A TRACE UPPER-BOUND CERTIFICATE WITH CLASSICAL AUXILIARIES ON AN ADAPTIVE DENSITY

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Abstract. An exactly specified rational piecewise-linear density gives $C_{86} \leq 1.80560940$ for the Schur–Siegel–Smyth trace constant. It uses 28 classical auxiliary polynomials, ten with nonreal roots, and 5,375 rational nodes. Their recorded list occurrences are: Flammang 2019 (all 28), Liang–Wu 2011 (26), Flammang 2009 (21), McKee 2011 (21), Tomar's public A22 pool (18), Aguirre–Peral 2007 (16), and Flammang 2024 (9). The density refines infrastructure originating in Tomar's adaptive-node certificate. We give the full deduction from finite potential and logarithmic-integral checks through Smith's arithmetic-measure criterion and converse theorem, and describe the rigorous continuous enclosures. The contribution is the explicit feasible measure and its smaller upper bound. This note is unrefereed, with no independent human review or formalization.

hexagon:2610.00695v3

1. Result and antecedents

For a totally positive algebraic integer α, its normalized trace is the mean of its conjugates, $\mathrm{Tr}(\alpha)/[\mathbb{Q}(\alpha) : \mathbb{Q}]$. Write $C_{86} = \lambda_{\mathrm{SSS}}$ for the smallest limiting normalized trace, with finitely many exceptional algebraic integers disregarded, as in Smith [1].

Theorem 1.

$$C_{86} \leq \frac{288897505229428473}{160000001279369600} < 1.80560940.$$

The method is Smith's sufficient criterion for arithmetic measures, used in the measure optimization of Orloski and Talebizadeh Sardari [1, 2]. The related work of Orloski, Talebizadeh Sardari and Smith develops the trace problem's measure and auxiliary-function framework [3]. Selected earlier upper bounds are listed below; the computational entries are unrefereed claims, not assertions of independent human validation.

Source	Upper bound
Smith [1]	1.89831
Orloski–Talebizadeh Sardari [2]	1.8216
Pinho, earlier step-density certificate [11]	1.8117251
Dennis Koshta [12]	1.8084990
Vidit Patankar, PR 270 [13]	1.8077686
Tejasvi Singh Tomar [14]	1.8070808326
Pinho, version 2 of this note	1.80697883
Pinho, classical half-cell certificate [15]	1.80580911

Tomar's exact certificate ratio is $81383685642728309283453/45035996273705240000000$; the displayed decimal is rounded upward. Tomar's piecewise-linear construction and adaptive nodes, Koshta's enlarged auxiliary pool and checking ideas, and Patankar's further pool and grid enlargement are computational antecedents [14, 12, 13].

Date: 2 November 2026; version 3; unrefereed preprint.

2020 *Mathematics Subject Classification.* 11R04, 11C08, 31A05.

Key words and phrases. totally positive algebraic integers, Schur–Siegel–Smyth trace constant, logarithmic potential, auxiliary polynomials, interval certificates.

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Version 2 used 23 auxiliaries and emphasized a degree-eight pair with nonreal roots. Those polynomials already occur in Flammang's Table 1, rows 22 and 23, and McKee's Table 1, rows 21 and 23 [5, 6]. Their discovery and the use of complex-root auxiliaries in the trace problem are prior work. The present measure instead combines more of the classical lists with a refined adaptive density. Recorded occurrences cover all 28 in Flammang's 2019 first trace list [7], 26 in Liang–Wu's 2011 Table 1 [8], 21 each in Flammang's 2009 and McKee's 2011 Table 1 [5, 6], 18 in Tomar's public A22 pool [14], 16 in Aguirre–Peral's 2007 Table 2 [9], and 9 in Flammang's 2024 lists [10]. We claim the frozen feasible certificate and bound, without claiming new polynomials, a new sufficiency principle, or optimality. The value is filed, with its unrefereed qualification, in PR 298 [17].

2. Why a finite certificate suffices

We use the positive-log convention

$$U_\mu(z) = \int \log|z - x|\, d\mu(x).$$

Smith's potential has the opposite sign. Let μ have a bounded, continuous, nonnegative piecewise-linear density, extended by zero, with compact support $S \subset (0, \infty)$. Put $m = \mu(\mathbb{R}) > 0$ and $T = \int x\, d\mu(x)$.

Proposition 2. *Let Q_j be distinct primitive irreducible integer polynomials with positive leading coefficients and $b_j > 0$. Suppose that*

$$\beta := \sum_j b_j \deg Q_j < m, \tag{1}$$

$$I_j := \int \log|Q_j(x)|\, d\mu(x) \geq 0 \quad \textit{for every } j, \tag{2}$$

$$F(x) := U_\mu(x) - \sum_j b_j \log|Q_j(x)| \geq 0 \quad \textit{for every } x \in S. \tag{3}$$

Then $C_{86} \leq T/m$. No real-rootedness condition is imposed on Q_j.

Proof. This is the sufficient-certificate argument of the repository's `PROOF.md` [16], spelling out Smith's resultant criterion [1, Proposition 5.15]. The bounded compact density makes U_μ finite and continuous: the logarithmic singularity is uniformly locally integrable against a bounded line density; away from it the kernel is continuous. Off S and the finitely many auxiliary roots, F is harmonic. At an auxiliary root, the corresponding negative logarithm tends to $+\infty$, whereas U_μ remains finite. Uniformly at infinity,

$$F(z) = (m - \beta)\log|z| - \sum_j b_j \log(\operatorname{lc} Q_j) + o(1) \longrightarrow +\infty.$$

Excise small disks about the auxiliary roots and truncate at a large circle. Their boundaries have positive values, and the boundary along S has nonnegative values by (3). The minimum principle for harmonic functions [4] gives $F \geq 0$ on the remaining domains. Letting the disks shrink and the circle expand proves this inequality throughout $\mathbb{C}$, including the real gaps of S.

Let $P \in \mathbb{Z}[x]$ be primitive irreducible, of positive leading coefficient c, distinct from every Q_j, with roots α_k. Factorization over $\mathbb{C}$ and the resultant identity give

$$\begin{aligned}\int \log|P|\, d\mu &= m\log c + \sum_k U_\mu(\alpha_k) \\ &\geq m\log c + \sum_j b_j \sum_k \log|Q_j(\alpha_k)| \\ &= (m - \beta)\log c + \sum_j b_j \log|\operatorname{Res}(P, Q_j)| \geq 0.\end{aligned}$$

Indeed $c \geq 1$ and every resultant is a nonzero integer. The identity $\operatorname{Res}(P, Q_j) = c^{\deg Q_j} \prod_k Q_j(\alpha_k)$ includes all leading-coefficient terms, even for nonmonic auxiliaries. Selected $P = Q_j$ use their separate checks (2). Integer content and factorization therefore give $\int \log|P|\, d\mu \geq 0$ for every nonzero integer polynomial P. All these integrals exist, since a bounded density integrates each logarithmic root singularity.

Consequently $\nu = \mu/m$ is an arithmetic probability measure. The weights in Smith's normalized criterion are b_j/m. Choose a compact positive interval Σ containing S and of length greater than four. Its logarithmic capacity is its length divided by four [4]. Smith's Theorem 1.5 [1] then provides infinitely many distinct irreducible monic integer polynomials, all roots in Σ, whose normalized root measures converge weakly to ν. Their roots are totally positive algebraic integers. Testing weak convergence against the continuous function x on this compact interval gives normalized traces tending to T/m. This proves the stated bound. □

3. Exact input and classical attribution

The fixed data are in `c86/adaptive-cone/certificate.json` at the commit pinned in [16]. Its fields `edges`, `node_densities`, `A` and `b` specify rational nodes e_i, rational heights h_i, integer coefficient arrays (constant coefficient first), and rational weights. On $[e_i, e_{i+1}]$ the density interpolates h_i, h_{i+1} linearly; it is zero elsewhere. The exact data give

$$m = \frac{50000000399803}{500000000000000}, \qquad T = \frac{288897505229428473}{1600000000000000000},$$
$$\beta = \frac{89509133300897}{1000000000000000}, \qquad m - \beta = \frac{10490867498709}{1000000000000000} > 0.$$

These follow by summing, with $\Delta_i = e_{i+1} - e_i$,

$$m = \sum_i \frac{\Delta_i}{2}(h_i + h_{i+1}),$$
$$T = \sum_i \frac{\Delta_i}{6}\big((2e_i + e_{i+1})h_i + (e_i + 2e_{i+1})h_{i+1}\big),$$

obtained by integrating the affine density and x times that density on each cell. Thus no assumed unit mass enters the bound. The exact check of the final upward rounding can also be written

$$1.80560940 - \frac{T}{m} = \frac{3376916571357}{5000000039980300000000} > 0.$$

There are 5,375 nodes and 3,791 cells with a positive endpoint height. Their closed union is S, whose hull is $[17/320, 5237/1000]$; this does not assert that its intervening gaps carry density. For Proposition 2 we may use $\Sigma = [1/1000, 13]$, of capacity $12999/4000 > 1$. All 28 selected polynomials are monic, primitive, distinct and irreducible, and every weight is positive. The largest degree is 21; ten polynomials have nonreal roots.

Table 1 gives recorded source rows: all 28 occur in Flammang's 2019 first trace list [7]; 26 in Liang–Wu's 2011 Table 1 [8]; 21 each in Flammang's 2009 and McKee's 2011 Table 1 [5, 6]; 18 in Tomar's public A22 pool [14]; and 16 in Aguirre–Peral's 2007 Table 2 [9]. Flammang's 2024 Rhin-measure list contains 9 of them, and its sector lists contain 3 (among those 9) [10]; Flammang's 2019 second and third lists contain 13 and 8, respectively. Liang–Wu's matches include the nonreal-root auxiliaries $j = 21, 22, 25, 26, 27$; only $j = 23, 24$ have no recorded occurrence outside Flammang's lists. The dated columns expose the earliest published occurrence recorded here, without asserting first discovery or exhaustive priority. The exact weights are integer multiples of 10^{-14}. Complete coefficient arrays and all source matches are in `adaptive_checked-provenance.json` [16].

TABLE 1. Active inputs and recorded list matches: AP07 (16), F09 (21), M11 (21), LW11 (26), F19 (28), T (18), F24 (9). AP07: Aguirre–Peral 2007 Table 2; F09: Flammang 2009 Table 1; M11: McKee 2011 Table 1; LW11: Liang–Wu 2011 Table 1; F19: Flammang 2019 first trace list; T: Tomar's public A22 pool. Entries are source rows (pool entries for T); a dash means no recorded match in that source. A plus in F24 marks Flammang 2024 Rhin-list membership. The index j starts at zero; r_j counts nonreal roots.

j	$\deg Q_j$	r_j	AP07	F09	M11	LW11	F19	T	F24	$10^{14}b_j$
0	1	0	1	1	1	1	1	1	+	4133119422362
1	1	0	2	2	2	2	2	2	+	3447544095751
2	1	0	3	3	3	3	3	3	+	2014805323618
3	2	0	4	4	4	4	4	4	+	2239752443239
4	3	0	7	7	7	7	7	5	+	1284972133134
5	4	0	11	11	11	11	11	6	+	865418825428
6	4	0	12	12	12	12	12	7	+	926684307119
7	5	0	17	16	14	16	15	8	+	465206989929
8	5	0	14	13	15	13	16	9	–	550600969104
9	5	0	16	15	16	15	13	10	–	180658152586
10	5	0	15	14	13	14	14	12	–	561133439478
11	10	0	36	27	27	29	28	13	–	434034647635
12	10	0	38	29	25	31	30	14	–	323476166442
13	10	0	37	28	26	30	29	15	–	234508271836
14	8	0	–	25	20	25	24	16	–	371665324449
15	8	0	–	24	22	24	23	17	–	249638717832
16	7	0	31	21	19	21	20	18	–	239541442220
17	2	0	6	6	5	6	6	19	+	719437546552
18	8	2	–	23	23	23	26	–	–	257823933007
19	12	4	–	33	29	35	33	–	–	701940569923
20	12	4	–	32	30	34	35	–	–	569338653568
21	14	4	–	–	–	49	47	–	–	403799833432
22	14	2	–	–	–	46	48	–	–	349477648447
23	14	4	–	–	–	–	54	–	–	321626320155
24	15	2	–	–	–	–	58	–	–	3148218
25	16	4	–	–	–	54	64	–	–	19981362
26	17	2	–	–	–	61	72	–	–	282758541948
27	21	4	–	–	–	77	83	–	–	21080332809

4. Continuous checks and enclosure formulas

This section gives the analytic basis of the separately written standalone checker in [16], following its `VERIFIER_NOTES.md`. Exact rational arithmetic checks the nodes, nonnegative heights, zero exterior heights, moments, weights and degree deficit. FLINT exact factorization checks irreducibility; certified complex root balls account for every root, including nonreal ones. The analytic checks use Arb ball arithmetic, with acceptance only from a strictly positive lower endpoint. Floating diagnostics do not decide feasibility.

4.1. **Potentials and direct logarithmic integrals.** Let $s_i = (h_{i+1} - h_i)/(e_{i+1} - e_i)$, and set exterior slopes to zero. The slope jumps $J_i = s_i - s_{i-1}$ satisfy $\rho'' = \sum_i J_i \delta_{e_i}$ in the distributional sense. For real t put

$$G(t) = \frac{t^2}{2}\log|t| - \frac{3t^2}{4}, \qquad G(0) = 0.$$

Differentiation gives $G'(t) = t(\log|t| - 1)$ and $G''(t) = \log|t|$ away from zero, with the latter singularity integrable. Twice integrating by parts, with vanishing exterior density and slope, gives the real-axis identity

$$U_\mu(x) = \sum_i J_i G(x - e_i). \tag{4}$$

The same integrations against 1, x, and $x^2/2$ give the exact checks $\sum_i J_i = \sum_i J_i e_i = 0$ and $\sum_i J_i e_i^2/2 = m$.

For a separate computation of I_j, factor $Q_j(x) = \ell_j \prod_\alpha (x - \alpha)$ over its certified roots. On one density cell write $z = x - \alpha$ and $\rho(x) = sz + k$, where k is the affine cell density evaluated at α. A primitive for $\rho(x)\log|x - \alpha|$ is

$$\operatorname{Re}\left[\left(\frac{sz^2}{2} + kz\right)\operatorname{Log} z - \frac{sz^2}{4} - kz\right]. \tag{5}$$

Its derivative is the desired integrand: the nonlogarithmic derivative terms cancel and $sz + k = \rho(x)$ is real on the integration segment. For nonreal α use a continuous logarithm on the horizontal segment; for real α use $\log|z|$ and the continuous limits of $z\log|z|$ and $z^2\log|z|$ at zero. Sum endpoint differences over cells and roots and add $m\log\ell_j$. This proves the formula used to enclose every I_j, with no numerical quadrature or omitted leading coefficient.

4.2. **Closed-interval lower bounds.** For a trial interval $[c - r, c + r]$, write $x = c + h$, $|h| \le r$, and use Taylor order $N = 8$ in the separately written checker. For a knot with $t = c - e_i$ and $|t| > 2r$, the coefficients for $G(t + h)$ are $G(t)$, $G'(t)$, $\log|t|/2$, and

$$\frac{(-1)^{n-3}}{n(n-1)(n-2)t^{n-2}} \quad (3 \le n \le N).$$

They follow by differentiating $G'''(t) = 1/t$. The Lagrange remainder has absolute value at most

$$\frac{r^{N+1}}{(N+1)N(N-1)(|t| - r)^{N-1}}, \tag{6}$$

since $|G^{(N+1)}(u)| = (N-2)!/|u|^{N-1}$ and $|u| \ge |t| - r$ along the Taylor segment. Multiply this bound by $|J_i|$.

Near a knot, $|t| \le 2r$, the checker requires $|t| + r < 1$. Concavity from $G''(v) = \log|v| \le 0$ gives

$$-2r^2(1 - \log r) \le G(t + h) - G(t) - hG'(t) \le 0, \tag{7}$$

where $G'(0) = 0$. To see the lower bound, the absolute integral remainder is at most r times the integral of $-\log|v|$ on a containing interval of length $2r$. That integral is largest on $[-r, r]$, where it equals $2r(1 - \log r)$. Thus a positive J_i incurs the negative error in (7); for a negative J_i, the affine tangent already gives a lower bound. This argument also covers knots at interval endpoints.

For an auxiliary root α with $d = |c - \alpha| > 2r$, expand

$$\log|c + h - \alpha| = \log d + \operatorname{Re}\sum_{n=1}^{N} \frac{(-1)^{n+1}h^n}{n(c - \alpha)^n} + R.$$

The logarithm's convergent power series bounds its tail by

$$|R| \le \frac{(r/d)^{N+1}}{(N+1)(1 - r/d)} : \tag{8}$$

each denominator in the tail is at least $N + 1$, after which the geometric series sums explicitly. For a nearer root, the triangle inequality instead gives

$$-b_j \log|x - \alpha| \ge -b_j \log(d + r).$$

It remains valid at the root, with the left side interpreted as $+\infty$. Include $-\sum_j b_j \log\ell_j$ separately.

Summing these lower enclosures gives a polynomial in h minus rigorous remainder bounds. Interval evaluation on the closed ball $[-r, r]$ yields a lower bound for F on the entire trial interval. If it is not positive, bisect at the exact rational midpoint. The separately written checker uses 192-bit Arb arithmetic for these bounds. It saves accepted leaves with their exact rational endpoints; for each original support cell it checks that sorted leaves start and end at the cell endpoints and that adjacent endpoints coincide. Thus every point of the closed support, including every endpoint, is covered. Unsuccessful enclosures would leave the certificate unproved rather than certify a floating approximation.

4.3. **Receipts and conclusion.** For the fixed certificate, the saved receipt of the separately written checker reports positive direct integrals for all 28 polynomials and 15,332 accepted closed leaves covering all 3,791 support cells. Conservative downward summaries of its saved ball enclosures are

$$\min_j I_j > 8.94 \cdot 10^{-8}, \qquad F(x) > 1.29 \cdot 10^{-10} \quad (x \in S). \tag{9}$$

The receipt is `adaptive_checked-independent.json`. These numbers are obtained by comparing each saved lower ball with the displayed rational decimal; they are enclosure floors, not estimates of the true minima. The separate first checker, `verify_linear.py`, has a positive receipt on 10,243 pieces. The retained closed-cover audit reports successful endpoint and hash checks; a further, separately written AI review checks the same bound and all active auxiliaries [16].

The certificate SHA-256, binding these data and receipts, is

`10d68ec249f47c0504182d419fd8ddd1ca1c917e42a4bba59d944b9d1de0dad5`.

The rational identities above establish (1) and the stated ratio, and the continuous enclosures establish (2) and (3). Proposition 2 therefore proves Theorem 1. This is a computer-assisted proof using the frozen data; the analytic receipts are existing runs, not new full checker runs made for this writing revision. Both standalone checkers can be run on the pinned JSON to regenerate the finite checks.

5. Scope and disclosure

The density search uses nonnegative rational triangular hats and adds knots and free heights to the inherited adaptive grid. Floating linear programming proposes a measure; the proof concerns only its exported rational node data and verified inequalities. Refining the density and changing the active classical set both enter the certificate. We do not infer a unique cause of the numerical gain or an optimal solution from finite LP diagnostics.

The prior-art checks on 2 November 2026 covered the live Tao-table entry and open contributions, identified certificate repositories, selected public AI-result repositories, recent papers, and indexed community searches. Tomar's certificate is the strongest earlier external upper bound identified in those checks. Searches and list transcriptions are incomplete: in particular the source review records inaccessible or unaccepted Wang–Wu–Wu, Liang–Wu's lists for second and third powers (distinct from their 2011 Table 1 for the first power), Aguirre–Peral's 2008 trace-chapter list (distinct from their 2007 Farey-interval paper's Table 2), Wang–Pang–Wu and scanned Smyth lists. No exhaustive priority, auxiliary-pool coverage, or optimality claim follows. The pinned `METHOD.md` and `review/REVIEW.md` distinguish these search limits from the hypotheses of the frozen bound [16].

This is an unrefereed preprint. There has been no independent human review and no formalization. The result was found and checked by AI agents under Ricardo Pinho's direction. The AI checks do not formalize Smith's theorem or the complete analytic argument.

Analysis and proof drafting by AI agents under my direction; I reviewed the results at summary level. AI agents also prepared the certificate, checking software, and publication materials. No line-by-line human verification is claimed.

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