
Noncommutative Lagrange-Fujiwara, Carmichael-Mason and Kuniyeda Bounds for Roots of Polynomials

K. Mahesh Krishna

School of Mathematics and Natural Sciences
Chanakya University Global Campus
NH-648, Haralur Village
Devanahalli Taluk, Bengaluru North District
Karnataka State, 562 110, India
Email: kmaheshak@gmail.com
Date: October 31, 2026

Abstract: Two centuries ago, Lagrange derived an upper bound for every root of a complex polynomial using the maximum of powers of coefficients. In 1916, Fujiwara independently found the Lagrange bound. In 1914, Carmichael and Mason derived an upper bound using the sum of squares of the absolute values of the coefficients of a polynomial. This bound was extended by Kuniyeda in 1916. We derive noncommutative versions of the Lagrange-Fujiwara, Carmichael-Mason and Kuniyeda bounds.

Keywords: Lagrange-Fujiwara bound, Carmichael-Mason bound, Kuniyeda bound, Banach algebra.

Mathematics Subject Classification (2020): 30C15, 46H05, 46L05.

1. INTRODUCTION

More than 200 years ago, Lagrange derived an upper bound for the zeros of complex polynomials [10, 13]. This bound was later found independently by Fujiwara [5] in 1916.

Theorem 1.1. [5, 6, 10, 13] (*Lagrange-Fujiwara Upper Bound*) Let

$$p(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + z^n \in \mathbb{C}[z].$$

If $\lambda \in \mathbb{C}$ satisfies $p(\lambda) = 0$, then

$$|\lambda| < 2 \max_{0 \leq j \leq n-1} |a_j|^{\frac{1}{n-j}}.$$

Corollary 1.2. [5, 6, 10, 13] (*Lagrange-Fujiwara Upper Bound*) Let

$$p(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n \in \mathbb{C}[z], \quad a_n \neq 0.$$

If $\lambda \in \mathbb{C}$ satisfies $p(\lambda) = 0$, then

$$|\lambda| < 2 \max_{0 \leq j \leq n-1} \left| \frac{a_j}{a_n} \right|^{\frac{1}{n-j}}.$$

Corollary 1.2 gives the following result.

Theorem 1.3. [5, 6, 10, 13] (*Lagrange-Fujiwara Lower Bound*)

$$p(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n \in \mathbb{C}[z], \quad a_0 \neq 0, a_n \neq 0.$$

If $\lambda \in \mathbb{C}$ satisfies $p(\lambda) = 0$, then

$$|\lambda| > \frac{1}{2 \max_{1 \leq j \leq n} \left| \frac{a_j}{a_0} \right|^{\frac{1}{j}}}.$$

In 1914, Carmichael and Mason derived the following beautiful upper bound (using Hadamard theorem for determinants) for every root of a complex polynomial using the sum of the squares of the absolute values of the coefficients [3].

Theorem 1.4. [3, 4, 8, 11] (*Carmichael-Mason Upper Bound*) Let

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \in \mathbb{C}[z].$$

If $\lambda \in \mathbb{C}$ satisfies $p(\lambda) = 0$, then

$$|\lambda| < \left(1 + \sum_{j=0}^{n-1} |a_j|^2 \right)^{\frac{1}{2}}.$$

In 1916, Kuniyeda improved Theorem 1.4 using the Holder inequality [7].

Theorem 1.5. [1, 2, 7-9, 11, 12] (*Kuniyeda Upper Bound*) Let

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \in \mathbb{C}[z].$$

If $\lambda \in \mathbb{C}$ satisfies $p(\lambda) = 0$, then

$$|\lambda| < \left(1 + \left(\sum_{j=0}^{n-1} |a_j|^s \right)^{\frac{1}{s-1}} \right)^{\frac{s-1}{s}}, \quad \forall s \in (1, \infty).$$

Corollary 1.6. [1, 2, 7-9, 11, 12] (*Kuniyeda Upper Bound*) Let

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \in \mathbb{C}[z], \quad a_n \neq 0.$$

If $\lambda \in \mathbb{C}$ satisfies $p(\lambda) = 0$, then

$$|\lambda| < \left(1 + \left(\sum_{j=0}^{n-1} \left| \frac{a_j}{a_n} \right|^s \right)^{\frac{1}{s-1}} \right)^{\frac{s-1}{s}} = \left(1 + \left(\frac{1}{|a_n|^s} \sum_{j=0}^{n-1} |a_j|^s \right)^{\frac{1}{s-1}} \right)^{\frac{s-1}{s}}, \quad \forall s \in (1, \infty).$$

Corollary 1.6 gives the following result.

Theorem 1.7. [1, 2, 7-9, 11, 12] (*Kuniyeda Lower Bound*) Let

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \in \mathbb{C}[z], \quad a_0 \neq 0, a_n \neq 0.$$

If $\lambda \in \mathbb{C}$ satisfies $p(\lambda) = 0$, then

$$|\lambda| > \frac{1}{\left(1 + \left(\frac{1}{|a_0|^s} \sum_{j=1}^n |a_j|^s \right)^{\frac{1}{s-1}} \right)^{\frac{s-1}{s}}}, \quad \forall s \in (1, \infty).$$

We naturally ask for versions of Theorems 1.1 and 1.5 for polynomials over Banach algebras (particularly, over C^* -algebras). In this article, we derive them.

Noncommutative Lagrange-Fujiwara, Carmichael-Mason and Kuniyeda Bounds for Roots of Polynomials

2. NONCOMMUTATIVE LAGRANGE-FUJIWARA, CARMICHAEL-MASON AND KUNIYEDA BOUNDS

Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{A}[z]$ be the set of all polynomials over $\mathcal{A}$. We derive a noncommutative version of Theorem 1.1 as follows.

Theorem 2.1. (*Noncommutative Lagrange-Fujiwara Upper Bound*) *Let $\mathcal{A}$ be a unital Banach algebra. Let*

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n \in \mathcal{A}[z].$$

Let $u \in \mathcal{A}$ be such that $p(u) = 0$ and

$$(1) \quad \|u^n\| = \|u\|^n.$$

Then

$$\|u\| < 2 \max_{0 \leq j \leq n-1} \|a_j\|^{\frac{1}{n-j}}.$$

Proof. Let $u \in \mathcal{A}$ satisfies $p(u) = 0$ and $\|u^n\| = \|u\|^n$. Define

$$M := \max_{0 \leq j \leq n-1} \|a_j\|^{\frac{1}{n-j}}.$$

Then we have

$$(2) \quad M^{n-j} \geq \|a_j\|, \quad \forall 0 \leq j \leq n-1.$$

If $\|u\| \leq M$, then the conclusion holds. Hence we assume that $\|u\| > M$. Since $p(u) = 0$, we have

$$u^n = -(a_0 + a_1 u + \cdots + a_{n-1} u^{n-1}).$$

By taking the norm and using Equation (1), we get

$$\begin{aligned} \|u\|^n &= \|u^n\| = \|-(a_0 + a_1 u + \cdots + a_{n-1} u^{n-1})\| \\ &= \|a_0 + a_1 u + \cdots + a_{n-1} u^{n-1}\| \\ &\leq \|a_0\| + \|a_1 u\| + \cdots + \|a_{n-1} u^{n-1}\| \\ &\leq \|a_0\| + \|a_1\| \|u\| + \cdots + \|a_{n-1}\| \|u\|^{n-1} \\ &\leq \|a_0\| + \|a_1\| \|u\| + \cdots + \|a_{n-1}\| \|u\|^{n-1}. \end{aligned}$$

Dividing by $\|u\|^n$ and by using Equation (2) we get

$$\begin{aligned}
 1 &\leq \frac{\|a_0\|}{\|u\|^n} + \frac{\|a_1\|}{\|u\|^{n-1}} + \cdots + \frac{\|a_{n-2}\|}{\|u\|^2} + \frac{\|a_{n-1}\|}{\|u\|^1} \\
 &\leq \frac{M^n}{\|u\|^n} + \frac{M^{n-1}}{\|u\|^{n-1}} + \cdots + \frac{M^2}{\|u\|^2} + \frac{M}{\|u\|^1} \\
 &= \frac{M}{\|u\|} \left(\left(\frac{M}{\|u\|} \right)^{n-1} + \left(\frac{M}{\|u\|} \right)^{n-2} + \cdots + \frac{M}{\|u\|} + 1 \right) \\
 &= \frac{M}{\|u\|} \left(\frac{1 - \left(\frac{M}{\|u\|} \right)^n}{1 - \frac{M}{\|u\|}} \right) \\
 &< \frac{M}{\|u\|} \left(\frac{1}{1 - \frac{M}{\|u\|}} \right) \\
 &= \frac{M}{\|u\| - M}.
 \end{aligned}$$

Hence

$$1 < \frac{M}{\|u\| - M}.$$

Rearranging completes the proof. $\square$

Note that we used the condition (1) in the proof of Theorem 2.1. We are unable to derive the result without this assumption.

Corollary 2.2. (Noncommutative Lagrange-Fujiwara Upper Bound) Let $\mathcal{A}$ be a unital Banach algebra. Let

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \in \mathcal{A}[z], \quad a_n \text{ is invertible.}$$

Let $u \in \mathcal{A}$ be such that $p(u) = 0$ and

$$\|u^n\| = \|u\|^n.$$

Then

$$\|u\| < 2 \max_{0 \leq j \leq n-1} \|a_n^{-1} a_j\|^{\frac{1}{n-j}}.$$

Corollary 2.3. Let $\mathcal{A}$ be a unital C^* -algebra. Let

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \in \mathcal{A}[z], \quad a_n \text{ is invertible.}$$

Let $u \in \mathcal{A}$ be such that $p(u) = 0$ and u is normal or self-adjoint. Then

$$\|u\| < 2 \max_{0 \leq j \leq n-1} \|a_n^{-1} a_j\|^{\frac{1}{n-j}}.$$

Corollary 2.4. (Noncommutative Lagrange-Fujiwara Lower Bound) Let $\mathcal{A}$ be a unital Banach algebra. Let

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \in \mathcal{A}[z], \quad a_0 \text{ and } a_n \text{ are invertible.}$$

Let $u \in \mathcal{A}$ be such that $p(u) = 0$, u is invertible and

$$\|(u^{-1})^n\| = \|u^{-1}\|^n.$$

Noncommutative Lagrange-Fujiwara, Carmichael-Mason and Kuniyeda Bounds for Roots of Polynomials

Then

$$\frac{1}{\|u^{-1}\|} > \frac{1}{2 \max_{1 \leq j \leq n} \|a_0^{-1} a_j\|^{\frac{1}{j}}}.$$

Proof. Since u is invertible, we have

$$0 = a_n + a_{n-1}u^{-1} + \cdots + a_1u^{-(n-1)} + a_0u^{-n}.$$

Define $v := u^{-1}$. Then

$$0 = a_n + a_{n-1}v + \cdots + a_1v^{n-1} + a_0v^n.$$

Result follows by applying Corollary 2.2 to v . □

We derive a noncommutative version of Theorem 1.5 as follows.

Theorem 2.5. (Noncommutative Kuniyeda Upper Bound) *Let $\mathcal{A}$ be a unital Banach algebra. Let*

$$p(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + z^n \in \mathcal{A}[z].$$

Let $u \in \mathcal{A}$ be such that $p(u) = 0$ and

$$(3) \quad \|u^n\| = \|u\|^n.$$

Then

$$\|u\| < \left(1 + \left(\sum_{j=0}^{n-1} \|a_j\|^s \right)^{\frac{1}{s-1}} \right)^{\frac{s-1}{s}}, \quad \forall s \in (1, \infty).$$

Proof. Let $s \in (1, \infty)$. Define

$$M := \left(\sum_{j=0}^{n-1} \|a_j\|^s \right)^{\frac{1}{s-1}}.$$

Let $u \in \mathcal{A}$ satisfies $p(u) = 0$ and $\|u^n\| = \|u\|^n$. We claim that

$$\|u\| < (1 + M)^{\frac{s-1}{s}}.$$

Let us suppose that the claim fails. Then

$$\|u\| \geq (1 + M)^{\frac{s-1}{s}} > 1.$$

Since $p(u) = 0$, we have

$$u^n = -(a_0 + a_1u + \cdots + a_{n-1}u^{n-1}).$$

By taking the norm, using the Holder inequality and using Equation (3), we get

$$\begin{aligned}
 \|u\|^n &= \|u^n\| = \|(a_0 + a_1u + \cdots + a_{n-1}u^{n-1})\| \\
 &= \|a_0 + a_1u + \cdots + a_{n-1}u^{n-1}\| \\
 &\leq \|a_0\| + \|a_1u\| + \cdots + \|a_{n-1}u^{n-1}\| \\
 &\leq \|a_0\| + \|a_1\|\|u\| + \cdots + \|a_{n-1}\|\|u\|^{n-1} \\
 &\leq \|a_0\| + \|a_1\|\|u\| + \cdots + \|a_{n-1}\|\|u\|^{n-1} \\
 &= \sum_{j=0}^{n-1} \|a_j\|\|u\|^j \\
 &\leq \left(\sum_{j=0}^{n-1} \|a_j\|^s \right)^{\frac{1}{s}} \left(\sum_{j=0}^{n-1} (\|u\|^j)^{\frac{s}{s-1}} \right)^{\frac{s-1}{s}} \\
 &= M^{\frac{s-1}{s}} \left(\sum_{j=0}^{n-1} (\|u\|^{\frac{s}{s-1}})^j \right)^{\frac{s-1}{s}} \\
 &= M^{\frac{s-1}{s}} \left(\frac{(\|u\|^{\frac{s}{s-1}})^n - 1}{\|u\|^{\frac{s}{s-1}} - 1} \right)^{\frac{s-1}{s}} \\
 &= M^{\frac{s-1}{s}} \left(\frac{\|u\|^{\frac{ns}{s-1}} - 1}{\|u\|^{\frac{s}{s-1}} - 1} \right)^{\frac{s-1}{s}}.
 \end{aligned}$$

Hence

$$\|u\|^{\frac{ns}{s-1}} \leq M \frac{\|u\|^{\frac{ns}{s-1}} - 1}{\|u\|^{\frac{s}{s-1}} - 1}.$$

Rearranging,

$$\|u\|^{\frac{s}{s-1}} - 1 \leq M \frac{\|u\|^{\frac{ns}{s-1}} - 1}{\|u\|^{\frac{s}{s-1}} - 1} = M \left(1 - \frac{1}{\|u\|^{\frac{ns}{s-1}}} \right) < M.$$

Therefore

$$(1 + M)^{\frac{s-1}{s}} \leq \|u\| < (1 + M)^{\frac{s-1}{s}}$$

which is a contraction. Hence claim holds which completes the proof. $\square$

Note that we used the condition (3) in the proof of Theorem 2.5. We are unable to derive the result without this assumption.

Corollary 2.6. (Noncommutative Kuniyeda Upper Bound) Let $\mathcal{A}$ be a unital Banach algebra. Let

$$p(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n \in \mathcal{A}[z], \quad a_n \text{ is invertible.}$$

Let $u \in \mathcal{A}$ be such that $p(u) = 0$ and

$$\|u^n\| = \|u\|^n.$$

Noncommutative Lagrange-Fujiwara, Carmichael-Mason and Kuniyeda Bounds for Roots of Polynomials

Then

$$\|u\| < \left(1 + \left(\sum_{j=0}^{n-1} \|a_n^{-1} a_j\|^s \right)^{\frac{1}{s-1}} \right)^{\frac{s-1}{s}} \leq \left(1 + \left(\|a_n^{-1}\|^s \sum_{j=0}^{n-1} \|a_j\|^s \right)^{\frac{1}{s-1}} \right)^{\frac{s-1}{s}}, \quad \forall s \in (1, \infty).$$

Corollary 2.7. (Noncommutative Kuniyeda Lower Bound) Let $\mathcal{A}$ be a unital Banach algebra. Let

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \in \mathcal{A}[z], \quad a_0 \text{ and } a_n \text{ are invertible.}$$

Let $u \in \mathcal{A}$ be such that $p(u) = 0$, u is invertible and

$$\|(u^{-1})^n\| = \|u^{-1}\|^n.$$

Then

$$\frac{1}{\|u^{-1}\|} > \frac{1}{\left(1 + \left(\sum_{j=1}^n \|a_0^{-1} a_j\|^s \right)^{\frac{1}{s-1}} \right)^{\frac{s-1}{s}}} \geq \frac{1}{\left(1 + \left(\|a_0^{-1}\|^s \sum_{j=1}^n \|a_j\|^s \right)^{\frac{1}{s-1}} \right)^{\frac{s-1}{s}}}.$$

Corollary 2.8. (Noncommutative Carmichael-Mason Upper Bound) Let $\mathcal{A}$ be a unital Banach algebra. Let

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \in \mathcal{A}[z], \quad a_n \text{ is invertible.}$$

Let $u \in \mathcal{A}$ be such that $p(u) = 0$ and

$$\|u^n\| = \|u\|^n.$$

Then

$$\|u\| < \left(1 + \sum_{j=0}^{n-1} \|a_n^{-1} a_j\|^2 \right)^{\frac{1}{2}} \leq \left(1 + \|a_n^{-1}\|^2 \sum_{j=0}^{n-1} \|a_j\|^2 \right)^{\frac{1}{2}}.$$

Corollary 2.9. (Noncommutative Carmichael-Mason Lower Bound) Let $\mathcal{A}$ be a unital Banach algebra. Let

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \in \mathcal{A}[z], \quad a_0 \text{ and } a_n \text{ are invertible.}$$

Let $u \in \mathcal{A}$ be such that $p(u) = 0$, u is invertible and

$$\|(u^{-1})^n\| = \|u^{-1}\|^n.$$

Then

$$\frac{1}{\|u^{-1}\|} > \frac{1}{\left(1 + \sum_{j=1}^n \|a_0^{-1} a_j\|^2 \right)^{\frac{1}{2}}} \geq \frac{1}{\left(1 + \|a_0^{-1}\|^2 \sum_{j=1}^n \|a_j\|^2 \right)^{\frac{1}{2}}}.$$

3. CONCLUSIONS

- (1) Over two centuries ago, Lagrange established an upper bound for the zeros of complex polynomials [10, 13]. In 1916, Fujiwara found this bound independently [5].
- (2) In 1914, Carmichael and Mason derived an upper bound for the zeros of complex polynomials [3].
- (3) In 1916, Kuniyeda improved Carmichael-Mason bound [7].
- (4) In this article, we derived noncommutative versions of Lagrange-Fujiwara and Kuniyeda bounds.

REFERENCES

- [1] Amer Abu-Omar. Notes on some bounds for the zeros of polynomials. *Math. Inequal. Appl.*, 21(2):481–487, 2018.
- [2] Peter Borwein and Tamás Erdélyi. *Polynomials and polynomial inequalities*. New York, NY: Springer-Verlag, 1995.
- [3] R. D. Carmichael and T. E. Mason. Note on the roots of algebraic equations. *Bull. Am. Math. Soc.*, 21:14–22, 1914.
- [4] M. Fujiwara. Über die Wurzeln der algebraischen Gleichungen. *Tôhoku Math. J.*, 8:78–85, 1915.
- [5] M. Fujiwara. Über die obere Schranke des absoluten Betrages der Wurzeln einer algebraischen Gleichung. *Tôhoku Math. J.*, 10:167–171, 1916.
- [6] Michał Góra. Results on the Schur stability of the Hadamard powers and the Hadamard product of complex polynomials. *Linear Multilinear Algebra*, 70(6):1047–1054, 2022.
- [7] M. Kuniyeda. Note on the roots of algebraic equations. *Tôhoku Math. J.*, 9:167–173, 1916.
- [8] Morris Marden. *Geometry of polynomials. 2nd ed.*, volume 3 of *Math. Surv.* American Mathematical Society (AMS), Providence, RI, 1966.
- [9] Maurice Mignotte and Doru Ștefănescu. *Polynomials: an algorithmic approach*. Singapore: Springer, 1999.
- [10] A. M. Ostrowski. *Solution of equations and systems of equation*, volume 9 of *Pure Appl. Math., Academic Press*. Academic Press, New York, NY, 1960.
- [11] Q. I. Rahman and G. Schmeisser. *Analytic theory of polynomials*, volume 26 of *Lond. Math. Soc. Monogr., New Ser.* Oxford: Oxford University Press, 2002.
- [12] Tikara Toya. Some remarks on Montel’s paper concerning upper limit of absolute values of roots of algebraic equations. *Sci. Rep. Tokyo Bunrika Daigaku, Sect. A*, 1:275–282, 1933.
- [13] Chee Keng Yap. *Fundamental problems of algorithmic algebra*. Oxford: Oxford University Press, 2000.