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RIORDAN LATTICE PATHS COUNTED BY AREA

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Abstract. We study Riordan paths enumerated by the area between the path and the axis. We show that the generating function satisfies a q-Riccati equation and can be written in a closed form as a ratio of basic hypergeometric series. The auxiliary series admits an algebraic representation as the motivic generating series of a symmetric two-node quiver. We obtain explicit formulas for the coefficients, the exponential asymptotics of the number of paths of given area, and a linear recurrence whose coefficients are signed counts of partitions into distinct parts. For a random path of area n we obtain a joint Gaussian limit law for the numbers of returns, horizontal steps and up steps. The height is of order $\sqrt{\log n}$ and concentrates on two adjacent integers, which we determine explicitly.

1. Introduction

Lattice paths weighted by area appear in polymer and vesicle models [1, 2], in the study of exclusion statistics and Hofstadter-type Hamiltonians [3, 4, 5], and in knot theory.[1] Namely, in the so-called knots-quivers correspondence [9, 10, 11] certain families of paths below lines of rational slope encode coloured HOMFLY-PT invariants of torus knots. For several directed-path models, q-difference equations and quotients of q-series provide a useful description of the generating functions. For further developments in the knot-theoretic direction see [12, 13, 14, 15] and the review [16].

In this paper we study Riordan paths at fixed area. Recall that a *Motzkin path* of length n runs from $(0,0)$ to $(n,0)$ with steps $U=(1,1)$, $D=(1,-1)$ and $H=(1,0)$ and never passes below the x-axis. A *Riordan path* is a Motzkin path with no horizontal steps on the axis, and a *Dyck path* has no horizontal steps at all. Enumeration by length gives the Riordan numbers $1,0,1,1,3,6,15,36,91,\dots$ and the Catalan numbers, respectively. Here we instead fix the area between the path and the axis, and we assign the weight x to each up step and the weight y to each horizontal step.

Let us mention that length-area generating functions for Motzkin meanders with boundary markers were obtained by Polychronakos [2]; setting the weight of horizontal steps on the floor to zero in his formulas gives the Riordan case. Counts with prescribed step profiles were also studied by Gan, Ouvry and Polychronakos [4]. Here we use these path decompositions to study the fixed-area ensemble. We give explicit coefficient formulas and a partition recurrence, and determine the exponential growth, additive statistics and height distribution at large area. We also express the auxiliary q-series as the generating series of a symmetric two-node quiver; this is an algebraic identity, without an associated knot invariant.

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[1]Let us also mention that Motzkin paths and Motzkin polynomials appear in integrable Hamiltonian systems arising from the exact renormalization group flow in matrix scalar field theory, where the conserved quantities are labelled by lattice paths [6]; see [7, 8] for further developments in this direction.

Let $S_n(x,y)$ be the weight polynomial of the paths of area n and let $F(x,y;q)=\sum_{n\geq 0} S_n(x,y)q^n$. As the point of departure one may regard the following functional equation,

$$F(x,y;q)-1 = q\,F(xq^2,yq;q)\,\big[(x+y)F(x,y;q)-y\big], \tag{1.1}$$

which follows from a decomposition of the path at its first horizontal step at height one or its first return to the axis. The substitutions $x\mapsto xq^2$ and $y\mapsto yq$ account for the area gained when a path is lifted by one level. A continued fraction expansion then leads to the closed form

$$F(x,y;q)=\frac{A(q)}{A(q)-xq\,A(q^2)}, \qquad A(z)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}p_n(x,y;q)}{(q;q)_n}\,z^n, \tag{1.2}$$

where p_n stand for homogeneous forms of the Rogers continuous q-Hermite polynomials, defined in Section 4. Here and below we use the standard notation

$$(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad (a;q)_\infty=\prod_{j\geq 0}(1-aq^j),$$

with $(a;q)_0=1$ and $|q|<1$. The identity (1.2) holds as a formal power series in q, and the quotient also gives the meromorphic continuation of F to the unit disc. For $y=0$ it reduces to the Rogers-Ramanujan type formula for Dyck paths counted by area [17, 18]. After the substitution $x=\alpha\beta$, $y=\alpha+\beta$, the series A turns out to be the motivic generating series of the quiver with adjacency matrix $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$. This allows one to compare (1.2) with the quiver-series expressions for area-weighted Schröder paths [10, 11], although by itself it does not give a knot interpretation of Riordan paths.

The rest of the paper is organized as follows. In Section 2 we define the area polynomials and list the first examples. In Section 3 we prove the functional equation (1.1) and derive a recurrence for the polynomials S_n. In Section 4 we obtain the continued fraction and the closed form (1.2), including the case of bounded height. In Section 5 we give explicit formulas for the coefficients in terms of path profiles. In Section 6 we derive the asymptotics of the number of paths and a linear recurrence with partition-theoretic coefficients. In Section 7 we study the limit laws of additive statistics and of the height. We conclude with remarks and open problems.[2]

2. Riordan paths and area polynomials

In this section we fix the notation and state the problem. Let us first explain how the area of a path is computed. A step from height h to height h' contributes $(h+h')/2$ to the area. In other words, an up or down step between heights h and $h+1$ contributes $h+\frac{1}{2}$, while a horizontal step at height h contributes h. Note that the total area is an integer, since the numbers of up and down steps are equal. Every allowed step contributes at least $\frac{1}{2}$, so a Riordan path of area n has length at most $2n$, and in particular there are only finitely many such paths.[3] For $n\geq 1$ we define

$$S_n(x,y)=\sum_{\mathrm{area}(P)=n} x^{\#U(P)}\,y^{\#H(P)}, \qquad S_0=1, \tag{2.1}$$

where the sum runs over Riordan paths of area n, and $\#U(P)$ and $\#H(P)$ stand for the numbers of up and horizontal steps of P. A path with i up steps and j horizontal steps has weight x^iy^j and length $2i+j$. One can easily compute the first polynomials,

$$S_1=x, \qquad S_2=x^2+xy, \qquad S_3=x^3+2x^2y+xy^2, \qquad S_4=x^4+3x^3y+3x^2y^2+xy^3+x^2,$$
$$S_5=x^5+4x^4y+6x^3y^2+4x^2y^3+xy^4+2x^3+2x^2y,$$

[2]Finite checks compare the polynomial recurrence and profile formula with direct enumeration through area 16, and the partition recurrence with step-by-step counts through area 1500. The Gaussian constants were evaluated independently and compared with exact moments at area 300; the height probabilities in Table 1 were also recomputed from exact counts. These checks complement the proofs below.

[3]This is the reason for excluding horizontal steps on the axis: a Motzkin path may contain arbitrarily many of them without changing the area.

and $S_6, \dots, S_{10}$ are listed in Appendix A. For instance, the path $UHUDD$ in Figure 1 has area 5 and weight x^2y; it is one of the two paths giving the term $2x^2y$ of S_5. The numbers $r_n = S_n(1,1)$ of Riordan paths of area n are, starting at $n = 1$,

$$1,\ 2,\ 4,\ 9,\ 20,\ 45,\ 101,\ 227,\ 511,\ 1150,\ 2589,\ 5829,\ 13124,\ 29550,\ \dots \tag{2.2}$$

with $r_0 = 1$. The Dyck sequence $S_n(1,0)$ begins, for $n \geq 1$, with $1, 1, 1, 2, 3, 4, 6, 9, 14, 21, \dots$ and is recorded as OEIS A143951 [19].[4]

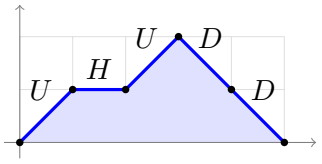


FIGURE 1. The path $UHUDD$, of area $\frac{1}{2} + 1 + \frac{3}{2} + \frac{3}{2} + \frac{1}{2} = 5$.

3. The functional equation

In this section we derive the functional equation (1.1). The key observation is very simple. If one lifts a path by one level, every step gains area 1, so the area increases by the length of the path. Since a path of weight x^iy^j has length $2i + j$, the lift is represented by the substitution $x \mapsto xq^2$, $y \mapsto yq$. In what follows we write $F^{\uparrow} = F(xq^2, yq; q)$.

Theorem 3.1. *The generating function satisfies*

$$F - 1 = q\, F^{\uparrow}\big[(x + y)F - y\big]. \tag{3.1}$$

For $x \neq 0$ this can be rearranged as

$$F(xq^2, yq; q) = \frac{F(x, y; q) - 1}{q\,[(x + y)F(x, y; q) - y]},$$

with the factor q cancelled formally. At $x = 0$ one has $F = 1$, and the undivided identity remains valid.

Proof. A nonempty Riordan path starts with U. Let us read it up to the first subsequent step which either returns to the axis or is horizontal at height 1. The segment $\hat{R}$ in between is a Riordan path lifted by one level and contributes $F^{\uparrow}$. If the distinguished step is D, one has $P = U\hat{R}DP'$ with an arbitrary Riordan path P', which contributes $xqF^{\uparrow}F$. If the distinguished step is H, one has $P = U\hat{R}HW$, and removing $\hat{R}H$ leaves the nonempty Riordan path UW; this case contributes $yqF^{\uparrow}(F - 1)$. Adding the two contributions one obtains (3.1). □

For nonzero x, (3.1) is an equation of q-Riccati type: the lifted function $F^{\uparrow}$ is a linear fractional transform of F. For $y = 0$ it reduces to the equation $F = 1 + xqF(xq^2)F$ for Dyck paths counted by area, which goes back to Flajolet [17], see also [1, 18]; the Motzkin analogue was obtained by Polychronakos [2].[5]

Extracting the coefficient of q^n in (3.1) one obtains the following recurrence for the area polynomials,

$$S_n = [t^{n-1}]\Big(\sum_{m<n} t^m\, S_m(xt^2, yt)\Big)\Big((x + y)\sum_{p<n} t^p\, S_p(x, y) - y\Big), \qquad n \geq 1. \tag{3.2}$$

[4]Let us mention that enumeration at fixed area differs from the total area of all paths of a fixed length, which was studied in [20, 21].

[5]If one allows horizontal steps on the axis with weight z, the generating function becomes $F/(1 - zF)$, since such steps separate Riordan components and contribute no area.

For example, the first factor is $1+xt^3+\cdots$ and the second one is $x+x(x+y)t+x(x+y)^2t^2+x(x+y)^3t^3+\cdots$, so that $S_4=x(x+y)^3+x^2$. Actually, the substitution $S_m(xt^2,yt)$ retains the information about the length, which is needed for the lift, and as a result (3.2) does not close on the numbers r_n alone. A scalar recurrence for r_n will be given in Theorem 6.3.

4. Continued fraction and closed form

In this section we obtain the continued fraction expansion of F and, from it, the closed form (1.2); we only need the simplest ingredients of the approach of Flajolet [17]. Let M_h be the generating function of Motzkin paths based at height $h\ge 1$, with the area measured from the x-axis. Such a path is a sequence of horizontal steps and excursions above height h, and therefore $M_h=1/(1-yq^h-xq^{2h+1}M_{h+1})$ and $F=1/(1-xqM_1)$. In other words,

$$F=\cfrac{1}{1-\cfrac{xq}{1-yq-\cfrac{xq^3}{1-yq^2-\cfrac{xq^5}{1-yq^3-\cdots}}}}. \tag{4.1}$$

4.1. **Paths of bounded height.** Let us first look at the convergents of (4.1). Truncating the continued fraction at level $k\ge 1$ one obtains the generating function $F^{[k]}$ of paths of height at most k; for $k=0$ one has $F^{[0]}=1$. We write $F^{[k]}=R_1^{(k)}/R_0^{(k)}$, where

$$R_{k+1}^{(k)}=1,\qquad R_k^{(k)}=1-yq^k,\qquad R_h^{(k)}=(1-yq^h)R_{h+1}^{(k)}-xq^{2h+1}R_{h+2}^{(k)}\quad(0\le h<k), \tag{4.2}$$

with the factor $1-yq^0$ replaced by 1 at $h=0$. Expanding this recurrence one recovers the standard interpretation of the convergents in terms of matchings, which also goes back to [17]. Namely, $R_h^{(k)}$ is the generating polynomial of monomer-dimer configurations on the path graph with vertices $h,\dots,k$, where a monomer on the vertex v has weight $-yq^v$ and a dimer on the edge $\{v,v+1\}$ has weight $-xq^{2v+1}$.[6] Since every occupied vertex v contributes q^v, one can write, for $1\le h\le k$,

$$R_h^{(k)}=\sum_{S\subseteq\{h,\dots,k\}} q^{\sum S}\prod_{I} m_{|I|}(x,y),\qquad m_r(x,y)=\sum_{d=0}^{\lfloor r/2\rfloor}\binom{r-d}{d}(-x)^d(-y)^{r-2d}, \tag{4.3}$$

where the product runs over the maximal runs I of consecutive integers in S, and m_r stands for the matching polynomial of the path with r vertices. At the boundary $h=0$ we use $R_0^{(k)}=R_1^{(k)}-xqR_2^{(k)}$. Equivalently, in the subset formula the run containing 0, if present, has weight $b_1=0$ and $b_r=-x\,m_{r-2}(x,y)$ for $r\ge 2$, since the vertex 0 must be covered by the dimer $\{0,1\}$.

Two special cases can be written in a closed form. The dimer-only part, valid also at $h=0$, is a q^2-binomial sum,

$$R_h^{(k)}\big|_{y=0}=\sum_{m=0}^{\lfloor (k-h+1)/2\rfloor}(-x)^m q^{m(2m+1)+2m(h-1)}\begin{bmatrix}k-h+1-m\\ m\end{bmatrix}_{q^2},$$

which at $h=1$ agrees with the formula obtained by Owczarek and Prellberg for Dyck paths of bounded height [1]. The monomer-only part is $(yq^h;q)_{k-h+1}$ for $h\ge 1$ and $(yq;q)_k$ for $h=0$. It seems that the mixed contributions do not factor in a similar way.

At $x=y=1$ the exponential growth rate of the number of paths of height at most k is $1/q_0^{(k)}$, where $q_0^{(k)}$ denotes the smallest positive zero of $R_0^{(k)}$. One finds

$$1/q_0^{(k)}=2,\quad 2.2469796\ldots,\quad 2.2516428\ldots,\quad 2.2516611\ldots\qquad(k=1,2,3,4).$$

[6] No monomer is allowed on the vertex 0, which reflects the absence of horizontal steps on the axis.

Paths of height one contribute $x(x+y)^{n-1}$ at area $n \geq 1$, which explains the first value. For height at most two, $1/q_0^{(2)} = 1 + 2\cos(2\pi/7)$ is the largest root of $t^3 - 2t^2 - t + 1$. The rapid convergence to the limit $1/q_0 = 2.2516612\ldots$ is a curious phenomenon, which will be explained in Theorem 7.3.

4.2. **The closed form.** Let us now take $k \to \infty$. For each fixed $h \geq 1$ one finds from (4.2) that $R_h^{(k)} \to A(q^h)$, where

$$A(z) = (1 - yz)A(zq) - xqz^2 A(zq^2), \qquad A(0) = 1. \tag{4.4}$$

We usually suppress the dependence on x, y, q and write $A(z)$ for $A(z; x, y; q)$. At the boundary one has $R_0^{(k)} \to A(q) - xqA(q^2)$. Comparing coefficients, the solution of (4.4) is the series A in (1.2), where the polynomials p_n are determined by

$$p_0 = 1, \qquad p_1 = y, \qquad p_n = y\,p_{n-1} - x(1 - q^{n-1})\,p_{n-2}. \tag{4.5}$$

Then $M_h = A(q^{h+1})/A(q^h)$ and $F = 1/(1 - xqM_1) = A(q)/(A(q) - xqA(q^2))$, which proves (1.2). Equivalently, $1/F = y + A(1)/A(q)$.

Lemma 4.1. *For fixed complex x, y, the series $A(z; x, y; q)$ is entire in z and holomorphic in q for $|q| < 1$. For each $h \geq 1$, the continuants $R_h^{(k)}$ converge to $A(q^h)$ locally uniformly in this disc. In particular, (1.2) defines a meromorphic function of q there.*

Proof. Fix $0 < r < 1$. The recurrence (4.5) gives $|p_n| \leq K^n$ for $|q| \leq r$, with K depending only on x, y. Also $|(q;q)_n| \geq \prod_{j \geq 1}(1 - r^j) > 0$. The factor $q^{\binom{n}{2}}$ therefore ensures local uniform convergence of the series for A in (z, q).

For the continuants, ignore the exclusion constraints in the matching expansion. The sum of the absolute weights of all configurations on the infinite path is bounded by

$$\prod_{v \geq h} \bigl(1 + |y|r^v + |x|r^{2v+1}\bigr) < \infty.$$

The weights of configurations involving vertices beyond k tend to zero uniformly on $|q| \leq r$. Hence the finite continuants converge locally uniformly. Their formal limit, already identified above, is $A(q^h)$. The denominator in (1.2) is holomorphic and equals 1 at $q = 0$, which proves the last assertion. □

Remark 4.2. Let us write $x = \alpha\beta$ and $y = \alpha + \beta$. Then the solution of (4.5) takes the form

$$p_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \alpha^{n-k}\beta^k,$$

which is a homogeneous form of the continuous q-Hermite polynomial of Rogers. Namely, one has $p_n = x^{n/2} H_n(y/(2\sqrt{x}) \mid q)$ for $x \neq 0$, with its polynomial extension at $x = 0$. Moreover, one can write

$$A(z) = (\alpha z; q)_\infty \, {}_1\phi_1\left(\begin{matrix} 0 \\ \alpha z \end{matrix}; q, \beta z \right),$$

which is a q-Bessel type function, see e.g. [22]. At apparent singularities of the product representation, the identity is understood by analytic continuation.

Remark 4.3 (An algebraic quiver-series representation). Let us introduce the so-called symmetric-quiver series

$$P_C(x_1, x_2) = \sum_{d_1, d_2 \geq 0} \frac{(-q^{1/2})^{d^T C d} x_1^{d_1} x_2^{d_2}}{(q;q)_{d_1}(q;q)_{d_2}}, \qquad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since $d^T C d = (d_1 + d_2)^2$ and $(-q^{1/2})^{n^2} = (-1)^n q^{n^2/2}$, grouping the terms with $d_1 + d_2 = n$ one obtains the formal identities

$$A(z) = P_C\bigl(\alpha z q^{-1/2},\, \beta z q^{-1/2}\bigr), \qquad \frac{1}{F} - y = \frac{P_C(\alpha q^{-1/2}, \beta q^{-1/2})}{P_C(\alpha q^{1/2}, \beta q^{1/2})}. \tag{4.6}$$

Ratios of quiver series at q-shifted arguments occur in the enumeration of area-weighted Schröder paths [10, 11], where the corresponding q-difference equations are related to the so-called quantum A-polynomials; combinatorial derivations of such equations appear in [14], and the coefficient formulas of [12] may be compared with Theorem 5.1 below. In the conventions of [11], the Schröder-path quiver of slope $1/f$ has the matrix $\left(\begin{smallmatrix} f & f \\ f & f+1 \end{smallmatrix}\right)$, so that our matrix differs from its $f=1$ case by one diagonal entry. For $y=0$ the specialization (4.7) below can similarly be written as $F(x,0;q) = P_{(2)}(-xQ^{1/2};Q)/P_{(2)}(-xQ^{-1/2};Q)$ with $Q=q^2$, and the one-node quiver (2) appears in the formulas for the reduced unknot in framing two [11]. We stress that these are comparisons of formal series and adjacency matrices, not of knot invariants.

For $y=0$, i.e. $\beta=-\alpha$, the closed form becomes the Rogers-Ramanujan type identity

$$F(x,0;q) = \frac{\sum_{k\geq 0}(-x)^k q^{k(2k+1)}/(q^2;q^2)_k}{\sum_{k\geq 0}(-x)^k q^{k(2k-1)}/(q^2;q^2)_k}, \tag{4.7}$$

in agreement with [1, 18]. Finally, taking $k\to\infty$ in (4.3) one obtains a second interpretation of the auxiliary series: for $h\geq 1$, $A(q^h)$ enumerates partitions into distinct parts at least h, with each maximal run of r consecutive parts weighted by $m_r(x,y)$. We will need this interpretation in Section 6.

5. Explicit formulas for the coefficients

In this section we give explicit formulas for the coefficients of S_n. Let us start with a simple observation. Deleting the horizontal steps of a Riordan path leaves its Dyck *skeleton*. Conversely, one can insert an arbitrary run of horizontal steps at each skeleton vertex v of positive height $h(v)$. Each inserted step has weight $yq^{h(v)}$, and therefore

$$F = \sum_{D\ \text{Dyck}} x^{\#U(D)} q^{\mathrm{area}(D)} \prod_{v} \frac{1}{1-yq^{h(v)}}, \tag{5.1}$$

where the product runs over the vertices of D of positive height. For fixed m, the coefficient $F_m=[x^m]F$ is a rational function of q whose denominator divides $\prod_{h=1}^{m}(1-yq^h)^{m+1-h}$. For example, $F_1 = q/(1-yq)$ and $F_2 = \frac{q^2}{(1-yq)^2} + \frac{q^4}{(1-yq)^2(1-yq^2)}$. In particular, the paths of height one contribute exactly $x(x+y)^{n-1}$ to S_n, which is the leading row visible in Appendix A.

In order to obtain a positive formula for all coefficients, we will need also the following profiles. The Dyck skeleton count used below is the lowest-floor case of [4, Eq. (18)]; that work also treats horizontal-step profiles. We include a direct proof to make the fixed-area constraint explicit. For a path P, let t_h be the number of its interior vertices at height h, and let u_h be the number of its up steps from height h to $h+1$. Summing the contributions of the endpoints of each step one finds

$$\mathrm{area}(P) = \sum_{h\geq 1} h\, t_h, \tag{5.2}$$

i.e. the area is the sum of the heights of the interior vertices. If P has height H, its up-profile $(u_0,\dots,u_{H-1})$ is a composition of $\#U(P)$, and its skeleton has $v_h = u_{h-1}+u_h$ vertices at each height $1\leq h\leq H$, where $u_H=0$.

Theorem 5.1. *Let $u=(u_0,\dots,u_{H-1})$ be a composition and let $t=(t_1,\dots,t_H)$ be a sequence of positive integers, and set $u_H=0$. Here a binomial coefficient is zero when its lower argument exceeds its nonnegative upper argument. The number of Riordan paths with up-profile u and vertex profile t is*

$$\prod_{h=1}^{H} \binom{u_{h-1}+u_h-1}{u_h} \binom{t_h-1}{u_{h-1}+u_h-1}.$$

Consequently,

$$F = 1 + \sum_{u} x^{|u|} \prod_{h=1}^{H} \binom{u_{h-1}+u_h-1}{u_h} \Big(\frac{q^h}{1-yq^h}\Big)^{u_{h-1}+u_h},$$

where the sum runs over all nonempty compositions u and H denotes the number of parts. For $m \geq 1$, the number of paths of area n with m up steps and j horizontal steps is

$$c(n,m,j) = \sum_{\substack{u \models m,\ t \\ \sum_h t_h = 2m-u_0+j,\ \sum_h h\,t_h = n}} \prod_{h=1}^{H} \binom{u_{h-1}+u_h-1}{u_h} \binom{t_h-1}{u_{h-1}+u_h-1}.$$

Proof. Let us first count the Dyck skeletons with profile u. At level h, the u_h excursions above height h are distributed among the u_{h-1} excursions rising from height $h-1$. This is a weak composition into u_{h-1} parts, counted by $\binom{u_{h-1}+u_h-1}{u_h}$, and applying this construction at successive levels one obtains the first factor.

Next, at height h, one distributes $t_h - v_h$ horizontal steps among the v_h skeleton vertices; there are $\binom{t_h-1}{v_h-1}$ choices, which gives the second factor. The area is then determined by (5.2). Since the skeleton has u_0 primitive components, exactly $u_0 - 1$ of its interior vertices lie on the axis, and therefore it has $2m - u_0$ vertices of positive height. Inserting j horizontal steps gives $\sum_{h=1}^{H} t_h = 2m - u_0 + j$, which proves the formula for $c(n,m,j)$. Summing over the insertions at each height one obtains the generating function, where the term 1 represents the empty path. □

6. Asymptotics and a linear recurrence

In this section we determine the growth of the numbers r_n and obtain a recurrence for them. The point of departure is the decomposition of a path into its *primitive* components, i.e. the pieces between consecutive returns to the axis. Every nonempty Riordan path decomposes uniquely into primitive paths, which touch the axis only at their endpoints. At $x = y = 1$ their generating function is

$$g(q) = \sum_{P \text{ primitive}} q^{\mathrm{area}(P)} = q + q^2 + q^3 + 2q^4 + 3q^5 + 5q^6 + \cdots.$$

Therefore $F(1,1;q) = 1/(1-g)$ and, by (1.2), $g = qA(q^2)/A(q)$. Each primitive component contributes one return to the axis, including the final one, so marking the returns by u one obtains $F(u;q) = 1/(1-ug) = F/(u+(1-u)F)$.

Theorem 6.1. *Let ρ be the radius of convergence of g. The equation $g(q) = 1$ has a unique root $q_0 \in (0,\rho)$, which is also the smallest positive root of $A(q) = qA(q^2)$ at $x = y = 1$. For some $\theta < 1$,*

$$r_n = C\, q_0^{-n}\big(1 + O(\theta^n)\big), \qquad C = \frac{1}{q_0\, g'(q_0)},$$

with

$$q_0 = 0.44411654605086\ldots, \quad q_0^{-1} = 2.25166121121160\ldots, \quad C = 0.34315827064\ldots$$

Proof. The series g has nonnegative coefficients and a positive radius of convergence, since a path of area n has length at most $2n$. By Lemma 4.1, g is meromorphic in $|q| < 1$. The primitive paths UH^jD give $g(t) \geq t/(1-t)$ wherever both sides converge, so $\rho \leq 1$. If $\rho < 1$, Pringsheim's theorem together with meromorphicity implies that ρ is a pole and $g(t) \to \infty$ as $t \uparrow \rho$. If $\rho = 1$, the same divergence follows from the lower bound. Since g is strictly increasing on $[0,\rho)$, it reaches 1 at a unique point $q_0 < \rho$.

For $|q| \leq q_0$ one has $|g(q)| \leq g(|q|) \leq 1$. On $|q| = q_0$, equality in the triangle inequality requires the terms of degrees 1 and 2 to have the same argument, and their positive coefficients force $q = q_0$. Thus $1 - g$ has a unique zero in the closed disc, and this zero is simple because $g'(q_0) > 0$. Analyticity in

a larger disc and isolation of zeros give an exponentially smaller remainder after subtraction of the simple pole, and the residue gives $C = 1/(q_0 g'(q_0))$.

It remains to rule out a cancellation in the quotient for g. Let us display the base parameter of A explicitly. If, for some $0 < t < 1$, both $A(t;1,1;t)$ and $A(t^2;1,1;t)$ vanished, the recurrence for $B_h = A(t^h;1,1;t)$,

$$B_h = (1-t^h)B_{h+1} - t^{2h+1}B_{h+2},$$

would force $B_h = 0$ for every $h \geq 1$, which contradicts $B_h \to A(0;1,1;t) = 1$. Consequently a zero of $A(t;1,1;t)$ cannot cancel in $g(t)$, so $A(t;1,1;t) > 0$ for $0 \leq t < \rho$. On this interval the denominator is $A(q)(1-g(q))$, and its first positive zero is q_0. □

Let us make a few comments. The leading asymptotics is already accurate at modest area; namely, $r_{22}/(Cq_0^{-22}) = 1 + 4.8722 \cdot 10^{-7}$. A further positive pole is found numerically at $q_1 = 0.77091704183\ldots$, and the argument principle applied numerically on the circles $|q| = 0.6,\ 0.7,\ 0.8$ gives $1, 1, 2$ zeros of $A(q) - qA(q^2)$. These computations suggest that any $\theta \in (q_0/q_1, 1)$ is admissible in Theorem 6.1; we do not use this sharper, uncertified estimate. We do not know closed forms for q_0 and C; the Dyck constants $q_D = 0.6629014851\ldots$ and $C_D = 0.3377611503\ldots$ recorded in OEIS A143951 [19] are likewise only known numerically. For positive weights x, y the same argument applies whenever the primitive generating function reaches 1 strictly inside its radius of convergence, which holds near $(x, y) = (1, 1)$.

6.1. **A recurrence for r_n.** Let us now come back to the partition interpretation of Section 4. We write $A(q)|_{x=y=1} = \sum_n \nu_n q^n$ and $A(q^2)|_{x=y=1} = \sum_n \nu'_n q^n$. Then ν_n counts partitions of n into distinct parts, with each maximal run of r consecutive parts weighted by $m_r(1,1) = \sum_d (-1)^{r-d}\binom{r-d}{d}$, and ν'_n counts the same partitions with the additional restriction that every part is at least 2. The run weights turn out to be periodic, namely $m_r(1,1) = 1, -1, 0$ for $r \equiv 0, 1, 2 \pmod 3$, respectively. The following simple cancellation explains this periodicity.

Lemma 6.2. *Let us write a perfect monomer-dimer cover of the path with r vertices as a word in the letters M and D, with sign $(-1)^{\#\text{letters}}$. The map exchanging a leading D with a leading MM, and fixing the words with neither prefix, is an involution which reverses the sign on its nonfixed pairs. Its fixed points are the words MDw, together with M and the empty word. Hence $m_r(1,1) = m_{r-3}(1,1)$ for $r \geq 3$, with the initial values $1, -1, 0$.*

Theorem 6.3. *Let $r_0 = 1$, set $\nu'_{-1} = 0$, and let $d_n = \nu_n - \nu'_{n-1}$ for $n \geq 0$. Then*

$$r_n = \nu_n - \sum_{k=1}^{n} d_k\, r_{n-k}, \qquad n \geq 1,$$

where $\nu_n = \sum_\lambda (-1)^{\#\{\text{runs of length} \equiv 1 \pmod 3\}}$, the sum being over partitions λ of n into distinct parts with no run of length $\equiv 2 \pmod 3$, and ν'_n is the same sum over partitions with parts ≥ 2. The first values are

$$(\nu_n)_{n\geq 0} = 1, -1, -1, -1, 0, 0, 2, 1, 2, 2, 1, 0, 1, -2, -2, -3, -3, -5, -3, -5, -3, \ldots$$
$$(\nu'_n)_{n\geq 0} = 1, 0, -1, -1, -1, -1, 0, 0, 1, 2, 2, 2, 3, 2, 1, 1, 0, -2, -2, -4, -5, \ldots$$
$$(d_n)_{n\geq 0} = 1, -2, -1, 0, 1, 1, 3, 1, 2, 1, -1, -2, -1, -5, -4, -4, -4, -5, -1, -3, 1, \ldots$$

Proof. This is the identity $F(1,1;q)\,(A(q) - qA(q^2)) = A(q)$ written out coefficientwise. □

It is worth mentioning that the coefficients of the recurrence are small compared with r_n, since their absolute values are bounded by partition counts; for example, $|\nu_n| \leq 1167$ for $n \leq 120$, whereas $r_{120} \approx 10^{42}$.

7. Limit laws

In this section we describe a typical Riordan path of large area. In a few words, such a path looks like a long sequence of small primitive components, and the statistics we are interested in are sums over the components. The point of departure is again the decomposition into primitive paths. Since $g(q_0) = 1$, one can assign the probability $q_0^{\mathrm{area}(P)}$ to each primitive path P; this defines a probability distribution on primitive paths, which we call the Boltzmann distribution. Let N be the area of a Boltzmann primitive path and let R be an additive statistic, namely its number of returns, of horizontal steps or of up steps. Note that $\mathbb{E}N = q_0 g'(q_0) = 1/C$. In other words, the constant C of Theorem 6.1 is the inverse of the mean area of a primitive component.

7.1. Additive statistics.

Let us first consider the statistics which are additive over the components. The following theorem says that they behave like sums of independent random variables, and that the constants are the ones of the classical renewal-reward theorem.

Theorem 7.1. *Let a Riordan path of area n be chosen uniformly at random, and let X_n, Y_n, Z_n be its numbers of returns to the axis, of horizontal steps and of up steps. Each of these is asymptotically Gaussian. Namely, for any one of them, denoted by W_n,*

$$\mathbb{E}W_n = \mu n + O(1), \qquad \mathrm{Var}(W_n) = \sigma^2 n + O(1), \qquad \frac{W_n - \mu n}{\sigma\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1),$$

where

$$\mu = \frac{\mathbb{E}R}{\mathbb{E}N}, \qquad \sigma^2 = \frac{\mathrm{Var}(R - \mu N)}{\mathbb{E}N} > 0. \tag{7.1}$$

Numerically,

$$\begin{array}{lll} X_n: & \mu_X = C \approx 0.343158270645, & \sigma_X^2 \approx 0.344199446374, \\ Y_n: & \mu_Y \approx 0.365093862590, & \sigma_Y^2 \approx 0.292149839727, \\ Z_n: & \mu_Z \approx 0.431620789070, & \sigma_Z^2 \approx 0.225725860430. \end{array}$$

For the returns one has $\sigma_X^2 = \mathrm{Var}(N)/(\mathbb{E}N)^3$.

Proof. Let $g(v;q)$ be the generating function of primitive paths with the statistic marked by v. Then the bivariate generating function is $1/(1-g(v;q))$, i.e. we are dealing with the so-called supercritical sequence schema of Flajolet and Sedgewick [18, Chapter IX]. Let us choose $r > q_0$ and $\varepsilon > 0$ with $(1+\varepsilon)r < \rho$. Each statistic satisfies $0 \le R \le N$, so the marked series is dominated by $g((1+\varepsilon)r)$ for $|q| \le r$, $|v| \le 1+\varepsilon$, and is therefore jointly analytic in a neighbourhood of $(1, q_0)$. As a result, the simple dominant pole persists for v near 1, with uniform asymptotics of the coefficients, and the quasi-powers theorem of Hwang [18, Chapter IX] gives a Gaussian law with $\mu = -q'(1)/q(1)$ and $\sigma^2 = -q''(1)/q(1) - q'(1)/q(1) + (q'(1)/q(1))^2$, where $q(v)$ is the root of $g(v;q(v)) = 1$ near $q(1) = q_0$, provided that $\sigma^2 \neq 0$. Differentiating $g(v;q(v)) = 1$ twice at $v = 1$ and writing the derivatives of g as Boltzmann moments one obtains (7.1). It remains to check that the variance is positive. Actually, this is clear, since $R - \mu N$ is not almost surely constant. Indeed, the primitive paths UD, UHD and $UUDD$ have $(N, R) = (1,1), (2,1), (4,1)$ for returns, $(1,0), (2,1), (4,0)$ for horizontal steps and $(1,1), (2,1), (4,2)$ for up steps, and in each case $R - \mu N$ takes at least two values. □

It is worth mentioning that the mean number of returns per unit area coincides with the prefactor C of Theorem 6.1, and that the mean length of a path of area n is $(2\mu_Z + \mu_Y)n + O(1) \approx 1.228\, n$.[7]

Corollary 7.2. *Let $\mathbf{W}_n = (X_n, Y_n, Z_n)^T$ and $\boldsymbol{\mu} = (\mu_X, \mu_Y, \mu_Z)^T$. For a Boltzmann primitive path, write $\mathbf{R} = (1, \#H, \#U)^T$. Then*

$$\frac{\mathbf{W}_n - n\boldsymbol{\mu}}{\sqrt{n}} \xrightarrow{d} \mathcal{N}_3(0, \Sigma), \qquad \mathrm{Cov}(\mathbf{W}_n) = n\Sigma + O(1),$$

[7] Exact moments at $n = 300$, divided by n, give $(0.346502, 0.342381)$, $(0.364715, 0.291206)$ and $(0.434149, 0.224821)$ for the (mean, variance) of X_n, Y_n, Z_n, in agreement with the $O(1/n)$ corrections.

where the matrix

$$\Sigma = \frac{\mathbb{E}\big[(\mathbf{R} - \boldsymbol{\mu} N)(\mathbf{R} - \boldsymbol{\mu} N)^T\big]}{\mathbb{E}N} \tag{7.2}$$

is positive definite.

Proof. Mark the three statistics by $\mathbf{v} = (v_1, v_2, v_3)$ in the primitive series $g(\mathbf{v}; q)$. Each component of $\mathbf{R}$ is at most N, so joint analyticity follows by choosing $r > q_0$ and $\varepsilon > 0$ with $(1+\varepsilon)^3 r < \rho$. The simple pole persists at an analytic root $q(\mathbf{v})$ near q_0, with uniform coefficient asymptotics. Consequently, for $\mathbf{s}$ near zero,

$$\mathbb{E}e^{\mathbf{s}^T \mathbf{W}_n} = \frac{c(e^{\mathbf{s}})}{C} \left(\frac{q_0}{q(e^{\mathbf{s}})} \right)^n \big(1 + O(\theta^n)\big),$$

where c is analytic, $c(\mathbf{1}) = C$, and $\theta < 1$. Differentiating $g(e^{\mathbf{s}}; q(e^{\mathbf{s}})) = 1$ gives the gradient $\boldsymbol{\mu}$ and the Hessian (7.2) of $\log(q_0/q(e^{\mathbf{s}}))$ at zero. The Taylor expansion at $\mathbf{s} = i\mathbf{t}/\sqrt{n}$ gives the joint Gaussian limit; differentiation gives the covariance estimate.

To check nondegeneracy, suppose $a + b\#H + c\#U = dN$ for every primitive path. Applying this relation to UD, UHD, $UUDD$ and $UUHDD$ gives

$$a + c = d, \qquad a + b + c = 2d, \qquad a + 2c = 4d, \qquad a + b + 2c = 6d.$$

These equations imply $a = b = c = d = 0$. Thus no nonzero linear combination of $\mathbf{R} - \boldsymbol{\mu} N$ has zero variance, and Σ is positive definite. □

7.2. **The height.** Let us now turn to the height H_n, which is not additive and behaves quite differently. The idea is the following. The height of a path is the maximum of the heights of its components, and a primitive component of height larger than k is very rare under the Boltzmann distribution, since the smallest such path is the pyramid $U^{k+1}D^{k+1}$ of area $(k+1)^2$. Therefore the height of a path of area n is determined by the balance between the number of components, which is of order n, and this small probability. To make this precise, for $k \geq 1$ let $g^{[k]}$ be the generating function of primitive paths of height at most k, and let $q_0^{(k)}$ be the positive root of $g^{[k]} = 1$ within its disc of convergence. We set $\lambda_k = \log(q_0^{(k)}/q_0)$ and $\tau_k = 1 - g^{[k]}(q_0)$, so that τ_k is the Boltzmann probability that a primitive path has height greater than k. The following theorem gives an exact formula for τ_k and the resulting law of the height.

Theorem 7.3. *Let $M_h = A(q^{h+1})/A(q^h)$ and let $M_h^{(k)}$ be its truncation at height k. At $x = y = 1$ one has, for $k \geq 1$,*

$$\tau_k = q_0^{(k+1)^2} M_{k+1}(q_0) \prod_{h=1}^{k} M_h(q_0) M_h^{(k)}(q_0) \sim \frac{q_0^{(k+1)^2}}{A(q_0)^2}, \tag{7.3}$$

and

$$\lambda_k = C\tau_k\big(1 + O(k^2\tau_k)\big).$$

There exist k_0, $\theta < 1$ and constants $c_k \to C$ such that, uniformly in $k \geq k_0$,

$$\mathbb{P}(H_n \leq k) = \frac{c_k}{C} e^{-\lambda_k n} + O(\theta^n).$$

Consequently, with $\Lambda = C/A(q_0)^2 = 3.9586573854\ldots$,

$$\sup_{k \in \mathbb{Z}_{\geq 0}} \left| \mathbb{P}(H_n \leq k) - \exp\Big(-\Lambda n q_0^{(k+1)^2}\Big) \right| \longrightarrow 0 \qquad (n \to \infty),$$

and

$$H_n = \sqrt{\frac{\log n}{\log(1/q_0)}} + O_{\mathbb{P}}(1).$$

The proof, which is based on an exact telescoping formula for τ_k, is somewhat technical and is given in Appendix B.

TABLE 1. Height probabilities for a random Riordan path of area n, from exact bounded-height counts.

n	$\mathbb{P}(H_n \le 2)$	$\mathbb{P}(H_n \le 3)$	$\mathbb{P}(H_n > 4)$
300	0.5451536770	0.9976892919	$1.598915753 \cdot 10^{-6}$
600	0.2919739476	0.9952539517	$3.345151784 \cdot 10^{-6}$
1500	0.0448558995	0.9879835406	$8.583841582 \cdot 10^{-6}$

Corollary 7.4. *Put*

$$b_n = \sqrt{\frac{\log(\Lambda n)}{\log(1/q_0)}}, \qquad m_n = \left\lfloor b_n + \tfrac{1}{2} \right\rfloor .$$

Then $\mathbb{P}\big(H_n \in \{m_n - 1, m_n\}\big) \to 1$ *as* $n \to \infty$.

Proof. Write $a = \log(1/q_0) > 0$. Since $|m_n - b_n| \le \frac{1}{2}$,

$$b_n^2 - (m_n - 1)^2 \ge b_n - \tfrac{1}{4}, \qquad (m_n + 1)^2 - b_n^2 \ge b_n + \tfrac{1}{4}$$

for all sufficiently large n. In the approximation of Theorem 7.3, the exponent $\Lambda n q_0^{(k+1)^2} = \exp\{a[b_n^2 - (k+1)^2]\}$ therefore tends to infinity at $k = m_n - 2$ and to zero at $k = m_n$. Hence $\mathbb{P}(H_n \le m_n - 2) \to 0$ and $\mathbb{P}(H_n \le m_n) \to 1$. □

Remark 7.5. Let us stress that the double-exponential form in Theorem 7.3 describes the distribution at integer thresholds only and should not be interpreted as a continuous Gumbel limit. Indeed, $(H_n + 1)^2 - L_n$ with $L_n = \log(\Lambda n)/\log(1/q_0)$ is not tight: choosing integers n_m with $L_{n_m} = m^2 + m + \frac{1}{2} + o(1)$, every integer square is at distance at least $m + \frac{1}{2} - o(1)$ from L_{n_m}.

Numerically, one finds $\lambda_k = 0.1185211,\ 2.08134 \cdot 10^{-3},\ 8.14655 \cdot 10^{-6}$ and $5.82080 \cdot 10^{-9}$ for $k = 1, \dots, 4$, and exact bounded-height counts give the probabilities in Table 1.

At area 1500, the probability of height exactly three is about 0.9431. The slow growth and two-height concentration differ sharply from the square-root height scale of uniformly chosen Motzkin paths of fixed length.

8. Concluding remarks

In this work, we have studied Riordan paths enumerated by area. We have shown that the enumeration is governed by a single auxiliary q-series. Its continued fraction representation gives the generating functions of paths of bounded height, while its expansion over partitions into distinct parts yields a linear recurrence for the area counts. The decomposition into primitive paths determines the dominant pole and the joint Gaussian law for additive statistics. It also gives the tail of the height distribution, whose leading term is fixed by the area $(k+1)^2$ of the smallest path reaching height $k+1$. The uniform estimate implies concentration on two adjacent heights.

The auxiliary series admits an algebraic representation as the motivic generating series of the symmetric quiver $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, after the substitution $x = \alpha\beta$, $y = \alpha + \beta$, and the Dyck specialization admits a one-node representation. These identities allow a comparison with the quiver generating series appearing in knot theory, but we do not identify a knot invariant for Riordan paths; it seems that this would require additional structure beyond the identities obtained here.

The path decomposition and the continued fraction extend to horizontal steps $H_\ell = (\ell, 0)$ with integer $\ell \ge 1$, forbidden on the axis; the case $\ell = 2$ gives Schröder-type paths. Such a step at height h has area ℓh. Writing $F_\ell^{\uparrow} = F_\ell(xq^2, yq^\ell; q)$, the functional equation takes the form

$$F_\ell - 1 = xqF_\ell^{\uparrow} F_\ell + yq^\ell F_\ell^{\uparrow}(F_\ell - 1),$$

and the continued fraction has $1-yq^{\ell h}$ in place of $1-yq^h$. The corresponding auxiliary function satisfies

$$A_\ell(z) = (1-yz^\ell)A_\ell(qz) - xqz^2A_\ell(q^2z), \qquad A_\ell(0)=1.$$

For general ℓ the q-Hermite and quiver representations require a separate analysis. At $x=y=1$ the growth constants for $\ell=1,2,3,4$ are 2.2516612, 1.8656543, 1.7182811 and 1.6413284; they decrease to the Dyck value 1.5085198…, since increasing ℓ decreases the primitive weights and the simple-root argument gives the convergence of the poles.

Several questions remain. The signed partition recurrence in Theorem 6.3 suggests a direct sign-reversing involution on suitable pairs of paths and partitions. For the additive statistics, local limit laws and quantitative error estimates would refine Corollary 7.2. Step sets with several jump sizes may require more general continued fractions. It also remains to understand whether the two-node quiver representation has a meaning beyond the algebraic identity established here.

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Appendix A. The polynomials $S_6,\dots,S_{10}$

Here we list the polynomials $S_6,\dots,S_{10}$; terms are grouped by decreasing total degree.

$$\begin{aligned}
S_6 &= x^6+5x^5y+10x^4y^2+10x^3y^3+5x^2y^4+xy^5+3x^4+6x^3y+3x^2y^2+x^2y\\
S_7 &= x^7+6x^6y+15x^5y^2+20x^4y^3+15x^3y^4+6x^2y^5+xy^6\\
&\quad +4x^5+12x^4y+12x^3y^2+4x^2y^3+2x^3y+2x^2y^2+x^3\\
S_8 &= x^8+7x^7y+21x^6y^2+35x^5y^3+35x^4y^4+21x^3y^5+7x^2y^6+xy^7\\
&\quad +5x^6+20x^5y+30x^4y^2+20x^3y^3+5x^2y^4+3x^4y+6x^3y^2+3x^2y^3+3x^4+3x^3y+x^2y^2\\
S_9 &= x^9+8x^8y+28x^7y^2+56x^6y^3+70x^5y^4+56x^4y^5+28x^3y^6+8x^2y^7+xy^8\\
&\quad +6x^7+30x^6y+60x^5y^2+60x^4y^3+30x^3y^4+6x^2y^5\\
&\quad +4x^5y+12x^4y^2+12x^3y^3+4x^2y^4+6x^5+12x^4y+8x^3y^2+2x^2y^3+2x^3y+x^3\\
S_{10} &= x^{10}+9x^9y+36x^8y^2+84x^7y^3+126x^6y^4+126x^5y^5+84x^4y^6+36x^3y^7+9x^2y^8+xy^9\\
&\quad +7x^8+42x^7y+105x^6y^2+140x^5y^3+105x^4y^4+42x^3y^5+7x^2y^6\\
&\quad +5x^6y+20x^5y^2+30x^4y^3+20x^3y^4+5x^2y^5\\
&\quad +10x^6+30x^5y+33x^4y^2+16x^3y^3+3x^2y^4+6x^4y+6x^3y^2+x^2y^3+3x^4+2x^3y
\end{aligned}$$

The numbers of paths of area n with $r=1,2,\dots$ returns are, for $n=1,\dots,8$: 1; 1, 1; 1, 2, 1; 2, 3, 3, 1; 3, 6, 6, 4, 1; 5, 11, 13, 10, 5, 1; 8, 20, 27, 24, 15, 6, 1; 13, 36, 54, 55, 40, 21, 7, 1.

Appendix B. Proof of Theorem 7.3

Proof of Theorem 7.3. Let us first prove the formula for τ_k. To this end we write $T_k(q)=g(q)-g^{[k]}(q)$ and $\delta_h = M_h - M_h^{(k)}$. Subtracting the recursions for M_h and $M_h^{(k)}$, with $M_{k+1}^{(k)}=0$, one finds

$$\delta_h = q^{2h+1}M_hM_h^{(k)}\delta_{h+1}, \qquad \delta_{k+1}=M_{k+1}.$$

Since $g = qM_1$, iteration gives

$$T_k(q) = q^{(k+1)^2} B_k(q), \qquad B_k(q) = M_{k+1}(q) \prod_{h=1}^{k} M_h(q) M_h^{(k)}(q).$$

Note that the exponent $1 + \sum_{h=1}^{k}(2h+1) = (k+1)^2$ is precisely the area of the pyramid of height $k+1$. On a sufficiently small complex neighbourhood of q_0 the factors are analytic and nonzero. Moreover, for some $r < 1$ one has $M_h = 1 + O(r^h)$ and $M_h^{(k)} = 1 + O(r^h)$ uniformly in $k \geq h$, and for each fixed h the latter converges locally uniformly to M_h. These summable bounds justify the convergence of the products and of their derivatives. Since

$$\prod_{h=1}^{\infty} M_h(q) = \lim_{H\to\infty} \frac{A(q^{H+1})}{A(q)} = \frac{1}{A(q)},$$

one obtains $B_k(q) \to A(q)^{-2}$ locally uniformly. Evaluating at q_0 one gets (7.3), and logarithmic differentiation gives

$$T_k'(q_0) = \tau_k \left(\frac{(k+1)^2}{q_0} + O(1) \right). \tag{B.1}$$

Next, let us compare the roots $q_0^{(k)}$ and q_0. The coefficients of $g^{[k]}$ increase to those of g, which gives uniform convergence on every closed disc strictly inside $|q| < \rho$. Let us choose $r \in (q_0, \rho)$ such that q_0 is the only zero of $1 - g$ in $|q| \leq r$. By Rouché's theorem, for all sufficiently large k the function $1 - g^{[k]}$ has exactly one zero in this disc. It is the positive simple root $q_0^{(k)}$, and $q_0^{(k)} \to q_0$. Put $\Delta_k = q_0^{(k)} - q_0$. The derivatives $(g^{[k]})'$ are uniformly bounded below near q_0 on the real axis, and the second derivatives are uniformly bounded there. The mean value theorem gives $\Delta_k = O(\tau_k)$. Taylor expansion and (B.1) then yield

$$\tau_k = (g^{[k]})'(q_0)\Delta_k + O(\Delta_k^2), \qquad (g^{[k]})'(q_0) = g'(q_0) + O(k^2\tau_k).$$

Thus $\Delta_k = \tau_k / g'(q_0)\,(1 + O(k^2\tau_k))$, and expanding $\log(1 + \Delta_k/q_0)$ one obtains the estimate for λ_k.

The uniform estimate for the probability is obtained in the same way as in Theorem 6.1. On $|q| = r$, $|1 - g^{[k]}(q)|$ is bounded below uniformly in large k. Subtracting the simple pole and applying Cauchy's formula one finds

$$r_n^{[k]} = c_k (q_0^{(k)})^{-n} + O(r^{-n}), \qquad c_k = \frac{1}{q_0^{(k)} (g^{[k]})'(q_0^{(k)})} \longrightarrow C,$$

uniformly in k, and it remains to divide by the asymptotic formula for r_n.

Finally, let us prove the distributional approximation. Put $a_k = \Lambda q_0^{(k+1)^2}$. We have $\lambda_k/a_k \to 1$ and $c_k/C \to 1$. For each fixed finite set of k, both the probabilities and e^{-na_k} tend to zero; indeed $q_0^{(k)} > q_0$, and the bounded-height generating function is analytic in a disc of radius larger than q_0. For large k one uses the uniform pole estimate and

$$\sup_{t\geq 0} |e^{-(1+\epsilon)t} - e^{-t}| \longrightarrow 0 \quad \text{as } \epsilon \to 0.$$

Splitting the supremum into these two ranges one obtains the claim; the case $k = 0$ is immediate, since $\mathbb{P}(H_n \leq 0) = 0$ for $n \geq 1$. For the last statement, let $b_n = \sqrt{\log(\Lambda n)/\log(1/q_0)}$. At $k = \lfloor b_n \rfloor - 2$ the exponential approximation tends to zero, whereas at $k = \lfloor b_n \rfloor + 1$ it tends to one, which proves the $O_{\mathbb{P}}(1)$ bound. □

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