Robin's Inequality at Colossally Abundant Numbers and Nicolas' Inequality at Their Largest Prime Factor

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Abstract

Robin's criterion states that the Riemann Hypothesis holds if and only if $\sigma(n) < e^{\gamma} \cdot n \cdot \log\log n$ for every $n > 5040$, and Nicolas' criterion states that it holds if and only if $c(q) = \log(q\#/\varphi(q\#)) - \gamma - \log\log\theta(q) > 0$ for every prime q, where $q\#$ is the primorial of q and θ is the Chebyshev function. Let N be a colossally abundant number, let $\xi(N)$ be the real number associated with N by Nicolas, and let q be the largest prime not exceeding $\xi(N)$. Writing $\Delta(N) = \log(\sigma(N)/N) - \gamma - \log\log\log N$, we prove without any hypothesis that, whenever $\log q \geq 31$, $1.353/(\sqrt{q}\log^2 q) \leq \Delta(N) - c(q) + 2\sqrt{2}/(\sqrt{q}\log q) \leq 3.970/(\sqrt{q}\log^2 q)$. Thus Robin's inequality at N and Nicolas' inequality at q differ by $2\sqrt{2}/(\sqrt{q}\log q)$ up to an explicit error of smaller order. As consequences, every colossally abundant counterexample to Robin's inequality yields a prime q with $c(q) > 2.70/(\sqrt{q}\log q)$; conversely, every prime q with $\log q \geq 31$ and $c(q) > 2\sqrt{2}(1 - 0.4787/\log q)/(\sqrt{q}\log q)$ yields a colossally abundant counterexample, so that Nicolas' oscillation theorem implies Robin's theorem on colossally abundant counterexamples; and under the Riemann Hypothesis every colossally abundant number with $\log q \geq 31$ satisfies $\Delta(N) < -0.654/(\sqrt{q}\log q)$, in accordance with the constant $2(\sqrt{2} - 1)$ of Ramanujan's bound. For superabundant counterexamples, using the exponent bound of Nazardonyavi and Yakubovich, which gives $\log n \geq \psi(p_k)$ for the largest prime factor p_k, we prove $c(p_k) > 0.9991/(\sqrt{p_k}\log p_k)$, and we show that the least colossally abundant number not smaller than the least counterexample n is itself a counterexample N with $\log N - \log\xi(N) \leq \log n$.

Keywords: Riemann Hypothesis, Robin's inequality, Nicolas' inequality, Colossally abundant numbers, Superabundant numbers, Primorial, Chebyshev function
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1. Introduction

The Riemann Hypothesis (RH) is the conjecture that the Riemann zeta function has its zeros only at the negative even integers and at complex numbers with real part $\frac{1}{2}$ [1]. It has been called the "Holy Grail of Mathematics" [2, 3], and numerous equivalent formulations exist [4]. One of particular interest here is Robin's criterion [5], which asserts that RH is equivalent to the inequality

$$\sigma(n) < e^{\gamma} \cdot n \cdot \log\log n \qquad \text{for every integer } n > 5040. \tag{1}$$

Here $\gamma \approx 0.57721$ is the Euler–Mascheroni constant, $\sigma(n) = \sum_{d|n} d$ is the sum-of-divisors function, and log denotes the natural logarithm. Grönwall's theorem [6] states that $\limsup_{n\to\infty} \sigma(n)/(n\log\log n) = e^{\gamma}$, so (1) asserts that this limiting value is never attained beyond 5040. A closely related elementary reformulation in terms of harmonic numbers is due to Lagarias [7], and the extremal integers relevant to both criteria have been studied computationally by Briggs [8] and structurally by Caveney, Nicolas and Sondow [9].

It is known that (1) holds for many classes of integers: odd integers exceeding 9 and squarefree integers exceeding 30 [10], sums of two squares exceeding 720 [11], and t-free integers for $t = 5, 7, 11$ and 20 [10, 12, 13, 14]. Akbary

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and Friggstad [15] showed that the least counterexample to (1), should one exist, must be a superabundant number, Nazardonyavi and Yakubovich [16] refined this to the smaller class of extremely abundant numbers, and the author proved that its largest prime factor exceeds $e^{31.018}$ [17]. Robin [5] proved that if RH is false, then (1) fails for infinitely many colossally abundant numbers.

A second criterion concerns the primorials $q\# = \prod_{p \le q} p$. For a real number $x \ge 2$ put

$$c(x) = \sum_{p \le x} \log \frac{p}{p-1} - \gamma - \log\log \theta(x),$$

where $\theta(x) = \sum_{p \le x} \log p$ is the Chebyshev function, so that $c(q) > 0$ is the inequality $\frac{q\#}{\varphi(q\#)} > e^{\gamma} \cdot \log\log(q\#)$, with φ Euler's totient function. Nicolas [18] proved that $c(x) > 0$ for every $x \ge 2$ if RH is true, and that if RH is false, then $c(x) = \Omega_{\pm}(x^{-b})$ for some $0 < b < \frac{1}{2}$. Under RH, Ramanujan's asymptotic upper bound for $\sigma(n)/n$ [19] contains the term $-\frac{2(\sqrt{2}-1)}{\sqrt{\log n}}$, and Nicolas [20] gave an effective form of this bound, using the structure of colossally abundant numbers and his estimate [21] of $c(x)$ under RH.

In this paper we compare Robin's inequality at a colossally abundant number with Nicolas' inequality at its largest prime factor, without assuming RH. For an integer $n \ge 16$ put

$$\Delta(n) = \log \frac{\sigma(n)}{n} - \gamma - \log\log\log n,$$

so that Robin's inequality holds for $n > 5040$ if and only if $\Delta(n) < 0$. To each colossally abundant number N Nicolas associates a real number $\xi(N)$ [20] [Definition 4.3]; every prime factor of N is at most $\xi(N)$ (Section 3). Our main result is the following.

Theorem 1.1. *Let N be a colossally abundant number, and let q be the largest prime with $q \le \xi(N)$. If $\log q \ge 31$, then*

$$\frac{1.353}{\sqrt{q} \cdot \log^2 q} \le \Delta(N) - c(q) + \frac{2\sqrt{2}}{\sqrt{q} \cdot \log q} \le \frac{3.970}{\sqrt{q} \cdot \log^2 q}.$$

Thus Robin's function at N and Nicolas' function at q differ by $\frac{2\sqrt{2}}{\sqrt{q}\log q}$, up to an explicit error that is smaller by a factor $\log q$. Remark 3.5 describes the relation of Theorem 1.1 to the work of Robin and Nicolas, and Remark 3.6 the size of its error term. Theorem 1.1 has the following consequences.

(1) Every colossally abundant counterexample to (1) yields a prime q with $\log q > 31.03$ and $c(q) > \frac{2.70}{\sqrt{q}\log q}$ (Corollary 4.2).

(2) Conversely, every prime q with $\log q \ge 31$ and $c(q) > \frac{2\sqrt{2}}{\sqrt{q}\log q}\left(1 - \frac{0.4787}{\log q}\right)$ yields a colossally abundant counterexample to (1) (Corollary 4.3).

(3) Nicolas' theorem $c(x) = \Omega_{+}(x^{-b})$ under the failure of RH implies, through (2), Robin's theorem that (1) then fails for infinitely many colossally abundant numbers (Corollary 4.4).

(4) Under RH, every colossally abundant number N with $\log q \ge 31$ satisfies $\Delta(N) < -\frac{0.654}{\sqrt{q}\log q}$ (Corollary 4.6); the constant $2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$ obtained from Theorem 1.1 and the main term of $c(q)$ is the constant of Ramanujan's bound.

(5) RH holds if and only if $c(q) < \frac{2.70}{\sqrt{q}\log q}$ for every prime $q \ge 10^9$ (Corollary 4.7). This equivalence also follows from Nicolas' theorem [18] together with his estimate under RH [21, 20], and holds with 2.70 replaced by any constant greater than $2 + \tau = 2.0461\ldots$; Corollary 4.7 shows that the primes q of item (1) violate it.

For superabundant counterexamples, which need not be colossally abundant, we prove the following (Section 5).

(6) Every superabundant number n with largest prime factor p_k satisfies $\log n \ge \psi(p_k)$ (Lemma 5.1, following [16]), and every superabundant counterexample satisfies $c(p_k) > \frac{0.9991}{\sqrt{p_k}\log p_k}$ (Theorem 5.4).

(7) If n is the least counterexample, then the least colossally abundant number $N \geq n$ is a counterexample, and $\log N - \log \xi(N) \leq \log n \leq \log N$ (Corollary 5.5).

The proofs use the description of colossally abundant numbers by Alaoglu and Erdős [22], Robin [5] and Nicolas [20], explicit estimates for the Chebyshev function of Rosser and Schoenfeld [23], Büthe [24] and Dusart [25], the estimate of Axler [26] for the Mertens product, and the numerical verification of (1) for all $5040 < n \leq 10^{10^{13.11485}}$ due to Morrill and Platt [14]. Section 2 collects these results, Section 3 proves Theorem 1.1, Section 4 derives its consequences, Section 5 treats superabundant counterexamples, and Section 6 describes the computer verification of the numerical constants.

2. Background

We write $I(n) = \frac{\sigma(n)}{n}$ for the abundancy index, and we say that *Robin's inequality holds for n* if $I(n) < e^{\gamma} \cdot \log\log n$. An integer $n > 5040$ for which Robin's inequality does not hold is called a *counterexample*. We write $\psi(x) = \sum_{p^j \leq x} \log p = \sum_{j \geq 1} \theta(x^{1/j})$ for the second Chebyshev function, so that $\psi(x) = \sum_{q \leq x} \left\lfloor \frac{\log x}{\log q} \right\rfloor \log q$ and $\psi(x) \geq \theta(x) + \theta(\sqrt{x})$. For a prime p we write $p\# = \prod_{q \leq p} q$, so that $\log(p\#) = \theta(p)$, and $p_1 = 2 < p_2 = 3 < \cdots$ denotes the sequence of primes. Since $\sum_{p \leq x} \log \frac{p}{p-1}$ and $\theta(x)$ change only at primes, $c(x) = c(q)$ when q is the largest prime with $q \leq x$.

2.1. Robin's Criterion and Counterexamples

Proposition 2.1. *Robin's inequality holds for all natural numbers $n > 5040$ if and only if the Riemann Hypothesis is true [5] [Theorem 1 pp. 188].*

Proposition 2.2. *For $n > 1$, $I(n) < \prod_{q \mid n} \frac{q}{q-1} = \frac{n'}{\varphi(n')}$, where n' is the product of the distinct primes dividing n [10] [(2.7) pp. 362].*

Proposition 2.3. *Robin's inequality holds for every integer n with $5040 < n \leq 10^{10^{13.11485}}$ [14] [Theorem 5].*

A natural number n is *superabundant* if $I(m) < I(n)$ for every $m < n$, and *colossally abundant* if there is $\varepsilon > 0$ with $\frac{\sigma(m)}{m^{1+\varepsilon}} \leq \frac{\sigma(n)}{n^{1+\varepsilon}}$ for every $m \geq 1$ [22]; every colossally abundant number is superabundant [22].

Proposition 2.4. *If n is superabundant with largest prime factor p_k, then $n = \prod_{i=1}^{k} p_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ [22] [Theorem 1 pp. 450].*

Proposition 2.5. *If $n > 5040$ is the smallest integer such that Robin's inequality does not hold for n, then n is superabundant [15] [Theorem 3, p. 273].*

Proposition 2.6. *If the Riemann Hypothesis is false, then Robin's inequality fails for infinitely many colossally abundant numbers [5] [Section 4, Proposition 1].*

2.2. Nicolas' Criterion

Proposition 2.7. *If the Riemann Hypothesis is true, then $c(x) > 0$ for every $x \geq 2$. If the Riemann Hypothesis is false, then there is b with $0 < b < \frac{1}{2}$ such that $c(x) = \Omega_{\pm}(x^{-b})$; more precisely, this holds for every b with $1 - \Theta < b < \frac{1}{2}$, where Θ is the supremum of the real parts of the zeros of the Riemann zeta function [18] [Theorem 3 (b), (c)], see also [21].*

In [18, 21] the statement is given for $\log f(x) = -c(x)$, where $f(x) = e^{\gamma} \cdot \log\theta(x) \cdot \prod_{p \leq x} \left(1 - \frac{1}{p}\right)$. Put

$$\tau = 2 + \gamma - \log(4\pi) = 0.0461914\ldots, \qquad S_1(x) = \sum_{\rho} \frac{x^{\rho-1}}{\rho(1-\rho)},$$

where ρ runs over the nontrivial zeros of the Riemann zeta function [20] [(1.3), (1.5)].

Proposition 2.8. *Assume the Riemann Hypothesis. Then* $|S_1(x)| \le \frac{\tau}{\sqrt{x}}$ *for* $x > 1$ *[20] [(1.4)], and for* $x \ge 10^9$ *[21] [(2.16)], [20] [(1.11)],*

$$\frac{0.055}{\sqrt{x}\log^2 x} \le -c(x) + \frac{2}{\sqrt{x}\log x} + \frac{S_1(x)}{\log x} \le \frac{2.062}{\sqrt{x}\log^2 x}.$$

Consequently $c(x) < \frac{2+\tau}{\sqrt{x}\log x} < \frac{2.0462}{\sqrt{x}\log x}$ *for every* $x \ge 10^9$.

Proof. The left inequality gives $c(x) \le \frac{2}{\sqrt{x}\log x} + \frac{S_1(x)}{\log x} - \frac{0.055}{\sqrt{x}\log^2 x} < \frac{2+\tau}{\sqrt{x}\log x}$, and $2 + \tau < 2.0462$. □

2.3. Explicit Estimates for Primes

Proposition 2.9. *The following hold.*

(a) $\theta(x) < x$ *for* $0 < x \le 10^{19}$, *and* $\theta(x) > x - 1.94\sqrt{x}$ *for* $1427 < x \le 10^{19}$ *[24] [Theorem 2], see also [20] [(2.1), (2.2)].*

(b) $|\theta(x) - x| < \frac{x}{\log^3 x}$ *for* $x \ge 89{,}967{,}803$ *[25] [Theorem 4.2].*

(c) For every $x \ge 89{,}693$ *there is a prime* p *with* $x < p \le x \cdot \left(1 + \frac{1}{\log^3 x}\right)$ *[25] [Proposition 5.4].*

(d) $\theta(x) \ge 0.945 \cdot x$ *for* $x \ge 853$ *[23] [Theorem 10], and* $\pi(x) < \frac{1.26 \cdot x}{\log x}$ *for* $x > 1$ *[23] [(3.6)], where* $\pi(x)$ *is the number of primes* $p \le x$.

(e) $\sum_{p \ge x} \frac{1}{p^3} \le \frac{0.086}{x^2}$ *for* $x \ge 1418$ *[20] [(2.15)].*

(f) For $x \ge 46{,}909{,}038$ *[26] [Proposition 6.1],*

$$\prod_{p \le x} \frac{p}{p-1} < e^{\gamma} \cdot \log x \cdot (1 - \varepsilon(\log x))^{-1}, \qquad \varepsilon(P) = \frac{1}{20P^3} + \frac{3}{16P^4}.$$

Lemma 2.10. *The following hold.*

(a) $\theta(t) < 1.000012 \cdot t$ *for every* $t > 0$.

(b) Let $\lambda \ge 31$. *Then* $|\theta(t) - t| \le \frac{0.022}{\lambda} \cdot t$ *for every* $t \ge 1.39838 \cdot e^{\lambda/2}$.

Proof. (a) For $t \le 10^{19}$ this is Proposition 2.9 (a); for $t > 10^{19}$, Proposition 2.9 (b) gives $\theta(t) < t \cdot \left(1 + \frac{1}{\log^3(10^{19})}\right) < 1.000012 \cdot t$.

(b) Let $t \ge 1.39838 \cdot e^{\lambda/2}$. If $t \le 10^{19}$, Proposition 2.9 (a) gives $|\theta(t) - t| < \frac{1.94}{\sqrt{t}} \cdot t$ and $\sqrt{t} \ge 1.18253 \cdot e^{\lambda/4}$, so $\lambda \cdot \frac{|\theta(t)-t|}{t} < \frac{1.94 \cdot \lambda \cdot e^{-\lambda/4}}{1.18253}$; the right-hand side is decreasing for $\lambda \ge 4$, and is less than 0.022 at $\lambda = 31$. If $t > 10^{19}$, Proposition 2.9 (b) gives $\lambda \cdot \frac{|\theta(t)-t|}{t} < \frac{\lambda}{\log^3 t}$ with $\log t > \max\left(\log(10^{19}), \frac{\lambda}{2}\right)$. If $\lambda \le 2\log(10^{19})$, this is at most $\frac{2\log(10^{19})}{\log^3(10^{19})} < 0.022$, and otherwise it is at most $\frac{8}{\lambda^2} < \frac{8}{31^2} < 0.022$. □

Lemma 2.11. *Let* $y > 1$ *and* $0 \le \delta < 1$ *be such that* $|\theta(t) - t| \le \delta t$ *for every* $t \ge y$. *Put* $A_j = \frac{1}{y \log^j y}$. *Then*

$$(1 - 3\delta) \cdot A_1 - A_2 \le \sum_{p > y} \frac{1}{p^2} \le (1 + 3\delta) \cdot A_1 - (1 + \delta) \cdot A_2 + 2(1 + \delta) \cdot A_3.$$

Proof. Put $J_j = \int_y^{\infty} \frac{dt}{t^2 \log^j t}$. Since $\frac{d}{dt}\left(-\frac{1}{t \log^j t}\right) = \frac{1}{t^2 \log^j t} + \frac{j}{t^2 \log^{j+1} t}$, we have $J_j = A_j - jJ_{j+1}$, hence $J_1 = A_1 - J_2$, $J_2 = A_2 - 2J_3$, and $0 < J_j \le A_j$. By Stieltjes integration,

$$\sum_{p > y} \frac{1}{p^2} = \int_y^{\infty} \frac{d\theta(t)}{t^2 \log t} = -\frac{\theta(y)}{y^2 \log y} + \int_y^{\infty} \theta(t) \cdot \frac{2\log t + 1}{t^3 \log^2 t}\, dt,$$

and $\int_y^{\infty} \frac{2\log t + 1}{t^2 \log^2 t}\, dt = 2J_1 + J_2 = 2A_1 - J_2$. Using $(1 - \delta)t \le \theta(t) \le (1 + \delta)t$ for $t \ge y$, the sum is at least $-(1 + \delta)A_1 + (1 - \delta)(2A_1 - A_2) \ge (1 - 3\delta)A_1 - A_2$, and at most $-(1 - \delta)A_1 + (1 + \delta)(2A_1 - A_2 + 2A_3)$, using $J_2 \ge A_2 - 2A_3$. □

3. Colossally Abundant Numbers

3.1. The Structure of Colossally Abundant Numbers

We recall the description given by Nicolas [20] [Section 4], which goes back to Alaoglu and Erdős [22] and Robin [5]. For real $t > 1$ and an integer $j \geq 1$ put

$$F(t, j) = \frac{\log\left(1 + \frac{1}{t+t^2+\cdots+t^j}\right)}{\log t},$$

which is decreasing in t. Let N be a colossally abundant number and κ its largest parameter. Then $\xi = \xi(N) > 1$ is defined by $F(\xi, 1) = \kappa$, and $\xi_j = \xi_j(N)$ for $j \geq 2$ by $F(\xi_j, j) = \kappa$, with $\xi_1 = \xi$; K is the largest j with $\xi_j \geq 2$ [20] [Definition 4.3]. Put $N_\kappa = \prod_{j=1}^{K} \prod_{\xi_{j+1} < p \leq \xi_j} p^j$.

Proposition 3.1. *With this notation, the following hold.*

(a) N_κ *is colossally abundant with largest parameter* κ*, and* N *divides* N_κ*; every prime dividing* N *is at most* ξ*, and a prime* $p \leq \xi$ *has exponent* j *in* N_κ *if* $\xi_{j+1} < p \leq \xi_j$ *[20] [(4.11), Proposition 4.2, Lemma 4.4], [22] [Theorem 10].*

(b) $\sum_{j=1}^{K} \theta(\xi_j) - \log \xi \leq \log N \leq \sum_{j=1}^{K} \theta(\xi_j)$*, and* $\frac{\xi}{\xi+1} \cdot I(N_\kappa) \leq I(N) \leq I(N_\kappa)$*, where*

$$I(N_\kappa) = \prod_{j=1}^{K} \prod_{\xi_{j+1} < p \leq \xi_j} \frac{1 - p^{-(j+1)}}{1 - p^{-1}}$$

[20] [Proposition 4.5].

(c) $\xi^{1/j} \leq \xi_j \leq (j\xi)^{1/j}$ *for every* $j \geq 2$ *[20] [(4.26)].*

(d) $\sqrt{2\xi} \cdot \left(1 - \frac{\log 2}{2\log \xi}\right) \leq \xi_2$ *for* $\xi \geq 1530$ *[5] [p. 190], [20] [(4.28)], and* $\xi_2 \leq \sqrt{2\xi} \cdot \left(1 - \frac{0.323}{\log \xi}\right)$ *for* $\xi \geq 10^9$ *[20] [(4.29)].*

(e) If $\xi > 10^9 + 7$*, then* $\sum_{j=3}^{K} \xi_j \leq 1.9769 \cdot \xi^{1/3}$ *[20] [Lemmas 4.8–4.10].*

(f) For every integer $m \geq 2$ *there are two consecutive colossally abundant numbers* $N' < N$ *with* $\frac{N}{\xi(N)} \leq N' < m \leq N$ *[20] [Proposition 4.6].*

(g) If $3 \leq N' < N$ *are consecutive colossally abundant numbers and* $N' \leq m \leq N$*, then*

$$\frac{I(m)}{\log\log m} \leq \max\left\{\frac{I(N')}{\log\log N'}, \frac{I(N)}{\log\log N}\right\}$$

[5] [Proposition 1], see also [16].

In (e), Nicolas states [20] [Lemma 4.10] for $N \geq N^{(0)}$, where $N^{(0)}$ is the colossally abundant number with $\xi(N^{(0)}) = 10^9 + 7$. If $\xi > 10^9 + 7$, then $\kappa < F(10^9 + 7, 1)$, and since the parameters of colossally abundant numbers decrease as the numbers increase [20] [(4.4)], we have $N > N^{(0)}$, so (e) applies.

3.2. Three Estimates

Throughout this subsection, N is a colossally abundant number with $\lambda = \log \xi \geq 31$, where $\xi = \xi(N)$, and we put

$$M_\xi = \frac{\sqrt{2}}{\sqrt{\xi} \cdot \lambda}, \qquad L(t) = \log\log t, \qquad S_2 = \sum_{\xi_2 < p \leq \xi} \frac{1}{p^2}.$$

By Proposition 3.1 (d), and since $1 - \frac{\log 2}{2 \cdot 31} > 0.98882$ and $\sqrt{2} \cdot 0.98882 > 1.39838$,

$$\xi_2^- := \sqrt{2\xi} \cdot \left(1 - \frac{\log 2}{2\lambda}\right) \leq \xi_2 \leq \xi_2^+ := \sqrt{2\xi} \cdot \left(1 - \frac{0.323}{\lambda}\right), \qquad \xi_2 \geq 1.39838 \cdot e^{\lambda/2}, \tag{2}$$

so Lemma 2.10 (b) applies to every $t \geq \xi_2$ with $\delta = \frac{0.022}{\lambda}$. Moreover $\log \xi_2 \leq \frac{1}{2}(\lambda + \log 2)$, and since $-\log\left(1 - \frac{\log 2}{2 \cdot 31}\right) < 0.011306$, also $\log \xi_2 \geq \frac{1}{2}(\lambda + \log 2) - 0.011306 \geq \frac{\lambda}{2} \cdot \left(1 + \frac{0.670535}{\lambda}\right)$.

Lemma 3.2. $M_\xi \cdot \left(1 - \frac{2.43616}{\lambda}\right) \le S_2 \le M_\xi \cdot \left(1 - \frac{1.920374}{\lambda}\right)$.

Proof. Put $\ell = \log \xi_2$ and $A_j = \frac{1}{\xi_2 \ell^j}$. Since $t \mapsto t \log t$ is increasing, the bounds on ξ_2 and ℓ give

$$A_1 \ge \frac{M_\xi}{\left(1 - \frac{0.323}{\lambda}\right)\left(1 + \frac{\log 2}{\lambda}\right)} \ge \frac{M_\xi}{1 + \frac{0.370148}{\lambda}} \ge M_\xi \cdot \left(1 - \frac{0.370148}{\lambda}\right)$$

and, since $\left(1 - \frac{\log 2}{2\lambda}\right)\left(1 + \frac{0.670535}{\lambda}\right) \ge 1 + \frac{0.316464}{\lambda}$ for $\lambda \ge 31$,

$$A_1 \le \frac{M_\xi}{1 + \frac{0.316464}{\lambda}} \le M_\xi \cdot \left(1 - \frac{0.316464}{\lambda} + \frac{0.316464^2}{\lambda^2}\right) \le M_\xi \cdot \left(1 - \frac{0.313233}{\lambda}\right).$$

Since $\frac{\lambda}{2} \le \ell \le \frac{\lambda}{2}\left(1 + \frac{\log 2}{\lambda}\right)$, we get $A_2 = \frac{A_1}{\ell} \le \frac{2M_\xi}{\lambda}$, $A_3 = \frac{A_2}{\ell} \le \frac{4M_\xi}{\lambda^2}$ and

$$A_2 \ge \frac{2M_\xi}{\lambda} \cdot \frac{1 - \frac{0.370148}{\lambda}}{1 + \frac{\log 2}{\lambda}} \ge \frac{2M_\xi}{\lambda} \cdot \left(1 - \frac{1.063296}{31}\right) > \frac{1.9314 \cdot M_\xi}{\lambda}.$$

By Lemma 2.11 with $y = \xi_2$ and $3\delta = \frac{0.066}{\lambda}$,

$$\sum_{p > \xi_2} \frac{1}{p^2} \ge \left(1 - \frac{0.066}{\lambda}\right)\left(1 - \frac{0.370148}{\lambda}\right) M_\xi - \frac{2M_\xi}{\lambda} \ge M_\xi \cdot \left(1 - \frac{2.436148}{\lambda}\right)$$

and

$$\sum_{p > \xi_2} \frac{1}{p^2} \le \left(1 + \frac{0.066}{\lambda}\right)\left(1 - \frac{0.313233}{\lambda}\right) M_\xi - \frac{1.9314 \cdot M_\xi}{\lambda} + \frac{8.006 \cdot M_\xi}{\lambda^2} \le M_\xi \cdot \left(1 - \frac{1.920374}{\lambda}\right),$$

where we used $2(1 + \delta) \cdot 4 < 8.006$ and $\frac{8.006}{\lambda} \le \frac{8.006}{31} < 0.258259$. By Lemma 2.11 with $y = \xi$, using $-(1 + \delta)A_2(\xi) + 2(1 + \delta)A_3(\xi) \le 0$,

$$0 \le \sum_{p > \xi} \frac{1}{p^2} \le \frac{1.0022}{\xi\lambda} = \frac{1.0022}{\sqrt{2} \cdot \sqrt{\xi}} \cdot M_\xi \le \frac{1.0022 \cdot 31 \cdot e^{-31/2}}{\sqrt{2}} \cdot \frac{M_\xi}{\lambda} < \frac{4.08 \cdot 10^{-6} \cdot M_\xi}{\lambda},$$

because $\lambda e^{-\lambda/2}$ is decreasing for $\lambda \ge 2$. Since $S_2 = \sum_{p > \xi_2} \frac{1}{p^2} - \sum_{p > \xi} \frac{1}{p^2}$, the claim follows. □

Lemma 3.3. $M_\xi \cdot \left(1 - \frac{0.369659}{\lambda}\right) \le L(\log N) - L(\theta(\xi)) \le M_\xi \cdot \left(1 - \frac{0.05272}{\lambda}\right)$.

Proof. Put $D = \log N - \theta(\xi)$. By Proposition 3.1 (c), $\xi_3 \ge \xi^{1/3} \ge e^{31/3} > 853$, so $K \ge 3$, and by Proposition 2.9 (d), $\theta(\xi_3) \ge 0.945 \cdot \xi^{1/3} > \lambda$, because $0.945 \cdot e^{s/3} > s$ for $s \ge 31$. Proposition 3.1 (b) and (e) and Lemma 2.10 then give

$$\theta(\xi_2) \le D \le \theta(\xi_2) + 1.000012 \cdot 1.9769 \cdot \xi^{1/3}.$$

By (2) and Lemma 2.10 (b),

$$\theta(\xi_2) \ge \left(1 - \frac{0.022}{\lambda}\right)\left(1 - \frac{\log 2}{2\lambda}\right)\sqrt{2\xi} \ge \left(1 - \frac{0.368574}{\lambda}\right)\sqrt{2\xi}$$

and

$$\theta(\xi_2) \le \left(1 + \frac{0.022}{\lambda}\right)\left(1 - \frac{0.323}{\lambda}\right)\sqrt{2\xi} \le \left(1 - \frac{0.301}{\lambda}\right)\sqrt{2\xi}.$$

Moreover

$$\frac{1.000012 \cdot 1.9769 \cdot \xi^{1/3}}{\sqrt{2\xi}} \le 1.397905 \cdot e^{-\lambda/6} \le \frac{1.397905 \cdot 31 \cdot e^{-31/6}}{\lambda} < \frac{0.24717}{\lambda},$$

because $\lambda e^{-\lambda/6}$ is decreasing for $\lambda \geq 6$. Hence

$$\left(1 - \frac{0.368574}{\lambda}\right)\sqrt{2\xi} \leq D \leq \left(1 - \frac{0.0538}{\lambda}\right)\sqrt{2\xi}.$$

By Proposition 2.9 (b), $\xi \cdot (1 - \lambda^{-3}) < \theta(\xi) < \xi \cdot (1 + \lambda^{-3})$. Hence $\log N = \theta(\xi) + D \leq \xi \cdot \left(1 + \lambda^{-3} + \sqrt{2}e^{-\lambda/2}\right) \leq \xi \cdot \left(1 + \frac{0.0010488}{\lambda}\right)$, so that $\log\log N \leq \lambda + \frac{0.0010488}{\lambda} \leq \lambda \cdot \left(1 + \frac{0.0000339}{\lambda}\right)$ and

$$\log N \cdot \log\log N \leq \xi\lambda \cdot \left(1 + \frac{0.001085}{\lambda}\right).$$

Also $\log\theta(\xi) \geq \lambda + \log(1 - \lambda^{-3}) \geq \lambda - 1.0001\lambda^{-3}$, so that

$$\theta(\xi) \cdot \log\theta(\xi) \geq \xi\lambda \cdot (1 - \lambda^{-3})(1 - 1.0001\lambda^{-4}) \geq \xi\lambda \cdot \left(1 - \frac{0.00108}{\lambda}\right).$$

Since $L'(t) = \frac{1}{t\log t}$ is decreasing for $t > 1$, the mean value theorem gives

$$\frac{D}{\log N \cdot \log\log N} \leq L(\log N) - L(\theta(\xi)) \leq \frac{D}{\theta(\xi) \cdot \log\theta(\xi)}.$$

With $\frac{\sqrt{2\xi}}{\xi\lambda} = M_\xi$, the lower bound is at least $M_\xi \cdot \frac{1-0.368574/\lambda}{1+0.001085/\lambda} \geq M_\xi \cdot \left(1 - \frac{0.369659}{\lambda}\right)$, and the upper bound is at most $M_\xi \cdot \frac{1-0.0538/\lambda}{1-0.00108/\lambda} \leq M_\xi \cdot \left(1 - \frac{0.05272}{\lambda}\right)$. □

Lemma 3.4. *Let q be the largest prime with $q \leq \xi$. Then*

$$\sum_{p\leq q} \log\frac{p}{p-1} - S_2 - \frac{1.0156 \cdot M_\xi}{\lambda} \leq \log I(N) \leq \sum_{p\leq q} \log\frac{p}{p-1} - S_2.$$

Proof. For $p \leq \xi$ let v_p be the exponent of p in N_κ. By Proposition 3.1 (a) and (b), the primes $p \leq \xi$ are the primes $p \leq q$, and

$$\log I(N_\kappa) = \sum_{p\leq q} \log\frac{p}{p-1} + \sum_{p\leq q} \log\left(1 - p^{-(v_p+1)}\right).$$

Every term of the second sum is negative, and $v_p = 1$ for $\xi_2 < p \leq \xi$; since $\log(1-u) \leq -u$, the second sum is at most $-S_2$, and the upper bound follows from $I(N) \leq I(N_\kappa)$.

For the lower bound we split the second sum into three parts. For $\xi_2 < p \leq \xi$, $-\log(1 - p^{-2}) \leq \frac{p^{-2}}{1-\xi_2^{-2}}$, so this part is at least $-S_2 - 2\xi_2^{-2}S_2$. For $\xi_3 < p \leq \xi_2$, $v_p = 2$, and by Proposition 2.9 (e), since $\xi_3 > 1418$, this part is at least $-\frac{1}{1-\xi_3^{-3}} \cdot \frac{0.086}{\xi_3^2} \geq -0.0861 \cdot \xi^{-2/3}$. For $p \leq \xi_3$, if $\xi_{j+1} < p \leq \xi_j$ with $j \geq 3$, then $p^{j+1} > \xi_{j+1}^{j+1} \geq \xi$ by Proposition 3.1 (c), so $-\log(1 - p^{-(j+1)}) \leq \frac{1}{\xi-1}$; by Proposition 2.9 (d) and Proposition 3.1 (c), the number of such primes is at most $\pi\left((3\xi)^{1/3}\right) < \frac{3\cdot 1.26\cdot 3^{1/3}\cdot\xi^{1/3}}{\lambda} < \frac{5.4518\cdot\xi^{1/3}}{\lambda}$, so this part is at least $-\frac{5.4518}{31} \cdot \frac{\xi^{1/3}}{\xi-1} \geq -0.17587 \cdot \xi^{-2/3}$. Together with $I(N) \geq \frac{\xi}{\xi+1} I(N_\kappa)$ and $\log\left(1 + \frac{1}{\xi}\right) \leq \frac{1}{\xi} \leq e^{-31/3} \cdot \xi^{-2/3}$,

$$\log I(N) \geq \sum_{p\leq q} \log\frac{p}{p-1} - S_2 - 2\xi_2^{-2}S_2 - 0.26201 \cdot \xi^{-2/3}.$$

Finally $\xi^{-2/3} = \frac{\lambda^2 e^{-\lambda/6}}{\sqrt{2}} \cdot \frac{M_\xi}{\lambda}$, and $\lambda^2 e^{-\lambda/6}$ is decreasing for $\lambda \geq 12$, so $0.26201 \cdot \xi^{-2/3} \leq \frac{0.26201\cdot 31^2\cdot e^{-31/6}}{\sqrt{2}} \cdot \frac{M_\xi}{\lambda} < \frac{1.0155\cdot M_\xi}{\lambda}$, while $2\xi_2^{-2}S_2 \leq 2\xi_2^{-2}M_\xi < \frac{10^{-4}\cdot M_\xi}{\lambda}$ by Lemma 3.2 and (2). □

3.3. Proof of Theorem 1.1

Proof of Theorem 1.1. Put $\xi = \xi(N)$, $\lambda = \log \xi$ and $u = \log q \geq 31$, so that $\lambda \geq u \geq 31$. Since $q \leq \xi$ and the primes $p \leq \xi$ are the primes $p \leq q$, we have $\theta(\xi) = \theta(q)$ and

$$\Delta(N) - c(q) = \left(\log I(N) - \sum_{p \leq q} \log \frac{p}{p-1} \right) - \left(L(\log N) - L(\theta(\xi)) \right).$$

By Lemmas 3.2, 3.3 and 3.4,

$$-M_\xi \cdot \left(2 - \frac{1.973094 - 1.0156}{\lambda}\right) \leq \Delta(N) - c(q) \leq -M_\xi \cdot \left(2 - \frac{0.369659 + 2.43616}{\lambda}\right),$$

that is, $-M_\xi \cdot \left(2 - \frac{0.957494}{\lambda}\right) \leq \Delta(N) - c(q) \leq -M_\xi \cdot \left(2 - \frac{2.805819}{\lambda}\right)$. Put $M_q = \frac{\sqrt{2}}{\sqrt{q} \cdot u}$. The prime following q exceeds ξ, so by Proposition 2.9 (c), $\xi < q \cdot (1 + u^{-3})$ and $\lambda < u + u^{-3}$. Hence $M_\xi \leq M_q$, $\frac{1}{\lambda} \geq \frac{1-u^{-4}}{u}$, and

$$\frac{M_\xi}{M_q} \geq \frac{1}{(1+u^{-3})^{1/2}(1+u^{-4})} \geq 1 - \frac{u^{-2}/2 + u^{-3}}{u} \geq 1 - \frac{0.000554}{u}.$$

Therefore

$$\Delta(N) - c(q) \geq -M_q \cdot \left(2 - \frac{0.957494 \cdot (1 - 31^{-4})}{u}\right) \geq -M_q \cdot \left(2 - \frac{0.9574}{u}\right)$$

and

$$\Delta(N) - c(q) \leq -M_q \cdot \left(1 - \frac{0.000554}{u}\right)\left(2 - \frac{2.805819}{u}\right) \leq -M_q \cdot \left(2 - \frac{2.8070}{u}\right).$$

Since $2M_q = \frac{2\sqrt{2}}{\sqrt{q}\log q}$, $0.9574 \cdot \sqrt{2} > 1.353$ and $2.8070 \cdot \sqrt{2} < 3.970$, the theorem follows. □

In the notation of the proof, Theorem 1.1 is the pair of inequalities

$$c(q) - \frac{2\sqrt{2}}{\sqrt{q}\log q}\left(1 - \frac{0.4787}{\log q}\right) \leq \Delta(N) \leq c(q) - \frac{2\sqrt{2}}{\sqrt{q}\log q}\left(1 - \frac{1.4035}{\log q}\right). \tag{3}$$

Remark 3.5. The decomposition of $\Delta(N) - c(q)$ used in the proof of Theorem 1.1 follows Nicolas' proof of [20] [Proposition 5.1], in which $\log(\sigma(N)/N)$ is written as $U = U_1 + U_2 + U_3 - U_4 - U_5$ [20] [(5.4)], with $-U_5 = \sum_{p \leq \xi} \log \frac{p}{p-1}$ the Mertens product; a comparison of $\sigma(N)/N$ with this product for colossally abundant numbers N also underlies Robin's proof of Proposition 2.6 [5] [Section 4]. In [20] [Proposition 5.1] the Mertens product is evaluated under RH by [21] [(2.16)], that is, by Proposition 2.8. In Theorem 1.1 it is kept in the form $c(q) + \gamma + \log\log\theta(q)$, so that no hypothesis on the zeros of the Riemann zeta function is used, and the remaining terms are bounded from above and from below with explicit constants for every colossally abundant number with $\log q \geq 31$.

Remark 3.6. The two bounds of Theorem 1.1 differ by $\frac{3.970-1.353}{\sqrt{q}\log^2 q}$. Consequently, Corollaries 4.2 and 4.3 below do not decide whether Robin's inequality holds at a colossally abundant number N when $c(q)\sqrt{q}\log q$ lies between $2\sqrt{2}\left(1 - \frac{1.4035}{\log q}\right)$ and $2\sqrt{2}\left(1 - \frac{0.4787}{\log q}\right)$. In the proof, the two sides differ by $(2.805819 - 0.957494) \cdot \frac{M_\xi}{\lambda} = 1.848325 \cdot \frac{M_\xi}{\lambda}$; of this, 1.0156 comes from Lemma 3.4 (the primes $p \leq \xi_2$, which have exponent at least 2), $2.43616 - 1.920374 = 0.515786$ from Lemma 3.2, and $0.369659 - 0.05272 = 0.316939$ from Lemma 3.3.

4. Consequences

Lemma 4.1. *Let n be a counterexample with largest prime factor p_k. Then $\log p_k > 31.0319$ and $\log\log n < 1.000002 \cdot \log p_k$.*

Proof. Put $L = \log\log n$ and $P = \log p_k$. By Proposition 2.3, $n > 10^{10^{13.11485}}$, so $L > 13.11485 \cdot \log 10 + \log\log 10 > 31.032$. By Proposition 2.2, $e^\gamma L \le I(n) < \prod_{p \le p_k} \frac{p}{p-1}$. If $p_k < 46{,}909{,}038$, then by Proposition 2.9 (f) at $x = 46{,}909{,}038$, this product is less than $e^\gamma \cdot 17.6638 \cdot 1.000011 < e^\gamma \cdot 31.032$, a contradiction. Hence $p_k \ge 46{,}909{,}038$, and Proposition 2.9 (f) gives $L < P \cdot (1 - \varepsilon(P))^{-1}$. Since ε is decreasing and $P \ge \log(46{,}909{,}038)$, $(1 - \varepsilon(P))^{-1} < 1.000011$, so $P > \frac{31.032}{1.000011} > 31.0316$; then $(1 - \varepsilon(P))^{-1} \le (1 - \varepsilon(31.0316))^{-1} < 1.000002$, so $L < 1.000002 \cdot P$ and $P > \frac{31.032}{1.000002} > 31.0319$. □

For the least counterexample, the author proved $\log p_k > 31.018$ [17] [Theorem 4.2]; Lemma 4.1 applies to every counterexample.

Corollary 4.2. *Let N be a colossally abundant counterexample, and q the largest prime with $q \le \xi(N)$. Then $\log q > 31.0319$ and*

$$c(q) > \frac{2\sqrt{2}}{\sqrt{q}\log q}\left(1 - \frac{1.4035}{\log q}\right) > \frac{2.70}{\sqrt{q}\log q}, \qquad \textit{that is,} \qquad \frac{q\#}{\varphi(q\#)} > e^\gamma \cdot \log\theta(q) \cdot \exp\left(\frac{2.70}{\sqrt{q}\log q}\right).$$

Proof. The largest prime factor p_k of N is at most $\xi(N)$ by Proposition 3.1 (a), so $p_k \le q$ and $\log q > 31.0319$ by Lemma 4.1. Since Robin's inequality fails for N, $\Delta(N) \ge 0$, and the upper bound in (3) gives the first inequality. The function $u \mapsto 2\sqrt{2}\left(1 - \frac{1.4035}{u}\right)$ is increasing, and its value at $u = 31.0319$ exceeds 2.70. □

Corollary 4.3. *Let q be a prime with $\log q \ge 31$ and*

$$c(q) > \frac{2\sqrt{2}}{\sqrt{q}\log q}\left(1 - \frac{0.4787}{\log q}\right).$$

Then every colossally abundant number N such that q is the largest prime with $q \le \xi(N)$ is a counterexample. In particular, the colossally abundant number $N_{F(q,1)}$ of Proposition 3.1 (a), for which $\xi = q$, is a counterexample.

Proof. The lower bound in (3) gives $\Delta(N) > 0$, that is, $I(N) > e^\gamma \log\log N$, and $N > 5040$ because q divides N_κ and $\log N \ge \theta(q) - \log q$ by Proposition 3.1 (b). For $\kappa = F(q,1)$, the number N_κ is colossally abundant with largest parameter κ by Proposition 3.1 (a), and $\xi(N_\kappa) = q$ because $F(\cdot, 1)$ is decreasing. □

By Corollaries 4.2 and 4.3, for colossally abundant numbers N with $\log q \ge 31$, Robin's inequality fails at N when $c(q)\sqrt{q}\log q > 2\sqrt{2}\left(1 - \frac{0.4787}{\log q}\right)$ and holds at N when $c(q)\sqrt{q}\log q < 2\sqrt{2}\left(1 - \frac{1.4035}{\log q}\right)$.

Corollary 4.4. *Proposition 2.7 and Corollary 4.3 imply Proposition 2.6: if the Riemann Hypothesis is false, then Robin's inequality fails for infinitely many colossally abundant numbers.*

Proof. If RH is false, Proposition 2.7 gives $b < \frac{1}{2}$ and $C > 0$ such that $c(x) > Cx^{-b}$ for arbitrarily large x. Let q be the largest prime with $q \le x$; then $c(q) = c(x)$ and $q > \frac{x}{2}$, so $c(q) > Cx^{-b} > \frac{4}{\sqrt{x}\log(x/2)} > \frac{2\sqrt{2}}{\sqrt{q}\log q}$ for all sufficiently large such x. Corollary 4.3 gives a colossally abundant counterexample $N_{F(q,1)}$ for each such q with $\log q \ge 31$, and these numbers are distinct because $\xi(N_{F(q,1)}) = q$. □

Remark 4.5. Corollary 4.4 recovers Proposition 2.6 of Robin [5], here obtained from Proposition 2.7. Corollary 4.6 below is an explicit inequality, for colossally abundant numbers with $\log q \ge 31$, of the type of Ramanujan's upper bound for $\sigma(n)/n$ under RH, of which [20] [Theorem 1.1] gives an effective form for every $n \ge 2$.

Corollary 4.6. *Assume the Riemann Hypothesis. Let N be a colossally abundant number, and q the largest prime with $q \le \xi(N)$. If $\log q \ge 31$, then*

$$\Delta(N) < -\frac{0.654}{\sqrt{q}\log q}, \qquad \textit{that is,} \qquad \frac{\sigma(N)}{N} < e^\gamma \cdot \log\log N \cdot \exp\left(-\frac{0.654}{\sqrt{q}\log q}\right).$$

Proof. By (3) and Proposition 2.8, since $q > 10^9$,

$$\Delta(N)\sqrt{q}\log q < 2 + \tau - 2\sqrt{2} + \frac{2\sqrt{2}\cdot 1.4035}{\log q} \le 2 + \tau - 2\sqrt{2} + \frac{2\sqrt{2}\cdot 1.4035}{31} < -0.654.$$

□

The number $2\sqrt{2} - 2 - \tau = 2(\sqrt{2} - 1) - \tau = 0.7822\ldots$ appearing in the proof is the constant obtained from Ramanujan's bound [19] together with $|S_1(x)| \le \frac{\tau}{\sqrt{x}}$, as in [20] [(1.8)]; here $\sqrt{q}\log q$ plays the role of $\sqrt{\log N}\cdot\log\log N$, since $\log N = q\cdot(1 + o(1))$ by the proof of Lemma 3.3 and Proposition 2.9 (c).

Corollary 4.7. *The Riemann Hypothesis is true if and only if*

$$\frac{q\#}{\varphi(q\#)} < e^{\gamma}\cdot\log\theta(q)\cdot\exp\left(\frac{2.70}{\sqrt{q}\log q}\right)$$

for every prime $q \ge 10^9$.

Proof. The inequality is $c(q) < \frac{2.70}{\sqrt{q}\log q}$. If RH is true, it holds for every prime $q \ge 10^9$ by Proposition 2.8. If RH is false, Proposition 2.6 gives a colossally abundant counterexample, and Corollary 4.2 gives a prime q with $\log q > 31.0319$, so $q > 10^9$, and $c(q) > \frac{2.70}{\sqrt{q}\log q}$. □

By Propositions 2.7 and 2.8, the equivalence of Corollary 4.7 also holds with 2.70 replaced by any constant greater than $2 + \tau$. Corollary 4.2 identifies primes that violate it: the primes q attached to colossally abundant counterexamples.

5. Superabundant Counterexamples

Lemma 5.1. *Let* $n = \prod_{i=1}^{k} p_i^{a_i}$ *be a superabundant number with largest prime factor* p_k. *Then* $a_i \ge \left\lfloor \frac{\log p_k}{\log p_i} \right\rfloor$ *for every* $1 \le i \le k$, *and consequently* $\log n \ge \psi(p_k)$.

Proof. The first claim is due to Nazardonyavi and Yakubovich [16]; we give the proof for completeness. For a prime p and an integer $b \ge 0$ put $g(p,b) = \sum_{j=0}^{b} p^{-j}$. For $i = k$ the claim is $a_k \ge 1$. Let $i < k$, write $q = p_i$ and $a = a_i \ge 1$, and suppose that $a \le \left\lfloor \frac{\log p_k}{\log q} \right\rfloor - 1$. Then $q^{a+1} \le p_k$, and since $q^{a+1} \ne p_k$, we have $q^{a+1} < p_k$. Put $m = \frac{n\cdot q^{a+1}}{p_k}$, which is an integer with $m < n$, and write $b = a_k \ge 1$. Since I is multiplicative,

$$\frac{I(n)}{I(m)} = \frac{g(q,a)}{g(q,2a+1)}\cdot\frac{g(p_k,b)}{g(p_k,b-1)} = \frac{1 + p_k^{-b}/g(p_k,b-1)}{1 + q^{-(a+1)}} \le \frac{1 + \frac{1}{p_k}}{1 + \frac{1}{q^{a+1}}} < 1,$$

where we used $g(q,2a+1) = g(q,a)\cdot(1 + q^{-(a+1)})$, $g(p_k,b-1) \ge 1$ and $q^{a+1} < p_k$. Thus $I(m) > I(n)$ with $m < n$, which contradicts the superabundance of n. Finally, by Proposition 2.4 the primes dividing n are exactly the primes $q \le p_k$, so

$$\log n = \sum_{q\le p_k} a_q \log q \ge \sum_{q\le p_k} \left\lfloor \frac{\log p_k}{\log q} \right\rfloor \log q = \psi(p_k),$$

where a_q denotes the exponent of q in n. □

Lemma 5.2. *Let* n *be a counterexample, and* n' *the product of the distinct primes dividing* n. *Then* $n' > 15$ *and*

$$\log\frac{n'}{\varphi(n')} - \gamma - \log\log\log n' > \log\log\log n - \log\log\log n' \ge 0.$$

Proof. By Proposition 2.2, and since Robin's inequality fails for n, $\frac{n'}{\varphi(n')} > I(n) \ge e^{\gamma}\cdot\log\log n$. By Proposition 2.3, $\log\log n > 31.032$, so $\frac{n'}{\varphi(n')} > 55$. Since $\frac{m}{\varphi(m)} \le 3$ for every $m \le 15$, we get $n' > 15$, hence $\log\log n' > 0$. Taking logarithms gives $\log\frac{n'}{\varphi(n')} > \gamma + \log\log\log n$, and subtracting $\log\log\log n'$ gives the first inequality. The second follows from $n \ge n'$. □

For a superabundant counterexample with largest prime factor p_k, Proposition 2.4 gives $n' = p_k\#$, so Lemma 5.2 gives

$$c(p_k) > \log\log\log n - \log\log\theta(p_k) \geq 0. \tag{4}$$

Lemma 5.3. *Let n be a superabundant counterexample with largest prime factor p_k. Then $\log\log n < \log p_k + \frac{0.0562}{\log^2 p_k}$ and $\log n \cdot \log\log n < 1.0000603 \cdot p_k \cdot \log p_k$.*

Proof. Put $P = \log p_k$ and $L = \log\log n$. By Lemma 4.1, $P > 31.0319$, and by Propositions 2.2 and 2.9 (f), $e^\gamma L \leq I(n) < e^\gamma P \cdot (1 - \varepsilon(P))^{-1}$. Hence

$$L - P < \frac{P \cdot \varepsilon(P)}{1 - \varepsilon(P)} = \frac{1}{P^2} \cdot \frac{\frac{1}{20} + \frac{3}{16P}}{1 - \varepsilon(P)} \leq \frac{1}{P^2} \cdot \frac{\frac{1}{20} + \frac{3}{16 \cdot 31.0319}}{1 - \varepsilon(31.0319)} < \frac{0.0562}{P^2}.$$

Therefore $\log n < p_k \cdot e^{0.0562/P^2}$ and $L < P \cdot \left(1 + \frac{0.0562}{P^3}\right)$, so that

$$\log n \cdot \log\log n < p_k \cdot P \cdot e^{\frac{0.0562}{31.03^2}} \cdot \left(1 + \frac{0.0562}{31.03^3}\right) < 1.0000603 \cdot p_k \cdot P.$$

□

Theorem 5.4. *Let n be a superabundant counterexample with largest prime factor p_k. Then*

$$c(p_k) > \log\log\psi(p_k) - \log\log\theta(p_k) > \frac{0.9991}{\sqrt{p_k} \cdot \log p_k}.$$

Proof. By (4) and Lemma 5.1, $c(p_k) > \log\log\log n - \log\log\theta(p_k) \geq \log\log\psi(p_k) - \log\log\theta(p_k)$. The function $t \mapsto \log\log t$ has the decreasing derivative $\frac{1}{t\log t}$ for $t > 1$, and $\theta(p_k) < \theta(p_k) + \theta(\sqrt{p_k}) \leq \psi(p_k) \leq \log n$. By the mean value theorem and Lemma 5.3,

$$\log\log\psi(p_k) - \log\log\theta(p_k) \geq \frac{\psi(p_k) - \theta(p_k)}{\psi(p_k) \cdot \log\psi(p_k)} \geq \frac{\theta(\sqrt{p_k})}{\log n \cdot \log\log n} > \frac{\theta(\sqrt{p_k})}{1.0000603 \cdot p_k \cdot \log p_k}.$$

By Lemma 4.1, $\sqrt{p_k} > e^{15.515} > 5.47 \cdot 10^6$, so Proposition 2.9 (a) gives $\theta(\sqrt{p_k}) > \left(1 - \frac{1.94}{\sqrt{5.47 \cdot 10^6}}\right)\sqrt{p_k} > 0.99917 \cdot \sqrt{p_k}$ if $\sqrt{p_k} \leq 10^{19}$, and Proposition 2.9 (b) gives $\theta(\sqrt{p_k}) > (1 - 0.000012)\sqrt{p_k}$ otherwise. Since $\frac{0.99917}{1.0000603} > 0.9991$, the second inequality follows. □

Corollary 5.5. *Suppose that the Riemann Hypothesis is false, and let n be the least counterexample. Let $N' < N$ be the consecutive colossally abundant numbers with $N' < n \leq N$. Then N is a counterexample,*

$$\log N - \log\xi(N) \leq \log N' < \log n \leq \log N,$$

and the largest prime $q \leq \xi(N)$ satisfies $c(q) > \frac{2.70}{\sqrt{q}\log q}$.

Proof. By Proposition 3.1 (f) applied to $m = n$, the consecutive colossally abundant numbers $N' < n \leq N$ satisfy $\frac{N}{\xi(N)} \leq N'$, which gives the displayed inequalities. The number 55440 is colossally abundant [20] [Figure 1], and $55440 < n$ by Proposition 2.3, so $N' \geq 55440 > 5040$. Since $N' < n$, Robin's inequality holds for N', that is, $\frac{I(N')}{\log\log N'} < e^\gamma$, whereas $\frac{I(n)}{\log\log n} \geq e^\gamma$. By Proposition 3.1 (g), $\frac{I(N)}{\log\log N} \geq \frac{I(n)}{\log\log n} \geq e^\gamma$, and $N \geq n > 5040$, so N is a counterexample. The last claim is Corollary 4.2. □

6. Experimental Verification

Every numerical inequality between explicit constants used in this paper was checked by a self-contained Python 3.12 script, `verify_robin_nicolas.py`, written with the help of Claude Opus 5.5 (Anthropic) and run by the author, as in the earlier version of this work. The script uses the arbitrary-precision library `mpmath` with 60 significant decimal digits, and exact rational arithmetic (Python's `fractions` module) for the inequalities that involve only decimal constants, some of which hold with equality. For each inequality it prints the computed value of its left-hand side. The script has three parts, and all 69 checks performed by the script passed: 14 in Part 1, 50 in Part 2 and 5 in Part 3. We now describe the output.

6.1. *Part 1: Counterexamples*

Part 1 checks the constants of Lemmas 4.1, 5.2 and 5.3 and of Theorem 5.4. The computed values are listed in Table 1. The starting point is $\log\log\left(10^{10^{13.11485}}\right) = 31.0320905\ldots$, the bound for $\log\log n$ coming from Proposition 2.3. At $x = 46{,}909{,}038$, where Proposition 2.9 (f) begins to apply, $\log x = 17.6637\ldots$ and $(1-\varepsilon(\log x))^{-1} = 1.0000109986\ldots$, which lies below 1.000011 by $1.4\cdot 10^{-9}$; this is the smallest margin in Part 1. The two steps of Lemma 4.1 give $\frac{31.032}{1.000011} = 31.03165\ldots$ and $\frac{31.032}{1.000002} = 31.031937\ldots$. In Lemma 5.3, the constant 0.0562 exceeds the computed value $0.0560422\ldots$, and the factor 1.0000603 exceeds the computed value $1.0000602505\ldots$ by $4.9\cdot 10^{-8}$.

Table 1: Values computed in Part 1 of the script.

Quantity	**Computed value**	**Bound used**
$\log\log\left(10^{10^{13.11485}}\right)$	$31.0320905521\ldots$	> 31.032
$\log(46{,}909{,}038)$	$17.6637209227\ldots$	< 17.6638
$(1-\varepsilon(\log 46{,}909{,}038))^{-1}$	$1.0000109986\ldots$	< 1.000011
$31.032/1.000011$	$31.0316586518\ldots$	> 31.0316
$(1-\varepsilon(31.0316))^{-1}$	$1.0000018754\ldots$	< 1.000002
$31.032/1.000002$	$31.0319379361\ldots$	> 31.0319
$\left(\frac{1}{20}+\frac{3}{16\cdot 31.0319}\right)/(1-\varepsilon(31.0319))$	$0.0560422746\ldots$	< 0.0562
$e^{0.0562/31.03^2}\left(1+\frac{0.0562}{31.03^3}\right)$	$1.0000602505\ldots$	< 1.0000603
$e^{15.515}$	$5471153.3376\ldots$	$> 5.47\cdot 10^6$
$1-1.94/\sqrt{5.47\cdot 10^6}$	$0.9991705159\ldots$	> 0.99917
$0.99917/1.0000603$	$0.9991097537\ldots$	> 0.9991
$e^{\gamma}\cdot 31.032$	$55.2702392751\ldots$	> 55
$\max_{m\le 15} m/\varphi(m)$	3	$= 3$

6.2. *Part 2: The Proof of Theorem 1.1*

Part 2 checks the constants of Lemmas 2.10, 3.2, 3.3 and 3.4 and of the proof of Theorem 1.1, all at the threshold $\lambda = 31$, where $e^{31} = 29048849665247.4\ldots$; each quantity checked at $\lambda = 31$ is monotone in λ in the direction stated in the proofs. Table 2 lists the main values. In Lemma 2.10 (b), the relative error $\frac{0.022}{\lambda}$ for $\theta(t)$ on $t \ge \xi_2$ comes from Büthe's bound, with computed value $0.0219063\ldots$ at $\lambda = 31$, whereas Dusart's bound contributes at most 0.00105. In Lemma 3.3, the contribution $1.397905\cdot 31\cdot e^{-31/6} = 0.2471636\ldots$ of the primes with exponent at least 3 reduces the coefficient 0.301 coming from ξ_2 to 0.0538. In Lemma 3.4, the primes with exponent 2 and those with exponent at least 3 contribute $0.262002530\ldots\cdot\xi^{-2/3}$, which is $1.0154801\ldots\cdot\frac{M_\xi}{\lambda}$ at $\lambda = 31$. Twelve of the checks in Part 2 are identities or inequalities between the decimal constants of the proofs, such as $0.066 + 0.370148 + 2 = 2.436148$ and $0.313233 - 0.066 + 1.9314 - 0.258259 = 1.920374$; they are verified in exact rational arithmetic. The smallest positive margins in Part 2 are those of $\frac{1}{2\cdot 31^2}+\frac{1}{31^3} = 0.000553858\ldots < 0.000554$, of $2\left(\frac{\log 2}{2}-0.011306\right) = 0.6705351805\ldots \ge 0.670535$, and of $2\left(1-\frac{0.370148+\log 2}{31}\right) = 1.9314003\ldots > 1.9314$, with differences $1.4\cdot 10^{-7}$, $1.8\cdot 10^{-7}$ and $3.1\cdot 10^{-7}$. The final constants of Theorem 1.1 are $0.9574\cdot\sqrt{2} = 1.3539680\ldots > 1.353$ and $2.8070\cdot\sqrt{2} = 3.9696974\ldots < 3.970$.

6.3. *Part 3: The Corollaries*

Part 3 checks the constants of Proposition 2.8 and Corollaries 4.2, 4.6 and 4.7. The script computes $\tau = 2+\gamma-\log(4\pi) = 0.0461914179\ldots$, so that $2+\tau = 2.0461914179\ldots < 2.0462$, and $2\sqrt{2}-2-\tau = 0.7822357068\ldots$, the constant discussed after Corollary 4.6. The constant of Corollary 4.2 is $2\sqrt{2}\left(1-\frac{1.4035}{31.0319}\right) = 2.7005040047\ldots > 2.70$, and the constant of Corollary 4.6 is $2+\tau-2\sqrt{2}+\frac{2\sqrt{2}\cdot 1.4035}{31} = -0.6541809497\ldots < -0.654$. The constant 2.70 of Corollary 4.7 exceeds the bound 2.0462 of Proposition 2.8 by 0.6538. Finally, $e^{31.0319} = 2.999\ldots\cdot 10^{13} > 10^9$, so the primes q of Corollary 4.2 lie in the range of Proposition 2.8.

Table 2: Values computed in Part 2 of the script, at $\lambda = 31$.

Quantity	Computed value	Bound used
$1 - \frac{\log 2}{62}$	$0.9888202067\ldots$	> 0.98882
$1.94 \cdot 31 \cdot e^{-31/4}/1.18253$	$0.0219062995\ldots$	< 0.022
$-\log\left(1 - \frac{\log 2}{62}\right)$	$0.0112427568\ldots$	< 0.011306
$0.670535 - \frac{\log 2}{2} - \frac{\log 2}{2} \cdot \frac{0.670535}{31}$	$0.3164649670\ldots$	> 0.316464
$0.316464 - 0.316464^2/31$	$0.3132333721\ldots$	> 0.313233
$2\left(1 - \frac{0.370148+\log 2}{31}\right)$	$1.9314003109\ldots$	> 1.9314
$1.0022 \cdot 31 \cdot e^{-31/2}/\sqrt{2}$	$4.0760229901\ldots \cdot 10^{-6}$	$< 4.08 \cdot 10^{-6}$
$1.397905 \cdot 31 \cdot e^{-31/6}$	$0.2471636095\ldots$	< 0.24717
$\frac{1}{31^2} + \sqrt{2} \cdot 31 \cdot e^{-31/2}$	$0.0010487169\ldots$	< 0.0010488
$0.0861 + 0.17587 + e^{-31/3}$	$0.2620025305\ldots$	< 0.26201
$0.26201 \cdot 31^2 \cdot e^{-31/6}/\sqrt{2}$	$1.0154801390\ldots$	< 1.0155
$\frac{1}{2 \cdot 31^2} + \frac{1}{31^3}$	$0.0005538585\ldots$	< 0.000554
$0.957494 \cdot (1 - 31^{-4})$	$0.9574929632\ldots$	> 0.9574
$0.9574 \cdot \sqrt{2}$	$1.3539680646\ldots$	> 1.353
$2.8070 \cdot \sqrt{2}$	$3.9696974696\ldots$	< 3.970

Statements and Declarations

Declaration of interests

The author declares no competing interests.

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Declaration of generative AI in scientific writing

The author used Claude Opus 5.5 to write the verification script `verify_robin_nicolas.py` of Section 6 and to generate parts of the text of this manuscript. The experiments were run by the author. After using this tool, the author reviewed and edited the code and the content as needed and takes full responsibility for the content of the publication.

Data availability

The Python script `verify_robin_nicolas.py` used in Section 6 is available in the preprint.

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