

Curvature-Induced Killing at Stochastic Algebraic Folds

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Abstract

We study the lifetime of a regular algebraic branch under continuous forcing and infinite-activity tangent jumps. For a parabolic fold, a first-order tangent kick decreases the discriminant by an exact quadratic amount. Symmetric stable marks of index $\alpha \in (0, 2)$ therefore generate a one-sided normal jump kernel of index $q = \alpha/2$, with a coefficient determined by curvature. The normal diffusion and jump amplitudes depend on the evolving tangential state. We compute the full killed generator on powers of the discriminant and obtain occupation and terminal-jump bounds for this same joint process. Uniformly subcritical normal drift gives finite censored boundary lifetime and finite cumulative killing rate; continuous contact and jump exit both have positive probability. At and above the uniform critical threshold, a logarithmic nonlocal barrier excludes censored boundary contact, and the original branch terminates at a jump almost surely. A comparison after a stochastic time change yields an explicit stretched-exponential upper bound for survival and hence all positive lifetime moments. In the subcritical regime, an optimized upper bound for the probability of jump termination tends to zero as the curvature coefficient or the initial discriminant tends to zero. The analysis combines power estimates for the jump generator with a logarithmic barrier at the critical threshold. Numerical calculations illustrate the dependence of terminal causes on curvature.

Keywords. Stochastic differential-algebraic equation; algebraic fold; tangent jump; killed jump diffusion; occupation measure; censored process.

1 Introduction

A differential-algebraic equation may have a smooth constraint manifold whose projection onto the differential variables develops a fold. On either regular side, the algebraic variable is a smooth reconstruction of the differential state. At the fold this reconstruction becomes singular. Under stochastic forcing, the lifetime of a chosen branch depends both on continuous boundary contact and on jumps that leave its feasible side.

Regular index-one stochastic differential-algebraic equations admit local reduction to stochastic differential equations [15, 13]. Deterministic work on singularity-induced bifurcation addresses the related loss of algebraic regularity [4, 11]. For stochastic dynamics, preservation of a closed feasible set and accessibility of its boundary from an open regular branch are distinct questions. Stochastic invariance criteria for diffusions and jump diffusions make explicit the role of the exact jump image [2, 1]. A finite tangent displacement can leave the feasible side of a curved constraint.

We study these two termination mechanisms in a family of constrained systems. The algebraic equation is

$$Y^2 + \kappa V^2 - U = 0, \quad \kappa > 0,$$

and the selected branch is $Y = \sqrt{H}$, where $H = U - \kappa V^2 > 0$. A mark z produces the displacement

$$(\Delta U, \Delta V) = (2\kappa V a(V)z, a(V)z).$$

It is tangent to every level set of H at the pre-jump state, but its exact endpoint satisfies

$$H_{\text{after}} = H_{\text{before}} - \kappa a(V)^2 z^2.$$

For a symmetric stable mark measure, the squared displacement has a one-sided stable radial density. The constraint curvature thus determines the normal jump mechanism and its exit rate.

The continuous coefficients extend as C^2 functions of the original differential and algebraic variables. After applying Itô's formula, the discriminant satisfies a square-root diffusion with negative finite-variation jumps. Its drift, diffusion amplitude, and jump amplitude depend on the evolving tangential variable. When δ is nonconstant, a common stochastic time change preserves its dependence on the tangential state. The occupation estimates below concern this complete joint dynamics.

The main analytical calculation is the action of the killed generator on H^p . Its continuous contribution has order H^{p-1} , while its jump contribution has order H^{p-q} , where $q = \alpha/2 < 1$. The coefficient of the jump term is explicit. Positive powers give an occupation estimate derived directly from the coefficients in the uniformly subcritical regime, both for the killed process and for its censored counterpart. Negative powers give quantitative approach and occupation bounds above the critical threshold. At the critical threshold, a logarithm with a lower-order correction yields a nonlocal barrier for the censored process.

Censoring is an established construction for stable processes [6, 7]. Decreasing censored subordinators and their finite boundary lifetimes are studied in [8, 5]. We use local interlacing to construct the censored joint process and retain the boundary lifetime in its Feynman–Kac representation. The additional step here is a coefficient-level determination of that lifetime and of the accumulated killing rate for the curved-fold system. Below the uniform critical threshold, the censored process reaches zero in finite time and accumulates a finite killing rate. At and above it, the censored process is conservative and the accumulated rate diverges. These statements identify the two terminal mechanisms of the original equation.

The occupation estimate also gives a curvature-dependent bound on the probability of eventual jump termination. Optimizing over a normal collar shows that this probability tends to zero with curvature in the subcritical regime. Above the critical threshold every positive curvature produces eventual jump termination, although the flat model retains its branch. A separate comparison with squared Bessel motion gives an explicit survival upper bound of stretched-exponential form for every bounded nonnegative dimension coefficient. The bound gives quantitative control of survival over the entire lifetime.

The power calculation, stochastic comparison, and exponential killing representation use standard methods from stochastic calculus and fractional generators [3, 10, 8]. Their role is to derive a joint-process lifetime result from the original constraint and coefficients. Classical Bessel kernels provide the numerical transitions for frozen continuous coefficients.

Section 2 specifies the system and states its lifetime classification. Section 3 constructs the killed and censored processes. Section 4 proves the full-generator occupation estimates. Sections 5 and 6 determine boundary contact, terminal causes, and quantitative lifetime bounds. Section 7 reports calculations for the same family. The appendices discuss curved graph constraints and the numerical stopping convention.

2 The fold equation and its lifetime

2.1 Coefficients and driving noises

Fix $\kappa > 0$. Work on a filtered probability space satisfying the usual conditions. Let W and B be independent standard Brownian motions and let N be an independent Poisson random

measure on $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ with compensator $\mathrm{d}t \nu(\mathrm{d}z)$, where

$$\nu(\mathrm{d}z) = c|z|^{-1-\alpha} \mathrm{d}z, \quad c > 0, \quad 0 < \alpha < 2, \quad q = \frac{\alpha}{2}. \quad (2.1)$$

This density is imposed at all nonzero mark sizes. Write $\tilde{N} = N - \mathrm{d}t \nu$ on $|z| \leq 1$.

Assumption 2.1. The functions $b, \eta, a, \sigma, \delta$ are C^2 on $\mathbb{R}$. The drift b is globally Lipschitz with at most linear growth. The functions η, a, σ, δ and their first derivatives are bounded, and

$$0 < a_- \leq a(v) \leq a_+, \quad 0 < \sigma_- \leq \sigma(v) \leq \sigma_+, \quad 0 \leq d_- \leq \delta(v) \leq d_+ < \infty. \quad (2.2)$$

Second derivatives are locally bounded. The initial state is deterministic, with $v_0 \in \mathbb{R}$ and $h_0 = u_0 - \kappa v_0^2 > 0$.

Define the tangential jump integral

$$\mathrm{d}J_t^a = \int_{|z| \leq 1} a(V_{t-}) z \tilde{N}(\mathrm{d}t, \mathrm{d}z) + \int_{|z| > 1} a(V_{t-}) z N(\mathrm{d}t, \mathrm{d}z). \quad (2.3)$$

The constrained system is

$$\begin{aligned} \mathrm{d}V_t &= b(V_t) \mathrm{d}t + \eta(V_t) \mathrm{d}B_t + \mathrm{d}J_t^a, \\ \mathrm{d}U_t &= [2\kappa V_t b(V_t) + \kappa \eta(V_t)^2 + \delta(V_t) \sigma(V_t)^2] \mathrm{d}t \\ &\quad + 2\kappa V_t \eta(V_t) \mathrm{d}B_t + 2\sigma(V_t) Y_t \mathrm{d}W_t + 2\kappa V_{t-} \mathrm{d}J_t^a, \\ 0 &= Y_t^2 + \kappa V_t^2 - U_t, \quad Y_t > 0. \end{aligned} \quad (2.4)$$

The coefficients in (2.4) extend as C^2 functions across $Y = 0$. The loss of regularity occurs in the reconstruction $Y = \sqrt{U - \kappa V^2}$.

The compensation in (2.3) is fixed throughout. The continuous tangent drift $\kappa \eta^2$ is the Itô curvature term associated with V^2 . At a jump, U changes by the linear tangent increment $2\kappa V a(V) z$.

2.2 Exact geometry of a tangent kick

Set

$$g(u, v, y) = y^2 + \kappa v^2 - u, \quad h(u, v) = u - \kappa v^2, \quad E = \{(u, v) : h(u, v) > 0\}. \quad (2.5)$$

Then $Dg = (-1, 2\kappa v, 2y)$ never vanishes, whereas $g_y = 2y$ and $g_{yy} = 2$. Thus $g = 0$ has a simple fold at $y = 0$. The base fold curve is $u = \kappa v^2$; its Euclidean curvature is $2\kappa(1 + 4\kappa^2 v^2)^{-3/2}$. The parameter κ is its quadratic curvature coefficient.

For $\Gamma(u, v, z) = (2\kappa v a(v) z, a(v) z)$,

$$Dh(u, v) \Gamma(u, v, z) = 0, \quad h((u, v) + \Gamma(u, v, z)) = h(u, v) - \kappa a(v)^2 z^2. \quad (2.6)$$

The first equality is a linear tangency statement. The second is an exact endpoint identity. A mark terminates the selected branch precisely when

$$|z| \geq \frac{\sqrt{h}}{\sqrt{\kappa} a(v)}. \quad (2.7)$$

A terminal proposal sends the process to the cemetery state before algebraic reconstruction.

Proposition 2.2 (The normal equation). *Up to branch termination, $H = U - \kappa V^2$ satisfies*

$$dH_t = \delta(V_t)\sigma(V_t)^2 dt + 2\sigma(V_t)\sqrt{H_t}dW_t - \kappa \int_{\mathbb{R} \setminus \{0\}} a(V_{t-})^2 z^2 N(dt, dz). \quad (2.8)$$

At a fixed pre-state (h, v) the normal jump size $s = \kappa a(v)^2 z^2$ has intensity

$$\Pi_v(ds) = C(v)s^{-1-q} ds, \quad C(v) = c\kappa^q a(v)^{2q}, \quad C_- = c\kappa^q a_-^{2q}, \quad C_+ = c\kappa^q a_+^{2q}. \quad (2.9)$$

The exact killing rate is

$$\lambda(h, v) = \frac{C(v)}{q} h^{-q} = \frac{2c}{\alpha} \kappa^{\alpha/2} a(v)^\alpha R^{-\alpha}, \quad R = \sqrt{h}. \quad (2.10)$$

Proof. Itô's formula gives

$$d(V^2) = 2V_- dV + \eta(V)^2 dt + \int a(V_-)^2 z^2 N(dt, dz).$$

Subtracting $\kappa d(V^2)$ from dU proves (2.8). The last integral is locally of finite variation: its small-mark first moment is finite because $\alpha < 2$, and its large marks have finite arrival rate. On each half-line the substitution $s = \kappa a(v)^2 z^2$ contributes $\frac{1}{2} c\kappa^q a(v)^{2q} s^{-1-q} ds$. Adding both signs proves (2.9), and integrating over $s \geq h$ gives (2.10). $\square$

In the original mark truncation, the compensated form of (2.8) has drift

$$\delta(v)\sigma(v)^2 - \frac{2c\kappa a(v)^2}{2 - \alpha}. \quad (2.11)$$

The finite-variation form (2.8) and (2.11) describe the same process. Subsequent generator calculations retain all of their jump terms.

2.3 Terminal events and main classification

Let ζ denote the minimal lifetime in $H > 0$, including a jump with proposed discriminant at most zero. A terminal jump has a strictly positive pre-jump discriminant and satisfies (2.7). A continuous contact is a finite lifetime with $H_t \rightarrow 0$ and no terminal mark. Infinitely many preceding accepted jumps are allowed in this definition. Section 3 shows that these are the only finite terminal events. Let J and C be the respective events.

The censored process $(\bar{H}, \bar{V})$ suppresses all terminal proposals. Its minimal boundary lifetime is $\bar{\tau}$. Define

$$A_t = \int_0^t \lambda(\bar{H}_s, \bar{V}_s) ds, \quad t < \bar{\tau}, \quad A_{\bar{\tau}} = \lim_{t \uparrow \bar{\tau}} A_t. \quad (2.12)$$

Superscript c on a probability or expectation indicates the censored law.

Theorem 2.3 (Lifetime and terminal cause). *Under Assumption 2.1, the system (2.4) has a pathwise unique solution on its selected branch up to ζ . Its lifetime has finite moments of every positive order. The following conclusions hold uniformly in the initial tangential coordinate.*

If $d_+ < 2$, then

$$\bar{\tau} < \infty, \quad 0 < A_{\bar{\tau}} < \infty \quad \mathbb{P}_{h_0, v_0}^c \text{-almost surely}, \quad (2.13)$$

and

$$0 < \mathbb{P}_{h_0, v_0}(\text{C}) = \mathbb{E}_{h_0, v_0}^c e^{-A_{\bar{\tau}}} < 1, \quad \mathbb{P}(\text{J}) = 1 - \mathbb{P}(\text{C}). \quad (2.14)$$

Moreover, $\mathbb{P}(\text{C}) \rightarrow 1$ as $h_0 \downarrow 0$ or as $\kappa \downarrow 0$, with the other model bounds fixed. Quantitative bounds are given in Theorem 5.2 and Corollary 6.1.

If $d_- \geq 2$, including equality, then

$$\bar{\tau} = \infty, \quad A_\infty = \infty \quad \mathbb{P}_{h_0, v_0}^c\text{-almost surely}, \quad \mathbb{P}_{h_0, v_0}(\mathbf{J}) = 1. \quad (2.15)$$

The occupation bounds in [Theorem 4.3](#) and the survival bound in [Theorem 6.3](#) refer to the original complete jump process.

Sections 3–6 prove the theorem. The coefficient bounds hold uniformly along the fold, while δ , σ , and a may vary with V . The state variable throughout the argument is the joint process (H, V) .

Example 2.4 (One tangentially modulated family). The calculations use

$$\begin{aligned} b(v) &= -v, & \eta(v) &= 0.4, & \sigma(v) &= 1 + 0.15 \tanh v, \\ a(v) &= 1 + 0.10 \tanh v, & \delta(v) &= \delta_0 + 0.20 \tanh v. \end{aligned} \quad (2.16)$$

The choices $\delta_0 = 1.2, 2.2$, and 3.0 give the subcritical regime, the regime with critical lower bound, and the strictly supercritical regime. At a common value of H , the coefficient ratio depends on v through $\delta(v)$. Consequently, a common stochastic clock cannot turn this family into a fixed-dimensional Bessel equation. The tangential process and the normal jump loss also share their marks.

3 The killed and censored joint processes

The survival law can be expressed through a process that suppresses terminal marks. We construct both processes locally, identify their generators, and derive the relation between the accumulated exit rate and terminal causes.

3.1 Local construction and absence of explosion

For a function f on $(0, \infty) \times \mathbb{R}$, write $f^\dagger(h, v) = 0$ when $h \leq 0$. The killed process is sent to a separate cemetery state, at which all test functions used below have value zero.

Proposition 3.1 (Existence and finite terminal events). *Under [Assumption 2.1](#), the original system has a pathwise unique maximal local strong solution on $H > 0$. There is also a pathwise unique maximal local censored process, obtained by suppressing precisely the marks in [\(2.7\)](#). Neither process can explode in its differential variables in finite time while its discriminant remains positive. A finite original lifetime is either a terminal mark or continuous contact at $H = 0$. A finite censored lifetime is continuous contact at $H = 0$.*

Proof. On every interior compact set the reconstructed coefficients are locally Lipschitz. Their small-mark values and derivatives have square-integrable envelopes, since $\int_{|z| \leq 1} z^2 \nu(dz) < \infty$. Large marks have finite rate. Local existence and pathwise uniqueness follow by small-jump SDE theory and interlacing of large arrivals [\[3, 14\]](#).

For censoring, localize first to $\varepsilon < h < L$ and $|v| < M$. Choose

$$z_* = \min \left\{ 1, \frac{\sqrt{\varepsilon}}{2\sqrt{\kappa} a_+} \right\}.$$

Every $|z| < z_*$ is accepted throughout this region. All marks with $|z| \geq z_*$ form a finite-rate proposal process. Between its arrivals, solve the locally Lipschitz equation driven by the smaller marks. At each proposal, apply the exact acceptance test $\kappa a(v)^2 z^2 < h$. This yields existence and uniqueness up to the localization boundary, without a Lipschitz requirement on the acceptance indicator. Nested localizations agree by uniqueness.

The small rejected-mark correction in the original coordinates is

$$\int_{\{|z| \leq 1, \kappa a(v)^2 z^2 \geq h\}} (2\kappa v a(v)z, a(v)z) \nu(dz) = 0.$$

For a fixed interior state this integral is absolutely convergent and vanishes by symmetry. Thus no additional tangential drift remains after censoring. Applying Itô's formula to the censored system gives

$$d\bar{H}_t = \delta(\bar{V}_t) \sigma(\bar{V}_t)^2 dt + 2\sigma(\bar{V}_t) \sqrt{\bar{H}_t} dW_t - \kappa \int a(\bar{V}_{t-})^2 z^2 \mathbf{1}_{\{\kappa a(\bar{V}_{t-})^2 z^2 < \bar{H}_{t-}\}} N(dt, dz). \quad (3.1)$$

In $\bar{V}$, the same indicator restricts the compensated small-mark and uncompensated large-mark integrals.

For either process the tangential small-jump martingale has quadratic variation in expectation bounded by a constant times time before localization. Its drift has linear growth, its diffusion coefficient is bounded, and its large marks have finite arrival rate. Localizing the largest large mark and applying the usual martingale and Gronwall estimates excludes finite tangential explosion; the large-mark localization is then removed.

Both discriminants have only negative jumps. Put $D = d_+ \sigma_+^2$. On the function h , the normal generator of either process is at most D . For censoring the jump drift is $-C(v)h^{1-q}/(1-q)$; for killing all jump increments of the nonnegative killed state are nonpositive. Optional stopping at the first upper level M and at the lifetime therefore gives

$$\mathbb{P}\{\tau_M \leq T, \tau_M < \text{lifetime}\} \leq \frac{h_0 + DT}{M}. \quad (3.2)$$

Here the upper crossing is continuous. Localize the lower and tangential boundaries first and use Fatou's lemma. Letting $M \rightarrow \infty$ excludes upward explosion.

On every bounded time interval the small normal losses have finite total variation, bounded by $\kappa a_+^2 \int_{|z| \leq 1} z^2 N$. The Brownian quadratic variation is bounded when H is bounded, and there are only finitely many large proposals. Hence the state has a finite limit at a finite minimal lifetime unless that lifetime is an actual terminal proposal. An interior limit would allow local continuation. The remaining limit has $H = 0$. The zero approach can be announced by lower-level stopping times, and a Poisson measure with time compensator dt has no mark at this predictable time. This proves the terminal-event classification. $\square$

The vanishing correction in this proposition follows from the symmetry of the exact rejected set. It is a property of this model and its specified truncation. In a general fold equation the corresponding correction need not vanish.

3.2 The generators

In (h, v) coordinates set $s_z = \kappa a(v)^2 z^2$. For $f \in C_c^2((0, \infty) \times \mathbb{R})$,

$$\begin{aligned} \mathcal{A}^\dagger f(h, v) &= \delta(v) \sigma(v)^2 f_h + b(v) f_v + 2\sigma(v)^2 h f_{hh} + \frac{1}{2} \eta(v)^2 f_{vv} \\ &\quad + \int_{\mathbb{R} \setminus \{0\}} [f^\dagger(h - s_z, v + a(v)z) - f(h, v) - \mathbf{1}_{\{|z| \leq 1\}} a(v)z f_v(h, v)] \nu(dz). \end{aligned} \quad (3.3)$$

The normal loss is uncompensated in this representation. This convention is legitimate because the normal small losses have a finite first moment. The tangential integral retains the compensation in (2.3).

The generator of the constructed censored process satisfies

$$\mathcal{A}^\dagger f = \mathcal{A}^c f - \lambda f. \quad (3.4)$$

Indeed, the terms over $s_z \geq h$ in (3.3) give $-\lambda f$ and the odd compensation integral already evaluated in Proposition 3.1. For a test function ψ depending only on the normal coordinate, the two operators reduce to

$$\mathcal{A}^\dagger \psi(h, v) = \sigma(v)^2 [\delta(v) \psi'(h) + 2h \psi''(h)] + C(v) \int_0^\infty [\psi^\dagger(h-s) - \psi(h)] s^{-1-q} ds, \quad (3.5)$$

$$\mathcal{A}^c \psi(h, v) = \sigma(v)^2 [\delta(v) \psi'(h) + 2h \psi''(h)] + C(v) \int_0^h [\psi(h-s) - \psi(h)] s^{-1-q} ds. \quad (3.6)$$

Both expressions evaluate the joint generators on functions of the normal coordinate.

3.3 Survival and terminal causes

Proposition 3.2 (Minimal killing representation). *Interpret $e^{-A_T} \mathbf{1}_{\{T < \bar{\tau}\}}$ as zero on $\{T \geq \bar{\tau}\}$. For every $T > 0$,*

$$\mathbb{P}(\zeta > T) = \mathbb{E}^c[e^{-A_T}; T < \bar{\tau}], \quad (3.7)$$

$$\mathbb{P}(\mathbf{J}, \zeta \leq T) = \mathbb{E}^c \int_0^{T \wedge \bar{\tau}} e^{-A_t} \lambda(\bar{H}_t, \bar{V}_t) dt = \mathbb{E} \int_0^{T \wedge \zeta} \lambda(H_t, V_t) dt, \quad (3.8)$$

$$\mathbb{P}(\mathbf{C}, \zeta \leq T) = \mathbb{E}^c[e^{-A_{\bar{\tau}}}; \bar{\tau} \leq T]. \quad (3.9)$$

The three probabilities sum to one. We use $e^{-\infty} = 0$.

Proof. On an interior compact set, the terminal-proposal rate is bounded. Kill the censored process when A_t exceeds an independent unit exponential random variable. Its generator is $\mathcal{A}^c - \lambda$, which equals (3.3). The local finite-proposal construction gives uniqueness of the killed law. Exhausting the open state space identifies the minimal lifetime as the earlier of this exponential clock and the censored boundary lifetime. Conditional on the censored path, survival to $T < \bar{\tau}$ is e^{-A_T} , and the conditional killing measure is $e^{-A_t} dA_t$. Monotone limits give (3.7)–(3.9), including an infinite accumulated rate at a finite boundary lifetime.

The last equality in (3.8) is Poisson compensation applied to the terminal count of the original process. The count is zero or one; compact localization and monotone convergence give the stated integral. Finally,

$$e^{-A_{T \wedge \bar{\tau}}} + \int_0^{T \wedge \bar{\tau}} e^{-A_t} dA_t = 1$$

with its limiting interpretation when $A_{\bar{\tau}} = \infty$. Splitting the first term according to $T < \bar{\tau}$ proves the probability identity. $\square$

Equations (3.7) and (3.8) express the terminal probabilities under the censored and original laws, respectively.

4 Nonlocal power estimates and occupation

Powers of the discriminant compare the continuous boundary drift with the normal jump loss. The resulting generator inequalities control time spent near the fold and the probability of a terminal jump from that region.

4.1 An exact fractional integral

For $p > q - 1$, define

$$m_q(p) = \frac{\Gamma(1-q)\Gamma(1+p)}{q\Gamma(1+p-q)}. \quad (4.1)$$

It is positive on this range, and $m_q(0) = 1/q$.

Lemma 4.1 (Powers of the killed discriminant). *Give h^p value zero at the cemetery state. For $h > 0$ and $p > q - 1$,*

$$\mathcal{A}^\dagger h^p(h, v) = \sigma(v)^2 p [\delta(v) + 2p - 2] h^{p-1} - C(v) m_q(p) h^{p-q}, \quad (4.2)$$

$$\mathcal{A}^c h^p(h, v) = \sigma(v)^2 p [\delta(v) + 2p - 2] h^{p-1} - C(v) [m_q(p) - 1/q] h^{p-q}. \quad (4.3)$$

For $p > 0$, $m_q(p) - 1/q > 0$.

Proof. The continuous part follows by differentiating h^p . In the jump integral substitute $s = hu$ to obtain h^{p-q} times

$$\int_0^1 [(1-u)^p - 1] u^{-1-q} du - \frac{1}{q} = -m_q(p). \quad (4.4)$$

For $p > 0$, integration by parts on $(0, \infty)$ gives

$$\int_0^\infty [1 - (1-u)_+^p] u^{-1-q} du = \frac{p}{q} \int_0^1 u^{-q} (1-u)^{p-1} du = m_q(p).$$

To include negative p , the left side of (4.4) is holomorphic on $\operatorname{Re} p > -1$: near zero its numerator is $O(u)$ uniformly on compact parameter sets, while the other endpoint has an integrable power and its logarithmic derivatives. The gamma quotient has the same holomorphic extension on this half-plane. The identity theorem extends the calculation, in particular to $p > q - 1$. Censoring removes the $-1/q$ tail term. The strict positivity for $p > 0$ also follows directly from $1 - (1-u)^p > 0$ on $(0, 1)$. $\square$

The different powers h^{p-1} and h^{p-q} are consequences of the square-root diffusion and the quadratic tangent loss. The jump term is retained in both identities, including the contribution of terminal marks.

4.2 A localization principle

Lemma 4.2 (Nonnegative test functions). *Suppose a nonnegative function F is C^2 in the interior, its jump increments are locally integrable, and the killed generator satisfies*

$$\mathcal{A}^\dagger F \leq -w,$$

where $w \geq 0$. Suppose its chosen value at termination is nonnegative. Then

$$\mathbb{E} \int_0^\zeta w(H_t, V_t) dt \leq F(h_0, v_0). \quad (4.5)$$

The same conclusion holds for the minimal censored process with $\mathcal{A}^c$, using its boundary localization. If a test function diverges at zero, the inequality is first applied before a lower-level stopping time, and its nonnegative terminal term is kept when bounding that stopping probability.

Proof. Stop first in a compact subset of the open state space and at a bounded time. Itô's formula and compensation show that

$$F(H_t, V_t) + \int_0^t w(H_s, V_s) ds$$

is a nonnegative local supermartingale up to that stopping time, and hence a supermartingale after the usual martingale localization. Dropping its nonnegative terminal value gives the integral estimate. Exhaust the compacts, then let the time horizon increase. Fatou's lemma for the terminal values and monotone convergence for the nonnegative integrals give (4.5).

For $F = h^p$ in Lemma 4.1, possible jump landings near zero are integrable since $p > q - 1 > -1$. Small normal increments have a finite first moment, and all derivatives of F are bounded before a fixed lower-level stop. Thus the stated localization applies to these unbounded powers without assuming square integrability of the terminal value. $\square$

4.3 Occupation of the actual jump process

There are two useful ranges of powers. In the subcritical case $d_+ < 2$, choose

$$0 < p < \min\{1 - d_+/2, 1 - q\}, \quad a_p = \sigma_-^2 p(2 - d_+ - 2p) > 0. \quad (4.6)$$

In the strictly supercritical case $d_- > 2$, choose

$$p = -\theta, \quad 0 < \theta < \min\{(d_- - 2)/2, 1 - q\}, \quad a_p = \sigma_-^2 \theta(d_- - 2 - 2\theta) > 0. \quad (4.7)$$

Both choices satisfy $q - 1 < p < 1 - q$ and

$$\sigma(v)^2 p[\delta(v) + 2p - 2] \leq -a_p.$$

Theorem 4.3 (Full-jump occupation bounds). *For either (4.6) or (4.7), the original killed process satisfies*

$$a_p \mathbb{E} \int_0^\zeta H_t^{p-1} dt + C_- m_q(p) \mathbb{E} \int_0^\zeta H_t^{p-q} dt \leq h_0^p. \quad (4.8)$$

In particular, for every $\varepsilon > 0$,

$$\mathbb{E} \int_0^\zeta \mathbf{1}_{\{0 < R_t \leq \varepsilon\}} dt \leq \frac{h_0^p}{a_p} \varepsilon^{2(1-p)}. \quad (4.9)$$

If $F_{\text{near}}(\varepsilon)$ is the probability that the terminal mark occurs with $0 < R_{\zeta-} \leq \varepsilon$, then

$$F_{\text{near}}(\varepsilon) \leq \frac{C_+ h_0^p}{q a_p} \varepsilon^{2(1-p-q)}. \quad (4.10)$$

Bounds larger than one in (4.10) may be replaced by one.

Under (4.6), the censored process also satisfies

$$\mathbb{E}^c \int_0^{\bar{\tau}} \bar{H}_t^{p-1} dt \leq \frac{h_0^p}{a_p}. \quad (4.11)$$

Proof. Equation (4.2) is bounded above by $-a_p h^{p-1} - C_- m_q(p) h^{p-q}$. Apply Lemma 4.2 to obtain (4.8). Since $p < 1$, on $0 < h \leq \varepsilon^2$,

$$1 \leq \varepsilon^{2(1-p)} h^{p-1},$$

which proves (4.9). Poisson compensation at the terminal jump gives

$$F_{\text{near}}(\varepsilon) = \mathbb{E} \int_0^\zeta \mathbf{1}_{\{H_t \leq \varepsilon^2\}} \frac{C(V_t)}{q} H_t^{-q} dt.$$

Because $1 - p - q > 0$, the integrand is at most $(C_+/q) \varepsilon^{2(1-p-q)} H_t^{p-1}$. This proves (4.10). For the positive-power censored process, both terms of (4.3) are nonpositive and its continuous term is at most $-a_p h^{p-1}$. A second application of Lemma 4.2 gives (4.11). $\square$

The constants in (4.8)–(4.10) depend on the model parameters, the coefficient bounds in Assumption 2.1, and the chosen power p . Equation (4.11) provides the corresponding estimate under the censored law, which is used next to control its accumulated killing rate.

Corollary 4.4 (Approach probability above the threshold). *Under (4.7), for $0 < \varepsilon^2 < h_0$,*

$$\mathbb{P}\{H_t \leq \varepsilon^2 \text{ for some } t < \zeta\} \leq \left(\frac{\varepsilon^2}{h_0}\right)^\theta. \quad (4.12)$$

Proof. Stop the nonnegative supermartingale associated with $h^{-\theta}$ at the first alive entry into $(0, \varepsilon^2]$, or at the lifetime and an upper localization. At an alive entry its value is at least $\varepsilon^{-2\theta}$. A terminal jump has cemetery value zero and is not counted as an entry. Retaining the terminal term in Lemma 4.2 and removing localization proves the bound. $\square$

5 Boundary contact and the two terminal mechanisms

The censored process determines whether continuous contact contributes to the original lifetime. Below the critical threshold, a squared-Bessel comparison gives finite censored contact and the occupation bound makes the accumulated killing rate finite. At and above the threshold, a corrected logarithmic barrier excludes contact.

5.1 Comparison under an adapted clock

Lemma 5.1 (A squared-Bessel upper comparison). *For the censored process, put*

$$S_t = \int_0^t \sigma(\bar{V}_s)^2 ds.$$

On a suitable extension past its minimal lifetime there is a squared-Bessel process Z with dimension d_+ and $Z_0 = h_0$ such that

$$\bar{H}_t \leq Z_{S_t}, \quad t < \bar{\tau}, \quad \sigma_-^2 t \leq S_t \leq \sigma_+^2 t. \quad (5.1)$$

The comparison is pathwise, with Z and the tangential state coupled through the adapted clock. If $d_+ < 2$, then $\bar{\tau} < \infty$ almost surely.

Proof. Extend the bounded positive predictable process $\sigma(\bar{V}_t)$ past $\bar{\tau}$, for example by a fixed positive value within its bounds. Before the lifetime, change time by the continuous, strictly increasing clock S . The martingale $\int_0^t \sigma(\bar{V}_s) dW_s$ becomes Brownian motion under this clock. The time-changed discriminant has continuous drift $\delta(\bar{V}_{S^{-1}(u)}) \leq d_+$, diffusion coefficient $2\sqrt{h}$, and only negative finite-variation jumps. Drive

$$dZ_u = d_+ du + 2\sqrt{Z_u} d\beta_u$$

by the same Brownian motion, using its nonnegative squared-Bessel continuation.

The scalar comparison follows by the standard smooth approximation to the positive part. At a positive difference, the drift difference is nonpositive; the common square-root diffusion satisfies $(\sqrt{x} - \sqrt{y})^2 \leq |x - y|$, so the local-time term at zero vanishes in the approximation. Negative jumps of the first process cannot increase its positive difference from the continuous second process. Localization and Gronwall's argument therefore give $\bar{H}_{S^{-1}(u)} \leq Z_u$ until the censored lifetime. This is the usual square-root comparison proof [10], and it allows the drift bound to be adapted.

The zero time of Z is finite almost surely when $d_+ < 2$. To see this, fix $0 < r < 1 - d_+/2$ and put $a = r(2 - d_+ - 2r) > 0$. Its generator on z^r is $-az^{r-1}$. For $0 < \varepsilon < h_0 < L$, stop at the first passage to ε or L and at a bounded time. Since $z^{r-1} \geq L^{r-1}$ before exit, the nonnegative terminal term gives

$$\mathbb{E}(\tau_\varepsilon^Z \wedge \tau_L^Z) \leq h_0^r L^{1-r}/a.$$

Continuity and monotone convergence as $\varepsilon \downarrow 0$ show that exit from $(0, L)$ has finite mean, and retaining the upper-exit term gives $\mathbb{P}(\tau_L^Z < \tau_0^Z) \leq (h_0/L)^r$. On $\{\tau_0^Z = \infty\}$ every such exit is through L , so letting $L \rightarrow \infty$ proves $\tau_0^Z < \infty$ almost surely. The comparison forces the censored process to reach its own boundary no later than this zero time in the adapted clock. Since $S_t \geq \sigma_-^2 t$, its physical lifetime is finite as well. $\square$

5.2 Subcritical contact and finite accumulated rate

Theorem 5.2 (Subcritical terminal causes). *Assume $d_+ < 2$ and choose p as in (4.6). Then (2.13) and (2.14) hold. More quantitatively, for $0 < h_0 < L$, let τ_L be the first alive upper-level passage of H to L . Then*

$$\mathbb{E}(\zeta \wedge \tau_L) \leq \frac{h_0^p L^{1-p}}{a_p}, \quad (5.2)$$

$$\mathbb{P}(\tau_L < \zeta) \leq (h_0/L)^p, \quad (5.3)$$

$$\mathbb{P}(\mathbf{J}, \zeta < \tau_L) \leq \frac{C_+ h_0^p}{q a_p} L^{1-p-q}, \quad (5.4)$$

$$\mathbb{P}(\mathbf{C}, \zeta < \tau_L) \geq 1 - (h_0/L)^p - \frac{C_+ h_0^p}{q a_p} L^{1-p-q}. \quad (5.5)$$

A negative lower bound is replaced by zero.

Proof. By Lemma 5.1, $\bar{\tau}$ is finite. Let $M = \sup_{t < \bar{\tau}} \bar{H}_t$. The absence of finite explosion and continuous zero limit imply $M < \infty$ almost surely. The nonnegative random integral in (4.11) has finite expectation, hence is finite almost surely. Because $1 - p - q > 0$,

$$A_{\bar{\tau}} \leq \frac{C_+}{q} M^{1-p-q} \int_0^{\bar{\tau}} \bar{H}_t^{p-1} dt < \infty \quad \text{almost surely.} \quad (5.6)$$

The lifetime is strictly positive from an interior initial state and the rate is strictly positive, so $A_{\bar{\tau}} > 0$. Applying Proposition 3.2 at $T \rightarrow \infty$ gives

$$\mathbb{P}(\mathbf{C}) = \mathbb{E}^c e^{-A_{\bar{\tau}}}, \quad \mathbb{P}(\mathbf{J}) = \mathbb{E}^c (1 - e^{-A_{\bar{\tau}}}).$$

Both are strictly between zero and one. This proves the qualitative classification from the coefficient assumptions.

For the quantitative bounds, stop the original process at $\tau = \zeta \wedge \tau_L$. Its positive-power supermartingale gives

$$\mathbb{E} H_{\tau}^p + a_p \mathbb{E} \int_0^{\tau} H_t^{p-1} dt \leq h_0^p,$$

with zero cemetery value. Since $H_t < L$ before τ , the integral is at least $L^{p-1} \mathbb{E} \tau$, giving (5.2). The upper crossing has $H_{\tau_L} = L$ because there are no positive normal jumps, so retaining that terminal contribution gives (5.3). Compensation of terminal marks before the upper exit gives

$$\mathbb{P}(\mathbf{J}, \zeta < \tau_L) = \mathbb{E} \int_0^{\tau} \frac{C(V_t)}{q} H_t^{-q} dt \leq \frac{C_+}{q} L^{1-p-q} \mathbb{E} \int_0^{\tau} H_t^{p-1} dt,$$

which is (5.4). The stopping time is finite by (5.2). Its possibilities are contact, terminal jump, or upper passage. Subtracting the last two probabilities proves (5.5). $\square$

The censored occupation estimate makes the exponential weight in (3.9) strictly positive at boundary contact. Under the original law, (5.4) bounds the competing jump probability within a normal collar. Section 6 optimizes this bound over the collar width.

5.3 The critical nonlocal barrier

Let

$$A_q = \int_0^1 [-\log(1-u)] u^{-1-q} du = \frac{\psi(1) - \psi(1-q)}{q}, \quad D_q = m_q(1-q) - \frac{1}{q} > 0, \quad (5.7)$$

where $\psi = \Gamma'/\Gamma$. The integrals are finite for $0 < q < 1$.

Theorem 5.3 (Conservative censoring at and above the threshold). *Assume $d_- \geq 2$. Choose $L > 0$ so that*

$$C_+ D_q L^{1-q} \leq \sigma_-^2 (1-q)^2, \quad K = \frac{C_+ A_q}{\sigma_-^2 (1-q)^2}. \quad (5.8)$$

Then

$$F(h) = \log(L/h) + K(L^{1-q} - h^{1-q}), \quad 0 < h \leq L, \quad (5.9)$$

satisfies $F \geq 0$, $F(L) = 0$, $F(h) \rightarrow \infty$ as $h \downarrow 0$, and $\mathcal{A}^c F \leq 0$ for every tangential state. The censored process has $\bar{\tau} = \infty$ almost surely.

Proof. Differentiating (4.3) at $p = 0$, or directly integrating the logarithmic increment, gives

$$\mathcal{A}^c(-\log h) = \frac{\sigma(v)^2(2 - \delta(v))}{h} + C(v)A_q h^{-q}. \quad (5.10)$$

The first term is nonpositive. For $r = 1 - q$, (4.3) gives

$$\begin{aligned} \mathcal{A}^c h^{1-q} &= \sigma(v)^2(1-q)[\delta(v) - 2q]h^{-q} - C(v)D_q h^{1-2q} \\ &\geq [2\sigma_-^2(1-q)^2 - C_+ D_q L^{1-q}]h^{-q} \geq \sigma_-^2(1-q)^2 h^{-q}. \end{aligned}$$

Combining this with (5.10) proves $\mathcal{A}^c F \leq 0$.

Stop $F(\bar{H})$ at its first lower-level entry into $(0, \varepsilon]$ or upper passage to L , and also localize time and the tangential state. At a lower entry its value is at least $F(\varepsilon)$, since F is decreasing. Optional stopping gives

$$\mathbb{P}_{h,v}^c\{\bar{\tau}_\varepsilon < \bar{\tau}_L\} \leq \frac{F(h)}{F(\varepsilon)}, \quad 0 < h < L.$$

A jump landing arbitrarily close to zero causes no integrability difficulty for the logarithm. Let $\varepsilon \downarrow 0$. Boundary contact before the upper level has probability zero.

From an arbitrary initial height, apply this result on each entry into $(0, L/2]$ after an upper passage to L , and at the initial entry if necessary. A finite zero contact would require an entry after which the process reaches zero before returning to L . Infinitely many crossings of the fixed interval $[L/2, L]$ cannot accumulate at a finite zero limit of a càdlàg path. The countable strong-Markov argument therefore excludes every finite zero contact. By Proposition 3.1, no other finite censored lifetime is possible. $\square$

This proof uses the accepted jump integral in (3.6). Its lower-order correction is essential at $d_- = 2$, where the leading continuous logarithmic drift can vanish. The accepted jump term therefore enters the boundary classification even at the critical threshold.

6 Curvature bounds and lifetime moments

We first optimize the subcritical collar estimate to obtain an explicit dependence on curvature. We then combine the killing representation with an upper comparison for the censored height to bound survival in all regimes covered by Assumption 2.1.

6.1 Optimizing the terminal-cause estimate

Corollary 6.1 (Curvature and eventual jump termination). *Assume (4.6). Then*

$$\mathbb{P}_{h_0, v_0}(\mathbf{J}) \leq 1 \wedge \left\{ h_0^p \frac{1-q}{1-p-q} \left[\frac{C_+(1-p-q)}{qa_p p} \right]^{p/(1-q)} \right\}. \quad (6.1)$$

The unconstrained optimal normal collar is

$$L_* = \left[\frac{qa_p p}{C_+(1-p-q)} \right]^{1/(1-q)}. \quad (6.2)$$

In particular, with h_0, v_0 and all coefficient bounds other than κ fixed,

$$\mathbb{P}_{h_0, v_0}(\mathbf{J}) = O\left(h_0^p \kappa^{qp/(1-q)}\right) = O\left(R_0^{2p} \kappa^{\alpha p/(2-\alpha)}\right). \quad (6.3)$$

The same estimate gives $\mathbb{P}(\mathbf{C}) \rightarrow 1$ as $h_0 \downarrow 0$, uniformly in v_0 .

Proof. A terminal jump either occurs before the first upper passage to L or after that passage. By (5.3) and (5.4), for $L > h_0$,

$$\mathbb{P}(\mathbf{J}) \leq h_0^p \left[L^{-p} + \frac{C_+}{qa_p} L^{1-p-q} \right].$$

Both exponents have the signs needed for a finite minimizer. Differentiation gives (6.2); substitution gives (6.1). If the displayed value is less than one, then necessarily $L_* > h_0$, so the minimizer is admissible. Otherwise the bound by one suffices. Finally $C_+ = c\kappa^q a_+^{2q}$ gives (6.3). $\square$

The probability bound gives a dynamical consequence of the exact geometric rate. A diverging instantaneous rate near $H = 0$ coexists with continuous contact of probability tending to one from sufficiently small H_0 . The occupation estimate controls the time over which that rate acts.

6.2 An explicit upper tail bound

Lemma 6.2 (A maximal estimate). *For every $T > 0$ and $M > 0$,*

$$\mathbb{P}^c \left\{ \sup_{0 \leq t < T \wedge \bar{\tau}} \bar{H}_t \geq M \right\} \leq 2^{d_+/2} \exp \left(\frac{h_0}{2\sigma_+^2 T} - \frac{M}{4\sigma_+^2 T} \right). \quad (6.4)$$

Proof. By Lemma 5.1, the supremum is bounded by $\sup_{u \leq \sigma_+^2 T} Z_u$. Fix $u > 0$ and $0 < \lambda < 1/(2u)$. The function

$$G(t, z) = [1 - 2\lambda(u - t)]^{-d_+/2} \exp \left(\frac{\lambda z}{1 - 2\lambda(u - t)} \right), \quad 0 \leq t \leq u,$$

satisfies the backward equation for squared-Bessel motion. Itô's formula makes $G(t, Z_t)$ a nonnegative local martingale. Localization and Fatou's lemma give

$$\mathbb{E} e^{\lambda Z_u} \leq (1 - 2\lambda u)^{-d_+/2} \exp \left(\frac{h_0 \lambda}{1 - 2\lambda u} \right). \quad (6.5)$$

In particular, the exponential moment is finite. Since $d_+ \geq 0$, the squared-Bessel process is a submartingale, and conditional Jensen's inequality shows that $e^{\lambda Z_t}$ is a nonnegative submartingale on $[0, u]$. Doob's maximal inequality with $u = \sigma_+^2 T$ and $\lambda = (4\sigma_+^2 T)^{-1}$ proves (6.4). $\square$

Theorem 6.3 (Lifetime moments for the full system). *Under [Assumption 2.1](#), set*

$$\gamma = \frac{1-q}{1+q}, \quad a_0 = \frac{C_-}{q}, \quad c_0 = a_0^{1/(1+q)} (4\sigma_+^2)^{-q/(1+q)}. \quad (6.6)$$

For every $T > 0$,

$$\mathbb{P}_{h_0, v_0}(\zeta > T) \leq \left[1 + 2^{d_+/2} \exp\left(\frac{h_0}{2\sigma_+^2 T}\right) \right] \exp(-c_0 T^\gamma). \quad (6.7)$$

Consequently $\zeta < \infty$ almost surely and $\mathbb{E}\zeta^m < \infty$ for every $m > 0$. If $d_- \geq 2$, then $A_\infty = \infty$ under the censored law, and original termination is by a jump almost surely.

Proof. On $\{T < \bar{\tau}, \sup_{t \leq T} \bar{H}_t \leq M\}$,

$$A_T \geq a_0 T M^{-q}.$$

Split the expectation in (3.7) according to this maximum and use [Lemma 6.2](#). For every $M > 0$,

$$\mathbb{P}(\zeta > T) \leq e^{-a_0 T/M^q} + 2^{d_+/2} \exp\left(\frac{h_0}{2\sigma_+^2 T} - \frac{M}{4\sigma_+^2 T}\right).$$

Take $M = (4\sigma_+^2 a_0)^{1/(1+q)} T^{2/(1+q)}$. The two negative exponents agree and give (6.7). For T bounded away from zero its prefactor is bounded, so integrating $mT^{m-1}\mathbb{P}(\zeta > T)$ proves all positive lifetime moments.

When $d_- \geq 2$, [Theorem 5.3](#) gives $\bar{\tau} = \infty$. Hence (3.7) becomes $\mathbb{E}^c e^{-A_T}$. The tail bound tends to zero; monotone convergence implies $e^{-A_\infty} = 0$ almost surely. There is no censored boundary contact, so [Proposition 3.2](#) assigns all terminal probability to a jump. $\square$

Remark 6.4 (The stable density at large marks). The all-scale stable density in (2.1) gives the lower rate $a_0 h^{-q}$ at every height, including large h . This lower rate is used in [Theorem 6.3](#). For bounded marks, the killing rate can vanish at sufficiently large heights. The assumption at all mark sizes is thus used specifically in the global lifetime estimate. The distinction between the continuous order h^{p-1} and the quadratic-jump order h^{p-q} determines the local barrier and occupation arguments.

6.3 Flattening the same constraint

Keep the coefficient functions and the initial discriminant fixed and let $\kappa \downarrow 0$. In the limit $h = U$, the tangent marks act only on V and the normal loss disappears. Under $d_+ < 2$, the comparison in [Lemma 5.1](#), applied with zero jump loss, gives almost sure continuous contact. For positive κ , both terminal causes occur, and (6.3) quantifies the approach to the flat continuous-contact law.

Under $d_- \geq 2$, the flat model has no boundary contact, as follows from the same logarithmic barrier with $C_+ = 0$. It retains its branch for all finite time. For every $\kappa > 0$, however, [Theorem 6.3](#) gives finite lifetime and almost sure jump termination. Thus the vanishing-curvature limit distinguishes persistence of the flat branch from eventual loss of a curved branch. The constant c_0 in (6.6) tends to zero with κ , so the survival estimate degenerates in the flat limit.

Completion of [Theorem 2.3](#). Existence and the terminal alternatives follow from [Proposition 3.1](#). The subcritical claims are [Theorem 5.2](#) and [Corollary 6.1](#). The critical and supercritical claims follow from [Theorems 5.3](#) and [6.3](#). The final theorem also gives all lifetime moments. Each result is uniform in v_0 because its constants use only the bounds in [Assumption 2.1](#). $\square$

7 Numerical calculations

7.1 Coefficients and analytic bounds

We use the tangentially varying family in [Example 2.4](#), with

$$\begin{aligned}\alpha &= 1.2, & c &= 0.12, & b(v) &= -v, & \eta(v) &= 0.4, \\ \sigma(v) &= 1 + 0.15 \tanh v, & a(v) &= 1 + 0.10 \tanh v, \\ \delta(v) &= \delta_0 + 0.20 \tanh v.\end{aligned}$$

The three choices $\delta_0 = 1.2, 2.2, 3$ have, respectively, $d_+ = 1.4$, $d_- = 2$, and $d_- = 2.8$. In particular, the first lies uniformly below the threshold and the last two satisfy the noncontact theorem. In the middle case, the infimum of δ is the critical value two, approached as $v \rightarrow -\infty$.

Nine independent quadrature evaluations of (4.4), including negative powers and $q = 0.85$, differ from the gamma expression by at most 1.93×10^{-13} . The code also evaluates the complete power and critical-barrier generators at 134226 pairs of normal and tangential states. These finite evaluations check the formulas used in the implementation; the uniform sign inequalities follow from their proofs.

Table 1 evaluates [Corollary 6.1](#) with $p = 0.2$. These analytical bounds apply to the infinite-activity process over its entire lifetime. Figure 1 displays the same expression over a continuous range of κ .

Table 1: Analytical bounds on eventual terminal causes, with $\delta_0 = 1.2$ and $p = 0.2$. The initial discriminant is held fixed as curvature varies.

h_0	κ	Jump upper bound	Contact lower bound	L_*
10^{-4}	0.01	0.22178	0.77822	5.9634
10^{-4}	0.0625	0.38432	0.61568	0.38166
10^{-4}	0.25	0.58252	0.41748	0.047707
10^{-4}	1	0.88294	0.11706	0.0059634
10^{-8}	0.01	0.03515	0.96485	5.9634
10^{-8}	0.0625	0.06091	0.93909	0.38166
10^{-8}	0.25	0.09232	0.90768	0.047707
10^{-8}	1	0.13994	0.86006	0.0059634

7.2 Joint-path calculations and numerical resolution

Table 2 reports a finite-cutoff, frozen-coefficient approximation of the same joint process. All groups start at $(h_0, v_0) = (0.05, 0)$, use horizon $T = 1$, and contain 20000 independent paths. Different rows use independent random seeds. Absolute arrival times of retained marks are sampled, and both V and H are updated using that same mark. The killed Bessel kernel handles contact within a frozen interval. [Appendix B](#) gives the precise construction.

For example, at $\delta_0 = 1.2$, $\rho = 0.02$ and $\Delta = 1/256$, increasing κ from 0.0625 to 0.25 and then to 1 gives estimated jump probabilities 0.04950, 0.10700 and 0.21525. Their respective Monte Carlo standard errors are 0.00153, 0.00219 and 0.00291. These estimates illustrate the variation of finite-horizon jump termination with curvature in the family (2.16).

The subcritical rows show the dependence on the mark cutoff and the maximum time step. In the critical and supercritical rows, the frozen Bessel kernels assign probability zero to continuous contact. The positive survival probabilities at $T = 1$ describe the finite-horizon distribution; eventual termination is governed by [Theorem 6.3](#).

The package saves all 200000 path records and their parameter files. The accompanying tests check the fractional coefficients, generator decomposition, jump geometry, curvature bounds,

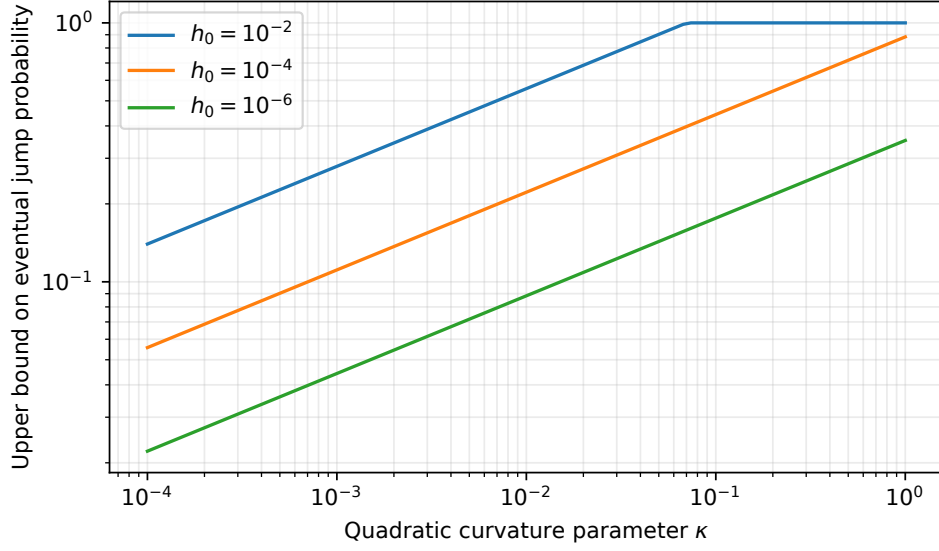


Figure 1: The upper bound (6.1), evaluated as a function of curvature for fixed initial discriminants.

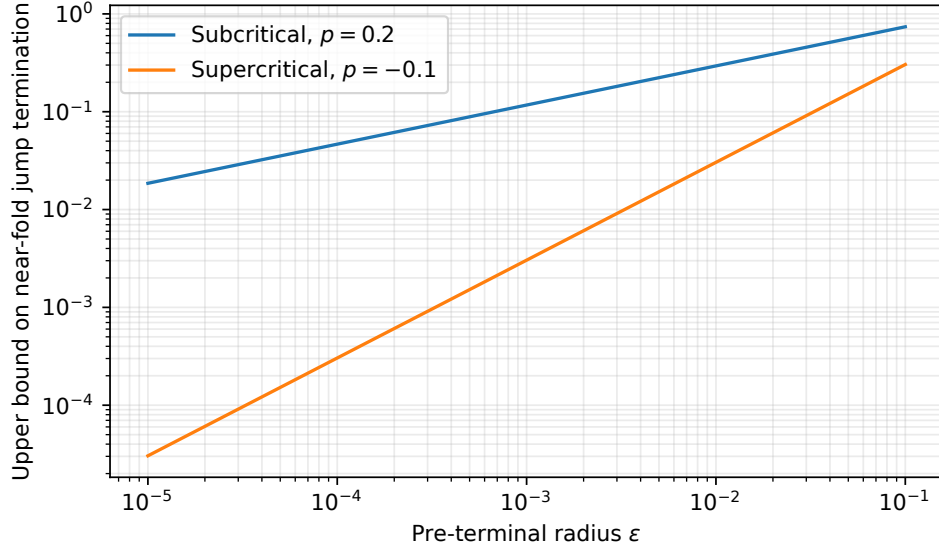


Figure 2: The bound (4.10) on termination at a mark whose pre-jump radius is at most ε , with $h_0 = 0.05$ and $\kappa = 0.25$. The positive and negative powers concern the respective full jump laws.

Table 2: Approximate probabilities at $T = 1$. Here ρ is the mark cutoff and Δ the maximum coefficient-freezing step. Each row sums to one and has 20000 paths.

δ_0	κ	ρ	Δ	Contact	Jump	Survival
1.2	0.25	0.08	1/128	0.69685	0.09860	0.20455
1.2	0.25	0.04	1/128	0.69185	0.10195	0.20620
1.2	0.25	0.04	1/256	0.69605	0.10395	0.20000
1.2	0.25	0.02	1/256	0.68910	0.10700	0.20390
2.2	0.25	0.04	1/256	0.00000	0.18135	0.81865
3	0.25	0.08	1/128	0.00000	0.13450	0.86550
3	0.25	0.04	1/256	0.00000	0.13060	0.86940
3	0.25	0.02	1/256	0.00000	0.12520	0.87480
1.2	0.0625	0.02	1/256	0.71500	0.04950	0.23550
1.2	1	0.02	1/256	0.63795	0.21525	0.14680

critical barriers, Bessel formulas and terminal-cause counts. The data files report sampling standard errors and bounds on the omitted small-mark input.

8 Conclusion

A tangent jump at the base of an algebraic fold can have a nonzero second-order normal effect. In the parabolic model that effect determines the normal Lévy kernel exactly. The same kernel enters the occupation estimate, the censored boundary classification, and the distribution of terminal causes. The coefficient conditions distinguish finite censored contact with finite cumulative killing from conservative censoring with eventual jump termination. The critical case is included through a logarithmic barrier, while an explicit survival bound gives finite moments of every positive order.

The classification is uniform along the fold, including the critical lower bound. For dimension coefficients that cross the critical value along the fold, a further classification must account for the tangential dynamics. General curved constraints introduce remainders into the normal jump kernel, for which the parabolic calculation provides an explicit reference. The lifetime ends when the selected regular branch ceases to exist.

A Tangent kicks for general graph folds

Let $\phi \in C^3(\mathbb{R})$ and consider $g(u, v, y) = y^2 + \phi(v) - u$. Replace the tangential drift, diffusion and jump coefficients in the U equation by

$$\phi'(V)b(V) + \frac{1}{2}\phi''(V)\eta(V)^2 + \delta(V)\sigma(V)^2, \quad \phi'(V)\eta(V), \quad \phi'(V_-)dJ_t^a,$$

respectively, retaining the normal diffusion $2\sigma(V)Y dW$. Itô's formula gives the normal loss

$$D_\phi(v, z) = \phi(v + a(v)z) - \phi(v) - \phi'(v)a(v)z. \quad (\text{A.1})$$

It is nonnegative when ϕ is convex. Uniformly on tangential compact sets,

$$D_\phi(v, z) = \frac{1}{2}\phi''(v)a(v)^2z^2 + O(|z|^3).$$

Proposition A.1. *Suppose ϕ'' is strictly positive on a neighbourhood of a compact tangential set K . With the stable mark law (2.1), uniformly in $v \in K$,*

$$\int \mathbf{1}_{\{D_\phi(v, z) \geq h\}} \nu(dz) \sim \frac{c}{q} \left(\frac{\phi''(v)}{2} \right)^q a(v)^{2q} h^{-q}, \quad h \downarrow 0. \quad (\text{A.2})$$

Proof. For any $\epsilon \in (0, 1)$ a fixed sufficiently small mark neighbourhood satisfies

$$(1 - \epsilon)^{\frac{1}{2}} \phi''(v) a(v)^2 z^2 \leq D_\phi(v, z) \leq (1 + \epsilon)^{\frac{1}{2}} \phi''(v) a(v)^2 z^2$$

uniformly in $v \in K$. Integrating the two corresponding quadratic thresholds gives the proposed constant multiplied by $(1 \pm \epsilon)^q$, up to bounded errors. Marks outside the neighbourhood have finite total intensity. Divide by h^{-q} , let $h \downarrow 0$, and then let $\epsilon \downarrow 0$. $\square$

For $\phi(v) = \kappa v^2$, the remainder vanishes at every mark size. This exact identity permits the global Mellin calculation and the large-height killing estimate in the main text. For a general graph, (A.2) describes the local exit rate near the fold.

The jump map specifies how a physical impulse follows the constraint. Replacing the first-order tangent increment $\Delta U = \phi'(v) a(v) z$ by the finite displacement $\Delta U = \phi(v + a(v) z) - \phi(v)$ preserves H exactly. These are distinct jump mechanisms. The model studied here uses the first-order tangent impulse.

B Numerical formulas and reproducibility

B.1 Independent evaluation of the power coefficient

For $p > q - 1$ the real integral for $m_q(p)$ is evaluated by subtracting its first-order term at zero:

$$m_q(p) = \frac{1}{q} + \frac{p}{1 - q} - \int_0^1 [(1 - u)^p - 1 + pu] u^{-1-q} du.$$

A quartic expansion is used only for the subtracted numerator at very small u . Adaptive quadrature on two subintervals handles the integrable singularity at $u = 1$ when $p < 0$. This value is compared with the gamma-function expression in Lemma 4.1.

B.2 Killed transitions on frozen normal intervals

For frozen dimension $d \in [1, 2)$ and variance coefficient s^2 , over an interval Δ first draw the reflecting squared-Bessel endpoint

$$H' = s^2 \Delta \chi_d'^2 \left(\frac{H}{s^2 \Delta} \right).$$

Conditional on the two positive radial endpoints, the probability of no zero contact is

$$\frac{I_{1-d/2}(z)}{I_{d/2-1}(z)}, \quad z = \frac{\sqrt{HH'}}{s^2 \Delta}.$$

This follows by dividing the killed transition kernel by the reflecting kernel [12, 10]. For $d \geq 2$ this probability is one. Scaled modified Bessel functions and the connection formula

$$I_{-\eta}(z) = I_\eta(z) + \frac{2}{\pi} \sin(\pi \eta) K_\eta(z)$$

avoid division of two exponentially growing quantities [9, Eq. 10.27.2]. The code returns the killed state when an independent uniform variate exceeds the conditional survival probability. For continuous contact, the recorded diffusion interval brackets the first-contact time.

For the retained mark process $|z| \geq \rho$, the arrival intensity is $2c\rho^{-\alpha}/\alpha$. Absolute arrival times are sampled from exponential waiting times and included in the time partition. The tangential Ornstein-Uhlenbeck transition is exact on each interval. The normal coefficients are frozen at the left endpoint, and the shared mark updates both coordinates. Accepted marks preserve positivity precisely when $H > \kappa a(V)^2 z^2$.

There are two approximation parameters. The time increment freezes the variable coefficients. The mark cutoff omits a small-jump tangential martingale and a finite-variation normal loss. Per unit time,

$$\int_{|z|<\rho} z^2 \nu(\mathrm{d}z) = \frac{2c}{2-\alpha} \rho^{2-\alpha}, \quad \int_{|z|<\rho} \kappa a(v)^2 z^2 \nu(\mathrm{d}z) \leq \frac{2c\kappa a_+^2}{2-\alpha} \rho^{2-\alpha}.$$

Multiplying the first expression by $a(v)^2$ gives the predictable quadratic variation per unit time of the omitted tangential martingale at state v ; the second expression bounds the omitted normal loss per unit time. The simulation approximates the joint dynamics through mark truncation and coefficient freezing. Its reported Monte Carlo standard errors quantify sampling uncertainty under that approximation; they exclude truncation and coefficient-freezing bias in stopping probabilities.

B.3 Computational files

The analytical formulas are implemented in `anc/code/fold_model.py`. Joint paths are generated by `anc/code/simulate_fold.py`. Reproduction and tests use `anc/code/reproduce.py` and `anc/tests/test_fold.py`. The reproduction script writes the tables and figures used here. Each simulated configuration has a CSV file containing terminal causes, contact-time intervals, pre-jump discriminants, terminal marks and tangential states, together with a JSON parameter record. The script also writes all values used in the quadrature and generator checks.

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