A CONTACT AND COEFFICIENT CORRECTION FOR KORENBLUM'S CONSTANT

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Abstract. We give an unrefereed computer-assisted proof that Korenblum's constant in unweighted Bergman space satisfies $C_{68} \geq 0.44685$. The new input retains the signed first contact harmonic in an annular coefficient estimate. An outer factor and square completion yield a correction of 19/200 to the linear coefficient, replacing the 7/100 correction of version 2. A rational measure verifies the resulting moment conditions. We include the analytic argument and describe the finite interval certificates and both infinite tails. The Green/flux input is V. Patankar's (PR 274); the coefficient and moment methods build on Wang, Pinho's earlier construction, and Wikström. There has been no independent human review or formalization.

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1. Statement, antecedents, and scope

Write $\mathbb{D} = \{z : |z| < 1\}$ and use normalized area $dx\,dy/\pi$. For $f(z) = \sum a_k z^k$, angular orthogonality followed by radial integration gives $\|f\|_{A^2}^2 = \sum_{k\geq 0} |a_k|^2/(k+1)$. Define C_{68} as the supremum of $c \in (0,1)$ for which

$$|f(z)| \leq |u(z)| \quad (c < |z| < 1), \quad f, u \in A^2(\mathbb{D}) \quad \Longrightarrow \quad \|f\|_{A^2} \leq \|u\|_{A^2}.$$

Theorem 1. *The radius $c = 8937/20000$ is admissible. Thus $C_{68} \geq 0.44685$.*

Wang's published lower bound is 0.3554 [2]; Wikström's moment method gives 0.4263 [4]. Pinho's earlier covering-strip Schwarz and hyperbolic Jung construction gives 0.432 [5]. V. Patankar's PR 274 announces 0.446, with a supplementary certificate at 0.44605 [1]. That Green/flux improvement is an essential input here and is credited to Patankar. Version 2 of this note gave 0.44667 [6]. The present change keeps a signed harmonic that the earlier contact reduction bounded below by its minimum. It raises the useful correction from 0.07 to 0.095, while retaining a correction only for coefficient one. The frozen measure below uses Wikström's atomic-plus-Lebesgue form and moment integration. Wang and Schuster [3] supply the annular-pseudodistance and coefficient precedents.

As checked on 2 November 2026, the official table still records 0.432 with an unrefereed marker, and PRs 274 and 299 are open; the latter now records the present 0.44685 claim [8, 9]. A bounded check of those sources, recent public repositories, named AI result catalogs, recent arXiv results, and indexed community sources located no stronger Bergman lower bound. Catalog scans and indexed searches do not cover all code, private work, or unindexed discussions. No exhaustive priority, optimality, or framework barrier is asserted.

2. Green function and Fourier input

All distances below have curvature -1: $d(a,b) = 2\operatorname{artanh}|(a-b)/(1-\bar{b}a)|$. Put

$$w = -\log c, \quad A = \pi^2/w, \quad q = e^{-A}, \quad \Psi = (1+q)^2/(8q), \quad \kappa = 3/25, \tag{1}$$

$$\sigma(t) = \sin^2\frac{\pi \log t}{2w}, \quad D(t) = \frac{\Psi}{2\sigma(t)}, \quad H(t) = \sqrt{1+D(t)}, \quad g(t) = \frac{2}{1+H(t)}, \quad \beta(t) = 1 - g(t).$$

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The certificate verifies $\Psi > 6$ and

$$\begin{aligned} \operatorname{sech}^2(A/2) \le 1/3, \qquad 16\sinh^2(A/4) \le 2(1-8\kappa)\Psi, \\ \tanh^2(A/8) \ge (1+8\kappa)\pi^2/64, \qquad \operatorname{sech}^2(A/8) \le (1-4\kappa)/2. \end{aligned} \tag{2}$$

We give the inherited flux argument explicitly, following the retained input derivations [10]; the basic Green-function minimum principle, Poisson representation, and compact exhaustion used here are classical potential theory [7].

Let $\Omega = \{c < |z| < 1\}$ and $\varphi(z) = (\log|z| + w)/w$. The strip $-w < \Re\zeta < 0$ covers Ω by exponentiation; $E(\zeta) = \exp(i\pi(\zeta + w)/w)$ maps it to the upper half-plane. Its positive Green function, normalized with singularity $-\log|\zeta - b|$, is

$$G_S(\zeta, b) = -\log\left|\frac{E(\zeta) - E(b)}{E(\zeta) - \overline{E(b)}}\right|.$$

For any finite set of distinct lifts b_j of a pole b, the annular Green function satisfies $G(e^\zeta, b) \ge \sum_j G_S(\zeta, b_j)$. Indeed the difference is superharmonic, has only nonnegative remaining singularities, and nonnegative boundary lower limits, including the strip ends; bounded-strip exhaustion and the minimum principle apply. The subtracted finite sum tends to zero at the ends, so the difference is bounded below. For upper-half-plane points Z, B,

$$1 - \left|\frac{Z-B}{Z-\bar B}\right|^2 = \frac{4\Im Z \Im B}{|Z-\bar B|^2}, \qquad -\log x \ge (1-x^2)/2 \quad (0 < x \le 1).$$

The latter follows by differentiating the difference; bounding the denominator by $(|Z| + |B|)^2$ therefore gives

$$G_S(\zeta, b) \ge \tfrac{1}{2}\sin(\pi\varphi(e^\zeta))\sin(\pi\varphi(e^b))\operatorname{sech}^2\frac{\pi(\Im\zeta - \Im b)}{2w}.$$

For the two adjacent lifts the arguments are $v, A - v$. Their sech^2 sum is at least $2\operatorname{sech}^2(A/2)$: the second derivative is nonnegative beyond $v_* = \operatorname{artanh}(1/\sqrt{3})$, so use convexity if both arguments exceed v_*; otherwise one term is at least $2/3$, which suffices by (2). Consequently

$$G(a, b) \ge \sin(\pi\varphi(a))\sin(\pi\varphi(b))\operatorname{sech}^2(A/2). \tag{3}$$

Suppose $\omega : \Omega \to \mathbb{D}$ is holomorphic, $\omega(a) = 0$, and $|a| = |z| = r$. Then the flux input is

$$-\log|\omega(z)| \ge G(z, a) + \sigma(r^2)\operatorname{sech}^2(A/2). \tag{4}$$

Here is the period argument. First work on a compact subannulus with zero-free boundary. The positive superharmonic function $-\log|\omega|$ decomposes as a sum of Green functions at its finitely many zeros, counted with multiplicity, plus a harmonic function h with nonnegative boundary values. Define flux as the derivative of the normalized circle mean with respect to log radius. The flux of $\log|\omega|$ on a zero-free circle is an integer, by the argument principle. The Green flux for a pole b is $1 - \varphi(b)$ below its radius and $-\varphi(b)$ above it, since its radial mean is piecewise linear with slope drop one. On a test circle just outside r, the selected zero contributes $-\varphi(r)$; the sum of the other fluxes and F_h is therefore congruent to $\varphi(r)$ modulo integers. Set $K = \sin(\pi\varphi(r))\operatorname{sech}^2(A/2)$. By (3), every other pole contributes at least $K|\sin(\pi F_b)|$ at z. Dividing (3) by logarithmic distance to either boundary gives an inward Green normal derivative at least $K\pi/w$. Poisson representation, with factor $1/(2\pi)$ on each logarithmic boundary, thus gives

$$h(z) \ge \frac{K\pi}{w}(m_{\text{out}} + m_{\text{in}}) \ge K\pi|F_h| \ge K|\sin(\pi F_h)|, \qquad F_h = (m_{\text{out}} - m_{\text{in}})/w.$$

The function $|\sin(\pi x)|$ is integer-periodic and subadditive by the sine addition formula. Summing proves (4). Increase the subannuli to Ω: Green functions increase to G, and widths and radial coordinates converge. The strict verified first inequality in (2) holds on sufficiently close subannuli. This justifies the passage to the limit even with infinitely many annular zeros.

For equal radii, direct substitution in G_S for the two adjacent lifts gives

$$G(a, z) \ge -\log p(v) - \log p(2A - v), \quad p(v)^2 = \frac{\sinh^2(v/2)}{\sinh^2(v/2) + \sigma(r^2)}, \quad v = A\theta/\pi,$$

where $\theta \in [0, 2\pi]$ is the angular difference. Postcompose an arbitrary disk-valued ω with the automorphism sending $\omega(a)$ to zero. From (4) its pseudohyperbolic separation δ is at most $p(v)p(2A - v)e^{-\sigma \operatorname{sech}^2(A/2)}$. Expanding the inverse square and using $e^x \geq 1 + x$ gives, with $s = v/2$,

$$\delta^{-2} - 1 \geq \sigma\{\operatorname{csch}^2 s + \operatorname{csch}^2(A - s) + 2\operatorname{sech}^2(A/2)\}.$$

Since $\cosh d - 1 = 2/(\delta^{-2} - 1)$, this proves

$$\cosh d(\omega(re^{i\theta}), \omega(re^{i\phi})) - 1 \leq 2DL(\theta - \phi), \quad L(\theta) = \frac{2\Phi(A\theta/(2\pi))}{\Psi}, \tag{5}$$

where $\Phi(s) = (\operatorname{csch}^2 s + \operatorname{csch}^2(A - s) + 2\operatorname{sech}^2(A/2))^{-1}$; angles are reduced modulo 2π and endpoint values follow by continuity.

The four hypotheses imply the Fourier majorant

$$L(\theta) \leq (1 - \cos\theta)/2 - \kappa(1 - \cos 2\theta). \tag{6}$$

To verify it, symmetry reduces to $0 \leq s \leq A/2$. We have $\Phi(s) \leq \sinh^2 s$ and $\Phi(A/2 + x) \leq (\Psi/2)\operatorname{sech}^2 x$. For the second bound the denominator is at least $8q\cosh(2x) + 8q/(1 + q)^2 \geq 16q\cosh^2 x/(1 + q)^2$; the first bound follows by dropping positive denominator terms. On $s \leq A/4$, the right side of (6) equals $\sin^2(\pi s/A)[1 - 8\kappa\cos^2(\pi s/A)]$ and is at least $4(1 - 8\kappa)s^2/A^2$ by the sine chord bound. The second hypothesis and increasing $\sinh s/s$ bound $2\sinh^2 s/\Psi$ by this expression. On $A/4 \leq s \leq A/2$, set $x = A/2 - s$, $y = \pi x/A$. The right side is $\cos^2 y(1 - 8\kappa\sin^2 y)$. For $x \leq A/8$ it is at least $1 - (1 + 8\kappa)y^2$; the third hypothesis and decreasing $\tanh x/x$ bound $\operatorname{sech}^2 x$ by that. For $A/8 \leq x \leq A/4$ the two factors are at least $1/2$ and $1 - 4\kappa$, so the last hypothesis applies. The monotonicities follow by differentiating: the relevant numerator derivatives are $s\sinh s \geq 0$ and $2x\operatorname{sech}^2 x\tanh x \geq 0$. The sine chord bound follows from concavity. This proves (6).

3. Contact geometry and the signed harmonic lemma

For the compact image of a circle choose its smallest enclosing hyperbolic disk, center α, radius $\rho > 0$. Existence follows from properness: centers escaping to infinity have unbounded maximum distance to any fixed image point. At the minimizing center the active unit contact directions balance zero. Otherwise separation and the first variation of distance give a motion strictly decreasing all active distances, and compactness gives a smaller enclosing disk. In the plane, a convex-hull representation can be reduced to at most three directions by eliminating an affine dependence among four or more weighted points. Choose balancing weights λ_i and domain preimages θ_i, and set $m_j = \sum_i \lambda_i e^{ij\theta_i}$. Then $|m_j| \leq 1$. These domain angles are not identified with the image contact directions. The hyperbolic cosine rule gives, with $h = \cosh\rho$,

$$\sum_{i,j} \lambda_i\lambda_j \cosh d(p_i, p_j) = h^2, \qquad \sum_i \lambda_i \cosh d(z, p_i) = h\cosh d(z, \alpha).$$

Indeed in polar coordinates about α the rule is $\cosh d(z, p) = \cosh d(z, \alpha)\cosh\rho - \sinh d(z, \alpha)\sinh\rho\, v_z \cdot v_p$; the direction terms cancel on averaging. This rule also follows directly from the Lorentz product of $(\cosh R, \sinh R\, v)$. Define

$$S = |m_1|^2 + \frac{6}{25}(1 - |m_2|^2), \qquad T(\theta) = \Re(e^{i\theta}\overline{m}_1) + \frac{6}{25}[1 - \Re(e^{2i\theta}\overline{m}_2)], \tag{7}$$
$$W(\theta) = \max(S, T(\theta)).$$

Averaging (5) and (6) twice gives $h^2 \leq H^2 - DS$. Averaging once gives, for $\zeta = \cosh d(\omega(re^{i\theta}), \alpha)$, $\zeta^2 \leq h\zeta \leq 1 + 2D\sum_i \lambda_i L(\theta - \theta_i)$. Since $\zeta \leq h$ and $1 - 2\sum_i \lambda_i L(\theta - \theta_i) \geq T(\theta)$,

$$H^2 - \zeta^2 \geq DW(\theta). \tag{8}$$

Lemma 2 (Signed contact harmonic)**.** *For arbitrary complex m_1, m_2 with $|m_1|, |m_2| \leq 1$, an analytic $U(z) = \sum U_k z^k$ on $\mathbb{D}$, and $0 < r < 1$,*

$$\mathrm{M}(W(\theta)|U(re^{i\theta})|^2) \geq \eta r^2|U_1|^2, \qquad \eta = 19/200, \tag{9}$$

where M *denotes integration against $d\theta/(2\pi)$.*

Proof. If $|m_1| \ge 31/100$ or $|m_2| \le 77/100$, then $S > \eta$: $(31/100)^2 > 19/200$ and $(6/25)(1-(77/100)^2) > 19/200$. Parseval suffices. Otherwise rotate so $m_2 = b \ge 0$; reflection and a π shift put the two components of m_1 in the first quadrant. Reflection is implemented by conjugating the coefficients of U, preserving analyticity and $|U_1|$. We need only cover the larger rectangle

$$\begin{gathered} 0 \le x, y \le 31/100, \quad 77/100 \le b \le 1, \\ W = \max\left(x^2 + y^2 + \tfrac{6}{25}(1-b^2),\, x\cos\theta + y\sin\theta + \tfrac{6}{25}(1 - b\cos 2\theta)\right). \end{gathered} \tag{10}$$

When $S > 0$, put $J = \log W$ and define

$$C(z) = \exp\left(\frac{1}{2}\,\mathrm{M}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}J(\theta)\right]\right).$$

The kernel expansion gives $C_0^2 = e^{\mathrm{M}\,J}$, $C_0 > 0$, and $d := C_1/C_0 = \mathrm{M}(Je^{-i\theta})$. The real part of $\log C$ is half the Poisson integral of J. Parseval for $C(z)U(rz)$ on $|z| = s$, retaining its first two coefficients, and then $s \uparrow 1$ yield

$$\mathrm{M}(W|U|^2) \ge C_0^2(|U_0|^2 + |rU_1 + dU_0|^2) \ge \frac{e^{\mathrm{M}\,J}}{1+|d|^2} r^2 |U_1|^2.$$

The limit is justified by uniform Poisson convergence for continuous J and uniform convergence of $U(rse^{i\theta})$; no boundary continuity of C's harmonic conjugate is needed. Poisson convergence follows by splitting the integral into a short arc, controlled by uniform continuity, and its complement, on which the Poisson kernel tends uniformly to zero. The last inequality is exactly the square completion

$$|v|^2 + |u + dv|^2 = (1+|d|^2)\left|v + \frac{\bar{d}u}{1+|d|^2}\right|^2 + \frac{|u|^2}{1+|d|^2}.$$

Thus it suffices to certify

$$e^{\mathrm{M}\log W} \ge \eta\{1 + |\,\mathrm{M}(\log W e^{-i\theta})|^2\}. \tag{11}$$

This is why retaining the sign is useful: C_1 need not vanish, and square completion absorbs its coupling to U_0. No simultaneous constant coefficient correction is asserted.

We handle a neighborhood of the singular case analytically. Put $a = \sqrt{x^2+y^2}$, $\delta_0 = 1/10000$. For $a \le \delta_0$, $b \le 99/100$, the weight is at least $p + q_0 \sin^2\theta$, where $p = 23/10000$ and $q_0 = 231/625$. Its even factor is $A_0 - B_0 z^2$, with $A_0 = (\sqrt{p} + \sqrt{p+q_0})/2$ and $B_0 = (\sqrt{p+q_0} - \sqrt{p})/2$. The certificate checks $A_0^2 > \eta$, so its first two coefficients and Parseval prove (9). For $b \ge 99/100$, put $\ell = 297/625$; then $W \ge h_a = \max(a^2, \ell\sin^2\theta - a)$. At $a = 0$ use the factor $\sqrt{\ell}(1-z^2)/2$, since $\ell/4 > \eta$. For $0 < a \le \delta_0$, put $X = \sin^2\theta$, $X_0 = a(1+a)/\ell$, $d_0 = X_0/2$. The checks $X_0 < 1$ and $4\ell > (1+a)^2$ imply

$$h_a \ge \ell(X - d_0)^2.$$

On $[0, X_0]$ use $|X - d_0| \le X_0/2$; on $[X_0, 1]$ the difference $\ell X - a - \ell(X-d_0)^2$ is concave and nonnegative at both endpoints (the value at one is $a^2[1-(1+a)^2/(4\ell)]$). The polynomial $z^4 - 2(1-2d_0)z^2 + 1$ has four unit-circle roots and squared modulus $16(X-d_0)^2$. The circle mean of the logarithm of the distance to a unit-circle root is zero, by the mean-value identity inside the circle and its integrable logarithmic limit. Hence $\mathrm{M}\log(X-d_0)^2 = -\log 16$.

Subdivide each successive interval with endpoints $2^{-11}, 2^{-10}, \dots, 2^{-1}, 1023/1024$ into 128 equal rational bins $[u, v]$. For all these disjoint bins the checks give $u > 2\delta_0/\ell$, $u > d_0$, and $\gamma_v = 1/v - \delta_0/(\ell v^2) > 1$. On a bin, $(X-d_0)^2 \le X^2$ and $1/X - \delta_0/(\ell X^2)$ is decreasing; thus $h_a/[\ell(X-d_0)^2] \ge \gamma_v$. The bin has angular mass $\frac{2}{\pi}(\arcsin\sqrt{v} - \arcsin\sqrt{u})$. Outside the bins the ratio is at least one, so

$$\mathrm{M}\log h_a \ge \log(\ell/16) + \sum_{[u,v]} \frac{2}{\pi}(\arcsin\sqrt{v} - \arcsin\sqrt{u})\log\gamma_v > \log\eta. \tag{12}$$

All logarithmic zeros are integrable. Since h_a is positive, continuous, and π-periodic for $a > 0$, its outer factor is even. Its linear coefficient vanishes, and the preceding Parseval argument proves the lemma there.

For the remaining parameters (11) is a finite interval proof. The rational binary tree in `signed-cover.json` reconstructs (10) by strictly interior coordinate splits. Its leaves are two boxes with $S > \eta$, one with $x^2 + y^2 \le \delta_0^2$, and 210 grid boxes. Each grid box uses $n = 4096$ angular cells of half-width $h_0 = \pi/n$ and centers $\theta_j = (2j+1)\pi/n$. On a cell, sine and cosine differ from their center values by at most h_0, and $\cos 2\theta$ by at most $2h_0$, by their derivative bounds. Intersect with $[-1, 1]$ and apply interval multiplication to (10). Taking the maximum of the two lower (respectively upper) endpoints gives $0 < w_j^- \le W \le w_j^+$ on the entire parameter box and angular cell. With directed bounds $l_j \le \log w_j^-$ and $u_j \ge \log w_j^+$, set $M_j = (l_j + u_j)/2$, $E_j = (u_j - l_j)/2$. Cellwise triangle inequalities give

$$\mathrm{M}\, J \ge L_- := n^{-1} \sum l_j, \qquad |\,\mathrm{M}(Je^{-i\theta})| \le \left| n^{-1} \sum M_j e^{-i\theta_j} \right| + n^{-1} \sum (E_j + |M_j| h_0) =: B_*.$$

Here $|e^{-i\theta} - e^{-i\theta_j}| \le h_0$. Outward Arb arithmetic verifies $e^{L_-} - \eta(1 + B_*^2) > 0$ on every grid leaf; its minimum exceeds $9.7372642 \cdot 10^{-5}$. For (12), the log margin exceeds 0.07905954587; the other small case has $A_0^2 > 0.1081733546$. Exact rational tree reconstruction proves coverage, including boundaries. These finite checks, implemented in `verify_signed_cover.py`, establish the lemma without numerical quadrature or a singular endpoint assumption. □

4. From the lemma to the coefficient inequality

Define

$$\epsilon(t) = \eta \frac{D(t)}{H(t)(1 + H(t))^2} = \frac{19}{400} g(t)(1 - 1/H(t)).$$

Set $g = \epsilon = 0$ at $t = c^2, 1$ by continuity. Since $\Psi > 6$, $H > 2$, $g < 2/3$, and $0 \le \epsilon < (19/400)g$, whence $0 \le \beta - \epsilon \le \beta \le 1$. If $u \equiv 0$ domination gives $f \equiv 0$. Otherwise annular zeros of u are canceled by zeros of f of at least the same order, so $\omega = f/u$ is holomorphic on Ω. A nonconstant quotient maps into $\mathbb{D}$ by the maximum principle. A zero-radius image circle would force constancy by the identity theorem. Thus the preceding contact construction applies for each r, with center α allowed to depend on r. Let

$$F = f - \alpha u, \quad U = u - \bar{\alpha} f, \quad v(\theta) = 1 - \left| \frac{\omega - \alpha}{1 - \bar{\alpha}\omega} \right|^2 = \frac{2}{1 + \zeta}.$$

From (8), $\zeta \le H$, and elementary rationalization,

$$v - g = \frac{2(H - \zeta)}{(1 + \zeta)(1 + H)} \ge \frac{DW}{H(1 + H)^2}.$$

Both F, U are analytic on the whole disk even if f/u has inner poles. On the circle $|F|^2 = (1 - v)|U|^2$, and coefficientwise cancellation gives

$$|U_k|^2 - |F_k|^2 = (1 - |\alpha|^2)(|b_k|^2 - |a_k|^2).$$

Put $x_k = (|a_k|^2 - |b_k|^2)_+$, $y_k = (|b_k|^2 - |a_k|^2)_+$, $X(t) = \sum_{k \ge 0} x_k t^k$, $Y(t) = \sum_{k \ge 0} y_k t^k$. The identity implies $|U_k|^2 \ge (1 - |\alpha|^2) y_k$. Parseval and Lemma 2 give

$$(1 - |\alpha|^2)(Y - X) = \mathrm{M}(v|U|^2) \ge g\, \mathrm{M}\, |U|^2 + \epsilon r^2 |U_1|^2 \ge (1 - |\alpha|^2)(gY + \epsilon y_1 t).$$

Cancel the positive factor to obtain

$$X(t) \le \beta(t) Y(t) - \epsilon(t) y_1 t. \tag{13}$$

A constant quotient gives $X = 0$ and a nonnegative right side, so it too satisfies (13). Continuity extends the inequality to $t = c^2$; no boundary value at one will be needed.

5. Rational measure, finite arithmetic, and infinite tails

The certificate specifies $R = 99/100$ and

$$\mu = \sum_{j=1}^{8} w_j \delta_{t_j} + \mathbf{1}_{[R,1]}\, dt.$$

The following decimals are exact, with no rounding:

j	t_j	w_j
1	0.1996749225	0.423285875499212157
2	0.610958991619666304	0.332543359249341696
3	0.838503758015159552	0.143257541084897584
4	0.931950605621513344	0.057076131069107536
5	0.969424952196794112	0.023082618805674884
6	0.984486839940487424	0.008926372797755483
7	0.989470036574182144	0.00166310149401066
8	0.99	0.000165

Exact arithmetic checks positive ordered nodes, $t_1 = c^2$, $t_8 = R$, and $\sum w_j = R$. Thus μ is a probability measure with no atom at one. The sufficient moments are

$$\int t^k \, d\mu \geq \frac{1}{k+1} \quad (k \geq 0), \qquad \int \beta t^k \, d\mu \leq \frac{1}{k+1} \quad (k \geq 2), \tag{14}$$

$$\int (\beta - \epsilon) t \, d\mu \leq \frac{1}{2}.$$

The index-zero upper condition follows from $\beta \leq 1$. For $k \geq 1$, define

$$A_k = \sum_j w_j t_j^k - \frac{R^{k+1}}{k+1},$$

$$B_k = \frac{R^{k+1}}{k+1} + \int_R^1 g(t) t^k \, dt - \sum_j w_j \beta(t_j) t_j^k + \mathbf{1}_{k=1} \sum_j w_j \epsilon(t_j) t_j.$$

Then $A_k \geq 0$, $B_k \geq 0$ suffice for (14); at $k = 1$ we discard the nonnegative continuous ϵ contribution.

Here are the full tail-enclosure and arithmetic rules. For a rational cell $[a, b] \subset [R, 1]$, let

$$U_* = -\frac{\pi \log a}{2w}, \quad B'_* = \frac{\pi}{2w}\left(1 + \frac{1-b}{2}\right)(1 - U_*^2/6), \quad z_* = \sqrt{2} B'_* (1-a), \quad \ell_{ab} = \frac{2\sqrt{2} B'_*}{z_* + \sqrt{z_*^2 + \Psi}}.$$

The checker verifies $0 < U_* < \sqrt{6}$. For $\Delta = 1 - t$ on the cell, $-\log t \geq \Delta + \Delta^2/2 \geq \Delta(1 + (1-b)/2)$, by the positive logarithmic series. Also $\sin x \geq x - x^3/6$ for $x \geq 0$, obtained by integrating $1 - \cos x \leq x^2/2$. Thus $\sqrt{\sigma} \geq B'_* \Delta$. Writing $g = 2\sqrt{2\sigma}/(\sqrt{2\sigma} + \sqrt{2\sigma + \Psi})$ shows it increases with $\sqrt{\sigma}$; substituting the lower bound and then increasing the denominator after division by Δ proves $g(t) \geq \ell_{ab}(1 - t)$. The endpoint at one follows by continuity. Round ℓ_{ab} down and integrate with the exact antiderivative $F_k(t) = t^{k+1}/(k+1) - t^{k+2}/(k+2)$. The supplied 3200 rational cells partition all of $[R, 1]$; strict ordering and endpoints are checked. This proves every integral lower bound used in B_k, without quadrature. All atomic values and moment recurrences are enclosed by outward Arb balls from exact rational inputs.

The finite checks cover $1 \leq k < 6000$ for A_k and $1 \leq k < 1800$ for B_k. Writing $Q_k = R^{k+1}/(k+1)$, the retained 320-bit receipt gives

$$\min A_k/Q_k > 4.9881480 \cdot 10^{-6}, \qquad \min B_k/Q_k > 4.6984076 \cdot 10^{-6},$$

attained at indices 406 and 650 respectively. The $k = 1$ sufficient margin is $B_1 > 2.4445383 \cdot 10^{-6}$. For every $k \geq 6000$, the exact check $w_8\, 6001 - R = 33/200000 > 0$ gives $\sum w_j t_j^k \geq w_8 R^k \geq R^{k+1}/(k+1)$. For the second infinite tail take $a = 999/1000$, $b = 1249/1250$ and a fresh slope ℓ_{ab}. The verified inequality is

$$\ell_{ab}(1-b)(b-a)a^{1800} - \sum_j w_j t_j^{1800} > 8.8990994 \cdot 10^{-10} > 0.$$

Since $t_j \leq R \leq a$, division by a^k makes each atomic ratio $(t_j/a)^k$ nonincreasing. The inequality therefore persists for all $k \geq 1800$. Integration on $[a, b]$ alone gives $\int_R^1 g t^k \, dt > \sum w_j t_j^k \geq \sum w_j \beta(t_j) t_j^k$, proving the whole second tail.

Finally integrate (13) against μ. The weights β for $k \neq 1$ and $\beta - \epsilon$ for $k = 1$ are nonnegative. Tonelli and (14) give

$$\sum_{k \geq 0} \frac{x_k}{k+1} \leq \int X \, d\mu \leq \int (\beta Y - \epsilon y_1 t) \, d\mu \leq \sum_{k \geq 0} \frac{y_k}{k+1} < \infty.$$

Bergman Parseval proves Theorem 1.

6. Verification and disclosure

The fixed packet [10] contains `certificate-44685.json`, `signed-cover.json`, the proofs, verifiers and retained receipts. The complete measure and scalar-cover verification was rerun at 320 bits for this version and passed. All displayed diagnostic lower bounds above are rounded down from interval lower endpoints. The separate AI review retained in the packet uses another scalar implementation and a finer moment mesh; its slightly different moment margins are not substituted for those of the supplied receipt. The programs check finite quantitative hypotheses and exact coverage, not complex analysis or arbitrary mathematical prose. The packet's mutation controls reject corrupted inputs. The supplied review is an AI review, not independent human review.

The scope of the lemma matters. We claim only a correction for U_1. For example the tempting three-coefficient extension with constant $7/100$ is false: take $m_1 = 0, m_2 = 1$ and $V(z) = 1 + (3/2)z^2 + (9/8)z^4 + (3/4)z^6 + (3/8)z^8$. Then $W = (3/25)|1 - z^2|^2$ on the circle, and multiplying out $(1 - z^2)V$ and applying Parseval gives $\mathrm{M}(W|V|^2) = 87/400 < 91/400 = (7/100)(|V_0|^2 + |V_1|^2 + |V_2|^2)$. For any interior radius use $U(z) = V(z/r)$. This says nothing about more careful matrix corrections or optimality.

This is an unrefereed preprint. There has been no independent human review and no formalization.

Analysis and proof drafting by AI agents under my direction; I reviewed the results at summary level. The result was found and checked by AI agents under my direction. AI agents also prepared certificates, checking software, and publication materials. No line-by-line human verification is claimed. AI is not an author.

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