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Abstract: We ask for ultrametric version of following three: (1) Bourgain-Figiel-Milman Theorem, (2) Enflo Type, (3) Mendel-Naor Cotype.

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1. ULTRAMETRIC BOURGAIN-FIGIEL-MILMAN PROBLEM

Let $\mathcal{X}$ and $\mathcal{Y}$ be finite dimensional normed linear spaces such that $\dim(\mathcal{X}) = \dim(\mathcal{Y})$. The **Banach-Mazur distance** between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$d_{BM}(\mathcal{X}, \mathcal{Y}) := \inf\{\|T\|\|T^{-1}\| : T : \mathcal{X} \rightarrow \mathcal{Y} \text{ is invertible linear operator}\}.$$

For $n \in \mathbb{N}$, let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be the standard Euclidean Hilbert space. Following wonderful result says that every finite dimensional Banach space has a subspace which is close (in the Banach-Mazur distance) to the Euclidean space.

Theorem 1.1. *[1, 6] (Dvoretzky-Milman Theorem) There is a universal constant $C > 0$ satisfying the following property: If $\mathcal{X}$ is any n -dimensional real Banach space and $0 < \varepsilon < \frac{1}{3}$, then for every natural number*

$$k \leq C \frac{\varepsilon^2}{|\log \varepsilon|} \log n,$$

there exists a k -dimensional Banach subspace $\mathcal{Y}$ of $\mathcal{X}$ such that

$$d_{BM}(\mathcal{Y}, (\mathbb{R}^k, \langle \cdot, \cdot \rangle)) < 1 + \varepsilon.$$

We refer [8–13, 22, 23, 26–28, 30–32, 34] for more information on Dvoretzky-Milman theorem. With Theorem 1.1, we ask about the metric space version of that. Let $\mathcal{M}$ and $\mathcal{N}$ be finite metric spaces with $o(\mathcal{M}) = o(\mathcal{N})$. Given a map $f : \mathcal{M} \rightarrow \mathcal{N}$, define

$$\|f\|_{\text{Lip}} := \sup \left\{ \frac{d(f(x), f(y))}{d(x, y)} : x, y \in \mathcal{M}, x \neq y \right\}.$$

The **Bourgain-Figiel-Milman distance** between $\mathcal{M}$ and $\mathcal{N}$ is defined as

$$d_{BFM}(\mathcal{M}, \mathcal{N}) := \inf\{\|f\|_{\text{Lip}}\|f^{-1}\|_{\text{Lip}} : f : \mathcal{M} \rightarrow \mathcal{N} \text{ is injective}\}.$$

In 1986, Bourgain, Figiel and Milman proved the following nonlinear Dvoretzky theorem [3].

Theorem 1.2. [3] (*Bourgain-Figiel-Milman Theorem*) For every $\varepsilon > 0$, there is a universal constant $C(\varepsilon) > 0$ satisfying the following: Every finite metric space $\mathcal{M}$ contains a subset $\mathcal{N}$ satisfying following conditions.

(i) There is an injective map $f : \mathcal{N} \rightarrow \ell^2(\mathbb{N}, \mathbb{R})$ such that

$$d_{BFM}(\mathcal{N}, f(\mathcal{N})) \leq 1 + \varepsilon.$$

(ii)

$$o(\mathcal{N}) \geq C(\varepsilon) \log(o(\mathcal{M})).$$

Further, following holds: If $\varepsilon < 1$, then we can take

$$C(\varepsilon) = \frac{c_1 \varepsilon}{\log(\frac{c_2}{\varepsilon})} \quad \text{for some } c_1 > 0, c_2 > 0.$$

We will formulate problem based on Theorem 1.2 to ultrametric spaces.

Definition 1.3. [33] Let $\mathcal{M}$ be a set. A map $d : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$ is said to be an **ultrametric** if it satisfies following conditions.

(i) $d(z, z) = 0$ for all $z \in \mathcal{M}$. If $x, y \in \mathcal{M}$ are such that $d(x, y) = 0$, then $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{M}$.

(iii) $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in \mathcal{M}$.

In this case, $(\mathcal{M}, d)$ is called as an **ultrametric space**.

Let $\mathbb{K}$ be a non-Archimedean valued field. We define

$$c_0(\mathbb{N}, \mathbb{K}) := \left\{ \{a_n\}_n : a_n \in \mathbb{K}, \forall n \in \mathbb{N}, \lim_{n \rightarrow \infty} a_n = 0 \right\}$$

equipped with the ultra-norm

$$\|\{a_n\}_n\| := \max_{n \in \mathbb{N}} |a_n|, \quad \forall \{a_n\}_n \in c_0(\mathbb{N}, \mathbb{K})$$

and the p-adic inner product

$$\langle \{a_n\}_n, \{b_n\}_n \rangle := \sum_{n=1}^{\infty} a_n b_n, \quad \forall \{a_n\}_n, \{b_n\}_n \in c_0(\mathbb{N}, \mathbb{K}).$$

Then $c_0(\mathbb{N}, \mathbb{K})$ is a non-Archimedean Banach space which is also a p-adic Hilbert space [17, 29]. We now ask for ultrametric version of Bourgain-Figiel-Milman theorem.

Problem 1.4. (*Ultrametric Bourgain-Figiel-Milman Problem*) $\mathbb{K}$ be a non-Archimedean valued field. For every $\varepsilon > 0$, whether there is a universal constant $C(\varepsilon, \mathbb{K}) > 0$ satisfying the following: Every finite ultrametric space $\mathcal{M}$ contains a subset $\mathcal{N}$ satisfying following conditions.

(i) There is an injective map $f : \mathcal{N} \rightarrow c_0(\mathbb{N}, \mathbb{K})$ such that

$$d_{BFM}(\mathcal{N}, f(\mathcal{N})) \leq 1 + \varepsilon.$$

(ii)

$$o(\mathcal{N}) \geq C(\varepsilon, \mathbb{K}) \log(o(\mathcal{M})).$$

2. ULTRAMETRIC ENFLO TYPE AND MENDEL-NAOR COTYPE PROBLEMS

Let $\mathcal{H}$ be a Hilbert space, $n \in \mathbb{N}$. Recall that for any n points $h_1, \dots, h_n \in \mathcal{H}$, we have

$$(1) \quad \frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} \left\| \sum_{j=1}^n \varepsilon_j h_j \right\|^2 = \sum_{j=1}^n \|h_j\|^2.$$

Equality (1) motivated the definition of Type and Cotype for Banach spaces.

Definition 2.1. [1] Let $1 \leq p \leq 2$. A Banach space $\mathcal{X}$ is said to be of **(Rademacher) Type p** if there exists $T_p(\mathcal{X}) > 0$ such that

$$\left(\frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p \right)^{\frac{1}{p}} \leq T_p(\mathcal{X}) \left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}}, \quad \forall x_1, \dots, x_n \in \mathcal{X}, \quad \forall n \in \mathbb{N}.$$

Definition 2.2. [1] Let $2 \leq q < \infty$. A Banach space $\mathcal{X}$ is said to be of **(Rademacher) Cotype q** if there exists $C_q(\mathcal{X}) > 0$ such that

$$\left(\sum_{j=1}^n \|x_j\|^q \right)^{\frac{1}{q}} \leq C_q(\mathcal{X}) \left(\frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^q \right)^{\frac{1}{q}}, \quad \forall x_1, \dots, x_n \in \mathcal{X}, \quad \forall n \in \mathbb{N}.$$

There is a vast literature on Type-Cotype of Banach spaces, see [1, 5, 14–16, 19–21, 24, 35].

In 1970, Enflo introduced the notion of Type for metric spaces [7, 25].

Definition 2.3. [7, 25] Let $p > 0$. A metric space $(\mathcal{M}, d)$ is said to be of **Enflo Type p** if there exists $E_p(\mathcal{M}) > 0$ satisfying following conditions: For every $n \in \mathbb{N}$ and for every $f : \{-1, 1\}^n \rightarrow \mathcal{M}$ it holds

$$\left(\frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} d(f(\varepsilon_1, \dots, \varepsilon_n), d(-\varepsilon_1, \dots, -\varepsilon_n))^p \right)^{\frac{1}{p}} \leq E_p(\mathcal{M}) \left(\sum_{j=1}^n \frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} d(f(\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n), d(\varepsilon_1, \dots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_n))^p \right)^{\frac{1}{p}}.$$

In 2008, Mendel and Naor introduced the notion of Cotype for metric spaces [25].

Definition 2.4. [25] Let $q > 0$. A metric space $(\mathcal{M}, d)$ is said to be of **Mendel-Naor Cotype q** if there exists $\Gamma_q(\mathcal{M}) > 0$ satisfying following conditions: For every $n \in \mathbb{N}$, there exists an even integer $m \in \mathbb{N}$ such that for every $f : \mathbb{Z}_m^n \rightarrow \mathcal{M}$ it holds

$$\begin{aligned} & \sum_{j=1}^n \frac{1}{m^n} \sum_{x \in \mathbb{Z}_m^n} d\left(f\left(x + \frac{m}{2} e_j, f(x)\right)\right)^q \\ & \leq \Gamma_q(\mathcal{M})^q m^q \frac{1}{m^n} \sum_{x \in \mathbb{Z}_m^n} \frac{1}{3^n} \sum_{\varepsilon \in \{-1, 0, 1\}^n} d(f(x + \varepsilon), f(x))^q, \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n), \end{aligned}$$

where $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbb{R}^n$.

We now ask for ultrametric versions of Enflo Type and Mendel-Naor Cotype.

Problem 2.5. (Ultrametric Enflo Type Problem) Whether there is a way to define ultrametric Enflo Type?

Problem 2.6. (*Ultrametric Mendel-Naor Cotype Problem*) *Whether there is a way to define ultrametric Mendel-Naor Cotype?*

Note that there are also notions of Type for metric spaces by Bourgain, Milman and Wolfson [4] and by Ball [2]. These lead to following problems.

Problem 2.7. (*Ultrametric Bourgain-Milman-Wolfson Type Problem*) *Whether there is a way to define ultrametric Bourgain-Milman-Wolfson Type?*

Problem 2.8. (*Ultrametric Markov/Ball Type Problem*) *Whether there is a way to define ultrametric Markov/Ball Type?*

Remark 2.9. *In 2026, we formulated non-Archimedean John, non-Archimedean Dvoretzky-Milman, non-Archimedean Type-Cotype and non-Archimedean Kwapien problems which are still open [18].*

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