hexagon:2610.00700v1

Shared Randomness and Normalization in One-Use Loop Communication

Daniel A. Pyle
Independent Researcher

Research draft — 30 October 2026

Abstract

We study a finite-message communication model defined by contracting one backward binary symmetric channel with one fixed, finite-dimensional forward quantum channel and then normalizing the resulting weights. The devices may share a finite classical seed that is independent of the message *before normalization*. The seed branches are mixed at the level of unnormalized weights, followed by a single normalization for each message. For $M \geq 2$ messages, backward crossover probability $0 < p < 1/2$, and forward channel Φ, we derive the exact worst-message success probability

$$S_{\mathrm{raw}}(\Phi, p, M) = \frac{2(1-p)}{M - (M-2)(1-2p)\delta(\Phi)}, \qquad \delta(\Phi) = \max_{\rho,\sigma} \frac{1}{2}\|\Phi(\rho) - \Phi(\sigma)\|_1.$$

The quantity δ is the established trace-distance contraction coefficient, not a new channel invariant. A cyclic construction with M seed values attains the bound. Crucially, normalization can make the posterior seed distribution message-dependent: the theorem does not describe an average of separately normalized branches or require success for every seed. We give an exact example separating these conventions. Matched depolarizing-qubit and classical binary symmetric forward channels consequently have equal optima in this resource model. The result is a restricted one-use optimization, not a physical backward-communication protocol or a general asymptotic capacity theorem.

1 Introduction

In a normalized communication model, a protocol first assigns nonnegative weights to outcomes and then divides by their sum. This last step is nonlinear. Consequently, introducing shared randomness requires specifying whether random branches are mixed before or after normalization. The distinction affects both the attainable probabilities and the meaning of a message-independent seed.

This note gives an exact optimization for one particular convention. There is one stipulated backward binary symmetric channel, one fixed noisy forward carrier, and a shared classical seed. The seed is sampled independently of the externally selected message at the raw level; its branches are mixed before a single per-message normalization. The performance criterion is the minimum success probability over messages after the seed has been marginalized. The proof reduces this minimax problem to an aggregate ratio for a single seed, reduces the decoder to two deterministic

labels, and then equalizes the bound by cyclic relabeling. The forward carrier enters only through its maximum output trace distance.

The surrounding frameworks are established. Ji, Lloyd, and Wilde analyze retrocausal channel communication using normalized contractions and discuss noisy forward memory as a broader rate-tradeoff problem [1]. Ji, Regula, and Wilde study worst-message conditional error in postselected communication with assisted resources [2]. Neither the normalization operation nor conditional success is introduced here. The present task fixes the forward channel and restricts assistance to a finite shared classical seed; Section 6 explains the scope of the comparison. The claimed result is the explicit optimization in this stated model. No claim of comprehensive originality certification is made.

2 The model and the order of normalization

Let $\Phi : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ be a completely positive trace-preserving map between finite-dimensional operator spaces. Write $\mathcal{D}(\mathcal{H})$ for the density operators on $\mathcal{H}$. For a Hermitian operator X, $\|X\|_1 = \operatorname{Tr}|X|$; for states, $\mathrm{TD}(\rho,\sigma) = \frac{1}{2}\|\rho-\sigma\|_1$. The channel parameter is

$$\delta(\Phi) = \max_{\rho_0,\rho_1\in\mathcal{D}(\mathcal{H}_A)} \mathrm{TD}\big(\Phi(\rho_0),\Phi(\rho_1)\big) \in [0,1]. \tag{1}$$

The maximum exists by compactness. This is the quantum Dobrushin, or trace-distance contraction, coefficient [3, 4]; a self-contained verification appears in Section 5.

There are $M \geq 2$ messages, labeled by $\mathbb{Z}_M$. The backward channel has input $a \in \{0,1\}$ and output $b \in \{0,1\}$, with

$$N_p(b \mid a) = \begin{cases} 1-p, & b = a, \\ p, & b \neq a, \end{cases} \qquad 0 < p < \tfrac{1}{2}. \tag{2}$$

For convenience put $A = 1-p$ and $k = 1-2p$, so $A = (1+k)/2$ and $0 < k < 1$.

A seed s belongs to a finite set $\mathcal{S}$ and has prescribed raw probabilities $\pi_s \geq 0$ with $\sum_s \pi_s = 1$. Both devices know s. These probabilities do not depend on the message or on b. For each s and b, the decoder chooses operators

$$\tau^s_{yb} \succeq 0 \quad (y \in \mathbb{Z}_M), \qquad \sum_y \operatorname{Tr}\tau^s_{yb} = 1. \tag{3}$$

The decoder outputs y and prepares the subnormalized forward state τ^s_{yb}. The carrier then undergoes Φ. For each message m and seed s, the encoder uses a binary effect $0 \preceq E^s_m \preceq I$, with $E^s_{0|m} = E^s_m$ and $E^s_{1|m} = I - E^s_m$. Its output is a.

These specifications define a loop contraction, not an ordinary causal circuit. The corresponding weights and normalizers are

$$w^s_m(y) = \sum_{a,b=0}^{1} N_p(b \mid a) \operatorname{Tr}\Big[E^s_{a|m}\Phi(\tau^s_{yb})\Big], \tag{4}$$

$$Z^s_m = \sum_y w^s_m(y). \tag{5}$$

No output label y or other side channel is forwarded to the encoder. There is no entanglement assistance or additional forward memory beyond the shared seed. The shared seed is an additional resource and is not charged against the input or output dimension of Φ.

2.1 Raw mixing followed by one normalization

The defining probability law is

$$P_{\mathrm{raw}}(y \mid m) = \frac{\overline{w}_m(y)}{\overline{Z}_m}, \qquad \overline{w}_m(y) = \sum_s \pi_s w_m^s(y), \quad \overline{Z}_m = \sum_s \pi_s Z_m^s. \tag{6}$$

All denominators are positive. Indeed, set $\rho_b^s = \sum_y \tau_{yb}^s$ and $\Delta^s = \Phi(\rho_0^s) - \Phi(\rho_1^s)$. Expanding the binary channel gives

$$Z_m^s = 1 + k \operatorname{Tr}(E_m^s \Delta^s). \tag{7}$$

Since Δ^s is traceless and Hermitian, its positive and negative parts have trace $d_s = \frac{1}{2}\|\Delta^s\|_1$. Therefore

$$1 - kd_s \le Z_m^s \le 1 + kd_s, \qquad 2p \le Z_m^s \le 2(1-p). \tag{8}$$

The success objective is

$$S_{\mathrm{raw}}(\Phi, p, M) = \sup_{\mathcal{S},\pi,\tau,E} \min_{m \in \mathbb{Z}_M} P_{\mathrm{raw}}(m \mid m), \tag{9}$$

where the supremum includes all finite seed alphabets and all choices in (3)–(4).

The raw seed law must not be confused with its law in the normalized ensemble. The latter is

$$\Pr_{\mathrm{norm}}(s \mid m) = \frac{\pi_s Z_m^s}{\overline{Z}_m}, \tag{10}$$

which can depend on m. In contrast, externally mixing already normalized branches would give

$$P_{\mathrm{external}}(y \mid m) = \sum_s \pi_s \frac{w_m^s(y)}{Z_m^s}. \tag{11}$$

Equation (11) is a different model. Nor does (9) impose a minimum over seeds: success is assessed only after marginalizing the seed as in (6).

3 Exact optimum

Theorem 1. *For every finite-dimensional CPTP map Φ, every integer $M \ge 2$, and every $0 < p < 1/2$, the raw-mixture model* (6) *satisfies*

$$\boxed{S_{\mathrm{raw}}(\Phi, p, M) = \frac{2(1-p)}{M - (M-2)(1-2p)\delta(\Phi)}.} \tag{12}$$

The supremum is attained with a uniform seed having M values. For $M = 2$, no seed is needed. The statement assumes mixing of unnormalized branches before one normalization for each message; it does not assert the same optimum under (11) *or under a success requirement conditional on every seed.*

3.1 Reduction to a single aggregate ratio

For any seeded protocol define

$$D = \sum_m \overline{w}_m(m), \qquad Q = \sum_m \overline{Z}_m. \tag{13}$$

Because the normalizers are strictly positive,

$$\min_m P_{\mathrm{raw}}(m \mid m) \le \frac{D}{Q} = \sum_m \frac{\overline{Z}_m}{Q} P_{\mathrm{raw}}(m \mid m). \tag{14}$$

For each seed put $D_s = \sum_m w_m^s(m)$ and $Q_s = \sum_m Z_m^s$. A second weighted-average identity yields

$$\frac{D}{Q} = \sum_s \frac{\pi_s Q_s}{Q} \frac{D_s}{Q_s} \le \max_{s:\pi_s>0} \frac{D_s}{Q_s}. \tag{15}$$

Thus an upper bound on every unseeded aggregate ratio

$$R = \frac{\sum_m w_m(m)}{\sum_m Z_m} \tag{16}$$

also bounds the seeded minimax objective. This step bounds a ratio of sums; it does not replace the objective with an unweighted average of conditional successes.

Lemma 2. *To upper-bound* (16)*, it suffices to consider decoders that, for each b, choose one deterministic label y_b and prepare one pure state ρ_b.*

Proof. Fix all encoder effects. For a fixed b, the feasible tuples $(\tau_{yb})_y$ in (3) form a compact convex set. Spectrally decomposing each block expresses any such tuple as a convex combination of tuples with exactly one nonzero block, that block being a rank-one unit-trace state. Conversely, each such tuple is extreme: positivity forces any convex decomposition to retain its only nonzero block, and a pure state has no nontrivial state decomposition. The feasible decoder set is the product of the two sets for $b = 0, 1$. Taking products of the two finite decompositions gives a convex decomposition into the stated decoders.

For fixed effects, the numerator N and denominator T of (16) are linear in the decoder blocks. If a decoder decomposes with weights λ_j, then $T_j > 0$ by (8) and

$$\frac{N}{T} = \sum_j \frac{\lambda_j T_j}{T} \frac{N_j}{T_j}.$$

At least one component has ratio no smaller than the original. An upper bound for all such components and all encoder effects is therefore an upper bound for every decoder. □

Fix a decoder from Lemma 2. Write

$$\sigma_b = \Phi(\rho_b), \qquad \Delta = \sigma_0 - \sigma_1, \qquad d = \tfrac{1}{2}\|\Delta\|_1 \le \delta(\Phi). \tag{17}$$

For each encoder effect,

$$Z_m = 1 + k \operatorname{Tr}(E_m \Delta), \qquad 1 - kd \le Z_m \le 1 + kd. \tag{18}$$

The endpoint values are attainable by the negative and positive spectral projectors of Δ, respectively. Its zero eigenspace may be assigned arbitrarily.

3.2 Two different output labels

Suppose $y_0 \neq y_1$ and relabel these messages as 0 and 1. Their correct weights are

$$w_0(0) = p + k \operatorname{Tr}(E_0\sigma_0), \qquad w_1(1) = A - k \operatorname{Tr}(E_1\sigma_1). \tag{19}$$

Every other correct weight is zero. For any $t \in [0,1]$,

$$w_0(0) - tZ_0 = p - t + k \operatorname{Tr}\big[E_0\big((1-t)\sigma_0 + t\sigma_1\big)\big] \leq A - t, \tag{20}$$

$$w_1(1) - tZ_1 = A - t - k \operatorname{Tr}\big[E_1\big(t\sigma_0 + (1-t)\sigma_1\big)\big] \leq A - t. \tag{21}$$

The operators inside both traces are positive and have trace one. The first trace is at most one and the second is nonnegative. The two inequalities hold with equality at $E_0 = I$ and $E_1 = 0$. For each remaining message, (18) gives

$$w_m(m) - tZ_m = -tZ_m \leq -t(1 - kd).$$

Summing these inequalities yields

$$\sum_m \big(w_m(m) - tZ_m\big) \leq 2(A - t) - t(M-2)(1-kd). \tag{22}$$

Choose

$$R_2(d) = \frac{2A}{2 + (M-2)(1-kd)} = \frac{2A}{M - (M-2)kd}. \tag{23}$$

This value is in $[0,1]$, since its denominator is at least 2 and $2A < 2$. Setting $t = R_2(d)$ makes the right-hand side of (22) zero. Dividing by $\sum_m Z_m > 0$ proves $R \leq R_2(d)$. Equality is attained by the two constant active effects above and the negative spectral projector of Δ for every inactive message. Finally, $R_2(d)$ is nondecreasing in d, so $R \leq R_2(\delta(\Phi))$.

3.3 One common output label

Suppose instead that $y_0 = y_1 = y_*$. The only nonzero correct weight is $w_{y_*}(y_*) = Z_{y_*}$. The ratio (16) is increasing in Z_{y_*} and decreasing in every other Z_m. These normalizers can be chosen independently through the message-dependent effects. Equation (18) therefore gives the exact maximum for these preparations as

$$R_1(d) = \frac{1 + kd}{(1 + kd) + (M-1)(1-kd)} = \frac{1 + kd}{M - (M-2)kd}. \tag{24}$$

Because $d \leq 1$ and $1 + k = 2A$, it follows that

$$R_1(d) \leq \frac{1 + k}{M - (M-2)kd} = R_2(d) \leq R_2(\delta(\Phi)). \tag{25}$$

Together with Lemma 2 and (14)–(15), this proves the upper bound in Theorem 1 for every finite seed alphabet.

3.4 A cyclic construction attaining the bound

Choose states ρ_0, ρ_1 attaining (1), and choose an effect H such that

$$\mathrm{Tr}\big[H\big(\Phi(\rho_0) - \Phi(\rho_1)\big)\big] = -\delta(\Phi). \tag{26}$$

The negative spectral projector is one choice. Let s be uniform on $\mathbb{Z}_M$. Given $b = 0$, the decoder outputs s and prepares ρ_0; given $b = 1$, it outputs $s + 1$ modulo M and prepares ρ_1. Set

$$E_m^s = \begin{cases} I, & m = s, \\ 0, & m = s + 1, \\ H, & m \notin \{s, s + 1\}. \end{cases} \tag{27}$$

For any fixed message m, exactly one seed places m in the first active role and exactly one in the second. In either active role, $w_m^s(m) = A$ and $Z_m^s = 1$. In each of the remaining $M - 2$ roles, $w_m^s(m) = 0$ and $Z_m^s = 1 - k\delta(\Phi)$. Thus

$$\overline{w}_m(m) = \frac{2A}{M}, \qquad \overline{Z}_m = \frac{2 + (M - 2)(1 - k\delta(\Phi))}{M}. \tag{28}$$

Every message attains the ratio in (12), proving equality and attainment. If $M = 2$, a single fixed seed with labels $0, 1$ and effects $I, 0$ already gives success A for both messages, so no seed is necessary.

The construction uses M seed values, with entropy $\log_2 M$. This is a sufficient size, not a minimality result. Storage in binary memory requires $\lceil \log_2 M \rceil$ cells. Exact uniform sampling from fair bits may use a variable number of bits when M is not a power of two; storage and random-bit generation are different resource costs.

4 Why the normalization convention matters

Take $M = 3$, $p = 1/10$, and a forward channel with $\delta(\Phi) = 1$. For each message, the cyclic construction has three equally likely raw seed branches with pairs

$$\big(w_m^s(m), Z_m^s\big) : \qquad (9/10, 1), \quad (9/10, 1), \quad (0, 1/5). \tag{29}$$

Mixing the weights first gives

$$P_{\mathrm{raw}}(m \mid m) = \frac{(9/10 + 9/10 + 0)/3}{(1 + 1 + 1/5)/3} = \frac{9}{11}. \tag{30}$$

By contrast, externally mixing the same three normalized branches gives

$$P_{\mathrm{external}}(m \mid m) = \frac{1}{3}\left(\frac{9/10}{1} + \frac{9/10}{1} + \frac{0}{1/5}\right) = \frac{3}{5}. \tag{31}$$

This compares two conventions for the *same construction*; it is not a calculation of the optimum under (11). The normalized seed probabilities in the raw model are $5/11, 5/11, 1/11$, rather than $1/3$ each. Which seed is suppressed depends on the message. Conditional on the inactive seed, success is zero.

There is an elementary postselection interpretation of this bookkeeping. Since $Z_m^s \le 2A$, one may define an abstract joint distribution in which a sampled seed is accepted with probability $Z_m^s/(2A)$ and, upon acceptance, outputs y with probability $w_m^s(y)/Z_m^s$. Conditioning on acceptance gives (6) and (10). This observation explains the change in the seed law. It does *not* implement the stipulated backward channel, establish a causal realization of the loop, or show that the abstract acceptance rule respects a physical communication constraint.

The theorem therefore applies only when the seed is included in the ensemble that is normalized. Requiring the seed distribution to remain message-independent after conditioning, or demanding a guarantee for every realized seed, changes the feasible resource model and requires a separate analysis.

5 The channel coefficient and concrete comparisons

The reduction to $\delta(\Phi)$ makes the dependence on the forward channel explicit. Its identification with an established contraction coefficient is important for both interpretation and attribution.

Proposition 3. *For an input space of dimension at least two,*

$$\delta(\Phi) = \sup_{\rho \neq \sigma} \frac{\mathrm{TD}(\Phi(\rho), \Phi(\sigma))}{\mathrm{TD}(\rho, \sigma)} = \max_{\langle u,v\rangle = 0} \mathrm{TD}\big(\Phi(|u\rangle\langle u|), \Phi(|v\rangle\langle v|)\big). \tag{32}$$

Here u, v are unit vectors. For a one-dimensional input space, $\delta(\Phi) = 0$.

Proof. For distinct states, write the Jordan decomposition as

$$\rho - \sigma = t(\omega_+ - \omega_-), \qquad t = \mathrm{TD}(\rho, \sigma) > 0,$$

where ω_+ and ω_- are states with orthogonal supports. Linearity shows that the ratio in (32) is $\mathrm{TD}(\Phi(\omega_+), \Phi(\omega_-)) \le \delta(\Phi)$. Conversely, if $\delta(\Phi) > 0$, a diameter-attaining pair has $0 < t \le 1$, so its contraction ratio is at least $\delta(\Phi)$. If $\delta(\Phi) = 0$, every numerator vanishes, giving the same equality.

For orthogonally supported states, spectral decompositions and convexity give

$$\mathrm{TD}(\Phi(\omega_+), \Phi(\omega_-)) \le \sum_{i,j} \alpha_i \beta_j \, \mathrm{TD}\big(\Phi(|u_i\rangle\langle u_i|), \Phi(|v_j\rangle\langle v_j|)\big),$$

where $\langle u_i, v_j\rangle = 0$. Hence the distance is no larger than the maximum over orthogonal pure pairs. The reverse inequality follows from (1); compactness supplies a maximum. □

Equation (32) is standard; see, in particular, the equivalent definitions in Fawzi, Müller-Hermes, and Shayeghi [3], and the study of contraction coefficients by Hiai and Ruskai [4]. The proof is included so that the present optimization does not depend on an unstated channel-geometric fact.

5.1 Qubit and classical channels

A qubit channel acts on Bloch vectors as $v \mapsto Tv + c$. Since the trace distance between two qubit states is half the Euclidean distance between their Bloch vectors, the translation cancels and

$$\delta(\Phi) = \|T\|_{2\to 2}. \tag{33}$$

Antipodal unit vectors in a largest right-singular direction attain the value. This geometry is also related to the qubit communication-value identity $\mathrm{cv}(\Phi) = 1 + \|T\|_{2\to 2}$ [5]. That identity concerns a different objective and is not asserted here for higher-dimensional channels.

For a classical stochastic channel K with columns $K(\cdot \mid x)$,

$$\delta(K) = \max_{x,x'} \frac{1}{2} \sum_z \big|K(z \mid x) - K(z \mid x')\big|. \tag{34}$$

Convexity bounds the distance between arbitrary output mixtures by this maximum, and deterministic inputs attain it. The preceding proof of Theorem 1 applies with probability vectors in place of density operators and diagonal effects in place of quantum effects. In particular, the attaining inputs and the negative-part decision rule are classical. Thus the same formula is the exact classical optimum in the corresponding raw-seed model.

Corollary 4. *Let $\Phi_\varepsilon(\rho) = (1-\varepsilon)\rho + \varepsilon I/2$ be a qubit depolarizing channel, $0 \le \varepsilon \le 1$, and let K_q be a classical binary symmetric forward channel, $0 \le q \le 1/2$. Then*

$$\delta(\Phi_\varepsilon) = 1 - \varepsilon, \qquad \delta(K_q) = 1 - 2q.$$

When $q = \varepsilon/2$, their raw-seed optima agree for every $M \ge 2$ and $0 < p < 1/2$.

This is a matched-diameter equality under the stated assistance. It is not a universal absence of quantum advantage. For example, a qubit amplitude-damping channel with damping parameter $0 < \gamma < 1$ has Bloch linear part

$$T = \mathrm{diag}(\sqrt{1-\gamma}, \sqrt{1-\gamma}, 1-\gamma),$$

and hence quantum diameter $\sqrt{1-\gamma}$. Restricting its inputs to the two computational-basis states and using its classical output populations instead gives diameter $1-\gamma$. The former is strictly larger; for $M > 2$, (12) is strictly increasing in the diameter. Thus a fixed-basis classical restriction need not match the optimized quantum carrier. Nor should a model with shared seed storage be equated with a model in which that resource is forbidden.

5.2 Boundary checks

The formula has the following special values:

$$\begin{aligned} M = 2: &\quad S_{\mathrm{raw}} = 1 - p, \\ \delta = 0: &\quad S_{\mathrm{raw}} = \frac{2(1-p)}{M}, \\ \delta = 1: &\quad S_{\mathrm{raw}} = \frac{1-p}{1+(M-2)p}. \end{aligned} \tag{35}$$

At $p = 1/2$, every normalizer equals one and the weights are independent of the message. Their minimum diagonal entry is at most $1/M$, attained by a uniform output distribution, so the result extends to $S_{\mathrm{raw}} = 1/M$. For $M = 1$, always outputting the unique message gives success one. The endpoint $p = 0$ is excluded: arbitrary protocols can then have zero normalizers, and a theorem at that endpoint needs an explicit admissibility convention. Taking a limit of (12) alone does not settle that question.

6 Relation to previous work and scope

Ji, Lloyd, and Wilde define retrocausal communication through normalized channel contractions [1]. Their architecture includes a decoder, forward memory, and an encoder; their treatment allows resources beyond the fixed noisy carrier and seed-only assistance considered here. In particular, their protocols do use shared randomness, so the present resource distinction must not be described as an absence of shared randomness in their framework. The noisy-forward-memory rate tradeoff raised there is a broader asymptotic problem. Equation (12) is a one-use, classical-message result with a binary classical backward channel and does not solve that general tradeoff.

Ji, Regula, and Wilde's postselected communication model already uses a worst-message ratio of correct conclusive weight to total conclusive weight [2]. Their principal assisted results concern entanglement and nonsignalling resources. Mixing a seed at the unnormalized level is consistent with the usual bookkeeping of a shared resource before conditioning, but consistency of normalization is not a resource-preserving equivalence between the two communication tasks. No such equivalence is assumed in this proof.

The individual tools used here are also established. The channel coefficient is the Dobrushin coefficient [3, 4]; binary trace-distance optimization and the qubit communication-value geometry are not new [5]. Frenkel and Weiner prove equality of classical and quantum expected rewards in an ordinary finite-level storage game [6], while Bowles, Quintino, and Brunner explicitly distinguish independent-device and shared-randomness assumptions in prepare-and-measure settings [7]. These results caution against interpreting a resource-specific equality as a general claim about classical and quantum communication. Their objectives and constraints are not identical to (9).

The contribution proposed in this note is therefore limited to the exact formula, its proof, and its finite attaining construction for the specified normalized-loop model. A bounded examination of the cited primary sources did not identify this exact statement. That examination is not an exhaustive priority search, and independent human assessment of originality remains pending.

7 Reproducibility and open questions

The analytic proof is supplemented by finite computational checks. Exact rational enumeration tests the unseeded aggregate optimum for four classical channels, $M \in \{2, 3, 4\}$, and $p \in \{1/10, 3/10, 1/2\}$, giving 36 parameter settings. These enumerations check all extreme classical encoder and decoder choices in their finite alphabets. Separate checks verify symbolic identities, 304 cyclic-construction settings, and the 9/11 versus 3/5 normalization example. An additional test implementation checks 220 exact construction cases and 1800 randomly generated quantum protocols. The recorded tests pass. Such checks can detect implementation or algebraic mistakes; they are not a proof, a physical simulation, or independent human review.

The theorem leaves several distinct questions open. The minimum necessary seed size is not determined. Optimizing the external-mixture model (11), or imposing seed-wise success or a message-independent posterior seed law, requires a different analysis. Larger backward alphabets, multiple channel uses, and restrictions on seed storage may introduce channel parameters beyond the binary output diameter. Any claim about physically realizable backward communication would require a separate physical model and evidence absent from this mathematical construction.

Code and data availability The companion archive contains the checking scripts and machine-readable results. It is publicly available as `shared_seed_loop_reproducibility (1).zip` in https://github.com/dapyle11/normalized-loop-memory. The deposited archive was verified against the final reproducibility package on 31 October 2026. The manuscript source archive is separate from the computational archive.

AI assistance and review status Lyra, an OpenAI AI assistant, carried out the technical derivations, code-based verification, literature screening, and manuscript drafting in an AI-assisted workflow directed by Daniel A. Pyle. It is disclosed as a tool and is not a coauthor. This document is a research draft; independent human review of the model, proofs, bibliography, and originality remains pending.

References

[1] K. Ji, S. Lloyd, and M. M. Wilde, "Retrocausal capacity of a quantum channel: Communicating through noisy closed timelike curves," *Physical Review Letters* **136**, 230801 (2026). doi:10.1103/znyd-npk5. arXiv:2509.08965v3.

[2] K. Ji, B. Regula, and M. M. Wilde, "Postselected communication over quantum channels," *International Journal of Quantum Information* **22**(5), 2440012 (2024). doi:10.1142/S0219749924400124. arXiv:2308.02583v3.

[3] O. Fawzi, A. Müller-Hermes, and A. Shayeghi, "A lower bound on the space overhead of fault-tolerant quantum computation," in *13th Innovations in Theoretical Computer Science Conference (ITCS 2022)*, LIPIcs **215**, 68:1–68:20 (Schloss Dagstuhl, 2022). doi:10.4230/LIPIcs.ITCS.2022.68.

[4] F. Hiai and M. B. Ruskai, "Contraction coefficients for noisy quantum channels," *Journal of Mathematical Physics* **57**(1), 015211 (2016). doi:10.1063/1.4936215.

[5] E. Chitambar, I. George, B. Doolittle, and M. Junge, "The communication value of a quantum channel," *IEEE Transactions on Information Theory* **69**(3), 1660–1679 (2023). doi:10.1109/TIT.2022.3218540.

[6] P. E. Frenkel and M. Weiner, "Classical information storage in an n-level quantum system," *Communications in Mathematical Physics* **340**(2), 563–574 (2015). doi:10.1007/s00220-015-2463-0.

[7] J. Bowles, M. T. Quintino, and N. Brunner, "Certifying the dimension of classical and quantum systems in a prepare-and-measure scenario with independent devices," *Physical Review Letters* **112**, 140407 (2014). doi:10.1103/PhysRevLett.112.140407.