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UNIFORMIZABILITY OF ELEMENTARY ANDERSON t-MOTIVES OF DIMENSION 2

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Abstract. Let $M(A)$, $A \in M_2(\mathbb{C}_\infty)$, be the elementary Anderson t-motive of dimension 2 and rank 4 defined by $Te = \theta e + A\tau e + \tau^2 e$. Using the affine equation of Ehbauer, Grishkov and Logachev, we give an explicit sufficient condition for $M(A)$ to be uniformizable, in terms of the valuations of the coefficients of that equation. It holds as soon as no cancellation occurs among the four summands of one of these coefficients; in particular $h^1(M(A)) = 4$ in the simple case where no cancellation occurs in any coefficient. This proves their Conjecture 3.7: $h^1(M(A)) = 4$ whenever $(\operatorname{ord} a_{11}, \operatorname{ord} a_{12}, \operatorname{ord} a_{21}, \operatorname{ord} a_{22})$ lies outside an explicit union of nine hyperplanes.

1. The affine equation

Let $q = p^s$, let ord be the valuation of $\mathbb{C}_\infty$ with $\operatorname{ord}\theta = -1$, and let $A = (a_{jk}) \in M_2(\mathbb{C}_\infty)$. The t-motive $M(A)$ is defined by $Te_* = \theta e_* + A\tau e_* + \tau^2 e_*$, $e_* = (e_1, e_2)^t$; it is pure of dimension 2 and rank 4 [EGL]. Assume $a_{21} \neq 0$. By [EGL, (3.0), (3.1), Theorem 3.3], $h^1(M(A))$ is the $\mathbb{F}_q[T]$-rank of the module of small solutions $X = \sum_{i\geq 0} x_i T^i$, $x_i \in \mathbb{C}_\infty$, $x_i \to 0$, of the system ($i \geq 0$, $x_{-1} = x_{-2} = 0$)

$$\sum_{k=0}^{4} a_k\, x_i^{q^k} + b_{14}\, x_{i-1}^{q^4} + b_{13}\, x_{i-1}^{q^3} + b_{12}\, x_{i-1}^{q^2} + b_{24}\, x_{i-2}^{q^4} = 0, \tag{1}$$

with

$$a_4 = \frac{\theta^{q^3+q^2}}{a_{21}^{q^2}}, \quad a_3 = \frac{a_{11}^{q^2}\theta^{q^2}}{a_{21}^{q^2}} + \frac{a_{22}^{q}\theta^{q^2}}{a_{21}^{q}}, \quad a_2 = \frac{\theta^q}{a_{21}} + \frac{\theta^{q^2}}{a_{21}^{q^2}} + \frac{a_{11}^q a_{22}^q}{a_{21}^q} - a_{12}^q,$$

$$a_1 = \frac{a_{11}}{a_{21}} + \frac{a_{22}^q}{a_{21}^q}, \quad a_0 = \frac{1}{a_{21}}, \quad b_{14} = -\frac{\theta^{q^3} + \theta^{q^2}}{a_{21}^{q^2}}, \quad b_{13} = -\frac{a_{11}^{q^2}}{a_{21}^{q^2}} - \frac{a_{22}^q}{a_{21}^q},$$

$$b_{12} = -\frac{1}{a_{21}} - \frac{1}{a_{21}^{q^2}}, \quad b_{24} = \frac{1}{a_{21}^{q^2}}.$$

Put $\alpha_k = \operatorname{ord} a_k$, $\beta_{jk} = \operatorname{ord} b_{jk}$ (with $\operatorname{ord} 0 = +\infty$) and $P_k = (q^k, \alpha_k)$. One has the exact relations

$$a_3 = -\theta^{q^2} b_{13}, \qquad \beta_{14} = \alpha_4 + q^2, \qquad \beta_{24} = \alpha_4 + q^3 + q^2. \tag{2}$$

Since $a_0 a_4 \neq 0$, the solutions of the equation $i = 0$ form a 4-dimensional $\mathbb{F}_q$-vector space S_0, and the valuations of its nonzero elements are the slopes, with sign changed, of the lower convex hull Π of the points P_k, an edge from abscissa q^j to q^l carrying $q^l - q^j$ of them. For every c, the elements of S_0 of valuation $> c$ form, with 0, an $\mathbb{F}_q$-subspace, of dimension l if the edges of slope $< -c$ end at abscissa q^l.

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2. Two elementary lemmas

For $g \in \mathbb{Q}$ put $W(g) = \max_k (g - \alpha_k)/q^k$, the maximum over the k with $a_k \neq 0$.

Lemma 1. *Let $c \in \mathbb{C}_\infty^\times$ with $\operatorname{ord} c = g$. The equation $\sum_k a_k x^{q^k} = -c$ has solutions, all of valuation $\leq W(g)$, and some of valuation exactly $W(g)$. The function W is strictly increasing.*

Proof. If $\operatorname{ord} x > W(g)$, every term $a_k x^{q^k}$ has valuation $> \alpha_k + q^k W(g) \geq g$, which is impossible. The Newton polygon of $c + \sum_k a_k x^{q^k}$ has its first edge from $(0, g)$ to the point P_k minimizing $(\alpha_k - g)/q^k$; its roots have valuation $W(g)$, and $\mathbb{C}_\infty$ is algebraically closed. □

Given x_{i-1}, x_{i-2} with valuations v, v', put $T_{14} = \beta_{14} + q^4 v$, $T_{13} = \beta_{13} + q^3 v$, $T_{12} = \beta_{12} + q^2 v$, $T_{24} = \beta_{24} + q^4 v'$, and $\Phi(v) = W(\beta_{12} + q^2 v)$.

Lemma 2. *$W(T_{14}) \geq v + q^{-2}$, $W(T_{13}) \geq v + q^{-1}$ (if $b_{13} \neq 0$), $W(T_{24}) \geq v' + (q+1)q^{-2}$, and $\Phi(v) \geq v + (\beta_{12} - \alpha_2)q^{-2}$ (if $a_2 \neq 0$).*

Proof. Bound W below by the term $k = 4$, $k = 3$, $k = 4$, $k = 2$ respectively and use (2). □

3. The criterion

For $k \in \{0, 1, 3, 4\}$ with $a_k \neq 0$ put $f_k = (\beta_{12} - \alpha_k)/(q^k - q^2)$, and let

$$I = \{v \in \mathbb{R} : \Phi(v) \leq v\}.$$

A direct computation shows that $v \in I$ if and only if $\alpha_2 \geq \beta_{12}$ and $\max(f_3, f_4) \leq v \leq \min(f_0, f_1)$ (indices with $a_k = 0$ omitted); geometrically, the line through (q^2, β_{12}) of slope $-v$ lies on or below all the points P_k.

Theorem 3. *Assume $a_{21} \neq 0$.*

1. *If $I = \emptyset$, then $h^1(M(A)) = 4$. This holds in particular if $\operatorname{ord} a_2 < \operatorname{ord} b_{12}$.*
2. *If $I \neq \emptyset$ and $a_1 \neq 0$, $f_1 < f_0$, then $h^1(M(A)) \geq 1$.*

Proof. Let $x_0 \in S_0$, $v_0 = \operatorname{ord} x_0$, and define $x_1, x_2, \ldots$ inductively: if the tail of (1) vanishes, take $x_i = 0$; otherwise take for x_i a solution of maximal valuation (Lemma 1). Since the valuation of the tail is at least $\min(T_{14}, T_{13}, T_{12}, T_{24})$, Lemma 1 gives, with $v_j = \operatorname{ord} x_j$,

$$v_i \;\geq\; \min\big(W(T_{14}), W(T_{13}), \Phi(v_{i-1}), W(T_{24})\big), \tag{3}$$

the terms with a zero coefficient being omitted (for $i = 1$ the term T_{24} is absent).

(1) The function $\Phi(v) - v$ is convex and piecewise linear, with slope $q^2 - 1 > 0$ for $v \gg 0$ and slope $q^{-2} - 1 < 0$ for $v \ll 0$; if $I = \emptyset$ it is positive, hence $\geq c$ for some $c > 0$. By (3) and Lemma 2, $v_i \geq \min(v_{i-1}, v_{i-2}) + \min(c, q^{-2})$, so $v_i \to +\infty$ and $X = \sum x_i T^i$ is a small solution. If $\alpha_2 < \beta_{12}$, then $\Phi(v) - v \geq (\beta_{12} - \alpha_2)q^{-2} > 0$ for all v by Lemma 2, so $I = \emptyset$. Applying this to an $\mathbb{F}_q$-basis $x_0^{(1)}, \ldots, x_0^{(4)}$ of S_0 gives small solutions $X^{(1)}, \ldots, X^{(4)}$. They are $\mathbb{F}_q[T]$-linearly independent: in a relation $\sum c_j X^{(j)} = 0$ with $c_j \in \mathbb{F}_q[T]$ not all zero we may assume, T being injective on $\mathbb{C}_\infty[[T]]$, that some $c_j(0) \neq 0$, and the constant term gives $\sum c_j(0) x_0^{(j)} = 0$. Hence $h^1(M(A)) \geq 4$, and $h^1(M(A)) \leq 4$ always.

(2) Let $v_+ = \max I = \min(f_0, f_1) = f_1$. The line ℓ through (q^2, β_{12}) of slope $-v_+$ passes through P_1, strictly below P_0 (as $f_1 < f_0$), and on or below the other points; hence $P_0 P_1$ is the first edge of Π, of slope $< -v_+$, and S_0 contains an element $x_0 \neq 0$ with $v_0 > v_+$. The function $\Phi(v) - v$ is convex, ≤ 0 on I and > 0 to the right of v_+, so it is nondecreasing on $[v_+, \infty)$ and $\Phi(v) - v \geq c_0 := \Phi(v_0) - v_0 > 0$ for $v \geq v_0$. By induction, (3) and Lemma 2 give $v_i \geq v_0$ and $v_i \geq \min(v_{i-1}, v_{i-2}) + \min(c_0, q^{-2})$, so X is a small solution and $h^1 \geq 1$. □

Corollary 4 (simple case)**.** *Assume $a_{21} \neq 0$. If the valuation of a_2 equals the minimum of the valuations of its four summands (in particular in the simple case of* [L26b]*, where the valuation of each of the nine coefficients equals the minimum of the valuations of its summands), then $h^1(M(A)) = 4$.*

Proof. The first two summands of a_2 have valuations $-q - \operatorname{ord} a_{21}$ and $-q^2 - q^2 \operatorname{ord} a_{21}$, smaller than the valuations $-\operatorname{ord} a_{21}$ and $-q^2 \operatorname{ord} a_{21}$ of the two summands of b_{12}. Hence $\alpha_2 < \min(-\operatorname{ord} a_{21}, -q^2 \operatorname{ord} a_{21}) \leq \beta_{12}$, and Theorem 3(1) applies. □

Remark 5. If at every step the minimum of $T_{14}, T_{13}, T_{12}, T_{24}$ is attained by a single term, the inequalities above become equalities, and one finds that the starts x_0 with $\operatorname{ord} x_0 \leq \max I$ stay of valuation $\leq \max I$; the value of h^1 is then 4, 1 or 0 according as $I = \emptyset$, $f_1 < f_0$, or neither. Values 2 and 3 can therefore only occur through cancellations in the tail.

4. Conjecture 3.7 of Ehbauer, Grishkov and Logachev

Write $(u_{11}, u_{12}, u_{21}, u_{22}) = (\operatorname{ord} a_{11}, \operatorname{ord} a_{12}, \operatorname{ord} a_{21}, \operatorname{ord} a_{22}) \in (\mathbb{Q} \cup \{+\infty\})^4$. In [EGL, Conjecture 3.7] (see also [L26a, Conjecture 4.2.3]) it is conjectured that $h^1(M(A)) = 4$ whenever this point lies outside a countable union of linear subspaces of dimension ≤ 3.

For $u_{21} < \infty$ put $s_1 = -q - u_{21}$, $s_2 = -q^2 - q^2 u_{21}$, $s_3 = q(u_{11} + u_{22} - u_{21})$, $s_4 = q u_{12}$, with $s_3 = +\infty$ if $u_{11} = +\infty$ or $u_{22} = +\infty$, and $s_4 = +\infty$ if $u_{12} = +\infty$; these are the valuations of the four summands of a_2.

Corollary 6. *Let $H \subset (\mathbb{Q} \cup \{+\infty\})^4$ be the set of points with $u_{21} < \infty$ at which two of the finite numbers $s_1, \ldots, s_4$ are equal. It is the union of the six sets defined by*

$$u_{21} = -\tfrac{q}{q+1}, \qquad u_{11} + u_{22} - \tfrac{q-1}{q} u_{21} + 1 = 0, \qquad u_{12} + \tfrac{1}{q} u_{21} + 1 = 0,$$
$$u_{11} + u_{22} + (q-1) u_{21} + q = 0, \qquad u_{12} + q\, u_{21} + q = 0, \qquad u_{11} + u_{22} - u_{21} - u_{12} = 0,$$

each equation being imposed with all the coordinates it involves finite and the other coordinates arbitrary in $\mathbb{Q} \cup \{+\infty\}$. Let H' be the image of H under $u_{11} \leftrightarrow u_{22}$, $u_{12} \leftrightarrow u_{21}$. This exchange maps the set given by $s_1 = s_4$ onto the set given by $s_2 = s_4$ and fixes the set given by $s_3 = s_4$, so $H \cup H'$ is the union of nine such sets. If $(u_{11}, u_{12}, u_{21}, u_{22}) \notin H \cup H'$, then $M(A)$ is uniformizable: $h^1(M(A)) = h_1(M(A)) = 4$.

Proof. Suppose first $u_{21} < \infty$. The equalities $s_j = s_k$ between finite s_j, s_k are the six equations defining H, and s_1, s_2 are always finite. Outside H the minimum of the s_j is attained once, so no cancellation can occur in a_2, and Corollary 4 gives $h^1 = 4$. If $u_{21} = \infty$ and $u_{12} < \infty$, the basis (e_2, e_1) gives $M(A) = M(JAJ)$ with $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which exchanges $a_{11} \leftrightarrow a_{22}$ and $a_{12} \leftrightarrow a_{21}$; apply the first case. If $a_{12} = a_{21} = 0$, $M(A)$ is the direct sum of two Drinfeld modules of rank 2, which are uniformizable. Finally $h^1 = 4$ if and only if $h_1 = 4$ by Anderson's theorem [EGL]. □

Remark 7. For antidiagonal A ($u_{11} = u_{22} = +\infty$), $H \cup H'$ meets the plane of (u_{12}, u_{21}) in the four lines $u_{21} = -\frac{q}{q+1}$, $u_{12} = -\frac{q}{q+1}$, $u_{21} = -q\, u_{12} - q$ and $u_{21} = -u_{12}/q - 1$. The first, third and fourth contain the rays R_1, R_2, R_3 of [EGL, Section 6], on which the non-uniformizable $M(A)$ of [EGL, Theorem 5.1] lie. On $H \cup H'$ uniformizability may fail only if the leading terms of the tied summands of a_2 cancel to the point that $I \neq \emptyset$. The uniformizable locus is open in families [BH].

5. Numerical checks

Theorem 5.1 of [EGL]. For antidiagonal A, the value of h^1 obtained by iterating (3) with equality coincides with [EGL, Theorem 5.1] at all 38 640 points of a grid in $(\operatorname{ord} a_{21}, \operatorname{ord} a_2)$, for $q = 2$ and $q = 3$.

The example of [GL21]. For $q = 2$ and $A = \left(\begin{smallmatrix} \theta & \theta^6 \\ \theta^{-2} & 0 \end{smallmatrix}\right)$ one has $s_2 = s_4 = -12$ (and $u_{22} = +\infty$): the point lies on H, and θ^{12} cancels in $a_2 = \theta^4$, so that $\alpha_2 = -4 \geq -8 = \beta_{12}$. This agrees with $h^1(M(A)) = 0$ and $h^1(M(A^t)) = 1$ [GL21, Theorems 4.1 and 4.7].

The example of [L26b]. For $q = 2$ and $A = \left(\begin{smallmatrix} \theta^3 & \theta^{-2} \\ \theta^{-1} & 1 \end{smallmatrix}\right)$ the nine coefficients are those of [L26b]; the two summands θ^8 of a_2 cancel ($s_2 = s_3 = -8$), and $a_2 = \theta^3 + \theta^{-4}$. Here $\beta_{12} = -4$ and $(f_0, f_1, f_3, f_4) = (1, 0, 4, 1)$, so $\max(f_3, f_4) > \min(f_0, f_1)$ and $I = \emptyset$ whatever the valuation of a_2. Since a_1, a_3, b_{12}, b_{13} have no ties among their summands, $h^1(M(A)) = 4$ for every A whose entries have the valuations $\left(\begin{smallmatrix} -3 & 2 \\ 1 & 0 \end{smallmatrix}\right)$, in accordance with [L26b]. The lower bounds (3) for the start of valuation $-1/2$ are $-1/2, 0, 1/2, 1, \ldots$, as in [L26b].

Exact computations. For 40 000 matrices A with entries in $\mathbb{F}_q[\theta, \theta^{-1}]$ ($q = 2, 3$), chosen so that cancellations in a_2 are frequent, the nine coefficients were computed exactly. In every case $I = \emptyset$ holds for A if and only if it holds for A^t, in accordance with $h^1(M(A)) = 4 \Leftrightarrow h^1(M(A^t)) = 4$.

The scripts used for these checks are not part of this deposit.

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