

The partition exponent in the graded pieces of the Claude 3-SAT algorithm is optimal for pieces of that shape

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Abstract

In Appendix B of *A $2^{o(n)}$ time algorithm for 3-SAT* (Claude, Anthropic, 2026) the gain parameter $\rho = 1/\sqrt{2}$ comes from the Hardy–Ramanujan growth $\log p(E) \sim c\sqrt{E}$ of the partition numbers, and Appendix H observes that growth of order E^α would give $\rho = 2^{\alpha-1}$ by the same calculation. We point out that the rank floor stated in the same Appendix H already excludes $\alpha < 1/2$ for every graded family of the same shape: the pieces at weight h have total rank at most $(4h+1)\dim V_{\leq 4h}$ in one of the two directions, and the crossing they must add up to has rank $(\dim V_h)^2$, so the construction needs $\log \dim V_{\leq 4h} \geq 2 \log \dim V_h - O(\log h)$, that is $\alpha \geq 1/2$. Restoring the rank with a larger weight costs $\rho \geq 2^{-\alpha}$ in the weight count, so the count never goes below $\max(2^{\alpha-1}, 2^{-\alpha}) \geq 1/\sqrt{2}$, with equality only at $\alpha = 1/2$; when the identity holds on the whole weight space, as Lemma 9 gives it, the bound sharpens to $2^\alpha/(2(4^\alpha - 1))$, which exceeds 1 for $\alpha < 0.357$; Theorem B of the paper is rank-tight up to a polynomial factor. The inequality is exact and we tabulate it: for the partition numbers it holds for every h , for partitions into squares ($\alpha = 1/3$) it fails for all $h \geq 5891$. We also record that the Heisenberg Fock spaces on a sparse set of modes, the natural candidates for slower growth, do not carry Lemma 9 of the paper, and that no choice of coefficients repairs it. The floor $\rho \geq 1/2$ remains open.

1 Setting

We use the notation of [1]; equation and lemma numbers refer to it. Fix $\lambda \geq 1$. The *crossing* is the function $[x = v][y = u]$ of four strings $x, y, u, v \in \{0, 1\}^\lambda$. A function $F(z, z')$ *factors through c bits* if $F(z, z') = \sum_{\theta \in \{0, 1\}^c} f(z, \theta) f'(z', \theta)$ (Definition 1); as a matrix with rows z and columns z' it then has rank at most 2^c . *Pieces with parameters* (ρ, c) (Definition 7) are, for every λ , integer functions A_i ($i \in I$, $|I| \leq (2\lambda)^c$) on four λ -bit strings with $\sum_i A_i = d[x = v][y = u]$, each factoring through c_i^{row} bits as a function of (x, y) and (u, v) and through c_i^{col} bits as a function of (x, u) and (y, v) , with $c_i^{\text{row}} + c_i^{\text{col}} \leq 4\rho\lambda + c \log(2\lambda)$, all computable fast. Theorem A turns pieces with parameters (ρ, c) into a $2^{O(m/(\log m)^{\alpha'})}$ algorithm for counting satisfying assignments, $\alpha' = \log(1/\rho)$; Theorem B provides pieces with $\rho = 1/\sqrt{2}$, hence the exponent $1/2$ of $\log m$ in the paper's bound in m .

The floor. Appendix H (“What limits the octonions”) gives the lower bound $\rho \geq 1/2$: the crossing is a permutation matrix of size $2^{2\lambda}$ in either arrangement of its arguments, so by subadditivity of rank some piece has rank at least $2^{2\lambda}/|I|$ row-wise, that is $c_i^{\text{row}} \geq 2\lambda - O(\log \lambda)$, which leaves it at most about $(4\rho - 2)\lambda$ bits column-wise. Definition 7 accordingly restricts ρ to $[\frac{1}{2}, 1)$.

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The graded pieces. Appendix B works on $V = \mathbb{Q}[t_1, t_2, \dots]$ graded by weight, V_E the span of the monomials t^μ with μ a partition of E , so $\dim V_E = p(E)$. For a polynomial a and $r \in \mathbb{Z}$ it defines operators $a[r]: V_E \rightarrow V_{E+r}$ (eq. (47)), and for $b \in V_h$ and a of weight at most h the operators

$$Q_s(a, b) = \sum_{i=0}^{2h} (-1)^i a^{(i)} [s - h + i] b [h - s - i], \quad s \in \mathbb{Z} \quad (1)$$

(eq. (48)). Lemma 9 states that on V_E : (a) $Q_s(a, b)$ is unchanged when the two factors are exchanged in all products at once; (b) $Q_s(a, b) = 0$ for $|s| > E + h$; (c) $\sum_{|s| \leq E+h} Q_s(a, b) = (a[-h]b) \cdot \text{Id}$. Lemma 10 gives polynomials a^μ, b^μ indexed by the partitions of h with $a^\mu[-h]b^\nu = h! [\mu = \nu]$. With h least such that $p(h) \geq 2^\lambda$ and $\mu(x)$ the x -th partition of h , the pieces are

$$A_s(x, y, u, v) = [t^{\mu(x)}] Q_s(a^{\mu(x)}, b^{\mu(v)}) t^{\mu(y)}, \quad s \in \{-2h, \dots, 2h\}, \quad (2)$$

and $\sum_s A_s = h! [x = v][y = u]$ by Lemma 9(b),(c) with $E = h$ and Lemma 10 (eq. (53)). Expanding the intermediate polynomial in monomials gives the two factorizations (eqs. (54), (55)): the product $a^{(i)} [s - h + i] b [h - s - i]$ applied to $t^{\mu(y)}$ passes through V_{2h-s-i} , which factors A_s as a function of (x, u) and (y, v) through $R_s^{\text{col}} = \sum_{i=0}^{2h} p(2h - s - i)$ terms; with the factors exchanged (Lemma 9(a)) it passes through V_{s+i} , which factors A_s as a function of (x, y) and (u, v) through $R_s^{\text{row}} = \sum_{i=0}^{2h} p(s + i)$ terms. By Hardy–Ramanujan (eq. (46)), $\lambda \approx c_{\text{HR}} \sqrt{h}$ and

$$\log R_s^{\text{row}} + \log R_s^{\text{col}} \approx c_{\text{HR}} \sqrt{2h + s} + c_{\text{HR}} \sqrt{2h - s} \leq 2c_{\text{HR}} \sqrt{2h} \approx 2\sqrt{2} \lambda, \quad (3)$$

which is eq. (77) and gives $\rho = 1/\sqrt{2}$. Appendix H then remarks: if $\log \dim V_E$ grew like E^α with $0 < \alpha \leq 1$, the same calculation would give $\rho = 2^{\alpha-1}$.

That remark is a correct calculation, and it suggests that a graded space with slower growth, $\alpha < 1/2$, would improve the exponent. The point of this note is that the floor argument of the same appendix shows no such space can carry the construction.

2 The rank count

Definition 2.1. Let $V = \bigoplus_{W \geq 0} V_W$ be a graded $\mathbb{Q}$ -vector space with finite-dimensional homogeneous pieces, and $h \geq 1$. A family of pieces of Theorem B's shape at weight h consists of a finite index set $\mathcal{X}$ and, for $x, v \in \mathcal{X}$, $s \in \mathbb{Z}$ and $0 \leq i \leq 2h$, linear operators $P_{x,s,i}$ and $R_{v,s,i}$ on V with

$$P_{x,s,i}(V_W) \subseteq V_{W+s-h+i}, \quad R_{v,s,i}(V_W) \subseteq V_{W+h-s-i} \quad \text{for all } W,$$

such that the operators $Q_s(x, v) = \sum_{i=0}^{2h} P_{x,s,i} R_{v,s,i}$ satisfy, on V_h ,

- (b) $Q_s(x, v) = 0$ unless $-2h \leq s \leq 2h$;
- (c) $\sum_{s=-2h}^{2h} Q_s(x, v) = d [x = v] \cdot \text{Id}_{V_h}$ for some integer $d \neq 0$.

The pieces of the family are $A_s(x, y, u, v) = \langle e_u, Q_s(x, v) e_y \rangle$ for a basis $(e_y)_{y \in \mathcal{Y}}$ of V_h , where $\langle e_u, w \rangle$ is the coefficient of e_u in w .

Theorem B's construction is of this shape with $P_{x,s,i} = (-1)^i a^{\mu(x)(i)} [s - h + i]$ and $R_{v,s,i} = b^{\mu(v)} [h - s - i]$, $\mathcal{X}$ the set of 2^λ strings or, equally well, the set of all $p(h)$ partitions of h : (b) is Lemma 9(b) with $E = h$, and (c) is Lemma 9(c) together with Lemma 10. The definition keeps only the weight bookkeeping and the identity; it says nothing about how the operators are built, and it does not use Lemma 9(a).

all parts ($\alpha = 1/2$)				squares ($\alpha = 1/3$)				cubes ($\alpha = 1/4$)			
h	$\log_2 p(h)$	ratio	slack	h	$\log_2 p_{\square}(h)$	ratio	slack	h	$\log_2 p_3(h)$	ratio	slack
2	1.00	4.46	5.43	17	3.17	2.55	9.72	33	2.58	2.38	11.44
9	4.91	2.88	9.16	94	9.78	2.07	11.89	375	10.47	1.82	13.89
38	14.67	2.42	12.92	133	11.87	2.01	12.02	1500	19.10	1.69	12.82
106	28.52	2.27	15.74	265	17.03	1.92	11.86	3000	25.03	1.65	11.35
300	53.04	2.17	18.66	1061	32.75	1.80	9.35	6000	32.30	1.61	9.08
600	78.60	2.13	20.62	4243	59.02	1.73	2.51	12000	41.19	1.58	5.88
				5891	67.40	1.72	-0.00				

Table 1: The count (4) for partitions into all parts, squares and cubes. “ratio” is $\log_2 p_S(4h)/\log_2 p_S(h)$, which tends to 4^α (2, 1.59, 1.41); “slack” is $\log_2 T_S(h) - 2\log_2 p_S(h)$, which must be ≥ 0 for the construction to exist at weight h with all partitions as strings. Exact integer arithmetic; the first failure for squares is at $h = 5891$, and the slack peaks at 12.04 bits at $h = 160$.

Theorem 2.2. *For every family of pieces of Theorem B’s shape at weight h on V ,*

$$|\mathcal{X}| \cdot \dim V_h \leq T_V(h) := \sum_{s=-2h}^{2h} \sum_{i=0}^{2h} \dim V_{s+i} \leq (4h+1) \dim V_{\leq 4h}, \quad (4)$$

where $\dim V_{\leq W} = \sum_{W' \leq W} \dim V_{W'}$. In particular $(\dim V_h)^2 \leq T_V(h)$ when $\mathcal{X}$ indexes a basis of V_h , and $2^{2\lambda} \leq 2^{\tilde{\lambda}} \dim V_h \leq T_V(h)$ when $|\mathcal{X}| = 2^\lambda \leq \dim V_h$ as in Theorem B.

Proof. Insert a basis of V_{2h-s-i} between the two factors:

$$A_s(x, y, u, v) = \sum_{i=0}^{2h} \sum_{\sigma} \langle e_u, P_{x,s,i} e_{\sigma} \rangle \langle e_{\sigma}, R_{v,s,i} e_y \rangle,$$

σ running over a basis of V_{2h-s-i} (an empty sum when $2h-s-i < 0$). This writes A_s as a sum of $R_s^{\text{col}} = \sum_i \dim V_{2h-s-i}$ products of a function of (x, u) and a function of (y, v) , so as a matrix with rows (x, u) and columns (y, v) it has rank at most R_s^{col} . By (b) and (c), $\sum_{s=-2h}^{2h} A_s = d[x=v][y=u]$, a nonzero multiple of a permutation matrix of size $|\mathcal{X}| \cdot \dim V_h$ in that arrangement, of full rank. Subadditivity of rank gives $|\mathcal{X}| \cdot \dim V_h \leq \sum_{s=-2h}^{2h} R_s^{\text{col}}$, which is $T_V(h)$ after $s \mapsto -s$. The last inequality of (4) bounds each inner sum by $\dim V_{\leq 2h+s} \leq \dim V_{\leq 4h}$. $\square$

Corollary 2.3. *Let $\dim V_W$ be nondecreasing in W and $\log \dim V_{\leq W} = (c + o(1))W^\alpha$ with $c > 0$, $0 < \alpha < 1/2$. Then families of pieces of Theorem B’s shape exist on V at finitely many weights h only.*

Proof. Monotonicity gives $\dim V_h \geq \dim V_{\leq h}/(h+1)$, so (4) implies

$$2(c + o(1))h^\alpha \leq (c + o(1))(4h)^\alpha + O(\log h),$$

that is $2 \leq 4^\alpha + o(1)$, false for large h when $\alpha < 1/2$. $\square$

For the partition numbers, $p(h)^2 \leq T(h)$ must hold at every h since the construction exists, and it does: Table 1 gives the slack $\log_2 T(h) - 2\log_2 p(h)$, which grows like $2\log_2 h$. Indeed $T(h) \approx \sum_{j \geq 1} j p(4h-j) \approx h p(4h)$, and $p(4h)/p(h)^2 \approx \sqrt{3} \cdot h$ by Hardy–Ramanujan’s $p(E) \sim e^{\pi\sqrt{2E/3}}/(4E\sqrt{3})$, so $T(h) = \Theta(h^2) p(h)^2$: the pieces of Theorem B have a total rank budget exceeding the rank of the crossing by a factor $O(h^2) = O(\lambda^4)$ only. *The construction is rank-tight up to a polynomial factor.* For partitions into squares, where $\ln p_{\square}(E) \sim 3.31 E^{1/3}$ by Wright [4], the slack peaks at 12.04 bits at $h = 160$ and the inequality fails for every $h \geq 5891$ (exact arithmetic; $\log_2 p_{\square}(5891) = 67.4$). For cubes it is 5.9 bits at $h = 12000$ and falling.

3 What the count says about ρ

Theorem 2.2 is a nonexistence statement and needs no assumption on costs. To speak about ρ we bound costs the way the construction does, by the dimensions of the weight spaces the two orders pass through: $c_s^{\text{row}} = \lceil \log R_s^{\text{row}} \rceil$, $c_s^{\text{col}} = \lceil \log R_s^{\text{col}} \rceil$. Call the resulting ρ the *weight count* of the family. The weight count is the only cost a proof via this shape provides; a family whose pieces had rank far below these dimensions would owe its gain to something other than the grading, and is not covered by what follows.

Write $f(W) = \log \dim V_{\leq W}$ and suppose $f(W) = (c + o(1))W^\alpha$. At the balanced piece $s = 0$ both orders pass through weight $2h$, and by concavity of $W \mapsto W^\alpha$ this is where $f(2h + s) + f(2h - s)$ is largest, as in (3); so the weight count is $\rho = f(2h)/(2\lambda)$ up to lower-order terms.

Proposition 3.1. *For families of Theorem B's shape with $f(W) = (c + o(1))W^\alpha$ and $\dim V_W$ nondecreasing:*

- (i) *at the least weight h with $\dim V_h \geq 2^\lambda$ the weight count is $\rho = 2^{\alpha-1} + o(1)$, and for $\alpha < 1/2$ the family does not exist there for large λ ;*
- (ii) *at any weight h at which the family exists, $\rho \geq 2^{-\alpha} - o(1)$;*
- (iii) *hence $\rho \geq \max(2^{\alpha-1}, 2^{-\alpha}) - o(1) \geq 2^{-1/2} - o(1)$, with the bound $2^{-1/2}$ attained only at $\alpha = 1/2$;*
- (iv) *if (c) holds on all of V_h , as Lemma 9(c) gives, then at any weight at which the family exists*

$$\rho \geq \frac{2^\alpha}{2(4^\alpha - 1)} - o(1),$$

which is $\geq 2^{-\alpha}$ for $\alpha \leq 1/2$ with equality only at $\alpha = 1/2$, and exceeds 1 for $\alpha < \log_2 \frac{1+\sqrt{17}}{4} = 0.357$: slow growth then gives no gain at all.

Proof. (i) At the least such h , $\dim V_{h-1} < 2^\lambda \leq \dim V_h$ and $f(h-1)/f(h) \rightarrow 1$, so $\lambda = f(h)(1 + o(1))$ and $f(2h)/(2\lambda) \rightarrow 2^\alpha/2$; nonexistence for $\alpha < 1/2$ is Corollary 2.3. (ii) Theorem 2.2 with $|\mathcal{X}| = 2^\lambda$ gives $f(4h) \geq 2\lambda - O(\log h)$, and $f(2h) = (2^{-\alpha} + o(1))f(4h)$, so $f(2h)/(2\lambda) \geq 2^{-\alpha} - o(1)$. (iii) combines the two; $\max(2^{\alpha-1}, 2^{-\alpha})$ is minimized where the two agree, at $\alpha = 1/2$. (iv) Theorem 2.2 with $|\mathcal{X}| = 2^\lambda$ gives $f(4h) \geq \lambda + \log \dim V_h - \log(4h+1) \geq \lambda + f(h) - O(\log h)$, and $f(4h) = (4^\alpha + o(1))f(h)$, so $f(h) \geq (1 - o(1))\lambda/(4^\alpha - 1)$ and $f(2h)/(2\lambda) = (2^\alpha + o(1))f(h)/(2\lambda) \geq 2^\alpha/(2(4^\alpha - 1)) - o(1)$. The comparison with $2^{-\alpha}$ is $4^\alpha \leq 2(4^\alpha - 1)$, i.e. $4^\alpha \geq 2$; and $2^\alpha > 2(4^\alpha - 1)$ exactly when $2^\alpha < (1 + \sqrt{17})/4$. $\square$

Slower growth makes the pieces too small to add up to the crossing; faster growth makes the balanced piece too expensive. The $\sqrt{E}$ of Hardy–Ramanujan is the optimum of this shape, not an accident that a better graded algebra could improve. At finite λ the counts are not yet asymptotic. For squares, the least weight h with $2^{2\lambda} \leq T_\square(h)$ gives a weight count of 0.728 at $\lambda = 68$, rising to 0.742 at $\lambda = 96$ and 0.750 at $\lambda = 120$, while Theorem B's family at the same λ gives 0.776, 0.760, 0.752, falling toward 0.707; the limits are $2^{-1/3} = 0.794$ against 0.707, and the order reverses between $\lambda = 120$ and $\lambda = 128$. So the optimality of $\alpha = 1/2$ is asymptotic, in the same sense as the paper's bound. Under the stronger condition $2^\lambda \dim V_h \leq T_\square(h)$ of part (iv), the least such h gives 0.732 at $\lambda = 68$ and 0.800 at $\lambda = 96$, on the way to the limit $2^{1/3}/(2(4^{1/3} - 1)) = 1.07$.

Remark 3.2 (Scope). Part (ii) uses only that the lowest piece passes through weight ≥ 0 , so the top piece reaches at most twice the balanced weight, and the proof of Theorem 2.2 uses only (b), (c) and the weight bookkeeping; so (ii) carries over to any weight-graded family with the same bookkeeping and regularly varying f , whatever the operators, coefficients or module. We have stated and proved

it for Theorem B's shape. Part (i) uses that the operators act on the weight space whose dimension carries λ bits ($E = h$ in Appendix B). Two shapes escape the count. First, operators of weight h smaller than the acting weight E , with $\lambda = f(E) > f(h)$: this needs 2^λ independent operators of weight h , hence an algebra much larger than the module it acts on; the count allows a weight count down to $2^{-\alpha}$ there, which is below $1/\sqrt{2}$ for $\alpha > 1/2$ (for $\alpha = 0.6$ and $h \approx 0.6E$, about 0.66). In a vertex-algebra-type construction the operators are the fields of vectors of the space itself, and the 2^λ vectors b_v must be linearly independent (the identity (c) is bilinear in (a_x, b_v)), so $\dim V_{\leq h} \geq 2^\lambda$ and part (i) applies; we know no algebra of the other kind. Second, dimension profiles that are not regularly varying: $f(4h) \geq 2f(h)$ together with $f(2h) \ll \sqrt{2}f(h)$ is consistent along a sparse sequence of h . Characters of rational C_2 -cofinite vertex operator algebras are modular (Zhu), hence regularly varying, which excludes the natural candidates. The count says nothing about constructions without a weight grading, such as the octonion pieces of the paper's main text, and nothing about the floor $1/2$.

Remark 3.3. With the operators of weight h acting on V_E , $E \neq h$, the same proof gives $2^{2\lambda} \leq \sum_{|s| \leq E+h} \sum_{i=0}^{2h} \dim V_{E+h-s-i} \leq (2E+2h+1)(2h+1) \dim V_{\leq 2E+2h}$, from which (ii) follows in the same way.

4 The sparse Heisenberg algebras

The candidate family for slower growth is the Fock space $V_S = \mathbb{Q}[t_s : s \in S]$ of the Heisenberg algebra on a sparse set of modes $S \subset \mathbb{Z}_{>0}$, graded by weight, with $\dim(V_S)_E = p_S(E)$ the number of partitions of E into parts from S ; for S the k -th powers, $\log p_S(E) \sim c_k E^{1/(k+1)}$ [4]. Theorem 2.2 already excludes it for the squares at the weights Theorem B's recipe uses, the least h with $p_\square(h) \geq 2^\lambda$, from $\lambda = 68$ on ($h = 6023$; at $\lambda = 67$, $h = 5804$, the count still holds), and Section 3 shows what larger weights would cost. It fails for a second, independent reason at the smallest weights: the operators of Appendix B do not satisfy Lemma 9 once modes are missing, and no change of coefficients helps.

For $n \in S$ the operator of eq. (47) is $t_n[r] = \binom{r-1}{n-1} \nabla_{-r}$ (∇_{-r} is multiplication by t_r for $r > 0$ and $|r| \partial/\partial t_{|r|}$ for $r < 0$; generalized binomial coefficients as in the paper), restricted on V_S to $|r| \in S$. Allow arbitrary coefficients: $t_n[r] = c_n(r) \nabla_{-r}$ for $|r| \in S$, and 0 otherwise, which is the only form eq. (47) admits for a one-part monomial.

Proposition 4.1. *Let $S \subseteq \mathbb{Z}_{>0}$ have infinite complement, $n \leq m$ in S , and let $Q_s(t_n, t_m)$ be formed as in (1) with the operators above and any coefficients c_n, c_m . If $Q_s(t_n, t_m)$ is unchanged by exchanging the factors in all products (Lemma 9(a)) for all s , then $c_n(r)c_m(-r) = 0$ for every r ; consequently $\sum_s Q_s(t_n, t_m) = 0$, and Lemma 9(c) can hold only if the pairing $t_n[-m]t_m$ vanishes. In particular Lemma 9 fails on V_S with the paper's coefficients.*

Proof. With $H = n + m$ (so $t_n^{(i)} = \binom{H}{i} t_n$), the difference between the two orders of $Q_s(t_n, t_m)$ is

$$\sum_{i=0}^H (-1)^i \binom{H}{i} [t_n[s-m+i], t_m[m-s-i]], \quad \text{where}$$

$$[t_n[r], t_m[-r]] = c_n(r)c_m(-r)[\nabla_{-r}, \nabla_r] = -rc_n(r)c_m(-r)[|r| \in S] =: g(r)$$

is a scalar. The condition for all s says that the H -th finite difference of the sequence g vanishes on $\mathbb{Z}$, so g is a polynomial of degree $< H$; it vanishes on the complement of $\pm S$, an infinite set, so $g \equiv 0$. Then every product $t_n[r]t_m[-r]$ is zero, $\sum_s Q_s(t_n, t_m) = 0$, and (c) forces $t_n[-m]t_m = 0$. For the paper's coefficients $c_n(r)c_m(-r) = \binom{r-1}{n-1} \binom{-r-1}{m-1} \neq 0$ for $r \geq n$, so $g \neq 0$ at every $r \in S$, $r \geq n$. (For $S = \mathbb{Z}_{>0}$, $g(r) = -r \binom{r-1}{n-1} \binom{-r-1}{m-1}$ is a polynomial of degree $H-1$, which is why the binomial coefficients work.) $\square$

A machine check agrees. We implemented V_S , $\nabla_{\pm k}$ for $k \in S$, $a[r]$ through the recursion (O2) of the paper with modes restricted to S (for $S = \mathbb{Z}_{>0}$ this is eq. (47)), and Q_s by (1), and tested

Lemma 9(a),(b),(c) on V_E for all monomial pairs $a = t^\mu$ ($|\mu| \leq h$), $b = t^\nu$ ($|\nu| = h$) with $E, h \leq 6$, and Lemma 10 as stated. For $S = \mathbb{Z}_{>0}$ every test passes (3878 triples (a, b, E) ; Lemma 10 through $h = 6$ with the pairing of full rank $p(h)$), which validates the implementation against the paper. For squares, cubes and odd numbers all three parts fail already at $a = b = t_1$ on V_0 : for squares, $Q_{-3}(t_1, t_1) 1 = \nabla_4 t_4 = 4$ in one order and 0 in the other (in the full algebra the modes 3 and 2 contribute $-6 + 2$ and the sum cancels), and $\sum_{|s| \leq 1} Q_s(t_1, t_1) 1 = -1$ against the pairing $t_1[-1]t_1 = 1$. Proposition 4.1 concerns one-part monomials; multi-part operators have more freedom, and for them the exclusion is Theorem 2.2.

5 What remains

The floor $\rho \geq 1/2$ and the value $1/\sqrt{2}$ stand where the paper left them. The note closes the route between them that Appendix H's last paragraph appears to open: within graded pieces, the exponent $1/2$ cannot be moved by changing the growth of the graded space. What is not excluded is listed in Remark 3.2: an algebra much larger than the module it acts on, a dimension profile that is not regularly varying, or pieces that are not graded at all.

Provenance. This note comes from a project that studies the two October 2026 refutations of the Exponential Time Hypothesis [1, 2] as instances of proof search. The observation, the computations and the text were produced by Claude (Anthropic) in a session directed by the author on 2026-10-29; the proofs and all numbers were independently rechecked by multiple agents with varied instructions. The author has checked the main argument and takes responsibility for the content. Three scripts in the ancillary files reproduce every number in the note: `anc/rank_count.py` (Table 1 and the values of Sections 3 and 4), `anc/lemma9_check.py` (the Lemma 9 and Lemma 10 checks of Section 4) and `anc/kernel_check.py` (the kernels behind Proposition 4.1); they need Python 3 only.

References

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