + 2x^3 + 2x^2y,$$

and $S_6, \dots, S_{10}$ are listed in Appendix A. For instance, the path $UHUDD$ in Figure 1 has area 5 and weight x^2y; it is one of the two paths giving the term $2x^2y$ of S_5. The numbers $r_n = S_n(1,1)$ of Riordan paths of area n are, starting at $n = 1$,

$$1,\ 2,\ 4,\ 9,\ 20,\ 45,\ 101,\ 227,\ 511,\ 1150,\ 2589,\ 5829,\ 13124,\ 29550,\ \dots \tag{2.3}$$

with $r_0 = 1$. We denote the generating function by $F(x,y;q) = \sum_{n\geq 0} S_n(x,y)q^n$, and we use the standard notation

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad (a;q)_\infty = \prod_{j\geq 0}(1-aq^j),$$

with $(a;q)_0 = 1$ and $|q| < 1$. For Dyck paths the numbers $S_n(1,0)$, $n \geq 1$, are $1,1,1,2,3,4,6,9,\dots$; this is the sequence A143951 of [28].[3]

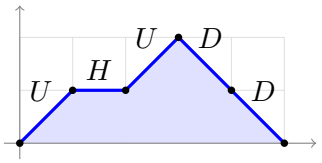


FIGURE 1. The path $UHUDD$, of area $\frac{1}{2} + 1 + \frac{3}{2} + \frac{3}{2} + \frac{1}{2} = 5$.

3. The generating function

In this section we recall the functional equation, the continued fraction and the closed form of F. These results are not new: as explained in Sections 3.2 and 3.6, they are elementary transforms of the results of Prellberg and Brak [2], Owczarek and Prellberg [19, 20, 21] and Polychronakos [3]. We include short derivations in order to keep the paper self-contained and to fix the notation needed in Sections 4 and 5.

3.1. **The functional equation.** If one lifts a path by one level, every step gains area 1, so the area increases by the length of the path. Since a path of weight x^iy^j has length $2i+j$, the lift is represented by the substitution $x \mapsto xq^2$, $y \mapsto yq$. In what follows we write $F^{\uparrow} = F(xq^2, yq; q)$.

Theorem 3.1. *The generating function satisfies*

$$F - 1 = q\,F^{\uparrow}\big[(x+y)F - y\big]. \tag{3.1}$$

For $x \neq 0$ this can be rearranged as

$$F(xq^2, yq; q) = \frac{F(x,y;q) - 1}{q\,[(x+y)F(x,y;q) - y]},$$

with the factor q cancelled formally. At $x = 0$ one has $F = 1$, and the undivided identity remains valid.

Proof. A nonempty Riordan path starts with U. We read it up to the first subsequent step which either returns to the axis or is horizontal at height 1. The segment $\hat{R}$ in between is a Riordan path lifted by one level and contributes $F^{\uparrow}$. If the distinguished step is D, one has $P = U\hat{R}DP'$ with an arbitrary Riordan path P', which contributes $xqF^{\uparrow}F$. If the distinguished step is H, one has $P = U\hat{R}HW$, and removing $\hat{R}H$ leaves the nonempty Riordan path UW; this case contributes $yqF^{\uparrow}(F-1)$. Adding the two contributions one obtains (3.1). □

[3]Enumeration at fixed area differs from the total area of all paths of a fixed length, which was studied in [29, 30].

For nonzero x, (3.1) is an equation of q-Riccati type: the lifted function $F^{\uparrow}$ is a linear fractional transform of F. For $y = 0$ it reduces to the equation $F = 1 + xqF(xq^2)F$ for Dyck paths counted by area, which goes back to Flajolet [31], see also [1, 22]. Nonlinear q-functional equations of this type for directed cluster models were studied systematically by Prellberg and Brak [2], and the Motzkin case with boundary weights was treated by Owczarek and Prellberg [19, 20, 21] and by Polychronakos [3].

Extracting the coefficient of q^n in (3.1) one obtains the following recurrence for the area polynomials,

$$S_n = [t^{n-1}]\Big(\sum_{m<n} t^m\, S_m(xt^2, yt)\Big)\Big((x+y)\sum_{p<n} t^p\, S_p(x,y) - y\Big), \qquad n \geq 1. \tag{3.2}$$

The recurrence (3.2) does not close on the numbers r_n alone, since the lift requires the information about the length. A scalar recurrence for r_n will be given in Theorem 4.3.

3.2. **Relation to staircase polygons.** Here we relate (3.1) to the known results on staircase polygons. Let

$$G(x,y;q) = 1 - \frac{1}{F(x,y;q)}$$

be the generating function of primitive Riordan paths, i.e. of the paths which touch the axis only at their endpoints. A primitive path has the form UWD, where W is a Motzkin path based at height one, and such a path W is a sequence of horizontal steps at height one and of lifted primitive paths. Therefore

$$G(x,y;q) = \frac{xq}{1 - yq - G(xq^2, yq; q)}, \tag{3.3}$$

which is equivalent to (3.1). On the other hand, let $S(a,b;q)$ be the generating function of staircase polygons, where a and b mark the horizontal and the vertical half-perimeter and q marks the area. It satisfies the functional equation of Prellberg and Brak [2, Eq. (3.14)],

$$S(a,b;q) = \big[S(qa,b;q) + qa\big]\big[b + S(a,b;q)\big]. \tag{3.4}$$

By the symmetry of S in a and b this can be brought to the form

$$S(a,b;q) = \frac{abq}{1 - q(a+b) - S(aq, bq; q)}.$$

Comparing with (3.3) one obtains the following dictionary,

$$G(ab, a+b; q) = S(a,b;q), \qquad F(ab, a+b; q) = \frac{1}{1 - S(a,b;q)}. \tag{3.5}$$

Under this dictionary, a staircase polygon corresponds to a primitive Riordan path with two colors for its horizontal steps. The numbers of cells on successive diagonals are the positive interior vertex heights. When the size of a diagonal changes, the corresponding rise and fall together have weight ab. When its size stays the same, the two possible horizontal steps carry weights a and b. The sum of the horizontal and vertical half-perimeters is the path length, and the area is given by (2.1).

More generally, for sequences of staircase polygons and of single horizontal steps on the axis, with weight c per component and weight $t(a+b)$ for a step on the axis, the ordered sequence construction gives $N_{c,t} = 1/(1 - c\,[S + t(a+b)])$. For $t = 1$ this is the generating function of area-weighted Motzkin paths of [21, Eq. (7)]; Riordan paths correspond to $N_{1,0}$ and Motzkin paths to $N_{1,1} = F/(1 - yF)$. Thus (3.1) should be regarded as recalled background.

3.3. **The continued fraction.** For the continued fraction expansion of F we only need the simplest ingredients of the approach of Flajolet [31]. Let M_h be the generating function of Motzkin paths based at height $h \geq 1$, with the area measured from the x-axis. Such a path is a sequence of horizontal steps and excursions above height h, and therefore $M_h = 1/(1 - yq^h - xq^{2h+1}M_{h+1})$ and $F = 1/(1 - xqM_1)$, i.e.

$$F = \cfrac{1}{1 - \cfrac{xq}{1 - yq - \cfrac{xq^3}{1 - yq^2 - \cfrac{xq^5}{1 - yq^3 - \cdots}}}}. \tag{3.6}$$

3.4. **Paths of bounded height.** Area-weighted Motzkin paths in a slit were solved in terms of q-orthogonal polynomials by Owczarek and Prellberg [20, Section 5] and, with boundary markers, by Polychronakos [3]; the formulas below are the Riordan case in our notation. Truncating the continued fraction at level $k \geq 1$ one obtains the generating function $F^{[k]}$ of paths of height at most k; for $k = 0$ one has $F^{[0]} = 1$. We write $F^{[k]} = R_1^{(k)}/R_0^{(k)}$, where

$$R_{k+1}^{(k)} = 1, \qquad R_k^{(k)} = 1 - yq^k, \qquad R_h^{(k)} = (1 - yq^h)R_{h+1}^{(k)} - xq^{2h+1}R_{h+2}^{(k)} \quad (0 \leq h < k), \tag{3.7}$$

with the factor $1 - yq^0$ replaced by 1 at $h = 0$. Expanding this recurrence one recovers the standard interpretation of the convergents in terms of matchings, which also goes back to [31]. Namely, $R_h^{(k)}$ is the generating polynomial of monomer-dimer configurations on the path graph with vertices $h, \dots, k$, where a monomer on the vertex v has weight $-yq^v$ and a dimer on the edge $\{v, v+1\}$ has weight $-xq^{2v+1}$.[4] Since every occupied vertex v contributes q^v, one can write, for $1 \leq h \leq k$,

$$R_h^{(k)} = \sum_{S \subseteq \{h,\dots,k\}} q^{\sum S} \prod_{I} m_{|I|}(x, y), \qquad m_r(x, y) = \sum_{d=0}^{\lfloor r/2 \rfloor} \binom{r-d}{d} (-x)^d (-y)^{r-2d}, \tag{3.8}$$

where the product runs over the maximal runs I of consecutive integers in S, and m_r stands for the matching polynomial of the path with r vertices. At the boundary $h = 0$ we use $R_0^{(k)} = R_1^{(k)} - xqR_2^{(k)}$.

At $x = y = 1$ the exponential growth rate of the number of paths of height at most k is $1/q_0^{(k)}$, where $q_0^{(k)}$ denotes the smallest positive zero of $R_0^{(k)}$. One finds

$$1/q_0^{(k)} = 2, \quad 2.2469796\dots, \quad 2.2516428\dots, \quad 2.2516611\dots \qquad (k = 1, 2, 3, 4).$$

Paths of height one contribute $x(x + y)^{n-1}$ at area $n \geq 1$, which explains the first value. The rapid convergence to the limit $1/q_0 = 2.2516612\dots$ will be explained in Theorem 5.3.

3.5. **The closed form.** Let us now take $k \to \infty$. For each fixed $h \geq 1$ one finds from (3.7) that $R_h^{(k)} \to A(q^h)$, where

$$A(z) = (1 - yz)A(zq) - xqz^2A(zq^2), \qquad A(0) = 1. \tag{3.9}$$

We usually suppress the dependence on x, y, q and write $A(z)$ for $A(z; x, y; q)$. At the boundary one has $R_0^{(k)} \to A(q) - xqA(q^2)$. Comparing coefficients, the solution of (3.9) is

$$A(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} p_n(x, y; q)}{(q; q)_n} z^n,$$

where the polynomials p_n are determined by

$$p_0 = 1, \qquad p_1 = y, \qquad p_n = y\,p_{n-1} - x(1 - q^{n-1})\,p_{n-2}. \tag{3.10}$$

[4] No monomer is allowed on the vertex 0, which reflects the absence of horizontal steps on the axis.

Then $M_h = A(q^{h+1})/A(q^h)$ and $F = 1/(1 - xqM_1)$, i.e.

$$F(x, y; q) = \frac{A(q)}{A(q) - xq\, A(q^2)}. \tag{3.11}$$

Equivalently, $1/F = y + A(1)/A(q)$. The identity (3.11) holds as a formal power series in q, and by the following lemma the quotient also gives the meromorphic continuation of F to the unit disc.

Lemma 3.2. *For fixed complex x, y, the series $A(z; x, y; q)$ is entire in z and holomorphic in q for $|q| < 1$. For each $h \geq 1$, the continuants $R_h^{(k)}$ converge to $A(q^h)$ locally uniformly in this disc. In particular, (3.11) defines a meromorphic function of q there.*

Proof. Fix $0 < r < 1$. The recurrence (3.10) gives $|p_n| \leq K^n$ for $|q| \leq r$, with K depending only on x, y. Also $|(q; q)_n| \geq \prod_{j\geq 1}(1 - r^j) > 0$. The factor $q^{\binom{n}{2}}$ therefore ensures local uniform convergence of the series for A in (z, q).

For the continuants, ignore the exclusion constraints in the matching expansion. The sum of the absolute weights of all configurations on the infinite path is bounded by

$$\prod_{v\geq h}\left(1 + |y|r^v + |x|r^{2v+1}\right) < \infty.$$

The weights of configurations involving vertices beyond k tend to zero uniformly on $|q| \leq r$. Hence the finite continuants converge locally uniformly. Their formal limit, already identified above, is $A(q^h)$. The denominator in (3.11) is holomorphic and equals 1 at $q = 0$, which proves the last assertion. □

Remark 3.3. Write $x = \alpha\beta$ and $y = \alpha + \beta$. Then the solution of (3.10) takes the form

$$p_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \alpha^{n-k}\beta^k,$$

which is a homogeneous form of the continuous q-Hermite polynomial of Rogers: one has $p_n = x^{n/2}H_n(y/(2\sqrt{x}) \mid q)$ for $x \neq 0$, with its polynomial extension at $x = 0$. Moreover, one can write

$$A(z) = (\alpha z; q)_\infty\, {}_1\phi_1\left(\begin{matrix} 0 \\ \alpha z \end{matrix}; q,\, \beta z\right),$$

which is a q-Bessel type function, see e.g. [32]. At apparent singularities of the product representation, the identity is understood by analytic continuation.

3.6. **Comparison with the literature.** We now compare (3.9), (3.10) and (3.11) with the known results. The passage from the nonlinear equation (3.1) to the linear q-difference equation (3.9) is the linearisation method of Prellberg and Brak [2, Section 4]. They write the solution of a q-Riccati equation as a ratio $H(qx)/H(x)$, see Eqs. (4.1) to (4.3) there, and for staircase polygons they obtain

$$S(a, b; q) = b\left[\frac{T(qa)}{T(a)} - 1\right], \qquad T(a) = \sum_{n\geq 0} \frac{(-qa)^n q^{\binom{n}{2}}}{(q; q)_n (qb; q)_n} = {}_1\phi_1\left(\begin{matrix} 0 \\ qb \end{matrix}; q,\, qa\right),$$

see [2, Eqs. (4.6) to (4.9)] and [21, Eqs. (9), (10)]. By the dictionary (3.5) this gives F as well. In our notation, with $x = ab$ and $y = a + b$, Remark 3.3 gives $A(q) = (bq; q)_\infty T(a)$, so that the denominators of the two representations agree up to an infinite product.

The equation (3.9) itself is Eq. (4.2) of Owczarek and Prellberg [19]. In their notation it reads $H(Z) + (Z - 1)H(qZ) + qZ^2v^2H(q^2Z) = 0$, and for $y \neq 0$ one has $A(z) = H(yz)$ with $v^2 = x/y^2$. Their coefficients d_n, defined in [19, Eqs. (4.5), (4.6)], are related to ours by $p_n = y^n d_n$, and (3.10) is their three-term recurrence. Finally, the double sum for A which follows from Remark 3.3,

$$A(z) = \sum_{n\geq 0} (-z)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \frac{\alpha^{n-k}\beta^k}{(q; q)_{n-k}(q; q)_k},$$

is the secular determinant F_∞ of Polychronakos for paths without a ceiling [3, Eq. (4.33)]. There the area of a step is defined in the same way as in Section 2, the up and down steps have weight z and the horizontal steps have weight $\mathsf{z}_h = \mathsf{z}(\omega + \omega^{-1})$. With $x = \mathsf{z}^2$ and $y = \mathsf{z}_h$, i.e. $\alpha = \mathsf{z}\omega$ and $\beta = \mathsf{z}\omega^{-1}$, one has

$$A(\zeta; x, y; q) = F_\infty(\zeta\mathsf{z}, \zeta\mathsf{z}_h; q),$$

so that $A(1)$ is the determinant itself and $A(q^h)$ is the determinant with both weights multiplied by q^h. Then (3.9) at $z = 1$ is the expansion [3, Eq. (3.16)] at infinite height, and the Motzkin generating function $F/(1 - yF) = A(q)/A(1)$ is [3, Eq. (3.12)].

Therefore the closed form (3.11) is a restatement of known results in a notation adapted to the Riordan boundary condition. What we use from it below is the following: the meromorphic continuation of Lemma 3.2, the ratios $M_h = A(q^{h+1})/A(q^h)$ at all levels, which enter the height estimate of Section 5, and the partition expansion of $A(q^h)$.

Remark 3.4. Since the exponent of q in the above double sum is quadratic in $(n - k, k)$, the series A can be read as the generating series of the symmetric quiver with adjacency matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, with the variables $\alpha z q^{-1/2}$ and $\beta z q^{-1/2}$, in the conventions of [8, 9]. This is an identity of formal series only, and we do not know a knot invariant related to Riordan paths.

For $y = 0$, i.e. $\beta = -\alpha$, the closed form becomes the Rogers-Ramanujan type identity

$$F(x, 0; q) = \frac{\sum_{k\geq 0}(-x)^k q^{k(2k+1)}/(q^2; q^2)_k}{\sum_{k\geq 0}(-x)^k q^{k(2k-1)}/(q^2; q^2)_k}, \tag{3.12}$$

in agreement with [1, 22]. Finally, taking $k \to \infty$ in (3.8) one obtains a second interpretation of the auxiliary series: for $h \geq 1$, $A(q^h)$ enumerates partitions into distinct parts at least h, with each maximal run of r consecutive parts weighted by $m_r(x, y)$. We will need this interpretation in Section 4.

4. Asymptotics and a linear recurrence

In this section we determine the growth of the numbers r_n and obtain a recurrence for them. The point of departure is the decomposition of a path into its *primitive* components, i.e. the pieces between consecutive returns to the axis. Every nonempty Riordan path decomposes uniquely into primitive paths, which touch the axis only at their endpoints. At $x = y = 1$ their generating function is

$$g(q) = \sum_{P \text{ primitive}} q^{\mathrm{area}(P)} = q + q^2 + q^3 + 2q^4 + 3q^5 + 5q^6 + \cdots .$$

Therefore $F(1, 1; q) = 1/(1 - g)$ and, by (3.11), $g = qA(q^2)/A(q)$. Each primitive component contributes one return to the axis, including the final one, so marking the returns by u one obtains $F(u; q) = 1/(1 - ug) = F/(u + (1 - u)F)$.

Theorem 4.1. *Let ρ be the radius of convergence of g. The equation $g(q) = 1$ has a unique root $q_0 \in (0, \rho)$, which is also the smallest positive root of $A(q) = qA(q^2)$ at $x = y = 1$. For some $\theta < 1$,*

$$r_n = C\, q_0^{-n}\big(1 + O(\theta^n)\big), \qquad C = \frac{1}{q_0\, g'(q_0)},$$

with

$$q_0 = 0.44411654605086\ldots, \quad q_0^{-1} = 2.25166121121160\ldots, \quad C = 0.34315827064\ldots$$

Proof. The series g has nonnegative coefficients and a positive radius of convergence, since a path of area n has length at most $2n$. By Lemma 3.2, g is meromorphic in $|q| < 1$. The primitive paths UH^jD give $g(t) \geq t/(1 - t)$ wherever both sides converge, so $\rho \leq 1$. If $\rho < 1$, Pringsheim's theorem together with meromorphicity implies that ρ is a pole and $g(t) \to \infty$ as $t \uparrow \rho$. If $\rho = 1$, the same divergence follows from the lower bound. Since g is strictly increasing on $[0, \rho)$, it reaches 1 at a unique point $q_0 < \rho$.

For $|q| \le q_0$ one has $|g(q)| \le g(|q|) \le 1$. On $|q| = q_0$, equality in the triangle inequality requires the terms of degrees 1 and 2 to have the same argument, and their positive coefficients force $q = q_0$. Thus $1 - g$ has a unique zero in the closed disc, and this zero is simple because $g'(q_0) > 0$. Analyticity in a larger disc and isolation of zeros give an exponentially smaller remainder after subtraction of the simple pole, and the residue gives $C = 1/(q_0 g'(q_0))$.

It remains to rule out a cancellation in the quotient for g. We display the base parameter of A explicitly. If, for some $0 < t < 1$, both $A(t; 1, 1; t)$ and $A(t^2; 1, 1; t)$ vanished, the recurrence for $B_h = A(t^h; 1, 1; t)$,

$$B_h = (1 - t^h)B_{h+1} - t^{2h+1}B_{h+2},$$

would force $B_h = 0$ for every $h \ge 1$, which contradicts $B_h \to A(0; 1, 1; t) = 1$. Consequently a zero of $A(t; 1, 1; t)$ cannot cancel in $g(t)$, so $A(t; 1, 1; t) > 0$ for $0 \le t < \rho$. On this interval the denominator is $A(q)(1 - g(q))$, and its first positive zero is q_0. □

The leading asymptotics is already accurate at modest area; namely, $r_{22}/(Cq_0^{-22}) = 1 + 4.8722 \cdot 10^{-7}$. A further positive pole is found numerically at $q_1 = 0.77091704183\ldots$, and the argument principle applied numerically on the circles $|q| = 0.6, 0.7, 0.8$ gives $1, 1, 2$ zeros of $A(q) - qA(q^2)$. These computations suggest that any $\theta \in (q_0/q_1, 1)$ is admissible in Theorem 4.1; we do not use this sharper, uncertified estimate. We do not know closed forms for q_0 and C; the Dyck constants $q_D = 0.6629014851\ldots$ and $C_D = 0.3377611503\ldots$ recorded in OEIS A143951 [28] are likewise only known numerically. For positive weights x, y the same argument applies whenever the primitive generating function reaches 1 strictly inside its radius of convergence, which holds near $(x, y) = (1, 1)$.

4.1. **A recurrence for r_n.** Let us now come back to the partition interpretation of Section 3.6. We write $A(q)|_{x=y=1} = \sum_n \nu_n q^n$ and $A(q^2)|_{x=y=1} = \sum_n \nu'_n q^n$. Then ν_n counts partitions of n into distinct parts, with each maximal run of r consecutive parts weighted by $m_r(1, 1) = \sum_d (-1)^{r-d}\binom{r-d}{d}$, and ν'_n counts the same partitions with the additional restriction that every part is at least 2. The run weights turn out to be periodic, namely $m_r(1, 1) = 1, -1, 0$ for $r \equiv 0, 1, 2 \pmod 3$, respectively. The following simple cancellation explains this periodicity.

Lemma 4.2. *Write a perfect monomer-dimer cover of the path with r vertices as a word in the letters M and D, with sign $(-1)^{\#\text{letters}}$. The map exchanging a leading D with a leading MM, and fixing the words with neither prefix, is an involution which reverses the sign on its nonfixed pairs. Its fixed points are the words MDw, together with M and the empty word. Hence $m_r(1, 1) = m_{r-3}(1, 1)$ for $r \ge 3$, with the initial values $1, -1, 0$.*

Theorem 4.3. *Let $r_0 = 1$, set $\nu'_{-1} = 0$, and let $d_n = \nu_n - \nu'_{n-1}$ for $n \ge 0$. Then*

$$r_n = \nu_n - \sum_{k=1}^{n} d_k\, r_{n-k}, \qquad n \ge 1,$$

where $\nu_n = \sum_\lambda (-1)^{\#\{\text{runs of length} \equiv 1 \pmod 3\}}$, the sum being over partitions λ of n into distinct parts with no run of length $\equiv 2 \pmod 3$, and ν'_n is the same sum over partitions with parts ≥ 2. The first values are

$$(\nu_n)_{n\ge 0} = 1, -1, -1, -1, 0, 0, 2, 1, 2, 2, 1, 0, 1, -2, -2, -3, -3, -5, -3, -5, -3, \ldots$$
$$(\nu'_n)_{n\ge 0} = 1, 0, -1, -1, -1, -1, 0, 0, 1, 2, 2, 2, 3, 2, 1, 1, 0, -2, -2, -4, -5, \ldots$$
$$(d_n)_{n\ge 0} = 1, -2, -1, 0, 1, 1, 3, 1, 2, 1, -1, -2, -1, -5, -4, -4, -4, -5, -1, -3, 1, \ldots$$

Proof. This is the identity $F(1, 1; q)\,(A(q) - qA(q^2)) = A(q)$ written out coefficientwise. □

Lemma 4.2 and Theorem 4.3 are related to the work of Polychronakos [3] as follows. With $x = \alpha\beta$ and $y = \alpha + \beta$ the run weights are

$$m_r(x, y) = (-1)^r\, \frac{\alpha^{r+1} - \beta^{r+1}}{\alpha - \beta},$$

and for $\alpha = z\omega$, $\beta = z\omega^{-1}$ this is the factor $(1-\omega^{2(r+1)})/(1-\omega^2)$ of the occupation-number expansion [3, Eqs. (4.34), (4.35)], up to a monomial. For ω a root of unity these weights are periodic in r and vanish periodically, see [3, Eq. (4.37)]; the point $x = y = 1$ is the case $\omega = e^{i\pi/3}$ of period three. Thus the signed partition expansion behind Theorem 4.3 is a reformulation of that determinant expansion. Lemma 4.2 gives a bijective proof of the periodicity, and the theorem records the resulting recurrence for the fixed-area counts.

The coefficients of the recurrence are small compared with r_n, since their absolute values are bounded by partition counts; for example, $|\nu_n| \le 1167$ for $n \le 120$, whereas $r_{120} \approx 10^{42}$.

5. Limit laws

In this section we describe a typical Riordan path of large area. Such a path looks like a long sequence of small primitive components, and the statistics we are interested in are sums over the components. The point of departure is again the decomposition into primitive paths. Since $g(q_0) = 1$, one can assign the probability $q_0^{\mathrm{area}(P)}$ to each primitive path P; this defines a probability distribution on primitive paths, which we call the Boltzmann distribution. Let N be the area of a Boltzmann primitive path and let R be an additive statistic, namely its number of returns, of horizontal steps or of up steps. One has $\mathbb{E}N = q_0 g'(q_0) = 1/C$, i.e. the constant C of Theorem 4.1 is the inverse of the mean area of a primitive component.

Let us first recall what is known about fixed-area ensembles. The perimeter of a staircase polygon chosen uniformly among the polygons of area n is asymptotically Gaussian, see [22, Chapter IX] and Section 3.6 of the survey of Richard [18]; the proof uses the ratio of two q-Bessel functions and the perturbation of a simple pole, which is also our tool in Theorem 5.1. In terms of the dictionary (3.5) that result concerns a single polygon with $a = b = 1$, i.e. the coefficients of $G(1, 2; q)$, whereas we consider sequences of primitive paths at $x = y = 1$, i.e. the coefficients of $F(1, 1; q)$. So the ensembles and the constants are different, but the method is the same. Moreover, a primitive Riordan path of area n is the same as a composition of n whose first and last parts are equal to 1 and whose adjacent parts differ by at most one; the parts are the heights of the interior vertices. Such objects belong to the locally restricted compositions of Bender and Canfield [23, 24], who proved, in a general setting, a central limit theorem for the number of parts and the order $\sqrt{\log n}$ for the largest part when the differences of adjacent parts are bounded. Thus the Gaussian laws and the scale of the height obtained below are expected from these general results. The much larger literature on paths of fixed length or polygons of fixed perimeter, where the area has an Airy-type limit [18], concerns a different ensemble and serves here only as a context.

5.1. Additive statistics.

We first consider the statistics which are additive over the components. The following theorem says that they behave like sums of independent random variables, and that the constants are the ones of the classical renewal-reward theorem.

Theorem 5.1. *Let a Riordan path of area n be chosen uniformly at random, and let X_n, Y_n, Z_n be its numbers of returns to the axis, of horizontal steps and of up steps. Each of these is asymptotically Gaussian. Namely, for any one of them, denoted by W_n,*

$$\mathbb{E}W_n = \mu n + O(1), \qquad \mathrm{Var}(W_n) = \sigma^2 n + O(1), \qquad \frac{W_n - \mu n}{\sigma\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1),$$

where

$$\mu = \frac{\mathbb{E}R}{\mathbb{E}N}, \qquad \sigma^2 = \frac{\mathrm{Var}(R - \mu N)}{\mathbb{E}N} > 0. \tag{5.1}$$

Numerically,

$$\begin{aligned} X_n:&\quad \mu_X = C \approx 0.343158270645, && \sigma_X^2 \approx 0.344199446374,\\ Y_n:&\quad \mu_Y \approx 0.365093862590, && \sigma_Y^2 \approx 0.292149839727,\\ Z_n:&\quad \mu_Z \approx 0.431620789070, && \sigma_Z^2 \approx 0.225725860430. \end{aligned}$$

For the returns one has $\sigma_X^2 = \mathrm{Var}(N)/(\mathbb{E}N)^3$.

Proof. Let $g(v;q)$ be the generating function of primitive paths with the statistic marked by v. Then the bivariate generating function is $1/(1-g(v;q))$, i.e. we are dealing with the so-called supercritical sequence schema of Flajolet and Sedgewick [22, Chapter IX]. Choose $r > q_0$ and $\varepsilon > 0$ with $(1+\varepsilon)r < \rho$. Each statistic satisfies $0 \le R \le N$, so the marked series is dominated by $g((1+\varepsilon)r)$ for $|q| \le r$, $|v| \le 1+\varepsilon$, and is therefore jointly analytic for $|q| < r$, $|v| < 1+\varepsilon$. The simple dominant pole persists for v near 1, with uniform asymptotics of the coefficients, for the following reason. By the proof of Theorem 4.1 and the isolation of zeros one can choose r such that q_0 is the only zero of $1-g(1;q)$ in the closed disc $|q| \le r$. Then $|1-g(1;q)| \ge \delta$ on the circle $|q| = r$ for some $\delta > 0$, and by continuity there is a neighbourhood V of $v = 1$ such that $|g(v;q) - g(1;q)| < \delta/2$ for $v \in V$ and $|q| = r$. By Rouché's theorem, for every $v \in V$ the function $1-g(v;q)$ has exactly one zero $q(v)$ in the disc. Since $\partial_q g(1;q_0) = g'(q_0) > 0$, the implicit function theorem shows that $q(v)$ is analytic and the zero is simple, after shrinking V if necessary. Subtracting the pole at $q(v)$ and applying Cauchy's formula on the common contour $|q| = r$, where $|1-g(v;q)| \ge \delta/2$, one finds

$$[q^n]\frac{1}{1-g(v;q)} = c(v)\,q(v)^{-n} + O(r^{-n}), \qquad c(v) = \frac{1}{q(v)\,\partial_q g(v;q(v))},$$

uniformly for $v \in V$. Shrinking V once more one has $|q(v)| \le r' < r$, so that the error term is exponentially small relative to the main term. The same argument applies to several marks. Then the quasi-powers theorem of Hwang [22, Chapter IX] gives a Gaussian law with $\mu = -q'(1)/q(1)$ and $\sigma^2 = -q''(1)/q(1) - q'(1)/q(1) + (q'(1)/q(1))^2$, where $q(v)$ is the root of $g(v;q(v)) = 1$ near $q(1) = q_0$, provided that $\sigma^2 \neq 0$. Differentiating $g(v;q(v)) = 1$ twice at $v = 1$ and writing the derivatives of g as Boltzmann moments one obtains (5.1). The variance is positive, since $R - \mu N$ is not almost surely constant. Indeed, the primitive paths UD, UHD and $UUDD$ have $(N,R) = (1,1),(2,1),(4,1)$ for returns, $(1,0),(2,1),(4,0)$ for horizontal steps and $(1,1),(2,1),(4,2)$ for up steps, and in each case $R - \mu N$ takes at least two values. □

The mean number of returns per unit area coincides with the prefactor C of Theorem 4.1, and the mean length of a path of area n is $(2\mu_Z + \mu_Y)n + O(1) \approx 1.228\,n$.[5]

Corollary 5.2. *Let $\mathbf{W}_n = (X_n, Y_n, Z_n)^T$ and $\boldsymbol{\mu} = (\mu_X, \mu_Y, \mu_Z)^T$. For a Boltzmann primitive path, write $\mathbf{R} = (1, \#H, \#U)^T$. Then*

$$\frac{\mathbf{W}_n - n\boldsymbol{\mu}}{\sqrt{n}} \xrightarrow{\;d\;} \mathcal{N}_3(0,\Sigma), \qquad \mathrm{Cov}(\mathbf{W}_n) = n\Sigma + O(1),$$

where the matrix

$$\Sigma = \frac{\mathbb{E}\big[(\mathbf{R} - \boldsymbol{\mu}N)(\mathbf{R} - \boldsymbol{\mu}N)^T\big]}{\mathbb{E}N} \tag{5.2}$$

is positive definite. In the order (X,Y,Z), its numerical value is

$$\Sigma \approx \begin{pmatrix} 0.34419945 & -0.03898214 & 0.26024597 \\ -0.03898214 & 0.29214984 & -0.11542887 \\ 0.26024597 & -0.11542887 & 0.22572586 \end{pmatrix}.$$

Proof. Mark the three statistics by $\mathbf{v} = (v_1, v_2, v_3)$ in the primitive series $g(\mathbf{v};q)$. Each component of $\mathbf{R}$ is at most N, so joint analyticity follows by choosing $r > q_0$ and $\varepsilon > 0$ with $(1+\varepsilon)^3 r < \rho$. As in the proof of Theorem 5.1, the simple pole persists at an analytic root $q(\mathbf{v})$ near q_0, with coefficient asymptotics which are uniform for $\mathbf{v}$ in a neighbourhood of $\mathbf{1}$. Consequently, for $\mathbf{s}$ near zero,

$$\mathbb{E}e^{\mathbf{s}^T\mathbf{W}_n} = \frac{c(e^{\mathbf{s}})}{C}\left(\frac{q_0}{q(e^{\mathbf{s}})}\right)^n \big(1 + O(\theta^n)\big),$$

[5]Exact moments at $n = 300$, divided by n, give $(0.346502, 0.342381)$, $(0.364715, 0.291206)$ and $(0.434149, 0.224821)$ for the (mean, variance) of X_n, Y_n, Z_n, in agreement with the $O(1/n)$ corrections.

where c is analytic, $c(\mathbf{1}) = C$, and $\theta < 1$. Differentiating $g(e^{\mathbf{s}}; q(e^{\mathbf{s}})) = 1$ gives the gradient $\boldsymbol{\mu}$ and the Hessian (5.2) of $\log(q_0/q(e^{\mathbf{s}}))$ at zero. The Taylor expansion at $\mathbf{s} = i\mathbf{t}/\sqrt{n}$ gives the joint Gaussian limit; differentiation gives the covariance estimate.

To check nondegeneracy, suppose $a + b\#H + c\#U = dN$ for every primitive path. Applying this relation to UD, UHD, $UUDD$ and $UUHDD$ gives

$$a + c = d, \qquad a + b + c = 2d, \qquad a + 2c = 4d, \qquad a + b + 2c = 6d.$$

These equations imply $a = b = c = d = 0$. Thus no nonzero linear combination of $\mathbf{R} - \boldsymbol{\mu}N$ has zero variance, and Σ is positive definite. □

5.2. **The height.** We now turn to the height H_n, which is not additive. The height of a path is the maximum of the heights of its components, and a primitive component of height larger than k is very rare under the Boltzmann distribution, since the smallest such path is the pyramid $U^{k+1}D^{k+1}$ of area $(k+1)^2$. Therefore the height of a path of area n is determined by the balance between the number of components, which is of order n, and this small probability. To make this precise, for $k \geq 1$ let $g^{[k]}$ be the generating function of primitive paths of height at most k, and let $q_0^{(k)}$ be the positive root of $g^{[k]} = 1$ within its disc of convergence. We set $\lambda_k = \log(q_0^{(k)}/q_0)$ and $\tau_k = 1 - g^{[k]}(q_0)$, so that τ_k is the Boltzmann probability that a primitive path has height greater than k. The following theorem gives an exact formula for τ_k and the resulting law of the height.

For a single primitive path of fixed area, the largest-part theorem of Bender and Canfield [23] gives the scale $\sqrt{\log n}$. The theorem below gives the height distribution in the full Riordan ensemble. The concentration of the maximum of independent integer-valued random variables on two adjacent integers is also a known phenomenon. It goes back to Anderson [25] and holds when the ratio of consecutive tail probabilities tends to zero, see Sethuraman [26] and Ottolini [27]. In our case $\tau_{k+1}/\tau_k \sim q_0^{2k+3} \to 0$, so that the two-height concentration of Corollary 5.4 is an instance of this mechanism. What we add is the following: the exact product formula and the constant in the tail (5.3), an approximation of the distribution function which is uniform in the threshold k under the fixed-area conditioning, where the number of components is random and the components are not independent, and the explicit identification of the two heights.

Theorem 5.3. *Let $M_h = A(q^{h+1})/A(q^h)$ and let $M_h^{(k)}$ be its truncation at height k. At $x = y = 1$ one has, for $k \geq 1$,*

$$\tau_k = q_0^{(k+1)^2} M_{k+1}(q_0) \prod_{h=1}^{k} M_h(q_0) M_h^{(k)}(q_0) \sim \frac{q_0^{(k+1)^2}}{A(q_0)^2}, \tag{5.3}$$

and

$$\lambda_k = C\tau_k\big(1 + O(k^2\tau_k)\big).$$

There exist $k_0 \in \mathbb{N}$, $0 < \theta < 1$ and constants $c_k \to C$ such that, uniformly in $k \geq k_0$,

$$\mathbb{P}(H_n \leq k) = \frac{c_k}{C} e^{-\lambda_k n} + O(\theta^n).$$

Consequently, with $\Lambda = C/A(q_0)^2 = 3.9586573854\ldots$,

$$\sup_{k \in \mathbb{Z}_{\geq 0}} \left| \mathbb{P}(H_n \leq k) - \exp\Big(-\Lambda n q_0^{(k+1)^2}\Big) \right| \longrightarrow 0 \qquad (n \to \infty),$$

and

$$H_n = \sqrt{\frac{\log n}{\log(1/q_0)}} + O_{\mathbb{P}}(1).$$

The proof, which is based on an exact telescoping formula for τ_k, is somewhat technical and is given in Appendix C.

TABLE 1. Height probabilities for a random Riordan path of area n, from exact bounded-height counts.

n	$\mathbb{P}(H_n \le 2)$	$\mathbb{P}(H_n \le 3)$	$\mathbb{P}(H_n > 4)$
300	0.5451536770	0.9976892919	$1.598915753 \cdot 10^{-6}$
600	0.2919739476	0.9952539517	$3.345151784 \cdot 10^{-6}$
1500	0.0448558995	0.9879835406	$8.583841582 \cdot 10^{-6}$

Corollary 5.4. *Put*

$$b_n = \sqrt{\frac{\log(\Lambda n)}{\log(1/q_0)}}, \qquad m_n = \left\lfloor b_n + \tfrac{1}{2} \right\rfloor.$$

Then $\mathbb{P}\big(H_n \in \{m_n - 1, m_n\}\big) \to 1$ *as* $n \to \infty$. *For any* $t > 0$, *if* $(n_j)_{j\ge 1}$ *is a sequence of positive integers such that* $\Lambda n_j q_0^{j^2} \to t$, *then*

$$\mathbb{P}(H_{n_j} = j - 1) \longrightarrow e^{-t}, \qquad \mathbb{P}(H_{n_j} = j) \longrightarrow 1 - e^{-t}.$$

Proof. Write $a = \log(1/q_0) > 0$. Since $|m_n - b_n| \le \frac{1}{2}$,

$$b_n^2 - (m_n - 1)^2 \ge b_n - \tfrac{1}{4}, \qquad (m_n + 1)^2 - b_n^2 \ge b_n + \tfrac{1}{4}$$

for all sufficiently large n. In the approximation of Theorem 5.3, the exponent $\Lambda n q_0^{(k+1)^2} = \exp\{a[b_n^2 - (k+1)^2]\}$ therefore tends to infinity at $k = m_n - 2$ and to zero at $k = m_n$. Hence $\mathbb{P}(H_n \le m_n - 2) \to 0$ and $\mathbb{P}(H_n \le m_n) \to 1$.

For the subsequence statement, one may take n_j to be the nearest positive integer to $(t/\Lambda) q_0^{-j^2}$. The condition on n_j gives $b_{n_j}^2 = j^2 + (\log t)/a + o(1)$, so $m_{n_j} = j$ for all sufficiently large j. Theorem 5.3 applied at $k = j - 1$ gives $\mathbb{P}(H_{n_j} \le j - 1) \to e^{-t}$. The two-height concentration then gives the asserted probabilities. □

Remark 5.5. The double-exponential form in Theorem 5.3 describes the distribution at integer thresholds only and should not be interpreted as a continuous Gumbel limit. Indeed, $(H_n + 1)^2 - L_n$ with $L_n = \log(\Lambda n)/\log(1/q_0)$ is not tight: choosing integers n_m with $L_{n_m} = m^2 + m + \frac{1}{2} + o(1)$, every integer square is at distance at least $m + \frac{1}{2} - o(1)$ from L_{n_m}.

Numerically, one finds $\lambda_k = 0.1185211,\ 2.08134 \cdot 10^{-3},\ 8.14655 \cdot 10^{-6}$ and $5.82080 \cdot 10^{-9}$ for $k = 1, \dots, 4$, and exact bounded-height counts give the probabilities in Table 1.

At area 1500, the probability of height exactly three is about 0.9431. The slow growth and two-height concentration differ sharply from the square-root height scale of uniformly chosen Motzkin paths of fixed length.

6. Concluding remarks

We determined the asymptotics of the number of Riordan paths of fixed area, the joint Gaussian law for the additive statistics and the law of the height, using the known closed form of the generating function and the decomposition into primitive paths. The leading term of the tail of the height distribution is fixed by the area $(k+1)^2$ of the smallest path reaching height $k+1$, and the uniform estimate implies the concentration on two adjacent heights.

The path decomposition and the continued fraction extend to horizontal steps $H_\ell = (\ell, 0)$ with integer $\ell \ge 1$, forbidden on the axis; the case $\ell = 2$ gives Schröder-type paths. Such a step at height h has area ℓh. Writing $F_\ell^{\uparrow} = F_\ell(xq^2, yq^\ell; q)$, the functional equation takes the form

$$F_\ell - 1 = xqF_\ell^{\uparrow} F_\ell + yq^\ell F_\ell^{\uparrow}(F_\ell - 1),$$

and the continued fraction has $1-yq^{\ell h}$ in place of $1-yq^h$. The corresponding auxiliary function satisfies

$$A_\ell(z) = (1-yz^\ell)A_\ell(qz) - xqz^2A_\ell(q^2z), \qquad A_\ell(0)=1.$$

For general ℓ the q-Hermite representation requires a separate analysis. At $x=y=1$ the growth constants for $\ell=1,2,3,4$ are 2.2516612, 1.8656543, 1.7182811 and 1.6413284; they decrease to the Dyck value $1.5085198\ldots$. To see this, we denote by g_ℓ and g_D the generating functions of primitive paths with steps H_ℓ and of primitive Dyck paths at $x=y=1$, and by q_ℓ and q_D the positive roots of $g_\ell=1$ and $g_D=1$. Increasing ℓ decreases the weight $q^{\ell h}$ of every horizontal step, so that $g_\ell(r)$ decreases and q_ℓ increases with ℓ. By the analogue of (B.1), a primitive Dyck skeleton with m up steps has $2m-1$ interior vertices, all of positive height, and each of them carries a factor $1/(1-r^{\ell h}) \le 1/(1-r^\ell)$. Since the area of the skeleton is at least $2m-1$, one obtains

$$g_D(r) \le g_\ell(r) \le g_D\left(\frac{r}{1-r^\ell}\right)$$

whenever the right-hand side converges. The first inequality gives $q_\ell \le q_D$. For a fixed $r<q_D$ one has $r/(1-r^\ell)\to r$ as $\ell\to\infty$, so the second inequality gives $g_\ell(r)<1$ and therefore $q_\ell>r$ for all sufficiently large ℓ. Hence $q_\ell\to q_D$.

Several questions remain. The signed partition recurrence in Theorem 4.3 suggests a direct sign-reversing involution on suitable pairs of paths and partitions. For the additive statistics, local limit laws and quantitative error estimates would refine Corollary 5.2. It would be interesting to obtain similar uniform height estimates for other fixed-area ensembles, e.g. for Motzkin paths with a weight for the steps on the axis and for paths with several jump sizes. It also remains to understand whether the quiver form of Remark 3.4 has a meaning beyond the algebraic identity.

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Appendix A. The polynomials $S_6,\ldots,S_{10}$

Here we list the polynomials $S_6,\ldots,S_{10}$; terms are grouped by decreasing total degree.

$$\begin{aligned}
S_6 &= x^6+5x^5y+10x^4y^2+10x^3y^3+5x^2y^4+xy^5+3x^4+6x^3y+3x^2y^2+x^2y\\
S_7 &= x^7+6x^6y+15x^5y^2+20x^4y^3+15x^3y^4+6x^2y^5+xy^6\\
&\quad +4x^5+12x^4y+12x^3y^2+4x^2y^3+2x^3y+2x^2y^2+x^3\\
S_8 &= x^8+7x^7y+21x^6y^2+35x^5y^3+35x^4y^4+21x^3y^5+7x^2y^6+xy^7\\
&\quad +5x^6+20x^5y+30x^4y^2+20x^3y^3+5x^2y^4+3x^4y+6x^3y^2+3x^2y^3+3x^4+3x^3y+x^2y^2\\
S_9 &= x^9+8x^8y+28x^7y^2+56x^6y^3+70x^5y^4+56x^4y^5+28x^3y^6+8x^2y^7+xy^8\\
&\quad +6x^7+30x^6y+60x^5y^2+60x^4y^3+30x^3y^4+6x^2y^5\\
&\quad +4x^5y+12x^4y^2+12x^3y^3+4x^2y^4+6x^5+12x^4y+8x^3y^2+2x^2y^3+2x^3y+x^3\\
S_{10} &= x^{10}+9x^9y+36x^8y^2+84x^7y^3+126x^6y^4+126x^5y^5+84x^4y^6+36x^3y^7+9x^2y^8+xy^9\\
&\quad +7x^8+42x^7y+105x^6y^2+140x^5y^3+105x^4y^4+42x^3y^5+7x^2y^6\\
&\quad +5x^6y+20x^5y^2+30x^4y^3+20x^3y^4+5x^2y^5\\
&\quad +10x^6+30x^5y+33x^4y^2+16x^3y^3+3x^2y^4+6x^4y+6x^3y^2+x^2y^3+3x^4+2x^3y
\end{aligned}$$

The numbers of paths of area n with $r = 1, 2, \dots$ returns are, for $n = 1, \dots, 8$: 1; 1, 1; 1, 2, 1; 2, 3, 3, 1; 3, 6, 6, 4, 1; 5, 11, 13, 10, 5, 1; 8, 20, 27, 24, 15, 6, 1; 13, 36, 54, 55, 40, 21, 7, 1.

Appendix B. Explicit formulas for the coefficients

In this appendix we give explicit formulas for the coefficients of S_n. Deleting the horizontal steps of a Riordan path leaves its Dyck *skeleton*. Conversely, one can insert an arbitrary run of horizontal steps at each skeleton vertex v of positive height $h(v)$. Each inserted step has weight $yq^{h(v)}$, and therefore

$$F = \sum_{D \text{ Dyck}} x^{\#U(D)} q^{\mathrm{area}(D)} \prod_{v} \frac{1}{1 - yq^{h(v)}}, \tag{B.1}$$

where the product runs over the vertices of D of positive height. For fixed m, the coefficient $F_m = [x^m]F$ is a rational function of q whose denominator divides $\prod_{h=1}^{m}(1-yq^h)^{m+1-h}$. For example, $F_1 = q/(1-yq)$ and $F_2 = \frac{q^2}{(1-yq)^2} + \frac{q^4}{(1-yq)^2(1-yq^2)}$. In particular, the paths of height one contribute exactly $x(x+y)^{n-1}$ to S_n, which is the leading row visible in Appendix A.

In order to obtain a positive formula for all coefficients, we will need also the following profiles. Theorem B.1 below is a specialisation of the results of Gan, Ouvry and Polychronakos [5]. The Dyck skeleton count is the lowest-floor case of [5, Eq. (18)], and the Motzkin profile formulas of [5, Sections 5.2 and 5.3] count periodic paths and paths starting from a given floor with prescribed numbers of up and horizontal steps at each level. Restricting their product to excursions with no horizontal steps on the lowest floor and passing from horizontal-step numbers to vertex numbers one obtains the product of Theorem B.1, which can be written equivalently as

$$\prod_{h=1}^{H} \frac{(t_h - 1)!}{(t_h - u_{h-1} - u_h)!\, u_h!\, (u_{h-1} - 1)!}\,.$$

We include a short direct proof in order to make the fixed-area constraint explicit. For a path P, let t_h be the number of its interior vertices at height h, and let u_h be the number of its up steps from height h to $h+1$. Recall that the area is given by (2.1). If P has height H, its up-profile $(u_0, \dots, u_{H-1})$ is a composition of $\#U(P)$, and its skeleton has $v_h = u_{h-1} + u_h$ vertices at each height $1 \le h \le H$, where $u_H = 0$.

Theorem B.1. *Let $u = (u_0, \dots, u_{H-1})$ be a composition and let $t = (t_1, \dots, t_H)$ be a sequence of positive integers, and set $u_H = 0$. Here a binomial coefficient is zero when its lower argument exceeds its nonnegative upper argument. The number of Riordan paths with up-profile u and vertex profile t is*

$$\prod_{h=1}^{H} \binom{u_{h-1} + u_h - 1}{u_h} \binom{t_h - 1}{u_{h-1} + u_h - 1}.$$

Consequently,

$$F = 1 + \sum_{u} x^{|u|} \prod_{h=1}^{H} \binom{u_{h-1} + u_h - 1}{u_h} \Big(\frac{q^h}{1 - yq^h}\Big)^{u_{h-1}+u_h},$$

where the sum runs over all nonempty compositions u and H denotes the number of parts. For $m \ge 1$, the number of paths of area n with m up steps and j horizontal steps is

$$c(n, m, j) = \sum_{\substack{u \models m,\ t \\ \sum_h t_h = 2m - u_0 + j,\ \sum_h h\, t_h = n}} \prod_{h=1}^{H} \binom{u_{h-1} + u_h - 1}{u_h} \binom{t_h - 1}{u_{h-1} + u_h - 1}.$$

Proof. We first count the Dyck skeletons with profile u. At level h, the u_h excursions above height h are distributed among the u_{h-1} excursions rising from height $h-1$. This is a weak composition into

u_{h-1} parts, counted by $\binom{u_{h-1}+u_h-1}{u_h}$, and applying this construction at successive levels one obtains the first factor.

Next, at height h, one distributes $t_h - v_h$ horizontal steps among the v_h skeleton vertices; there are $\binom{t_h-1}{v_h-1}$ choices, which gives the second factor. The area is then determined by (2.1). Since the skeleton has u_0 primitive components, exactly $u_0 - 1$ of its interior vertices lie on the axis, and therefore it has $2m - u_0$ vertices of positive height. Inserting j horizontal steps gives $\sum_{h=1}^{H} t_h = 2m - u_0 + j$, which proves the formula for $c(n, m, j)$. Summing over the insertions at each height one obtains the generating function, where the term 1 represents the empty path. □

Appendix C. Proof of Theorem 5.3

Proof of Theorem 5.3. We first prove the formula for τ_k. Write $T_k(q) = g(q) - g^{[k]}(q)$ and $\delta_h = M_h - M_h^{(k)}$. Subtracting the recursions for M_h and $M_h^{(k)}$, with $M_{k+1}^{(k)} = 0$, one finds

$$\delta_h = q^{2h+1} M_h M_h^{(k)} \delta_{h+1}, \qquad \delta_{k+1} = M_{k+1}.$$

Since $g = qM_1$, iteration gives

$$T_k(q) = q^{(k+1)^2} B_k(q), \qquad B_k(q) = M_{k+1}(q) \prod_{h=1}^{k} M_h(q) M_h^{(k)}(q).$$

The exponent $1 + \sum_{h=1}^{k}(2h+1) = (k+1)^2$ is precisely the area of the pyramid of height $k+1$. To make the uniformity explicit, we fix r with $q_0 < r < \rho$; recall that $\rho \le 1$. At $x = y = 1$ the series M_h and $M_h^{(k)}$ have nonnegative coefficients, and lifting a path can only increase its area, so that coefficientwise $M_h^{(k)} \le M_h$ and $M_h(r) \le M_1(r) = g(r)/r =: L < \infty$. Hence

$$|M_h(q)| \le L, \qquad |M_h^{(k)}(q)| \le L \qquad \text{for } |q| \le r$$

and all $1 \le h \le k$, and all these functions are analytic in this disc. The recursion $M_h(1 - q^h - q^{2h+1} M_{h+1}) = 1$ and its truncated analogue show that they do not vanish there, and writing them in the form $M_h - 1 = M_h\,(q^h + q^{2h+1} M_{h+1})$ one obtains

$$|M_h(q) - 1| \le (L + L^2)\, r^h, \qquad |M_h^{(k)}(q) - 1| \le (L + L^2)\, r^h \qquad \text{for } |q| \le r,$$

uniformly in $k \ge h$. Moreover, for each fixed h the coefficients of $M_h^{(k)}$ increase to those of M_h, so that $M_h^{(k)} \to M_h$ uniformly on $|q| \le r$. Since the bounds are summable in h and do not depend on k, the products converge uniformly on $|q| \le r$, the product over the truncated factors converges to the product over the full ones, and by Cauchy's estimates the same holds for the derivatives on any smaller disc. Since

$$\prod_{h=1}^{\infty} M_h(q) = \lim_{H \to \infty} \frac{A(q^{H+1})}{A(q)} = \frac{1}{A(q)},$$

one obtains $B_k(q) \to A(q)^{-2}$ locally uniformly. Evaluating at q_0 one gets (5.3), and logarithmic differentiation gives

$$T_k'(q_0) = \tau_k \left(\frac{(k+1)^2}{q_0} + O(1) \right). \tag{C.1}$$

Next, let us compare the roots $q_0^{(k)}$ and q_0. The coefficients of $g^{[k]}$ increase to those of g, which gives uniform convergence on every closed disc strictly inside $|q| < \rho$. Choose $r \in (q_0, \rho)$ such that q_0 is the only zero of $1 - g$ in $|q| \le r$. By Rouché's theorem, for all sufficiently large k the function $1 - g^{[k]}$ has exactly one zero in this disc. It is the positive simple root $q_0^{(k)}$, and $q_0^{(k)} \to q_0$. Put $\Delta_k = q_0^{(k)} - q_0$. The derivatives $(g^{[k]})'$ are uniformly bounded below near q_0 on the real axis, and the

second derivatives are uniformly bounded there. The mean value theorem gives $\Delta_k = O(\tau_k)$. Taylor expansion and (C.1) then yield

$$\tau_k = (g^{[k]})'(q_0)\Delta_k + O(\Delta_k^2), \qquad (g^{[k]})'(q_0) = g'(q_0) + O(k^2\tau_k).$$

Thus $\Delta_k = \tau_k/g'(q_0)\,(1 + O(k^2\tau_k))$, and expanding $\log(1 + \Delta_k/q_0)$ one obtains the estimate for λ_k.

The uniform estimate for the probability is obtained in the same way as in Theorem 4.1. On $|q| = r$, $|1 - g^{[k]}(q)|$ is bounded below uniformly in large k. Subtracting the simple pole and applying Cauchy's formula one finds

$$r_n^{[k]} = c_k (q_0^{(k)})^{-n} + O(r^{-n}), \qquad c_k = \frac{1}{q_0^{(k)}(g^{[k]})'(q_0^{(k)})} \longrightarrow C,$$

uniformly in k, and it remains to divide by the asymptotic formula for r_n.

Finally, let us prove the distributional approximation. Put $a_k = \Lambda q_0^{(k+1)^2}$. We have $\lambda_k/a_k \to 1$ and $c_k/C \to 1$. For each fixed finite set of k, both the probabilities and e^{-na_k} tend to zero; indeed $q_0^{(k)} > q_0$, and the bounded-height generating function is analytic in a disc of radius larger than q_0. For large k one uses the uniform pole estimate and

$$\sup_{t \geq 0} |e^{-(1+\epsilon)t} - e^{-t}| \longrightarrow 0 \quad \text{as } \epsilon \to 0.$$

Splitting the supremum into these two ranges one obtains the claim; the case $k = 0$ is immediate, since $\mathbb{P}(H_n \leq 0) = 0$ for $n \geq 1$. For the last statement, let $b_n = \sqrt{\log(\Lambda n)/\log(1/q_0)}$. At $k = \lfloor b_n \rfloor - 2$ the exponential approximation tends to zero, whereas at $k = \lfloor b_n \rfloor + 1$ it tends to one, which proves the $O_{\mathbb{P}}(1)$ bound. □

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