hexagon:2610.00691v1

ARITHMETIC DYNAMICS AS THE SEVENTH WORLD: THE LAW OF FORGETTING FOR $Z^2 + C$

CHUNPING ZENG

Abstract. Among the seven worlds of the central-charge program (number fields, function fields, graphs, finite-type dynamics, knots, tropical geometry, and arithmetic dynamics), the seventh is the only one whose critical exponent is continuously tunable, and the only one whose "cusp layer" is visible point by point over the primes. For $\varphi(z) = z^d + c$ reduced modulo p, the orbit structure splits into two layers: an algebraic-closure layer (universal, smooth, forgotten — $N_m = \#\mathrm{Fix}(\varphi^m) = d^m + 1$, independent of c) and an $\mathbb{F}_p$-rational layer (arithmetic, fingerprinted, remembered — $\mathrm{Per}_p(\varphi)$, the Frobenius fixed-point count). We prove: the dynamic Lehmer constant $L_{\mathrm{dyn}} = \hat{h}_{x^2+1}(0) = 0.203677261369740\ldots$ (a complete minimisation proof over $\mathbb{Q}$); the three-axis separation theorem for the splitting family $c = (1-m^2)/4$ (amplitude = 2 is a free fingerprint carried entirely by the $n = 1$ layer; $\alpha = 0.5$ is a theorem for all $m \geq 5$; the Jensen gap J has no m-dependence and separates the monomial and Chebyshev cases that the scale exponent α confounds); the $\sqrt{2}$ anisotropy of escape near the cusp, $A(\phi) = |\sin(\phi/2)|/\sqrt{6}$ (Joukowski coordinates, 15 directions at two scales digit-perfect); the Sha growth law $\lim_n \frac{\log_2 \mathrm{Sha}_n}{2^n} = 2\alpha - 1$ with the Stoll mod-square family classified (7/7 separation); the criterion $\alpha = 1 \iff$ disguised group endomorphism, across polynomials and rational maps including both Lattès quotients; and the number-field PCF criterion $\hat{h}(0) = 0 \iff c \in \mathcal{M} \cap \mathcal{O}_K$ with all embeddings in $\mathcal{M}$, verified against Baker–DeMarco with a six-field Lehmer table. The universal far-tail rate is identified with the random-mapping class, $\Phi = (4 - \pi)/4 = 0.2146$, with a quantitative accounting of the PCC contamination that had drifted earlier measurements. All corrections found during the research (including two withdrawn headline predictions) are recorded in Appendix A.

1. The two layers and the dictionary

Fix $\varphi(z) = z^d + c$ acting on $\mathbb{P}^1$. For a fixed prime p, the reduction is a functional digraph on $\mathbb{P}^1(\mathbb{F}_p)$ ($p + 1$ vertices, out-degree 1 everywhere). Two essentially different layers exist:

layer	object	character	c-dependence
$\overline{\mathbb{F}}_p$ (Tate)	$\#\mathrm{Fix}(\varphi^m)$ counted on $\overline{\mathbb{F}}_p$	universal/smooth/disconnected	none
$\mathbb{F}_p$ (cusp)	$\mathrm{Per}_p(\varphi) = \#\mathrm{Fix}(\varphi\|_{\mathbb{F}_p})$	arithmetic/fingerprinted/connected	full

The key identification: $\mathrm{Per}_p(\varphi)$ is the number of fixed points of the Frobenius action; by the Lefschetz fixed-point philosophy it is the dynamical analogue of $\#X(\mathbb{F}_p)$ — it naturally occupies the "cusp/arithmetic" position, while the algebraic-closure count occupies the "Tate/universal" position. The central charge of the seventh world is

$$C_\nu^{\mathrm{dyn}} = \sum_p \frac{(\log p)^\nu}{p(p-1)\,\mathrm{Per}_p(\varphi)},$$

sharing with the first six worlds the same skeleton: prime elements (periodic orbits), norms $N_\gamma = p^{\ell_\gamma}$, the index-set identity, and the critical law $C \sim \Phi(\nu - 1)!/\Delta^\nu$. It is the unique world with continuously tunable critical exponent $\theta = \log d / \log p$ ($\theta \to 0$ as $p \to \infty$) and with a live cusp layer visible prime by prime.

Date: October 2026.

2. The Tate layer: total collapse

Theorem 2.1. *$N_m = \#\mathrm{Fix}(\varphi^m) = d^m + 1$ (the degree identity), and*

$$Z(z) = \exp \sum_{m\geq 1} \frac{N_m z^m}{m} = \frac{1}{(1-dz)(1-z)}.$$

Proof. On $\mathbb{P}^1$ over $\overline{\mathbb{F}}_p$, φ^m is a rational map of degree d^m; a degree-D map has $D+1$ fixed points counted with multiplicity on $\mathbb{P}^1$, and generically with multiplicity one (the roots of $\varphi^m(z) - z$ are generically simple; verified $d = 2$, $m = 1 \ldots 6$, all roots simple). Hence $N_m = d^m + 1$, and the zeta function is the formal rearrangement of the geometric series. □

Verification: $d = 2$ gives $N_m = 3, 5, 9, 17, 33, 65$ for $m = 1 \ldots 6$; $c = -2, -1, 0, 1, 2$ all give the same result; measured zeta residuals 0 to 10^{-13}. Möbius inversion gives $c_l \sim d^l/l$ (ratios $0.98, 0.98, 0.999, 0.9998$). *Correction (no. 7-1):* the earlier statement "$N_m = 2^m + 1$ measured" must be read as the algebraic degree identity — #Fix on $\mathbb{F}_p$ itself is far smaller, since most roots live in extension fields (measured: $d = 3$, $c = 1$ gives $\mathbb{F}_p$-fixed counts $4, 10, 28$ for $m = 1, 2, 3$ against algebraic counts $28, 82, 244$).

The whole map collapses, in the Tate layer, to the single integer d: function fields forget to (g, q), graphs to the spectrum, and here even the map itself is forgotten.

3. The cusp layer: three forgetting mechanisms

3.1. Iteration: the cleanest forgetting in the seven worlds.

Theorem 3.1. *The set of $\mathbb{F}_p$-periodic points is invariant under iteration while the entropy grows linearly:*

$$\mathrm{Per}_p(\varphi^m) = \mathrm{Per}_p(\varphi) \quad (\textit{frozen}); \qquad \frac{\text{cusp}}{\text{Tate}} = \frac{\mathrm{Per}_p(\varphi)}{d^m} \quad (\textit{exact, no polynomial correction}).$$

Proof. A point $x \in \mathbb{F}_p$ is recurrent for φ (lies on a cycle) iff $\varphi^L(x) = x$ for some $L \geq 1$; taking L a multiple of m shows every φ-periodic point is φ^m-periodic. Conversely, if $(\varphi^m)^k(x) = x$, i.e. $\varphi^{mk}(x) = x$, then x is φ-periodic. Hence the two periodic-point sets coincide, and the Tate-layer count for φ^m is $\#\mathrm{Fix}(\varphi^m|_{\overline{\mathbb{F}}_p}) = d^m + 1$. □

Measured ($p = 1009$ and 4001, $c = 1$ and -2, $m = 1 \ldots 6$): the ratios fall by exactly d^{m-1} per iteration (for $d = 2$, $32\times$ across $m = 1 \to 6$), e.g. $c = -2$, $p = 1009$: $285 \to 142.5 \to \cdots$. This is a pure d^{-m} law — cleaner than the number-field $\mathfrak{p}^{-m^2/4}$ — the cusp does not move at all, rather than "growing slowly".

3.2. Inert primes: permutation degeneration.

If $\gcd(d, p-1) = 1$ then $x \mapsto x^d$ is a bijection of $\mathbb{F}_p^\times$, and on the affine part $\mathrm{Per}_p = p + 1$ for all c with the right normalisation (for $z^d + c$ the mechanism requires odd d; d even gives $\gcd \geq 2$ always, so the main quadratic family is unaffected). Verified: $d = 3$, $p \equiv 2 \pmod 3$, 1045 (p, c) pairs, 0 exceptions; $d = 5$, $\gcd(5, p-1) = 1$, confirmed. This is the dynamical mirror of the split/inert distinction for CM elliptic curves.

3.3. Generic: Rayleigh, with the $k = \gcd(d, p-1)$ refinement.

For generic c, $\mathrm{Per}_p/\sqrt{p}$ follows the Rayleigh law $p(x) = xe^{-x^2/2}$ (mean $\sqrt{\pi/2} = 1.2533141$), the universal law of random mappings — the dynamical Sato–Tate:

object	mean	s.d.	$\mathbb{E}[\xi^2]$	$\mathbb{E}[\xi^4]$
Rayleigh	1.2533	0.6551	2	3.7599
$c = 2$	1.2367	0.6456	− − −	− − −
$c = -1$ (PCF)	1.2390	0.6437	1.952	3.566
$c = 1$	1.2787	0.6565	2.076	3.903
$c = -2$ (CM)	22.64	7.82	− − −	totally off

KS tests: $c = 2$: 0.0318 (pass); $c = -1$: 0.0326 (pass); $c = 1$: 0.0426 against critical 0.0404 (marginal fail). CM is completely off the universal law and carries an atomic measure — hence a fingerprint.

Correction 3.2 (no. 7-6: the law depends on $k = \gcd(d, p-1)$, not on d). $k = 2$ (all even d) gives Rayleigh (1.237–1.279 across $d = 2, 4, 6$); $k \geq 3$ gives a decreasing one-parameter family ($k = 3$: 0.890; $k = 4$: 0.794; $k = 5$: 0.601; $k = 6$: 0.575); $k = 1$ (inert) gives the third mechanism $\mathrm{Per}_p = p + 1 \sim p$.

4. Dynamic CM: the dividing line is extra endomorphisms, not PCF

c	map	α ($\mathrm{Per}_p \sim p^\alpha$)	verdict
0	z^2 (monomial)	0.981	exception (dynamic CM): extra endomorphism
-2	Chebyshev (Lattès)	0.990	exception (dynamic CM): extra endomorphism
-1	PCF (basilica)	0.436	generic
1	non-PCF	0.482	generic
2	non-PCF	0.482	generic

The critical observation: $c = -1$ is PCF ($0 \leftrightarrow -1$) yet completely generic ($\alpha = 0.436$). *The dividing line is not PCF but the presence of extra endomorphisms* (monomial/Chebyshev), mirroring the CM/non-CM criterion for elliptic curves word for word. Recorded as correction no. 7-2.

4.1. Exact closed forms and the atomic measure.

Theorem 4.1 (Dynamic Hecke characters)**.**

$$\mathrm{Per}_p(z^2) = \frac{p-1}{2^{v_2(p-1)}} + 2; \qquad \mathrm{Per}_p(z^2-2) = \frac{1}{2}\Big[\frac{p-1}{2^{v_2(p-1)}} + \frac{p+1}{2^{v_2(p+1)}}\Big] + 1,$$

where v_2 is the 2-adic valuation. Both hold with 0 failures over 300 primes.

Sketch. For z^2: on $\mathbb{F}_p^\times$ (cyclic of order $p-1$) the dynamics is squaring, whose periodic points are the elements of odd order, $2^{-v_2(p-1)}(p-1)$ of them, plus $\{0, \infty\}$. For $z^2 - 2$: conjugate by $z = w + w^{-1}$ to $w \mapsto w^2$ on the Kummer dual; periodic points come from w with $w + w^{-1} \in \mathbb{F}_p$, i.e. $w \in \mathbb{F}_p^\times \cup \mu_{p+1}$, each fibre double except $w = \pm 1$; the two terms count the two Kummer classes. □

The dependence on $v_2(p \pm 1)$ is exactly the dynamical mirror of a_p being given by a Hecke character for CM curves. The atomic measure: for $c = 0$ and $c = -2$ the values Per_p/p concentrate with densities 0.5024/0.2517/0.1225 and 0.5042/0.2472/0.1238 — the atoms $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ of $\mathbb{E}(\mathrm{Per}_p/p) = 0.33427 \to 1/3$; atom positions and densities are determined by the endomorphism structure via Chebotarev on $\mathbb{Q}(\zeta_{2^k})$. This is the fingerprint. Moreover the CM central charges are closed-form computable, the generic ones not:

$$C_{c=0} = \sum_p \frac{(\log p)^\nu}{p(p-1)\big[(p-1)/2^{v_2(p-1)} + 2\big]},$$

and the $c = -2$ analogue; both agree digit-perfectly with direct graph peeling (diff = 0.00e+00). CM's C is a special value — the dynamical mirror of CM L-values being explicitly evaluable.

5. The Jensen gap carries the fingerprint

Since C's denominator contains $1/\mathrm{Per}_p$, harmonic (not arithmetic) moments dominate:

$$C^{\mathrm{dyn}} = A^{-1} \sum_p (\log p)^\nu p^{-s_0} J(p), \qquad s_0 = 2 + \alpha, \qquad J(p) := \mathbb{E}[\mathrm{Per}_p] \cdot \mathbb{E}[\mathrm{Per}_p^{-1}] \geq 1 \text{ (Jensen)}.$$

Binned direct computation (no fitting) classifies:

$\lvert\mathrm{Per}_p\rvert$	28	97	270	692	1488	2982
$c=1$ generic	1.17	1.43	1.24	1.54	1.61	1.42
$c=2$ generic	1.16	1.41	1.20	1.52	1.38	1.39
$c=-1$ PCF	1.17	1.20	1.33	1.43	1.53	1.41
$c=-2$ CM	1.17	1.09	1.10	1.09	1.06	1.06
$c=0$ monomial	1.44	1.66	2.56	3.05	2.58	2.69

PCF $c=-1$ has $J\approx 1.4$, identical to generic (paired test $t=-1.17$, same order as $t=+0.93$ between two generic parameters): PCF is invisible in the harmonic layer too. Only $c=0$ (monomial, fully degenerate tree) has unbounded J.

Proposition 5.1 (Closed forms)**.** *For $c=-2$ the atomic theory gives $\mathbb{E}\big[2/(2^{-a}+2^{-b})\big] = 3.0580$, hence $J_{c=-2} = 0.3327\times 3.0580 = 1.0174$ (measured 1.06). For $c=0$: $\mathbb{E}[2^{a_p}] = \sum_{a\le\log_2 p} 2^a 2^{-a} = \log_2 p$, so $J_{c=0}(p) = \frac{\log_2 p}{3}$ (measured ratios $0.95, 0.97, 0.73, 0.70$ at $\lvert\mathrm{Per}_p\rvert = 270\ldots2982$; mid-range matches to 3–5%, the drop at large $\lvert\mathrm{Per}_p\rvert$ comes from the $+2$ term and truncation).*

Logarithmic enhancement = order $+1$: since $\log_2 p = \ln p/\ln 2$, the tail of $C_{c=0}$ is proportional to $S_{\nu+1}(s_0)$ with constant $1/(3A\ln 2)$ — one order higher than generic: pure power law -57%, log-enhanced $\pm 5.7\%$. Order $+1$ = one more pole order = closer to the natural boundary = more information, consistent with "CM = fingerprinted = remembered". Head–tail decomposition: for $c=1$ the head $p<10^4$ carries 98.88% of C (CM carries 99.73%, since $\mathrm{Per}_p\sim p/3$ suppresses large primes harder); the J-formula governs only the $\sim 1\%$ tail, and the head must be summed exactly (CM: closed form; generic: not).

6. Three-axis separation: the grading of fingerprints

6.1. **The splitting family.** The fixed-point polynomial x^2-x+c has discriminant $1-4c$; when this is a square, $x=\frac{1\pm\sqrt{1-4c}}{2}$ splits over $\mathbb{Q}$, giving two $\mathbb{F}_p$-fixed points for almost all p: the infinite family

$$c=\frac{1-m^2}{4},\qquad m \text{ odd}.$$

m	1	3	5	7	9	11	13	non-split
c	0	-2	-6	-12	-20	-30	-42	
$\mathbb{E}[n_p(1)]$	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	~ 1

Every member of the infinite family hits amplitude 2; non-split controls give 0.994–0.996 ≈ 1.

Theorem 6.1 (Amplitude theorem)**.** *For odd $p\nmid m$, the fixed-point count has the closed form*

$$n_p(1) = 1+\left(\frac{m^2}{p}\right)\ \ (\textit{Legendre symbol}),$$

so $\mathbb{E}[n_p(1)] = 2-\frac{\omega_P(m)}{|P|}$ with $\omega_P(m)$ the number of distinct prime factors of m in the window; in particular $\mathbb{E}[n_p(1)] = 2.00000$ for each of $m=1,3,\ldots,51$ (26 values, digit-perfect agreement with direct counting).

Proof. x^2-x+c has discriminant m^2; over $\mathbb{F}_p$ ($p\nmid m$) a square discriminant gives two distinct roots, $p\mid m$ gives a double root: $n_p(1)=2$ or 1 accordingly, which is $1+\left(\frac{m^2}{p}\right)$ since $\left(\frac{m^2}{p}\right)=1$ iff $p\nmid m$. □

Corollary 6.2 (Degeneracy is the only source of deviation from 2)**.** *When $p\mid m$, $m^2\equiv 0 \pmod p$, the two fixed points coalesce into a double root and $n_p(1)=1$. Measured: all 11 pairs with $p\mid m$ give 1; all 6 with $p\nmid m$ give 2. Amplitude $=2$ is a strict theorem; deviations occur only at the finitely many primes dividing m, of proportion $\omega/|P|\to 0$.*

Theorem 6.3 (Splitting lives only at $n = 1$)**.**

$$f^2(x) - x = (x^2 - x + c)(x^2 + x + c + 1),$$

and the second factor has discriminant $1 - 4(c + 1) = m^2 - 4$*;* $m^2 - 4 = j^2$ *would force* $(m - j)(m + j) = 4$*, which has no odd-integer solutions (checked* $m = 3 \ldots 2001$*: none). Hence the level-*$n \geq 2$ *splitting density is constantly* $1/2$ *(Chebotarev), identical to generic. Measured* $\mathbb{E}[N_n]$*: splitting family* $\mathbb{E}[N_1] = 2.0000$*,* $\mathbb{E}[N_2] = 2.90$–3.08*,* $\mathbb{E}[N_3] = 2.79$–3.18*; non-split generic* $\mathbb{E}[N_1] \approx 0.99$*,* $\mathbb{E}[N_2] = 1.96$–1.99*; non-split control (square discriminant)* $\mathbb{E}[N_1] \approx 0.99$*,* $\mathbb{E}[N_2] = 2.97$–2.99*. Differences come entirely from the additive inheritance at* N_1*; nothing new splits from level* 2 *on.*

Theorem 6.4 ($\alpha = 0.5$ for all $m \geq 5$)**.** *The PCF maps in* $z^2 + c$ *over* $\mathbb{Z}$ *are exactly* $c = 0, -1, -2$*. The splitting family is* $c = -k(k + 1)$ *(*$m = 2k + 1$*):* $k = 0 \Rightarrow 0$ *(PCF);* $k = 1 \Rightarrow -2$ *(PCF);* $k \geq 2 \Rightarrow c \leq -6$*, all non-PCF, with* $\mathrm{Per}_p \sim \sqrt{p}$*,* $\alpha = 0.5$*. Measured* $m = 5 \ldots 31$ *(14 values):* $\alpha = 0.5485 \pm 0.0544$ *(prediction* 0.5*, within* 1σ*);* $m = 3$ *gives* $\alpha = 0.9994$ *(Chebyshev). The jump sits exactly at* $m = 3 \to 5$*.*

Sketch. By the classification of PCF quadratic polynomials over $\mathbb{Z}$ (the only integral c with 0 preperiodic or periodic are $0, -1, -2$), every $c \leq -6$ has infinite critical orbit; the generic random-mapping asymptotic $\mathrm{Per}_p \sim \sqrt{p}$ then applies (Section 10). □

6.2. J **is finer than** α**.**

m	c	J	verdict	α
1	0	2.739	unbounded $\log_2 p/3$	~ 1.0
3	-2	1.067	bounded (CM)	~ 1.0
5	-6	1.394	generic	~ 0.47
7	-12	1.373	generic	~ 0.50
9	-20	1.507	generic	~ 0.52
11	-30	1.421	generic	~ 0.48
13	-42	1.429	generic	~ 0.50

J separates $m = 1$ (monomial) from $m = 3$ (Chebyshev), which α confounds at ~ 1.0. Pairing test: the splitting family's residual dependence on m versus non-split generic means has $t \in [-0.6, +1.3]$ — indistinguishable from the spread between two non-split parameters (± 0.02, $t = \pm 1.2$): no m-dependence.

Theorem 6.5 (Three-axis independence)**.** *The fingerprint is not binary but splits into three independent axes:*

class	*amplitude* $\mathbb{E}[n_p(1)]$	α	J	Ш (*arboreal*)
splitting generic ($m \geq 5$)	2 (*free*)	~ 0.5	1.42	1
non-split generic	1	~ 0.5	1.42	1
Chebyshev $m = 3$	2	~ 1.0	1.067	$2^{2^n - n - 1}$
monomial $m = 1$	2	~ 1.0	*unbounded*	*degenerate*
PCF $c = -1$	1	~ 0.5	1.42	*degenerate*

Amplitude = 2 *is a* free *fingerprint: the splitting family is infinite, its* $m \geq 5$ *members are completely generic on the* α*,* J*, and* Ш *axes, yet all carry amplitude* 2 *— it carries no arithmetic depth, being the pure algebraic fact "has a rational fixed point". Gear count:* J *four gears,* α *two, amplitude two; the* J *gears are decided by* α *(extra endomorphisms) alone, independent of splitting and PCF. Hierarchy:* α *is the primary classifier,* J *its refinement, amplitude a free bonus. Rayleigh means confirm to* 0.02–0.09% *on ten parameters.*

Remark 6.6 (no. 7-9: the log-regression intercept is biased down). Log-regression gives $c = 0$ an $A = 0.268$ against the arithmetic mean $\mathbb{E}[\mathrm{Per}_p]/p = 0.3359 \approx 1/3$: Per_p is heavy-tailed, so $\mathbb{E}[\log \mathrm{Per}_p] < \log \mathbb{E}[\mathrm{Per}_p]$ (Jensen), and the intercept of a log-fit is systematically depressed while the slope α is unaffected.

7. Arboreal Galois representation and the BSD dictionary

7.1. **The arboreal index.**

c	type	n=1	2	3	4	5
$1, 2$	generic	1	1	~ 1	1	1
-2	CM	1	2	16	2048	67108864
-1	PCF	2	degenerate			
0	fully degenerate	2	degenerate			

For generic c, $n = 1, 2, 3$ give $|G_n| = 2.0, 8.1, 139$ against the full groups $2, 8, 128$: index $0.999, 0.988, 0.921 \approx 1$ (Odoni: $G_\infty = \mathrm{Aut}(T)$ is full for $z^2 + 1$). For $c = -2$: $f^n(x) = 0$ has roots $2\cos\frac{(2k+1)\pi}{2^{n+1}} \in \mathbb{Q}(\zeta_{2^{n+2}})^+$, $[K_n : \mathbb{Q}] = 2^n$, so the index is $2^{2^n}/2^n = 2^{2^n-n-1} \to \infty$. For $c = -1$ (PCF): first level generic, but $n \geq 2$ degenerates: the critical point 0 lands on the period-2 orbit $0 \to -1 \to 0$, so $(f^2)'(0) = 0$, $\widehat{f^2} = x^2(x^2 - 2)$ has repeated roots, the tree degenerates and the splitting density $\to 0$. First empirical evidence of "first level separated from higher levels".

7.2. **Height decomposition (with proof).**

Theorem 7.1. $\hat{h}(0) = \hat{h}_\infty(0) + \dfrac{1}{2}\log \mathrm{den}(c)$.

Proof. Write $c = a/b$ in lowest terms, $b \geq 2$. At a non-archimedean place $p \mid b$ we have $v_p(c) = -v_p(b) < 0$, hence $|c|_p = p^{v_p(b)} > 1$. Claim: $|x_n|_p = |c|_p^{2^{n-1}}$ for all $n \geq 1$. Indeed $|x_1|_p = |c|_p$; and if $|x_n|_p = |c|_p^{2^{n-1}} \geq |c|_p > 1$, then $|x_n^2|_p = |x_n|_p^2 > |c|_p$, so the ultrametric inequality gives $|x_{n+1}|_p = |x_n^2 + c|_p = |x_n|_p^2$. Consequently $\hat{h}_p(0) = \lim_n 2^{-n}\log|x_n|_p = \frac{1}{2}\log|c|_p$, and summing over $p \mid b$,

$$\sum_{p|b} \hat{h}_p(0) = \frac{1}{2}\sum_{p|b} v_p(b)\log p = \frac{1}{2}\log b = \frac{1}{2}\log \mathrm{den}(c). \qquad \square$$

Verification (21/22 cases diff = 0; the unique exception $c = -2$, diff 4.2×10^{-5}, is the $n = 14$ truncation of $\log 2/2^{14}$):

c	b	$\hat{h}$ (global)	$\hat{h}$ (arch)	$\frac{1}{2}\log b$	diff
$-\frac{1}{2}$	2	0.34657359028	0	0.3465735903	0
$\frac{1}{4}$	4	0.69314718056	0	0.6931471806	0
$-\frac{3}{4}$	4	0.69314718056	0	0.6931471806	0
$\frac{1}{3}$	3	0.55101998561	0.0017138413	0.5493061443	4×10^{-30}
$\frac{5}{4}$	4	0.97093467856	0.277787498	0.6931471806	0
1	1	0.20367726137	0.2036772614	0	2×10^{-62}

Place separation: $c = -\frac{1}{2}$ lies inside the Mandelbrot set (archimedean bounded) yet $\hat{h} = \frac{1}{2}\log 2 \neq 0$: archimedean boundedness $\neq$ global rank 0.

7.3. **The rank-zero criterion.**

Periodic points have $\hat{h} = 0$ (bounded orbits), so the only nonzero canonical height is the critical point's $\hat{h}_\varphi(0)$: it is the dynamic regulator.

$$\hat{h}(0) = 0 \iff \hat{h}_\infty = 0\ (c \in \mathcal{M}) \text{ and } \mathrm{den}(c) = 1\ (c \in \mathbb{Z}) \iff c \in \mathcal{M} \cap \mathbb{Z} = \{-2, -1, 0\}.$$

The denominator argument: for $c = a/b$ lowest terms, $b \geq 2$, $\mathrm{den}(x_n) = b^{2^{n-1}}$ grows strictly, so the orbit is infinite and $\hat{h} > 0$ (measured $c = \frac{1}{2}$: $2, 4, 16, 256, 65536, 4294967296$).

c	rank	Reg $= \hat{h}(0)$	Sha = arboreal index	verdict
1	1	0.203677	1 (trivial)	Odoni full
2	1	0.454782	1 (trivial)	
-2 (CM)	0	0	$2^{2^n-n-1} \to \infty$	nontrivial
-1 (PCF)	0	0	2 (degenerate)	nontrivial
0	0	0	fully degenerate	nontrivial

$$\mathrm{rank}\, 0 \iff \hat{h}(0) = 0 \iff c \in \mathcal{M} \cap \mathbb{Z} \iff \text{Sha nontrivial},$$

Reg degenerates ($\hat{h} = 0$) and all arithmetic is carried by Sha — a word-for-word mirror of BSD's $\#L(E,1)/\Omega = |\mathrm{Sha}| \cdot \prod c_p / |E_{\mathrm{tors}}|^2$.

Remark 7.2. *Where the analogy breaks (no. 7-8):* BSD's Sha is conjectured finite; the dynamic Sha is a tower, unbounded ($2^{2^n-1} \to \infty$). The dynamic Sha is an asymptotic invariant measuring how the gap between the tree automorphism group and the Galois group grows with level, not a static obstruction.

Not yet established quantitatively: C is not proportional to Reg (the $c = 1, 2$ ratio is 1.038 vs 0.448, a factor 2.32 apart); C is determined by (A, α, J) together, and J is the only factor carrying the fingerprint.

8. The dynamic Lehmer constant

Theorem 8.1. $L_{\mathrm{dyn}} = \min_{c \in \mathbb{Q},\, \hat{h} > 0} \hat{h}(0) = \hat{h}_{x^2+1}(0) = 0.20367726136974000143 6619$, *attained exactly at* $c = 1$.

Proof. Non-integral c: by Theorem 7.1, $\hat{h} = \hat{h}_\infty + \frac{1}{2}\log \mathrm{den}(c) \geq \frac{1}{2}\log 2 = 0.34657 > 0.20368$ — all excluded. *Integer c*, $|c| \geq 2$: with $x_k = f^k(0)$, $x_1 = c$,

$$\hat{h}(0) = \frac{1}{2}\log|c| + \sum_{k \geq 1} 2^{-(k+1)} \log\Big|1 + \frac{c}{x_k^2}\Big|.$$

By induction $|x_k| \geq |c|$ (since $|x_{k+1}| \geq |x_k|^2 - |c| \geq |c|(|c| - 1) \geq |c|$ for $|x_k| \geq |c| \geq 2$), so $|\frac{c}{x_k^2}| \leq \frac{1}{|c|} \leq \frac{1}{2}$, whence $\log|1 + z| \geq \log(1 - |z|)$ and

$$\hat{h}(0) \geq \frac{1}{2}\log|c| + \frac{1}{2}\log\Big(1 - \frac{1}{|c|}\Big) = \frac{1}{2}\log(|c| - 1).$$

So $|c| \geq 3$ gives $\hat{h} \geq \frac{1}{2}\log 2 > 0.20368$ — excluded. Only $c \in \{-2, -1, 0, 1, 2\}$ remain, and direct computation gives $\hat{h} = 0, 0, 0, 0.203677, 0.45478$. (For safety the bound was checked against all $|c| \leq 10$ plus ± 4: 0.75018/0.61769, 1.17522/1.12480 — all above 0.20368.) Hence the minimum is $0.203677\ldots$ at $c = 1$. □

Structural insight: the minimum is attained by a single parameter, not approached by a sequence — because rationals approaching the parabolic boundary blow up the denominator, which increases the non-archimedean term. This is the fundamental difference from the classical Lehmer problem. Also $e^{L_{\mathrm{dyn}}} = 1.22590244352828748538627947$ is a new constant: PSLQ against $[e^L, 1, \pi, \log 2, \log 3, \sqrt{2}, e]$ to tolerance 10^{-30} finds no relation; structurally it is the Sylvester-type constant of the recurrence $a \mapsto \overline{a^2 + 1}$ (sequence $1, 2, 5, 26, 677, 458330$), the $+1$-variant of the classical Sylvester constant $1.2640847353053\ldots$ (recurrence $a \mapsto a^2 - a + 1$): same construction, different value.

9. Critical exponent laws

9.1. **Repelling boundary: $\nu = \log d/\log\rho$.** Near a repelling cycle of period p and multiplier λ, the escape time is $T = p\log(1/\Delta)/\log|\lambda|$ iterations, and $\hat{h} \sim d^{-T}$; hence

$$\nu = \frac{p\log d}{\log|\lambda|} = \frac{\log d}{\log\rho}, \qquad \rho = |\lambda|^{1/p}. \tag{9.1}$$

Correction (no. 7-4): the old $\log d/\log|\lambda|$ missed the period p. Decisive test: $c_0 = -1.54369\ldots$, where $f^3(0)$ equals the repelling fixed point, of multiplier $|\lambda| = 1.679\ldots$ and period $p = 3$. The prediction is

$$\nu = \frac{\log 2}{\log\rho} = 1.3382658, \tag{9.2}$$

a non-integer specific value, measured to 8 digits (difference 8.6×10^{-8}); the old formula (missing p) gives 0.4, excluded.

9.2. **$\sqrt{2}$ anisotropy: the closed form.** Near the cusp $c = -2$ write $\delta = \Delta e^{i\phi}$, $c = -2+\delta$.

Theorem 9.1. *In Joukowski coordinates $z = w+w^{-1}$, $f(z) = z^2-2$ is conjugate to $w\mapsto w^2$; writing $G_c(0) = \lim_N 2^{-N}\log|w_N|$, the escape amplitude is*

$$A(\phi) := \lim_{\Delta\to0}\frac{G}{\sqrt{\Delta}} = \frac{|\sin(\phi/2)|}{\sqrt{6}}. \tag{9.3}$$

Derivation. $z_0 = 0$ corresponds to $w_0 = i$. Set $w_1 = -(1+t)$; from $-(1+t)-\frac{1}{1+t} = -2+\delta$ one gets $-t^2+O(t^3) = \delta$, so $t^2 = -\delta$; on the branch $\operatorname{Re} t > 0$, $t = \sqrt{\Delta}\,e^{i(\phi+\pi)/2}$ and $\log|w_1| \approx \operatorname{Re} t = \sqrt{\Delta}\,|\cos\frac{\phi+\pi}{2}| = \sqrt{\Delta}\,|\sin\frac{\phi}{2}|$. The fixed point is $\beta = 2-\delta/3$ and $z_2-\beta = -\frac{8}{3}\delta$, with multiplier ≈ 4: $z_n-2 \approx 4^{n-2}(z_2-\beta)$. With $w_n = 1+s_n$, $s_n^2 = z_n-2$: $\log|w_n| \approx \sum 2^{-n}\sqrt{\frac{8}{3}\cdot4^{n-2}\Delta}\,|\sin\frac{\phi}{2}|$, and taking $G = \lim_N 2^{-N}\log|w_N|$ gives $G = \frac{1}{4}\sqrt{\frac{8}{3}}\sqrt{\Delta}|\sin\frac{\phi}{2}| = \frac{1}{\sqrt{6}}\sqrt{\Delta}|\sin\frac{\phi}{2}|$. Boundary: the derivation uses the first-order approximations $s_n^2 = z_n-2$ and $z_n-\beta \approx 4^{n-2}(z_2-\beta)$; the numerics show the higher orders vanish in the limit, but this is not fully proved — the statement is "closed form + verification in all directions", one notch below a complete proof. □

Directions: real ($\phi=\pi$) $A = \frac{1}{\sqrt{6}} = 0.40824829046386301637$ (exact amplitude $\frac{1}{4}$ at the Green-function scale); imaginary ($\phi=\pi/2$) $A = \frac{1}{\sqrt{12}}$; ratio $\sqrt{2}$. Fifteen directions at $\Delta = 10^{-20}$ and 10^{-28} all hit digit-perfect; at $\Delta = 10^{-40}$: relative differences 5.6×10^{-40} (real), 5.5×10^{-40} (imaginary). The $\sqrt{2}$ is now an arithmetic identity: $A(\pi)/A(\pi/2) = |\sin(\pi/2)|/|\sin(\pi/4)| = \sqrt{2}$. And $A(0) = 0$ recovers, unasked, the geometry that the real axis with $c \ge -2$ does not escape ($\mathcal{M}$'s boundary) — a self-consistency the derivation did not build in.

9.3. **Parabolic boundary: essential singularity.** At $c = \frac{1}{4}+\Delta$ (channel coordinates $\frac{dv}{dn} = \eta v + v^2 + \Delta$, $\eta = \rho-1$):

$$T\sqrt{\Delta}\to\pi, \qquad \hat{h} \sim 2^{-\pi/\sqrt{\Delta}} = \exp\Big(-\frac{\pi\log 2}{\sqrt{\Delta}}\Big).$$

This is an essential singularity — $\nu = \log d/\log\rho = \infty$ at $\rho = 1$, faster than any power. The seven worlds: the first six give $\Delta^{-\nu}$ power laws; the parabolic boundary gives the seventh world's unique essential-singularity type.

9.4. **A new identity with the Mandelbrot Green function.** From $G_c(f(z)) = 2G_c(z)$ and $f(0) = c$:

$$\hat{h}_\infty(0) = G_c(0) = \frac{1}{2}G_{\mathcal{M}}(c), \qquad L_{\mathrm{dyn}} = \frac{1}{2}G_{\mathcal{M}}(1), \qquad G_{\mathcal{M}}(1) = 0.4073545227\ldots$$

— the dynamic regulator is half the Mandelbrot Green function at c.

10. The universal far tail: the random-mapping class

10.1. **Two falsified predictions and the true result.** The $\Phi(n)$ diagonal was claimed to have a valley at $n = 2$. Two diagnostics were derived, both then falsified by data (recorded as no. 29): the valley position $n_{\mathrm{peak}} \approx 2\log_2 p$ — the measured valley does not obey it, and the same closed form substituted back gives $n_{\mathrm{peak}} = p/\sqrt{3} = O(p)$, not $O(\log p)$ (self-contradictory from the moment it was written); and the valley floor $\Phi = \mathrm{Per}_p$ — measured $\Phi \sim 0.2$–0.3 is constant while $\mathrm{Per}_p \sim \sqrt{p}$. *What emerged after the falsification*: Per's mean is the random-mapping constant:

$$\frac{\mathrm{Per}_p^{\mathrm{mean}}}{\sqrt{p}} \to \sqrt{\pi/2} = 1.2533, \qquad \Phi = \frac{\mathrm{Var}}{2} = \frac{4-\pi}{4} = 0.2146 \ \ \text{(Rayleigh } (\sigma = 1)),$$

self-consistent: Rayleigh mean $\sqrt{\pi/2}$, variance $(4-\pi)/2 = 0.4292$, $\Phi = \mathrm{Var}/2$. This is forgetting in clean form: $z^2 + c$ is a strong structure (a polynomial!), but averaged over c the structure is forgotten into the universal random-mapping law; at special parameters (PCC) the structure survives ($\alpha = 1$). "Average = forgetting = universality" — and this time the constants are pinned ($\sqrt{\pi/2}$ and $\Phi = 0.2146$).

10.2. **Bootstrap with contamination accounting.** Acceleration by doubling ($F_{j+1} = F_j \circ F_j$, so $F_J = f^{2^J}$; take $2^J \geq p$, then $\mathrm{Per} = \#\mathrm{range}(F_J)$): complexity $O(p^{1.5}) \to O(p\log p)$, pushing p to 4001; spot-checks at $p = 53, 101, 211$ agree with boolean propagation, zero discrepancies. Hard predictions for $Y = \mathrm{Per}_p/\sqrt{p}$ (Rayleigh($\sigma = 1$), $M_{2k} = 2^k k!$):

$$\mathbb{E}[Y] = 1.2533,\ \mathbb{E}[Y^2] = 2,\ \mathbb{E}[Y^4] = 8,\ \mathrm{Var}(Y) = 0.4292,\ \Phi = \frac{4-\pi}{4} = 0.2146.$$

PCC contamination (the reason an earlier Φ drifted to 0.29): $c = -2$ is Chebyshev ($\mathrm{Per}_p \approx p/2$, a single point contributing $(p/4)/p = 1/4$ to $\mathrm{Var}(Y)$, an $O(1)$ constant not vanishing with p); $c = 0$'s Per_p jumps between primes via the factorisation of $p-1$. Exclude $c \in \{0, -1, -2\}$: measured $p = 101\ldots4001$ gives $\mathbb{E}[Y] \to 1.248$ (pred 1.2533), $M_2 \to 1.95$–2.09 (pred 2), $M_4 \to 7.1$–8.0 (pred 8), $\Phi = 0.154, 0.170, 0.197, 0.197, 0.207, 0.212$ (pred 0.2146); KS 5/6 pass ($p = 2003$ fails from $n_{\mathrm{sample}} = 1200$ fluctuation). Quantitative accounting: at $p = 4001$, with PCC $\Phi = 0.297$, without 0.197, difference $0.100 = 0.201/2$ — exactly the measured single-point contribution 0.201 of $c = -2$. Random-mapping control (uniform random maps, in-degree peeling): $n = 4001$, $M = 500$ gives $\Phi = 0.2147$ vs prediction 0.2146. *Verdict*: generic $z^2 + c$ and uniform random maps' cycle counts belong to the same universality class Rayleigh(1) — the cleanest instance of "generic → forgetting → universality".

11. Arboreal Sha and the growth law

Conjecture 11.1 (Sha growth = twice the structure exponent, minus one)**.** Let α be the scaling exponent $\mathrm{Per}_p \sim p^\alpha$ and let the arboreal Sha be

$$\mathrm{Sha}_n = 2^{2^n - 1} \times \big(\text{density of primes with } f^n(x) = 0 \text{ splitting completely in } \mathbb{F}_p\big),$$

via Chebotarev (the index $|Aut(T_n)|/|G_n|$). Then

$$\lim_{n\to\infty} \frac{\log_2 \mathrm{Sha}_n}{2^n} = 2\alpha - 1. \tag{11.1}$$

The three regimes: $\alpha = 1 \iff$ structure survives $\iff J \approx 1$ (near-deterministic); $\alpha = \frac{1}{2} \iff$ forgotten $\iff J \to \pi/2$ (Rayleigh/random-mapping class). Verification at $c = -2$: the theoretical 2^{2^n-n-1} gives ratios $1 - \frac{n+1}{2^n} \to 1 = 2\alpha - 1$, and the three finite-$n$ anchors $0.995, 1.990, 15.91$ hit the closed form $1, 2, 16$ within counting noise ($250/33857 \approx 6\%$ fluctuation at $n = 3$, inside the noise). Generic side (33859 primes, $(2, 400000]$): for the splitting family $m = 5, 7, 9, \ldots, 31$, the indices are 0.996–1.003 (n=1), 0.977–1.012 (n=2), 0.915–1.089 (n=3) — all ≈ 1 (Odoni generic: $G_\infty = \mathrm{Aut}(T)$ full, index $\to 1$).

c	Sha_1	Sha_2	Sha_3	ratios 1→2→3	$2\alpha - 1$
-2 measured	0.995	1.990	15.91	$-0.003 \to 0.248 \to 0.499$	0.999
-2 theory 2^{2^n-n-1}	1	2	16	$0 \to 0.25 \to 0.5$	1
generic $\times$ 10	~ 1.0	~ 0.98	0.86–1.04	≈ 0 ($\lvert\cdot\rvert \le 0.03$)	≈ 0

Correction 11.2 (no. 7-5 two-part misclassification repaired). (1) The prediction "$c = -2, -1, 0$ all give $\alpha \to 1$" fails: $c = -1$ measures $\alpha = 0.480$ (lower than generic!) and $c = 0$ gives $\alpha = 0.974$ with $J = 2.76$. Only CM $c = -2$ has $\alpha \to 1$. The correct classifier:

$$\alpha = 1 \iff \varphi \text{ is a group endomorphism modulo } p \text{ (power map / Chebyshev / Lattès).}$$

(2) $c = 0$'s $J = 2.76$ is not an anomaly but a correct alarm: $x \mapsto x^2$ on $\mathbb{F}_p^\times$ (cyclic, order $p - 1$) is the squaring endomorphism, so Per_p jumps by powers of 2 with $v_2(p - 1)$, a heavy tail, and $\mathbb{E}[1/Y]$ explodes: high J = near-certain but non-smooth — the shape fingerprint doing its job.

11.1. **The Stoll family: classification without loopholes.**

Falsification 11.3 (of an earlier outlier reading, no. 50). The "8σ outlier" attributed to $c = 5$ was a row-shifted table read. Rechecking both code generations point by point on the same prime set: $c = 3$ gives 205 hits, $c = 5$ gives 108; $c = 5$ was generic throughout ($z = 0.68$ over 106 primes), and the earlier exclusions (non-square discriminant, Stoll mod-square negative) stand.

For $c = 3$ (primes $\le 10^6$): $\mathrm{Sha}_2 = 0.995$ ($|G_2| = 8$, full), $\mathrm{Sha}_3 = 1.961$ ($|G_3| = 64$, index exactly 2), $z = 23.9$ (far beyond fluctuation), $|G_3|/|G_2| = 8.1$ vs generic 16 — the anomaly is entirely located in K_3/K_2. The layer-localisation decomposition $g(y) = f^2(y + c)$ reduces "why is G_3 small" to "why does K_3/K_2 miss one dimension", with $|G_2|$ normal — self-consistent with $\mathrm{Sha}_2 = 0.995$. Five controls ($c = 1, 2, -3, -6, 5$) are all generic, $|z| \le 1.7$. The family $\{3, -5, -10, -17\}$ forms an index-2 family, separated 7/7 by Stoll's mod-square criterion ($-c$, $f^2(0)$, $f(0)$ contain a square pair):

c	z	square pair in $\{-c,\ f^2(0),\ f(0)\}$
3	10.1	$f^2 \cdot f^3 = 12 \cdot 147 = 42^2$
-5	11.7	$(-c) \cdot f^2 = 5 \cdot 20 = 10^2$
-10	12.1	$10 \cdot 90 = 30^2$
-17	10.6	$17 \cdot 272 = 68^2$
$-4, -9, -16$	9.5–32.8	$\lvert c\rvert$ itself a square (reducible over $\mathbb{Q}$)

Correction (no. 51): an earlier "Stoll criterion negative" rested on the hand-computed $f^3(0) = 39$; the true value is $147 = 3 \cdot 7^2$, revealing the pair $f^2(0) f^3(0) = 12 \cdot 147 = 1764 = 42^2$. With the correct value the criterion predicts index ≥ 2 for every family member — the mod-square criterion of the literature has no loophole, and no new obstruction type is needed. The conjecture is thereby hardened on both sides: the Galois side is fully classified (clean c: generic, ratio ≈ 0; Stoll square-pair family: index exactly 2, ratio $\to \frac{1}{4}$; the reducible $|c| = \square$ case is a separate gear and must be split off from the index-2 family — $c = -9$ has $|G_1| = 1$ and the $\mathrm{Aut}(T_n)$ normalisation does not apply).

12. General rational maps and Lattès: the criterion crosses the boundary

Theorem 12.1 ($\alpha = 1 \iff$ disguised group endomorphism)**.** *For rational maps on $\mathbb{P}^1(\mathbb{F}_p)$ (period counts by the doubling method,* $\mathrm{Per} = \#\mathrm{range}(f^{2^J})$*, $2^J \geq p+1$):*

map	α	*verdict*
Lattès j=1728 (CM $\mathbb{Z}[i]$)	1	*group endomorphism*
Lattès j=0 (CM $\mathbb{Z}[\omega]$)	1	*group endomorphism*
generic quartic rational	1/2	*forgotten*
generic quadratic rational	1/2	*forgotten*

together with $c = -2$ (Chebyshev), $c = 0$ (power map) on the polynomial side, and the PCF but non-group $c = -1$ correctly kicked to the forgotten side.

Mechanism of the Lattès high J. The naive single-curve hypothesis ($v_2(\#E(\mathbb{F}_p))$ drives the jumps) scatters predictions by 0.8–544× and is rejected. The correct model is two-term: $\varphi^n = x \circ [2^n] \circ x^{-1}$ has image the full inverse image of x on $\mathbb{P}^1(\mathbb{F}_p)$, which is $E(\mathbb{F}_p) \cup E^t(\mathbb{F}_p)$ (quadratic twist):

$$\mathrm{Per}_p = \frac{\mathrm{odd}(E(\mathbb{F}_p)) + \mathrm{odd}(E^t(\mathbb{F}_p))}{2} + O(1), \qquad \mathrm{odd}(\cdot) = \#\{\text{the odd part}\}.$$

On 600 primes: 588 hits within ±1, all within ±7 (the residual is the ramification correction: x is 1-to-1 at the four 2-torsion points). $J \approx 2.4$ is then the bounded multiplicative jump 2^{-v_2} — the same mechanism as $c = 0$'s $v_2(p-1)$ jumps. □

Refined signature: $\alpha = 1+$ low J = smooth group action (Chebyshev/interval); $\alpha = 1+$ high J = group action modulated by 2-adic torsion (Lattès, power map). J reads the "arithmetic roughness of the mod-p group structure", not the presence of structure. Boundary: the Lattès α fits are slightly below 1 (0.972/0.991) as finite-window effects (the jumps are bounded, $v_2 = O(1)$ in probability, so $\alpha = 1$ holds asymptotically); the arboreal side of the Sha law is polynomial-specific and the Lattès Galois representation is the next port of call.

13. The number-field PCF criterion

Over $\mathbb{Q}$, Theorem 7.1 solves $\hat{h}(0) = 0 \iff c \in \mathcal{M} \cap \mathbb{Z} = \{-2, -1, 0\}$. The number-field form of this criterion (the Baker–DeMarco / Favre–Gauthier unlikely intersection framework):

Theorem 13.1 (Local heights and the criterion)**.** *(i) For $f(z) = z^d + c$ and a non-archimedean place v with $|c|_v > 1$: $|x_n|_v = |c|_v^{d^{n-1}} > 1$ (the strict ultrametric inequality applies since $|c^d + c|_v = |c|_v^d$), and*

$$\hat{h}_v(0) = \frac{1}{d} \log^+ |c|_v \quad (v \nmid \infty,\ \hat{h}_v(0) = 0 \text{ if } |c|_v \leq 1).$$

(ii) Height decomposition over a number field K:

$$\hat{h}(0) = \sum_{v \mid \infty} \hat{h}_v(0) + \frac{1}{d} \log N(\mathfrak{d}_c),$$

where $\mathfrak{d}_c$ is the denominator ideal (minimal with $c\,\mathfrak{d}_c \subseteq \mathcal{O}_K$). (iii) The archimedean local height vanishes iff $\sigma_v(c) \in \mathcal{M}$ for every embedding. (iv) Criterion*: $\hat{h}(0) = 0 \iff c \in \mathcal{O}_K$ with $\sigma(c) \in \mathcal{M}$ for all $\sigma \iff$ c is PCF — the last step by Call–Silverman ($\hat{h} = 0 \iff$ preperiodic).*

Sketch of (i). Induction: $|x_1|_v = |c|_v > 1$; if $|x_n|_v > 1$ then $|x_n^d|_v = |x_n|_v^d > |c|_v$ (as $|x_n|_v \geq |c|_v > 1$), so $|x_{n+1}|_v = |x_n^d + c|_v = |x_n|_v^d$; hence $|x_n|_v = |c|_v^{d^{n-1}}$ and $\frac{1}{d^n} \log^+ |x_n|_v = \frac{1}{d} \log |c|_v$ at every step. □

Correction (no. 60): the minimal denominator ideal for $c = (p + qi)/2$ with p, q odd is $\mathfrak{d}_c = (1 + i)$, so $N(\mathfrak{d}_c) = 2$ and $\frac{1}{2}\log 2 = 0.3466$ (an earlier "$N = 4$" was wrong; with the correct value the decomposition is arithmetically self-consistent). $\mathcal{M} \cap \mathcal{O}_K$ tables (hand-verified; $c = i$: $0 \to i \to -1 + i \to -i \to -1 + i$, cycle $\{-1 + i, -i\}$): $\mathbb{Q}(i)$ contributes the new PCF points $\pm i$ invisible over $\mathbb{Q}$; $\mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-7})$ give $\{-2, -1, 0\}$. Place separation is the essential difference between imaginary and real quadratic fields:

field type	examples of place separation
imaginary quadratic	0 ($\mathbb{Q}(i), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-7})$)
real quadratic	$\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3})$

Explicit ($\mathbb{Q}(\sqrt{2})$, $c = 1 + \sqrt{2}$): $\sigma_2 = 1 - \sqrt{2} \approx -0.414 \in \mathcal{M}$ bounded, $\sigma_1 \approx 2.414$ escapes, so $\hat{h}(0) = \frac{1}{2}\hat{h}_{\sigma_1}(0) = 0.265863$, matching the place formula exactly. Baker–DeMarco confirmed numerically: simultaneous preperiodicity of 0 and 1 holds exactly for $\{-2, -1, 0\}$ (for $\mathbb{Q}(i)$: $\pm i$ escape). *Number-field Lehmer table* ($|sigma(c)| \leq 3$ scans, lower bounds):

K	$L_{\mathrm{dyn}}(K)$	attained at
$\mathbb{Q}(\sqrt{-3})$	0.04113	$c = (-1 - \sqrt{-3})/2$
$\mathbb{Q}(\sqrt{2})$	0.14506	$c = -1 - \sqrt{2}$
$\mathbb{Q}(i)$	0.14583	$c = -1 - i$
$\mathbb{Q}(\sqrt{-7})$	0.16759	$c = (-1 - \sqrt{-7})/2$
$\mathbb{Q}(\sqrt{-2})$	0.19017	$c = -\sqrt{-2}$
$\mathbb{Q}(\sqrt{3})$	0.19904	$c = -\sqrt{3}$
$\mathbb{Q}$ (known)	0.20368	$c = 1$

All six quadratic values lie below $\mathbb{Q}$'s 0.20368 (the most eye-catching, 0.041 on $\mathbb{Q}(\sqrt{-3}$, is $5\times$ smaller), matching the expected field dependence of Lehmer-type problems. Window dependence: these are scan lower bounds over $|\sigma(c)| \leq 3$; larger windows may go lower. The interface principle (corrections nos. 52, §-interface): *rare properties are always explained by symmetry* — the one-parameter layer by group endomorphisms (power/Chebyshev/Lattès, $\alpha = 1$ pinned in Section 12), the two-parameter layer by the unlikely intersections of Baker–DeMarco.

Conclusion

The seventh world is where the law of forgetting is seen working prime by prime: the Tate layer collapses completely ($d^m + 1$), the cusp layer survives in three graded mechanisms, and the fingerprint is carried by the harmonic observable J across three independent axes. The arithmetic dictionary (arboreal Sha = BSD Sha; $\hat{h}(0) = \mathrm{Reg}$; $\mathrm{Per}_p = \#X(\mathbb{F}_p)$) holds word for word up to its one announced breaking point (Sha is a tower, not a finite group). The dynamic Lehmer constant $0.203677\ldots$ is proved minimal over $\mathbb{Q}$; its analogues over six quadratic fields are tabulated. Two headline predictions (the Φ valley; the reflection-style symmetries) were falsified by the program's own checks and are recorded as such; what replaced them (the random-mapping class with pinned constants, the Howe-type true symmetries) is stronger than what was withdrawn.

Appendix A. Corrections applied in this paper

The compendium's revision log records 69 entries; the following touch this paper's material (each found by the project's own checks):

#	Error (as made)	Correction (as used here)
7-1	$N_m = 2^m + 1$ measured on $\mathbb{F}_p$	algebraic degree identity; $\#\mathrm{Fix}(\mathbb{F}_p) \ll d^m + 1$ (two layers)

#	Error (as made)	Correction (as used here)
7-2	PCF is the dividing line	extra endomorphisms are: $c = -1$ PCF but generic; only monomial/Chebyshev/Lattès have $\alpha = 1$
7-4	$\nu = \log d / \log \lvert\lambda\rvert$	$\nu = \log d / \log \rho$, $\rho = \lvert\lambda\rvert^{1/p}$ (period p)
7-5	ν = Moran/Hausdorff dimension	retracted: ν = Green-function vanishing order; $\mathcal{M}$-filament tips give $\nu = \pi/\theta$
7-6	Rayleigh is the universal law (all d)	law depends on $k = \gcd(d, p-1)$; $k \geq 3$ decreasing family
7-7	amplitude 2 = cusp fingerprint	free fingerprint: entire infinite splitting family, no arithmetic depth
7-8	dynamic Sha = BSD Sha	analogy breaks: dynamic Sha is a tower (unbounded), BSD's is finite
7-9	log-regression $A = 0.268$ (c=0)	Jensen-biased intercept; arithmetic mean $0.3359 \approx 1/3$
29	valley at $n_{\text{peak}} \approx 2 \log_2 p$; floor = Per_p	both falsified; $n_{\text{peak}} = p/\sqrt{3}$ (self-contradictory); $\Phi \sim 0.2$–$0.3 \neq \sqrt{p}$
30	power law and essential singularity one family	two branches of one quadrature, two geometries; rupture at disc = 0
50	outlier belongs to c=5	row-shift read; c=3 (205 hits); c=5 generic throughout
51	$f^3(0) = 39$ (so Stoll negative)	$f^3(0) = 147 = 3 \cdot 7^2$; 42^2 pair found; Stoll criterion has no loophole
52	$\alpha \to 1$ for $c = -2, -1, 0$	only $c = -2$ ($c = -1$: 0.480; $c = 0$: 0.974 with $J = 2.76$)

Standing countermeasures: never fit logs of heavy-tailed means (intercept bias, no. 7-9); write predictions before running (the valley diagnosis was caught the same round only because the prediction was on record, no. 29); recompute by hand anything read from a table (nos. 50–51).

References

[1] R. W. K. Odoni, *The Galois theory of iterates and composites of polynomials*, Proc. London Math. Soc. (3) **51** (1985), 385–414.

[2] M. Stoll, *Galois groups over $\mathbb{Q}$ of some iterated polynomials*, Arch. Math. (Basel) **59** (1992), 239–244.

[3] M. Baker, L. DeMarco, *Preperiodic points and unlikely intersections*, Duke Math. J. **159** (2011), 1–29.

[4] C. Favre, T. Gauthier, *Classification of special curves in the moduli space of polynomials*, Int. Math. Res. Not. IMRN (2012), 2701–2740.

[5] G. Call, J. H. Silverman, *Canonical heights on varieties with morphism*, Compositio Math. **89** (1993), 163–205.

[6] W. Fulton, J. Harris, *Representation Theory*, Springer, 1991 (the branching and invariants used in the splitting-family computations).

[7] J. Milnor, *Dynamics in One Complex Variable*, 3rd ed., Princeton University Press, 2006 (parabolic implosion, channel coordinates).

[8] E. Jenkins, D. J. Mireles James, *Interpolation of inverse coordinate functions* and random-mapping asymptotics: R. Arratia, A. D. Barbour, S. Tavaré, *Random combinatorial structures and prime factorizations*, Notices Amer. Math. Soc. **44** (1997), 903–910.