ISOMETRIC EMBEDDINGS OF PARALLELIZABLE AND STABLY PARALLELIZABLE RIEMANNIAN MANIFOLDS

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ABSTRACT. Let $s_n = n(n+1)/2$. We prove that every smooth parallelizable Riemannian n-manifold, $n \geq 3$, admits a smooth isometric embedding into $\mathbb{R}^{s_n+1}$ whose image lies in an arbitrarily small ball. For closed stably parallelizable manifolds, each of whose connected components has vanishing Euler characteristic, we obtain small-image isometric embeddings into $\mathbb{R}^{s_n+5}$. The proof combines explicit partial isometries along commuting Hopf flows, an iterative scheme for exact metric correction using oscillatory perturbations constructed from the second fundamental form, and a transversality argument for metric-preserving perturbations.

hexagon:2610.00690v1

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1. INTRODUCTION

1.1. **Background.** A smooth map $F\colon M \to \mathbb{R}^N$ induces the symmetric tensor $F^*\delta$, where δ is the Euclidean metric; in local coordinates

$$(F^*\delta)_{ij} = \langle \partial_i F, \partial_j F \rangle. \tag{1.1}$$

The isometric embedding problem asks for F with $F^*\delta = g$ for a prescribed Riemannian metric g. Throughout we write

$$s_n = \frac{n(n+1)}{2}, \qquad q_n = s_{n-1} = \frac{n(n-1)}{2}.$$

The system (1.1) consists of s_n equations, so $N = s_n$ is the determined dimension. By the Cartan–Janet theorem [J26, C27], real-analytic metrics are locally induced by analytic embeddings into $\mathbb{R}^{s_n}$. Globally, Nash [N56, Theorem 2] showed that every closed Riemannian n-manifold embeds isometrically into an arbitrarily small ball of $\mathbb{R}^{n(3n+11)/2}$. Nash's argument perturbs *free* maps, i.e. maps whose first and second derivatives are linearly independent; this already requires $N \geq s_n + n$.

2020 *Mathematics Subject Classification.* 53C42, 53C24, 57R25.

Key words and phrases. Isometric embedding, parallelizable manifold, second fundamental form, Hopf action, metric correction, h-principle.

Günther [Gün91] (see also [Yan98]) recast the perturbation step as a contraction and obtained the dimension

$$N = \max\Big\{\frac{n(n+5)}{2}, \frac{n(n+3)}{2} + 5\Big\} = s_n + \max\{2n,\, n+5\}.$$

In this paper we show that, for manifolds with trivial tangent bundle, one extra dimension beyond the determined case suffices.

1.2. **Main results.** Manifolds are smooth, Hausdorff and second countable, and have no boundary unless stated otherwise; a closed manifold is compact without boundary. Write $\underline{\mathbb{R}}^j$ for the trivial real vector bundle of rank j and $B_r(p)$ for the open Euclidean ball.

An immersion $F\colon M \to \mathbb{R}^N$ is *phase regular* if at every point x there is a tangent hyperplane $P \subset T_xM$ such that the normal second fundamental form of F is injective on $\operatorname{Sym}^2 P$; see Theorem 2.1.

Theorem A. *Let M be a parallelizable n-manifold, $n \geq 3$, possibly noncompact, disconnected, or with smooth boundary, and let g be a smooth Riemannian metric on M, smooth up to the boundary. For every $p \in \mathbb{R}^{s_n+1}$ and $r > 0$ there is a smooth phase-regular embedding*

$$F\colon M \longrightarrow \mathbb{R}^{s_n+1}, \qquad F^*\delta = g, \qquad F(M) \subset B_r(p).$$

If M is noncompact and has no boundary, F can moreover be chosen so that $F(x) \neq p$ for all x, $F(x) \to p$ as x leaves every compact set, and F is a proper embedding into $\mathbb{R}^{s_n+1} \setminus \{p\}$.

For instance, every closed orientable 3-manifold is parallelizable (Stiefel [Sti35]; see also [Kir89]), so every closed orientable Riemannian 3-manifold embeds isometrically into an arbitrarily small ball of $\mathbb{R}^7$. Products $S^1 \times N$ with N stably parallelizable, and all Lie groups, are further examples.

For tangent bundles that are only stably trivial we obtain:

Theorem B. *Let M be a closed n-manifold, $n \geq 3$, with $TM \oplus \underline{\mathbb{R}}^k \cong \underline{\mathbb{R}}^{n+k}$ for some $k \geq 0$, and assume that every component of M has Euler characteristic zero. Then every smooth Riemannian metric on M is induced by a smooth phase-regular embedding into an arbitrarily small ball of $\mathbb{R}^{s_n+5}$.*

For example, the spheres S^{2m+1}, $m \geq 1$, and the products $S^{2m+1} \times N$ with N closed and stably parallelizable satisfy the hypotheses of Theorem B. On manifolds all of whose components are noncompact, stable triviality of the tangent bundle already implies triviality (Theorem 8.7), so Theorem A applies:

Corollary C. *Let M be an n-manifold without boundary, $n \geq 3$, all of whose components are noncompact, and suppose $TM \oplus \underline{\mathbb{R}}^k \cong \underline{\mathbb{R}}^{n+k}$ for some $k \geq 0$. Then the conclusions of Theorem A, including the properness statement, hold for every smooth Riemannian metric on M.*

No completeness, finite topology, or bounded geometry is assumed in the noncompact case.

1.3. **Phase regularity.** Let F be an immersion and $x \in M$. The normal second fundamental form is a linear map $B_F(x)\colon \operatorname{Sym}^2 T_xM \to \nu_x$ into the normal space, of dimension $N - n$. A nonzero covector ξ is *good* if $B_F(x)$ is injective on $\operatorname{Sym}^2 \ker \xi$, a space of dimension q_n. A good covector can exist only if $N \geq s_n$, and when it exists, the space of *admissible normals*

$$E_\xi = \nu_x \ominus B_F(x)\big(\operatorname{Sym}^2 \ker \xi\big)$$

has dimension $N - s_n$. In the dimension $N = s_n + 1$ of Theorem A, E_ξ is a line. Freeness, by contrast, is the injectivity of B_F on all of $\operatorname{Sym}^2 T_xM$, which needs $N \geq s_n + n$.

Admissible normals are what make metric correction possible. Let ψ be a local function with $\mathrm{d}\psi$ good, let $\nu \in E_{\mathrm{d}\psi}$ be a unit normal field, and add the oscillation $U = \tau\delta b\, \nu \cos(\psi/\tau)$ with $0 < \delta \leq \tau \ll 1$. The quadratic part of the change of metric has mean $\frac{1}{2}\delta^2 b^2\, \mathrm{d}\psi^2$, which is the increment we want. The linear part, $-2\langle B_F, U\rangle$ to leading order, vanishes on $\ker \mathrm{d}\psi \times \ker \mathrm{d}\psi$ precisely because ν is admissible. The components that do involve the direction $\mathrm{d}\psi$ are absorbed by tangential corrections, because conjugation by $e^{\mathrm{i}\psi/\tau}$ turns the operator $u \mapsto \mathrm{d}\psi \odot u$ into a

dominant term of size τ^{-1}; a finite Neumann series cancels them to any prescribed order in τ divided by the scale of the data. Iterating such explicit increments, with smoothing, produces exact solutions. The method needs $E_\xi \neq 0$, hence $N \geq s_n + 1$.

1.4. **Strategy.** The proof of Theorem A for closed M has four steps.

Step 1: model partial isometries (Section 3). On a product $X = \prod_{i=1}^{n} S^{2C-1}$ of Hopf spheres with commuting period-one Hopf fields $Y_1, \dots, Y_n$, we construct maps $S_k \colon X \to \mathbb{R}^{s_n+1}$, $k \to \infty$, with

$$\langle Y_i S_k, Y_j S_k \rangle = \delta_{ij} + O(k^{-\infty}), \qquad \|S_k\|_{C^j} \lesssim k^{\max(j-1,0)},$$

whose normal second form along $\operatorname{Sym}^2 \operatorname{span}(Y_1, \dots, Y_{n-1})$ has smallest singular value $\gtrsim k^{-1/2}$. They are obtained from explicit curves on Hopf spheres, matrix-valued primitives with vanishing mean, formal power-series corrections, a normal collar, and a final wrapping along the last factor.

Step 2: exact metric correction (Section 4). If an immersion into $\mathbb{R}^N$, $N \geq s_n+1$, has uniformly elliptic induced metric, C^{119} norm $O(\epsilon^{-A})$ and phase margin $\gtrsim \epsilon^{\kappa}$, then for small ϵ every metric within $O(\epsilon^K)$ of its induced metric in C^{117}, with $K = K(A, \kappa, b)$, is realized exactly by a C^2-perturbation of size ϵ^b. All estimates are uniform over atlases of bounded overlap, with no dependence on the number of charts.

Step 3: local phases (Section 5). Given an orthonormal coframe $\alpha_1, \dots, \alpha_n$ of (M, g), we build maps $z_\ell \colon M \to S^{2C-1}$ which near each point of a fine net look like $e^{2\pi i T \lambda(x)}$ with $\mathrm{d}\lambda \approx \alpha_\ell$, glued by a colouring argument. Then $S_k(z_1, \dots, z_n)$ has induced metric $T^2 g$ up to a polynomially small error; Step 2 makes it exact, and division by T gives a phase-regular isometric immersion into a small ball. The coframe need not be closed.

Step 4: separation of points (Section 6). Exact-metric perturbations with prescribed first variations of values and second forms (*probes*) are produced by the iteration of Step 2. A dimension estimate for spaces of quadratic forms shows that at least $n + 1$ admissible normal value directions exist, except along a rank stratum and an exceptional locus of second forms, which are removed by adding their defining equations. Sard's theorem and a Baire argument then give an embedding with the same metric.

For Theorem B (Section 7), a nonvanishing vector field splits $TM = H \oplus \underline{\mathbb{R}}^1$ with $H \oplus \operatorname{Sym}^2 H$ trivial; the model map of Step 1 is replaced by a partially free map produced from this bundle data by an h-principle for open ample relations and by Günther-type local metric additions, which cost the four extra dimensions. For noncompact manifolds (Section 8), a conformal change $h = a^{-2} g$ to bounded geometry (Müller–Nardmann) and the weighted derivative $\mathrm{d} + \mathrm{d} \log a$ make every estimate uniform; the decay $a \to 0$ gives properness.

1.5. **Organization.** Section 2 fixes notation for second forms, partial geometry, Hopf actions and uniform atlases. Sections 3 to 6 carry out Steps 1–4 and prove Theorem A for closed manifolds. Section 7 proves Theorem B, and Section 8 treats noncompact manifolds and boundary, completing Theorem A and Theorem C.

2. Preliminaries

2.1. **Second forms and phase regularity.** For a map $F \colon M \to \mathbb{R}^N$ we write $\gamma(F) = F^* \delta$. If F is an immersion, Π_F denotes the orthogonal projection of $\mathbb{R}^N$ onto the normal space $\nu_F = \mathrm{d}F(TM)^{\perp}$; in coordinates $\Pi_F = I - \mathrm{d}F\, \gamma(F)^{-1} \mathrm{d}F^{\mathsf{T}}$.

Definition 2.1 (Second form, phase regularity)**.** The *normal second form* of an immersion F is $B_F(v, w) = \Pi_F(\nabla^2 F(v, w))$, where ∇ is any torsion-free connection on TM; changing ∇ adds a tangent vector to $\nabla^2 F$, so B_F is well defined, and in coordinates $B_{ij} = \Pi_F \partial_i \partial_j F$. We regard $B_F(x)$ as a linear map $\operatorname{Sym}^2 T_x M \to \nu_x$.

(i) A nonzero $\xi \in T^*_x M$ is *good* for F if $B_F(x)|_{\mathrm{Sym}^2 \ker \xi}$ is injective. The immersion is *phase regular* if every point has a good covector.
(ii) For a good ξ, the space of *admissible normals* is $E_\xi = E_\xi(F,x) = \nu_x \cap B_F(x)(\mathrm{Sym}^2 \ker \xi)^\perp$; it has dimension $N - s_n$.
(iii) Given an auxiliary norm on TM, the *phase margin* is

$$\mu(F,x) = \max_{|\xi|=1} \sigma_{\min}\big(B_F(x)|_{\mathrm{Sym}^2 \ker \xi}\big),$$

where $\sigma_{\min}$ is the smallest singular value.

If $\gamma(F) = h$, the Gauss formula identifies B_F with the full Hessian of F for the Levi-Civita connection of h:

$$B_{ij} = \partial_i \partial_j F - \Gamma(h)^k_{ij} \partial_k F. \tag{2.1}$$

Lemma 2.2 (Stability of second forms)**.** *Let F be an immersion of a coordinate ball with $cI \le \gamma(F)$ and $\|F\|_{C^2} \le C$. There are $\rho, C' > 0$, depending only on c, C, N, such that $\|V\|_{C^1} \le \rho$ implies that $F + V$ is an immersion and*

$$|B_{F+V} - B_F| \le C'\big(\|V\|_{C^2} + \|V\|_{C^1}\|F\|_{C^2}\big).$$

In particular good covectors, phase regularity, and lower bounds on the phase margin persist under C^2-small perturbations, with explicit loss.

Proof. The formula $\Pi = I - \mathrm{d}F\gamma^{-1}\mathrm{d}F^{\mathsf{T}}$ shows $|\Pi_{F+V} - \Pi_F| \le C'\|V\|_{C^1}$ once $\gamma(F+V) \ge cI/2$. Write $B_{F+V} - B_F = \Pi_{F+V}\partial^2 V + (\Pi_{F+V} - \Pi_F)\partial^2 F$. □

Remark 2.3 (Dimensions). A good covector forces $N - n \ge q_n$, that is $N \ge s_n$. The constructions of this paper add oscillations along admissible normals, so they need $N \ge s_n + 1$. In the dimension $N = s_n + 1$, a good covector ξ determines the admissible normal line E_ξ, and $B_F(x)$ has rank q_n or $q_n + 1$.

2.2. **Partial geometry.** Let $D \subset TX$ be an integrable distribution of rank m. A *flat structure* on D is a covering of X by open sets carrying frames $(Y_1, \dots, Y_m)$ of D by commuting vector fields, such that on overlaps the transition matrices are constant along the leaves of D. A partial metric h_D is *flat* if these frames can be chosen h_D-orthonormal.

Definition 2.4 (Partial jets)**.** Let D carry a flat structure and let $H\colon X \to \mathbb{R}^N$ be smooth.

(i) The *partial metric* is $\gamma_D(H) = (\mathrm{d}H|_D)^*\delta$, so $\gamma_D(H)_{\alpha\beta} = \langle Y_\alpha H, Y_\beta H\rangle$.
(ii) The *partial Hessian* $\nabla^2_D H \in \Gamma(\mathrm{Sym}^2 D^* \otimes \mathbb{R}^N)$ is $\nabla^2_D H(Y_\alpha, Y_\beta) = Y_\alpha Y_\beta H$; this is independent of the flat frame since transition matrices are constant along leaves.
(iii) The *partial jet* is the bundle map

$$J_D(H)\colon D \oplus \mathrm{Sym}^2 D \to X \times \mathbb{R}^N, \qquad (v, A) \mapsto \mathrm{d}H(v) + \nabla^2_D H(A).$$

H is *D-free* if $J_D(H)$ is injective. Its adjoint $T_H\colon \mathbb{R}^N \to D^* \oplus \mathrm{Sym}^2 D^*$ is the *pairing map*; given a partial metric, $C_H = J_D(H)(J_D(H)^* J_D(H))^{-1}$ is the minimum-norm right inverse of T_H.
(iv) If $\gamma_D(H) > 0$, the *partial normal second form* is $B^D_H = \Pi^D_H \nabla^2_D H$, where Π^D_H is the orthogonal projection off $\mathrm{d}H(D)$.

Freeness does not depend on the choice of torsion-free partial connection: another choice adds a $\mathrm{d}H(D)$-valued term to the Hessian. If $\gamma_D(H)$ is constant in a flat orthonormal frame, then $\langle Y_\alpha Y_\beta H, Y_\gamma H\rangle = 0$: the coefficients are symmetric in (α, β) and, by differentiating the constant Gram matrix, antisymmetric in (β, γ), hence zero.

The main example is the following.

Definition 2.5 (Coefficient distributions)**.** Let $Z = \prod_{\ell=1}^{p} S^{2C_\ell-1}$ carry the Hopf actions of Section 2.3, with commuting generators $X_1, \dots, X_p$. For a manifold B and a subbundle $W \subset B \times \mathbb{R}^p$ of rank m, the *coefficient distribution* on $B \times Z$ is

$$D_W = \Big\{ Y_w = \sum_{\ell=1}^{p} w_\ell X_\ell \ : \ w \in W_b \Big\}, \qquad h_W(Y_w, Y_{w'}) = \langle w, w' \rangle.$$

For a local orthonormal frame (w_α) of W, the fields Y_{w_α} commute, because their coefficients depend only on b and the X_ℓ commute and are tangent to Z. Their transition matrices depend only on b. Hence D_W is integrable, it carries a canonical flat structure, and the *coefficient metric* h_W is flat. When $W = B \times \mathbb{R}^p$ we recover the span of all Hopf generators.

2.3. **Hopf actions, averages and primitives.** On $S^{2C-1} \subset \mathbb{C}^C$ we use the period-one circle action $R_t z = e^{2\pi i t} z$ with generator X, and on products the corresponding commuting actions. For a smooth function h (scalar, vector or matrix valued) put

$$\operatorname{av}(h)(z) = \int_0^1 h(R_t z)\, \mathrm{d}t, \qquad I_X h(z) = \int_0^1 t\, h(R_t z)\, \mathrm{d}t.$$

Lemma 2.6. $\operatorname{av}(h)$ *is R-invariant, and* $X \operatorname{av}(h) = 0$*. If* $\operatorname{av}(h) = 0$*, then* $X I_X h = h$ *and* $\operatorname{av}(I_X h) = 0$*, and $I_X h$ is the unique primitive of h along X with zero mean. Consequently, if Φ commutes with the action and* $\operatorname{av}(h) = 0$*, then* $I_X(h \circ \Phi) = (I_X h) \circ \Phi$*; in particular, if* $h \circ R_{1/2} = \pm h$ *then* $I_X h \circ R_{1/2} = \pm I_X h$*. These operations are smooth, continuous in every C^j norm, and commute with derivatives in parameters on which the action does not depend.*

Proof. Along an orbit, $X I_X h = \int_0^1 t\, \partial_t [h(R_t z)]\, \mathrm{d}t = h(z) - \operatorname{av}(h)(z)$ by parts, and $\operatorname{av}(I_X h) = \int_0^1 t \operatorname{av}(h)\, \mathrm{d}t = 0$. Two primitives differ by an invariant function, which equals its own mean. The equivariance statements follow from uniqueness. □

We repeatedly use the antipodal map $z \mapsto -z = R_{1/2} z$; a function is *even* or *odd* if it is invariant or anti-invariant under it. The mean of an odd function vanishes.

2.4. **Uniform atlases and weighted norms.**

Setup 2.7 (Uniform atlases)**.** A *uniform atlas* on an n-manifold is a finite or countable, locally finite family of charts $\phi_i \colon \widehat{U}_i \to B_3(0) \subset \mathbb{R}^n$ whose cores $\phi_i^{-1}(B_1(0))$ cover the manifold, such that each $\widehat{U}_i$ meets at most Λ others and all transition maps satisfy uniform bounds at every order. The C^j norm of a map or tensor is the supremum over charts of its coordinate C^j norm on $B_3(0)$; ellipticity $cI \le \gamma \le CI$ and singular values refer to coordinate norms. When the atlas is countable we only consider maps and tensors all of whose C^j norms are finite. A compact parameter space is included by using product charts and full derivative norms in source and parameter; only source derivatives enter metrics and second forms.

All constants below depend on the atlas only through n, N, Λ and the transition bounds, never on the number of charts. For $0 < a \le 1$ and integers $m \ge 0$ we use the weighted norms

$$|u|_{m,a} = \sum_{j=0}^{m} a^j \|u\|_{C^j}, \qquad |u|_{m,a}^{(2)} = \sum_{j=0}^{m} a^j \|u\|_{C^{j+2}}.$$

A function oscillating at frequency a^{-1} with slowly varying amplitude has $|\cdot|_{m,a}$ comparable to its supremum, which is why these norms are adapted to the oscillatory constructions.

A family e_ϵ depending smoothly on ϵ near 0 is *flat at* $\epsilon = 0$ if $\|e_\epsilon\|_{C^j} = O(|\epsilon|^R)$ for every j and every R.

3. Model partial isometries on products of Hopf spheres

In this section all distributions are spanned by Hopf generators, with the flat structure of Theorem 2.5. The goal is Theorem 3.10: approximately isometric maps of $\prod_{i=1}^{n} S^{2C_i-1}$, along the Hopf generators, into $\mathbb{R}^{s_n+1}$.

3.1. Curves on Hopf spheres.

Lemma 3.1 (Orbit reparametrization)**.** *Let $a > 0$ be smooth on S^{2C-1}, put $L = \operatorname{av}(a)$, $h = I_X(a/L - 1)$ and $\Phi(z) = R_{h(z)}z$. Then Φ is a diffeomorphism mapping each orbit onto itself. If p is smooth with $|Xp| = a$, then $\widetilde{p} = (p \circ \Phi^{-1})/L$ satisfies $|X\widetilde{p}| = 1$, and at corresponding points $\operatorname{span}(X\widetilde{p}, X^2\widetilde{p}) = \operatorname{span}(Xp, X^2p)$. If a is even, Φ commutes with $z \mapsto -z$.*

Proof. On the orbit $t \mapsto R_t z_0$ the map Φ is $t \mapsto \phi(t) = t + h(R_t z_0)$, with $\phi' = 1 + Xh = a/L > 0$ and $\phi(t+1) = \phi(t) + 1$; so it is a diffeomorphism of the orbit. Since $\mathrm{d}\Phi(X) = (a/L)X$ and $\mathrm{d}\Phi$ induces the differential of $R_{h(z)}$ transversally to the orbits, $\mathrm{d}\Phi$ is invertible, and Φ is a bijective local diffeomorphism. Differentiating $\widetilde{p}(R_{\phi(t)}z_0) = p(R_t z_0)/L$ gives $\phi'\, X\widetilde{p} = Xp/L$, hence $|X\widetilde{p}| = 1$, and a second derivative gives the statement about spans. If a is even, so is h (Theorem 2.6), and $\Phi(-z) = R_{h(z)}(-z) = -\Phi(z)$. □

Lemma 3.2 (Curves on Hopf spheres)**.** *Let $C \geq 1$.*

(a) There is a smooth $f\colon S^{2C-1} \to \mathbb{R}^2$ with

$$|Xf| = 1, \qquad f(-z) = \operatorname{diag}(1,-1)f(z), \qquad \operatorname{av}(f_1 X f_2 - f_2 X f_1) = 0.$$

(b) There is a smooth $c\colon S^{2C-1} \to \mathbb{R}^3$ with $|Xc| = 1$ such that Xc and X^2c are linearly independent everywhere. Then $b = Xc \times X^2c/|Xc \times X^2c|$ is a smooth unit field perpendicular to Xc and X^2c.

Proof. (a) Let $k_j = 2j-1$ and $u(z) = \sum_{j=1}^{C} \operatorname{Re}(z_j^{k_j})$. Along an orbit u is a trigonometric polynomial with distinct frequencies k_j, so

$$A = \operatorname{av}((Xu)^2) = 2\pi^2 \sum_{j=1}^{C} k_j^2 |z_j|^{2k_j} > 0,$$

and A is invariant. Put $v = u/\sqrt{A}$ and $q = Xv$, so $\operatorname{av}(q^2) = 1$, and

$$p = (I_X(1-q^2),\, v), \qquad Xp = (1-q^2,\, q), \qquad |Xp|^2 = \left(q^2 - \tfrac{1}{2}\right)^2 + \tfrac{3}{4}.$$

Since the k_j are odd, v and q are odd, so p_1 is even and p_2 is odd. Apply Theorem 3.1 with $a = |Xp|$, which is even, and put $f = \widetilde{p}$. Then $|Xf| = 1$; the antipodal map commutes with Φ, so $f(-z) = \operatorname{diag}(1,-1)f(z)$. The integrand $f_1 X f_2 - f_2 X f_1$ is odd, so its mean vanishes.

(b) Let $w = Xf$ and $c_0 = (f, |f|^2/2)$. Then

$$Xc_0 = (w, \langle f, w\rangle), \qquad X^2 c_0 = (Xw,\, 1 + \langle f, Xw\rangle),$$

using $|w| = 1$. If $Xw \neq 0$, the planar parts $w \perp Xw$ are independent; if $Xw = 0$, the second vector is $(0,1)$ while $w \neq 0$. Hence Xc_0, X^2c_0 are independent everywhere. Theorem 3.1, applied with $a = |Xc_0|$, gives c. □

Lemma 3.3 (Partial immersions)**.** *For $X_m = \prod_{i=1}^{m} S^{2C_i-1}$ with Hopf generators $Y_1, \ldots, Y_m$, there is a smooth $A_m\colon X_m \to \mathbb{R}^{m+1}$ such that $Y_1 A_m, \ldots, Y_m A_m$ are linearly independent everywhere.*

Proof. For $m = 1$ take f from Theorem 3.2(a). Given A_m, let ν_m be the unit normal in $\mathbb{R}^{m+1}$ to $Y_1 A_m, \ldots, Y_m A_m$ (the normalized generalized cross product). With $(a, b) = f$ on the new factor, set

$$A_{m+1}(x, z) = \left(A_m(x) + \epsilon a(z)\nu_m(x),\ \epsilon b(z)\right).$$

The old columns are $(Y_iA_m + \epsilon a Y_i\nu_m, 0)$ and the new one is $\epsilon\big((Xa)\nu_m, Xb\big)$, of length ϵ. They are orthogonal, since $\langle Y_iA_m, \nu_m\rangle = \langle Y_i\nu_m, \nu_m\rangle = 0$. By compactness the old columns remain independent for small $\epsilon > 0$. □

3.2. **Matrix primitives.** The next proposition provides the displacement used in the inductive step of Theorem 3.8. Its essential property is that the mixed metric terms it creates have zero mean along the new orbit.

Lemma 3.4 (Uniform oscillatory decay)**.** *Let $\eta(z) = \sum_{j=1}^{C} \operatorname{Re}(z_j^{2j-1})$ on S^{2C-1}. For every smooth function a there is C_a with*

$$\|\operatorname{av}(a\, e^{\mathrm{i}T\eta})\|_{C^0} \le C_a T^{-1/(2C)} \qquad (T \ge 1).$$

Proof. Along an orbit, $\partial_t^j \eta(R_t z) = \sum_{l=1}^{C} \operatorname{Re}\big((2\pi \mathrm{i} k_l)^j z_l^{k_l} e^{2\pi \mathrm{i} k_l t}\big)$ with $k_l = 2l-1$. If these vanished for $j = 1, \dots, 2C$ at some t, then, writing the real parts as sums over the $2C$ distinct nonzero nodes $\pm 2\pi \mathrm{i} k_l$, the invertibility of the Vandermonde matrix would force all $z_l = 0$. By compactness there is $c > 0$ with $\max_{1\le j\le 2C} |X^j \eta| \ge c$ on the sphere. Cover $(\mathbb{R}/\mathbb{Z}) \times S^{2C-1}$ by finitely many sets $I_\alpha \times U_\alpha$, I_α an interval, on which a fixed derivative $\partial_t^{j_\alpha}\eta(R_t z)$ has absolute value at least $c/2$, and take a subordinate smooth partition of unity $\rho_\alpha(t, z)$. By van der Corput's lemma with amplitude [Ste93, Chapter VIII, Section 1.2, Corollary], for $j_\alpha \ge 2$,

$$\Big|\int_{I_\alpha} \rho_\alpha(t,z)\, a(R_t z)\, e^{\mathrm{i}T\eta(R_t z)}\, \mathrm{d}t\Big| \le C_{j_\alpha} (cT)^{-1/j_\alpha} \int_{I_\alpha} |\partial_t(\rho_\alpha a)|\, \mathrm{d}t,$$

the amplitude vanishing at the endpoints. For $j_\alpha = 1$, one integration by parts using $|\partial_t \eta| \ge c/2$ and the bound on $\partial_t^2 \eta$ gives $O(T^{-1})$. The constants are uniform in z, and $j_\alpha \le 2C$. □

Proposition 3.5 (Matrix primitives)**.** *For all $C \ge 1$ and $d \ge 2$ there is a smooth $P\colon S^{2C-1} \to \operatorname{Mat}_{d\times d}(\mathbb{R})$ such that*

$$Q = XP \text{ is invertible everywhere}, \qquad \operatorname{av}(P^{\mathsf{T}} Q) = 0. \tag{3.1}$$

Proof. Block-diagonal matrices inherit both properties from their blocks, and every $d \ge 2$ is a sum of 2's and at most one 3. Let f be as in Theorem 3.2(a), $v = Xf = (s, t)$ and $u = J_2 v = (-t, s)$.

$d = 2$. Put $P_2 = f_1 I + f_2 J_2$, so $Q_2 = sI + tJ_2 \in \mathrm{SO}(2)$ and $P_2^{\mathsf{T}} Q_2 = \langle f, Xf\rangle\, I + (f_1 X f_2 - f_2 X f_1)\, J_2$. The first coefficient is $\frac{1}{2} X|f|^2$ and the second has zero mean by Theorem 3.2(a).

$d = 3$. Regard f, v, u as vectors in the first two coordinates of $\mathbb{R}^3$, let η be as in Theorem 3.4, and for $T \ge 1$ set

$$Q_0 = \big[\, v,\ e_3 \cos(T\eta) - u \sin(T\eta),\ -e_3 \sin(T\eta) - u \cos(T\eta)\,\big].$$

Its columns are orthonormal and $\det Q_0 = \det[v, u, e_3] = 1$, so $Q_0 \in \mathrm{SO}(3)$. Let $M_T = \operatorname{av}(Q_0)$. Its first column is $\operatorname{av}(Xf) = 0$ and, by Theorem 3.4, $\|M_T\|_{C^0} \to 0$ as $T \to \infty$. Put $Q_1 = Q_0 - M_T$ and $P_1 = I_X Q_1$, so $XP_1 = Q_1$ and the first column of P_1 is $p_1 = f - \operatorname{av}(f)$.

Let $R = \operatorname{diag}(1, -1, 1)$ and $D = \operatorname{diag}(1, 1, -1)$. Under $z \mapsto -z$ we have $v \mapsto Rv$, $u \mapsto -Ru$ and $\eta \mapsto -\eta$, hence $Q_0(-z) = RQ_0(z)D$. Since M_T is invariant, $M_T = RM_TD$, so $Q_1(-z) = RQ_1(z)D$ and, by Theorem 2.6, $P_1(-z) = RP_1(z)D$. The matrix $\Omega = \operatorname{av}(P_1^{\mathsf{T}} Q_1)$ is skew, because $\operatorname{av} X(P_1^{\mathsf{T}} P_1) = 0$, and satisfies $\Omega = D\Omega D$. Therefore only $\omega = \Omega_{12} = -\Omega_{21}$ can be nonzero. Since $p_1 \perp e_3$ and $\operatorname{av}(p_1) = 0$,

$$\omega = \operatorname{av}\langle p_1, e_3 \cos(T\eta) - u \sin(T\eta) - M_T e_2\rangle = -\operatorname{av}\big(\langle p_1, u\rangle \sin(T\eta)\big),$$

which tends to 0 uniformly by Theorem 3.4. Let P be P_1 with its second column replaced by $p_2 + \omega v$. As ω is invariant, $XP = Q_1 + \omega\,(Xv)e_2^{\mathsf{T}}$. Again $\operatorname{av}(P^{\mathsf{T}} XP)$ is skew and D-symmetric, and its $(1,2)$ entry is

$$\omega + \omega \operatorname{av}\langle p_1, Xv\rangle = \omega - \omega \operatorname{av}\langle Xp_1, v\rangle = \omega - \omega \operatorname{av}|v|^2 = 0.$$

Hence $\operatorname{av}(P^{\mathsf{T}}XP) = 0$. Finally $\|XP - Q_0\|_{C^0} \le \|M_T\|_{C^0} + \|\omega\|_{C^0}\|Xv\|_{C^0} < 1$ for T large, so XP is invertible. □

3.3. **Formal corrections.**

Lemma 3.6 (Borel lemma with parameters)**.** *Let P be a compact manifold, possibly with boundary or corners, and let $a_j \in C^\infty(P, \mathbb{R}^N)$, $j \ge 0$. There is $a \in C^\infty((-1,1) \times P, \mathbb{R}^N)$ whose Taylor series at $\epsilon = 0$ is $\sum_j \epsilon^j a_j$. If Λ is a continuous linear map from $C^\infty(P, \mathbb{R}^N)$ to a Fréchet space (for instance, restriction of some derivatives to a submanifold) with $\Lambda a_j = 0$ for all j, then a can be chosen with $\Lambda a(\epsilon, \cdot) = 0$ for every ϵ.*

Proof. Fix $\chi \in C_c^\infty(-1,1)$ equal to 1 near 0. For $j \ge 2$ choose $\lambda_j \ge 1$ so large that all derivatives of order at most $j-1$ of $\chi(\lambda_j\epsilon)\epsilon^j a_j(x)$ are bounded by 2^{-j}; this is possible because each such derivative carries a factor ϵ^i with $i \ge 1$ on the support $|\epsilon| \le \lambda_j^{-1}$. Then $a = a_0 + \epsilon a_1 + \sum_{j\ge 2} \chi(\lambda_j\epsilon)\epsilon^j a_j$ converges in C^∞ and has the required Taylor series. The cutoffs do not depend on x, so Λ passes through the sum. □

For a family of maps P and a displacement U, in a flat frame of D,

$$\gamma_D(P+U)_{\alpha\beta} = \gamma_D(P)_{\alpha\beta} + Y_\alpha\langle Y_\beta P, U\rangle + Y_\beta\langle Y_\alpha P, U\rangle - 2\langle Y_\alpha Y_\beta P, U\rangle + \gamma_D(U)_{\alpha\beta}. \tag{3.2}$$

In particular, if $\langle \mathrm{d}_D P, U\rangle = 0$, the linear part of the change of partial metric is $-2\langle \nabla_D^2 P, U\rangle$.

Lemma 3.7 (Formal correction of partial metrics)**.** *Let X be compact, D an integrable distribution with a flat structure, and h_D a partial metric. Let P_ϵ be a smooth family with*

$$\gamma_D(P_\epsilon) - h_D = \epsilon^{2L+1} S_\epsilon, \qquad L \ge 1,$$

and suppose the pairing map T_{P_ϵ} has, for $\epsilon \ne 0$, a right inverse C_ϵ such that $\widetilde{C}_\epsilon = \epsilon^L C_\epsilon$ extends smoothly through $\epsilon = 0$. Then there is a smooth family $\widehat{U}_\epsilon$ such that, with $U_\epsilon = \epsilon^{L+1}\widehat{U}_\epsilon$, the error $\gamma_D(P_\epsilon + U_\epsilon) - h_D$ is flat at $\epsilon = 0$.

Proof. By continuity $T_{P_\epsilon}\widetilde{C}_\epsilon = \epsilon^L \operatorname{Id}$ for all ϵ. Write $U = \epsilon^L v$ and consider

$$v = \widetilde{C}_\epsilon(0, \tfrac{1}{2}(\epsilon S_\epsilon + \gamma_D(v))). \tag{3.3}$$

Seek $v = \sum_{j\ge 1} \epsilon^j v_j$. Since $\gamma_D(v)$ is quadratic and $v = O(\epsilon)$, the coefficient of ϵ^j on the right depends only on $v_1, \dots, v_{j-1}$, through finitely many derivatives and products. Hence all v_j are determined and smooth, and Theorem 3.6 gives a smooth $v = \epsilon\widehat{v}$ satisfying (3.3) up to a flat error. Applying T_{P_ϵ}, we obtain modulo flat terms $\langle \mathrm{d}_D P, U\rangle = 0$ and $2\langle \nabla_D^2 P, U\rangle = \epsilon^{2L+1} S + \gamma_D(U)$. Inserting this in (3.2) shows that $\gamma_D(P+U) - h_D$ is flat. □

3.4. **Free families with flat metric error.**

Let $X_a = \prod_{i=1}^a S^{2C_i - 1}$ with Hopf generators $Y_1, \dots, Y_a$ and D their span. The partial jet of $H\colon X_a \to \mathbb{R}^N$ has the $a + s_a$ *columns* $Y_i H$ and $Y_i Y_j H$ $(i \le j)$. Give the columns $Y_i H$ and $Y_i Y_i H$ weight 0, and $Y_i Y_j H$ with $i < j$ weight $j - 1$.

Theorem 3.8 (Free families)**.** *For $a \ge 1$ and $C_1, \dots, C_a \ge 1$ there are $\epsilon_a > 0$ and smooth families $H_\epsilon\colon X_a \to \mathbb{R}^{s_a + a + 1}$, $|\epsilon| < \epsilon_a$, such that:*

- *(i) $\gamma_D(H_\epsilon) - I_a$ is flat at $\epsilon = 0$;*
- *(ii) each column divided by ϵ^{weight} extends smoothly to $\epsilon = 0$, and these* normalized columns *are linearly independent for all $|\epsilon| < \epsilon_a$.*

Consequently, for $\epsilon \ne 0$, H_ϵ is D-free; the unit normal ν_ϵ to the normalized columns is smooth through 0 and perpendicular to the partial jet; and the minimum-norm right inverse $C_{H,\epsilon}$ of the pairing map is such that $\epsilon^{a-1} C_{H,\epsilon}$ extends smoothly through 0.

Proof. The consequences follow from (ii): the number of columns is $a+s_a = N-1$, and if M_ϵ is the matrix of normalized columns and Δ_ϵ the diagonal matrix of the powers ϵ^{weight}, then $T_{H_\epsilon} = \Delta_\epsilon M_\epsilon^{\mathsf T}$ and $C_{H,\epsilon} = M_\epsilon(M_\epsilon^{\mathsf T}M_\epsilon)^{-1}\Delta_\epsilon^{-1}$; the largest weight is $a-1$.

For $a=1$ take $H=c$ from Theorem 3.2(b), independent of ϵ, with $\nu = b$. Assume the statement for $a=m$ and put $L=m$. Let c, b be the curve and binormal of Theorem 3.2(b) on the new factor $S^{2C_{m+1}-1}$, with generator $X = Y_{m+1}$. In $\mathbb{R}^N = \mathbb{R}^{s_m+m+1}\times\mathbb{R}^3\times\mathbb{R}^{m-1}$, $N = s_{m+1}+m+2$, set

$$F_{0,\epsilon} = (H_\epsilon, c, 0), \qquad T_\epsilon = \big[(\nu_\epsilon,0,0),(0,b,0),(0,0,e_1),\dots,(0,0,e_{m-1})\big],$$

an orthonormal $N\times(m+1)$ frame. Choose $A = A_m\colon X_m \to \mathbb{R}^{m+1}$ from Theorem 3.3, P and $Q = XP$ from Theorem 3.5 with $d = m+1$, and define the displacement

$$Z_\epsilon = T_\epsilon\, P\, A.$$

Each column of T depends on one factor and the columns lie in orthogonal blocks; hence

$$T^{\mathsf T}T = I, \quad T^{\mathsf T}Y_iT = 0, \quad T^{\mathsf T}XT = 0, \quad (Y_iT)^{\mathsf T}XT = 0, \quad Y_iXT = 0. \tag{3.4}$$

The columns of T are perpendicular to all first and second columns of F_0, and the mixed columns Y_iXF_0 vanish. By (3.2) the linearization of γ_D at F_0 kills Z, and (3.4) gives

$$\langle Y_iZ, XZ\rangle = \langle PY_iA, QA\rangle, \qquad T^{\mathsf T}Y_iXZ = QY_iA. \tag{3.5}$$

The first expression has zero mean along X by (3.1), because A does not depend on the new factor; the vectors QY_iA are linearly independent.

Second-order cancellation. Put $e_{ij} = \langle Y_iZ, Y_jZ\rangle$, $e = |XZ|^2$ and $w_i = -I_X\langle Y_iZ, XZ\rangle$. For $\epsilon\neq 0$ use $C_{H,\epsilon}$ to choose W_H with

$$\langle Y_iH, W_H\rangle = w_i, \qquad 2\langle Y_iY_jH, W_H\rangle = Y_iw_j + Y_jw_i + e_{ij},$$

and set $W = \big(W_H,\ \frac{e}{2|X^2c|^2}X^2c,\ 0\big)$. Since $\langle Xc, X^2c\rangle = 0$, a direct computation with (3.2) shows that the linearization of γ_D at F_0 maps W to $-\gamma_D(Z)$: the (ij), (iX) and (XX) entries are $-e_{ij}$, $Xw_i = -\langle Y_iZ, XZ\rangle$ and $-e$. By the inductive hypothesis $\widetilde{W}_\epsilon = \epsilon^{m-1}W_\epsilon$ is smooth through 0. The family

$$P_\epsilon = F_{0,\epsilon} + \epsilon^L Z_\epsilon + \epsilon^{L+1}\widetilde{W}_\epsilon$$

therefore satisfies

$$\gamma_D(P_\epsilon) - I_{m+1} = \operatorname{diag}\big(\gamma_D(H_\epsilon) - I_m, 0\big) + \epsilon^{2L+1}S_3 + \epsilon^{2L+2}S_4$$

with S_3, S_4 smooth, and the first term is flat.

Independence. Keep the old weights and give the new mixed columns Y_iXP_ϵ weight L. At $\epsilon=0$ the normalized columns of P_ϵ are the old normalized columns (in the first block), Xc and X^2c (in the second block), and Y_iXZ_0. The orthogonal complement of the first two groups is spanned by the columns of T_0, and by (3.5) the projection of Y_iXZ_0 onto it is T_0QY_iA. Hence all $N-1$ normalized columns are independent at $\epsilon=0$, so on an interval, and as in the first paragraph the pairing map of P_ϵ has a right inverse C_ϵ with ϵ^LC_ϵ smooth. Theorem 3.7 (the flat error is divisible by ϵ^{2L+1}) gives $U_\epsilon = O(\epsilon^{L+1})$ with flat metric error. Since all weights are at most L, adding U_ϵ changes the normalized columns by $O(\epsilon)$. Thus $H_{m+1,\epsilon} = P_\epsilon + U_\epsilon$ satisfies (i) and (ii) on a smaller interval. □

3.5. **Normal collars and wrapping.**

Proposition 3.9 (Wrapping)**.** *Let X be compact, D an integrable distribution of rank $m\geq 1$ with a flat structure and a flat partial metric h_D. Let $H_\epsilon\colon X\to\mathbb{R}^q$ and ν_ϵ be smooth through $\epsilon=0$, with $\gamma_D(H_\epsilon) - h_D$ flat at 0, and assume that for $\epsilon\neq 0$ the map H_ϵ is D-free, ν_ϵ is a unit vector perpendicular to its partial jet, and $\epsilon^KC_{H,\epsilon}$ extends smoothly through 0 for some integer $K\geq 0$. On*

$\widehat{X} = X \times S^{2C-1}$ let $\widehat{D} = D \oplus \mathbb{R}X_{\mathrm{new}}$ and $\widehat{h} = h_D + (X^*_{\mathrm{new}})^2$. Then for all large integers k there are maps $S_k \colon \widehat{X} \to \mathbb{R}^{q+1}$ such that, for all j and R,

$$\|S_k\|_{C^j} \le C_j k^{\max(j-1,0)}, \tag{3.6}$$

$$\|\gamma_{\widehat{D}}(S_k) - \widehat{h}\|_{C^j} \le C_{j,R}\, k^{-R}, \tag{3.7}$$

and the partial normal second form of S_k along $\widehat{D}$, restricted to $\mathrm{Sym}^2 D$, has smallest singular value at least $c\,k^{-K/(2K+1)} \ge ck^{-1/2}$.

Proof. Let $M = 2K+1$, let $(a,b) = f$ be the planar curve of Theorem 3.2(a) on S^{2C-1}, and fix $\bar{a}$ with $|a| \le \bar{a}$. Work in flat orthonormal frames Y_α of D.

Step 1: a normal collar. On $X \times [-\bar{a}, \bar{a}]$ we seek

$$G_\epsilon(x,s) = H_\epsilon(x) + \epsilon^M s\, \nu_\epsilon(x) + \epsilon^{2M-K} W(\epsilon, x, s), \qquad W|_{s=0} = \partial_s W|_{s=0} = 0.$$

In the variable $t = \epsilon^M s$ we have $G_t = \nu_\epsilon + \epsilon^{M-K} W_s$ and $G_{tt} = \epsilon^{-K} W_{ss}$. Let $T_G v = (\langle v, G_t\rangle, \langle v, \mathrm{d}_D G\rangle, \langle v, \nabla^2_D G\rangle)$ and $R_H(r,y) = r\nu_\epsilon + C_{H,\epsilon} y$. Since $C_{H,\epsilon}$ takes values in the span of the partial jet, R_H is a right inverse of T_G at $(G, G_t) = (H_\epsilon, \nu_\epsilon)$. The differences of the entries of T_G from these values are $O(\epsilon^{M-K})$ and $O(\epsilon^M)$, while $R_H = O(\epsilon^{-K})$; hence $A_G = T_G R_H = I + \epsilon^{M-2K}\Theta = I + \epsilon\Theta$, with Θ smooth in ϵ, x, s and the 2-jet of W. Consider

$$W_{ss} = \epsilon^K R_H A_G^{-1}(0, 0, \langle \mathrm{d}_D G_t, \mathrm{d}_D G_t\rangle). \tag{3.8}$$

Every occurrence of W on the right carries a positive power of ϵ, so the coefficient of ϵ^j determines $\partial_s^2 W_j$ from $W_0, \dots, W_{j-1}$; integrating twice with zero initial data gives smooth coefficients, and Theorem 3.6 realizes them by a smooth W with both initial conditions. Then (3.8) holds up to a flat error, so the physical residual $R = G_{tt} - R_H A_G^{-1}(0,0,\langle \mathrm{d}_D G_t, \mathrm{d}_D G_t\rangle)$ is flat.

Let $E_{tt} = |G_t|^2 - 1$, $E_{\alpha t} = \langle Y_\alpha G, G_t\rangle$, $E_{\alpha\beta} = \langle Y_\alpha G, Y_\beta G\rangle - h_{\alpha\beta}$ and $\theta_\alpha = \langle R, Y_\alpha G\rangle$. The three components of $T_G G_{tt} = (0,0,\langle \mathrm{d}_D G_t, \mathrm{d}_D G_t\rangle) + T_G R$ give

$$\partial_t E_{tt} = 2\langle G_t, R\rangle, \qquad \partial_t E_{\alpha t} = \tfrac{1}{2} Y_\alpha E_{tt} + \theta_\alpha, \qquad \partial_t^2 E_{\alpha\beta} = Y_\alpha \theta_\beta + Y_\beta \theta_\alpha - 2\langle R, Y_\alpha Y_\beta G\rangle.$$

At $s = 0$ we have $E_{tt} = E_{\alpha t} = 0$, $\partial_t E_{\alpha\beta} = -2\langle \nu_\epsilon, Y_\alpha Y_\beta H_\epsilon\rangle = 0$, and $E_{\alpha\beta}$ is flat. Since $\partial_s = \epsilon^M \partial_t$ and s ranges over a fixed interval, integration shows that all three errors are flat, with all derivatives in x and s.

Step 2: power maps. Let $P_k(z) = (z_\ell^k)_\ell / \big(\sum_\ell |z_\ell|^{2k}\big)^{1/2}$. Then $P_k(R_t z) = R_{kt} P_k(z)$, so $X(\phi \circ P_k) = k(X\phi) \circ P_k$. Moreover $\|P_k\|_{C^j} \le C_j k^j$: at a point where $m_z = \max_\ell |z_\ell| \ge C^{-1/2}$, the numerator has norm at least m_z^k and its derivatives of order j are at most $C_j k^j m_z^{k-j}$.

Step 3: wrapping. Let $\epsilon = k^{-1/M}$, so $\epsilon^M = k^{-1}$, and

$$S_k(x,z) = \big(G_\epsilon(x, a(P_k z)),\ k^{-1} b(P_k z)\big).$$

Every derivative of G_ϵ in s carries the factor $\epsilon^M = k^{-1}$, while j derivatives of $a \circ P_k$ or $b \circ P_k$ cost k^j; this gives (3.6). With $\varrho = (Xa) \circ P_k$ and $\varsigma = (Xb) \circ P_k$ we have $\varrho^2 + \varsigma^2 = 1$, $Y_\alpha S_k = (Y_\alpha G, 0)$ and $X_{\mathrm{new}} S_k = (\varrho G_t, \varsigma)$, so the entries of $\gamma_{\widehat{D}}(S_k) - \widehat{h}$ are $E_{\alpha\beta}$, $\varrho E_{\alpha t}$ and $\varrho^2 E_{tt}$ evaluated at $s = a(P_k z)$. Flatness in ϵ absorbs every power of k produced by the chain rule, proving (3.7).

Margin. Compare the first columns $(Y_\alpha G, 0)$, $(\varrho G_t, \varsigma)$ and the second columns $(Y_\alpha Y_\beta G, 0)$ with $(Y_\alpha H_\epsilon, 0)$, $(\varrho \nu_\epsilon, \varsigma)$ and $(Y_\alpha Y_\beta H_\epsilon, 0)$; the differences are $O(\epsilon^{M-K})$ and $O(\epsilon^M)$. The comparison second columns are perpendicular to $(\varrho\nu_\epsilon, \varsigma)$, so their projection off the comparison first columns is $(\Pi^D_{H_\epsilon} \nabla^2_D H_\epsilon, 0)$. For unit $\Xi \in \mathrm{Sym}^2 D$, $|\Pi^D_H \nabla^2_D H(\Xi)| = \min_v |J_D(H)(-v, \Xi)| \ge \sigma_{\min}(J_D(H)) = \|C_{H,\epsilon}\|^{-1} \ge c\epsilon^K$. The comparison Gram matrices are uniformly positive and all second jets are bounded, so the projector formula changes the normal second form by $O(\epsilon^{M-K}) = O(\epsilon^{K+1})$. Hence the margin is at least $\frac{c}{2}\epsilon^K = \frac{c}{2} k^{-K/(2K+1)}$. ∎

Corollary 3.10 (Model maps)**.** *Let $n \geq 2$, $C_1, \dots, C_n \geq 1$, and let $Y_1, \dots, Y_n$ be the Hopf generators of $X_n = \prod_{i=1}^{n} S^{2C_i-1}$. For all large k there are maps $S_k \colon X_n \to \mathbb{R}^{s_n+1}$ satisfying (3.6), (3.7) with $\widehat{h} = I_n$, whose partial normal second form has smallest singular value at least $ck^{-1/2}$ on $\operatorname{Sym}^2 \operatorname{span}(Y_1, \dots, Y_{n-1})$.*

Proof. Theorem 3.8 with $a = n-1$ gives families into $\mathbb{R}^{s_{n-1}+n} = \mathbb{R}^{s_n}$ with $K = n-2$; apply Theorem 3.9. □

4. Exact metric correction

This section proves Theorem 4.7: an immersion with polynomially controlled jets and phase margin, whose metric error is polynomially small of sufficiently high order, can be corrected to an exact isometric immersion. The basic step (Theorem 4.4) adds a prescribed positive tensor to the metric by explicit oscillations along admissible normals; an iteration with smoothing (Theorem 4.5) produces exact solutions. Throughout, norms refer to a uniform atlas as in Theorem 2.7, and C denotes constants depending only on the indicated data.

4.1. Smoothing.

Lemma 4.1 (Smoothing operators)**.** *For every integer $r \geq 1$ there are linear operators S_ℓ, $0 < \ell \leq \ell_0$, acting on maps and on tensors through their chart components, such that for all m:*

(i) $|S_\ell u|_{m,\ell} \leq C_{m,r}\|u\|_{C^0}$ and $|S_\ell G|^{(2)}_{m,\ell} \leq C_{m,r}\|G\|_{C^2}$;

(ii) if $\tau \leq \ell \leq t \leq 1$ and $|u|_{r,t} \leq \Lambda_0$, then $|u - S_\ell u|_{m,\tau} \leq C_{m,r}\big[\Lambda_0(\ell/t)^r + (\tau/t)^r |u|_{m,t}\big]$, and the same holds for $|\cdot|^{(2)}$ in place of $|\cdot|$.

Proof. Let κ be an even bump supported in the unit ball and $\kappa_\lambda(x) = \lambda^{-n}\kappa(x/\lambda)$. Choose $r+1$ distinct $\lambda_a \in [\frac{1}{2}, 1]$ and solve the Vandermonde system $\sum_a c_a = 1$, $\sum_a c_a \lambda_a^p = 0$ $(1 \leq p \leq r)$; then $\rho = \sum_a c_a \kappa_{\lambda_a}$ has $\int \rho = 1$ and vanishing moments of orders $1, \dots, r$. Let $\{\zeta_i\}$ be a partition of unity subordinate to the charts, with $\operatorname{supp}\zeta_i \subset B_2$ and uniform bounds, and put $S_\ell u = \sum_i \zeta_i\,(\rho_\ell * u_i)$, where u_i is the chart representative and $\rho_\ell(x) = \ell^{-n}\rho(x/\ell)$. Differentiating the kernel gives (i). For (ii), the vanishing moments and Taylor's formula give $\|u - S_\ell u\|_{C^j} \leq C\ell^{r-j}\|u\|_{C^r} \leq C\Lambda_0 \ell^{r-j} t^{-r}$ for $j \leq r$, while for $j > r$ we use $\|u - S_\ell u\|_{C^j} \leq C\|u\|_{C^j}$ and $\tau^j \leq (\tau/t)^r t^j$. Only suprema and the bounded overlap of the charts enter. □

4.2. Oscillatory modes.

For an immersion F let $L_F(V) = \langle \mathrm{d}F, \mathrm{d}V\rangle + \langle \mathrm{d}V, \mathrm{d}F\rangle$ be the linearized metric, so that

$$\gamma(F+V) = \gamma(F) + L_F(V) + \gamma(V). \tag{4.1}$$

For a covector field $\xi = \mathrm{d}\psi$ that is good for F, let $\beta_\psi \colon \nu_F \to \operatorname{Sym}^2(\ker \mathrm{d}\psi)^*$, $v \mapsto \langle B_F, v\rangle|_{\ker \mathrm{d}\psi}$; it is onto, its kernel is $E_{\mathrm{d}\psi}$, and we write $J_\psi = \beta_\psi^*(\beta_\psi\beta_\psi^*)^{-1}$.

Lemma 4.2 (Oscillatory modes)**.** *Let F be an immersion of B_3 with $cI \leq \gamma(F)$, let ψ be a function with $|\mathrm{d}\psi| \geq c$ such that $\mathrm{d}\psi$ is good for F with margin at least c, and let $0 < \tau \leq \ell \leq 1$. Assume $|F|^{(2)}_{m,\ell} + |\psi|_{m+2,\ell} \leq K_m$ for all m. For each integer $p \geq 1$ there is a linear differential operator $\mathcal{Z}_{p,\tau}$, whose coefficients are polynomials in τ with coefficients depending smoothly on finitely many derivatives of F and ψ (and on a fixed partition of unity), such that for a complex symmetric tensor h and a complex section v_f of $E_{\mathrm{d}\psi}$, both supported in a set $S \subset B_2$, the field $Z = \mathcal{Z}_{p,\tau}(h, v_f)$ is supported in S and*

$$L_F\big(Ze^{\mathrm{i}\psi/\tau}\big) = (h + \mathfrak{e})\, e^{\mathrm{i}\psi/\tau},$$

where, with $|(h,v_f)|_m = |h|_{m,\ell} + |v_f|_{m,\ell}$,

$$|Z|_{m,\ell} \le C_m |(h,v_f)|_{m+p}, \qquad |\mathfrak{e}|_{m,\ell} \le C_m \Big(\frac{\tau}{\ell}\Big)^p |(h,v_f)|_{m+p},$$
$$\big|Z - v_f + \tfrac{1}{2} J_\psi(h|_{\ker \mathrm{d}\psi})\big|_{m,\ell} \le C_m \frac{\tau}{\ell} |(h,v_f)|_{m+p}.$$

The constants depend on m, p, c *and finitely many* K_j.

Proof. By linearity and a fixed partition of unity we may assume that S lies in a ball of fixed radius carrying coordinates with $x^1 = \psi$; their bounds depend only on c and the K_j. Write $g = \gamma(F)$, $Z = \mathrm{d}F(g^{-1}u) + v$ with $u_i = \langle \partial_i F, Z\rangle$ and v normal. Then

$$L_F(Z)_{ij} = \partial_i u_j + \partial_j u_i - 2\Gamma^k_{ij} u_k - 2\langle B_{ij}, v\rangle,$$

and conjugation by $e^{\mathrm{i}\psi/\tau}$ adds $(\mathrm{i}/\tau)(\delta_{i1}u_j + \delta_{j1}u_i)$. Indices a, b run from 2 to n. Put $T(u)_{ab} = \frac{1}{2}(\partial_a u_b + \partial_b u_a) - \Gamma^k_{ab} u_k$ and $v = v_f + J_\psi(T(u) - \frac{1}{2}h_H)$, $h_H = (h_{ab})$; then the (ab) equations hold exactly. The remaining equations become $(\mathrm{i}/\tau + \mathcal{L})u = k$, where

$$\begin{aligned} &(\mathcal{L}u)_1 = \partial_1 u_1 - \Gamma^k_{11} u_k - \langle B_{11}, J_\psi T(u)\rangle, && k_1 = \tfrac{1}{2}h_{11} + \langle B_{11}, v_f\rangle - \tfrac{1}{2}\langle B_{11}, J_\psi h_H\rangle, \\ &(\mathcal{L}u)_a = \partial_1 u_a + \partial_a u_1 - 2\Gamma^k_{1a} u_k - 2\langle B_{1a}, J_\psi T(u)\rangle, && k_a = h_{1a} + 2\langle B_{1a}, v_f\rangle - \langle B_{1a}, J_\psi h_H\rangle. \end{aligned} \tag{4.2}$$

A residual e in this system produces the metric residual $2e_1$ in the (11) entry, e_a in the $(1a)$ entries, and 0 in the (ab) entries. Set

$$u = \frac{\tau}{\mathrm{i}} \sum_{l=0}^{p-1} \Big(-\frac{\tau}{\mathrm{i}}\mathcal{L}\Big)^l k,$$

whose residual is $(-1)^{p-1}(\tau/\mathrm{i})^p \mathcal{L}^p k$. Each application of the first-order operator $\mathcal{L}$ costs one derivative, i.e. a factor ℓ^{-1} in the slow norms; this gives all three estimates. □

4.3. **Phase data.** The increments need phases that are good, together with all the phases created by quadratic interactions.

Definition 4.3 (Phase data)**.** Let F_0 be an immersion and h_0 a positive tensor. *Phase data* for (F_0, h_0) with constant $c_* > 0$ consist of:

(a) functions χ_i, $i \in I$, with $\sum_i \chi_i^2 = 1$ and uniform bounds, each supported in a ball B_2 of some chart, and a colouring of I by at most Λ' colours such that equally coloured supports are disjoint;
(b) for each i, functions $\phi_{i,1}, \dots, \phi_{i,s_n}$ with uniform bounds whose differentials are constant in the chart of χ_i and form a *rank-one basis*: every symmetric matrix is uniquely $A = \sum_r \varpi_{i,r}(A)\, \mathrm{d}\phi_{i,r}^2$, with linear functionals $\varpi_{i,r}$ satisfying $\varpi_{i,r}(h_0) \ge c_*$ on $\operatorname{supp}\chi_i$;
(c) for every *label* $\lambda = (i,r)$, and every pair of distinct labels λ, λ' whose supports meet, the covectors $\mathrm{d}\phi_\lambda$, $2\mathrm{d}\phi_\lambda$ and $\mathrm{d}\phi_\lambda \pm \mathrm{d}\phi_{\lambda'}$ have length at least c_* and are good for F_0 with margin at least c_* on the corresponding supports;
(d) for each label a constant *seed* vector $a_\lambda \in \mathbb{R}^N$ whose orthogonal projection onto $E_{\mathrm{d}\phi_\lambda}(F_0)$ has length at least c_* on $\operatorname{supp}\chi_i$.

By Theorem 2.2, conditions (c) and (d) persist, with $c_*/2$, for all immersions F with $\|F - F_0\|_{C^2} \le \rho_0$, where $\rho_0 > 0$ depends on c_* and $\|F_0\|_{C^2}$; for such F we use the unit normals $n_\lambda(F)$ obtained by projecting a_λ onto $E_{\mathrm{d}\phi_\lambda}(F)$ and normalizing. Likewise (b) persists, with $c_*/2$, for tensors A with $\|A - h_0\|_{C^0} \le \eta_0$.

Lemma 4.4 (Finite positive increments)**.** *Fix phase data for* (F_0, h_0) *and* ρ_0, η_0 *as above, and an integer* $q \ge 1$. *Let* F *and* H *satisfy* $\|F - F_0\|_{C^2} \le \rho_0$, $\|H - h_0\|_{C^0} \le \eta_0/2$ *and, for some* $0 < \ell \le 1$, $|F|^{(2)}_{m,\ell} + |H|_{m,\ell} \le K_m$ *for all* m. *If* $0 < \delta \le \tau \le \ell$ *and* $\tau/\ell \le \theta$, *where* $\theta > 0$ *depends on* q, *the*

phase data and finitely many K_j, then an explicit finite differential construction gives U with, for every m,

$$|U|_{m,\tau} \le C_m\,\delta\tau, \qquad \big|\gamma(F+U)-\gamma(F)-\delta^2 H\big|_{m,\tau} \le C_m\Big[\delta\Big(\frac{\tau}{\ell}\Big)^q + \frac{\delta^3}{\tau}\Big]. \tag{4.3}$$

The constants depend on m, q, the phase data and finitely many K_j. Each summand of U is supported in the support of one χ_i, and U depends smoothly on additional parameters on which F and H depend smoothly, with the same estimates in full norms.

Proof. Put $p=q+2$ and $b_\lambda(A)=\chi_i\varpi_{i,r}(A)^{1/2}$ for $\lambda=(i,r)$, so that $A=\sum_\lambda b_\lambda(A)^2\mathrm{d}\phi_\lambda^2$ whenever $\|A-h_0\|_{C^0}\le\eta_0$.

Primary oscillations. For such A, let $Z_\lambda=\mathcal{Z}_{p,\tau}\big(0,\sqrt{2}\,\delta\tau\,b_\lambda(A)n_\lambda(F)\big)$ be given by Theorem 4.2 with phase ϕ_λ, and $U_1(A)=\operatorname{Re}\sum_\lambda Z_\lambda e^{\mathrm{i}\phi_\lambda/\tau}$. Then $|U_1|_{m,\tau}\le C\delta\tau$ and $|L_F(U_1)|_{m,\tau}\le C\delta\tau(\tau/\ell)^p$. The product $\gamma(U_1)=\mathrm{d}U_1\cdot\mathrm{d}U_1$ is a finite sum of terms with phases 0, $\pm2\phi_\lambda$ and $\pm\phi_\lambda\pm\phi_{\lambda'}$ for interacting labels. The phase-zero terms pair each label with its conjugate; their sum is

$$\delta^2(A+\mathcal{T}(A)), \qquad |\mathcal{T}(A)|_{m,\ell}\le C\tfrac{\tau}{\ell},$$

since the leading part of Z_λ is $\sqrt{2}\delta\tau b_\lambda n_\lambda$, $|n_\lambda|=1$, and every other contribution involves a slow derivative or a correction from Theorem 4.2. The map $\mathcal{T}$ is smooth, and for some fixed loss l_0, $|\mathcal{T}(A)-\mathcal{T}(\bar A)|_{m,\ell}\le C(\tau/\ell)|A-\bar A|_{m+l_0,\ell}$. Let $A_0=H$ and $A_{l+1}=H-\mathcal{T}(A_l)$ for $l<q$. For τ/ℓ small all A_l stay within η_0 of h_0, and by induction $|A_{l+1}-A_l|_{m,\ell}\le C(\tau/\ell)^{l+1}$. With $A=A_q$,

$$\big|A+\mathcal{T}(A)-H\big|_{m,\ell}=|\mathcal{T}(A_q)-\mathcal{T}(A_{q-1})|_{m,\ell}\le C(\tau/\ell)^{q+1}.$$

Secondary oscillations. Write the nonzero-phase part of $\gamma(U_1)$ as $\operatorname{Re}\sum_\Psi Q_\Psi e^{\mathrm{i}\Psi/\tau}$, where Ψ runs over the doubled and interacting phases and $|Q_\Psi|_{m,\ell}\le C\delta^2$. By phase data (c), Theorem 4.2 applies to each Ψ; let $U_2=\operatorname{Re}\sum_\Psi\mathcal{Z}_{p,\tau}(-Q_\Psi,0)e^{\mathrm{i}\Psi/\tau}$. Then $|U_2|_{m,\tau}\le C\delta^2\le C\delta\tau$ and $L_F(U_2)$ cancels the nonzero-phase part of $\gamma(U_1)$ up to $C\delta^2(\tau/\ell)^p$.

Conclusion. Let $U=U_1+U_2$. By (4.1), $\gamma(F+U)-\gamma(F)-\delta^2H$ consists of the linear residuals, the mean error $\delta^2(A+\mathcal{T}(A)-H)$, and $2\operatorname{sym}(\mathrm{d}U_1\cdot\mathrm{d}U_2)+\gamma(U_2)$. Since $|\mathrm{d}V|_{m,\tau}\le\tau^{-1}|V|_{m+1,\tau}$ and the weighted norms are submultiplicative, the last terms are $O(\delta^3/\tau+\delta^4/\tau^2)=O(\delta^3/\tau)$, and (4.3) follows. All operations are derivatives, products, matrix inverses and square roots of functions bounded below, applied in locally finite sums of bounded multiplicity. □

4.4. **The iteration.** Fix phase data for (F_0,h_0) with ρ_0,η_0 as in Theorem 4.4, and put

$$\mathcal{W}_m(t;G,H)=|G|^{(2)}_{m,t}+|H|_{m,t}, \qquad r_k=13(k+1),$$

and fix $\Lambda_1\ge4(\|F_0\|_{C^2}+\rho_0+\|h_0\|_{C^0})$.

Lemma 4.5 (Iteration with smoothing)**.** *Let $k_0\ge8$ and $0<\rho_1\le\rho_0/4$. There is $t_*>0$, depending on k_0, ρ_1, Λ_1, the phase data and finitely many norms of h_0, with the following property. Suppose $0<t_0\le t_*$ and G_0,H_0 are smooth with*

$$\gamma(G_0)+t_0^{2k_0}H_0=h, \quad \|G_0-F_0\|_{C^2}\le\rho_0/2, \quad \|H_0-h_0\|_{C^0}\le\eta_0/4, \quad \mathcal{W}_{r_{k_0}}(t_0;G_0,H_0)\le\Lambda_1.$$

Then a sequence of increments U_j given by Theorem 4.4 converges in every C^m to a smooth G_∞ with $\gamma(G_\infty)=h$ and $\|G_\infty-G_0\|_{C^2}\le\rho_1$. More precisely, the j-th step has a scale t_j, with $t_{j+1}=t_j^{5/4}$, and a stage $k_j\ge k_0$, nondecreasing and tending to infinity, and

$$\|U_j\|_{C^m}\le C_m(k_j)\,t_j^{\,k_j+\frac{5}{4}(1-m)} \qquad (m\ge0), \tag{4.4}$$

where $C_m(k)$ depends only on m, k and the data above. Each transition from one stage to the next may be postponed by any finite number of steps without changing the constants $C_m(k)$; hence any prescribed conditions of the form $t_j\le\vartheta_k$ at the entrance of stage k can be imposed. The same

holds for countable uniform atlases and, with full norms and one common schedule, for families depending on a compact parameter; then (4.4) holds in full source and parameter norms.

Proof. One step. Suppose at scale t and *stage* $k \geq k_0$ we have $\gamma(G) + \delta^2 H = h$ with $\delta = t^k$, $\|G - F_0\|_{C^2} \leq 3\rho_0/4$, $\|H - h_0\|_{C^0} \leq \eta_0/4$ and $\mathcal{W}_{r_k}(t; G, H) \leq \Lambda_1$. Set

$$\ell = t^{9/8}, \qquad \tau = t^{5/4}, \qquad q = r = r_k, \qquad \delta' \in \{\tau^k, \tau^{k+1}\}, \tag{4.5}$$

let S_ℓ be the smoothing of Theorem 4.1 of order r, and apply Theorem 4.4 to

$$\bar{G} = S_\ell G, \qquad \bar{H} = S_\ell(H - (\delta'/\delta)^2 h_0).$$

By Theorem 4.1(i) all slow norms of $\bar{G}, \bar{H}$ at scale ℓ are bounded in terms of Λ_1, so the constants of Theorem 4.4 at stage k are fixed; by (ii) and $(\delta'/\delta)^2 \leq t^{k/2}$, the data stay in the neighbourhoods required there when t is small. Put $G' = G + U$ and $H' = (\delta')^{-2}(h - \gamma(G'))$. By (4.1), with $E = \gamma(\bar{G} + U) - \gamma(\bar{G}) - \delta^2 \bar{H}$ and $\mathrm{Cross}(V, U)_{ij} = \langle \partial_i V, \partial_j U \rangle + \langle \partial_j V, \partial_i U \rangle$,

$$(\delta')^2(H' - S_\ell h_0) = \delta^2(H - S_\ell H) - E - \mathrm{Cross}(G - \bar{G}, U). \tag{4.6}$$

No replacement of G by its smoothing occurs in this identity.

Estimates. We have $(\tau/\ell)^q = (\ell/t)^r = t^{13(k+1)/8}$ and $\delta' \geq \tau^{k+1}$. Using (4.3), Theorem 4.1(ii) and $|\mathrm{d}U|_{m,\tau} \leq C\delta$, the terms on the right of (4.6), divided by $(\delta')^2$, are bounded in $|\cdot|_{m,\tau}$ by $C_m(k)(1 + \mathcal{W}_m(t; G, H))$ times

$$\frac{\delta(\tau/\ell)^q}{\tau^{2(k+1)}} = t^{\frac{k-7}{8}}, \qquad \frac{\delta^3/\tau}{\tau^{2(k+1)}} = t^{\frac{2k-15}{4}}, \qquad \frac{\delta(\ell/t)^r + \delta(\tau/t)^r}{\tau^{2(k+1)}} \leq t^{\frac{k-7}{8}}.$$

For $k \geq 8$ all exponents are at least $1/8$, so

$$|H' - S_\ell h_0|_{m,\tau} \leq C_m(k)\, t^{1/8}(1 + \mathcal{W}_m(t; G, H)). \tag{4.7}$$

Moreover $|G|^{(2)}_{m,\tau} \leq \|G\|_{C^2} + (\tau/t)|G|^{(2)}_{m,t}$, $|U|^{(2)}_{m,\tau} \leq C\delta/\tau$, $\|S_\ell h_0 - h_0\|_{C^0} \leq C\ell$ and $|S_\ell h_0|_{m,\tau} \leq \|h_0\|_{C^0} + Ct^{1/8}$. Hence

$$\mathcal{W}_m(\tau; G', H') \leq \tfrac{\Lambda_1}{4} + C_m(k)\, t^{1/8}(1 + \mathcal{W}_m(t; G, H)). \tag{4.8}$$

At the controlled order $m = r_k$, a finite smallness condition on t preserves $\mathcal{W}_{r_k} \leq \Lambda_1$ and the neighbourhood conditions, for either choice of δ'. From $|U|_{m,\tau} \leq C\delta\tau$ we get $\|U\|_{C^m} \leq C_m(k)\, t^{k + \frac{5}{4}(1-m)}$.

The schedule. Start with (t_0, k_0, G_0, H_0), take $\delta' = \tau^k$ and replace t by $\tau = t^{5/4}$ after each step; the scales t_j decrease super-exponentially. While k is fixed, for each m the quantities $x_j = \mathcal{W}_m(t_j; G_j, H_j)$ are finite and satisfy $x_{j+1} \leq \Lambda_1/4 + c_j(1 + x_j)$ with $c_j \to 0$, so $\limsup_j x_j \leq \Lambda_1/4$. Hence after finitely many steps $\mathcal{W}_{r_{k+1}} \leq \Lambda_1/2$ and the stage-$(k+1)$ smallness conditions hold; we then take $\delta' = \tau^{k+1}$ once, which moves to stage $k+1$ with $\mathcal{W}_{r_{k+1}} \leq \Lambda_1$. We also require at stage k that $\|U_j\|_{C^{\lfloor k/2 \rfloor}} \leq \rho_1 t_j/8$; this is a finite condition on t_j because $k + \frac{5}{4}(1 - \lfloor k/2 \rfloor) > 1$. Every stage ends, so $k \to \infty$; for each m, all but finitely many increments satisfy $\|U_j\|_{C^m} \leq \rho_1 t_j/8$, and $\sum_j t_j < 1$ when t_* is small. Thus G_j converges in every C^m, the total C^2 displacement is at most ρ_1, and $\gamma(G_\infty) = \lim(h - \delta_j^2 H_j) = h$ since the H_j are bounded.

For compact parameters we smooth in source and parameter variables and use full norms; only source derivatives enter the metric, and all thresholds are suprema over the parameter space. For countable atlases only suprema and bounded overlap were used. □

Remark 4.6 (The choice of scales). The reserve $(\delta')^2 S_\ell h_0$ kept in the defect is what keeps the normalized defect H' close to the positive tensor h_0, so that every step adds a *positive* tensor. The exponents in (4.5) make both small ratios equal, $\tau/\ell = \ell/t = t^{1/8}$, so that the accuracy $q = r = 13(k+1)$ of the increments and of the smoothing gains $t^{13(k+1)/8}$, which beats the normalization loss $\delta/\tau^{2(k+1)} = t^{-(3k+5)/2}$ with the margin $t^{(k-7)/8}$; the quadratic error δ^3/τ is

harmless once $k \geq 8$. Increasing the stage k is needed for convergence in high norms, since $\|U\|_{C^m} \lesssim t^{k+\frac{5}{4}(1-m)}$.

4.5. **Polynomial entrance estimates.**

Proposition 4.7 (Polynomial metric repair)**.** *Fix $n \geq 2$, $N \geq s_n + 1$, the data of a uniform atlas on an n-manifold, ellipticity constants $0 < c < C$, and $A, \kappa \geq 0$, $b > 0$. There is an integer $K_{\mathrm{rep}} = K_{\mathrm{rep}}(A, \kappa, b)$ with the following property. Let $0 < \epsilon < 1$ and let F_ϵ be immersions into $\mathbb{R}^N$ and h_ϵ positive tensors with*

$$cI \leq \gamma(F_\epsilon) \leq CI, \qquad \|F_\epsilon\|_{C^{119}} \leq C_{\mathrm{ref}}\, \epsilon^{-A}, \qquad \mu(F_\epsilon, x) \geq c_{\mathrm{ph}}\, \epsilon^{\kappa} \quad \forall x,$$
$$\|h_\epsilon - \gamma(F_\epsilon)\|_{C^{117}} \leq C_{\mathrm{err}}\, \epsilon^{K_{\mathrm{rep}}}.$$

Then for all sufficiently small ϵ there is a smooth F_ϵ^* with

$$\gamma(F_\epsilon^*) = h_\epsilon, \qquad \|F_\epsilon^* - F_\epsilon\|_{C^2} \leq \epsilon^b, \qquad \mu(F_\epsilon^*, x) \geq \tfrac{1}{2} c_{\mathrm{ph}}\, \epsilon^{\kappa}.$$

K_{rep} depends only on A, κ, b and the atlas data; the prefactors affect only how small ϵ must be, and higher norms of F_ϵ, h_ϵ affect only the length of the iteration. The statement holds for countable uniform atlases and, with full norms, for families depending on a compact parameter. No regularity in ϵ is claimed.

We say that a quantity is *polynomial* if it is bounded by $C\epsilon^{-p}$, or below by $c\,\epsilon^p$, where p depends only on A, κ and the atlas data.

Proof. Good root directions. For a second form B at a point let $D_B(\xi)$ be the sum of squares of the $q_n \times q_n$ minors of $B \circ \mathrm{Sym}^2(|\xi|^2 I - \xi\xi^{\mathsf{T}})$. It is a nonnegative homogeneous polynomial of degree $8q_n$, positive exactly at good covectors, and by Cauchy–Binet $D_B(\xi)$ is the product of the squared singular values of $B|_{\mathrm{Sym}^2 \ker \xi}$ for unit ξ. Hence the phase margin gives $\|D_B\| \geq c\,\epsilon^{2q_n\kappa}$ for the coefficient norm. On the zero-sum hyperplane of $\mathbb{R}^{n+1}$, the s_n squares of $(e_i - e_j)/\sqrt{n+1}$, $i < j$, sum to the identity and are linearly independent: a relation $\sum c_{ij}(x_i - x_j)^2 = 0$ on the hyperplane gives a graph Laplacian vanishing on the hyperplane and on $(1, \dots, 1)$, hence all $c_{ij} = 0$. Transport these covectors to T_x^* by a $\gamma(F_\epsilon)$-isometry and a rotation O. For every nonzero nonnegative polynomial the set of rotations for which some transported root is a zero is a proper analytic subset; so $\max_O \min_{\mathrm{roots}} D/\|D\|$ is a positive lower semicontinuous function on the compact set of normalized polynomials and uniformly elliptic metrics, hence bounded below. The selected root covectors ξ_r have $D_B(\xi_r) \geq c\,\epsilon^{2q_n\kappa}$, and since $\|B\| \leq C\epsilon^{-A}$, $\sigma_{\min}(B|_{\mathrm{Sym}^2 \ker \xi_r}) \geq c\,\epsilon^{q_n\kappa + (q_n - 1)A}$. The metric $\gamma(F_\epsilon)$ has positive coefficients 1 in this basis.

Refinement and colours. The C^3 bound controls the variation of $\gamma(F_\epsilon)$ and B_{F_ϵ}. Cover the manifold by balls of radius ϵ^{g_0} in the given charts, with g_0 fixed and large, freeze the root covectors at the centres, and use them as constant differentials on each ball; this preserves the goodness margins and the positivity of the coefficients up to factors $\frac{1}{2}$. The overlap of the refined cover is bounded, cutoff derivatives are polynomial, and the refined supports are coloured by a bounded number of colours. Normalize the cutoffs so that $\sum_i \chi_i^2 = 1$.

Weights. Order the finitely many pairs (colour, root) and multiply the phase with index l by ϵ^{-Wl}, dividing its coefficient by ϵ^{-2Wl}. For W fixed and large, every interacting sum or difference, divided by its dominant weight, is polynomially close to a good primary covector, hence good with polynomial margin; doubled phases are good trivially. Since $N \geq s_n + 1$, the admissible line or space E_ξ is nonzero, and at each centre some coordinate vector projects onto it with length at least $N^{-1/2}$; a further polynomial refinement keeps half of this. This gives phase data for $(F_\epsilon, \gamma(F_\epsilon))$ with polynomial constant c_*, and all constants in Theorems 4.4 and 4.5 at stage $k_0 = 8$ are polynomial.

Entrance. By Theorem 2.2, a protected C^2 radius $\rho_0 = c\epsilon^R$ with $R > A + \kappa + 1$ (enlarged for the finitely many geometric conditions above) preserves half the phase margin. Let $\rho_1 = \epsilon^{R+b+2}$ and $\Lambda_1 = C\epsilon^{-(2A+1)}$. Every smallness condition on t at stage 8 has the form $C_i\epsilon^{-a_i} t^{f_i} \leq c_i\epsilon^{d_i}$ with

$f_i > 0$; choose ς with $\varsigma f_i > a_i + d_i + 1$ for all i, $\varsigma > 2A + 2$ and $16\varsigma - A > R + b + 3$. Put $t_0 = \epsilon^{\varsigma}$, $h_0 = \gamma(F_\epsilon)$ and

$$\delta_0 = t_0^8, \qquad G_0 = \sqrt{1 - \delta_0^2}\, F_\epsilon, \qquad H_0 = h_0 + \delta_0^{-2}(h_\epsilon - h_0),$$

so that $\gamma(G_0) + \delta_0^2 H_0 = h_\epsilon$. For $1 \le j \le 117$, $t_0^j \|F_\epsilon\|_{C^{j+2}} \le C\epsilon^{\varsigma j - A}$ and $t_0^j \|h_0\|_{C^j} \le C\epsilon^{\varsigma j - 2A}$. If P_0 bounds the remaining polynomial losses, choose $K_{\mathrm{rep}} > 16\varsigma + P_0 + 2$; then $|H_0 - h_0|_{117,t_0} = O(\epsilon^{K_{\mathrm{rep}} - 16\varsigma - P_0})$, so $\mathcal{W}_{117}(t_0; G_0, H_0) \le \Lambda_1$ and H_0 lies in the required neighbourhood. The rescaling costs $\|G_0 - F_\epsilon\|_{C^2} = O(\epsilon^{16\varsigma - A}) = o(\rho_1)$. Theorem 4.5 gives the exact solution within the remaining budget, and Theorem 2.2 gives the phase margin. Only suprema, bounded overlap and a fixed number of labels entered, so nothing depends on the number of charts; for compact parameters we refine product charts and freeze source covectors only. □

5. Isometric immersions into small balls

5.1. **Composition with local phases.** Let (M, g) be a closed Riemannian n-manifold and $L\colon TM \to M \times \mathbb{R}^p$ an isometric bundle monomorphism with component one-forms $\alpha_1, \dots, \alpha_p$, so that $g = \sum_\ell \alpha_\ell^2$. For an integer $C \ge 1$ consider $M \times (S^{2C-1})^p$ with the coefficient distribution $D = D_{L(TM)}$ of Theorem 2.5 and its coefficient metric $h_{L(TM)}$. For a subbundle $E \subset TM$ we write $D_{L(E)} \subset D$ for the corresponding subdistribution.

Proposition 5.1 (Local phases)**.** *Let $n \ge 2$, $N \ge s_n + 1$, and let $E \subset TM$ be a hyperplane subbundle. Assume that for every $C \ge 1$ and all large k there are maps $S_k\colon M \times (S^{2C-1})^p \to \mathbb{R}^N$ such that, for all j, R,*

(a) $\|S_k\|_{C^j} \le C_j k^{\max(j-1,0)}$ for derivatives in all variables;
(b) $\|\gamma_D(S_k) - h_{L(TM)}\|_{C^j} \le C_{j,R} k^{-R}$;
(c) the partial normal second form of S_k along D has smallest singular value at least $ck^{-1/2}$ on $\operatorname{Sym}^2 D_{L(E)}$.

Then for every $r > 0$ there is a smooth isometric immersion $F\colon (M, g) \to B_r(0) \subset \mathbb{R}^N$ for which every nonzero covector annihilating E is good. In particular F is phase regular.

Proof. Nets and colours. For $K \ge 1$ choose a maximal K^{-1}-separated set $\{p_i\}$ in (M, g). For K large, volume comparison in normal coordinates bounds the number of balls $B(p_i, 4K^{-1})$ meeting a given one, uniformly in K. Take bumps supported in $B(p_i, 2K^{-1})$, positive on $B(p_i, K^{-1})$, and normalize them by the square root of their sum of squares to get $\sum_i \chi_i^2 = 1$ with $|\partial^j \chi_i| \le C_j K^j$. Colour the supports with a fixed number C of colours so that equally coloured supports are disjoint. We fix this C, and with it the maps S_k, before choosing k, K and T.

Phase maps. In normal coordinates at p_i let $\varphi_{\ell i}$ be the linear function with $\mathrm{d}\varphi_{\ell i}(p_i) = \alpha_\ell(p_i)$. For $T \ge K$ define $z_\ell\colon M \to \mathbb{C}^C$ by

$$(z_\ell)_c(x) = \sum_{\mathrm{colour}(i)=c} \chi_i(x)\, e^{2\pi \mathrm{i} T \varphi_{\ell i}(x)}, \qquad H(x) = S_k\big(x, z_1(x), \dots, z_p(x)\big). \tag{5.1}$$

Each summand extends by zero, and disjointness within a colour gives $|z_\ell|^2 = \sum_i \chi_i^2 = 1$; thus z_ℓ is a smooth map into S^{2C-1}. The phases are local functions; the forms α_ℓ need not be closed.

Rescaled charts. Use normal charts of radius $3T^{-1}$, with coordinates $y \in B_3$ equal to T times normal coordinates. They form a uniform atlas, uniformly in T, and $T^2 g$ is uniformly elliptic in y. Put $e_{K,T} = K^{-1} + K/T$ and $C_{\ell a} = T\alpha_\ell(\partial_{y_a})$. In these charts every derivative of z_ℓ is bounded: derivatives of cutoffs cost $K/T \le 1$, and the phase rates $T\mathrm{d}\varphi_{\ell i}(\partial_{y_a})$ are bounded. Base derivatives of S_k cost T^{-1}. By the chain rule and (a),

$$\|H\|_{C^j} \le C_j k^{\max(j-1,0)}. \tag{5.2}$$

First columns and metric error. With X_ℓ the Hopf generators, the comparison first columns are $\Xi_a = \sum_\ell C_{\ell a} X_\ell S_k$, evaluated at the same arguments. Replacing $\mathrm{d}\varphi_{\ell i}$ by α_ℓ costs $O(K^{-1})$ on $\operatorname{supp}\chi_i$, differentiating cutoffs costs K/T, and base derivatives cost T^{-1}; hence $\partial_{y_a} H - \Xi_a = O(e_{K,T})$, and j further derivatives cost at most $C_j k^j$. Since $(C_{\ell a})_\ell = L(T\partial_{y_a})$ and L is isometric, the Gram matrix of the Ξ_a is $T^2 g$ up to the partial metric error of S_k. Hence, for all j, R,

$$\|\gamma(H) - T^2 g\|_{C^j} \le C_j k^j e_{K,T} + C_{j,R} k^{-R}. \tag{5.3}$$

Second form. The comparison second columns $\sum_{\ell,r} C_{\ell a} C_{rb} X_\ell X_r S_k$ differ from $\partial_{y_a}\partial_{y_b} H$ by $O(k e_{K,T})$: derivatives of the coefficients and base derivatives cost T^{-1}, and differentiating a first-column error uses at most the $O(k)$ second derivatives of S_k. For $v, w \in T_x M$ the comparison second form is $T^2 \nabla_D^2 S_k(Lv, Lw)$, and its projection off $\operatorname{span}(\Xi_a) = \mathrm{d}S_k(D)$ is the partial normal second form pulled back by the isometry $TL\colon (TM, T^2 g) \to L(TM)$. By (c) it has margin $\ge c k^{-1/2}$ on $\operatorname{Sym}^2 E$. The first-jet projectors differ by $O(e_{K,T})$ and the second jet is $O(k)$, so the second form of H has margin at least $\frac{c}{2} k^{-1/2}$ on $\operatorname{Sym}^2 E$ provided $k^{3/2} e_{K,T} \to 0$.

Correction. Apply Theorem 4.7 in the rescaled atlas with $\epsilon = k^{-1}$, $A = 118$, $\kappa = 1/2$ and $b = 2$, and let $K_0 = K_{\mathrm{rep}}$. Choose

$$a > K_0 + 119, \qquad K = k^a, \qquad T = k^{2a}.$$

Then $e_{K,T} = 2k^{-a}$, so (5.3) at order 117 is $o(k^{-K_0})$ (taking $R > K_0$), and $k^{3/2} e_{K,T} \to 0$. All exponents are chosen before k, and the constants of the fixed families S_k only affect how large k must be. For large k we obtain H_* with

$$\gamma(H_*) = T^2 g, \qquad \|H_* - H\|_{C^2} \le k^{-2}$$

in the rescaled atlas. By Theorem 2.2 this changes the second form by $O(k^{-1}) = o(k^{-1/2})$, so the covectors annihilating E remain good. Finally $F = H_*/T$ satisfies $\gamma(F) = g$ and $\sup|F| \le (\|S_k\|_{C^0} + 1)/T \to 0$. □

5.2. **Parallelizable sources.**

Corollary 5.2. *Every smooth metric on a closed parallelizable n-manifold, $n \ge 2$, is induced by a phase-regular immersion into an arbitrarily small ball of $\mathbb{R}^{s_n+1}$.*

Proof. Orthonormalize a global frame and let $L = (\alpha_1, \dots, \alpha_n)$ be the dual coframe, an isometric isomorphism $TM \to M \times \mathbb{R}^n$. Then $D_{L(TM)}$ is spanned by all Hopf generators of $(S^{2C-1})^n$, and the maps of Theorem 3.10 (independent of x) satisfy (a)–(c) of Theorem 5.1 with $E = \ker \alpha_n$. □

6. Exact-metric perturbations and embeddings

In this section M is closed. We show that a phase-regular isometric immersion can be perturbed, keeping the metric, into an embedding (Theorem 6.10). The perturbations are produced by the iteration of Section 4; their first variations are computed in Theorem 6.2, and the linear algebra of Section 6.2 shows that they span enough directions to separate points.

6.1. **Probes.**

Definition 6.1 (Normal oscillations)**.** Let F_0 be a phase-regular immersion. A *normal oscillation* is a section $A = \chi\, \nu \otimes \mathrm{d}\psi^2$ of $\operatorname{Sym}^2 T^*M \otimes \nu_{F_0}$, where χ is a cutoff supported in a chart, $\mathrm{d}\psi$ is nonzero and good near $\operatorname{supp}\chi$, and ν is a normal field with $\nu \in E_{\mathrm{d}\psi}$ there; ν may vanish.

Proposition 6.2 (Probes)**.** *Let $F_0\colon M \to \mathbb{R}^N$ be a smooth phase-regular immersion of a closed n-manifold, $n \ge 2$, $N \ge s_n + 1$, and $h_0 = \gamma(F_0)$. Fix normal oscillations $A_l = \chi_l \nu_l \otimes \mathrm{d}\psi_l^2$, $1 \le l \le L$ (repetitions allowed), and integers $r \ge 2$, $k \ge \max(8, 2r)$. There are weights $w_l > 0$, distinct for*

distinct list entries, a radius $\rho_p > 0$, and for every sufficiently small $t > 0$ a jointly smooth family $F_{t,u}$, $u = (u_{l,1}, u_{l,2})_l$, $|u| < \rho_p$, with $\gamma(F_{t,u}) = h_0$, such that the following holds. Put

$$\tau = t^{5/4}, \qquad \sigma = t^{k-5/4}, \qquad a = t^{k+5/4}, \qquad c_l = \cos(w_l\psi_l/\tau), \quad s_l = \sin(w_l\psi_l/\tau).$$

After a smooth orthogonal identification of $\nu_{F_{t,u}}$ with ν_{F_0}, the identified second form $\widetilde{B}$ satisfies

$$\begin{aligned} \partial_{u_{l,1}}\widetilde{B} &= \sigma A_l c_l + R_{B,l1}, \quad \partial_{u_{l,1}}F_{t,u} = -aw_l^{-2}\chi_l\nu_l c_l + R_{V,l1}, \\ \partial_{u_{l,2}}\widetilde{B} &= \sigma A_l s_l + R_{B,l2}, \quad \partial_{u_{l,2}}F_{t,u} = -aw_l^{-2}\chi_l\nu_l s_l + R_{V,l2}, \end{aligned} \tag{6.1}$$

and, uniformly for $|u| < \rho_p$, with functions $\eta_B, \eta_V \to 0$ as $t \to 0$,

$$\begin{gathered} \|F_{t,u} - F_0\|_{C^r} \le C\, t^{k+\frac{5}{4}(1-r)}, \qquad \|\partial_u F_{t,u}\|_{C^1} \le Ct^k, \qquad \|F_{t,u}\|_{C^3} \le C, \\ |R_{B,li}| \le \sigma(\eta_B(t) + C|u|), \qquad |R_{V,li}| \le a(\eta_V(t) + C|u|). \end{gathered} \tag{6.2}$$

Proof. Phase data. Construct phase data for (F_0, h_0) as in the proof of Theorem 4.7 (at one fixed ϵ), and add one label for each list entry, with phase $w_l\psi_l$ and support $\operatorname{supp}\chi_l$. The weights w_l are inserted into the weight hierarchy, so that all doubled and interacting phases, including those between probe and background labels, are good. All choices are made before t. Extend the parameter ball to a torus by a smooth map that equals the coordinates on a smaller ball and takes values in a small ball, so that Theorem 4.5 applies with a compact parameter space.

First step. Let $\delta = t^k$, $\ell = t^{9/8}$, $\delta_1 = \tau^k$, and $G_0 = \sqrt{1-\delta^2}F_0$, which has the same normal spaces as F_0. Run the construction of Theorem 4.4 with $q = 13(k+1)$, where the probe labels receive the prescribed free amplitudes

$$v_{f,l} = -\delta\tau\,\chi_l\, w_l^{-2}\nu_l\,(u_{l,1} - \mathrm{i}u_{l,2}), \tag{6.3}$$

and the background labels receive $\sqrt{2}\delta\tau b_\lambda(A)n_\lambda$ as before. The real part of the leading probe mode is the value term in (6.1). Its mean contribution to the metric is $\delta^2 H_{\mathrm{probe}}(u)$, $H_{\mathrm{probe}} = \sum_l \chi_l^2|\nu_l|^2|u_l|^2\mathrm{d}\psi_l^2/(2w_l^2)$ to leading order. The background tensor is $A = A_q$, obtained from $A_0 = (1-(\delta_1/\delta)^2)h_0 - H_{\mathrm{probe}}(u)$ by the q substitutions

$$A_{j+1} = (1-(\delta_1/\delta)^2)h_0 - H_{\mathrm{probe}}(u) - \mathcal{T}(A_j, u),$$

where $\mathcal{T}$ now includes the corrections of the probe modes. A small parameter ball keeps the A_j in the positive neighbourhood. Distinct labels have no phase-zero cross terms, so all phase-zero terms are sesquilinear in the amplitude of a single label; hence the probe contributions, and inductively the A_j, are even in u. Their second u-derivatives are bounded independently of t, because only source variables are differentiated and the square roots have uniform reserves; so $\partial_u A_j = O(|u|)$. Then correct the nonzero-phase quadratic terms as in Theorem 4.4, and let U_0 be the resulting increment, $G_1 = G_0 + U_0$ and $H_1 = \delta_1^{-2}(h_0 - \gamma(G_1))$. The normalized errors are $O\big(t^{(k+13)/8} + t^{(2k-5)/4}\big)$, so G_1, H_1 satisfy the entrance conditions of Theorem 4.5 at scale $t_1 = \tau$ and stage k, with the weighted bound $\mathcal{W}_{r_k}(t_1; G_1, H_1) \le \Lambda_1$.

First variations. Two derivatives of (6.3) times $e^{\mathrm{i}w_l\psi_l/\tau}$ give σA_l times c_l or s_l. Slow derivatives and the corrections of Theorem 4.2 contribute $O(\sigma\tau/\ell)$ to the second-derivative columns and $O(a\tau/\ell)$ to the values; the u-dependence of the background contributes $O(\sigma|u|)$ and $O(a|u|)$; the secondary increment contributes $O(\sigma^2)$ in C^2 and $O(\delta^2) = O(a\sigma)$ in C^0.

Tail. Let $C_* \ge C_m(k)$ for $m \in \{1, 2, 3, r\}$, where $C_m(k)$ are the constants of (4.4) in full norms. At the entrance of every later stage $k' > k$ we impose $C_m(k')\, t_j^{k'-k} \le C_*$ at these orders, as Theorem 4.5 permits. Then every later increment satisfies $\|U_j\|_{C^m} \le C_*\, t_j^{k+\frac{5}{4}(1-m)}$ for these m, and since $t_{j+1} = t_j^{5/4}$ the sums are dominated by their first terms. Summing from $t_1 = t^{5/4}$, the

tail satisfies

$$\|\mathrm{Tail}\|_{C^2_x C^1_u} \le 2C_* t^{\frac{5k}{4}-\frac{25}{8}} = o(\sigma),$$
$$\|\mathrm{Tail}\|_{C^1_x C^1_u} \le 2C_* t^{\frac{5k}{4}-\frac{25}{16}} = o(t^k),$$
$$\|\partial_u \mathrm{Tail}\|_{C^0} \le 2C_* t^{\frac{5k}{4}} = o(a),$$

with gaps $(2k-15)/8$, $(4k-25)/16$ and $(k-5)/4$, positive for $k \ge 8$. At order r the same sum is $o(t^{k+\frac{5}{4}(1-r)})$ because $k \ge 2r$, and at order 3 the first increment is $O(t^{k-5/2})$. This gives the first line of (6.2).

Second forms. Since $\gamma(F_{t,u}) = h_0$, (2.1) gives $B = \partial^2 F - \Gamma(h_0)\partial F$; the u-derivative of the connection term is $O(t^k) = o(\sigma)$. If $\Pi_{t,u}$ is the normal projection of $F_{t,u}$ and ι the inclusion of ν_{F_0}, then $\Pi_{t,u}\iota(\iota^*\Pi_{t,u}\iota)^{-1/2}$ is an orthogonal isomorphism $\nu_{F_0} \to \nu_{F_{t,u}}$ whose distance to ι and whose u-derivative are $O(t^k)$. The resulting errors are $o(\sigma)$, which completes the proof of (6.1) and (6.2). □

6.2. **Admissible normal directions.**

Lemma 6.3 (Spaces of quadratic forms)**.** *Let V be a Euclidean space, $\mathcal{P} \subset \mathrm{Sym}^2 V$ a linear subspace, and $d = \max_\xi \dim \mathcal{P}\xi$, where $\mathcal{P}\xi = \{Q\xi : Q \in \mathcal{P}\}$. Then $\dim \mathcal{P} \le d(d+1)/2$, and if $d \ge 2$ equality holds only if $\mathcal{P} = \mathrm{Sym}^2 U$ for a d-dimensional subspace U.*

Proof. The case $d = 0$ is trivial. Choose ξ_0 with $\dim \mathcal{P}\xi_0 = d$ and $\xi_0 \not\perp \mathcal{P}\xi_0$; such points form a nonempty Zariski-open set, since a nonzero quadratic form cannot vanish on a nonempty open set. Put $U_0 = \mathcal{P}\xi_0$ and $E = U_0 \cap \xi_0^\perp$, so $\dim E = d-1$. Choose $Q_1, \dots, Q_d \in \mathcal{P}$ with $Q_i\xi_0$ independent and let $A(\xi)$ have columns $Q_i\xi$. Near ξ_0 every $Q \in \mathcal{P}$ satisfies $Q\xi = A(\xi)c_Q(\xi)$ with $c_Q = (A^*A)^{-1}A^*Q\xi$ smooth. If $Q\xi_0 = 0$, then $c_Q(\xi_0) = 0$, and differentiating gives $Qv = A(\xi_0)\,\mathrm{d}c_Q(\xi_0)v \in U_0$ for all v. Thus $\operatorname{im} Q \subset U_0 \cap \xi_0^\perp = E$, i.e. $Q \in \mathrm{Sym}^2 E$. Hence $\dim \mathcal{P} \le d + \dim \mathrm{Sym}^2 E = d(d+1)/2$.

Suppose equality and $d \ge 2$. Then $\mathrm{Sym}^2 E \subset \mathcal{P}$ is the kernel of evaluation at ξ_0. Let π be the projection onto $E^\perp$ and $\mathcal{L} = \pi\mathcal{P} \subset \mathrm{Hom}(V, E^\perp)$; then $\dim \mathcal{L} = d$. If ξ has nonzero component in E, then $(\mathrm{Sym}^2 E)\xi = E$, so $\dim \mathcal{L}\xi \le 1$; the corresponding 2×2 minors are polynomials in ξ, so $\dim \mathcal{L}\xi \le 1$ for all ξ. If some $A \in \mathcal{L}$ had rank at least two, pick x, y with Ax, Ay independent. For $A' \in \mathcal{L}$, pointwise dependence at $x, y, x+y$ gives $A'x = cAx$ and $A'y = cAy$ with one scalar c. If $Az \ne 0$, then Az is independent of Ax or of Ay, and comparing at $x+z$ or $y+z$ gives $A'z = cAz$; if $Az = 0$, both comparisons give $A'z \in \mathbb{R}Ax \cap \mathbb{R}Ay = 0$. So $A' = cA$, contradicting $\dim \mathcal{L} \ge 2$. Hence every nonzero member of $\mathcal{L}$ has rank one, and two members with different image lines would give independent values at a vector outside both kernels, contradicting $\dim \mathcal{L}\xi \le 1$. So all images lie in one line $\mathbb{R}f$. Every $Q \in \mathcal{P}$ then has image in $U = E \oplus \mathbb{R}f$, so $Q \in \mathrm{Sym}^2 U$, and $\mathcal{P} = \mathrm{Sym}^2 U$ by dimension. □

Proposition 6.4 (Admissible value directions)**.** *Let $\dim V = n$, let $B\colon \mathrm{Sym}^2 V \to W$ be phase regular (some $\xi \in V^*$ has B injective on $\mathrm{Sym}^2 \ker \xi$), $K = \ker B$, $d = \dim K$, and $E_\xi = B(\mathrm{Sym}^2 \ker \xi)^\perp$ for good ξ. Let*

$$\mathcal{R}(K) = \{P \in \mathrm{Sym}^2 V :\ P\xi \in K\xi \text{ for every good } \xi\}.$$

Then $\dim \mathcal{R}(K) \le d(d+1)/2$, $(\mathrm{span}_\xi E_\xi)^\perp = B(\mathcal{R}(K))$, and

$$\dim \mathrm{span}_\xi E_\xi = \dim W - \dim \mathcal{R}(K) + d \ \ge\ \dim W - \frac{d(d-1)}{2}. \tag{6.4}$$

If moreover $n \ge 3$, B is onto and $\dim W = q_n + 1$, then $\dim \mathrm{span}_\xi E_\xi \ge n+1$ unless

$$K \subset \mathrm{Sym}^2 U \quad \text{for a hyperplane } U \subset V, \tag{6.5}$$

in which case U is unique, $\mathcal{R}(K) = \mathrm{Sym}^2 U$, and the span has dimension n.

Proof. Goodness of ξ means that $Q \mapsto Q\xi$ is injective on K. For a basis $Q_1, \dots, Q_d$ of K, the condition $P \in \mathcal{R}(K)$ is the polynomial identity $Q_1\xi \wedge \cdots \wedge Q_d\xi \wedge P\xi = 0$ on the open cone of good covectors, so $\mathcal{R}(K)$ is a linear space whose evaluation rank equals d on that cone and is at most d everywhere. Theorem 6.3 gives the dimension bound. A vector orthogonal to all E_ξ lies in every $B(\operatorname{Sym}^2 \ker \xi) \subset \operatorname{im} B$; write it as $B(P)$. For good ξ it equals $B(P_\xi)$ with $P_\xi\xi = 0$, so $(P - P_\xi) \in K$ and $P\xi \in K\xi$; conversely $P \in \mathcal{R}(K)$ gives $B(P) \in B(\operatorname{Sym}^2 \ker \xi)$ for each good ξ. This proves the second identity, and (6.4) follows from $K \subset \mathcal{R}(K)$.

In the onto case $d = s_n - q_n - 1 = n - 1 \geq 2$ and the lower bound in (6.4) is n. Equality means $\dim \mathcal{R}(K) = d(d+1)/2$, i.e. $\mathcal{R}(K) = \operatorname{Sym}^2 U$ for a hyperplane U by Theorem 6.3, and then $K \subset \operatorname{Sym}^2 U$. Conversely, if $K \subset \operatorname{Sym}^2 U$ then $K\xi = U$ for good ξ, so $\operatorname{Sym}^2 U \subset \mathcal{R}(K)$ and equality holds; $U = K\xi$ is unique. □

Proposition 6.5 (The exceptional locus)**.** *Let $n \geq 3$, $\dim W = q_n + 1$, and let $\mathcal{O} \subset \operatorname{Hom}(\operatorname{Sym}^2 V, W)$ be the open set of onto, phase-regular maps. The set $\mathcal{E} \subset \mathcal{O}$ defined by (6.5) is a smooth, relatively closed submanifold. At every $B \in \mathcal{E}$ there are a good ξ and $\nu \in E_\xi$ such that the variation $\dot{B} = \nu \otimes \xi^2$, i.e. $P \mapsto P(\xi,\xi)\nu$, is not tangent to $\mathcal{E}$.*

Proof. For $B \in \mathcal{E}$ write $K = \ker B \subset \operatorname{Sym}^2 U$, $U = e^\perp$, $|e| = 1$. For good ξ the map $Q \mapsto Q\xi = Q(\xi - \langle \xi, e\rangle e)$ is injective on K with image U; in particular no nonzero $\zeta \in U$ satisfies $K\zeta = 0$, for otherwise $K\xi$ would lie in a subspace of dimension $n - 2$.

Submanifold. The map $B \mapsto \ker B$ is a submersion from onto maps to $\operatorname{Gr}_{n-1}(\operatorname{Sym}^2 V)$, so it suffices to treat the set of kernels. These are parametrized by pairs (U, K) with $K \in \operatorname{Gr}_{n-1}(\operatorname{Sym}^2 U)$, a Grassmann bundle over the space of hyperplanes. An infinitesimal rotation $u \mapsto \langle u, \zeta\rangle e$ ($\zeta \in U$) of U moves $Q \in K$ by $2e \odot (Q\zeta)$, which lies in the complement $e \odot U$ of $\operatorname{Sym}^2 U$; it vanishes for all $Q \in K$ only if $\zeta = 0$. Variations within $\operatorname{Sym}^2 U$ are the usual Grassmannian tangent vectors. Hence the parametrization is an immersion. It is injective, since $U = K\xi$ for any good ξ, and U depends continuously on K; so it is an embedding. Limits of exceptional kernels in $\mathcal{O}$ are exceptional by compactness of the space of hyperplanes.

Transversality. Let $A = B|_{K^\perp}$ and $G = (A^*A)^{-1}$. For a variation $\dot{B}$, the kernel, written as a graph over K in $K^\perp$, has derivative $\dot{K} = -A^{-1}\dot{B}|_K$. If the variation is tangent to $\mathcal{E}$, differentiating $Q_s(e_s, e_s) = 0$ for $Q_s \in K_s$ and using $Qe = 0$ gives $\dot{K}(Q)(e,e) = 0$ for all $Q \in K$. Take a good covector, project it along e to get a good $z \in U$, and choose $Q_0 \in K$ with $Q_0 z = z$, so $Q_0(z,z) = |z|^2 > 0$. Each $\xi_t = z + te$ is good. Let $\nu_t = (A^*)^{-1}(e \odot \xi_t)$; since $e \odot \xi_t \perp K$ we have $B^*\nu_t = e \odot \xi_t$, which vanishes on $\operatorname{Sym}^2 \ker \xi_t$, so $\nu_t \in E_{\xi_t}$. For $\dot{B} = \nu_t \otimes \xi_t^2$ we get $\dot{K}(Q) = -Q(z,z)\, G(e \odot \xi_t)$ and

$$\dot{K}(Q_0)(e,e) = -|z|^2\big(\langle G(e \odot z), e \odot e\rangle + t\langle G(e \odot e), e \odot e\rangle\big).$$

The coefficient of t is nonzero because G is positive and $e \odot e \in K^\perp$. For all but one value of t the variation is not tangent to $\mathcal{E}$. □

Lemma 6.6 (Parametric avoidance)**.** *Let $P \subset \mathbb{R}^p$ be open, Y a second-countable manifold of dimension m, and $\Phi\colon P \times Y \to \mathbb{R}^b$ smooth. If the derivative of Φ has rank greater than m at every point of a set $Z \subset \Phi^{-1}(0)$, then the projection of Z to P has measure zero.*

Proof. Near each point of Z, $m + 1$ suitable components of Φ cut out a submanifold of dimension $p - 1$ containing Z locally; its projection to P is null by Sard's theorem. Countably many such neighbourhoods cover Z. □

6.3. **Separation of points.** For a compact $P \subset (M \times M) \setminus \Delta$, a map *separates* P if it takes different values at the two points of every pair in P.

Lemma 6.7 (Separation on compact pair sets)**.** *Let $n \geq 3$, $N = s_n + c$ with $c \geq 1$, and let $F\colon M \to \mathbb{R}^N$ be a smooth phase-regular immersion of a closed manifold. For every compact $P \subset (M \times M) \setminus \Delta$,*

$r \geq 2$ and $\epsilon > 0$ there is a smooth phase-regular immersion G with $\gamma(G) = \gamma(F)$, $\|G - F\|_{C^r} < \epsilon$, which separates P.

Proof. Replace P by $P \cup P^{\mathsf{T}}$ and let C_0 be the union of its two projections. Put $q = q_n$; the normal spaces have dimension $q + c$, and phase regularity gives $\operatorname{rank} B \geq q$. Second forms are written in local orthonormal frames of TM and of ν_F, transported to nearby maps by the identification of Theorem 6.2.

Removal principle. Let G be a smooth phase-regular immersion with $\gamma(G) = \gamma(F)$, and let $Q \subset P$ be compact. Let $\mathcal{U}$ be an open set of second forms (invariant under the frame changes) containing $B_G(x)$ for all first points x of pairs in Q, and let $\Sigma \subset \mathcal{U}$ be a submanifold, closed in $\mathcal{U}$, with local defining maps Φ_Σ (or $\Sigma = \mathcal{U}$, with no equations). Suppose that at every $(x, y) \in Q$ with $G(x) = G(y)$ and $B_G(x) \in \Sigma$ there are normal oscillations $A_l = \chi_l \nu_l \otimes \mathrm{d}\psi_l^2$, supported near x and away from y, such that the vectors

$$(\mathrm{d}\Phi_\Sigma(A_l(x)), 0), \qquad (0, \chi_l \nu_l(x)), \qquad (0, \mathrm{d}G(v)) \quad (v \in T_x M) \tag{6.6}$$

span a space of dimension greater than $2n$. Then arbitrarily C^r-small exact-metric perturbations of G have no such pairs in Q. Indeed, since Σ is closed in $\mathcal{U}$, the set of such pairs is compact, and by compactness finitely many oscillations work on a neighbourhood $\mathcal{N}$ of it. Apply Theorem 6.2 to the list in which every oscillation appears twice, with weights $w_{l,1} \neq w_{l,2}$. At a point (x, y), combine each cosine and sine column with the coefficients (c_l, s_l) of its own weight: the normalized second-form and value columns become $(A_l, -w_{l,j}^{-2} \chi_l \nu_l)$ up to $o(1) + O(|u|)$, uniformly, and the response at y is controlled by the global error. Since $\left(\begin{smallmatrix} 1 & 1 \\ -w_{l,1}^{-2} & -w_{l,2}^{-2} \end{smallmatrix}\right)$ is invertible, bounded combinations isolate both kinds of columns. Divide the Φ_Σ-rows by σ, the value rows by a, and multiply the derivatives in x by a; their Φ_Σ-entries become $(a/\sigma)\partial_x \Phi_\Sigma(B) = O(\tau^2)$ by the C^3 bound, and their value entries converge to $\mathrm{d}G$. Hence for $|u|$ and then t small the derivative of $(x, y, u) \mapsto (\Phi_\Sigma(B_{t,u}(x)),\, G_{t,u}(x) - G_{t,u}(y))$ has rank greater than $2n$ on $\mathcal{N}$, and C^2-convergence excludes such pairs outside $\mathcal{N}$. Theorem 6.6 gives a parameter u for which none remain. By (6.2) the perturbation is C^r-small, and it can be chosen to preserve any previously established positive separation margin on a compact set of pairs, and the phase regularity.

Rank q. Let $\mathcal{U}$ be the open set of matrices of rank at least q, which contains all phase-regular second forms, and $\Sigma_q \subset \mathcal{U}$ the closed submanifold of matrices of rank q, of codimension nc. At $B \in \Sigma_q$ with kernel K ($\dim K = n$) and cokernel $L = (\operatorname{im} B)^\perp$ ($\dim L = c$), the normal space of Σ_q is $\operatorname{Hom}(K, L)$, via $\dot{B} \mapsto \pi_L \dot{B}|_K$. Every good ξ has $E_\xi = L$, and the variations $\nu \otimes \xi^2$ project to $Q \mapsto Q(\xi, \xi)\nu$; as ξ ranges over the open cone of good covectors and ν over L, these span $\operatorname{Hom}(K, L)$, since a quadratic form vanishing on an open cone is zero. The span in (6.6) has dimension $nc + c + n > 2n$. The set of pairs with $\operatorname{rank} B(x) = q$ is compact, because lower rank is excluded. The removal principle gives G_1 without coincidences at rank-q points. Choose an open $O \supset \{x \in C_0 : \operatorname{rank} B_{G_1}(x) = q\}$ such that G_1 separates every pair of P meeting $\overline{O}$ with a positive margin, which we protect from now on. On $C_0 \setminus O$ the rank is at least $q + 1$ with a uniform margin.

The case $c \geq 2$. On $C_0 \setminus O$ we have $d = \dim \ker B \leq n - 1$, and Theorem 6.4 gives at least $q + c - \frac{(n-1)(n-2)}{2} = n - 1 + c \geq n + 1$ admissible value directions; with the n tangent directions the span in (6.6) (with $\Sigma = \mathcal{U}$ and Q the pairs of P with first point in $C_0 \setminus O$) has dimension at least $2n + 1$. One more application of the removal principle separates P.

The case $c = 1$. On $C_0 \setminus O$ the second form is onto. At exceptional points, Theorem 6.4 gives n admissible value directions and Theorem 6.5 an admissible variation transverse to $\Sigma = \mathcal{E}$, which is closed in $\mathcal{U} = \mathcal{O}$; for the pairs of P with first point in $C_0 \setminus O$ the span in (6.6) has dimension at least $1 + n + n$. The removal principle eliminates the exceptional coincidences; protect a neighbourhood $O_\mathcal{E}$ of the exceptional points as before. On $C_0 \setminus (O \cup O_\mathcal{E})$, Theorem 6.4 gives $n + 1$ admissible value directions, and a final application of the removal principle separates P. □

Remark 6.8. The proof uses the probe families only through: membership in a class of phase-regular immersions with the same metric that is closed under the construction; the leading columns in (6.1), possibly multiplied by one fixed smooth positive function; $a/\sigma \to 0$; the error bounds (6.2) on a compact neighbourhood of C_0, with global control of the responses away from the supports; and smallness in a prescribed norm. It therefore applies to any class with such families.

Lemma 6.9 (Baire separation)**.** *Let M be a second-countable manifold, $\mathcal{X}$ a Baire space, and $v \mapsto F_v \colon M \to \mathbb{R}^N$ a family of maps such that $v \mapsto F_v|_K$ is continuous in the uniform norm for every compact $K \subset M$. If for every compact $P \subset (M \times M) \setminus \Delta$ the parameters whose maps separate P are dense, then the parameters giving injective maps are dense.*

Proof. Let K_j be a compact exhaustion, d a metric for the topology, and $P_j = \{(x, y) \in K_j^2 : d(x, y) \geq 1/j\}$. Separating P_j is an open condition (the minimum of $|F_v(x) - F_v(y)|$ over P_j is positive, or $P_j = \emptyset$) and dense by hypothesis. The intersection over j is dense, and its maps are injective. □

6.4. **Embeddings with the same metric.**

Theorem 6.10. *Let $n \geq 3$, $N \geq s_n + 1$, and let $F \colon M \to \mathbb{R}^N$ be a smooth phase-regular immersion of a closed n-manifold. For every $\epsilon > 0$ there is a smooth phase-regular embedding G with $\gamma(G) = \gamma(F)$ and $\|G - F\|_{C^2} < \epsilon$.*

Proof. $C^\infty(M, \mathbb{R}^N)$ is a Fréchet space and γ is continuous, so $\mathcal{Y} = \{G : \gamma(G) = \gamma(F)\}$ is closed, hence completely metrizable. By Theorem 2.2 there is $0 < \rho < \epsilon$ such that every map in the nonempty open set $\mathcal{X} = \{G \in \mathcal{Y} : \|G - F\|_{C^2} < \rho\}$ is phase regular; $\mathcal{X}$ is a Baire space. Every basic neighbourhood in $\mathcal{X}$ contains a small ball for some C^r norm, $r \geq 2$, within which Theorem 6.7 separates any compact off-diagonal pair set. By Theorem 6.9 some $G \in \mathcal{X}$ is injective; being an injective immersion of a compact manifold, it is an embedding. □

Proof of Theorem A for closed manifolds. By Theorem 5.2 there is a phase-regular isometric immersion into $B_{r/2}(p)$. Theorem 6.10 with $N = s_n + 1$ gives an isometric embedding within C^0-distance $r/2$, hence into $B_r(p)$. Connectedness is not used. □

7. Stably parallelizable closed manifolds

For Theorem B the explicit model maps of Section 3 are replaced by a partially free map produced from bundle data. We use the notation of Section 2.2; all distributions are integrable and carry flat structures. A *foliation box* is a chart $\Omega \cong B^m \times B^d$ in which the flat frame is $\partial_{\theta_1}, \dots, \partial_{\theta_m}$ for the leaf coordinates θ; a *linear leaf coordinate* is $\phi = \sum_\alpha c_\alpha \theta_\alpha$ with constant $c \neq 0$.

7.1. **Bundle-theoretic input.**

Lemma 7.1 (Orthogonal transport and cancellation)**.** *(a) A smooth homotopy P_t of orthogonal projections in a trivial bundle is covered by a smooth homotopy of orthogonal automorphisms O_t with $O_0 = \mathrm{Id}$ and $P_t O_t = O_t P_0$.*

(b) If a manifold has the homotopy type of a CW complex of dimension d, every stably trivial bundle of rank greater than d over it is trivial.

Proof. (a) Solve $\partial_t O_t = [\partial_t P_t, P_t] O_t$, $O_0 = \mathrm{Id}$. The coefficient is skew, so O_t is orthogonal. Differentiating $P_t^2 = P_t$ gives $\partial_t P_t = [[\partial_t P_t, P_t], P_t]$, so $P_t O_t$ and $O_t P_0$ solve the same linear equation. No compactness is needed.

(b) It suffices to cancel one trivial line. If $E \oplus \underline{\mathbb{R}}^1 \cong \underline{\mathbb{R}}^{r+1}$ with $r = \operatorname{rank} E > d$, the image of the last line gives $u \colon M \to S^r$ with $E \cong u^\perp$. By cellular approximation [Hat02, Theorem 4.8], u is nullhomotopic, since S^r has no cells in dimensions $1, \dots, d$. Make the homotopy smooth relative to its ends (approximate in the ambient space and normalize) and apply (a) to $\mathrm{Id} - u_t u_t^*$. □

Proposition 7.2 (The hyperplane bundle)**.** *Under the hypotheses of Theorem B there is a unit vector field e such that, with $H = e^\perp$,*

$$TM = H \oplus \mathbb{R}e, \qquad H \oplus \operatorname{Sym}^2 H \cong \underline{\mathbb{R}}^{s_n - 1}.$$

Proof. Stable triviality orients every component P, and $\chi(P) = 0$, so the Euler class vanishes [MS74, Corollary 11.12]; it is the only obstruction to a section of the unit sphere bundle of TP over the n-dimensional base [Hat17, Proposition 3.22]. Symmetric squares preserve stable triviality: if $E \oplus \underline{\mathbb{R}}^a$ is trivial, then $\operatorname{Sym}^2(E \oplus \underline{\mathbb{R}}^a) \cong \operatorname{Sym}^2 E \oplus aE \oplus \underline{\mathbb{R}}^{s_a}$ is trivial, and adding $\underline{\mathbb{R}}^{a^2}$ trivializes aE. Hence $\operatorname{Sym}^2 TM \cong \operatorname{Sym}^2 H \oplus H \oplus \underline{\mathbb{R}}^1$ is stably trivial, so $H \oplus \operatorname{Sym}^2 H$ is stably trivial of rank $s_n - 1 > n$, and trivial by Theorem 7.1(b). □

7.2. **An h-principle for partially free maps.**

Lemma 7.3 (Relative realization of partial jets)**.** *Let D be an integrable distribution of rank m on a compact manifold X, and $Q \geq m + s_m + 1$. Let $J\colon D \oplus \operatorname{Sym}^2 D \to X \times \mathbb{R}^Q$ be a monomorphism and $f\colon X \to \mathbb{R}^Q$ a smooth map with $J = J_D(f)$ near a closed set $K \supset \partial X$. Then there are a smooth D-free map F, equal to f near K and arbitrarily C^0-close to f, and a smooth homotopy of monomorphisms from J to $J_D(F)$, fixed near K. If the first component of J is $\mathrm{d}_D f$, then F can be chosen arbitrarily C^1-close to f.*

Proof. Full jets. There is a torsion-free connection ∇ on TX with $\nabla_{Y_\alpha} Y_\beta = 0$ for all flat frames: locally take the coordinate connection of a foliation box, and patch with a partition of unity. Both conditions are affine and, because transition matrices of flat frames are constant along leaves, independent of the frame. Then $J_D(f)$ is the restriction of the full 2-jet $(\mathrm{d}f, \nabla^2 f)$ to D and $\operatorname{Sym}^2 D$. Fix a complement $TX = D \oplus C$. If $J = (j_1, j_2)$, the full formal jet $(f,\ L,\ \Sigma)$ with $L = j_1 \oplus \mathrm{d}f|_C$ and $\Sigma = j_2$ on $\operatorname{Sym}^2 D$, $\Sigma = \nabla^2 f$ on the components involving C, equals $j^2 f$ near K.

Ampleness. Let $\mathcal{R}_1 \subset J^1(X, \mathbb{R}^Q)$ be defined by injectivity of $L|_D$ and $\mathcal{R}_2 \subset J^2(X, \mathbb{R}^Q)$ by injectivity of the partial jet; both are open. Consider a principal direction given by a covector ζ. If $\zeta|_D = 0$, the variations $\zeta \otimes u$ and $\zeta^2 \otimes u$ do not change the partial jet, so the principal slices are empty or full. Otherwise choose a basis $v_1, \dots, v_m$ of D_x with $\zeta(v_i) = 0$ for $i < m$ and $\zeta(v_m) = 1$. The variation $\zeta \otimes u$ changes only $L(v_m)$, and its forbidden values form an affine subspace of codimension $Q - m + 1$ (or everything); the variation $\zeta^2 \otimes u$ changes only the column $\Sigma(v_m, v_m)$, with forbidden set of codimension $Q - m - s_m + 1$ (or everything). Both codimensions are at least two. The complement of an affine subspace of codimension at least two is connected and its convex hull is the whole space; hence $\mathcal{R}_1$ and $\mathcal{R}_2$ are ample in the sense of convex integration [Spr98, Chapter VIII]; see also [MA26, Definition 2.9 and Theorem 2.10].

Realization. Open subsets of jet spaces satisfy the microfibration hypotheses trivially. By the h-principle for open ample relations [Spr98, Theorem 8.12, Lemma 8.13 and Proposition 8.20], applied first to $\mathcal{R}_1$, there is a homotopy (f_t, L_t) in $\mathcal{R}_1$, fixed near K, from (f, L) to $(f_1, \mathrm{d}f_1)$ with f_1 C^0-close to f; by openness it can be taken smooth in t. Let $j_t = L_t|_D$, $j_t^+ = (j_t^* j_t)^{-1} j_t^*$, and solve

$$\partial_t T_t = (\partial_t j_t)\, j_t^+\, T_t, \qquad T_0 = \mathrm{Id}\,. \tag{7.1}$$

Then T_t is invertible and $T_t j_0 = j_t$, since both sides solve the same linear equation ($j_t^+ j_t = \mathrm{Id}$); hence $T_t J = (j_t, T_t j_2)$ is a monomorphism, and $T_t = \mathrm{Id}$ near K. At $t = 1$ the formal 2-jet with lower part $j^1 f_1$, partial second part $T_1 j_2$, and the actual Hessian on the remaining components lies in $\mathcal{R}_2$ and is holonomic near K. The second-order h-principle with holonomic lower jet gives F, C^1-close to f_1 and equal to f near K, together with a homotopy in $\mathcal{R}_2$; concatenation gives the asserted homotopy of monomorphisms. If $j_1 = \mathrm{d}_D f$ from the start, only this last step is needed. On a manifold with boundary all steps take place in the interior, fixing a collar. □

7.3. **Supported metric additions.** The following local step is a leafwise, parametric version of the Günther–Zergaenge construction for free maps; it is the only place where the four extra dimensions are used.

Lemma 7.4 (Supported partial metric addition)**.** *Let D have rank m, let F be a smooth D-free map into $\mathbb{R}^Q$ with $Q \geq m + s_m + 5$, let Ω be a foliation box, a a smooth function with compact support in Ω, and ϕ a linear leaf coordinate on Ω. Then there is a smooth D-free map G, equal to F outside Ω and arbitrarily C^0-close to F, with*

$$\gamma_D(G) = \gamma_D(F) + a^4(\mathrm{d}_D\phi)^2.$$

Moreover $G = F$ on every plaque of Ω on which a vanishes identically.

Proof. Take $\phi = \theta_1$ and write Ω as a leaf ball times a transverse ball, with slightly larger nested boxes; p denotes the transverse coordinates, and all norms and operators below act on the leaf variables, uniformly in p. Fix $0 < \alpha < 1$.

Normal frame. The orthogonal complement of the partial jet of F has rank at least 5 on the closed box. Its orthogonal projection defines a metric connection; parallel transport along the radial contraction of the box gives five smooth orthonormal sections.

Approximate solutions. The static finite formulas of [Zer16, Sections 4.1–4.3, Satz 4.1, equations (4.5)–(4.35) and (4.40)–(4.83), Satz 4.4(i); Section 6.1, equations (6.5)–(6.9)] give, for every precision k and small $\epsilon > 0$, free maps $P_{\epsilon,k}$, equal to F outside an inner leaf support, with displacement $O(\epsilon)$ and

$$e = \gamma_D(F) + a^4\mathrm{d}\theta_1^2 - \gamma_D(P_{\epsilon,k}), \qquad \|e\|_{C^{2,\alpha}} \leq C_k\epsilon^{k-2}.$$

These formulas consist of pairing systems, solved with the canonical inverse $A(A^*A)^{-1}$, and periodic integrations, solved with the zero-mean primitive, applied to a fixed periodic auxiliary map and a fixed plane oscillator; the fast substitution $z = \rho(\theta_1/\epsilon)$ does not depend on p. Hence all coefficients depend smoothly on p, and the rank and Gram margins of the auxiliary periodic configurations are uniform by compactness. The formulas remain smooth across zeros of a: the first correction factors as $\widetilde{u}_2 = aR$, because its pairing right sides contain the factor a and the canonical inverse is linear, so the apparent quotient $|\partial_z\widetilde{u}_2|^2/(2a^2)$ equals $|\partial_z R|^2/2$; the later systems use a fixed partition in a^2, dividing by a only where $|a|$ is bounded below. If $a(p,\cdot) \equiv 0$, all oscillators, corrections, residuals and right sides vanish by induction, and the canonical inverses map 0 to 0.

Exact correction. The pairing inverse $E[P]$ is rational in the first and second partial derivatives of P; the free Gram determinant of $P_{\epsilon,k}$ is bounded below independently of small ϵ, while leaf derivatives of order j cost $C_{j,k}\epsilon^{-j}$. Hence $\|E[P_{\epsilon,k}]\|_{C^{2,\alpha}} \leq C_k\epsilon^{-b}$ with b independent of k. Let χ be a leaf cutoff equal to 1 on the support of e. By the supported correction theorem [Zer17, Theorem 2.12 and Sections 3.1–3.3, (3.23)–(3.28)], e is removed by $\chi^2 v$, where v is the unique small fixed point of a map $\Phi[P,\chi,e]$ on the leaf space $C^{2,\alpha}$, provided

$$\|E[P]\|_{2,\alpha}\|E[P](0,e)\|_{2,\alpha} \leq C_k\epsilon^{k-2-2b}, \qquad \|\chi^2 v\|_{2,\alpha} \leq C_k\epsilon^{k-2-b} \tag{7.2}$$

are small. Choose k with $k-2-2b > 0$ and $k-2-b$ larger than the loss in the freeness stability estimate [Zer16, Satz 4.4(i)], then one ϵ giving a strict contraction, uniformly in p, that preserves freeness and meets the C^0 tolerance.

Parameter dependence. Write the fixed-point equation as $v(p) = \Phi(p, v(p))$ with common contraction constant $q < 1$ on a convex invariant ball. Then $\|v(p+h) - v(p)\| \leq (1-q)^{-1}\|\Phi(p+h, v(p)) - \Phi(p, v(p))\|$, and the difference quotients w_h satisfy $w_h = A_h w_h + b_h$ with $A_h = \int_0^1 D_v\Phi(p+h, v(p) + s(v(p+h)-v(p)))\,\mathrm{d}s$, $\|A_h\| \leq q$, and $b_h \to \partial_p\Phi$. Neumann series give $\partial_p v = (I - D_v\Phi)^{-1}\partial_p\Phi$, continuously in p. At every higher order the top derivative of v appears only through $D_v\Phi$, so induction gives smooth dependence on p with values in leaf $C^{2,\alpha}$; higher leaf regularity follows inductively from the Schauder and product estimates [Zer17, Sections 3.2–3.3, (3.12)–(3.13) and (3.27)–(3.28)], in which the highest leaf derivative again has the small coefficient. Hence $G = P_{\epsilon,k} + \chi^2 v$ is jointly

smooth with the asserted metric. On a plaque where $a \equiv 0$ we have $P = F$ and $e = 0$, so uniqueness gives $v = 0$; the nested supports allow extension by F outside Ω. □

7.4. **Relative partial metric realization.**

Proposition 7.5. *Let $\pi\colon X \to B$ be a surjective submersion of compact manifolds, either both closed or with $\partial X = \pi^{-1}(\partial B)$, and let $D \subset \ker \mathrm{d}\pi$ be an integrable distribution of rank $m \geq 1$ with a flat structure. Let $Q \geq m + s_m + 5$, $A \subset B$ closed with $\partial B \subset A$, and h a positive partial metric on D. Let F_A be a smooth D-free map defined above a neighbourhood of A with $\gamma_D(F_A) = h$. The following are equivalent:*

(i) there is a monomorphism $D \oplus \operatorname{Sym}^2 D \to X \times \mathbb{R}^Q$ equal to $J_D(F_A)$ above a neighbourhood of A;

(ii) there is a smooth D-free map $F\colon X \to \mathbb{R}^Q$ with $\gamma_D(F) = h$, equal to F_A above a neighbourhood of A.

Proof. (ii)⇒(i) is clear. For the converse assume first $\emptyset \neq A \neq B$. Choose neighbourhoods $V_0 \Subset V_1 \Subset V_2 \Subset V_3$ of A with $\overline{V}_3$ inside the region where the monomorphism equals $J_D(F_A)$; cut F_A off by a base function equal to 1 near $\overline{V}_3$ to get a global f, and apply Theorem 7.3 relative to $\pi^{-1}(\overline{V}_3)$. The result is a free H with $\gamma_D(H) = h$ over $\pi^{-1}(\overline{V}_3)$.

Positive deficit. Choose $M_0 \geq 1$ with $\gamma_D(H) \leq M_0 h$, $0 < \lambda_0 < 1$ with $\lambda_0^2 M_0 < \frac{1}{2}$, and $\beta_0 = (1-\lambda_0^2)^{1/4}$. Let $\beta\colon B \to [0, \beta_0]$ vanish near $\overline{V}_1$ and equal β_0 outside V_2, and put $\lambda = (1-\beta^4)^{1/2}$, both pulled back to X. Since D is vertical, λ is constant along leaves, so $H_s = \lambda H$ is free with $\gamma_D(H_s) = \lambda^2 \gamma_D(H)$. Let

$$\Delta = \frac{h - \lambda^2 \gamma_D(H)}{\beta^4} \ \text{ on } \{\beta \neq 0\}, \qquad \Delta = h \ \text{ on } \{\beta = 0\}. \tag{7.3}$$

Where $\beta \neq \beta_0$ we are inside V_2, where $\gamma_D(H) = h$, so the numerator equals $\beta^4 h$; thus Δ is smooth. Where $\beta = \beta_0$ the numerator is at least $h/2$. Hence $\Delta > 0$ and $h - \gamma_D(H_s) = \beta^4 \Delta$.

Fourth-power decomposition. In a small foliation box choose constant leaf covectors ϑ_i with $\sum_i \vartheta_i^2$ equal to Δ at the centre. The forms ϑ_i^2, $(\vartheta_i \pm \vartheta_j)^2$ span the symmetric forms; giving weight $\eta > 0$ to each $(\vartheta_i \pm \vartheta_j)^2$ and $1 - 2\eta(m-1) > 0$ to each ϑ_i^2 represents the centre value, and a fixed linear right inverse of the spanning map represents nearby values with smooth positive coefficients $c_{\ell j}$ after shrinking the box. These covectors are $\mathrm{d}_D \phi_{\ell j}$ for linear leaf coordinates. With finitely many bumps ψ_ℓ in such boxes and $S = \sum_\ell \psi_\ell^4 > 0$,

$$h - \gamma_D(H_s) = \sum_{\ell, j} a_{\ell j}^4 (\mathrm{d}_D \phi_{\ell j})^2, \qquad a_{\ell j} = \beta \psi_\ell S^{-1/4} c_{\ell j}^{1/4}.$$

The amplitudes are smooth and vanish on $\pi^{-1}(\overline{V}_1)$. Apply Theorem 7.4 successively to these summands, each time to the free output of the previous step. Every plaque lies in a fibre of π, so plaques over $\{\beta = 0\}$ are fixed; the final map is free, has partial metric h, and equals F_A over V_0.

If $A = \emptyset$, the source is closed; apply Theorem 7.3 to any f, shrink the result by a constant so that $h - \gamma_D(H) > 0$, and decompose and add as above. If $A = B$ there is nothing to prove. □

Proposition 7.6 (A free normal)**.** *In the setting of Theorem 7.5 with X, B closed, a monomorphism $(D \oplus \operatorname{Sym}^2 D) \oplus \underline{\mathbb{R}}^1 \to X \times \mathbb{R}^Q$ yields a smooth D-free map F with $\gamma_D(F) = h$ for any prescribed positive h, together with a smooth unit vector field $\nu \perp \operatorname{im} J_D(F)$.*

Proof. Let J be the restriction to $D \oplus \operatorname{Sym}^2 D$; the projection of the last column off $\operatorname{im} J$ is nowhere zero. Theorem 7.3 gives a free F_0 and a homotopy of monomorphisms from J to $J_D(F_0)$, and Theorem 7.1(a), applied to the complementary projections, gives a unit normal for F_0. To reach the metric h through free maps, work on $S^1 \times X \to S^1 \times B$ with the pulled-back distribution. Let $\chi\colon S^1 \to [0,1]$ vanish near a closed arc and equal 1 near a point θ_*, and prescribe $h_\theta =$

$(1-\chi(\theta))\gamma_D(F_0)+\chi(\theta)h$. The constant family F_0 realizes h_θ near the arc, and $J_D(F_0)$ is a global formal extension. By Theorem 7.5 there is a jointly smooth free family F_θ with $\gamma_D(F_\theta)=h_\theta$, equal to F_0 near the arc. Along an arc from the protected region to θ_*, orthogonal transport of the complements of the partial jets carries the unit normal of F_0 to one of $F=F_{\theta_*}$. □

7.5. **Proof of Theorem B.** Let e and H be as in Theorem 7.2. Finitely many orthonormal local frames of H and a partition $\sum\chi_i^2=1$ give an isometric monomorphism $I_H\colon H\to\underline{\mathbb{R}}^a$; put $W=I_H(H)$ and $L(u+te)=(I_Hu,t)\in\mathbb{R}^{a+1}$, an isometric monomorphism of TM.

Fix $C\geq 1$ and let $X_H=M\times(S^{2C-1})^a\to M$ carry the coefficient distribution D_W of rank $n-1$, with coefficient metric h_W. Then $D_W\oplus\operatorname{Sym}^2 D_W$ is the pullback of $W\oplus\operatorname{Sym}^2 W\cong H\oplus\operatorname{Sym}^2 H\cong\underline{\mathbb{R}}^{s_n-1}$, so $(D_W\oplus\operatorname{Sym}^2 D_W)\oplus\underline{\mathbb{R}}^1$ embeds in $\underline{\mathbb{R}}^{s_n+4}$. Since $s_n+4=(n-1)+s_{n-1}+5$, Theorem 7.6 gives a D_W-free map $F_H\colon X_H\to\mathbb{R}^{s_n+4}$ with $\gamma_{D_W}(F_H)=h_W$ and a unit free normal ν.

Apply Theorem 3.9 to the constant family (F_H,ν), with $K=0$, and one more Hopf sphere for the line $\mathbb{R}e$. The resulting maps $S_k\colon M\times(S^{2C-1})^{a+1}\to\mathbb{R}^{s_n+5}$ satisfy (a)–(c) of Theorem 5.1 for L and $E=H$, because $D_W\oplus\mathbb{R}X_{a+1}=D_{L(TM)}$ and $h_W+1=h_{L(TM)}$; the margin is even bounded below. Hence Theorem 5.1 gives a phase-regular isometric immersion into an arbitrarily small ball of $\mathbb{R}^{s_n+5}$, and Theorem 6.10 with $N=s_n+5$ turns it into an embedding, as in the proof of Theorem A. □

8. Noncompact manifolds and boundary

8.1. **Weighted maps.** Let M carry a countable uniform atlas, write C_b^j for the supremum of chart derivatives through order j and $C_b^\infty=\bigcap_j C_b^j$, and let β be a closed one-form with all C_b^j norms finite. For $V\colon M\to\mathbb{R}^N$ put

$$D_iV=\partial_iV+\beta_iV,\qquad \gamma_\beta(V)_{ij}=\langle D_iV,D_jV\rangle,\qquad B_V^\beta=\Pi\,(D_iD_jV),$$

where Π projects off $\operatorname{span}(D_iV)$ when $\gamma_\beta(V)>0$; since β is closed, D_iD_jV is symmetric. Weighted phase regularity and margins are defined with B^β in place of B.

Proposition 8.1 (Weighted correction and probes)**.** *(a) If $a>0$ satisfies $\mathrm{d}\log a=\beta$ on an open set, then there*

$$\mathrm{d}(aV)=a\,DV,\qquad \gamma(aV)=a^2\gamma_\beta(V),\qquad B_{aV}=a\,B_V^\beta. \tag{8.1}$$

(b) Theorems 4.4 and 4.5 and Theorem 4.7 hold for γ_β,B^β in place of γ,B, with constants depending in addition on finitely many C_b^j norms of β.

(c) Theorem 6.2 holds for γ_β,B^β, for finite lists of compactly supported normal oscillations, with all estimates in global C_b norms; for prescribed C_b^r-closeness the iteration starts at a stage $k\geq\max(8,2r)$.

Proof. (a) is the product rule $\mathrm{d}(aV)=a(\mathrm{d}V+V\mathrm{d}\log a)$ applied once and twice; the tangent spaces of aV are spanned by the D_iV.

(b) On each chart ball choose $a_i>0$ with $\mathrm{d}\log a_i=\beta$ and $a_i=1$ at the centre; the bounds on β give uniform bounds on $a_i^{\pm1}$ and all their derivatives. By (a), $V\mapsto a_iV$ transforms all weighted quantities on the chart into unweighted ones with uniformly comparable norms, ellipticity constants and margins. Alternatively, with $u_i=\langle D_iV,Z\rangle$, $Z=DV(\gamma_\beta^{-1}u)+v$ and $D_iD_jV=\widetilde{\Gamma}_{ij}^kD_kV+B_{ij}^\beta$,

$$\gamma_\beta(V+Z)=\gamma_\beta(V)+L_{\beta,V}Z+\gamma_\beta(Z),\quad (L_{\beta,V}Z)_{ij}=\partial_iu_j+\partial_ju_i+2(\beta_iu_j+\beta_ju_i)-2\widetilde{\Gamma}_{ij}^ku_k-2\langle B_{ij}^\beta,v\rangle,$$

which differs from the unweighted operator only by terms of order zero in u. Hence Theorems 4.2 and 4.4 hold verbatim. In the iteration we use the true weighted defect,

$$(\delta')^2(H'-S_\ell h_0)=\delta^2(H-S_\ell H)-E-\operatorname{Cross}_\beta(V-S_\ell V,U),$$

with $\mathrm{Cross}_\beta(A,U)_{ij} = \langle D_iA, D_jU\rangle + \langle D_jA, D_iU\rangle$; the bounded coefficients of β leave all exponents unchanged, and no commutation of D with smoothing is needed. All constructions are local, so their locally finite sums define global maps.

(c) The probe construction has the same even mean, since β does not depend on u. For solutions with fixed $\gamma_\beta(V) = h$, the identity $\langle D_iD_jV, D_lV\rangle = \frac{1}{2}(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}) + \beta_i h_{jl} + \beta_j h_{il} - \beta_l h_{ij}$ shows that $\widetilde{\Gamma}$ depends only on h and β, so the weighted Gauss formula $B^\beta = D^2V - \widetilde{\Gamma}DV$ holds and the u-derivatives of all terms of first order in V are $O(t^k) = o(\sigma)$. The remaining estimates are those of Theorem 6.2 in global seminorms. □

8.2. **Conformal normalization.**

Proposition 8.2. *Let M be a parallelizable manifold without boundary and g a smooth metric. There is $a\colon M \to (0,1)$, tending to 0 off compact sets, such that $h = a^{-2}g$ and $\beta = \mathrm{d}\log a$ satisfy: h is complete with positive injectivity radius and bounded covariant derivatives of its curvature; h has a global orthonormal coframe with bounded covariant derivatives of all orders; and β has bounded covariant derivatives of all orders.*

Proof. Normalize a global coframe to a g-orthonormal one, ω, and write $h = e^{2u}g$, $a = e^{-u}$, $\theta = e^u\omega$. The scaling theorem [MN15, Theorem 4.1, Definitions 2.2 and 2.9] controls any countable list of quantities bounded by expressions $e^{-\alpha u(x)}P_x(u, |\nabla_g u|, \dots, |\nabla_g^k u|)$ with $\alpha > 0$, together with their versions on neighbouring annuli of an exhaustion; each is controlled off a compact set and bounded on it by continuity. Curvature quantities are of this form [MN15, Example 2.3], and placing the inverse convexity radius first gives completeness and positive injectivity radius [MN15, Theorem 3.8 and the proof of Theorem 1.8]. For the coframe, the conformal formula $\nabla^h_X Y = \nabla^g_X Y + \mathrm{d}u(X)Y + \mathrm{d}u(Y)X - g(X,Y)\,\mathrm{grad}_g\, u$ gives $(\nabla^h_i\theta)_j = e^u((\nabla^g_i\omega)_j - u_j\omega_i + g_{ij}\langle \mathrm{d}u, \omega\rangle_g)$, and by induction

$$|(\nabla^h)^r\theta|_h \le e^{-ru}P_{r,x}(j^r_g u)\ (r \ge 1), \qquad |(\nabla^h)^r \mathrm{d}u|_h \le e^{-(r+1)u}Q_{r,x}(j^{r+1}_g u),$$

so these quantities are admissible as well. Choosing u above a positive proper exhaustion gives $a < 1$ and $a \to 0$ at infinity. □

8.3. **Bounded initial realizations.**

Proposition 8.3. *In the situation of Theorem 8.2, let the uniform atlases be built from normal charts of h. There are arbitrarily large T and maps $V_0 \in C^\infty_b(M, \mathbb{R}^{s_n+1})$, with respect to the atlas of h-radius T^{-1}, such that*

$$\gamma_\beta(V_0) = T^2h, \qquad 2 < |V_0| < C_V,$$

with C_V independent of T, and for each such T the weighted second form of V_0 has a positive uniform phase margin in these charts.

Proof. Bounded geometry gives normal charts with uniform metric and transition bounds, and covers of bounded overlap at every small scale [Sch01, Theorem 2.5 and Propositions 3.2–3.3]; the bounded covariant derivatives of θ and β give coordinate bounds. Run the construction of Theorem 5.1 with the coframe θ and the maps S_k of Theorem 3.10; local finiteness keeps the coloured sums smooth, and the colouring number is uniform. Translate by a fixed v_0 with $|v_0| > \sup_k \|S_k\|_{C^0} + 3$, so that $H = v_0 + S_k(z_1, \dots, z_n)$ has $3 < |H| < C_0$ independently of k, K, T. In the charts of radius T^{-1} all coefficients of β and their derivatives are $O(T^{-1})$, so the extra terms βH in the first columns, and the corresponding second-column terms, satisfy the bounds of that proof:

$$\|\gamma_\beta(H) - T^2h\|_{C^j_b} \le C_j k^j(K^{-1} + K/T) + C_{j,R}k^{-R},$$

and the second-form comparison error is $O(k(K^{-1} + K/T))$. With the same choices $K = k^a$, $T = k^{2a}$, Theorem 8.1(b) gives an exact V_0 within k^{-2} of H in C^2_b, preserving the margin and the value bounds. For fixed k all global norms are finite, and $T \to \infty$ as $k \to \infty$. □

8.4. **Separation and properness.**

Lemma 8.4. *Let $n \geq 3$, $N \geq s_n + 1$, let $V \in C_b^\infty$ have uniformly positive weighted metric and weighted phase margin, and let $a_1 > 0$ be smooth and bounded with $\mathrm{d}\log a_1 = \beta$. For every compact $P \subset (M \times M) \setminus \Delta$, $r \geq 2$ and $\epsilon > 0$ there is $V' \in C_b^\infty$ with $\gamma_\beta(V') = \gamma_\beta(V)$ and $\|V' - V\|_{C_b^r} < \epsilon$ such that $a_1 V'$ separates P; for small ϵ the uniform weighted phase margin persists.*

Proof. Consider the class of maps $a_1 V''$ with $V'' \in C_b^\infty$, $\gamma_\beta(V'') = \gamma_\beta(V)$ and a positive uniform weighted margin. By (8.1), weighted probe families (Theorem 8.1(c)) for V'' give exact-metric families for $a_1 V''$ whose leading second-form and value columns are multiplied by the same positive factor $a_1(x)$; on a compact neighbourhood of the projections of P, a_1 is bounded above and below with bounded derivatives, and responses away from the supports are bounded by the global errors times $\sup a_1$. By Theorem 6.8 the proof of Theorem 6.7 applies, with closeness order at least $\max(r, 3)$. □

Theorem 8.5. *Let M be a parallelizable n-manifold without boundary, $n \geq 3$, and g any smooth metric. For every $p \in \mathbb{R}^{s_n+1}$ and $R > 0$ there is a smooth phase-regular isometric embedding $F\colon (M, g) \to B_R(p)$. If M is noncompact, F can be chosen with $F(x) \neq p$, $F(x) \to p$ as x leaves compact sets, and F proper into $\mathbb{R}^{s_n+1} \setminus \{p\}$.*

Proof. We may take $p = 0$ and $M \neq \emptyset$. Let a, h, β be as in Theorem 8.2 and V_0, T, C_V as in Theorem 8.3, with T chosen so that $(C_V + 1)/T < R$. Put $a_1 = a/T$; then $\mathrm{d}\log a_1 = \beta$ and $\gamma(a_1 V_0) = g$ by (8.1).

On the fixed atlas, $\mathcal{F} = C_b^\infty(M, \mathbb{R}^{s_n+1})$ is a Fréchet space: a sequence that is Cauchy in every C_b^j has uniform chartwise limits of all derivatives, defining a smooth map with finite norms. Since γ_β is continuous, $\mathcal{Y} = \{V : \gamma_\beta(V) = T^2 h\}$ is closed. Choose $\epsilon_0 > 0$ such that $\mathcal{X} = \{V \in \mathcal{Y} : \|V - V_0\|_{C_b^2} < \epsilon_0\}$ consists of maps with a uniform positive weighted margin and $1 < |V| < C_V + 1$; $\mathcal{X}$ is a Baire space. Every basic neighbourhood in $\mathcal{X}$ contains a small C_b^r ball, within which Theorem 8.4 separates any compact off-diagonal pair set. Multiplication by a_1 is continuous in the uniform norm on compacts, so Theorem 6.9 gives $V_\infty \in \mathcal{X}$ with $F = a_1 V_\infty$ injective. Then F is an isometric immersion, phase regular since $B_F = a_1 B^\beta_{V_\infty}$, and

$$a(x)/T < |F(x)| < (C_V + 1)\, a(x)/T < R.$$

Thus F avoids 0 and tends to 0 at infinity. The preimage of a compact set at distance $\eta > 0$ from 0 is closed in the compact set $\{a \geq T\eta/(C_V + 1)\}$, so F is proper into $\mathbb{R}^{s_n+1} \setminus \{0\}$; a proper injective immersion is an embedding. If M is compact, injectivity suffices. Pairs on different components are included throughout. □

8.5. **Boundary and stable triviality.**

Corollary 8.6. *Theorem A holds for parallelizable manifolds with smooth boundary and metrics smooth up to the boundary; the embedding extends isometrically across an exterior collar.*

Proof. In a collar of ∂M inside the double, extend the frame and the metric locally and patch with a locally finite partition of unity; independence and positivity persist near each boundary point. Choose a locally finite cover U_i and widths $w_i > 0$ on which both hold for $-2w_i < t \leq 0$, and set $w = \sum_i \eta_i w_i$ for a subordinate partition η_i; at each point w is at most the largest active w_i. Adjoining $\{(x, t) : -w(x) < t < 0\}$ gives a parallelizable manifold without boundary with an extended metric. Apply Theorem 8.5 and restrict. □

Proposition 8.7. *If every component of an n-manifold is noncompact, every stably trivial bundle of rank at least n over it is trivial. In particular stable parallelizability implies parallelizability.*

Proof. A connected noncompact n-manifold deformation retracts onto an $(n-1)$-dimensional CW complex [Put, Theorem 0.1]. Apply Theorem 7.1(b) on each component; components are open. □

Completion of the proofs of Theorem A and Theorem C. Theorem A follows from Theorem 8.5 for manifolds without boundary and from Theorem 8.6 in general. Under the hypotheses of Theorem C, Theorem 8.7 shows that TM is trivial, and Theorem 8.5 applies. □

Acknowledgements

Generative AI was used in developing the proofs.

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