DERIVATIVE CONTROLS AND BOOLEAN COMPLETION IN THE RELEVANT-VARIABLE RECURRENCE

RICARDO PINHO

Abstract. We give the unrefereed bound $C_{44} < 2.0417086280$ for the number of relevant variables of a Boolean function of real degree d, $r(f) \le C_{44}2^d$. The argument bounds the number of coordinates controlling a ternary derivative, exploits its dictator branches and Boolean completion, and propagates local-degree caps. The small-degree inputs are $R_4 = 22$, $R_5 \le 50$, and $R_6 \le 109$. The quartic proof uses an exact calculation on Rasoolzadeh's published cubic catalogue; the remaining finite arithmetic is rational. The lower bound remains 3/2. There has been no independent human review or formalization.

hexagon:2610.00694v3

1. Result, provenance, and scope

For sign Boolean functions on uniform independent sign inputs, let $r(f)$ be the number of relevant coordinates and $\deg f$ the real multilinear Fourier degree. Put $R_d = \max_{\deg f \le d} r(f)$ and $C_{44} = \sup_{d\ge 1} R_d/2^d$.

Theorem 1. *The argument and exact finite checks below give*

$$R_4 = 22, \qquad 46 \le R_5 \le 50, \qquad 94 \le R_6 \le 109.$$

$$\frac{3}{2} \le C_{44} \le \frac{8088041149216617370543733610 7}{3961408125713216879677197 5168} < 2.0417086280.$$

This is a large move for an elementary argument. Please read it with suspicion and scrutinize the proof, especially the classification bridge, the completion inequalities, and the finite case coverage.

The weighted restriction method, including its potential recurrence, is due to Chiarelli, Hatami and Saks (CHS) [4]; Wellens developed sharper recurrences and bounds [12]. Wellens's published bound is 4.394; BakeLab/Épi's subsequent unrefereed value is 4.3654674227, using exact polynomial certificates through degree 48 [13]. The author's earlier unrefereed values were 4.3576549226, 4.1225323803 (version 2 of this note), 3.9819073803, and 2.1974348818. The current claim is filed in Tao-table PR 294 [16]; it is not a claim of refereed acceptance.

The contribution relative to version 2 is the derivative-control argument, the cubic energy envelope and quartic equality exclusions, and the use of Boolean completion to sharpen the subsequent caps. The proof below follows the frozen PROOF files [14, 15]; it uses their `simple` mode. The earlier 52-case quartic/quintic control classification is compatible with the result but is unnecessary for this mode. The old block-sensitivity certificates are also unnecessary for this endpoint. The historical block-sensitivity route also credits D. Koshta, I. Stepanovs, and Alexander Richard Towell; see [14]. The lower construction is CHS's, also observed by Shinkar and Tal. Minimum-support structure is due to Valyuzhenich; the binary low-weight spectrum to Berlekamp–Sloane and Kasami–Tokura; the lattice gap argument to Nebe, building on Barnes–Wall and Broué–Enguehard. Related ternary energy arguments occur in Potapov [9]. No priority or optimality is asserted. The bounded searches recorded in the packet's PRIOR_ART files and refreshed on 2 November 2026

Date: 2 November 2026; version 3; unrefereed preprint.

2020 *Mathematics Subject Classification.* 06E30, 68Q25, 68R05.

Key words and phrases. Boolean functions, relevant variables, local degree, ternary derivatives, Reed–Muller codes.

ORCID: 0009-0006-2744-2599. Licensed under CC BY 4.0.

located no superseding applicable claim; authenticated global code search and private discussions were outside coverage.

2. Potential and derivative controls

Write $f = \sum_S \widehat{f}(S) x_S$, $x_S = \prod_{i\in S} x_i$. For relevant i let $d_i(f) = \max\{|S| : i \in S, \widehat{f}(S) \neq 0\}$ and put $\mathcal{D}(f) = \sum_i 2^{-d_i(f)}$, $W_d = \max_{\deg f \le d} \mathcal{D}(f)$. A nonzero degree-at-most-k polynomial has support density at least 2^{-k}: write $P = x_j A + B$ with $A \neq 0$ of degree at most $k-1$; on the support of A, at least one of $B + A, B - A$ is nonzero. Induction, starting with constants, proves the assertion. Thus the ternary derivative $A_i = (f_{i=1} - f_{i=-1})/2$ has influence $p_i = \mathbb{E}A_i^2 \ge 2^{1-d_i}$. Fourier orthogonality gives

$$I(f) = \sum_i p_i = \sum_S |S|\widehat{f}(S)^2 \le \deg f, \qquad \mathcal{D}(f) \le I(f)/2. \tag{1}$$

In particular $R_d \le d2^{d-1}$, so both maxima exist after removing dummy coordinates and relabeling.

We use the CHS identity $W_d = R_d/2^d$ [4, Corollary 2.2]. Here is its extremal argument. In a potential maximizer duplicate all coordinates L of local degree below d, sharing the other coordinates Y, and form $h = (1+z)f(L,Y)/2 + (1-z)f(L',Y)/2$. Degree-$d$ terms use only Y and cancel in the selector difference, so $\deg h \le d$. Each duplicated coordinate increases its local degree by one; its two copies retain its total weight. Shared coordinates retain local degree d, and the relevant selector adds positive weight. Hence L was empty. The reverse inequality follows by taking a count maximizer. A count maximizer separately has all local degrees d: replace a bit coordinate of smaller local degree by the AND of two fresh bits. Degree stays at most d, every old dependence survives, and the count increases.

For any fixed H and any chosen subset Z of coordinates outside H,

$$\sum_{i\in Z} 2^{-d_i(f)} \le \mathbb{E}_\alpha \mathcal{D}_Z(f_\alpha). \tag{2}$$

Here absent coordinates contribute zero. For a single restriction, if i survives both branches its local degrees cannot increase; if it survives only one, its derivative factors as $(1 \pm x_j)/2$ times the surviving branch derivative, and its local degree falls by exactly one. In either case its mean weight cannot decrease. Iterate this statement for each $i \in Z$, then sum. In particular $\mathcal{D}(f) \le \sum_{j\in H} 2^{-d_j(f)} + \mathbb{E}\mathcal{D}(f_\alpha)$.

Let A be ternary of degree at most $k \ge 1$, support density p, and relevant count h. Each nonzero $\Delta_j A = (A_{j=1} - A_{j=-1})/2$ has nonzero magnitude at least $1/2$. The support bound and orthogonality give

$$\mathbb{E}(\Delta_j A)^2 \ge 2^{-k-1}, \qquad h2^{-k-1} \le \sum_j \mathbb{E}(\Delta_j A)^2 = \sum_S |S|\widehat{A}(S)^2 \le kp. \tag{3}$$

Take a count maximizer f, $N = R_d$, $d \ge 2$, and choose i of minimum influence. Set $k = d-1$, $M = 2^k$, $\lambda = Mp_i$, and let H be the relevant coordinates of A_i, $h = |H|$. Then

$$1 \le \lambda \le dM/N, \qquad h \le \lfloor 2k\lambda \rfloor. \tag{4}$$

Write $f = x_i A_i(H) + B(H,Y)$. On active assignments $A_i = \pm 1$, Booleanity forces $B = 0$ identically in Y; the restriction is a signed dictator. Moreover H hits every degree-d monomial: a degree-d coefficient of B missing H survives restriction unchanged, contradicting an active assignment, and a top monomial involving i meets H because $d \ge 2$. Thus every restriction has degree at most $d-1$. For $c \ge W_{d-1}$, $c \ge 1/2$, the potential inequality gives

$$N \le 2^d c + h + (1-2c)\lambda. \tag{5}$$

We record the exact possible-weight relaxation. Integer-valued functions have integer bit coefficients by finite differences: $a_T = \sum_{U\subseteq T} (-1)^{|T|-|U|} A(1_U)$. Apply this to $(1-f)/2$ and substitute

bit $= (1-x)/2$: the sign Fourier coefficients of f, hence of A_i, lie in $M^{-1}\mathbb{Z}$. Write them z_S/M. Since $\sum z_S = MA_i(\mathbf{1})$ and M is even, $\sum z_S^2$ is even, giving $\lambda \in (2/M)\mathbb{Z}$. Reducing the integer bit coefficients of A_i itself modulo two gives its support indicator in $\mathrm{RM}(k,n)$, without squaring it. The published below-twice-minimum weight theorem [5, 3] says that $\lambda < 2$ must be $2-2^{-j}$, $j \geq 0$; dummy inputs ensure $n \geq k+2$. The energy grid excludes $j \geq k$. We therefore allow

$$\{2-2^{-j} : 0 \leq j < k\} \quad \text{below } 2, \qquad (2/M)\mathbb{Z} \quad \text{at or above } 2. \tag{6}$$

Parameter restrictions have only been relaxed. No higher-range Kasami–Tokura–Azumi table or binary affine degree-preservation assertion is used.

3. The quartic input: a complete equality argument

The published cubic result is $R_3 = 10$, due to Tarannikov–Kirienko, as stated for arbitrary cubic Boolean functions in [6, Section 3]; hence $W_3 = 5/4$. That source also credits Zverev's computer-free cubic argument. We need one further finite consequence of Rasoolzadeh's published complete classification [10]:

$$\mathcal{D}(g) \leq 1 + E_3(g)/4, \qquad E_3(g) = \sum_{|S|=3} \widehat{g}(S)^2 \quad (\deg g \leq 3). \tag{7}$$

Here is the exact calculation specifying that consequence. Pad g to ten inputs and multiply by full parity. This complements Fourier masks and identifies the resulting functions with the 6-resilient catalogue; input permutations/flips and output complementation preserve the quantities in question. The primary catalogue has 761 representatives. For each bit row $b(x)$ compute the integers

$$w_S = \sum_{x \in \{0,1\}^{10}} (-1)^{b(x)+|x|+\sum_{i\in S} x_i}.$$

Check $w_S = 0$ for $|S| > 3$ and $\sum w_S^2 = 2^{20}$; determine local degrees from nonzero masks. If $a = \sum_{i:d_i>0} 2^{3-d_i}$ and $T = \sum_{|S|=3} w_S^2$, the check for (7) is precisely $(a-8)2^{20} \leq 2T$. Direct integer character sums on every row pass. The equality data are

catalogue line	E_3	$\mathcal{D}$	relevant count, common local degree
757	0	1	$4, 2$
716	1/2	9/8	$9, 3$
700	1	5/4	$10, 3$

The data and direct checker are pinned in [15], `r4-exact/sources/F10T6.txt` and `r4-exact/verify_catalogue.cpp`. The data SHA-256 is

`d0ca4026c3bfa61f7c40945e4627feecd9264d7854114806c533d64e06836ba0`.

This is a finite computational proof of the envelope and equality table; completeness of the catalogue remains the published classification premise.

Choose a quartic count maximizer. If no derivative has influence 1/8, (6) gives $N \leq \lfloor 64/3 \rfloor = 21$. Otherwise choose one with influence 1/8. Valyuzhenich's minimum-support classification [11, Theorem 2(2)] describes its controls as disjoint point factors and blocks of size $m \geq 2$ supported on the two constant strings. Even blocks have opposite signs, odd blocks equal signs; their degree cost is $m-1$, and a point factor has cost one. Thus $r + \sum(m-1) = 3$, $h = r + \sum m \leq 6$. For $h \leq 4$, (5) gives $N/16 \leq 45/32$, so $N \leq 22$. The only remaining types are a point and two pairs, a pair and a triple, and three pairs.

Let $Q_\alpha(Y)$ be the homogeneous cubic part after fixing H. Its coefficients are affine in H and vanish on the active set; it is independent of i. Write $e = \mathbb{E}_\alpha \|Q_\alpha\|_2^2$. Apply (7) on inactive assignments and use potential 1/2 on the active fraction 1/8. The restriction inequality yields

$$N \leq h + 15 + 4e. \tag{8}$$

All vector norms below are in the Hilbert space of cubic polynomials in Y.

For a point and two pairs, normalize the active conditions to $u = 1, s = t, v = w$. Affine vanishing gives $Q = (1-u)P_0/2 + (s-t)P_1/2 + (v-w)P_2/2$. On the full-mismatch slice, averaging $\|Q\|^2 \le 1$ over independent signs gives $\sum \|P_j\|^2 \le 1$; the full average is half that sum. Thus $e \le 1/2$ and (8) gives $N \le 22$.

For a pair and a triple, normalize to $s = t, u = v = w$. Then $Q = (s-t)P/2 + uC_1 + vC_2 + wC_3$, with $C_1 + C_2 + C_3 = 0$. On nonconstant triple assignments and an unequal pair, averaging signs gives $\|P\|^2 + 4\|C_j\|^2 \le 1$ for each j. Hence $e = \|P\|^2/2 + \sum \|C_j\|^2 \le 3/4 - \|P\|^2/4 \le 3/4$. This gives $N \le 23$. Equality forces $P = 0$, $\|C_j\|^2 = 1/4$. Each $-2C_j$ is then sign Boolean by Parseval on its restriction, but their sum is zero, impossible for three signs. Thus $N \le 22$.

For three pairs, $Q = \sum_{j=1}^{3}(s_j - t_j)P_j/2$. Full-mismatch averaging gives $\sum e_j \le 1$, $e_j = \|P_j\|^2$, and $e = \sum e_j/2 \le 1/2$; again $N \le 23$. If $N = 23$, all inequalities are equalities. All eight full-mismatch norms equal one, so expanding them gives $\langle P_j, P_l\rangle = 0$ for $j \ne l$. On a mismatch set K the cubic energy is $\sum_{j\in K} e_j$. The equality table on singleton mismatches forces $e_j \in \{0, 1/2, 1\}$ and $\sum e_j = 1$.

If one $e_j = 1$, then on $s = -t$ the entire function is $sP_j(Y)$. We use the following pair argument. If a degree-at-most-d Boolean F satisfies $F(s, -s, Z) = sg(Z)$, write $F = U + sV + tW + stT$. Then $U = T$, $V - W = g$, $\deg U \le d-2$, and $h(s, Z) = sF(s, s, Z) = V + W + 2sU$ is Boolean of degree at most $d-1$. Every relevant Z occurs in g or h, so $r(F) \le 2 + 2R_{d-1}$. Here this is 22, a contradiction. If two $e_j = 1/2$, each singleton mismatch is an equality cubic with nine variables all of local degree three; hence each P_j uses nine Y coordinates. Since $|Y| = 23 - 6 - 1 = 16$, they share some y. On either single mismatch (probability $1/4$ each), y has local degree three; on a double mismatch it survives for at least half the signs, since its derivatives in $P_1 + P_2$ and $P_1 - P_2$ cannot both vanish. Its expected weight is therefore at least $(1/4 + 1/4 + 1/8)/8 = 5/64 > 1/16$, contradicting equality in the coordinatewise restriction inequality. This proves $R_4 \le 22$.

For attainment and later lower bounds use the CHS/Shinkar–Tal family

$$\Xi_1 = x, \qquad \Xi_d = \frac{s+t}{2}\Xi_{d-1}(X) + \frac{s-t}{2}\Xi_{d-1}(Y) \tag{9}$$

on disjoint inputs. Exactly one summand is active. All Fourier masks have size d, every variable remains relevant, and $L_1 = 1, L_d = 2L_{d-1} + 2$, whence $L_d = 3 \cdot 2^{d-1} - 2$. This proves $R_4 = 22$, the lower counts $46, 94$, and $C_{44} \ge 3/2$.

4. Equality and the integer energy gap

Two refinements of (3) are needed. First, for $k \ge 3$,

$$\lambda \notin \mathbb{Z} \quad \Longrightarrow \quad h \le \lceil 2k\lambda \rceil - 1. \tag{10}$$

Indeed equality in (3) makes A homogeneous of degree k and each partial minimum-supported with nonzero magnitude $1/2$. In Valyuzhenich's tensor classification a homogeneous minimum polynomial has only two-coordinate factors: point factors and blocks of size at least three have distinct Fourier degrees that cannot cancel in disjoint products. Each partial therefore has Fourier masks the transversals of $k-1$ disjoint pairs. The pure simplicial complex generated by the k-masks of A has every vertex link a cross-polytope boundary. Masuda–Panov's published Lemma 4.7 [7] implies that each connected component is a cross-polytope boundary, on $2k$ vertices. Hence h is a multiple of $2k$, forcing $\lambda = h/(2k)$ integral. This proves (10); away from equality the integer ceiling is immediate.

Lemma 2 (Integer energy gap)**.** *For a nonzero integer-valued polynomial Q of real degree at most r, $\mathbb{E}Q^2 = 2^{-r}$ or $\mathbb{E}Q^2 \ge (3/2)2^{-r}$.*

Proof. We give the lattice proof expanded in the packet, crediting Nebe [8], so the recent gap is not an unexplained premise. On n bit coordinates let b_T be the bit-monomial truth vector. The classical Barnes–Wall bases are $2^{\lfloor |T|/2 \rfloor} b_T$ for Λ_n and $2^{\lceil |T|/2 \rceil} b_T$ for Δ_n; their minimum squared norms are

$a_n = 2^{n-1}$ and $2a_n$ [1, Theorems 3.1(ii), 3.2]. We also use the classical pair-preserving minimum-vector transitivity [2, Theorem II.4] (with the dimension-eight stabilizer convention specified in [8, Theorem 2.5]).

The map $S(u,v) = (u+v, u-v)$ on the last-coordinate branches satisfies $S\Lambda_n = \Delta_n$, $S\Delta_n = 2\Lambda_n$, and $S^2 = 2I$. For a monomial not using the last bit, S sends its equal branches to $2b_T(1-b_n)$; for one using it, S sends $(0, b_T)$ to $b_T - 2b_n b_T$. Comparing the floor and ceiling exponents proves both containments, and $S^2 = 2I$ proves equality. Thus the normalized norm spectra agree. The bases likewise give

$$\Lambda_{n+1} = \{(\ell + \delta, \ell) : \ell \in \Lambda_n,\ \delta \in \Delta_n\}.$$

A minimum-vector coset $\ell + \Delta_n$ has no norm strictly between a_n and $2a_n$. For odd n, transitivity permits $\ell = 2^{(n-1)/2} e_0$. If $x = \ell + \delta$ has largest common coordinate power 2^j with $j \geq (n-1)/2$, norm divisibility proves the claim. Otherwise the odd positions of $x/2^j$ equal those of $\delta/2^j$. The bit-coefficient divisibilities in Δ_n make this nonzero binary residue have degree at most $2j$. Its support has at least 2^{n-2j} positions, so $\|x\|^2 \geq 2^{2j} 2^{n-2j} = 2a_n$. The binary support bound follows by branching induction: a zero branch costs one degree, and two nonzero branches add their lower weights. For even n use $(\Delta_n, 2\Lambda_n)$ and minimum vector $2^{n/2} e_0$. The corresponding residue degree is at most $2j-1$, giving norm at least 2^{n+1}; $j = 0$ would make the residue zero. Transport by S gives the same coset claim for (Λ_n, Δ_n).

Every Λ_n norm is divisible by $2^{\lceil n/2 \rceil}$ for $n \geq 2$: basis-vector norms are 2^n or 2^{n-1}, and twice the inner product for T, U has exponent $1 + \lfloor |T|/2 \rfloor + \lfloor |U|/2 \rfloor + n - |T \cup U| \geq \lceil n/2 \rceil$ (split at $|T| + |U| = n$). For $2 \leq n \leq 5$ this gives the $3/2$ gap; $n = 1$ is the integer lattice. Induct in the displayed recursion, ordering the branches by norm. If the smaller branch is zero, use the gap for Δ_n; if nonminimal, both branches have norm at least $3a_n/2$; if minimal, the coset claim makes the larger branch minimal or of norm at least $2a_n$. Thus any total norm above $2a_n$ is at least $3a_n$, proving the gap.

Finally append dummy inputs so $n \geq \max(r, 2)$. Integer bit coefficients put $2^{(r-1)/2} Q$ in Λ_n for odd r, and $2^{r/2} Q$ in Δ_n for even r. In both cases its squared norm divided by the lattice minimum is $2^r \mathbb{E} Q^2$. The norm gap proves the lemma. □

For a relevant partial of our ternary A, apply the lemma to $Q = 2\Delta_j A$ of degree at most $k-1$. With $e_0 = 2^{-k-1}$,

$$\mathbb{E}(\Delta_j A)^2 = e_0 \quad \text{or} \quad \mathbb{E}(\Delta_j A)^2 \geq 3e_0/2. \tag{11}$$

Since total derivative energy is at most $2k\lambda e_0$, the condition $3h > 4k\lambda$ forces a minimum partial. Equality in its support and integer-magnitude bounds makes its values $0, \pm 1/2$.

5. Boolean completion and the next two degrees

Restrict H and apply (2) only to Y. Active branches have no Y dependence; inactive branches have degree at most $d-1$ and potential at most c. Since $|Y| = N - h - 1$, this gives the improved budget

$$N \leq O_d(h, \lambda) := 2^d c + h + 1 - 2c\lambda. \tag{12}$$

Now assume $d \geq 4$, $\lambda > 1$ and $3h > 4k\lambda$; choose a minimum partial j from (11). Among j-edges indexed by $H \setminus \{j\}$, let q be the fraction with one active endpoint and r the fraction with two. Double-active signs agree, since the partial never has magnitude one. Thus $q = 4e_0 = 2^{1-k}$ and $r = p - q/2 = (\lambda - 1)2^{-k} > 0$. An affine function vanishing on the active set therefore has zero x_j coefficient, by subtracting its values on a double-active edge.

Set $K = H \setminus \{j\}$. A degree-d monomial of B missing K has Y-size d or $d-1$. Its constant or affine control coefficient vanishes on the active set; the preceding edge argument kills its possible

j coefficient. An x_iA term missing K uses only i, j and has degree at most two. Hence every restriction to K still has degree at most $d-1$.

For a section on (i, j, Y), no active endpoint gives no i dependence and $\mathcal{D}_{j,Y} \le c$; two active endpoints give the dictator $\pm i$ and $\mathcal{D}_{j,Y} = 0$. On a one-active edge change signs to write $F = (1+j)i/2 + (1-j)g(Y)/2$. If $t = \deg g \ge 1$, then $t \le d-2$ because the top j-multiple cannot cancel, and $\mathcal{D}_{j,Y}(F) = 2^{-t-1} + \mathcal{D}(g)/2$. For constant g, it equals $1/4$. With already proved caps $c_t \ge W_t$ put

$$b_d = \max\left(\frac{1}{2}, \max_{1\le t\le d-2}\left(\frac{1}{4} + 2^{-t-1} + \frac{c_t}{2}\right)\right). \tag{13}$$

Apply (2) to $\{j\} \cup Y$, of size $N-h$. The mean section potential is at most $c(1-q-r) + (b_d - 1/4)q$. Using $2^d q = 4$ gives the second budget

$$N \le T_d(h, \lambda) := 2^d c + h - 2c\lambda + 4\left(b_d - \frac{c + 1/2}{2}\right). \tag{14}$$

At degree five $c = 11/8, b_5 = 15/16$. For $\lambda = 1$, $h \le 8$ and (12) gives $201/4$, hence $N \le 50$. For $\lambda = 3/2$, (10) gives $h \le 11$. For $h \le 8$ use (12), at most $391/8$; for $9 \le h \le 11$ use (14), at most $407/8$. Both imply $N \le 50$. All remaining allowed weights are at least $7/4$, so (4) gives $N \le \lfloor 80/(7/4) \rfloor = 45$. Thus $R_5 \le 50$ and $c_5 = 25/16$. At degree six $b_6 = 31/32$ and the correction in (14) is $-1/4$. For $\lambda = 1$ the ordinary maximum is $863/8$. For $\lambda = 3/2$, $h \le 14$; the maxima for $h \le 10$ and $11 \le h \le 14$ are $1701/16$ and $1745/16$. All imply $N \le 109$. For $\lambda \ge 7/4$, the influence bound gives $N \le \lfloor 192/(7/4) \rfloor = 109$. Hence $R_6 \le 109$.

6. Exact propagation and the infinite tail

Start with $c_1 = 1/2, c_2 = 1, c_3 = 5/4, c_4 = 11/8$; the first two are exact by (1) and (9). At each $d \ge 5$ use only earlier caps, $c = c_{d-1}$ and (13). For a candidate excluded count n, test the weights (6) in $[1, d2^{d-1}/n]$. Let $h_{\max} = \lfloor 2k\lambda \rfloor$, demoted by one when λ is nonintegral and $2k\lambda$ integral, as required by (10). At each integer $0 \le h \le h_{\max}$ use O_d and, when $\lambda > 1, 3h > 4k\lambda$, also T_d. Take the smaller applicable budget and maximize over h. Both ranges increase with slope one in h, so it suffices to test $h_{\max}$ and $\min(h_{\max}, \lfloor 4k\lambda/3 \rfloor)$. If the maximum is strictly below n for every weight, all $N \ge n$ are excluded by (4); set $c_d = (n-1)/2^d$ and apply the CHS identity.

Here is the complete finite cover, avoiding an exponentially long grid. Write $G = 2^{d-2}$, $\lambda = z/G$. Below two enumerate (6). At or above two test $z = 2G$, each cell start $\lceil aG/(2k) \rceil$ and its next integer, and each start $\lceil 3aG/(4k) \rceil$ of a cell for $\lfloor 4kz/(3G) \rfloor$. Retain indices in the tested interval. These are all joint floor changes; the extra next point covers a demoted equality endpoint. Between changes each applicable budget has slope $-2c$ in λ, hence is nonincreasing. No untested interior point can be worse.

The exact certificate `round8/completion_certificate.json` in [15] records every exclusion through degree 96. The two supplied checkers, replayed for this version, agree in both modes. One checks rational next-count exclusions; the other derives integer maxima before comparing with the certificate and scans the full grid through degree 16. Selected ceiling rows in the simple mode are

d	5	6	7	8	9	10	11	12
$2^d c_d$	50	109	233	487	1001	2037	4117	8287

They verify nondecreasing $c_d \le d/2$ and obtain exactly

$$c_{96} = \frac{8088041149216617370543733 1353}{396140812571321687967719751 68}.$$

Feasible scalar points stored with the rows are not Boolean witnesses; only the strict exclusions are needed for this upper bound.

To cover all larger degrees, (9) gives $R_d \geq 2^d$ for $d \geq 2$, so its least-influence derivative has $\lambda \leq d/2$. Apply the weaker (5) with the actual $c = W_{d-1}$ and $h \leq 2(d-1)\lambda$. Since $1/2 \leq W_{d-1} \leq (d-1)/2$,

$$W_d \leq W_{d-1} + \frac{\lambda(2d - 1 - 2W_{d-1})}{2^d} \leq W_{d-1} + \frac{d(d-1)}{2^d}.$$

Differentiating the geometric series gives $\sum_{j\geq 1}(1, j, j^2)2^{-j} = (1, 2, 6)$; expanding $(H+j)(H+j-1)$ yields

$$\sum_{d>H} \frac{d(d-1)}{2^d} = \frac{H^2 + 3H + 4}{2^H}.$$

The nondecreasing finite caps cover every $d \leq 96$. Telescoping beyond 96 adds $9508/2^{96}$ to c_{96}, exactly the rational in Theorem 1. Its comparison with the printed decimal is a strict rational comparison, rounded upward. This completes the argument.

7. Evidence and disclosure

All code, catalogue data and certificates are linked at the single pinned revision in [14, 15]; Hexagon carries this TeX note. The finite calculation supports the written proof and its named published dependencies; it does not independently prove catalogue completeness or the classical lattice group theorem. The packet includes AI reviews of each stage. This version replayed the two scalar checkers and direct cubic character-sum checker; it did not rerun the historical large exhaustive controls or searches.

This is an unrefereed preprint. There has been no independent human review and no formalization. The programs check finite arithmetic, not all prose or the cited external theorems. No exact value of R_5, R_6, or C_{44} is claimed. The short derivative recursion may already exist; the source search was bounded, and expert scrutiny is requested. The arguments were found and checked by AI agents under Ricardo Pinho's direction.

Analysis and proof drafting by AI agents under my direction; I reviewed the results at summary level. AI agents also prepared certificates, checking software, and publication materials. No line-by-line human verification is claimed.

References

[1] E. S. Barnes and G. E. Wall, Some extreme forms defined in terms of Abelian groups, *J. Austral. Math. Soc.* 1 (1959), 47–63; doi:10.1017/S1446788700025064.

[2] M. Broué and M. Enguehard, Une famille infinie de formes quadratiques entières; leurs groupes d'automorphismes, *Ann. Sci. École Norm. Sup.* (4) 6 (1973), 17–51; doi:10.24033/asens.1241.

[3] C. Carlet, Identifying codewords in general Reed–Muller codes and determining their weights, *AIMS Mathematics* 9 (2024), 10609–10637, Section 2; doi:10.3934/math.2024518.

[4] J. Chiarelli, P. Hatami and M. Saks, An asymptotically tight bound on the number of relevant variables in a bounded degree Boolean function, *Combinatorica* 40 (2020), 237–244; `https://arxiv.org/abs/1801.08564`.

[5] T. Kasami and N. Tokura, On the weight structure of Reed–Muller codes, *IEEE Trans. Inform. Theory* 16 (1970), 752–759; doi:10.1109/TIT.1970.1054545. The below-two weight result is also credited to E. R. Berlekamp and N. J. A. Sloane.

[6] D. S. Krotov and A. A. Valyuzhenich, On degree-3 and $(n-4)$-correlation-immune perfect colorings of n-cubes, *Discrete Math.* 347 (2024), 114138, Section 3 (crediting Yu. Tarannikov and D. Kirienko, *Spectral analysis of high order correlation immune functions*, ePrint 2000/050); `https://arxiv.org/abs/2311.05566`.

[7] M. Masuda and T. E. Panov, Semifree circle actions, Bott towers and quasitoric manifolds, *Sb. Math.* 199 (2008), 1201–1223, Lemma 4.7; doi:10.1070/SM2008v199n08ABEH003959.

[8] G. Nebe, The second minimum of Barnes–Wall lattices, preprint (2026), Theorems 2.5 and 3.3; `https://arxiv.org/abs/2603.23133`.

[9] V. N. Potapov, On the number of relevant variables for discrete functions, *Cryptogr. Commun.* 17 (2025), 989–998; doi:10.1007/s12095-025-00797-4.

[10] S. Rasoolzadeh, Classification of all t-resilient Boolean functions with $t+4$ variables, *IACR Trans. Symmetric Cryptol.* 2023(3), 213–226; doi:10.46586/tosc.v2023.i3.213-226.

[11] A. Valyuzhenich, Optimal functions with spectral constraints in hypercubes, *Discrete Math.* 347 (2024), 113816, Theorem 2(2); `https://arxiv.org/abs/2303.10995`.
[12] J. Wellens, Relationships between the number of inputs and other complexity measures of Boolean functions, *Discrete Analysis* 2022:20; `https://discreteanalysisjournal.com/article/57741`.
[13] BakeLab, *Épi-results*, exact Wellens certificates, revision `97fa6a420f417458e11723c8bceb70f1529e447a`; Retained data at the pinned math-certificates revision.
[14] R. Pinho, *Math certificates*, revision `36325fade4b73afb787c37ee8cc4c90ae4adf856`, derivative-control base; `c44/reed-muller-caps`.
[15] R. Pinho, *Math certificates*, same revision, quartic proof and completion packet; `c44/small-degree-classification`.
[16] R. Pinho, Tao optimization table, PR 294 (open, unrefereed; read 2 November 2026); `https://github.com/teorth/optimizationproblems/pull/294`.

Independent researcher