A TRACE UPPER-BOUND CERTIFICATE WITH COMPLEX-ROOT AUXILIARIES

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Abstract. A rational piecewise-linear measure with 23 auxiliary polynomials gives $C_{86} \leq 1.80697883$ for the Schur–Siegel–Smyth trace constant. The proof uses Smith's arithmetic-measure criterion and converse theorem. Two degree-eight auxiliaries have nonreal roots; both polynomials already occur in Flammang (2009) and McKee (2011). Their use in this upper-bound certificate, rather than their discovery, is the change described here. This note is unrefereed, with no independent human review or formalization.

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1. Statement and provenance

Let C_{86} be the smallest limit point of $\mathrm{Tr}(\alpha)/[\mathbb{Q}(\alpha):\mathbb{Q}]$ over totally positive algebraic integers α.

Theorem 1.

$$C_{86} \leq \frac{135523412047086487}{75000000000013500} < 1.80697883.$$

The construction follows Smith, Orloski–Talebizadeh Sardari, and Orloski–Sardari–Smith [1, 2, 5], using enlarged polynomial sets from Koshta and David Niedbala Giraudin's PR 270 [3]. Tomar's public piecewise-linear certificate gives $1.8070808325882437\ldots$ [4]; the supplied review reproduced that certificate and the smaller value above. This is a comparison with identified prior work, not an exhaustive priority claim.

The two auxiliaries in question are

$$Q_1(x) = x^8 - 14x^7 + 78x^6 - 221x^5 + 339x^4 - 277x^3 + 111x^2 - 19x + 1,$$
$$Q_2(x) = x^8 - 14x^7 + 78x^6 - 222x^5 + 345x^4 - 289x^3 + 120x^2 - 21x + 1.$$

Each has six real roots and one nonreal conjugate pair. Both occur in Flammang (2009), Table 1, rows 22 and 23, and McKee (2011), Table 1, rows 21 and 23 [6, 7]. Complex-root auxiliaries in this subject are therefore not new. Their earlier use for lower bounds differs from their role in this feasible upper-bound measure. No optimizer formula requiring real-rooted auxiliaries is invoked.

2. A sufficient certificate and its proof

Let μ have a bounded continuous nonnegative piecewise-linear density supported on a compact subset S of the positive real axis. Put

$$m = \mu(\mathbb{R}) > 0, \quad T = \int x\, d\mu(x), \quad U_\mu(z) = \int \log|z - x|\, d\mu(x).$$

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Let Q_j be distinct primitive irreducible integer polynomials with positive leading coefficients, and $b_j \geq 0$. It suffices to check

$$\beta := \sum_j b_j \deg Q_j < m, \qquad \int \log|Q_j|\, d\mu \geq 0 \quad \text{for every } j, \tag{1}$$

$$F(x) := U_\mu(x) - \sum_j b_j \log|Q_j(x)| \geq 0 \quad (x \in S). \tag{2}$$

At auxiliary roots the inequality has its extended-real meaning. The logarithmic potential is continuous because the density is bounded. Off S and the auxiliary roots, F is harmonic; at an auxiliary root with positive weight it tends to $+\infty$. At infinity $F(z) = (m-\beta)\log|z| + O(1) \to +\infty$. Excising small disks about those roots and truncating at a large circle, the minimum principle extends (2) to every complex z. This includes gaps in S and nonreal roots.

For a primitive irreducible integer polynomial P, distinct from the Q_j, with leading coefficient $c > 0$ and roots α_k, resultants give

$$\begin{aligned}\int \log|P|\, d\mu &= m\log c + \sum_k U_\mu(\alpha_k) \\ &\geq (m-\beta)\log c + \sum_j b_j \log|\operatorname{Res}(P, Q_j)| \geq 0.\end{aligned}$$

Each resultant is a nonzero integer. This identity includes the leading coefficient of every Q_j in $\log|Q_j|$, so no monicity assumption has been suppressed. The selected polynomials satisfy (1); factorization and integer content cover every nonzero integer polynomial. Thus μ/m is an arithmetic probability measure in Smith's terminology [1, Proposition 5.15].

In the present certificate $S \subset [27/500, 5227/1000]$. Take the ambient interval $[1/100, 6]$, whose logarithmic capacity is $599/400 > 1$. Smith's Theorem 1.5 gives distinct totally positive algebraic integers whose conjugate measures converge weakly to μ/m in this interval. The continuous test function x shows that their normalized traces converge to T/m. Hence $C_{86} \leq T/m$.

3. The finite input and what has been checked

The exact rational node heights, edges, 23 polynomials and weights are in `complex-auxiliaries/certificate.json` of [8]. Their stored exact values are

$$m = \frac{50000000000009}{50000000000000}, \qquad T = \frac{135523412047086487}{75000000000000000},$$
$$\beta = \frac{43742583595997}{50000000000000}.$$

The certificate SHA-256 is

`f59be9a136ad96930f20d4f9a7a6ff019d9a30ab9c9081c0c05582a3f8e0ea43`.

The supplied `verify_linear.py` checks the rational structure and irreducibility, encloses complex roots, evaluates logarithmic integrals analytically in ball arithmetic, and covers every closed support cell. Its retained receipt records positive integrals for all 23 polynomials and a positive potential lower bound on 5587 pieces. A separate AI review used another implementation and an endpoint/adjacency audit; its refined 9764-piece cover proves $F > 10^{-6}$ on all 1556 nonzero density cells. These are stored results, not new verifier runs for this note. They prove feasibility, not optimality of the auxiliary pool or density.

The certificate citation is pinned to a fixed public commit.

This is an unrefereed preprint. There has been no independent human review and no formalization. The AI checks do not formalize Smith's theorem or the complete analytic argument.

Analysis and proof drafting by AI agents under my direction; I reviewed the results at summary level. AI agents also prepared the certificate, checking software, and publication materials. No line-by-line human verification is claimed.

References

[1] A. Smith, Algebraic integers with conjugates in a prescribed distribution, *Annals of Mathematics* 200 (2024), Theorem 1.5 and Proposition 5.15; `https://arxiv.org/html/2111.12660v2`.

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[5] B. Orloski, N. Talebizadeh Sardari and A. Smith, trace-bound construction, *Mathematics of Computation* 94 (2025), 2005–2040; `https://arxiv.org/abs/2401.03252`.

[6] V. Flammang, Trace of totally positive algebraic integers and integer transfinite diameter, *Mathematics of Computation* 78 (2009), 1119–1125, Table 1, rows 22–23; `https://doi.org/10.1090/S0025-5718-08-02120-0`.

[7] J. McKee, Computing totally positive algebraic integers of small trace, *Mathematics of Computation* 80 (2011), 1041–1052, Table 1, rows 21 and 23; `https://doi.org/10.1090/S0025-5718-2010-02424-X`.

[8] R. Pinho, *C86 trace certificate*, directory `complex-auxiliaries`, commit `e4d9c64`; `https://github.com/rpinho/c86-trace-certificate/tree/e4d9c640ebaf922b8869a81f275c798c09ab89b1/complex-auxiliaries`.

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