

Arithmetic Expression Geometry 0

Arithmetic Expression Spaces: Expansion and Motion, Eikonal Equation, Horocycle
Pullbacks, Commutativization, Contact Structure, and Tearing

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Draft notice. This manuscript is a working draft in the editorial and publication sense; here “draft” is not a mathematical claim-status label and does not alter the status of any theorem, proposition, computation, conjecture, or open problem. It replaces the earlier prelude of the same number and absorbs the geometric chapters of Paper I; the migration is recorded in `governance/00b-paper-0-geometric-foundation-amendment.md`.

Abstract

An arithmetic expression is a word of operations; a motion is a flow on a manifold. For expressions whose evaluation has a single internal spine, the two are one structure: affine for addition and multiplication, projective once the second operand slot is divided, contact-geometric in the order defect. A regular arithmetic expression space (AES) is an oriented Riemannian surface (M, g) with a smooth assignment $a: M \rightarrow \mathbb{R}$ satisfying the eikonal equation $|\nabla a|_g^2 = \mu^2 + \lambda^2 a^2$; the four operations then act as motions translating and scaling a . In the basic model $\mathfrak{E}_0$, the upper half-plane with the left-invariant metric of $\text{Aff}^+(1, \mathbb{R})$ (curvature $-\lambda^2$), addition and positive multiplication are affine translations and dilations obeying a Baumslag–Solitar law, and a is a Laplace–Beltrami eigenfunction of eigenvalue $2\lambda^2$. Left-expanded words stay in the affine Borel cell fixing ∞ ; the right-expanded language, once second-slot division is admitted, generates all of $\text{PGL}_2(\mathbb{R})$ through the rank-one Bruhat decomposition, and pulling standard horocycles back along one operator word gives the dual reading, ripple geometry. Commutativizing the motions by total charge yields the accumulative commutative space, where the order defect of two histories is an exact weighted area, the tearing. The same defect is the constant $\mu\lambda$ in the bracket $[D_u, D_v] = \mu\lambda\partial_a$ of the horizontal fields of $\alpha = da - \mu du - \lambda a dv$, a contact form whenever $\mu\lambda \neq 0$, and it is the curvature of that horizontal connection; for $\mathfrak{E}_0$ the basic model is proved to be the Reeb quotient of the contact model, and fixed-charge tearing and fixed-state

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Reeb drift are Legendre dual readings of the same curvature. The generators ∂_a , $a\partial_a$, and $a \log a \partial_a$ span an infinite-dimensional Lie algebra, so no finite-dimensional Lie group carries all three ranks, and no observable of the charge endpoint detects tearing. For every finite puncture count a conformally compactifiable punctured AES is constructed, with an explicit four-circle zero locus and exact reconstruction from three-ended pants blocks. All results are elementary, planar, or local to a three-dimensional contact model.

Keywords: arithmetic expression geometry; eikonal equation on surfaces; positive affine group; Baumslag–Solitar relation; hyperbolic upper half-plane and horocycles; affine cocycles; contact structures and Reeb quotients; Legendre duality; punctured Riemann surfaces and pants decompositions; infinite-dimensional Lie algebras

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1 Introduction

The earliest presentations of Arithmetic Expression Geometry began with a simple picture: a nested arithmetic expression can be read as a sequence of updates of one current value, and that sequence can be drawn as a path. This picture was developed for one comb orientation. The opposite comb orientation was visible in the syntax but was not given an equally explicit geometry. Division shows that the omission is substantive. Dividing the evolving value by a constant remains affine, whereas dividing a constant by the evolving value introduces inversion, a pole, and the projective point at infinity.

This paper develops the geometry that the picture requires. It is the first paper of the series, and it is the canonical source for three things that later papers import: the space in which arithmetic operations act as motions, the dual reading of those motions, and the commutativization against which their order defect is measured. The order of the chapters follows the order in which the objects are needed.

Expansions of an expression. Section 2 fixes the syntax. A binary expression whose internal operations have a unique temporal order can be written in two extreme ways: the active subtree occupies the first operand slot, or the second. We call these the left- and right-expanded combs, and we prove the two elementary facts on which everything else depends: each has a unique internal evaluation order, and the two grammars are exchanged by passing to opposite operations. The chapter then isolates the distributive law, which is the one algebraic identity that relates two *different* expansions of the same expression, and shows what it does geometrically: distributivity moves additive charge through multiplicative charge, and it therefore produces two expansions with the same operator and different charges. That observation is the seed of the tearing discussed in Section 8. Mechanical expansion of expressions, in the tradition of Wu Wen-tsun’s mathematics mechanization [16], uses this law as a means of eliminating the process; here the expansions themselves are the object of study.

Arithmetic expression spaces and arithmetic motion. Section 3 introduces the geometric object of the series. An arithmetic expression space (AES) carries an assignment function, whose value at a point is the current arithmetic value, and a metric, which measures the cost of moving. A *regular* AES is exactly a pair (g, a) on an oriented surface in which the two arithmetic directions are the two summands of one eikonal equation, $|\nabla a|_g^2 = \mu^2 + \lambda^2 a^2$ (Definition 3.1); that single identity determines the canonical oriented frame in which the motions are expressed, and it is what converts an algebraic choice of operations into a Riemannian structure. The four operations then act as *motions*: addition and subtraction move the value by a constant, multiplication and division scale it, and the motions that realize them transport the assignment in exactly that way. The basic model $\mathfrak{E}_0$ is the upper half-plane with the invariant metric of the positive affine group, of curvature $-\lambda^2$; its additive and multiplicative motions are affine maps of the plane that satisfy an elementary Baumslag–Solitar-type relation, which says that a multiplicative motion transports the additive ruler, and its assignment is a Laplace–Beltrami eigenfunction, $\Delta_{g_{\mu,\lambda}} a = 2\lambda^2 a$ (Proposition 3.5). The chapter also gives the continuous form of the motion, its rectification, and the first model with a puncture, $\mathfrak{E}_1$, which is a punctured disc whose metric extends across the puncture while its assignment does not. One subsection (Subsection 3.6) then asks whether a composed arithmetic word also determines a *length*, and answers in the affine sector: the multiplicative coordinate is the translation length of the induced motion up to sign, the additive coordinate is not a function of that motion at all, and the assignment to lengths is consequently not a homomorphism.

The projective dual. Section 4 reads the same operator word in the opposite direction. Instead of pushing one input forward, pull an entire output condition backward; for a fractional linear map the pullback of a standard horocycle is a line or a circle tangent at the operator’s pole, and the family of such pullbacks is what we call a ripple pencil. Taken as a whole — pole, pulled-back family, and the fixed points of the same matrix — we call this dual reading *ripple geometry*. The name is descriptive and provisional: the constructions are proved, their interpretation as a geometry intrinsic to the arithmetic word is not, and Section 4.5 states exactly what is and is not claimed. Sections 5 and 6 complete the elementary picture, treating reciprocals, poles, infinity, and iterative fixed points, and then placing paths, poles, fixed points, and ripple pencils in one matrix representation.

Commutativization, tearing, and contact. Section 7 computes the two translation coordinates of an affine history, and Section 8 constructs the commutativization of the arithmetic motions: keep the total additive and multiplicative charges of a history, forget the order in which they were accumulated, and the result is a plane — the accumulative commutative space, ACS. Two histories with the same charge bound a region of that plane, and the exact value of their order defect is a weighted area of that region. We read this as a statement about the AES: the space tears by that amount, where *tearing* names a defect that is invisible in either motion separately and becomes an area only in the commutativization (Definition 8.10). Section 9 gives the same defect a local form. The three coordinates (additive direction, multiplicative direction, current value) carry a contact form whose kernel is a non-integrable horizontal distribution, and the bracket of the two horizontal fields is the constant $\mu\lambda$ that also appears as the density of the tearing. This is the sense in which the contact structure describes the tearing rather than merely accompanying it: it shows that no local calibration can make both arithmetic directions straight at once. Two further readings of the same three-dimensional carrier are proved in that chapter. The arithmetic flow is the horizontal lift of an Ehresmann connection over the direction plane, and the constant $\mu\lambda$ is its curvature (Proposition 9.6); and the two quotients of the carrier — projecting out the assignment, and quotienting along the Reeb field — are Legendre conjugate, so that a fixed-charge comparison and a fixed-state comparison of the same pair of histories measure one curvature on two dual sides (Theorem 9.9).

Beyond linear language. Section 10 is methodological, and its two propositions are the technical reason for it. The three generators of the arithmetic tower, $A = \partial_a$, $M = a\partial_a$, and $P = a \log a \partial_a$ — translation, scaling, and exponentiation of the assignment — do not close: the iterated brackets $\text{ad}_A^n(P)$ are the independent fields $a^{1-n}\partial_a$, so the generated Lie algebra is infinite-dimensional and no finite-dimensional group carries all its motions (Proposition 10.1); and every observable of the commutativized charges agrees on charge-compatible histories, so no such observable can see tearing at all (Proposition 10.2). We therefore use manifolds, metrics, and Lie groups where they are proved to apply, and we do not adopt them as the ontology of the subject. The chapter records the alternative vocabulary that a companion programme, process geometry, has begun to develop, together with the exact calibrations in which that programme meets the laws proved here. The proposals are labelled as proposals.

Holed spaces and the language of tearing. Section 11 develops the punctured models: a definition of punctured arithmetic expression spaces, an exact computation of the local shape of a puncture, and a register of six computed witnesses about punctured configurations, each with its own status. Their interpretive reading — that a hole is where a comparison must be performed — is stated as a motivation and not as a theorem, and the list of what the witnesses do not establish

is part of the chapter.

Status boundary. Everything proved in this paper is elementary algebra, upper-half-plane geometry, one-dimensional dynamics, or a contact computation on a three-dimensional local model. Ordinary division remains a partial operation even when its Möbius continuation is well defined on the projective line. No convergence statement about infinite expressions is made beyond explicit analytic or formal hypotheses. The paper does not classify sequential trees, does not develop function theory, and does not classify singular zero sets; those belong to Paper I, Paper II, and Paper III respectively, and Section 13 records the interfaces. The migration that gives this paper its present scope — the geometric chapters of Paper I now developed here — is recorded in the amendment `governance/00b-paper-0-geometric-foundation-amendment.md`.

Finite-type compactification and interfaces. Section 12 extends the geometric foundation to every positive finite puncture count and reconstructs the original AES from three-ended blocks with full collar contracts. Theorem 12.3 is an existence result, not a classification by puncture number. Proposition 12.7 shows why an assignment identity need not be a full-state identity. The port and program-substitution layer belongs to Paper IV; Appendix E records the precise circle and branched-sheet comparisons motivating this extension.

2 Arithmetic Expansions and the Distributive Law

Let K be a field and let

$$\Omega = \{+, -, \times, /\}$$

be the ordinary binary arithmetic signature. Division is partial in the affine chart: a denominator must be nonzero at the moment at which the operation is performed.

An arithmetic expression is a planar binary tree. When its internal vertices form a single chain, its operations acquire a unique temporal order, and the expression can be written in two extreme ways: the already-computed subtree may occupy the first operand slot or the second. This section fixes that convention, proves the elementary facts on which every later chapter depends, and isolates the one algebraic law that relates the two expansions of a common expression.

2.1 Chronological comb conventions

Definition 2.1 (Left- and right-expanded one-hole expressions). Fix constants $c_1, \dots, c_n \in K$ and operation labels $\omega_1, \dots, \omega_n \in \Omega$. Starting from the hole z , define recursively

$$L_0[z] = z, \quad L_k[z] = \omega_k(L_{k-1}[z], c_k), \quad (1)$$

$$R_0[z] = z, \quad R_k[z] = \omega_k(c_k, R_{k-1}[z]). \quad (2)$$

The expression L_n is a *left-expanded comb*; R_n is a *right-expanded comb*. The index k is the evaluation time: the innermost operation has index 1 and is evaluated first. These recursions first define syntax. Ordinary evaluation is a partial interpretation of that syntax: a history is admissible at an input only if every operation is defined at its actual operands, including all intermediate denominators.

Convention 2.2 (Scope of the words “left” and “right”). Throughout this paper, “left-expanded” and “right-expanded” refer only to the recursion (1)–(2): the active subtree occupies operand slot (1) in the first case and operand slot (2) in the second. The index is chronological. Every later use of the words must be reducible to (1)–(2); the planar direction in which a tree happens to be drawn is not part of the convention.

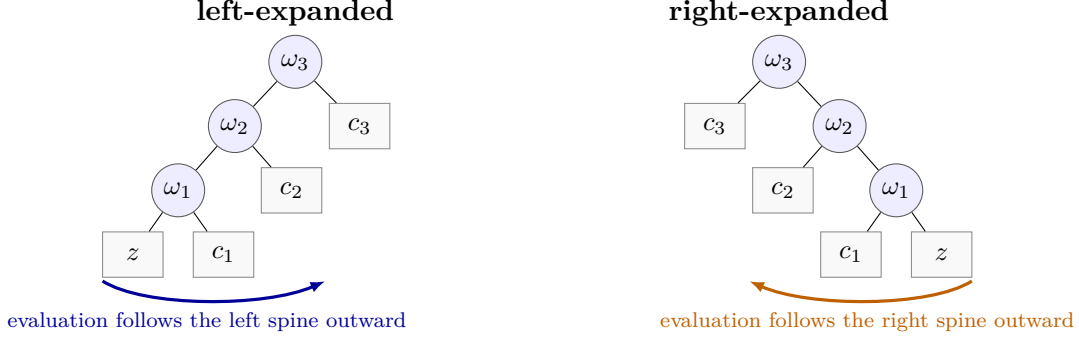


Figure 1: The two pure combs under the chronological indexing convention. Both have one chain of internal dependencies; the active subtree occupies opposite operand slots.

Thus, in fully parenthesized form,

$$L_n[z] = \omega_n(\cdots \omega_2(\omega_1(z, c_1), c_2) \cdots, c_n),$$

whereas

$$R_n[z] = \omega_n(c_n, \omega_{n-1}(c_{n-1}, \cdots \omega_1(c_1, z) \cdots)).$$

In the first expression the active subtree is the left child at every internal vertex. In the second it is the right child.

Proposition 2.3 (Unique internal order). *Every finite left- or right-expanded comb has a unique legal order of internal evaluation, namely the chronological order $1, 2, \dots, n$.*

Proof. At stage k , the only internal child of the k th vertex is the subtree formed at stage $k - 1$; the other child is the leaf c_k . Hence the internal vertices form one dependency chain

$$v_1 < v_2 < \cdots < v_n.$$

A finite chain has a unique linear extension. □

This proposition is deliberately weaker than the full sequential-tree classification of Paper I. Here the combs are given by explicit grammars; Paper I proves intrinsically that every binary expression tree with a unique internal evaluation order has one such spine after operand slots are recorded.

2.2 One-hole sections and opposite operations

For $\omega \in \Omega$ and $c \in K$, write

$$C_{\omega, c}^{(1)}[z] = \omega(z, c), \quad C_{\omega, c}^{(2)}[z] = \omega(c, z).$$

Then L_n is a chronological word of pure slot-(1) sections and R_n a word of pure slot-(2) sections.

Define the opposite binary operation by

$$\omega^{\text{op}}(u, v) := \omega(v, u).$$

Proposition 2.4 (Comb opposition). *With chronological indexing, a right-expanded comb for operations $\omega_1, \dots, \omega_n$ is exactly a left-expanded comb for the opposite operations:*

$$R_k[z] = \omega_k^{\text{op}}(R_{k-1}[z], c_k).$$

Consequently, the two comb grammars are isomorphic after replacing every operation by its opposite and exchanging the operand slots. Their planar trees retain those slots. For commutative operations with symmetric domains, their partial operator interpretations agree; this does not identify the planar trees or the marked histories.

Proof. By definition,

$$\omega_k(c_k, R_{k-1}[z]) = \omega_k^{\text{op}}(R_{k-1}[z], c_k).$$

Apply this identity at every stage. If ω_k is commutative, then $\omega_k^{\text{op}} = \omega_k$ as partial operations, including their domains. Interchanging the slots is the corresponding mirror of the planar comb. It changes syntax even when it preserves the interpreted map. $\square$

The result is combinatorial, not an assertion that the two arithmetic maps are always equal. Their elementary one-hole maps are:

Operation	active value in slot (1)	active value in slot (2)
addition	$z + c$	$c + z = z + c$
subtraction	$z - c$	$c - z$
multiplication	cz	cz
division	z/c	c/z

The first-slot division z/c is defined on all of K if $c \neq 0$ and nowhere if $c = 0$. Second-slot division c/z has ordinary domain $K \setminus \{0\}$ for every c , including $c = 0$: cancelling a zero numerator does not restore the missing input. Every division guard belongs to the original history. A later simplification or multiplication by zero cannot make an earlier undefined division admissible.

All first-slot maps other than division are affine self-maps; first-slot division is affine when its constant denominator is nonzero. An affine self-map $az + b$ belongs to the affine group only when $a \neq 0$. Thus multiplication by zero is a permitted arithmetic expression and a constant affine self-map, but not an invertible affine motion. Second-slot subtraction has multiplier $-1 \neq 0$ in any field. For $c \neq 0$, second-slot division has an invertible projective continuation with pole 0; it sends 0 to ∞ and ∞ to 0. When $c = 0$, the ordinary partial map is zero on its domain and has no invertible Möbius continuation. The projective statement does not declare an ordinary division by zero admissible.

Corollary 2.5 (Where the algebraic symmetry breaks). *If the operation alphabet is restricted to $+$ and $\times$, pure left and pure right expansion have the same one-hole operator algebra of affine self-maps. Restricting multiplicative constants to be nonzero gives affine group elements; addition and subtraction preserve this group boundary. First-slot division with nonzero constant denominator also remains inside that group. Once second-slot division c/z with $c \neq 0$ is admitted and projectively continued, the right-expanded language leaves the affine stabilizer of infinity. It contains inversion itself when $c = 1$ is allowed.*

Proof. The first two assertions follow from the table. For $c \neq 0$,

$$\frac{c}{z} = D_c \circ J(z), \quad D_c(z) = cz, \quad J(z) = \frac{1}{z}.$$

The map J exchanges 0 and ∞ , so it does not stabilize infinity. In homogeneous coordinates the same transformation is

$$[X : Y] \mapsto [cY : X], \quad \begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix}, \quad \det = -c \neq 0.$$

For $c = 0$, the representative $(0, 1)$ is sent to $(0, 0)$, so no projective transformation is defined by this matrix. The nonzero condition is therefore necessary. The non-affine assertion concerns this projective transformation, not interpolation of its values on a finite subset of K . $\square$

2.3 The distributive law as an expansion rewrite

Two different expansions may induce the same arithmetic operator. The law used here to relate two such expansions is distributivity, and it is worth isolating because every later chapter compares expansions that share an operator or a value. Other ring identities (associativity, the unit laws) also relate expansions with a common operator; what is special about distributivity is that it moves an additive step across a multiplicative one, which is why it, and not the others, produces the charge transport computed in Example 2.8.

Definition 2.6 (Distributive expansion). Let $p, k \in K$. The *distributive expansion* of the two-step slot-(1) word

$$z \mapsto k(z + p) = (kz + kp) \tag{3}$$

is the slot-(1) word obtained by reading the right-hand side as a multiplication followed by a translation:

$$z \mapsto kz \mapsto kz + kp.$$

The two words have the same length, the same operation alphabet, and the same induced operator $z \mapsto kz + kp$ on K . Their marked chronological data are respectively

$$((+, 1, p), (\times, 1, k)), \quad ((\times, 1, k), (+, 1, kp)),$$

where the middle entry records the active operand slot. Forgetting operation labels and slots gives the *chronological scalar projection* (p, k) and (k, kp) . This projection is not a complete marked history; it also differs from grouping constants by their additive or multiplicative role. A zero multiplicative constant is allowed here as arithmetic data, without making the associated constant map an affine group element.

Proposition 2.7 (Distributivity is operator-preserving and expansion-changing). *For all $p, k \in K$ the induced operators of the two words in (3) agree:*

$$(z \mapsto z + p) \text{ followed by } (z \mapsto kz) = (z \mapsto kz) \text{ followed by } (z \mapsto z + kp). \tag{4}$$

The two marked words are distinct for every p, k , since their first operation labels differ. Their chronological scalar projections coincide precisely when

$$(p, k) = (0, 0) \quad \text{or} \quad (p, k) = (1, 1).$$

In particular, if $p \neq 0$ and $k \neq 1$, then the two expansions share their operator and differ even after projecting to chronological scalar data.

Proof. Both sides send z to $kz + kp$, which is the distributive law in K . Marked-word equality would require $+ = \times$ at the first position in the operation signature, which is impossible, independently of any accidental equality of interpreted maps over a particular field. Equality of the scalar projections requires $p = k$ and $k = kp$. If $k = 0$, the first condition gives $p = 0$. If $k \neq 0$, cancellation in $k \cdot 1 = k \cdot p$ gives $p = 1$, and then $k = 1$. Conversely both listed pairs satisfy the two equations. The final assertion excludes both cases. Thus operator equality, scalar-projection equality and marked-history equality are separate judgments. $\square$

Example 2.8 (Distributivity transports the additive charge). Take $K = \mathbb{R}$, $p = 1$, $k = e^q$ with $q \neq 0$, and compare the two words of Proposition 2.7. Both induce $z \mapsto e^q z + e^q$. Their accumulated charges, in the sense fixed in Section 8, are

$$(A, M) = (1, q) \quad \text{and} \quad (A, M) = (e^q, q).$$

The multiplicative charge is the same because the two words differ only in where the additive step is performed; the additive charge differs by exactly the scale e^q that separates the two positions. In the notation of Definition 8.1, the weighted integrals agree,

$$\int_{C_\gamma} e^{-M} dA = 1 \cdot e^0 = 1, \quad \int_{C_\delta} e^{-M} dA = e^q \cdot e^{-q} = 1,$$

so that Proposition 8.2 returns the common value $e^q(x + 1)$ along both charge paths.

The two histories of Example 2.8 share their motion and differ in their charge. The difference is an instance of the transport elements of Theorem 8.6: the chronological word of γ followed by the inverse of δ evaluates to the identity motion while carrying additive charge $p(1 - e^q)$, which is the general phenomenon that section names and measures.

Example 2.8 is the elementary core of the later discussion of tearing. Distributivity does not merely reorganize symbols: it moves additive charge through multiplicative charge, and the moved amount is exactly the scale accumulated in between. An algebraic identity of operators therefore becomes a statement about two different charge histories.

Remark 2.9 (Where expansion stops). Distributivity is a law of the ring $(\times, +)$ and not a universal law of the whole signature Ω . Its analogues for the two remaining operations behave differently, and the difference is instructive.

1. For slot-(2) subtraction,

$$(c - z)k = kc - kz = (-k)z + kc.$$

This is the distributive law again, and the transported constant is kc ; the resulting one-hole map is still affine, but its multiplier is $-k$. Expansion therefore remains inside the affine self-maps; its map belongs to the affine group when $k \neq 0$.

2. For slot-(2) division,

$$\frac{c}{z}k = \frac{kc}{z}, \quad z \neq 0.$$

This is an identity of partial functions, not an expansion: the right-hand side has the original excluded input $z = 0$, even if $kc = 0$. If $kc \neq 0$, its projective continuation is inversion followed by an invertible scaling, outside the affine group. If $kc = 0$, it is a constant partial map with the same excluded input, not an invertible projective generator. It is already an ordinary slot-(2) comb; what requires a generator outside the affine group is the non-degenerate projective continuation.

Consequently the invertible expansion calculus of this section is an affine calculus, and the non-degenerate projective step c/z , $c \neq 0$, is where that group boundary ends. Section 4 reads that step contravariantly instead, as a condition on outputs rather than a motion of the value.

2.4 Expansion and mechanical computation

The distributive law is what makes mechanical expansion possible at all. A symbolic engine that reduces a polynomial expression to a normal form does so by distributing products over sums and collecting like terms; this is the algebraic core of the mechanization of mathematics, whose systematic programme was set out by Wu Wen-tsun [16]. In that tradition expansion is a means: the process is eliminated in favour of a canonical object, and the mechanical content is transferred to an algorithm acting on that object.

The present paper keeps the expansions. What Example 2.8 shows is that the expansions which distributivity relates are distinguishable objects: they carry different one-hole words, different intermediate values, and different charges, while sharing one operator and one endpoint. This is the sense in which AEG studies process rather than result. The value of an expression is one coordinate of its expansions, not the object itself; the geometry developed in the following sections is the geometry of the expansions.

2.5 Point propagation and condition propagation

The same one-hole word supports two elementary readings. Given maps $F_1, \dots, F_n$ and a point z_0 , the prefix operators

$$G_k = F_k \circ \dots \circ F_1$$

push the point forward:

$$z_k = G_k(z_0).$$

This is the *point propagation* used for paths. For ordinary partial maps, put $D_0 = K$ and $G_0 = \text{id}$, and recursively set

$$D_k = \{z \in D_{k-1} : G_{k-1}(z) \in \text{dom } F_k\}.$$

Then G_k is evaluated only on D_k . This prefix-domain rule retains every original division condition even when a simplified final formula has a larger natural domain.

Alternatively, given a family of output level sets $\{\Sigma_t\}$, the same prefix operator pulls each level set back:

$$\Sigma_{k,t} = G_k^{-1}(\Sigma_t).$$

For partial maps this means $\Sigma_{k,t} = \{z \in D_k : G_k(z) \in \Sigma_t\}$; it is an inverse image, not a claim that G_k has an inverse function. This is *condition propagation*. It is contravariant: an outer condition is transported toward the input by inverse image. Sections 3 and 4 show that the first reading yields a path in a space of arithmetic motions, while the second yields what we shall call ripple geometry.

2.6 Evaluable expressions and effective presentations

The early formulation of AEG began with rational arithmetic and explicitly restricted its subsequent discussion to *evaluable expressions* [17]. A later foundational question connected this restriction with the difficulty of deciding whether a real expression has a value [19]. The distinction matters: defining an expression, defining its partial interpretation, computing its value, and deciding its admissibility are different tasks.

Definition 2.10 (Evaluability at an input). For a finite one-hole arithmetic expression $e[z]$ over K , let $D_e \subseteq K$ be the set of inputs at which every node admits ordinary arithmetic evaluation, including every intermediate division guard. Write

$$\nu_e : D_e \longrightarrow K$$

for that evaluation. The expression is *evaluable at x* precisely when $x \in D_e$. A closed expression is *evaluable* when its entire recursive evaluation is defined. These are semantic conditions on an already generated syntax tree; they do not assert the existence of a decision procedure.

For a comb, D_e is exactly the final prefix domain defined above. A syntactically well-formed tree may have an empty domain: the expression $z/0$ is one example. This is partiality of evaluation, rather than failure of the tree grammar or incompleteness of a proof system. Alternatively, one may require evidence of defined operands and a nonzero denominator in the formation rule for division; such a rule defines a language of certified evaluable expressions. Checking supplied evidence and automatically finding it remain separate tasks.

Example 2.11 (A cancelled reciprocal retains its excluded input). Let $e[z] = 1/(1/z)$. Its first reciprocal requires $z \neq 0$; for such z , the value $1/z$ is nonzero, so the second reciprocal is defined and returns z . Thus

$$D_e = K \setminus \{0\}, \quad \nu_e(z) = z \quad (z \in D_e).$$

The expression z itself has domain K . The two expressions agree on D_e but do not define the same partial map. Recording the pair (D_e, ν_e) preserves this difference after algebraic cancellation. The missing input is not a pole of the simplified identity function. Composing the two projective reciprocals instead permits an intermediate infinity and gives the identity on $\mathbb{P}^1(K)$; that is a different interpretation, not an ordinary evaluation at zero. Neither exclusion by itself asserts a singularity of the regular AES structure introduced in the next section.

The choice $K = \mathbb{Q}$ supplies a useful effective starting point when constants and inputs are given as integer fractions with nonzero denominators. With unbounded exact integer arithmetic, evaluation of a finite closed tree, or of a finite tree at a supplied rational input, terminates with either a rational value or a division-by-zero failure: there are finitely many nodes, and each arithmetic operation and zero test is effective. This does not give every syntax tree a value. A bounded implementation must also distinguish a resource refusal from arithmetic undefinedness.

The rational carrier is being used here for its effective presentation, not for a discrete topology: $\mathbb{Q}$ is not discrete in the topology inherited from $\mathbb{R}$. The algebraic definitions of this section allow an arbitrary field K without an effective presentation; finite notation cannot, however, name every real constant over a fixed finite alphabet. The real geometric models impose their own smoothness hypotheses. Passing to such a model supplies neither exact numerical algorithms nor a general test for admissibility.

Richardson's theorem concerns decision problems in a richer function language. With rational constants, π , $\log 2$, the variable, exponential and sine, closed under addition, subtraction, multiplication and composition, the existence of a negative value is undecidable; adding absolute value also makes functional identity with zero undecidable [12]. This is not a prohibition on defining real expressions, nor a theorem that every language of closed real constants has undecidable zero tests. The signature, presentation and question must be specified before transferring such an obstruction.

Adva develops the finite construction viewpoint by retaining typed program structure and construction history, and separating exact structural checks from later numerical observations [25]. The rational examples above explain one effective starting point for that viewpoint. Finite syntax

and checkable evidence do not entail that every semantic question is decidable, and a bounded checker does not replace the continuous geometry. Keeping domains and histories explicit allows these discrete constructions and continuous interpretations to be studied together.

3 Arithmetic Expression Spaces and Arithmetic Motion

This section introduces the geometric object of the whole series. An arithmetic expression space is a space together with an assignment function and a metric on which the four arithmetic operations act as *motions*: each elementary operation moves the current arithmetic value, and the effect of that motion on points is part of the structure. We give the axioms, the canonical frame they determine, the basic hyperbolic model, the four elementary motions, the point paths they generate, and the continuous form of the same motion.

3.1 Arithmetic motions

Let M be a smooth surface with a Riemannian metric g , and let

$$a : M \longrightarrow \mathbb{R}$$

be a smooth function, called the *assignment*: $a(p)$ is the arithmetic value carried at the point p . An arithmetic operation is to be realized as a motion of M whose effect on a is that operation. For instance, adding a constant s should be a map X_s with

$$a \circ X_s = a + s,$$

and scaling by a positive constant k should be a map Y_k with $a \circ Y_k = ka$. Two things are deliberately kept apart.

1. The *motions* record arithmetic propagation. They are required to transport the assignment in a prescribed way, and they need not preserve the metric.
2. The *metric* measures the infinitesimal cost of moving. It is part of the structure, and the axioms below constrain it through the assignment.

The elementary model of Section 3.3 makes both visible at once: its additive and multiplicative motions are affine maps of the upper half-plane, and the metric is the invariant hyperbolic metric of the positive affine group.

3.2 Regular arithmetic expression spaces

Definition 3.1 (Regular arithmetic expression space). A *regular arithmetic expression space*, or regular AES, is a tuple

$$\mathfrak{E} = (M, g, a; \mu, \lambda)$$

such that M is an oriented smooth surface, possibly with boundary, g is a smooth Riemannian metric, $a \in C^\infty(M, \mathbb{R})$, $\mu \in \mathbb{R} \setminus \{0\}$, $\lambda \in \mathbb{R}$, and

$$|\nabla a|_g^2 = \mu^2 + \lambda^2 a^2 \tag{5}$$

everywhere on M . The function a is called the *assignment function*.

The metric in Definition 3.1 is part of the structure. Equation (5) is therefore an eikonal equation on a specified Riemannian surface, not a metric-free identity. Its right-hand side records the two arithmetic directions: the constant term μ^2 is the additive intensity and the assignment term $\lambda^2 a^2$ the multiplicative one.

Let J denote rotation by $+\pi/2$ in the oriented tangent planes. Since $\mu \neq 0$, the quantity

$$q = \sqrt{\mu^2 + \lambda^2 a^2}$$

is everywhere positive. The next proposition shows that the eikonal definition contains exactly the oriented orthonormal directional data used in the flow equation.

Proposition 3.2 (Canonical arithmetic frame). *For an oriented Riemannian surface (M, g) , a smooth function a , and constants $\mu \neq 0$ and λ , the following are equivalent.*

1. *The eikonal identity (5) holds.*
2. *There is a unique positively oriented orthonormal frame (X_u, X_v) satisfying*

$$X_u a = \mu, \quad X_v a = \lambda a. \quad (6)$$

When these conditions hold, put $E_a = \nabla a / q$. The frame is

$$\boxed{\begin{aligned} X_u &= \frac{\mu}{q} E_a - \frac{\lambda a}{q} J E_a, \\ X_v &= \frac{\lambda a}{q} E_a + \frac{\mu}{q} J E_a. \end{aligned}} \quad (7)$$

Proof. Assume the eikonal identity. Then $|\nabla a| = q > 0$, so E_a and $J E_a$ form a smooth positively oriented orthonormal frame. The coefficient matrix in (7) is

$$\frac{1}{q} \begin{pmatrix} \mu & \lambda a \\ -\lambda a & \mu \end{pmatrix},$$

whose columns are orthonormal and whose determinant is one. Hence (X_u, X_v) is positively oriented and orthonormal. Since $da(E_a) = q$ and $da(J E_a) = 0$, the two identities in (6) follow.

For uniqueness, write any positively oriented orthonormal frame in the form

$$X_u = c E_a + s J E_a, \quad X_v = -s E_a + c J E_a, \quad c^2 + s^2 = 1.$$

The two directional identities force $qc = \mu$ and $-qs = \lambda a$. Thus $c = \mu/q$ and $s = -\lambda a/q$, which gives (7).

Conversely, if an orthonormal frame satisfies (6), then

$$|\nabla a|^2 = (X_u a)^2 + (X_v a)^2 = \mu^2 + \lambda^2 a^2.$$

□

The frame is generally non-coordinate. Applying its bracket to the assignment function gives

$$[X_u, X_v]a = X_u(\lambda a) - X_v(\mu) = \mu \lambda. \quad (8)$$

Thus, when $\lambda \neq 0$, the fields cannot commute. If (η^u, η^v) is their dual coframe, the valid intrinsic Pfaffian statement is

$$da = \mu \eta^u + \lambda a \eta^v;$$

the one-forms η^u, η^v are not presumed exact. Equation (8) is the first appearance of a quantity that returns in Sections 8 and 9: the additive direction changes the multiplicative datum by $\mu \lambda$ per unit area of the arithmetic frame.

3.3 The basic hyperbolic model

Let

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$$

with its standard orientation, and fix $\mu, \lambda > 0$. Put

$$g_{\mu, \lambda} = \frac{1}{y^2} \left(\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2} \right), \quad a(x, y) = -\frac{x}{y}, \quad (9)$$

and record the assignment separately, since every later computation refers to it:

$$a(x, y) = -\frac{x}{y}. \quad (10)$$

We call

$$\mathfrak{E}_0(\mu, \lambda) = (\mathbb{H}^2, g_{\mu, \lambda}, a; \mu, \lambda)$$

the *basic hyperbolic model*. In earlier drafts of the series and in Paper I this model was written $\mathfrak{E}_{\text{hyp}}(\mu, \lambda)$; the identification is stated once, here, and the letter $\mathfrak{E}_0$ is used from now on (see also Subsection 3.8).

The metric admits the group-theoretic description that explains why addition and multiplication are the natural motions. Write

$$G = \text{Aff}^+(1, \mathbb{R}) = \{(x, y) : x \in \mathbb{R}, y > 0\}$$

with multiplication

$$(x, y)(x', y') = (x + yx', yy'). \quad (11)$$

The element (x, y) represents the affine map $t \mapsto x + yt$. Left translation by (x_0, y_0) is

$$L_{(x_0, y_0)}(x, y) = (x_0 + y_0x, y_0y).$$

For fixed $\mu, \lambda > 0$, consider the left-invariant vector fields

$$X_u = -\mu y \partial_x, \quad X_v = -\lambda y \partial_y. \quad (12)$$

Both signs are chosen so that the frame is positively oriented for $dx \wedge dy$ and has the action on the assignment stated below.

Proposition 3.3 (Invariant affine metric). *The Riemannian metric for which the frame (12) is orthonormal is*

$$g_{\mu, \lambda} = \frac{1}{y^2} \left(\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2} \right). \quad (13)$$

It is invariant under every left translation of G .

Proof. The coframe dual to (X_u, X_v) is

$$-\frac{dx}{\mu y}, \quad -\frac{dy}{\lambda y},$$

whose sum of squares is (13). Under $L_{(x_0, y_0)}$, one has

$$x \mapsto x_0 + y_0x, \quad y \mapsto y_0y, \quad dx \mapsto y_0dx, \quad dy \mapsto y_0dy.$$

Substitution in (13) shows $L_{(x_0, y_0)}^* g_{\mu, \lambda} = g_{\mu, \lambda}$. □

Theorem 3.4 (Basic hyperbolic AES). *For every $\mu, \lambda > 0$, the tuple*

$$\mathfrak{E}_0(\mu, \lambda) = (\mathbb{H}^2, g_{\mu, \lambda}, a; \mu, \lambda)$$

is a regular arithmetic expression space. Its canonical arithmetic frame is the left-invariant frame (12). Moreover, $(\mathbb{H}^2, g_{\mu, \lambda})$ is complete and has constant Gaussian curvature

$$K = -\lambda^2.$$

Proof. Direct differentiation gives

$$a_x = -\frac{1}{y}, \quad a_y = \frac{x}{y^2} = -\frac{a}{y}.$$

Therefore the frame (12) satisfies

$$X_u a = \mu, \quad X_v a = \lambda a.$$

It is positively oriented and orthonormal by Proposition 3.3, and the frame equivalence of Proposition 3.2 proves the regular-AES identity. Equivalently, using

$$g_{\mu, \lambda}^{-1} = \mu^2 y^2 \partial_x \otimes \partial_x + \lambda^2 y^2 \partial_y \otimes \partial_y,$$

one obtains directly

$$|\nabla a|_{g_{\mu, \lambda}}^2 = \mu^2 y^2 a_x^2 + \lambda^2 y^2 a_y^2 = \mu^2 + \lambda^2 a^2.$$

For completeness and curvature, introduce the global coordinate

$$X = \frac{\lambda}{\mu} x.$$

Then

$$g_{\mu, \lambda} = \frac{1}{\lambda^2} \frac{dX^2 + dy^2}{y^2}. \tag{14}$$

The metric in parentheses is the complete standard upper-half-plane metric of curvature -1 . Multiplication of a Riemannian metric by λ^{-2} preserves completeness and multiplies Gaussian curvature by λ^2 . Hence $g_{\mu, \lambda}$ is complete and has curvature $-\lambda^2$. $\square$

The parameters have different geometric roles. The change $X = (\lambda/\mu)x$ removes μ from the metric, so μ fixes the additive normalization of the assignment frame. The parameter λ sets the curvature scale. No uniqueness claim for regular AESs follows from this normalization.

Horocyclic coordinates and assignment contours. Set $y = e^{\lambda v}$ and $u = x$. Then

$$g_{\mu, \lambda} = e^{-2\lambda v} \frac{du^2}{\mu^2} + dv^2, \quad a(u, v) = -ue^{-\lambda v}. \tag{15}$$

The curves $y = \text{constant}$ are horocycles based at the ideal point ∞ , and the vertical lines $x = \text{constant}$ are their orthogonal geodesics. The assignment contour of value c is

$$a^{-1}(c) = \{x = -cy, \ y > 0\}.$$

In particular, the zero locus in this model is the vertical geodesic $\{x = 0, \ y > 0\}$.

The Laplace eigenfunction. We use the divergence-of-gradient sign convention

$$\Delta_g f = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} g^{ij} \partial_j f \right). \quad (16)$$

Proposition 3.5 (Laplace eigenfunction). *The assignment function of the basic hyperbolic AES satisfies*

$$\Delta_{g_{\mu,\lambda}} a = 2\lambda^2 a. \quad (17)$$

Proof. Since

$$\sqrt{\det g_{\mu,\lambda}} = \frac{1}{\mu\lambda y^2},$$

the products $\sqrt{\det g} g^{xx} = \mu/\lambda$ and $\sqrt{\det g} g^{yy} = \lambda/\mu$ are constant. Thus

$$\Delta_{g_{\mu,\lambda}} f = \mu^2 y^2 f_{xx} + \lambda^2 y^2 f_{yy}.$$

For $a = -x/y$, one has $a_{xx} = 0$ and $a_{yy} = -2x/y^3 = 2a/y^2$. Substitution gives (17). $\square$

3.4 The four operations as motions

On the basic model, define for $s \in \mathbb{R}$ and $k > 0$

$$\mathsf{X}_s(x, y) = (x - sy, y), \quad \mathsf{Y}_k(x, y) = \left(x, \frac{y}{k}\right). \quad (18)$$

Proposition 3.6 (Arithmetic action on the assignment). *The maps in (18) satisfy*

$$a \circ \mathsf{X}_s = a + s, \quad a \circ \mathsf{Y}_k = ka.$$

Consequently, translation and positive scaling of the current arithmetic value can be represented by motions of points in $\mathfrak{E}_0$.

Proof. Direct substitution gives

$$a(x - sy, y) = -\frac{x - sy}{y} = a(x, y) + s,$$

and

$$a\left(x, \frac{y}{k}\right) = -\frac{x}{y/k} = ka(x, y).$$

$\square$

These transformations are assignment-compatible; they are not generally isometries of $g_{\mu,\lambda}$. The distinction is the one announced in Subsection 3.3: the motions record arithmetic propagation, whereas the metric measures its infinitesimal geometry.

The remaining two operations are obtained from the same two families and a reflection.

Warning 3.7 (The motions are partial). The maps below are defined only where the arithmetic value stays in the range carried by the model, and they are not asserted to form a global group action even there. Concretely: the scaling motion Y_k is defined for $k > 0$ and moves the assignment from a to ka , so a word whose intermediate values leave the range of a on M has no realization on M ; a model whose assignment misses a half-axis, such as the punctured model $\mathfrak{E}_1$ of Example 3.19, cannot carry globally the motions that would move the value into the missing half-axis; and a motion of a punctured model may run into a puncture, as Section 11 records. Only the basic model $\mathfrak{E}_0$ carries the two affine motions globally, and only there do they generate the full positive affine group action. Everywhere else “the four operations act as motions” means: each operation is realized by a locally defined map wherever its assignment condition holds.

operation	motion of $\mathfrak{E}_0$	effect on a
addition of s	X_s	$a \mapsto a + s$
subtraction of s	X_{-s}	$a \mapsto a - s$
multiplication by $k > 0$	Y_k	$a \mapsto ka$
division by $k > 0$	$Y_{1/k}$	$a \mapsto a/k$
reflection (sign change)	$(x, y) \mapsto (-x, y)$	$a \mapsto -a$

Subtraction is thus the inverse of the additive motion, and division by a nonzero constant is the inverse of the multiplicative motion. The reflection is not by itself one of the four operations; it appears because the assignment of the model is signed, and it is the elementary instance of the second-slot subtraction $c - z$ of Section 2. Multiplying by a negative constant is realized as the reflection composed with a positive scaling; the model therefore carries every nonzero affine multiplier of a real history, while the *positive* scalings are the motions that stay in the identity component and are the ones used in the grid, in the flow, and in the ACS. Over a general field the affine cocycles of Section 7 again allow every nonzero multiplier, and the analytic statements of this paper are made over the positive component.

For an integer $m \geq 2$, the two moves satisfy the exact relation

$$Y_m^{-1} \circ X_s^m \circ Y_m = X_s. \quad (19)$$

Indeed, $X_s^m = X_{ms}$, and conjugation by Y_m divides the additive displacement by m . Equation (19) is the elementary Baumslag–Solitar-type relation visible in the grid [4]; no complexity conclusion is inferred from it. Its arithmetic content is that a multiplicative motion transports the additive ruler: after rescaling the space, the additive unit that produces the same effect on a is $1/m$ of the original one. The same transport, made infinitesimal and measured against a commutative shadow, is the tearing discussed in Section 8.

Division by the evolving value. The motions above are affine, and so is every word in them. The remaining elementary context of Section 2, second-slot division c/z , is not affine and cannot be realized by a motion of $\mathfrak{E}_0$ that preserves the assignment line. It is realized projectively: on the projective line it is an element of $\mathrm{PGL}_2(K)$, and it moves the arithmetic value $z \mapsto c/z$ on the ordinary domain $z \neq 0$. Section 4 reads this operation not as a motion of the value but as a transport of conditions, and Section 6 places both readings in one matrix.

3.5 Point paths

The left-expanded comb becomes geometric once its one-hole sections are restricted to translations and nonzero scalings. Let a left-expanded affine expression have chronological sections

$$F_i(z) = \alpha_i z + \beta_i, \quad \alpha_i \neq 0.$$

For an initial arithmetic value z_0 , define

$$z_i = F_i(z_{i-1}), \quad i = 1, \dots, n.$$

Choose a point $p_0 \in \mathbb{H}^2$ with $a(p_0) = z_0$, and realize each translation or positive scaling by the corresponding map in (18). The successive points

$$p_0, p_1, \dots, p_n$$

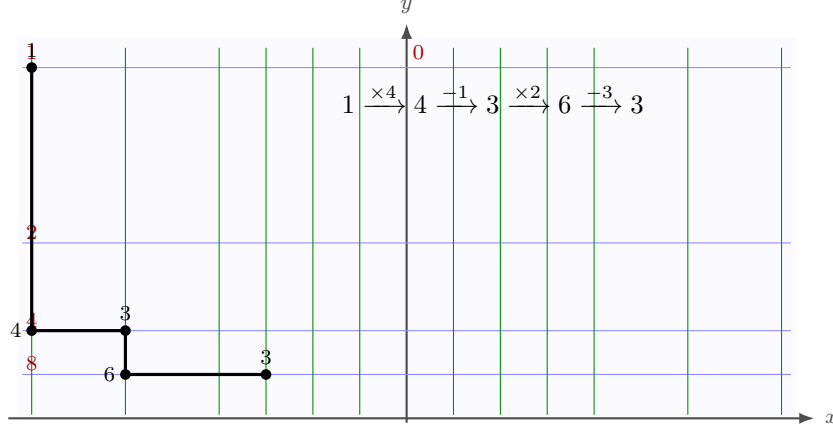


Figure 2: A schematic point path in the basic hyperbolic arithmetic grid. The black polyline retains the intermediate propagation that the endpoint value forgets.

form the *point path* of the bounded expression in this chosen realization. By Proposition 3.6,

$$a(p_i) = z_i.$$

Thus the geometry records every intermediate value, not only the endpoint. The word “point path” is a descriptive name for this orbit; it asserts no new structure beyond the displayed construction.

Example 3.8 (A four-step path). Start from $z_0 = 1$ and apply

$$z \mapsto 4z, \quad z \mapsto z - 1, \quad z \mapsto 2z, \quad z \mapsto z - 3.$$

The value history is

$$1 \mapsto 4 \mapsto 3 \mapsto 6 \mapsto 3.$$

In the grid this becomes two transverse scale moves and two horocyclic translation moves. The final value 3 alone does not determine this path.

The affine algebra behind the path. Every affine prefix has a unique form

$$G_i(z) = A_i z + B_i, \quad A_i \neq 0.$$

If $F_i(z) = \alpha_i z + \beta_i$, then

$$A_i = \alpha_i A_{i-1}, \quad B_i = \alpha_i B_{i-1} + \beta_i, \quad (A_0, B_0) = (1, 0). \quad (20)$$

The point path is therefore the evaluation of an affine matrix path

$$\begin{pmatrix} A_i & B_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ 1 \end{pmatrix}.$$

The lower-left entry remains zero at every stage. Projectively this means that all prefixes stabilize the same point ∞ . Section 4 shows what changes when right-slot division makes that entry nonzero.

3.6 The operation-to-length assignment in the affine sector

Subsection 3.5 assigns to a bounded affine expression the affine matrix product (20), and Proposition 7.1 condenses a chronological history to two real coordinates. Neither statement says whether a composed arithmetic word determines a *length*, that is, a displacement of the model. The affine sector answers that question completely, and the answer separates the two published coordinates of one history. Three extensions are not available here and are listed at the end of the subsection.

Length of an affine motion.

Proposition 3.9 (Translation length of a positive dilation). *Let $k > 0$ and let $D_k(z) = kz$ act on $\mathbb{H}^2 = \{z : \text{Im } z > 0\}$ with the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$. Then the imaginary axis $i\mathbb{R}_{>0}$ is invariant under D_k , it is a complete geodesic, and the translation length of D_k along it is*

$$\ell(D_k) = |\ln k|. \quad (21)$$

Proof. The map D_k sends iy to iky , so $i\mathbb{R}_{>0}$ is invariant and the restriction of D_k to it is the translation $y \mapsto ky$. The hyperbolic length of the segment joining iy to iky is $|\int_y^{ky} dt/t| = |\ln k|$, and it does not depend on y . The displacement is therefore $|\ln k|$, and it is attained on an invariant geodesic, which is the defining property of the translation length. $\square$

Proposition 3.10 (The length of an affine motion is its multiplicative charge, up to sign). *Let $\varphi \in \text{Aff}^+(1, \mathbb{R})$ be written $\varphi(z) = \Phi z + \xi$ with $\Phi > 0$, and put $M(\varphi) := \ln \Phi$. Then the translation length of the action of φ on $\mathbb{H}^2$ is*

$$\ell(\varphi) = |M(\varphi)|, \quad (22)$$

and consequently, for every history $\gamma \in G$,

$$\ell(\nu(\gamma)) = |M(\gamma)|, \quad (23)$$

where $\nu : G \rightarrow \text{Aff}^+(1, \mathbb{R})$ is the evaluation of Proposition 8.4 and $M = \text{pr}_2 \circ c = \log \circ L \circ \nu$ is the multiplicative charge of Theorem 8.6(3).

Proof. If $\Phi = 1$ then φ is the translation $z \mapsto z + \xi$; it has no fixed point in $\mathbb{H}^2$, it is parabolic, and its translation length is $0 = |\ln 1|$. If $\Phi \neq 1$ then φ has the fixed point $z^* = \xi/(1 - \Phi)$ on the boundary circle, and conjugating by the translation $\psi(z) = z - z^*$, which is an isometry of $\mathbb{H}^2$, gives $\psi\varphi\psi^{-1}(w) = \Phi w = D_\Phi(w)$. The translation length is invariant under conjugation by isometries, so Proposition 3.9 gives $|\ln \Phi|$. The second display is the first read on histories through Theorem 8.6(3). $\square$

Two data the length does not carry.

- Proposition 3.11** (The length is not a homomorphism, and it is not injective on histories). 1. $\ker \nu \subsetneq \ker M$. Every nonempty pure additive history lies in $\ker M$ and induces a non-identity motion, so it is not in $\ker \nu$; hence the multiplicative charge, and with it the length, is a function of the induced motion, while the history-to-motion assignment is not injective.
2. By Theorem 8.6(4) the additive charge does not factor through ν . Two histories with the same induced motion, and therefore the same length, may differ in additive charge, and no function of the induced motion, the length included, recovers that difference.

3. M is a homomorphism $G \rightarrow (\mathbb{R}, +)$, whereas $\ell = |M|$ is not. Writing M_q for the dilation generator,

$$\ell(M_q M_{-q}) = 0 \neq 2|q| = \ell(M_q) + \ell(M_{-q}) \quad (q \neq 0). \quad (24)$$

Equivalently, ℓ forgets the sign of M , that is, which of the two eigenvalues $e^{\pm M/2}$ of the induced affine motion is the expanding one.

Proof. (1) For $p \neq 0$ the history A_p satisfies $\nu(A_p) = T_p \neq \text{id}$ and $M(A_p) = 0$. (2) is the statement of Theorem 8.6(4). (3) $M_q M_{-q} = M_0$ has multiplicative charge 0, while $\ell(M_{\pm q}) = |q|$ by Proposition 3.9. For the eigenvalue claim: in homogeneous coordinates φ has the matrix $\begin{pmatrix} \Phi & \xi \\ 0 & 1 \end{pmatrix}$ with eigenvalues Φ and 1, so the eigenvalues of its normalised lift to $\text{SL}_2(\mathbb{R})$ are $e^{\pm M/2}$. $\square$

Proposition 3.12 (The scale-compatible relative defect changes no length). *Let γ, δ be scale-compatible affine histories with common scale Φ . Then the relative affine defect $\mathcal{K}_{\text{aff}}(\gamma, \delta)$ of Definition 7.5 has multiplier 1, and its translation length is zero:*

$$M(\mathcal{K}_{\text{aff}}(\gamma, \delta)) = 0, \quad \ell(\mathcal{K}_{\text{aff}}(\gamma, \delta)) = 0. \quad (25)$$

It is therefore parabolic whenever it is nontrivial, it fixes exactly one boundary point, and it preserves every translation length.

Proof. By (72) the defect is the translation $x \mapsto x + (\xi_\gamma - \xi_\delta)/\Phi$, whose homogeneous matrix $\begin{pmatrix} 1 & (\xi_\gamma - \xi_\delta)/\Phi \\ 0 & 1 \end{pmatrix}$ has multiplier 1 and trace 2. Apply Proposition 3.10 in the case $\Phi = 1$. $\square$

Proposition 3.13 (Trace classification and length of a projective motion). *Let $M \in \text{PSL}_2(\mathbb{R})$ act on $\mathbb{H}^2$ and let $\tilde{M} \in \text{SL}_2(\mathbb{R})$ be any lift, so that $\tau(M)^2 := \text{tr}(\tilde{M})^2$ is well defined. Then:*

1. *if $\tau^2 > 4$, M is hyperbolic, it has two fixed points on the boundary circle, it preserves the geodesic joining them, and its translation length is*

$$\ell(M) = 2 \text{arccosh}(|\tau(M)|/2); \quad (26)$$

2. *if $\tau^2 = 4$, M is parabolic or the identity; a nontrivial parabolic motion has one fixed point on the boundary circle, preserves no geodesic, and has translation length 0;*

3. *if $\tau^2 < 4$, M is elliptic, it has one fixed point in $\mathbb{H}^2$ and none on the boundary circle, it is the rotation about that point by $2 \arccos(|\tau(M)|/2)$, and its translation length is 0.*

Consequently the trichotomy is decided by the sign of $\tau^2 - 4$, and on the affine slice (26) agrees with Proposition 3.10.

Proof. The lift has determinant 1, so its eigenvalues are λ and λ^{-1} and $\tau = \lambda + \lambda^{-1}$. The three cases are the three positions of the discriminant $\tau^2 - 4$ of the characteristic polynomial. If $\tau^2 > 4$ then λ is real and not ± 1 ; conjugating M to the diagonal lift $\text{diag}(e^{\ell/2}, e^{-\ell/2})$, which is possible in $\text{SL}_2(\mathbb{R})$ for the appropriate $\ell > 0$, gives $\tau = 2 \cosh(\ell/2)$, and the translation length is ℓ by Proposition 3.9 together with the invariance of the length under conjugation by isometries. If $\tau^2 = 4$ then $\lambda = \pm 1$ and the lift is unipotent, $\tilde{M} = \pm(I + N)$ with $N^2 = 0$ and $N \neq 0$ unless M is the identity; such an element fixes exactly one boundary point. If $\tau^2 < 4$ then $\lambda = e^{\pm i\theta}$ with $\theta \notin \{0, \pi\}$, the lift is conjugate in $\text{SL}_2(\mathbb{R})$ to a rotation by 2θ , and $\tau = 2 \cos \theta$. For the last claim, an affine motion $\varphi(z) = \Phi z + \xi$ with $\Phi > 0$ has the lift $\Phi^{-1/2} \begin{pmatrix} \Phi & \xi \\ 0 & 1 \end{pmatrix}$, whose trace is $\Phi^{1/2} + \Phi^{-1/2} = 2 \cosh(\frac{1}{2} \ln \Phi)$; so (26) returns $|\ln \Phi|$, which is Proposition 3.10. $\square$

Proposition 3.14 (The two layers of the affine conjugacy classification). *Work in $\text{Aff}^+(1, \mathbb{R})$, write $\varphi(z) = \Phi z + \xi$ with $\Phi > 0$, and write $T_c(z) = z + c$ for the translation by c .*

1. *For every $\Phi > 0$ one has $D_\Phi T_c D_\Phi^{-1} = T_{\Phi c}$. Hence all nontrivial translations of a fixed sign are mutually conjugate, the stabiliser in $\text{Aff}^+(1, \mathbb{R})$ of T_c is the translation subgroup, and the conjugacy class $\{T_{\Phi c} : \Phi > 0\}$ is a torsor under the dilations.*
2. *A dilation D_Φ is conjugate to $D_{\Phi'}$ if and only if $\Phi = \Phi'$; its conjugacy class is therefore determined by the multiplier, so the length $|\ln \Phi|$ is a function of the class.*

Consequently the conjugacy class carries one continuous invariant in the dilation layer and none in the translation layer, where only the sign survives.

Proof. (1) $D_\Phi \circ T_c \circ D_\Phi^{-1}(z) = D_\Phi(z/\Phi + c) = z + \Phi c$. Conjugation by a translation fixes T_c , so the stabiliser is the translation subgroup, and $\text{Aff}^+(1, \mathbb{R})$ modulo that subgroup is $\mathbb{R}_{>0}$; the orbit is therefore a torsor under the dilations. (2) dilations commute, so $D_\Phi D_{\Phi'} D_\Phi^{-1} = D_{\Phi'}$ and conjugacy forces $\Phi = \Phi'$. The length is $|\ln \Phi|$ by Proposition 3.10. $\square$

Remark 3.15 (Status: structural proposal — the two coordinates of one history). Proposition 7.1 gives one chronological history two real coordinates, the accumulated scale Φ_n of (57) and the target affine cocycle ξ_n of (58). By Proposition 3.10 the quantity $\log|\Phi_n|$ is the translation length of the induced motion, and by Proposition 3.11(2) the additive data that ξ_n records is not a function of the induced motion at all. We propose reading $(\log \Phi_n, \xi_n)$ as one assignment with two sides: a side that is a function of the induced motion, and a side that is a function of the history only. This is a proposal about how to organise statements already proved above; it adds no theorem, it does not identify ξ_n with any geometric coordinate, and it makes no claim about the Fenchel–Nielsen coordinates of Section 12. One negative point can be recorded without leaving the affine sector: neither ξ_n nor the relative translation $(\xi_\gamma - \xi_\delta)/\Phi$ can be a Fenchel–Nielsen twist, because by Proposition 3.12 both are parabolic of length zero, whereas the holonomy of a closed geodesic of length $\ell > 0$ is hyperbolic of trace $2 \cosh(\ell/2) > 2$; a twist changes no length, and no rescaling of a coefficient changes a conjugacy type. What does transfer is the compatibility condition: “scale-compatible” in Definition 7.5 is exactly the requirement that a glued curve carry one agreed length. The length-zero verdict should be read with Proposition 3.14: a translation layer whose class carries no continuous invariant is not an empty layer but the layer in which the only datum is the choice of conjugate, i.e. a torsor. So the relative affine defect is not a failed length; it is the stratum on which a twist, if this geometry carries one, would have to live. Whether that torsor coincides with a resolution torsor of the kind used in a through-crossing theory is not decided here, and the series architecture gives Paper 0 no route to say so.

What this subsection does not do.

1. It does not pass from $\text{Aff}^+(1, \mathbb{R})$ to $\text{PGL}_2(K)$ for a general field K . Over $\mathbb{R}$ the passage is now made: Proposition 3.13 classifies a projective motion by the sign of $\tau^2 - 4$ and gives its translation length, so second-slot division, whose affine/projective break is Corollary 2.5, is covered in the real case. What remains outside is a general K : for a field with no order the sign of $\tau^2 - 4$ is not available, and the classification must be stated through the discriminant’s multiplicative character together with the number of rational fixed points. Paper 0 does not develop that form, and it asserts no length outside $\mathbb{R}$.

2. It does not pass from left- and right-expanded combs to arbitrary binary expression trees. The source of the assignment is the comb grammar of Definition 2.1, with the unique internal order of Proposition 2.3. The sequential-tree classification is a Paper I result, cited here prospectively and not restated.
3. It does not settle the category in which the assignment lives. Whether the source should be a free category of partial contexts, an action groupoid, a path groupoid, or a rewriting 2-category is OQ-047, deferred to Paper IV. The affine-sector instance stated here is compatible with each of those choices and selects none of them.

3.7 Continuous arithmetic motion

The discrete motions of Subsection 3.5 have an infinitesimal counterpart. Translation and dilation generate the Lie algebra of the positive affine group, and their linear combinations give the continuous propagation law. The formulation below is intrinsic: the symbols u and v label arithmetic directions, but need not be coordinates.

The affine Lie algebra and its flow. On the assignment line with coordinate z , let

$$E = \partial_z, \quad H = z\partial_z.$$

These are respectively the infinitesimal generators of translation and positive dilation, and they satisfy $[E, H] = E$. Fix constants $\mu, \lambda \in \mathbb{R}$, with $\mu \neq 0$. The generator selected by an angle θ is

$$\Omega_\theta = \mu \cos \theta E + \lambda \sin \theta H. \quad (27)$$

Consequently, an integral curve $z = z(s)$ satisfies

$$\frac{dz}{ds} = \mu \cos \theta + \lambda z \sin \theta. \quad (28)$$

This Lie-algebra calculation is a derivation of the continuous form of the arithmetic motion, and it fixes the convention: the translation coefficient is $\mu \cos \theta$, while the dilation coefficient is $\lambda \sin \theta$.

When θ is constant, set $b = \lambda \sin \theta$. The solution through $z(0) = z_0$ is

$$z(s) = \begin{cases} e^{bs} z_0 + \mu \cos \theta \frac{e^{bs} - 1}{b}, & b \neq 0, \\ z_0 + \mu s \cos \theta, & b = 0. \end{cases} \quad (29)$$

In particular, the positive arithmetic axes give

$$\theta = 0 : \quad z(s) = z_0 + \mu s, \quad \theta = \frac{\pi}{2} : \quad z(s) = e^{\lambda s} z_0.$$

The opposite axes give subtraction and inverse dilation.

The Pfaffian propagation constraint. There is a useful three-dimensional local state space with coordinates (u, v, z) . On it define

$$\alpha = dz - \mu du - \lambda z dv \quad (30)$$

and the two horizontal vector fields

$$D_u = \partial_u + \mu \partial_z, \quad D_v = \partial_v + \lambda z \partial_z. \quad (31)$$

A curve tangent to $\cos \theta D_u + \sin \theta D_v$ satisfies (28). Thus the frequently used notation

$$dz = \mu du + \lambda z dv$$

means that α vanishes on admissible tangent vectors. It is not an identity asserting that z is a function of two independent coordinates whose ordinary partial derivatives are simultaneously μ and λz . Indeed,

$$[D_u, D_v] = \mu \lambda \partial_z. \quad (32)$$

When $\mu \lambda \neq 0$, this bracket already rules out such an integrable coordinate interpretation. The same bracket is computed intrinsically in (8), and Proposition 9.8 proves that the local model here and the frame there are two descriptions of one object for the basic model: the horizontal fields descend to the canonical frame and the curvature form is the hyperbolic area form. Its geometric reading as a contact structure is taken up in Section 9.

Theorem 3.16 (Continuous affine flow). *Let $\mathfrak{E} = (M, g, a; \mu, \lambda)$ be a regular AES with its canonical arithmetic frame. If a unit-speed curve γ has tangent*

$$\dot{\gamma} = \cos \theta X_u + \sin \theta X_v,$$

where θ may vary along the curve, then

$$\frac{d}{ds}(a \circ \gamma) = \mu \cos \theta + \lambda(a \circ \gamma) \sin \theta. \quad (33)$$

For constant θ , its solutions are given by (29).

Proof. By the chain rule and (6),

$$\frac{d}{ds}(a \circ \gamma) = da(\dot{\gamma}) = \cos \theta X_u a + \sin \theta X_v a = \mu \cos \theta + \lambda a \sin \theta.$$

For constant θ , this is the linear equation already integrated in (29). $\square$

Rectification. For $\lambda \neq 0$, define

$$r = \operatorname{arcsinh}\left(\frac{\lambda a}{\mu}\right). \quad (34)$$

The chain rule and (5) give

$$|\nabla r| = |\lambda|. \quad (35)$$

Thus r has constant gradient magnitude. When $\lambda = 0$, the original equation already reduces to $|\nabla a| = |\mu|$, and a/μ is a unit-gradient rectifying function. Rectification of the assignment is possible; rectification of *both* arithmetic directions at once is not, by (32). This asymmetry is the discrete root of the tearing language of Section 11.

The same affine convention fixes the sign of the elementary local order defect. For small increments h and k , let

$$A_h(z) = z + \mu h, \quad M_k(z) = e^{\lambda k} z.$$

Then

$$\begin{aligned} (M_k \circ A_h)(z) - (A_h \circ M_k)(z) &= \mu h(e^{\lambda k} - 1) \\ &= \mu \lambda h k + O(\|(h, k)\|^3). \end{aligned} \quad (36)$$

This is an infinitesimal two-order comparison, not an exact area formula for an arbitrary finite region. The exact finite statement, and its interpretation as a defect measured in the commutativized space, occupy Section 8.

Affine motion inside projective motion. An infinitesimal projective transformation of the line has the Riccati form

$$\dot{z} = \beta + \alpha z + \kappa z^2.$$

The theory developed in this section is the affine slice $\kappa = 0$, with $\beta = \mu \cos \theta$ and $\alpha = \lambda \sin \theta$. Results proved for regular affine AESs do not automatically extend to the quadratic projective flow.

3.8 Model names: E_0 , E_1 , and later examples

The historical drafts of this programme used the labels E_0 and E_1 inconsistently; in some notes the hyperbolic half-plane is called E_0 , in others E_1 . Because later chapters introduce spaces that are *not* smooth, we fix the convention now, once, and record it as a convention rather than a theorem.

Convention 3.17 (Model labels). $\mathfrak{E}_0(\mu, \lambda)$ denotes the basic regular hyperbolic model of Subsection 3.3. For $k \geq 1$, the label $\mathfrak{E}_k$ is used descriptively for an arithmetic expression space whose regular locus is a surface with k declared punctures: points removed from the regular structure and carrying no regular AES data. No classification by k is asserted; the label records the number of declared punctures and nothing else. The reserved historical labels E_k and $E_{\log}$ of earlier drafts are not model classes, and this paper defines neither of them.

Two examples make the convention concrete; both are developed later, and the first one is the only example in this paper whose punctured model is verified in detail.

Example 3.18 (The basic model, $k = 0$). $\mathfrak{E}_0(\mu, \lambda)$ is the upper half-plane with the metric (9) and the assignment $a = -x/y$. It has no puncture, its assignment vanishes along one complete geodesic, and by Theorem 3.4 it is complete with constant curvature. This is the model in which the arithmetic motions of Subsection 3.5 are read as point paths.

Example 3.19 (A one-puncture model, $k = 1$). Let $\mathbb{D} = \{\rho < 1\}$ be the unit disc with the complete Poincaré metric $g_{\mathbb{D}} = 4(d\xi^2 + d\eta^2)/\lambda^2(1 - \rho^2)^2$ and the radial assignment

$$a_{\mathbb{D}}(\xi, \eta) = \begin{cases} \frac{2\mu}{\lambda} \frac{\rho}{1 - \rho^2}, & \rho \neq 0, \\ 0, & \rho = 0. \end{cases}$$

On the punctured disc $\mathbb{D} \setminus \{0\}$ this is a regular AES with $|\nabla a_{\mathbb{D}}|^2 = \mu^2 + \lambda^2 a_{\mathbb{D}}^2$, while at the centre the assignment fails to be C^1 although the metric extends smoothly. The regular locus is therefore $\mathbb{D} \setminus \{0\}$ and the model carries exactly one puncture. Paper I verifies the details of this model as its isolated-zero example; the global invariant $\lambda a_{\mathbb{D}} = \mu \sinh(\lambda s)$ along hyperbolic distance s from the centre shows that the assignment, not the metric, is what fails at the puncture.

Example 3.19 is the first holed arithmetic expression space, and Section 11 returns to it. Its puncture is not a hole in the metric; it is a point at which the regular structure stops being defined, and it is where the arithmetic motions of the punctured part can no longer be extended.

4 Ripple Geometry: The Projective Dual

A point path follows one chosen input through a sequence of motions. The projective operation c/z admits no such motion of the value, and the observation of Subsection 3.5 was that its matrix has a nonzero lower-left entry. There is nevertheless a geometry attached to it, and it is obtained by reversing the direction of the reading: instead of pushing a point forward, pull an entire

output condition backward through the same maps. The upper half-plane makes this dual reading elementary, because Möbius maps carry generalized circles to generalized circles.

We call the resulting picture *ripple geometry*: the family of conditions obtained by pulling output conditions back through a projective arithmetic operator. The name is descriptive, and Subsection 4.5 states precisely what it does and does not assert.

4.1 Poles and standard horocycles

Let

$$F(z) = \frac{Az + B}{Cz + D} \quad (37)$$

be a real fractional linear map with

$$\Delta = AD - BC > 0.$$

The positivity convention ensures that F maps the upper half-plane to itself. The projective pole is

$$p_F = F^{-1}(\infty) = \begin{cases} -D/C, & C \neq 0, \\ \infty, & C = 0. \end{cases}$$

For $t > 0$, denote the standard horocycle based at infinity by

$$\mathcal{H}_t = \{w \in \mathbb{H} : \operatorname{Im} w = t\}.$$

Definition 4.1 (Ripple pencil). The *ripple pencil* of F is the one-parameter family

$$\mathcal{R}(F) = \{F^{-1}(\mathcal{H}_t) : t > 0\}.$$

It is a family of horocycles based at the pole p_F . The word “ripple” is a visual name for this standard Möbius pullback family.

Theorem 4.2 (Line-to-circle transition). *Let F be as in (37).*

1. *If $C = 0$, every member of $\mathcal{R}(F)$ is a horizontal line; the pencil is based at ∞ .*
2. *If $C \neq 0$, the member with parameter t is the Euclidean circle*

$$\left(x + \frac{D}{C}\right)^2 + \left(y - \frac{\Delta}{2tC^2}\right)^2 = \left(\frac{\Delta}{2tC^2}\right)^2. \quad (38)$$

All these circles are tangent to the real boundary at the common point $p_F = -D/C$ and are nested as t varies.

Proof. For $z = x + iy$, direct calculation gives the standard identity

$$\operatorname{Im} F(z) = \frac{\Delta y}{|Cz + D|^2}. \quad (39)$$

The inverse image of $\mathcal{H}_t$ is therefore

$$\Delta y = t((Cx + D)^2 + C^2 y^2).$$

If $C = 0$, this reduces to a horizontal line. If $C \neq 0$, divide by tC^2 and complete the square in y :

$$\left(x + \frac{D}{C}\right)^2 + y^2 - \frac{\Delta}{tC^2}y = 0,$$

which is exactly (38). The center lies vertically above $-D/C$ at a height equal to the radius, so the circle is tangent to the boundary there. The radius is proportional to t^{-1} , proving nesting. $\square$

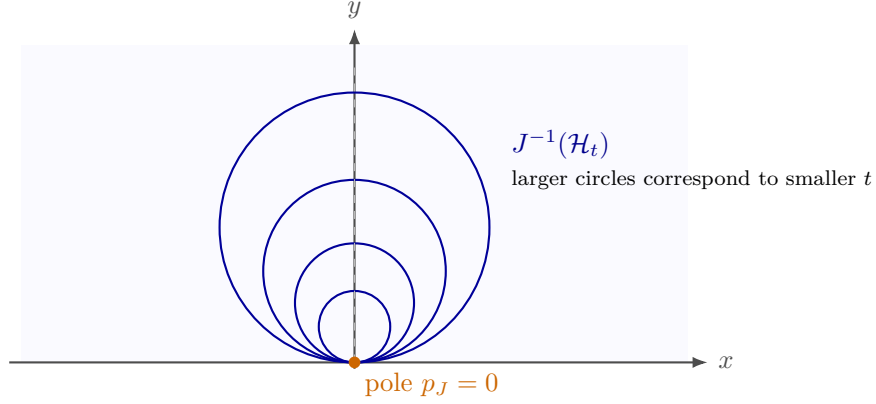


Figure 3: The ripple pencil of inversion. Every circle is a horocycle tangent at the projective pole 0; the affine horizontal-line pencil is recovered when the pole is moved to infinity.

The theorem gives a geometric interpretation of the lower-left matrix entry. The affine condition $C = 0$ places the pole at infinity and degenerates the ripple circles to parallel lines. A projective step with $C \neq 0$ moves the pole to a finite boundary point and curves the same horocyclic family into visible ripples.

Example 4.3 (Arithmetic inversion). For

$$J(z) = -\frac{1}{z}, \quad J \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

one has $p_J = 0$ and

$$x^2 + \left(y - \frac{1}{2t}\right)^2 = \left(\frac{1}{2t}\right)^2. \quad (40)$$

Thus inversion turns the horizontal horocycles based at infinity into a nested family of circles tangent at zero. The map exchanges the two ideal points 0 and ∞ .

Remark 4.4 (Which upper half-plane, and with which metric). The complex coordinate used throughout this section is the Möbius continuation chart, and it is not literally the point coordinate of Section 3. With the metric $g_{\mu,\lambda}$ of (9), the chart in which the horocycles are the Euclidean horizontal lines and the metric is the standard one of curvature -1 is

$$Z = \frac{\lambda}{\mu}x + iy,$$

as follows from (14). When $\mu = \lambda$ this is the identity, which is why the two roles of $x + iy$ are usually identified in the literature; here they are kept apart, and all Möbius statements below are made in the Z -chart. Only when a statement concerns the assignment a rather than the metric do we return to (x, y) .

4.2 Zero–pole pencils in the complex chart

There is a second elementary pencil attached to the same operator. Suppose $A, C \neq 0$ and write its zero and pole as

$$q_F = -\frac{B}{A}, \quad p_F = -\frac{D}{C}.$$

Then the output level set $|F(z)| = r$ is equivalent to

$$|z - q_F| = r \left| \frac{C}{A} \right| |z - p_F|. \quad (41)$$

For $r|C/A| \neq 1$ these are Apollonius circles; for $r|C/A| = 1$ the locus is the perpendicular bisector line of the segment $q_F p_F$, that is, a generalized circle of infinite radius. The horocyclic ripple pencil and the Apollonius pencil encode different output observables, $\text{Im } F$ and $|F|$, but both show that a projective one-hole operator naturally carries more than a point orbit: it carries a zero, a pole, and a family of generalized circles organized by them.

4.3 Ripple histories of right-expanded expressions

Let a right-expanded expression determine nondegenerate real Möbius sections $F_1, \dots, F_n$ of *positive* determinant, in chronological order, and write

$$G_k = F_k \circ \dots \circ F_1, \quad G_k(z) = \frac{A_k z + B_k}{C_k z + D_k}.$$

At stage k , the arithmetic value of a chosen input follows the point orbit $G_k(z_0)$. Simultaneously, the output horocycles pull back to

$$\mathcal{R}_k = \mathcal{R}(G_k).$$

The tangency point

$$p_k = G_k^{-1}(\infty) = -\frac{D_k}{C_k}$$

when $C_k \neq 0$ is the current pole of the prefix operator.

Thus a general right-expansion history should not be pictured as one fixed family of concentric circles. Each prefix has its own ripple pencil, and the common tangency point of that pencil may move with k . The appropriate geometric state is

$$(G_k, G_k(z_0), p_k, \mathcal{R}_k) :$$

the operator, the selected point, the pole, and the pulled-back wavefront family.

Proposition 4.5 (Covariant–contravariant duality). *For every prefix operator G_k and every output horocycle $\mathcal{H}_t$,*

$$z_0 \in G_k^{-1}(\mathcal{H}_t) \iff G_k(z_0) \in \mathcal{H}_t.$$

Hence the point path and the ripple history are two readings of the same operator word: one pushes the input forward, the other pulls the output condition backward.

Proof. This is the defining property of inverse image. □

The elementary nature of the proposition is precisely its value. No new geometry is inserted between the two pictures. Their unity is already present in the action of one Möbius matrix.

4.4 The pole of a prefix: covariance and rigidity

Section 4 introduced the pole $p_F = F^{-1}(\infty)$ of a single projective operator, and the ripple history of a word $F_1, \dots, F_n$ carries the pole sequence $p_k = G_k^{-1}(\infty)$ of the prefixes $G_k = F_k \circ \dots \circ F_1$. It was observed there that *the common tangency point of the pencil may move with k* . The two elementary statements below make that observation exact: they record how a pole transforms under composition, and they identify the histories along which it does not move. Neither statement is new mathematics; they are recorded because the mobility of the tangency point is otherwise left as a picture.

Throughout, a Möbius map is as in (37), and $B_\infty = \text{Stab}_{\text{PGL}_2(K)}(\infty)$ is the affine Borel sector of Section 6.

Lemma 4.6 (Covariance of the pole). *Let P, Q be nondegenerate Möbius maps and put $p_P := P^{-1}(\infty)$. Then*

$$p_{P \circ Q} = Q^{-1}(p_P). \quad (42)$$

Consequently

$$p_{P \circ Q} = p_Q \iff p_P = \infty \iff P \in B_\infty. \quad (43)$$

Proof. $(P \circ Q)^{-1} = Q^{-1} \circ P^{-1}$, so $p_{P \circ Q} = (Q^{-1} \circ P^{-1})(\infty) = Q^{-1}(p_P)$, which is (42). For the criterion: since Q^{-1} is injective on the projective line, $Q^{-1}(p_P) = Q^{-1}(\infty)$ holds exactly when $p_P = \infty$, and $P^{-1}(\infty) = \infty$ holds exactly when $P(\infty) = \infty$, that is $C_P = 0$, that is $P \in B_\infty$. $\square$

Lemma 4.6 says that the pole moves by the action of the *next* operator on the projective line: the new pole is the old one moved by Q^{-1} . It also says that a step leaves the pole unchanged exactly when the step is affine. Applied to a history, the two statements give the rigidity criterion.

Proposition 4.7 (Rigidity of the pole along a history). *Let $F_1, \dots, F_n$ be nondegenerate Möbius sections in chronological order, $G_k = F_k \circ \dots \circ F_1$, and $p_k = G_k^{-1}(\infty)$. The following are equivalent:*

1. $p_k = p_{k-1}$ for $k = 1, \dots, n$;
2. $F_k \in B_\infty$ for $k = 1, \dots, n$, i.e. every lower-left entry vanishes;
3. $p_k = \infty$ for $k = 1, \dots, n$.

Equivalently, the pole sequence is not constantly ∞ exactly when the history leaves the affine sector, and then the first index at which $C_k \neq 0$ is also the first index at which the tangency point is finite.

Proof. Apply Lemma 4.6 with $P = F_k$ and $Q = G_{k-1}$: the step $G_{k-1} \rightarrow G_k$ leaves the pole unchanged exactly when $F_k \in B_\infty$. This gives (1) $\Leftrightarrow$ (2). Since $G_0 = \text{id}$ and $p_0 = G_0^{-1}(\infty) = \infty$, the equivalence (2) $\Leftrightarrow$ (3) follows by induction: if every F_k is affine then every prefix is affine, so $C_k = 0$ and $p_k = \infty$; conversely if some $C_k \neq 0$ then $p_k = -D_k/C_k \neq \infty$ by the definition of the pole in Section 4. $\square$

Remark 4.8 (The pole is read off one factor of the factorization). When $C_k \neq 0$, the factorization (54) of G_k reads

$$G_k = T_{A_k/C_k} \circ D_{\Delta_k/C_k^2} \circ J \circ T_{D_k/C_k},$$

and the pole is $p_k = G_k^{-1}(\infty) = -D_k/C_k$, i.e. the negative of the translation parameter of the innermost factor. Thus of the four factors, the one that determines the tangency point of the pencil is the rightmost, while the value orbit $G_k(z_0)$ depends on the whole word. This is the precise sense in which the pole and the point are independent data of one prefix, and it is the reading of the cell decomposition (55) in terms of tangency: the pole is ∞ exactly in the affine cell, and it is finite exactly in the projective cell.

Remark 4.9 (Prospective interface). When a history leaves B_∞ , consecutive pencils $\mathcal{R}_k$ and $\mathcal{R}_{k+1}$ are based at different points of $\partial_\infty \mathbb{H}^2$ and therefore have no common tangency point; the families cannot be nested, and the pair carries a nonzero relative datum. Quantifying that datum is not part of this paper, whose scope fixes it to the poles and pencils themselves. A stabilizer-valued transport on the projective quotient that measures it is constructed in Paper IV; this remark is a prospective interface only and is not used anywhere in the present paper.

4.5 The status of ripple geometry

The preceding constructions are elementary and proved. The name “ripple geometry” is not, and it is important to separate the two.

- *What is proved.* A real orientation-preserving Möbius map pulls the standard horocycle $\mathcal{H}_t$ back to a line (when its pole is infinity) or to a circle tangent to the boundary at its pole, and these circles are nested; the pole is the inverse image of infinity; the same prefix matrix determines the point orbit, the pole, the fixed points, and the ripple pencil.
- *What is descriptive.* “Ripple geometry” names the dual reading in which the operator acts on conditions instead of points. It does not assert a new isomorphism class of hyperbolic or projective geometry, a new invariant of arithmetic words, or the existence of a projective arithmetic expression space.
- *What is open.* Whether the ripple family, taken together with its moving pole, is intrinsic to the arithmetic word rather than to a chosen upper-half-plane chart; which path or ripple data descend through relations among history words; and whether a punctured model in the sense of Section 11 carries a ripple structure that survives at the puncture.

The reason for keeping the name is that the dual reading is what makes division homogeneous with the other operations at the level of *conditions*: addition, subtraction, and multiplication by a constant move the value and hence move the contours of the assignment, while second-slot division moves the contours without moving the value affinely. The reason for keeping it provisional is that no intrinsic formulation of that observation is available yet; the current statement of it depends on the standard horocycle $\mathcal{H}_t$ and therefore on one chart.

5 Reciprocals, Poles, Infinity, and Fixed Points

Right-slot division makes three elementary notions meet: reciprocal arithmetic, a pole in an affine chart, and the projective point at infinity. Iteration adds a fourth notion: a finite or infinite expression can be the closure of a recurrence at one of its fixed points. We keep the algebraic fixed-point equation, analytic convergence, and projective continuation distinct.

5.1 Arithmetic reciprocals and projective poles

For $c \neq 0$, the ordinary map

$$J_c(z) = \frac{c}{z}$$

is defined on $K \setminus \{0\}$. Its matrix is

$$J_c \leftrightarrow \begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix},$$

so projectively

$$J_c(0) = \infty, \quad J_c(\infty) = 0.$$

This continuation does not make ordinary division by zero admissible.

The reciprocal coordinate centered at a finite point q is

$$w_q(z) = -\frac{1}{z - q}. \quad (44)$$

The sign is chosen so that w_q has positive determinant and therefore maps the upper half-plane to itself; the positive chart $1/(z - q)$ has determinant -1 , maps the upper half-plane to the lower one, and is not used for the horocycle discussion below. The coordinate sends q to infinity and infinity to zero. The rational functions whose only projective pole is q —equivalently, those regular on $\mathbb{P}^1 \setminus \{q\}$ and at the old point infinity—form

$$K[w_q] = K\left[\frac{1}{z - q}\right];$$

the sign does not change this ring. The usual polynomial chart $K[z]$ is the corresponding ring when the chosen pole is ∞ . This is a coordinate statement, not a claim that one pole gauge is globally preferred.

By Theorem 4.2, whose positivity hypothesis w_q now satisfies, the same q is the common boundary tangency point of the horocycles pulled back through w_q . The denominator zero and the ripple center are therefore two readings of one projective datum.

Warning 5.1 (Negative-determinant prefixes). Not every non-degenerate real Möbius prefix has positive determinant: the elementary contexts c/z with $c > 0$ and $c - z$ have determinant $-c$ and -1 respectively. The ripple statements of Section 4 therefore apply to the positive-determinant prefixes; for a general prefix one either conjugates by the reflection $z \mapsto -z$, which is the sign change of the assignment listed in Subsection 3.5, or extends the action anti-holomorphically by $z \mapsto (A\bar{z} + B)/(C\bar{z} + D)$, which preserves the upper half-plane and realizes the same fractional linear arithmetic on the real boundary. No ripple statement is made here for the negative-determinant case beyond this remark.

5.2 A reciprocal from an infinite expansion

Let

$$S_0 = 0, \quad S_{n+1} = 1 + rS_n. \quad (45)$$

Then

$$S_n = 1 + r(1 + r(\cdots(1 + r \cdot 0) \cdots)) = \sum_{k=0}^{n-1} r^k.$$

Scalar multiplication is commutative, so this example does not by itself separate left and right slots; it shows how a reciprocal can arise as recursive closure.

Proposition 5.2 (Finite and infinite geometric expansion). *If $r \neq 1$, then*

$$S_n = \frac{1 - r^n}{1 - r}. \quad (46)$$

If $K = \mathbb{R}$ or $\mathbb{C}$ and $|r| < 1$, then

$$S_n \longrightarrow \frac{1}{1 - r}.$$

Formally, $(1 - r)^{-1} = \sum_{k \geq 0} r^k$ in $K[[r]]$.

Proof. Multiplication of the finite sum by $1 - r$ telescopes to $1 - r^n$. Under $|r| < 1$, $r^n \rightarrow 0$. The formal identity follows coefficient by coefficient. $\square$

The convergence hypothesis is active. For $r = -1$ the sequence oscillates, and for $|r| > 1$ it generally does not converge to the algebraically existing reciprocal.

5.3 Fixed-point closure and escape to infinity

Put

$$T_r(s) = 1 + rs.$$

For $r \neq 1$, its finite fixed point is

$$q_r = \frac{1}{1 - r}. \quad (47)$$

Thus, when $|r| < 1$, the geometric-series truncations converge to the fixed point of their generating update.

For $r \neq 0$, the affine update is invertible and has projective matrix

$$M_r = \begin{pmatrix} r & 1 \\ 0 & 1 \end{pmatrix};$$

it fixes ∞ . The exceptional value $r = 0$ gives the constant update $T_0(s) = 1$, whose matrix is singular and which is not an element of PGL_2 . This degenerate case still has the finite fixed point $q_0 = 1$ but carries no invertible projective dynamics.

As $r \rightarrow 1$, the finite point q_r approaches ∞ in $\mathbb{P}^1$. At $r = 1$, $T_1(s) = s + 1$ has no finite fixed point and has ∞ as its repeated projective fixed point. The pole of $r \mapsto 1/(1 - r)$ is therefore exactly the parameter at which finite fixed-point closure becomes parabolic closure at infinity.

Proposition 5.3 (Affine iteration). *Let $T(z) = az + b$ with $a \neq 0$. If $a \neq 1$, put $q = b/(1 - a)$. Then*

$$T(z) = q + a(z - q), \quad T^n(z) = q + a^n(z - q). \quad (48)$$

Over $\mathbb{R}$ or $\mathbb{C}$, every finite orbit converges to q when $|a| < 1$. If $a = 1$ and $b \neq 0$, then $T^n(z) = z + nb$ and the only projective fixed point is ∞ .

Proof. The relation $(1 - a)q = b$ gives the centered form, and induction gives its n th iterate. $\square$

Algebraic existence, attraction, and projective fixed-point structure are different statements; they coincide only under additional hypotheses.

5.4 Right reciprocal iteration and continued fractions

A stationary right reciprocal update is

$$F(z) = a + \frac{b}{z}, \quad b \neq 0, \quad (49)$$

with matrix

$$M_F = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}.$$

Ordinary iteration is defined only while every intermediate denominator is nonzero. Its finite truncations are generalized continued fractions; the matrix product remains projectively meaningful

when a truncation reaches infinity. This is the standard matrix mechanism behind continued fractions [11]; the arithmetic reading of the same truncations used here is the specialization to two-step words.

Proposition 5.4 (Reciprocal fixed points). *The finite fixed points of (49) solve*

$$z^2 - az - b = 0. \quad (50)$$

At a nonzero fixed point q ,

$$F'(q) = -\frac{b}{q^2}.$$

Over $\mathbb{C}$, q is locally attracting when $|F'(q)| < 1$ and locally repelling when $|F'(q)| > 1$.

Proof. Multiply $q = a + b/q$ by q and differentiate F . □

Example 5.5 (The golden-ratio expansion). For $F(z) = 1 + 1/z$ and $z_0 = 1$, the orbit is

$$1, \quad 2, \quad \frac{3}{2}, \quad \frac{5}{3}, \quad \frac{8}{5}, \dots$$

The fixed points are

$$\phi = \frac{1 + \sqrt{5}}{2}, \quad -\phi^{-1} = \frac{1 - \sqrt{5}}{2}.$$

Since $|F'(\phi)| = \phi^{-2} < 1$, the positive fixed point is attracting on the relevant positive branch. The infinite continued fraction

$$1 + \frac{1}{1 + \frac{1}{1 + \dots}}$$

denotes ϕ only after this convergence has been established; the fixed-point equation alone does not select an attracting root in every field or domain.

For nonstationary maps

$$F_k(z) = a_k + \frac{b_k}{z}, \quad M_k = \begin{pmatrix} a_k & b_k \\ 1 & 0 \end{pmatrix},$$

the n th truncation is represented by $M_n \cdots M_1$. The nested expression, the recurrence, and matrix multiplication are the same finite calculation.

5.5 General projective fixed points

For

$$F(z) = \frac{Az + B}{Cz + D}, \quad AD - BC \neq 0,$$

a finite fixed point satisfies

$$Cz^2 + (D - A)z - B = 0. \quad (51)$$

In homogeneous coordinates, fixed points are the eigenlines of the representing matrix. Infinity is fixed exactly when $C = 0$. Over an algebraic closure a nonidentity projective map has two fixed points counted with multiplicity; over the base field they may be distinct, repeated, or visible only after a field extension.

The elementary chain is now complete:

$$\text{reciprocal} \longrightarrow \text{pole} \longrightarrow \infty \longrightarrow \text{fixed-point closure} \longrightarrow \text{matrix eigenline}.$$

6 Matrix and Projective Unification

The two geometric pictures now have a common algebraic carrier. A one-hole arithmetic section is represented by a 2×2 matrix; chronological composition is matrix multiplication, point paths are matrix orbits, ripple pencils are inverse images under the same products, poles are inverse images of infinity, and fixed points are eigenlines.

6.1 Elementary arithmetic matrices

Let K be a field. In homogeneous coordinates, write

$$z = [z : 1], \quad \infty = [1 : 0],$$

and let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

act by

$$M \cdot z = \frac{Az + B}{Cz + D}.$$

Nonzero scalar multiples represent the same element of $\mathrm{PGL}_2(K)$.

The nondegenerate elementary sections have the following representatives.

one-hole section	affine formula	matrix
$C_{+,c}^{(1)} = C_{+,c}^{(2)}$	$z + c$	$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$
$C_{-,c}^{(1)}$	$z - c$	$\begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}$
$C_{-,c}^{(2)}$	$c - z$	$\begin{pmatrix} -1 & c \\ 0 & 1 \end{pmatrix}$
$C_{\times,c}^{(1)} = C_{\times,c}^{(2)}$	cz	$\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$
$C_{/,c}^{(1)}$	z/c	$\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$
$C_{/,c}^{(2)}$	c/z	$\begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix}$

Here $c \neq 0$ whenever invertibility requires it. The table has two immediate consequences.

1. Every pure slot-(1) section is affine and fixes ∞ .
2. The projectively new primitive is slot-(2) division. Up to a nonzero scaling, it is inversion.

Ordinary arithmetic and projective continuation remain separate. A matrix in $\mathrm{PGL}_2(K)$ acts everywhere on $\mathbb{P}^1(K)$, while the corresponding ordinary word may be undefined at an intermediate denominator zero.

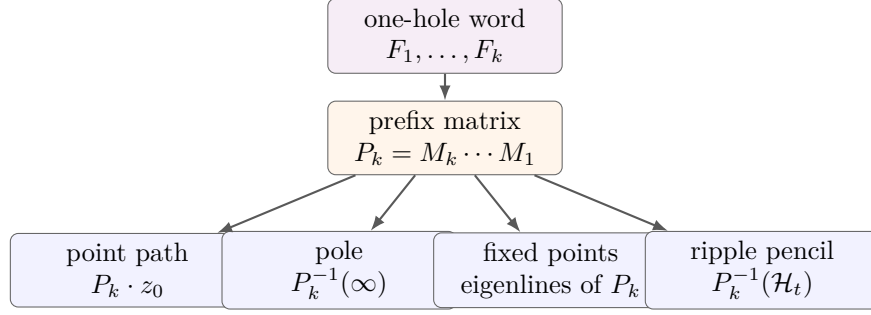


Figure 4: Four geometric readings of the same chronological matrix product.

6.2 Chronological products

Let $F_1, \dots, F_n$ be nondegenerate sections with matrix representatives $M_1, \dots, M_n$, and suppose F_1 is applied first. Define

$$P_k = M_k M_{k-1} \cdots M_1, \quad G_k = F_k \circ \cdots \circ F_1. \quad (52)$$

Then P_k represents G_k .

Theorem 6.1 (One matrix, four readings). *Write*

$$P_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}.$$

The same prefix matrix determines:

1. *the point-path state*

$$z_k = G_k(z_0) = \frac{A_k z_0 + B_k}{C_k z_0 + D_k};$$

2. *the projective pole*

$$p_k = G_k^{-1}(\infty) = -D_k/C_k$$

when $C_k \neq 0$, and $p_k = \infty$ when $C_k = 0$;

3. *the fixed-point equation*

$$C_k z^2 + (D_k - A_k)z - B_k = 0;$$

4. *over $\mathbb{R}$, when a lift has positive determinant, the ripple pencil $G_k^{-1}(\mathcal{H}_t)$ from Theorem 4.2.*

Proof. The first formula is the matrix action. The pole solves $C_k z + D_k = 0$. The fixed-point equation comes from $z(C_k z + D_k) = A_k z + B_k$. The final statement is the definition of the ripple pencil together with formula (39). $\square$

Thus *path* and *ripple* are not two unrelated representations. They are the covariant and contravariant geometries of one chronological matrix product.

6.3 The affine chart as the stabilizer of infinity

Let

$$B_\infty = \text{Stab}_{\text{PGL}_2(K)}(\infty).$$

A matrix fixes infinity exactly when $C = 0$. After projective rescaling, every such element has a unique representative

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a \in K^\times, \quad b \in K,$$

and acts as $z \mapsto az + b$. Hence

$$B_\infty \cong \text{Aff}(1, K). \quad (53)$$

This identifies the path geometry of Section 3 as a pole-at-infinity gauge. In that gauge:

- the denominator is constant;
- the ripple pencil is a parallel-line pencil;
- fixed-point recursion is affine;
- the lower-left matrix entry stays zero.

Choosing a finite pole q amounts to conjugating this affine subgroup by a projective map sending q to infinity. The coordinate $w_q = -1/(z - q)$ from (44), whose sign is fixed by the positivity convention of Section 4, is the simplest such gauge.

6.4 Generation of the projective group

Introduce

$$T_s(z) = z + s, \quad D_q(z) = qz \quad (q \neq 0), \quad J(z) = -\frac{1}{z}.$$

They arise respectively from addition, multiplication, and second-slot division.

Theorem 6.2 (Elementary projective generation). *For every field K , translations T_s , nonzero scalings D_q , and inversion J generate $\text{PGL}_2(K)$. Equivalently, the projective evaluations of nondegenerate left- and right-slot arithmetic sections have image $\text{PGL}_2(K)$.*

Proof. Let

$$F(z) = \frac{Az + B}{Cz + D}, \quad AD - BC \neq 0.$$

If $C = 0$, then $A, D \neq 0$ and

$$F = T_{B/D} \circ D_{A/D}.$$

If $C \neq 0$, direct calculation gives

$$F = T_{A/C} \circ D_{(AD-BC)/C^2} \circ J \circ T_{D/C}. \quad (54)$$

All scale parameters are nonzero, so every factor is available. These two cases exhaust $\text{PGL}_2(K)$. $\square$

The proof also yields the rank-one Bruhat decomposition

$$\text{PGL}_2(K) = B_\infty \sqcup B_\infty J B_\infty. \quad (55)$$

The first cell consists of operators whose pole is infinity. The second consists of operators with a finite pole in the chosen affine chart. Thus the transition from a point-path grid to a curved ripple pencil is the elementary geometric face of crossing from the affine cell to the projective cell.

6.5 Left and right representations in one projective line

We can now answer the algebraic comparison of the two combs precisely.

Theorem 6.3 (Projective unification of the two combs). *The left- and right-expanded combs have the same abstract chain of internal dependencies and are exchanged by the opposite-operation involution of Proposition 2.4. Their one-hole operator images have the following relation:*

1. *the pure left-expanded nondegenerate language lies in $B_\infty \cong \text{Aff}(1, K)$;*
2. *the pure right-expanded language already contains translations, nonzero scalings, reflections, and inversion, and therefore generates $\text{PGL}_2(K)$;*
3. *after embedding both languages into $\text{PGL}_2(K)$, their point paths, poles, fixed points, and ripple pencils are all read from the same matrix representation.*

Proof. The dependency and opposition statements were proved in Section 2. The elementary table shows that every slot-(1) generator fixes infinity. The pure slot-(2) language contains T_s because addition is commutative, contains D_q because multiplication is commutative, and contains J through second-slot division. Apply Theorem 6.2. The last assertion is Theorem 6.1. $\square$

The theorem does not say that a left-expanded word and a right-expanded word with the same constants have the same operator. It says that their apparently different elementary geometries live in one projective action and can be compared there without erasing operand-slot information.

7 Affine Cocycles and the Relative Defect

We now restrict attention to the affine motions of Section 3. Every evaluated history then has the form

$$x \mapsto \Phi x + \xi, \quad \Phi \in K^\times, \quad \xi \in K.$$

The accumulated scale Φ and translation ξ retain more of the process than an endpoint value. Their composition law forces two natural translation coordinates. The target-frame coordinate ξ weights each translation by the scales that occur after it; the source-normalized coordinate $\hat{\xi} = \Phi^{-1}\xi$ discounts it by the scales accumulated up to that point. These two coordinates are the discrete data whose commutativized shadow is constructed in Section 8.

Throughout Subsections 7.1–7.4, K is an arbitrary field and every affine multiplier is nonzero. The differential calculation in Subsection 7.3 is over $\mathbb{R}$.

7.1 Discrete affine evaluation

Let

$$f_i(x) = s_i x + t_i, \quad s_i \in K^\times, \quad t_i \in K,$$

and regard $(f_1, \dots, f_n)$ as a chronological history: f_1 acts first. Write

$$f_n \circ \dots \circ f_1(x) = \Phi_n x + \xi_n. \tag{56}$$

For the empty history we set $\Phi_0 = 1$ and $\xi_0 = 0$.

Proposition 7.1 (Affine cocycle formulas). *For a chronological affine history $(f_1, \dots, f_n)$,*

$$\Phi_n = \prod_{i=1}^n s_i, \quad (57)$$

$$\xi_n = \sum_{i=1}^n t_i \prod_{j=i+1}^n s_j. \quad (58)$$

The empty product is 1. Defining the source-normalized translation by

$$\hat{\xi}_n = \Phi_n^{-1} \xi_n, \quad (59)$$

one also has

$$\hat{\xi}_n = \sum_{i=1}^n \frac{t_i}{\prod_{j=1}^i s_j}. \quad (60)$$

Proof. We prove the first two identities simultaneously by induction on n . They hold for $n = 0$. Suppose

$$f_n \circ \dots \circ f_1(x) = \Phi_n x + \xi_n.$$

Postcomposing chronologically by f_{n+1} gives

$$f_{n+1}(\Phi_n x + \xi_n) = s_{n+1} \Phi_n x + (s_{n+1} \xi_n + t_{n+1}).$$

Therefore

$$\Phi_{n+1} = s_{n+1} \Phi_n = \prod_{i=1}^{n+1} s_i.$$

Using the induction hypothesis for ξ_n ,

$$\begin{aligned} \xi_{n+1} &= s_{n+1} \sum_{i=1}^n t_i \prod_{j=i+1}^n s_j + t_{n+1} \\ &= \sum_{i=1}^n t_i \prod_{j=i+1}^{n+1} s_j + t_{n+1} \prod_{j=n+2}^{n+1} s_j, \end{aligned}$$

which is exactly (58) for $n + 1$.

Every s_i is nonzero, so Φ_n is invertible. Dividing the i th term of (58) by $\Phi_n = \prod_{j=1}^n s_j$ gives

$$\frac{t_i \prod_{j=i+1}^n s_j}{\prod_{j=1}^n s_j} = \frac{t_i}{\prod_{j=1}^i s_j}.$$

Summation proves (60). □

Formula (58) is a future-weighting rule. A translation t_i is expressed in target coordinates only after every later scale $s_{i+1}, \dots, s_n$ has acted. Formula (60) is a past-normalization rule: the same translation is transported back through the scale accumulated from the source through step i . Neither formula is an additional geometric postulate.

Example 7.2 (Three steps). For

$$f_1(x) = s_1 x + t_1, \quad f_2(x) = s_2 x + t_2, \quad f_3(x) = s_3 x + t_3,$$

direct substitution gives

$$f_3 f_2 f_1(x) = s_3 s_2 s_1 x + s_3 s_2 t_1 + s_3 t_2 + t_3.$$

Here and below juxtaposition of the displayed functions means ordinary functional composition $f_3 \circ f_2 \circ f_1$, not chronological word concatenation. The formula makes the future scales attached to each translation visible.

7.2 Cocycle laws under chronological concatenation

For an affine history γ , write

$$\rho(\gamma)(x) = \Phi_\gamma x + \xi_\gamma, \quad \widehat{\xi}_\gamma = \Phi_\gamma^{-1} \xi_\gamma.$$

Recall that $\gamma\delta$ means first γ and then δ , so $\rho(\gamma\delta) = \rho(\delta)\rho(\gamma)$.

Proposition 7.3 (Chronological affine cocycle law). *For affine histories γ and δ ,*

$$\Phi_{\gamma\delta} = \Phi_\delta \Phi_\gamma, \tag{61}$$

$$\xi_{\gamma\delta} = \Phi_\delta \xi_\gamma + \xi_\delta, \tag{62}$$

$$\widehat{\xi}_{\gamma\delta} = \widehat{\xi}_\gamma + \Phi_\gamma^{-1} \widehat{\xi}_\delta. \tag{63}$$

If γ is invertible, then

$$\Phi_{\gamma^{-1}} = \Phi_\gamma^{-1}, \quad \xi_{\gamma^{-1}} = -\Phi_\gamma^{-1} \xi_\gamma = -\widehat{\xi}_\gamma. \tag{64}$$

Proof. Apply $\rho(\delta)$ to $\rho(\gamma)(x)$:

$$\rho(\delta)(\Phi_\gamma x + \xi_\gamma) = \Phi_\delta \Phi_\gamma x + \Phi_\delta \xi_\gamma + \xi_\delta.$$

This proves (61) and (62). Since K is commutative,

$$\begin{aligned} \widehat{\xi}_{\gamma\delta} &= (\Phi_\delta \Phi_\gamma)^{-1} (\Phi_\delta \xi_\gamma + \xi_\delta) \\ &= \Phi_\gamma^{-1} \xi_\gamma + \Phi_\gamma^{-1} \Phi_\delta^{-1} \xi_\delta, \end{aligned}$$

which is (63). Finally, solving $y = \Phi_\gamma x + \xi_\gamma$ for x yields

$$\rho(\gamma)^{-1}(y) = \Phi_\gamma^{-1} y - \Phi_\gamma^{-1} \xi_\gamma,$$

and hence (64). □

These formulas show why the translation coordinate is a cocycle rather than an additive homomorphism. The semidirect-product action of scale on translation is part of the arithmetic composition law.

7.3 Left and right affine differentials

We now pass to the positive real affine group and write

$$g(M, \xi) = \begin{pmatrix} e^M & \xi \\ 0 & 1 \end{pmatrix}. \tag{65}$$

The symbol M denotes accumulated logarithmic scale; it is distinct from the fixed infinitesimal intensity λ used in later flow equations.

Proposition 7.4 (Left and right affine differentials). *For the matrix g in (65), the translation components of the left and right Maurer–Cartan forms are*

$$(g^{-1} dg)_{\text{tr}} = e^{-M} d\xi, \tag{66}$$

$$(dg g^{-1})_{\text{tr}} = d\xi - \xi dM. \tag{67}$$

If $\widehat{\xi} = e^{-M} \xi$, then

$$d\xi - \xi dM = e^M d\widehat{\xi}. \tag{68}$$

Proof. We have

$$g^{-1} = \begin{pmatrix} e^{-M} & -e^{-M}\xi \\ 0 & 1 \end{pmatrix}, \quad dg = \begin{pmatrix} e^M dM & d\xi \\ 0 & 0 \end{pmatrix}.$$

Multiplication on the two sides gives

$$g^{-1}dg = \begin{pmatrix} dM & e^{-M}d\xi \\ 0 & 0 \end{pmatrix}$$

and

$$dg g^{-1} = \begin{pmatrix} dM & d\xi - \xi dM \\ 0 & 0 \end{pmatrix}.$$

The upper-right entries prove (66) and (67). Differentiating $\hat{\xi} = e^{-M}\xi$ gives

$$d\hat{\xi} = e^{-M}(d\xi - \xi dM),$$

which is equivalent to (68). $\square$

The left Maurer–Cartan form $g^{-1}dg$ measures velocity in body or source coordinates; the right form $dg g^{-1}$ measures it in spatial or target coordinates. This left–right group distinction is not the same as operand-slot chirality. Relating a slot choice to left or right group multiplication requires an additional rule specifying how a new context is inserted into a represented history. No such identification is assumed here.

There are likewise two separate coordinate statements. The matrix coordinate ξ is the target-frame translation in $x \mapsto e^M x + \xi$, whereas $\hat{\xi} = e^{-M}\xi$ is its source-normalized value. Equations (66)–(68) relate their differentials but do not make the two coordinates interchangeable.

7.4 Relative affine defect

The affine cocycle permits a comparison of two histories before any global torsion theory is introduced.

Definition 7.5 (Relative affine defect). For two affine histories γ and δ , the *relative affine defect* of γ with reference δ is

$$\mathcal{K}_{\text{aff}}(\gamma, \delta) = \rho(\delta)^{-1}\rho(\gamma) \in \text{Aff}(1, K). \quad (69)$$

The pair is *scale-compatible* when $\Phi_\gamma = \Phi_\delta$.

The order of the arguments in (69) is part of the convention. Interchanging γ and δ inverts the group-valued defect.

Proposition 7.6 (Coordinates of the relative affine defect). *For arbitrary affine histories γ and δ ,*

$$\mathcal{K}_{\text{aff}}(\gamma, \delta)(x) = \frac{\Phi_\gamma}{\Phi_\delta}x + \frac{\xi_\gamma - \xi_\delta}{\Phi_\delta}. \quad (70)$$

If the pair is scale-compatible with common scale Φ , then

$$\nu_x(\gamma) - \nu_x(\delta) = \xi_\gamma - \xi_\delta, \quad (71)$$

$$\mathcal{K}_{\text{aff}}(\gamma, \delta)(x) = x + \frac{\xi_\gamma - \xi_\delta}{\Phi}. \quad (72)$$

In particular, the endpoint difference is then independent of x .

Proof. The inverse of $y \mapsto \Phi_\delta y + \xi_\delta$ is $y \mapsto \Phi_\delta^{-1}(y - \xi_\delta)$. Applying it after $x \mapsto \Phi_\gamma x + \xi_\gamma$ gives (70). If $\Phi_\gamma = \Phi_\delta = \Phi$, direct subtraction of the two endpoint formulas gives (71), and substituting the common scale into (70) gives (72). $\square$

For unequal scales, the endpoint difference contains $(\Phi_\gamma - \Phi_\delta)x$ and depends on the initial value. In that case the group-valued comparison remains defined, but there is no base-point-independent scalar translation defect of the form (71).

Example 7.7 (The elementary affine order defect). Fix $\mu \in K$ and $q \in K^\times$. Let γ first add μ and then scale by q , while δ first scales by q and then adds μ . Then

$$\rho(\gamma)(x) = q(x + \mu) = qx + q\mu, \quad \rho(\delta)(x) = qx + \mu.$$

The histories have the same scale q , and hence

$$\nu_x(\gamma) - \nu_x(\delta) = \mu(q - 1), \quad \mathcal{K}_{\text{aff}}(\gamma, \delta)(x) = x + \mu(1 - q^{-1}). \quad (73)$$

Over the positive real group, writing $q = e^M$ gives the endpoint defect $\mu(e^M - 1)$.

Equation (73) is an open comparison of two chronological histories. It is not a closed commutator-loop holonomy, a comparison with path inverse, or a mirror invariant. Section 8 places compatible two-history defects in the global torsion framework.

7.5 Propagation of a change in the initial accumulator

The accumulated scale also controls sensitivity to the initial value.

Proposition 7.8 (Exact affine perturbation propagation). *Let γ be an affine history with scale Φ_γ . For any $x, \tilde{x} \in K$,*

$$\nu_{\tilde{x}}(\gamma) - \nu_x(\gamma) = \Phi_\gamma(\tilde{x} - x). \quad (74)$$

When $\tilde{x} \neq x$,

$$\frac{\nu_{\tilde{x}}(\gamma) - \nu_x(\gamma)}{\tilde{x} - x} = \Phi_\gamma. \quad (75)$$

Proof. The affine normal form gives

$$\nu_{\tilde{x}}(\gamma) = \Phi_\gamma \tilde{x} + \xi_\gamma, \quad \nu_x(\gamma) = \Phi_\gamma x + \xi_\gamma.$$

Subtracting cancels the translation cocycle and proves (74). Division by the nonzero element $\tilde{x} - x$ proves (75). $\square$

This is an exact discrete identity, not a linear approximation. The continuous arithmetic motion of Section 3 is obtained by differentiating the same semidirect-product action. Supplementary expansions and convention checks appear in Appendix C; Subsection C.1 recovers the elementary alternating-expression derivation from which this viewpoint developed.

8 Commutativization: The ACS and Arithmetic Tearing

The arithmetic motions of Section 3 do not commute: addition and multiplication transport each other's rulers by the Baumslag–Solitar-type relation (19), and the affine cocycle of Section 7 records the resulting order dependence in two translation coordinates. This section constructs a commutativization of that non-commutative motion, and the level at which commutation is imposed must be stated exactly.

Take the history language itself: the free monoid generated by the two families of one-step histories, additive steps and multiplicative steps. Impose that generators of the two families commute. Equivalently — and this is the form used below — keep the total additive charge and the total logarithmic multiplicative charge of a history and forget the order in which they were accumulated; the charges satisfy the imposed commutation by construction. The resulting object is a plane with two coordinates, and we call it the *accumulative commutative space*, ACS. The construction has two distinct uses, and they must not be conflated. For a positive add–scale history it records the two total charges; and it supplies a weighted integral along the charge path whose value restores the translation part that the charge endpoint alone forgets. The ACS is therefore a *shadow* of the presented history: it is a commutative quotient object attached to a non-commutative one, not another presentation of the full arithmetic syntax.

This is *not* the abelianization of the group of affine motions, and the difference is worth recording once. In $\text{Aff}^+(1, \mathbb{R})$ one has $D_k T_p D_k^{-1} T_p^{-1} = T_{(k-1)p}$, so the commutator subgroup is the translation subgroup and the abelianization keeps the scale alone:

$$\text{Aff}^+(1, \mathbb{R})_{\text{ab}} \cong \mathbb{R}_{>0} \cong (\mathbb{R}, +).$$

The ACS keeps the additive charge as well, because it is attached to the presented histories *before* evaluation rather than to their images in the affine group. The hierarchy — literal histories, then charges, then evaluated motions — is accordingly not a convenience of exposition; collapsing it changes the object. Subsection 8.6 states what the ACS forgets, and the distributivity example of Subsection 2.3 already shows why the charge map cannot be pushed through the operator quotient.

The geometric use is the reason for the name of this chapter. Two histories with the same charge endpoint have charge paths that together bound a region of the ACS, and the exact value of their order defect is a weighted area of that region (Theorem 8.9). In Section 8.7 we read that identity as a statement about the original AES: the arithmetic space carries a defect that cannot be seen in any single motion, and the geometry in which it becomes an area is the commutativized charge geometry of the presented histories.

Throughout this section an elementary additive step and an elementary scaling step are denoted by

$$A_p(x) = x + p, \quad M_q(x) = e^q x, \quad p, q \in \mathbb{R}.$$

For a history $\gamma = (g_1, \dots, g_n)$, the entries are performed from left to right, so that

$$\nu_x(\gamma) = g_n \circ \dots \circ g_1(x).$$

Negative values of p and q are allowed. Thus the same notation covers inverse translations and inverse positive scalings. In this section an *add–scale history* means a finite word in these two families of generators. Such a word evaluates in $\text{Aff}^+(1, \mathbb{R})$. The ACS construction is attached to this presented word; it is not asserted for negative multipliers or for an arbitrary affine syntax before that syntax has been written in add–scale form.

8.1 The charge path

Definition 8.1 (Accumulative commutative space and charge path). The *accumulative commutative space*, abbreviated ACS, is the oriented plane

$$\mathcal{A} = \mathbb{R}_{(A,M)}^2, \quad \text{with orientation } dA \wedge dM.$$

An additive step A_p has charge increment $(p, 0)$, and a scaling step M_q has charge increment $(0, q)$. Starting at $(A_0, M_0) = (0, 0)$, accumulate these increments in chronological order. The resulting oriented broken path is denoted by C_γ , and its terminal point by

$$(A_\gamma, M_\gamma) = \left(\sum_{g_i=A_{p_i}} p_i, \sum_{g_i=M_{q_i}} q_i \right).$$

The terminal charge is insensitive to order, whereas the broken path generally is not. In particular, two histories may have the same ACS endpoint while inducing different affine maps. The missing information is recovered by the following integral formula.

Proposition 8.2 (Direct ACS evaluation). *For every add-scale history γ and every initial value x ,*

$$\nu_x(\gamma) = e^{M_\gamma} \left(x + \int_{C_\gamma} e^{-M} dA \right). \quad (76)$$

Proof. Let an additive step A_{p_i} occur when the cumulative multiplicative charge is M_{i-1} . Along the corresponding horizontal edge of C_γ ,

$$\int e^{-M} dA = p_i e^{-M_{i-1}}.$$

After that addition, the remaining multiplicative steps have total charge $M_\gamma - M_{i-1}$. Consequently the contribution of p_i to the final value is

$$p_i e^{M_\gamma - M_{i-1}} = e^{M_\gamma} p_i e^{-M_{i-1}}.$$

The initial value contributes $e^{M_\gamma} x$, and vertical ACS edges contribute nothing to the dA -integral. Summing over the additive steps gives (76). The argument uses oriented edge integrals, so it also covers negative increments. $\square$

The kernel e^{-M} is therefore forced by the affine cocycle: it normalizes an additive contribution to the source frame, before the common terminal scale e^{M_γ} returns it to the target frame.

8.2 The history group and the descent of the charges

The preceding subsections attached charges to words. To say precisely what the ACS is, and in particular what it is *not*, it is useful to say which words are being identified. The identifications used below are only the ones the paper has already made implicitly.

Convention 8.3 (The history group). Let

$$G = (\mathbb{R}, +) * (\mathbb{R}, +) = \langle A_p, M_q \mid A_p A_{p'} = A_{p+p'}, M_q M_{q'} = M_{q+q'}, A_0 = M_0 = 1 \rangle$$

be the free product of two copies of the additive group of $\mathbb{R}$, written multiplicatively, with generators A_p (additive steps) and M_q (multiplicative steps). A *history* is an element of G together with a chosen word representing it. Two words differing only by regrouping consecutive steps of one family represent the same element of G and have the same charge chain and the same evaluation, so nothing is lost by this convention; words which interleave the two families are *not* identified.

Proposition 8.4 (Charge endpoint and evaluation are homomorphisms). *The charge endpoint and the evaluation extend uniquely to homomorphisms*

$$\begin{aligned} c : G &\longrightarrow \mathbb{R}^2, & A_p &\longmapsto (p, 0), & M_q &\longmapsto (0, q), \\ \nu : G &\longrightarrow \text{Aff}^+(1, \mathbb{R}), & A_p &\longmapsto T_p, & M_q &\longmapsto D_{e^q}, \end{aligned}$$

where $T_p(z) = z + p$ and $D_k(z) = kz$. Both are surjective. Moreover the charge chain C_γ is a function of the element of G and not only of the word: regrouping consecutive steps of one family replaces consecutive edges of one charge path by the single edge with the same endpoints.

Proof. Each family is generated by elements satisfying the only relations imposed within it, so a homomorphism is determined by its values on the generators and is well defined; the images stated are themselves homomorphisms of $(\mathbb{R}, +)$. Surjectivity of c is clear, and surjectivity of ν holds because translations and positive dilations generate $\text{Aff}^+(1, \mathbb{R})$. For the last claim, a regrouping $A_p A_{p'} \mapsto A_{p+p'}$ joins two consecutive horizontal edges of C_γ into the horizontal edge with the same endpoints and the same orientation, and the vertical case is identical. $\square$

Proposition 8.5 (The ACS endpoint is the abelianization of the history group). *The charge homomorphism c is the abelianization of G : its kernel is the commutator subgroup,*

$$\ker c = [G, G], \quad \text{so} \quad G_{\text{ab}} \cong \mathbb{R}^2$$

and the charge plane of Definition 8.1 is the abelianization of the history group.

Proof. For any groups F, H one has $(F * H)_{\text{ab}} \cong F_{\text{ab}} \oplus H_{\text{ab}}$, because abelianization is left adjoint to the inclusion of abelian groups and therefore preserves coproducts. With $F = H = (\mathbb{R}, +)$, which is abelian, this gives $G_{\text{ab}} \cong \mathbb{R} \oplus \mathbb{R}$. Since c is a surjection onto the abelian group $\mathbb{R}^2$, it factors through G_{ab} as an isomorphism, so $\ker c = [G, G]$. $\square$

Now the question the review of this paper raised: does the charge pass to the *evaluated* motion? The next theorem answers it, and the answer is asymmetric.

Theorem 8.6 (Only the multiplicative charge descends). *Fix $q \neq 0$ and $p \neq 0$ and put*

$$h(p, q) = M_q A_p M_{-q} A_{-e^{-q}p} \in G. \quad (77)$$

Then:

1. $h(p, q)$ is a nontrivial element of $\ker \nu$; that is, $h(p, q)$ evaluates to the identity motion while being a nonempty history;
2. its charges are

$$c(h(p, q)) = (p(1 - e^{-q}), 0);$$

3. the multiplicative charge $M : G \rightarrow \mathbb{R}$, $M = \text{pr}_2 \circ c$, factors through ν : it equals $\log \circ L \circ \nu$, where $L : \text{Aff}^+(1, \mathbb{R}) \rightarrow \mathbb{R}_{>0}$ is the linear part ($z \mapsto \Phi z + \xi \mapsto \Phi$);
4. the additive charge $A : G \rightarrow \mathbb{R}$, $A = \text{pr}_1 \circ c$, does not factor through ν : there is no homomorphism $\varphi : \text{Aff}^+(1, \mathbb{R}) \rightarrow \mathbb{R}^2$ with $c = \varphi \circ \nu$. Equivalently, the additive charge is a property of the history and not of the motion it induces.

Proof. (1) Apply the motions of $h(p, q)$ in chronological order to an arbitrary z :

$$z \xrightarrow{M_q} e^q z \xrightarrow{A_p} e^q z + p \xrightarrow{M_{-q}} z + e^{-q} p \xrightarrow{A_{-e^{-q}p}} z.$$

Hence $\nu(h(p, q)) = \text{id}$. It is nontrivial in G because a reduced word of a free product is trivial only if it is empty, and $q \neq 0, p \neq 0$ make (77) reduced.

(2) Add the charge increments in the same chronological order:

$$(0, q) + (p, 0) + (0, -q) + (-e^{-q}p, 0) = (p(1 - e^{-q}), 0).$$

(3) L is a homomorphism, and on generators $\log L(T_p) = 0 = M(A_p)$ and $\log L(D_{eq}) = q = M(M_q)$; since both sides are homomorphisms on G agreeing on generators, they agree on G .

(4) If $c = \varphi \circ \nu$ for some homomorphism φ into the abelian group $\mathbb{R}^2$, then c would vanish on $\ker \nu$. But by (1) and (2), $h(p, q) \in \ker \nu$ while $c(h(p, q)) = (p(1 - e^{-q}), 0) \neq (0, 0)$ for $p \neq 0, q \neq 0$. Contradiction. (The structural reason is Proposition 8.5 together with the computation $\text{Aff}^+(1, \mathbb{R})_{\text{ab}} \cong \mathbb{R}_{>0}$ recorded in the opening of this section: every homomorphism from the motion group to an abelian group is one-dimensional, whereas the charges are two-dimensional. The explicit element $h(p, q)$ is what makes the failure a proof rather than a dimension count.) $\square$

Three consequences deserve to be stated together, because they are the precise version of statements that earlier drafts left informal.

1. *The ACS is the commutativization of the history language, and the sense of that phrase is now fixed.* It is the abelianization of G (Proposition 8.5), not of the motion group. The motion group's abelianization is one-dimensional and keeps the scale alone; the additive charge does not survive evaluation at all (Theorem 8.6).
2. *What the ACS retains beyond its abelianized endpoint is the path.* Two histories with the same charge endpoint may have different charge chains; the elementary rectangle $\gamma = (A_p, M_q)$ and $\delta = (M_q, A_p)$ is the smallest example, with equal endpoints (p, q) and different chains. Proposition 8.2 uses the chain, not the endpoint, and that is why it recovers the evaluated motion from commutative data. A charge *endpoint* observable therefore cannot see the difference, exactly as Proposition 10.2 states, while the endpoint together with the path can.
3. *The two failure directions are dual.* Histories with equal motion and different charges are the elements of $\ker \nu$ of Theorem 8.6; histories with equal charges and different motions are the pairs measured by the relative torsion of Definition 8.7. The first failure is why the charge map cannot be pushed through evaluation; the second is why the evaluation cannot be recovered from the charge endpoint. The ACS is the object in which both become computable, because it carries the endpoint (first failure visible) and the path (second failure measured by the weighted area of Theorem 8.9).

8.3 Compatibility and relative torsion

There are two useful compatibility conditions, and they should not be conflated. Histories γ and δ are *scale-compatible* if

$$M_\gamma = M_\delta.$$

They are *charge-compatible* if, in addition,

$$A_\gamma = A_\delta.$$

Scale compatibility makes their affine maps have the same linear part. Charge compatibility makes their ACS paths have both endpoints in common and hence makes $C_\gamma - C_\delta$ a closed oriented one-chain.

Definition 8.7 (Relative arithmetic torsion). For scale-compatible add-scale histories γ and δ , their *relative arithmetic torsion* is

$$\tau(\gamma, \delta) := \xi_\gamma - \xi_\delta, \quad (78)$$

where ξ_γ and ξ_δ are the target-frame translation terms of the two affine operators. This relative translation defect is the primary torsion invariant used below.

Proposition 8.8. *For every initial value x , relative arithmetic torsion equals the endpoint difference*

$$\tau(\gamma, \delta) = \nu_x(\gamma) - \nu_x(\delta),$$

and is therefore independent of x . If $M_\gamma = M_\delta = M_$, then*

$$\tau(\gamma, \delta) = e^{M_*} \left(\int_{C_\gamma} e^{-M} dA - \int_{C_\delta} e^{-M} dA \right). \quad (79)$$

Proof. Apply Proposition 8.2 to both histories. Their initial terms are respectively $e^{M_\gamma}x$ and $e^{M_\delta}x$, which cancel under scale compatibility. The remaining terms give (79). $\square$

Temporal reversal gives one possible comparison, but it is not built into the definition. The pair (γ, δ) may consist of any two scale-compatible add-scale histories. This extension is important because order defects occur between histories that are neither reversals nor planar mirrors of one another.

8.4 The torsion–Stokes formula

Suppose now that γ and δ are charge-compatible, with common terminal charge (A_*, M_*) . Define the target-normalized one-form

$$\eta_* := e^{M_* - M} dA \quad (80)$$

on the ACS. Relative torsion is then the integral of η_* over the closed one-chain $C_\gamma - C_\delta$. The chosen order of the two paths fixes the sign.

Theorem 8.9 (Weighted torsion–Stokes theorem). *Let γ and δ be charge-compatible add-scale histories with common terminal multiplicative charge M_* . Let $\Sigma_{\gamma, \delta}$ be any oriented piecewise smooth singular two-chain in the ACS satisfying*

$$\partial \Sigma_{\gamma, \delta} = C_\gamma - C_\delta.$$

Then

$$\tau(\gamma, \delta) = \int_{C_\gamma - C_\delta} \eta_* \quad (81)$$

$$= \int_{\Sigma_{\gamma, \delta}} e^{M_* - M} dA \wedge dM. \quad (82)$$

The value of the area integral is independent of the chosen filling chain.

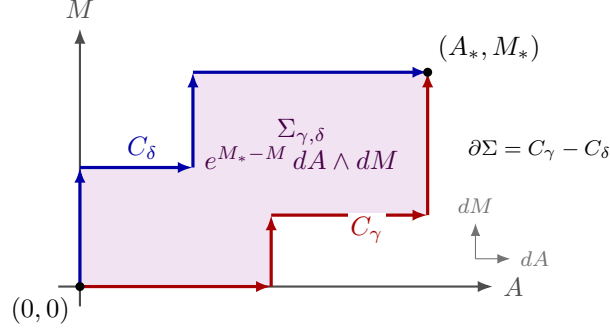


Figure 5: Two charge-compatible ACS histories. Both paths run from $(0,0)$ to the common charge endpoint (A_*, M_*) . Traversing C_γ forward and C_δ backward gives the positive boundary of the displayed filling for the orientation $dA \wedge dM$; the density is weighted by e^{M_*-M} .

Proof. Equation (79) and charge compatibility give

$$\tau(\gamma, \delta) = \int_{C_\gamma - C_\delta} e^{M_*-M} dA = \int_{C_\gamma - C_\delta} \eta_*.$$

With the orientation $dA \wedge dM$, direct differentiation yields

$$d\eta_* = d(e^{M_*-M}) \wedge dA = -e^{M_*-M} dM \wedge dA = e^{M_*-M} dA \wedge dM.$$

Stokes' theorem for piecewise smooth chains therefore gives

$$\int_{C_\gamma - C_\delta} \eta_* = \int_{\partial \Sigma_{\gamma, \delta}} \eta_* = \int_{\Sigma_{\gamma, \delta}} d\eta_*,$$

which is (82). Such a filling exists because the ACS plane is contractible. Finally, if Σ and Σ' have the same boundary, applying Stokes' theorem to each shows

$$\int_{\Sigma} d\eta_* = \int_{\partial \Sigma} \eta_* = \int_{\partial \Sigma'} \eta_* = \int_{\Sigma'} d\eta_*.$$

This also proves filling independence. The chain formulation remains valid when the broken paths self-intersect or carry signed increments; no informal notion of an enclosed region is required. $\square$

Charge compatibility is needed for the closed-chain and area formulas, but not for Definition 8.7. A merely scale-compatible pair still has a well-defined translation defect, even when the endpoints of its ACS paths have different A -coordinates.

Figure 5 depicts the chain orientation used in Theorem 8.9. It is a schematic of the theorem's data rather than an argument for filling independence.

8.5 Explicit order comparisons

The elementary rectangle. Let

$$\gamma = (A_p, M_q), \quad \delta = (M_q, A_p).$$

The path $C_\gamma - C_\delta$ is the positively oriented boundary of the rectangle $[0, p] \times [0, q]$ when $p, q > 0$. Direct evaluation and the weighted-area formula agree:

$$\begin{aligned}\tau(\gamma, \delta) &= e^q(x + p) - (e^q x + p) \\ &= p(e^q - 1) \\ &= \int_0^p \int_0^q e^{q-M} dM dA.\end{aligned}$$

Thus “add, then multiply” minus “multiply, then add” is positive under the orientation fixed in Definition 8.1. In the flow notation used in the next chapter, $p = \mu h$ and $q = \lambda k$, so the defect is

$$\mu h(e^{\lambda k} - 1).$$

A four-step staircase. For

$$\gamma = (A_p, M_r, A_q, M_s), \quad \delta = (M_r, A_p, M_s, A_q),$$

the histories have the same total charge $(p + q, r + s)$, but δ is not the temporal reversal of γ . Their evaluations are

$$\begin{aligned}\nu_x(\gamma) &= e^{r+s}x + pe^{r+s} + qe^s, \\ \nu_x(\delta) &= e^{r+s}x + pe^s + q.\end{aligned}$$

Consequently

$$\tau(\gamma, \delta) = pe^s(e^r - 1) + q(e^s - 1). \quad (83)$$

For positive p, q, r, s , both terms are positive weighted strips in an ACS filling. Formula (83) also shows how a later scale changes the weight of an earlier commutation defect.

Equal charges without reversal. The pair

$$(A_p, M_r, A_q), \quad (A_q, M_r, A_p)$$

has common charge $(p + q, r)$, and its relative torsion is

$$(p - q)(e^r - 1).$$

It may be positive, negative, or zero. In particular, equal charge does not determine torsion, while zero torsion does not imply equality of marked histories.

8.6 What the ACS forgets

At the formal level, the map

$$\gamma \mapsto (A_\gamma, M_\gamma)$$

is an order-forgetting charge homomorphism from concatenated marked histories to the additive group $\mathbb{R}^2$. It is therefore analogous to an abelianization map. This statement concerns the formal charge language only. It does not identify the symbolic history space with the evaluated affine group, and it does not assert that either object is a free group without an additional choice of formal generators and relations.

The content of Theorem 8.9 is precisely what survives this loss of order: the endpoint charges specify a common frame, while the weighted filling measures the translation defect between two histories. Section 9 retains the current value as a third coordinate and identifies the infinitesimal form of the same defect with contact curvature.

8.7 Tearing

We can now name the phenomenon that the three preceding chapters have been describing from three sides. The name is new here, the mathematics is not: it is the content of Definition 8.7 and Theorem 8.9 read geometrically.

Definition 8.10 (Tearing of arithmetic motion). Let γ and δ be two add-scale histories of a regular AES that are scale-compatible in the sense of Section 8, and let $\tau(\gamma, \delta)$ be their relative torsion. The pair is said to *tear* the AES by $\tau(\gamma, \delta)$: the two histories induce the same linear part, hence agree on the multiplicative content of the motion, while their additive content differs by exactly $\tau(\gamma, \delta)$.

The identification of that scalar defect with an area requires a second hypothesis, and the two must not be merged. By Theorem 8.9, if γ and δ are in addition *charge-compatible*, then $\tau(\gamma, \delta)$ equals the integral of the weighted area form over any oriented filling of the closed chain $C_\gamma - C_\delta$. For a merely scale-compatible pair the chain $C_\gamma - C_\delta$ is not closed, because the two paths have different additive endpoints, so there is no such filling; the scalar defect (78) remains defined, but it is not yet an area.

Three properties justify treating tearing as a geometric notion rather than a bookkeeping defect.

1. *It exists only relative to the commutativization.* In the AES itself, γ and δ are simply two motions; there is no region bounded by them, because the AES is where the two rulers fail to commute. The region appears only after passing to the ACS, and the weight of the filling is forced by the evaluation formula (76).
2. *For charge-compatible pairs it is measured, not merely detected.* There $\tau(\gamma, \delta)$ is an exact weighted area (82); for the elementary rectangle it is $\mu h(e^{\lambda k} - 1)$, which is the discrete Baumslag–Solitar transport of (19) read as a defect.
3. *It is infinitesimally rigid.* Its density is the constant $\mu\lambda$ of (36), which is also the bracket coefficient of the arithmetic frame (8). Section 9 shows that this coefficient is the curvature of the specified horizontal connection, hence that tearing cannot be removed by any local recalibration of the two arithmetic directions in which both remain horizontal.

The word *tearing* is chosen because the two-dimensional picture is that of a surface cut along the two motions: an additive edge and a multiplicative edge that close up on the level of total charge but not on the level of motion. The standard description of a rip in a material is also a pair of paths that agree far away and separate inside; the arithmetic version of that picture is Definition 8.10. As with ripple geometry, the terminology is descriptive and asserts nothing beyond the displayed identities; what it adds is a single name for the phenomenon whose exact measure is the ACS area, whose infinitesimal density is $\mu\lambda$, and whose carrier in a punctured model is the subject of Section 11.

9 The Contact Structure of Tearing

Section 8 measured tearing by a weighted area in the commutativization of the arithmetic motions. That measurement is exact at every finite scale, but it is stated in a space that has forgotten the current value. This section recovers the local picture: a three-dimensional state space that retains the value and distinguishes the two infinitesimal arithmetic directions, carrying an Ehresmann connection over the two-dimensional direction plane and, in the nondegenerate case, a contact form. Tearing then appears a third time, with the same density: as the curvature of that connection.

Fix real constants μ and λ , and let

$$\mathcal{C} = \mathbb{R}_{(u,v,a)}^3, \quad \pi : \mathcal{C} \longrightarrow \mathbb{R}_{(u,v)}^2.$$

Here a is the current assignment value, while u and v parametrize additive and multiplicative motion. We orient $\mathcal{C}$ by

$$du \wedge da \wedge dv.$$

9.1 The arithmetic contact form

Define

$$\alpha := da - \mu du - \lambda a dv. \quad (84)$$

The equation $\alpha = 0$ says that a lifted base motion obeys the affine propagation law.

Proposition 9.1 (Contact nondegeneracy). *With the orientation $du \wedge da \wedge dv$,*

$$\alpha \wedge d\alpha = \mu\lambda du \wedge da \wedge dv. \quad (85)$$

Hence α is a contact form if and only if $\mu\lambda \neq 0$.

Proof. Since μ and λ are constant,

$$d\alpha = -\lambda da \wedge dv.$$

It follows that

$$\begin{aligned} \alpha \wedge d\alpha &= (da - \mu du - \lambda a dv) \wedge (-\lambda da \wedge dv) \\ &= \mu\lambda du \wedge da \wedge dv. \end{aligned}$$

This three-form is nowhere zero exactly when $\mu\lambda \neq 0$. □

The contact-isomorphism class is the standard local one; the arithmetic content lies in the coordinates and in the preferred horizontal frame introduced next. Even in a degenerate case, $\ker \alpha$ remains a well-defined horizontal distribution, although it is then not contact.

9.2 Horizontal lifts and the affine flow

Set

$$D_u := \partial_u + \mu \partial_a, \quad D_v := \partial_v + \lambda a \partial_a. \quad (86)$$

Then

$$\alpha(D_u) = \alpha(D_v) = 0,$$

and the two fields project respectively to ∂_u and ∂_v . Thus

$$\mathcal{D} := \ker \alpha = \text{span}\{D_u, D_v\}$$

is an Ehresmann connection for π . Its additive and multiplicative flows are

$$\Phi_u^h(u, v, a) = (u + h, v, a + \mu h), \quad (87)$$

$$\Phi_v^k(u, v, a) = (u, v + k, e^{\lambda k} a). \quad (88)$$

For a direction angle θ , define

$$D_\theta = \cos \theta D_u + \sin \theta D_v.$$

If $(u(s), v(s), a(s))$ is tangent to D_θ , then

$$\frac{du}{ds} = \cos \theta, \quad \frac{dv}{ds} = \sin \theta,$$

and

$$\frac{da}{ds} = \mu \cos \theta + \lambda a \sin \theta. \quad (89)$$

Thus the horizontal lift reproduces the continuous affine flow rather than postulating a separate dynamics.

9.3 Horizontal curvature

Theorem 9.2 (Curvature of the specified horizontal connection). *The preferred horizontal fields satisfy*

$$[D_u, D_v] = \mu\lambda\partial_a. \quad (90)$$

In particular, when $\mu\lambda \neq 0$, the horizontal distribution is not integrable. Its bracket defect is vertical and has constant coefficient $\mu\lambda$ in the arithmetic frame.

Proof. Using (86),

$$\begin{aligned} [D_u, D_v] &= [\partial_u + \mu\partial_a, \partial_v + \lambda a\partial_a] \\ &= \mu\partial_a(\lambda a)\partial_a - \lambda a\partial_a(\mu)\partial_a \\ &= \mu\lambda\partial_a. \end{aligned}$$

All remaining coordinate brackets vanish. Since ∂_a is vertical and is not contained in $\ker \alpha$, the bracket is the vertical failure of horizontal closure. $\square$

The sign in (90) agrees with the chronological convention of the preceding chapter. Equivalently,

$$\alpha([D_u, D_v]) = \mu\lambda, \quad d\alpha(D_u, D_v) = -\mu\lambda.$$

The two formulas have opposite signs by the identity $d\alpha(X, Y) = -\alpha([X, Y])$ for horizontal fields X, Y .

9.4 Open defects and closed horizontal loops

The bracket is infinitesimal. At finite scale there are two related, but different, quantities. First compare the two open lifts from (u, v) to $(u + h, v + k)$:

$$\begin{aligned} \Phi_v^k \circ \Phi_u^h(u, v, a) &= (u + h, v + k, e^{\lambda k}(a + \mu h)), \\ \Phi_u^h \circ \Phi_v^k(u, v, a) &= (u + h, v + k, e^{\lambda k}a + \mu h). \end{aligned}$$

Their assignment difference, with “add then multiply” taken first, is

$$T_{\text{open}}(h, k) = \mu h(e^{\lambda k} - 1). \quad (91)$$

This is exactly the elementary ACS rectangle torsion with charge increments $(\mu h, \lambda k)$.

Now perform the chronological four-step history

$$A_h, \quad M_k, \quad A_{-h}, \quad M_{-k},$$

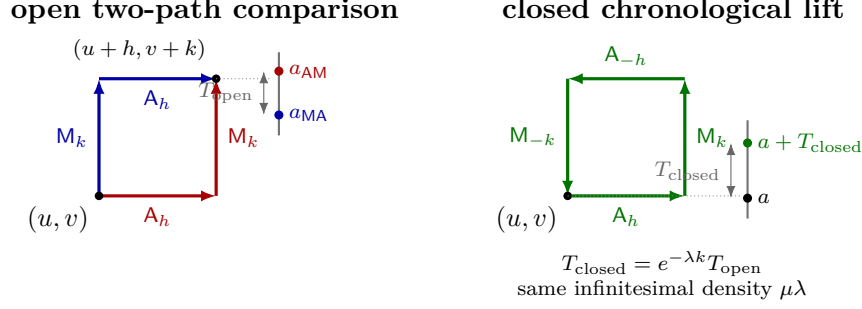


Figure 6: Open defect and closed holonomy in the contact model. The two open horizontal lifts have the same base endpoint but different assignment values. The four-step closed base loop returns to its starting base point with a raw vertical drift. These finite quantities differ by the displayed scale factor; converting the closed drift to the target frame gives the open defect, while their infinitesimal densities agree as stated in the text.

where A_h denotes the flow Φ_u^h and M_k denotes Φ_v^k . The base point returns to (u, v) , and direct substitution gives

$$\begin{aligned} a &\mapsto a + \mu h \mapsto e^{\lambda k}(a + \mu h) \\ &\mapsto e^{\lambda k}(a + \mu h) - \mu h \mapsto a + \mu h(1 - e^{-\lambda k}). \end{aligned}$$

The closed-loop vertical drift is therefore

$$T_{\text{closed}}(h, k) = \mu h(1 - e^{-\lambda k}) = e^{-\lambda k} T_{\text{open}}(h, k). \quad (92)$$

Thus the finite open comparison and the finite closed holonomy are not equal. Here T_{closed} is the source-normalized relative translation. Converting it to the target frame gives $e^{\lambda k} T_{\text{closed}} = T_{\text{open}}$ exactly.

Figure 6 keeps the two finite constructions visually separate. The square is drawn in the (u, v) base; the short vertical segments represent assignment fibers, not additional base edges.

9.5 The scalar horizontal differential

For a smooth scalar field $F \in C^\infty(\mathcal{C})$, define

$$\delta_H F := dF - (\partial_a F) \alpha = (D_u F) du + (D_v F) dv. \quad (93)$$

This is the ordinary differential restricted to horizontal directions and written on the base coframe. In particular,

$$\delta_H a = \mu du + \lambda a dv.$$

For scalar fields only, define the antisymmetrized second horizontal derivative by

$$\mathcal{R}_H(F) := (D_u D_v F - D_v D_u F) du \wedge dv. \quad (94)$$

This notation deliberately avoids presupposing an extension of δ_H to a graded complex.

Proposition 9.3 (Curvature on scalar fields). *For every $F \in C^\infty(\mathcal{C})$,*

$$\mathcal{R}_H(F) = \mu \lambda (\partial_a F) du \wedge dv. \quad (95)$$

In particular,

$$\mathcal{R}_H(a) = \mu \lambda du \wedge dv.$$

Proof. By definition and Theorem 9.2,

$$\mathcal{R}_H(F) = [D_u, D_v](F) du \wedge dv = \mu \lambda (\partial_a F) du \wedge dv.$$

Taking $F = a$ proves the final formula. $\square$

One may use $\delta_H^2 F$ as shorthand for $\mathcal{R}_H(F)$ after declaring this antisymmetrization convention. It does not mean that δ_H has been defined on horizontal forms, nor does it assert the existence of a nilpotent differential complex. No such graded calculus is required for the foundational result.

9.6 Three levels of the same order defect

Proposition 9.4 (Finite and infinitesimal comparison). *For the elementary rectangle with additive side μh and logarithmic multiplicative side λk , the following statements hold.*

1. *The open affine defect equals the ACS weighted area exactly:*

$$T_{\text{open}}(h, k) = \int_{[0, \mu h] \times [0, \lambda k]} e^{\lambda k - M} dA \wedge dM = \mu h (e^{\lambda k} - 1),$$

with oriented-chain interpretation when a side is negative.

2. *The closed chronological lift has exact drift*

$$T_{\text{closed}}(h, k) = e^{-\lambda k} T_{\text{open}}(h, k).$$

Thus T_{closed} is the source-normalized relative translation; its target-frame conversion agrees with the open defect, but the raw finite values generally differ.

3. *Both defects have the common infinitesimal density*

$$\lim_{(h, k) \rightarrow (0, 0)} \frac{T_{\text{open}}(h, k)}{hk} = \lim_{(h, k) \rightarrow (0, 0)} \frac{T_{\text{closed}}(h, k)}{hk} = \mu \lambda,$$

where the quotients are taken for $hk \neq 0$. This is the coefficient of $[D_u, D_v]$ and of $\mathcal{R}_H(a)$.

Proof. The first identity is Theorem 8.9 applied to the elementary rectangle, and its integral evaluates to

$$\mu h \int_0^{\lambda k} e^{\lambda k - M} dM = \mu h (e^{\lambda k} - 1).$$

The second identity is (92). Finally,

$$\frac{e^{\lambda k} - 1}{k} \longrightarrow \lambda, \quad \frac{1 - e^{-\lambda k}}{k} \longrightarrow \lambda,$$

which proves the two limits. The curvature identifications follow from Theorem 9.2 and Proposition 9.3. $\square$

The synthesis has three layers, not one finite equality: exact target-frame open affine torsion is measured by ACS weighted area; source-normalized closed horizontal holonomy differs by an exact frame-conversion factor; and contact curvature is their common infinitesimal limit. The contact distribution alone supplies neither a preferred complex structure nor a full analytic function theory. The present paper uses only the horizontal connection, its scalar differential, and its curvature.

Remark 9.5 (What the coefficient $\mu\lambda$ is and is not). The number $\mu\lambda$ is the bracket coefficient of the *specified* horizontal connection in the *specified* arithmetic frame, and it is not a contact-isomorphism invariant: by the Darboux theorem every contact form on a three-manifold is locally equivalent to $dz - y dx$, and Appendix D exhibits that coordinate change explicitly for α . The arithmetic content of (84) therefore lies in the coordinates, the projection, the assignment, and the preferred horizontal frame, not in a new local contact-isomorphism class. In particular $\mu\lambda$ must not be confused with the Gaussian curvature $K = -\lambda^2$ of the basic model (Theorem 3.4): the two numbers have different dimensions, different carriers, and different invariance groups. What (90) does assert is *intrinsic relative to the declared structure*: the horizontal distribution is not integrable, so no choice of coordinates can make both arithmetic directions straight while keeping them horizontal.

A standard account of the local theory of contact structures, used only for this comparison, is [8]; the elementary Darboux computation needed here is carried out in the appendix and does not depend on it.

Proposition 9.6 (The curvature form of the horizontal distribution). *Let $\mathcal{D} = \ker \alpha$ be the horizontal distribution of the arithmetic contact model, and let $\pi(u, v, a) = (u, v)$ be the charge projection, whose vertical bundle $\ker d\pi$ is spanned by ∂_a . For horizontal vector fields X, Y define $\omega(X, Y)$ by the congruence*

$$[X, Y] \equiv \omega(X, Y) \partial_a \pmod{\mathcal{D}}. \quad (96)$$

Then:

1. ω is a well-defined smooth 2-form on $\mathcal{D}$: it is bilinear, alternating, and $C^\infty(\mathcal{C})$ -linear in each argument;
2. in the arithmetic frame, $\omega(D_u, D_v) = \mu\lambda$;
3. ω is unchanged when α is replaced by $\kappa\alpha$ for $\kappa \neq 0$, and it is invariant under every diffeomorphism of $\mathcal{C}$ that preserves the distribution, the charge projection, and the orientation of the vertical line;
4. if a horizontal metric h is declared and h is rescaled to $\rho^2 h$, then an h -orthonormal positively oriented frame scales by ρ^{-1} and the number $\omega(e_u, e_v)$ scales by ρ^{-2} , while the form ω itself is unchanged.

Proof. (1) For $f \in C^\infty(\mathcal{C})$ one has $[fX, Y] = f[X, Y] - Y(f)X$, and $Y(f)X$ is horizontal, so the class of $[X, Y]$ modulo $\mathcal{D}$ is unchanged; alternation is immediate. The vertical bundle is trivialized by ∂_a , so the congruence determines a single smooth coefficient. (2) is Theorem 9.2: $[D_u, D_v] = \mu\lambda\partial_a$, whose class is already vertical, so $\omega(D_u, D_v) = \mu\lambda$. (3) Replacing α by $\kappa\alpha$ does not change $\ker \alpha$, and no factor of α appears in (96); a diffeomorphism preserving the distribution maps horizontal fields to horizontal fields and preserves the trivialization of $\ker d\pi$ up to the orientation sign. (4) If $h' = \rho^2 h$ then $e'_u = \rho^{-1}e_u$ and $e'_v = \rho^{-1}e_v$ are h' -orthonormal and positively oriented, so $\omega(e'_u, e'_v) = \rho^{-2}\omega(e_u, e_v)$. $\square$

Remark 9.7 (What the curvature scalar is invariant under). Proposition 9.6 answers precisely the question that Remark 9.5 raised. The coefficient $\mu\lambda$ is not a contact-isomorphism invariant, but neither is it a gauge artifact of coordinates. Two statements hold, and they concern different invariance groups.

1. On the contact side the canonical object is the 2-form ω of (96): it is determined by the distribution and the charge projection, and it is insensitive to the normalization of α . The

number $\mu\lambda$ is the value of ω on the h -orthonormal frame for the declared horizontal metric $h = du^2 + dv^2$; it is therefore invariant under every transformation preserving the triple $(\mathcal{D}, h, \pi)$ together with the orientation and the horizontal area normalization, and its only residual freedom is the rescaling of h , with the explicit law ρ^{-2} of (4). A general contact transformation need not preserve h , and that is exactly why the number is not a contact invariant.

2. On the arithmetic side the same number is a functional of the AES data alone: the canonical frame satisfies

$$da([X_u, X_v]) = X_u(\lambda a) - X_v(\mu) = \mu\lambda$$

by (8), an identity that mentions no chart, no projection, and no normalization of any contact form. Here the freedom is the choice of the pair (μ, λ) itself, which is part of the structure in Definition 3.1 rather than a choice of frame.

So the correct phrase is not “the curvature of the contact structure” but “the curvature scalar of the declared arithmetic frame and horizontal metric”. Rescaling the arithmetic units changes (μ, λ) : for instance $(u, v, a) \mapsto (cu, dv, \kappa a)$ replaces (μ, λ) by $(\mu c/\kappa, \lambda d)$ and multiplies $\mu\lambda$ by cd/κ . Such a rescaling changes the AES data, hence the model, rather than expressing one model in another frame, which is the sense in which the theorem of Section 9 concerns a specified connection.

9.7 The contact model as a quotient of the basic model

Sections 3 and 9 describe one object from two sides: intrinsically, by the frame and metric of a regular AES, and locally, by the three coordinates (u, v, a) and the contact form. The following five relations make the correspondence exact for the basic model. (The comparison was proposed in a review of this paper’s first version; it is stated here with proofs, and it supersedes the earlier informal remark that the two descriptions “describe one object”.)

Define

$$Q : \mathcal{C} = \mathbb{R}_{(u,v,a)}^3 \longrightarrow \mathbb{H}^2, \quad Q(u, v, a) = (-ae^{-\lambda v}, e^{-\lambda v}) = (x, y). \quad (97)$$

Proposition 9.8 (The basic model is the Reeb quotient of the contact model). *Let $\mu, \lambda > 0$, and let $\alpha, D_u, D_v, \mathcal{C}$ be as in (84) and (86). Then:*

1. assignment compatibility: $a_{\mathfrak{C}_0} \circ Q = a$, that is, $-x/y = a$ along Q ;
2. frame descent: $Q_* D_u = X_u$ and $Q_* D_v = X_v$, where $X_u = -\mu y \partial_x$ and $X_v = -\lambda y \partial_y$ are the canonical frame (12);
3. horizontal length:

$$Q^* g_{\mu,\lambda} = \frac{(da - \lambda a dv)^2}{\mu^2} + dv^2, \quad \text{so} \quad Q^* g_{\mu,\lambda}|_{\ker \alpha} = du^2 + dv^2;$$

4. the fibers of Q are the Reeb lines: $\ker dQ = \text{span}(\partial_u) = \text{span}(R_\alpha)$ with $R_\alpha = -\frac{1}{\mu} \partial_u$, and Q is surjective with global section $(x, y) \mapsto (0, -\frac{1}{\lambda} \log y, -\frac{x}{y})$;
5. curvature is the hyperbolic area form:

$$d\alpha = -\mu\lambda Q^*(d\text{vol}_{g_{\mu,\lambda}}), \quad d\text{vol}_{g_{\mu,\lambda}} = \frac{dx \wedge dy}{\mu\lambda y^2}.$$

Proof. (1) is immediate: $-x/y = ae^{-\lambda v}/e^{-\lambda v} = a$.

(2) Since Q is independent of u and $\partial_a Q = (-e^{-\lambda v}, 0)$, $\partial_v Q = (a\lambda e^{-\lambda v}, -\lambda e^{-\lambda v})$, we get

$$Q_* D_u = Q_*(\partial_u + \mu \partial_a) = \mu(-e^{-\lambda v}, 0) = (-\mu y, 0) = X_u,$$

and

$$Q_* D_v = Q_*(\partial_v + \lambda a \partial_a) = (a\lambda e^{-\lambda v}, -\lambda e^{-\lambda v}) + \lambda a(-e^{-\lambda v}, 0) = (0, -\lambda e^{-\lambda v}) = X_v.$$

(3) Differentiating (97),

$$dx = e^{-\lambda v}(-da + \lambda a dv) = y(-da + \lambda a dv), \quad dy = -\lambda e^{-\lambda v} dv = -\lambda y dv.$$

Substituting into $g_{\mu,\lambda} = (dx^2/\mu^2 + dy^2/\lambda^2)/y^2$ gives the first identity. On $\ker \alpha$ one has $da = \mu du + \lambda a dv$, hence $da - \lambda a dv = \mu du$ and the restriction is $du^2 + dv^2$.

(4) The map Q does not depend on u , and the pair $(\partial_v Q, \partial_a Q)$ is independent because $\det \begin{pmatrix} a\lambda e^{-\lambda v} & -\lambda e^{-\lambda v} \\ -e^{-\lambda v} & 0 \end{pmatrix} = \lambda e^{-2\lambda v} \neq 0$. Hence $\ker dQ = \text{span}(\partial_u)$. Since $\alpha(R_\alpha) = -\mu \cdot (-1/\mu) = 1$, R_α spans the same line, and it is the Reeb field of α . Surjectivity and the displayed section follow by solving $x = -ae^{-\lambda v}$, $y = e^{-\lambda v}$.

(5) By (3), $dx \wedge dy = y(-da + \lambda a dv) \wedge (-\lambda y dv) = \lambda y^2 da \wedge dv$. Hence

$$Q^*(d\text{vol}_{g_{\mu,\lambda}}) = \frac{dx \wedge dy}{\mu \lambda y^2} = \frac{da \wedge dv}{\mu},$$

and $-\mu\lambda$ times it is $-\lambda da \wedge dv = d\alpha$, as computed in the proof of Proposition 9.1. $\square$

Three points about the proposition deserve emphasis. First, Q is not the charge projection $\pi(u, v, a) = (u, v)$ of the ACS discussion: Q forgets the Reeb direction u , whereas the ACS forgets the assignment a . The two projections have different fibers and different images, and neither factors through the other. Second, relation (2) says that the horizontal flow of the contact model and the arithmetic motion of the basic model are the *same* curves, which is what makes the two derivations of $\frac{da}{ds} = \mu \cos \theta + \lambda a \sin \theta$ — the intrinsic one in Theorem 3.16 and the coordinate one in (89) — statements about one object rather than two analogies. Relation (3) adds that the horizontal metric pulled back to the base coordinates is Euclidean, so the direction angle θ has the same meaning in both derivations. Third, the proposition is proved for the basic model; it must not be read as asserting that every regular AES is a Reeb quotient of a contact three-manifold, or that every AES carries such a quotient.

9.8 Legendre duality between the charge and Reeb quotients

Proposition 9.8 places the charge projection $\pi(u, v, a) = (u, v)$ of Section 8 and the Reeb quotient $Q(u, v, a) = (-ae^{-\lambda v}, e^{-\lambda v}) = (x, y)$ on a common three-dimensional carrier $\mathcal{C} = \mathbb{R}_{(u,v,a)}^3$. The two maps forget complementary directions: π forgets the assignment fiber ∂_a , whereas Q forgets the Reeb direction ∂_u . We now record the exact sense in which they form a conjugate Legendre pair for horizontal lifts of arithmetic histories.

In the natural charge coordinates $A = \mu u$ and $M = \lambda v$, the arithmetic contact form (84) and its rescaled companion $e^{-M}\alpha$ satisfy the two exact decompositions

$$\alpha = da - dA - a dM, \tag{98}$$

$$e^{-M}\alpha = d(e^{-M}a) - e^{-M}dA. \tag{99}$$

Since $e^{-M} > 0$ everywhere on $\mathcal{C}$, both one-forms define the same horizontal distribution $\mathcal{D} = \ker \alpha$. Along any piecewise smooth horizontal curve $\tilde{\gamma}$ in $\mathcal{C}$, both (98) and (99) vanish identically. Notice that the quantity $\hat{a} := e^{-M}a = -x$ in (99) is simultaneously the source-normalized assignment and minus the horizontal coordinate of the basic hyperbolic model $\mathfrak{E}_0$ under Q .

Theorem 9.9 (Dual tearing and Reeb drift). *Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be piecewise smooth horizontal lifts in $\mathcal{C}$ starting at a common initial state (u_0, v_0, a_0) , with projected charge chains $C_i = \pi \circ \tilde{\gamma}_i$ in $\mathcal{A}$ and projected state paths $\bar{\gamma}_i = Q \circ \tilde{\gamma}_i$ in $\mathbb{H}^2$.*

1. (Fixed-charge comparison in $\mathcal{A}$). *Suppose $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are charge-compatible, so that C_1 and C_2 share their terminal point (A_*, M_*) . Then their terminal states lie in the same π -fiber and their vertical assignment defect is*

$$a(\tilde{\gamma}_1(1)) - a(\tilde{\gamma}_2(1)) = e^{M_*} \int_{C_1 - C_2} e^{-M} dA = \int_{\Sigma_{12}} e^{M_* - M} dA \wedge dM, \quad (100)$$

for any oriented piecewise smooth two-chain Σ_{12} in $\mathcal{A}$ with $\partial \Sigma_{12} = C_1 - C_2$.

2. (Fixed-state comparison in $\mathbb{H}^2$). *Suppose instead that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are state-compatible under Q , so that $\bar{\gamma}_1$ and $\bar{\gamma}_2$ reach the same terminal point $p_* = (x_*, y_*) \in \mathbb{H}^2$, equivalently the same terminal pair (M_*, a_*) . Then their terminal states lie in the same Reeb fiber $Q^{-1}(p_*)$, and their additive charge drift $\Delta A := A(\tilde{\gamma}_1(1)) - A(\tilde{\gamma}_2(1)) = \mu(u(\tilde{\gamma}_1(1)) - u(\tilde{\gamma}_2(1)))$ satisfies*

$$\Delta A = - \int_{\bar{\gamma}_1 - \bar{\gamma}_2} a dM = - \int_{\Omega_{12}} da \wedge dM = -\mu\lambda \int_{\Omega_{12}} d\text{vol}_{g_{\mu,\lambda}}, \quad (101)$$

for any oriented piecewise smooth two-chain Ω_{12} in $\mathbb{H}^2$ with $\partial \Omega_{12} = \bar{\gamma}_1 - \bar{\gamma}_2$.

Proof. (1) Integrating $e^{-M}\alpha = 0$ in the form (99) along each horizontal lift gives

$$e^{-M_*}a(\tilde{\gamma}_i(1)) - e^{-M_0}a_0 = \int_{C_i} e^{-M} dA, \quad i = 1, 2.$$

Subtracting the two identities cancels the initial term, and multiplying by e^{M_*} gives the first equality in (100). The second equality follows by Stokes' theorem as in Theorem 8.9.

- (2) Integrating $\alpha = 0$ in the form (98) along each horizontal lift gives

$$A(\tilde{\gamma}_i(1)) - A_0 = (a(\tilde{\gamma}_i(1)) - a_0) - \int_{\tilde{\gamma}_i} a dM, \quad i = 1, 2.$$

Under state compatibility in $\mathbb{H}^2$, both lifts terminate at the same $y_* = e^{-M_*}$ and the same $a_* = -x_*/y_*$, so the boundary term $a_* - a_0$ cancels upon subtraction. This proves the line-integral formula in (101). Applying Stokes' theorem on $\mathbb{H}^2$ gives the integral of $-da \wedge dM = -\lambda da \wedge dv$. By Proposition 9.8(5), $-\lambda da \wedge dv = d\alpha = -\mu\lambda Q^*(d\text{vol}_{g_{\mu,\lambda}})$, which yields the hyperbolic area identity. $\square$

Remark 9.10 (Unification of Theorem 8.6 and Theorem 8.9). Theorem 9.9 shows that the two basic non-commutativity phenomena of Section 8 are Legendre-conjugate readings of the same contact curvature:

- Holding the commutative charge endpoint (A_*, M_*) fixed projects the order defect onto the assignment fiber $\ker d\pi = \text{span}(\partial_a)$, recovering the weighted ACS area of Theorem 8.9.

- Holding the evaluated hyperbolic state $p_* \in \mathbb{H}^2$ fixed projects the order defect onto the Reeb fiber $\ker dQ = \text{span}(\partial_u)$, measuring the failure of the additive charge to descend to the motion group. For the elementary kernel word $h(p, q) = M_q A_p M_{-q} A_{-e^{-q}p} \in \ker \nu$ of Theorem 8.6, the two prefixes $\gamma_1 = (M_q, A_p)$ and $\gamma_2 = (A_{e^{-q}p}, M_q)$ connect $(0, a_0)$ to the same state $(q, e^q a_0 + p)$ in $\mathbb{H}^2$, and (101) evaluates their Reeb drift $-\int_{\tilde{\gamma}_1 - \tilde{\gamma}_2} a dM = p(1 - e^{-q})$ as the hyperbolic area of the enclosed strip.

Notice also the structural complement to Proposition 11.10: whereas a charge-compatible rectangle remains open on $\mathbb{H}^2$, a state-compatible pair closes on $\mathbb{H}^2$ and opens instead along the Reeb fiber in $\mathcal{C}$.

10 Beyond Linear Language

The picture developed so far is deliberately written in the language of surfaces, metrics, vector fields, and contact forms. That language is convenient, and it is also, in a precise sense, too small. This section records two elementary obstructions that explain why we do not take linear language (manifolds, tangent spaces, Lie groups, and character theory built on modules) as the foundation of AEG, and it records the alternative that a companion programme, process geometry, has begun to develop. The obstructions are theorems; the alternative is a structural proposal, and the two are kept apart throughout.

10.1 Two obstructions to linear language

The first obstruction concerns the Lie-algebraic reading of the arithmetic tower. Addition and multiplication are the first two ranks of that tower: an additive step moves the assignment by a constant, a multiplicative step scales it. On the positive assignment line $a > 0$ these two kinds of motion are generated by

$$A = \partial_a, \quad M = a\partial_a.$$

Together with the third rank, multiplication viewed as an endomorphism of addition, they generate more than a three-dimensional Lie algebra.

Proposition 10.1 (The arithmetic generators do not close). *Let*

$$A = \partial_a, \quad M = a\partial_a, \quad P = a \log a \partial_a$$

be defined for $a > 0$. Then

$$[A, M] = A, \quad [M, P] = M, \quad [A, P] = (1 + \log a)\partial_a.$$

The last field does not lie in the real span of A, M, P , and the Lie algebra generated by A, M, P is infinite-dimensional.

Proof. The three brackets are direct: for any smooth f ,

$$\begin{aligned} [A, M]f &= \partial_a(af') - a\partial_a f' = f', \\ [M, P]f &= a\partial_a(a \log a f') - a \log a \partial_a(af') = af', \\ [A, P]f &= \partial_a(a \log a f') - a \log a \partial_a f' = (1 + \log a)f'. \end{aligned}$$

For the last claim, suppose first that $(1 + \log a)\partial_a = \alpha A + \beta M + \gamma P$ for constants α, β, γ . Equivalently $1 + \log a = \alpha + \beta a + \gamma a \log a$ for all $a > 0$. Letting $a \rightarrow 0^+$ makes the left-hand side

tend to $-\infty$ while the right-hand side tends to α , a contradiction. Hence the generated Lie algebra is strictly larger than $\text{span}\{A, M, P\}$.

More is true, and the argument is shorter than the one just given: it suffices to iterate ad_A on P . The derived algebra is closed under ad_A , and $P = a \log a \partial_a$ belongs to it, so it contains

$$\text{ad}_A^n(P) = \frac{d^n}{da^n}(a \log a) \partial_a = (-1)^n(n-2)! a^{1-n} \partial_a, \quad n \geq 2,$$

where the derivative is computed for $a > 0$. Hence the derived algebra contains $a^{-1}\partial_a, a^{-2}\partial_a, a^{-3}\partial_a, \dots$, and these fields are real-linearly independent on every nonempty open subinterval of $a > 0$: multiplying a finite linear relation by a sufficiently high power of a turns it into the vanishing of a polynomial. Therefore the Lie algebra generated by A and P alone is infinite-dimensional, and so is the algebra generated by A, M, P ; the dilation M is not needed for this conclusion. $\square$

The three generators have the following arithmetic readings, which fix the calibration of this section. A generates translation of the assignment ($a \mapsto a + s$). M generates multiplication by a positive constant ($a \mapsto e^s a$), so its parameter is the logarithm of the multiplier, which is why the multiplicative data enter the ACS as a logarithmic charge. P generates the one-parameter group of power maps: its flow solves $\dot{a} = a \log a$, hence

$$a(t) = a(0)^{e^t}.$$

Thus P moves the *exponent* of the assignment, and it is the infinitesimal form of the third rank of the arithmetic tower, in which multiplication is iterated. We use “third rank” here as a name for this calibration only; no classification of arithmetic ranks is attempted in this paper.

The consequence for this paper is narrow but decisive. A finite-dimensional Lie group acting locally near the assignment line has a finite-dimensional Lie algebra of fundamental vector fields. If the rank-one, rank-two, and rank-three motions above were all fundamental vector fields of one such action, then their Lie closure — infinite-dimensional by Proposition 10.1 — would have to be finite-dimensional. Hence no finite-dimensional Lie group action realizes all three motions simultaneously.

Two weaker statements must not be confused with that one, and the distinction matters for everything that follows. First, a finite-dimensional *manifold* can carry infinitely many vector fields: the three fields above are defined on the one-dimensional half-line $a > 0$, and every smooth manifold has an infinite-dimensional space of vector fields. What is excluded is a finite-dimensional Lie *algebra of fundamental fields*, not a finite-dimensional carrier. Second, linear language is not thereby refuted. The affine sector has the faithful two-dimensional matrix representation $\begin{pmatrix} \Phi & \xi \\ 0 & 1 \end{pmatrix}$ used throughout Section 7, and it does detect order defects; the non-abelian representation theory of that group is not touched by Proposition 10.1. The boundary established here is the boundary of finite-dimensional Lie-group closure, and it says only that the arithmetic tower cannot be *linearized into a finite-dimensional Lie algebra without loss*.

The second obstruction concerns what an observable of the *commutativized* data can see. It is the obstruction that makes the tearing of Section 8 irreducible in that class of descriptions.

Proposition 10.2 (Observables of the charge endpoint cannot see tearing). *Let $f : \mathbb{R}^2 \rightarrow X$ be any function whatsoever of the charge endpoint, and let γ, δ be charge-compatible add-scale histories with common charge (A_*, M_*) . Then*

$$f(A_\gamma, M_\gamma) = f(A_\delta, M_\delta),$$

while their evaluation defect is the relative torsion

$$\nu_x(\gamma) - \nu_x(\delta) = \tau(\gamma, \delta),$$

which is independent of x and need not vanish. Consequently no observable that factors through the charge endpoint can detect tearing.

Proof. The first display is immediate from charge compatibility. The second is Proposition 8.8. Nonvanishing is witnessed by the elementary rectangle: for $\gamma = (A_p, M_q)$ and $\delta = (M_q, A_p)$ one has $\tau(\gamma, \delta) = p(e^q - 1)$, which is nonzero whenever $p \neq 0$ and $q \neq 0$. $\square$

Characters of the charge group are the special case $X = S^1$ with f multiplicative, so Proposition 10.2 subsumes them; but the statement is deliberately stated for arbitrary functions, because its content is not about representation theory. It is about a forgetting map: the charge endpoint is the image of the history under the map (76)'s charges, and any observable of the image loses whatever the map forgot. Two consequences should be stated together to avoid overreading the proposition.

First, the proposition does *not* say that no linear description can see tearing. On the contrary, the affine matrix $\begin{pmatrix} \Phi & \xi \\ 0 & 1 \end{pmatrix}$ of Section 7 is faithful on the affine sector, the target-frame translation ξ is precisely the coordinate in which the two-history defect is measured, and ξ is *not* a function of the charge endpoint: the charge endpoint gives the scale coordinate alone. What is excluded is a description that first passes to the commutative charges and then reads a (scalar or vectorial) observable of them.

Second, the measure of what such a description loses is exactly the ACS area of Theorem 8.9: for charge-compatible pairs the defect equals the weighted area between the two charge paths. So the correct summary is not that characters fail, but that the commutativization has a price, and the price is computable.

10.2 What process geometry proposes instead

The companion programme process geometry develops a vocabulary for exactly this situation, and the vocabulary is quotable [26]. Its central methodological statement is that geometry is not supplied in advance but generated by the process and by what its observers can distinguish; and that no familiar structure may be imported into a *native* description merely because it is familiar:

- geometry enters "because process generation and observer-relative distinguishability acquire stable relational, local, and compositional structure—not because a manifold, metric, coordinate system, or arithmetic model was supplied in advance";
- "no level should be imported merely because the next level is mathematically familiar";
- the public vocabulary deliberately contains *no* group or Lie-group hierarchy, *no* general representation hierarchy, and *no* spectrum or eigenmode ontology; those "may become justified later, but only after an executable vignette requires it";
- where a classical mechanism (power series, ordinary polynomial algebra, matrix linearization, Fourier or spectral methods, generic computer algebra) is used, it must be declared as such and accompanied by a *lowering witness* scoped to the exact task and lane, rather than silently acquiring the status of native structure.

Two features of that programme are relevant to AEG, and only two are claimed here.

First, it introduces structure without presupposing a linear carrier. A process is a generation rule; a history is what that rule produces; a task selects which distinctions matter; and the resulting quotient, when it is stable, may be *objectified* into a new primitive of the next rank. The vocabulary is deliberately non-linear: it speaks of presentations, semantic compression, rank raising, and rank lowering, and its stated obligations are compositional (a higher-rank composition must remain interpretable at the lower rank) rather than algebraic in the module-theoretic sense.

Second, it records an obstruction of the same shape as Proposition 10.2. Its finite-family layer introduces families with a parameter composition law, together with one-dimensional response objects, and then states plainly that such one-dimensional scalar response is *not* sufficient to encode the additive–multiplicative interaction while retaining a nontrivial translation response; the layer stops at that obstruction instead of enlarging the ambient linear structure. The shape is the same as ours: a *scalar* observable of a commutative label fails, while a non-scalar description succeeds. In our setting the non-scalar description is the affine matrix of Section 7; the finite-family layer’s declaration that its object “does not require inverses, topology, Haar measure, Lie algebra, commutativity, or even a group structure” is a statement about what that layer deliberately does *not* assume, not evidence that such structures are unavailable.

10.3 Exact calibrations that meet this paper’s laws

The proposals above would be of little use if they did not meet the identities proved here. They do, and in one case the match is literal.

1. *The discrete transport law.* In the calibration of the companion programme, the objectified multiplication acts on translation by

$$D_k T_a = T_{ka} D_k,$$

and the mixed language generated by both is the positive affine monoid $\mathbb{Z} \rtimes \mathbb{N}_{>0}$. This is exactly the relation (19) of Section 3, read in the equivalent direction convention:

$$Y_k X_s Y_k^{-1} = X_{ks},$$

with $D_k = Y_k$ and $T_a = X_a$. The two programmes therefore meet at the same discrete law, one derived from the arithmetic grid and the other from a rank-lowering obligation.

2. *The infinitesimal transport law.* In the continuous form the same structure appears as $S_s T_t = T_{est} S_s$ together with the algebra relation $[A, M] = A$ for $A = \partial_a$ and $M = \partial_v + a\partial_a$. This is the affine structure equation already used in Section 3. It must not be confused with the horizontal bracket (32): the latter is the curvature of the horizontal distribution in the three-dimensional contact model of Section 9, whereas $[A, M] = A$ is a bracket on the base algebra of the two arithmetic motions. The two are related — both have the same coefficient $\mu\lambda$ in the arithmetic frame — but they are different objects, and the identification is not made here.
3. *Non-integrability in a second calibration.* The companion programme also records the exact local obstruction

$$\vartheta = dz - x dy, \quad X = \partial_x, \quad Y = \partial_y + x\partial_z, \quad [X, Y] = \partial_z, \quad \vartheta([X, Y]) = 1.$$

Since $d\vartheta = -dx \wedge dy$, one has $\vartheta \wedge d\vartheta = \pm dx \wedge dy \wedge dz \neq 0$: ϑ is a contact form whose kernel does not close, exactly as for the arithmetic contact form (84). The two forms are not the same object — in α the multiplier of the multiplicative direction is the assignment a itself, whereas in ϑ the coefficient x is an independent coordinate — but the non-integrability pattern is identical: a Pfaffian form on a three-dimensional space whose kernel is spanned by two fields that fail to close, with a nonzero constant bracket component. The same pattern therefore appears in a calibration where no arithmetic assignment is present at all, which is evidence that the phenomenon belongs to the two directions rather than to the assignment.

These three items are exact computations, not proposals. Their status is nevertheless local to their own calibration: they are recorded in a research repository whose own maturity labels are, for the items cited, of the kind "precise conjecture" and "research calibration", and this paper does not promote them.

10.4 Status and non-claims

For the avoidance of doubt, this section claims the following and nothing more.

- Proposition 10.1 and Proposition 10.2 are proved above, by elementary computation, in this paper.
- The vocabulary of Subsection 10.2 is a *structural proposal*: it is a way of organizing the construction of non-linear descriptions, imported from a companion programme and used here as a language, not as a theorem about AEG.
- This section does not claim that manifolds, metrics, Lie groups, modules, or characters are illegitimate in AEG. Sections 3, 8, and 9 use all of them, and use them with proofs. What is refused is their status as ontology: no linear structure is imported into a foundational claim without a derivation of its transport at the arithmetic rank in question.
- This section does not claim any universality of arithmetic-generated geometry, and it does not import the companion programme's own open conjectures. Where that programme states that a claim is not established, the same claim is not established here.
- No efficiency, complexity, or approximation statement is made. The two propositions above are structural obstructions, and an obstruction is not a lower bound: no conclusion about computational cost follows from them, in either direction.

11 Holed Arithmetic Expression Spaces and the Language of Tearing

The geometry of Section 3 is developed on the basic model, which is complete and unpunctured: $\mathfrak{E}_0$ has no point at which the arithmetic structure fails. (The one-puncture model $\mathfrak{E}_1$ of Example 3.19 already leaves the regular category, and is the case this chapter generalizes.) This chapter develops the opposite situation through explicit punctured models and the vocabulary proposed for them. Its exact local constructions are kept distinct from a register of six computed witnesses, each with its own status. The finite-type construction and reconstruction interface follow in Section 12. The interpretive reading of those witnesses — the language of tearing — is presented as a structural proposal, and Subsection 11.6 lists what is not established.

11.1 Punctured arithmetic expression spaces

Definition 11.1 (Punctured arithmetic expression space). Let $\overline{M}$ be an oriented smooth surface, let $S \subset \overline{M}$ be a finite set of points, and put $M = \overline{M} \setminus S$. Suppose given a smooth Riemannian metric g on M , a function $a : M \rightarrow \mathbb{R}$ that is smooth on M , and constants $\mu \neq 0$, λ , such that

1. $(M, g, a; \mu, \lambda)$ is a regular AES in the sense of Definition 3.1, and
2. for every $p \in S$, no neighbourhood U of p in $\overline{M}$ admits extensions of $g|_{U \cap M}$ and $a|_{U \cap M}$ to smooth data $\tilde{g}, \tilde{a}$ on U for which $(U, \tilde{g}, \tilde{a}; \mu, \lambda)$ is a regular AES.

Then $(\overline{M}, S, g, a; \mu, \lambda)$ is called a *punctured arithmetic expression space*, and the points of S are its *punctures*. When $|S| = k$ we write $\mathfrak{E}_k$ for it, in the descriptive sense of Convention 3.17.

The ambient surface and the data are deliberately separated: the regular AES is given only on the punctured surface M , so that condition (ii) is a genuine requirement rather than a contradiction. A puncture is not any point that has been deleted, but a point at which the regular structure genuinely cannot be continued. This is the finite-point case of the singular arithmetic expression space of Paper I; the general singular theory, the classification of singular germs, and everything concerning tubes, braids, and knot invariants belong to Paper I and Paper III. The notion is specialized here for one reason only: the arithmetic motions of Subsection 3.5 are defined on the punctured surface, and the question of what happens to them at a puncture is a question about AES.

The first example is the one-puncture model of Example 3.19: the unit disc with the complete Poincaré metric and the radial assignment $a_{\mathbb{D}} = \frac{2\mu}{\lambda} \rho / (1 - \rho^2)$, whose metric extends smoothly across the centre while the assignment does not. There $S = \{0\}$ and the regular locus is $\mathbb{D} \setminus \{0\}$. Two features of that example are worth recording here, because they are typical of punctured models. First, the extended metric is complete on $\mathbb{D}$ while its restriction to $\mathbb{D} \setminus \{0\}$ is *not* complete: the puncture is at finite distance from every interior point, so the incompleteness is caused by the deletion and not by the geometry. Second, the assignment has a finite limit at the puncture (here 0) but no C^1 extension, so the failure is a failure of differentiability rather than of continuity. Both statements are verified in Paper I together with the model.

11.2 The local shape of a puncture

A puncture is not merely a missing point; at least in the example that has been computed in detail, it carries a definite local shape. The following statement is exact.

Proposition 11.2 (Tangent cone of the four limiting circles). *Fix $\rho > 0$ and consider the four circles through the origin*

$$C_{\pm} : x^2 + y^2 \mp 2\rho y = 0, \quad D_{\pm} : x^2 + y^2 \mp 2\rho x = 0.$$

Write $S_2 = x^2 + y^2$ and let

$$P = (S_2^2 - 4\rho^2 y^2)(S_2^2 - 4\rho^2 x^2)$$

be the defining polynomial of their union. Then

$$P = S_2^4 - 4\rho^2 S_2^3 + 16\rho^4 x^2 y^2,$$

so the lowest-degree homogeneous part of P is $16\rho^4 x^2 y^2$. Consequently the tangent cone of the union at the origin is

$$(xy)^2 = 0,$$

that is, the two coordinate axes, each with multiplicity two.

Proof. The four circles are $S_2 = \pm 2\rho y$ and $S_2 = \pm 2\rho x$, whose defining polynomials are $S_2^2 - 4\rho^2 y^2$ and $S_2^2 - 4\rho^2 x^2$ respectively; the product is P . Expanding,

$$(S_2^2 - 4\rho^2 y^2)(S_2^2 - 4\rho^2 x^2) = S_2^4 - 4\rho^2 S_2^2(x^2 + y^2) + 16\rho^4 x^2 y^2 = S_2^4 - 4\rho^2 S_2^3 + 16\rho^4 x^2 y^2,$$

since $S_2^2(x^2 + y^2) = S_2^3$. The terms have degrees eight, six, and four, so the lowest-degree part is $16\rho^4 x^2 y^2$, whose zero set is $xy = 0$ counted twice. $\square$

Two remarks, both used below. First, the two circles of the same family are tangent at the origin, so the origin is a degenerate point of each family, while circles of different families cross there with tangent lines $x = 0$ and $y = 0$; the tangent cone is the union of those two lines. Second, the tangent cone is *reducible with multiplicity*: it is the square of a product of two distinct linear forms. The computation is recorded in the exploration register of the programme [23]; the identity above is verified here directly.

11.3 An explicit punctured model with four zero branches

The review of this paper observed, correctly, that the tangent cone above was a statement about a curve configuration with no arithmetic expression space attached to it. The gap is now closed: the configuration is exactly the zero locus of an explicit punctured AES. We need one general fact first.

Lemma 11.3 (The gradient never vanishes in a regular AES). *In a regular AES $\mathfrak{E} = (M, g, a; \mu, \lambda)$ one has $\nabla a \neq 0$ everywhere on M ; indeed $|\nabla a|_g^2 = \mu^2 + \lambda^2 a^2 \geq \mu^2 > 0$. Hence, when M is boundaryless, every level set of a is a smooth embedded curve. The regular-zero theorem of Paper I is the case of level zero.*

Proof. Both sides of the eikonal identity are non-negative and the right-hand side is at least $\mu^2 > 0$, so ∇a does not vanish. The regular-value theorem then applies at every level, not only at 0. $\square$

The next theorem gives a conditional answer to the canonicity question. Both smooth extension of the assignment and extension of the conformal background are required. The latter hypothesis repairs the statement printed in the September 22 version: constructing a new AES metric does not by itself extend the prescribed metric.

Theorem 11.4 (Critical punctures with an extending conformal background). *Let $(\overline{M}, S, g, a; \mu, \lambda)$ be a punctured AES in the sense of Definition 11.1, and suppose that a extends to a smooth function $\tilde{a}$ on $\overline{M}$. Suppose in addition that, near each $p \in S$, the given metric g is conformal on $U \setminus S$ to a smooth Riemannian metric g_0 on a neighbourhood U of p , with g_0 non-degenerate at p . Then*

$$S = \{p \in \overline{M} : d\tilde{a}(p) = 0\}. \quad (102)$$

Within this conformal-background class, the puncture set is determined by the assignment and its cardinality is the number of critical points of $\tilde{a}$ in $\overline{M}$.

Proof. For the inclusion $S \subseteq \{d\tilde{a} = 0\}$, let $p \in S$ and suppose $d\tilde{a}(p) \neq 0$. Use the extending background metric g_0 supplied by the hypothesis on a neighbourhood U of p . Writing $g = f g_0$ on $U \setminus S$, the eikonal identity forces

$$f = \frac{|d\tilde{a}|_{g_0}^2}{\mu^2 + \lambda^2 \tilde{a}^2}.$$

Thus the extension constructed below agrees with the *given* metric. After shrinking U we have $|d\tilde{a}|_{g_0}^2 > 0$ on U , because $|d\tilde{a}|_{g_0}^2$ is continuous and positive at p , and also $\mu^2 + \lambda^2 \tilde{a}^2 > 0$ on U , because $\mu \neq 0$. Define

$$\tilde{g} = \frac{|d\tilde{a}|_{g_0}^2}{\mu^2 + \lambda^2 \tilde{a}^2} g_0 \quad \text{on } U.$$

Then $\tilde{g}$ is a smooth Riemannian metric on U , and

$$|d\tilde{a}|_{\tilde{g}}^2 = \frac{\mu^2 + \lambda^2 \tilde{a}^2}{|d\tilde{a}|_{g_0}^2} |d\tilde{a}|_{g_0}^2 = \mu^2 + \lambda^2 \tilde{a}^2,$$

so $(U, \tilde{g}, \tilde{a}; \mu, \lambda)$ is a regular AES extending (g, a) across p . This contradicts condition (ii) of Definition 11.1. Hence $d\tilde{a}(p) = 0$.

For the reverse inclusion, let $p \in \overline{M}$ with $d\tilde{a}(p) = 0$ and suppose $p \notin S$. Then $p \in M$ and the eikonal identity at p reads

$$0 = |da(p)|_g^2 = \mu^2 + \lambda^2 a(p)^2 \geq \mu^2 > 0,$$

a contradiction. Hence $p \in S$. □

Remark 11.5 (The hypothesis is active). The theorem requires the assignment to extend smoothly. The isolated-zero disc model imported from Paper I (Example 3.19) does not satisfy the hypothesis: there the metric extends smoothly across the centre while the assignment is only continuous, and not C^1 , so $d\tilde{a}$ does not exist at the puncture and (102) has no meaning. The two verified punctured models are therefore of different kinds: in the disc model the *assignment* fails at the puncture, while in the model below the assignment is smooth everywhere and the *metric* is what cannot be non-degenerately extended. The critical-set conclusion applies to the second kind only with the theorem's extending-conformal-background hypothesis. Without it even a smoothly extending assignment need not determine the punctures. For example, on a punctured disc, $a = x$, $\mu = 1$, $\lambda = 0$, and

$$g = dx^2 + (x^2 + y^2)^{-1} dy^2$$

satisfy $|da|_g^2 = 1$, while g cannot extend at the origin and da never vanishes. In every smoothly extending case only $\text{Crit}(\tilde{a}) \subseteq S$ holds without further hypotheses.

Theorem 11.6 (A punctured AES whose zero locus is the four-circle configuration). *Fix $\rho > 0$ and $0 < r < \rho$, let*

$$\overline{M} = D_r = \{(x, y) : x^2 + y^2 < r^2\}, \quad S = \{0\}, \quad M = \overline{M} \setminus S,$$

let P be the union polynomial of Proposition 11.2, put $a = P|_M$, fix $\mu \neq 0$ and λ , and define the conformal metric

$$g = \frac{|\nabla a|^2}{\mu^2 + \lambda^2 a^2} (dx^2 + dy^2) \quad \text{on } M. \tag{103}$$

Then:

1. g is a smooth Riemannian metric on M , and $(M, g, a; \mu, \lambda)$ is a regular AES;
2. the zero locus of a in $\overline{M}$ is exactly the union of the four circles, so the configuration is the zero set of the assignment and not an auxiliary curve family;
3. the origin is locally essential: no neighbourhood of 0 admits a smooth non-degenerate extension of (g, a) to a regular AES;

4. consequently $(\overline{M}, S, g, a; \mu, \lambda)$ is a punctured arithmetic expression space in the sense of Definition 11.1, whose zero locus has four branches meeting at the puncture;
5. the metric is incomplete on M , and the puncture lies at finite distance.

Proof. Throughout, write $S_2 = x^2 + y^2$ and

$$f(t) = S_2^2(S_2 - 3\rho^2) + 4\rho^4 t^2,$$

so that, differentiating $P = S_2^4 - 4\rho^2 S_2^3 + 16\rho^4 x^2 y^2$,

$$\partial_x P = 8x f(y), \quad \partial_y P = 8y f(x). \quad (104)$$

(1) The denominator $\mu^2 + \lambda^2 a^2$ is bounded below by $\mu^2 > 0$ on M and $|\nabla a|^2$ is smooth, so g is smooth and positive definite. Its eikonal identity is immediate:

$$|\nabla a|_g^2 = \frac{\mu^2 + \lambda^2 a^2}{|\nabla a|^2} |\nabla a|^2 = \mu^2 + \lambda^2 a^2.$$

We must also check that ∇a does not vanish on M , so that the conformal factor is strictly positive there. Suppose $\nabla P = 0$ at a point of $\overline{M}$. If $x = 0$ then $S_2 = y^2 > 0$ at a punctured point and $f(0) = y^4(y^2 - 3\rho^2)$, which is nonzero because $y^2 = S_2 < r^2 < \rho^2 < 3\rho^2$; then $\partial_y P = 8y f(0) \neq 0$, a contradiction. The case $y = 0$ is symmetric. If $x \neq 0$ and $y \neq 0$, then (104) forces $f(x) = f(y) = 0$. Here $S_2 < 3\rho^2$, so f is strictly increasing in t^2 for fixed S_2 ; hence $f(x) = f(y)$ gives $x^2 = y^2$, so $S_2 = 2x^2$ and

$$0 = f(x) = 4x^2 [x^2(2x^2 - 3\rho^2) + \rho^4],$$

that is $2u^2 - 3u + 1 = 0$ for $u = x^2/\rho^2$, whose roots are $u = 1$ and $u = \frac{1}{2}$. These give $S_2 = 2\rho^2$ and $S_2 = \rho^2$, both at least r^2 , so no such point lies in $\overline{M}$. Therefore $\nabla P \neq 0$ on M .

(2) P factors as $(S_2^2 - 4\rho^2 y^2)(S_2^2 - 4\rho^2 x^2)$, and each quadratic factor splits into two circles through the origin:

$$S_2 = 2\rho y \iff x^2 + (y - \rho)^2 = \rho^2, \quad S_2 = -2\rho y \iff x^2 + (y + \rho)^2 = \rho^2,$$

and similarly with x and y exchanged. Inside $\overline{M}$ the only common point of any two of these circles is the origin: two circles of the same family are tangent there, and two of different families meet again only at radius $\sqrt{2}\rho > r$, as one checks by solving $S_2 = 2\rho y$, $S_2 = -2\rho x$ or $S_2 = 2\rho y$, $S_2 = 2\rho x$. Hence the zero locus in $\overline{M}$ consists of four arcs, two tangent to the x -axis and two to the y -axis, meeting only at the origin — which is exactly the tangent cone computed in Proposition 11.2.

(3) Let U be a neighbourhood of the origin and let $\tilde{g}, \tilde{a}$ be smooth extensions to U of $g|_{U \cap M}$ and $a|_{U \cap M}$ for which $(U, \tilde{g}, \tilde{a}; \mu, \lambda)$ is a regular AES. Since $\tilde{a}$ is continuous on U and agrees with a off the origin, while a extends continuously to the origin with value $P(0) = 0$, we have $\tilde{a}(0) = 0$, and $\nabla \tilde{a}(0) = \nabla a(0) = 0$ because a is smooth on $\overline{M}$. Evaluating the eikonal identity at the origin by continuity gives

$$0 = \tilde{g}^{ij}(0) \partial_i \tilde{a}(0) \partial_j \tilde{a}(0) = \mu^2 + \lambda^2 a(0)^2 = \mu^2 \neq 0,$$

a contradiction. So no such extension exists, which is condition (ii) of Definition 11.1. (Equivalently: by Lemma 11.3 a regular AES has no critical point of its assignment, whereas the origin is a critical point of P .)

(4) Conditions (i) and (ii) of Definition 11.1 are (1) and (3), so the tuple is a punctured AES. By (2) its zero locus is the four-circle configuration, and by the local structure computed there the four arcs meet only at the puncture.

(5) On the x -axis and away from the origin, $\partial_y P = 0$ and $f(0) = x^4(x^2 - 3\rho^2)$, so $|\nabla P| = 8|x||f(0)| = 8|x|^5|3\rho^2 - x^2|$, which is $24\rho^2|x|^5 + O(|x|^7)$ as $x \rightarrow 0$. Hence along the axis

$$\sqrt{g_{xx}} dx = \frac{|\nabla P|}{\sqrt{\mu^2 + \lambda^2 P^2}} dx = \frac{24\rho^2}{|\mu|} |x|^5 dx (1 + o(1)),$$

whose integral from 0 to ε is finite; so the puncture lies at finite distance and the metric on M is incomplete. The distance is not zero: on the annulus $\varepsilon/2 \leq S_2^{1/2} \leq r$ the conformal factor is bounded below by a positive constant, so every path from a point of radius ε to the puncture has length bounded below. $\square$

The theorem answers the question the review posed, and it is worth noting what it does *not* do. Its puncture set is not stipulated: by Theorem 11.4, applied to the extending Euclidean background of this construction, the set S must equal the critical set, and the proof above computes that critical set on $\overline{M} = D_r$ to be $\{0\}$ alone, so the model's puncture is forced by a . What the theorem does not do is attach the configuration to the arithmetic motions rather than to the assignment, or treat more than one puncture. Finally it makes the interface with Paper I visible rather than implicit: the zero locus of a regular AES is a smooth curve (Lemma 11.3), the complete-splitting theorem of Paper I forbids branching on a connected complete boundaryless regular space, and here branching occurs exactly at the puncture — with the metric incomplete, which is one of the ways the theorem's hypotheses can fail.

Corollary 11.7 (Puncture counts in the conformal-background class). *Under all the hypotheses of Theorem 11.4, $|S| = |\text{Crit}(\tilde{a})|$. Conversely, if a smooth function $\tilde{a}$ on an oriented smooth surface $\overline{M}$ has a finite critical set S , then for every smooth background metric g_0 the formula*

$$g = \frac{|d\tilde{a}|_{g_0}^2}{\mu^2 + \lambda^2 \tilde{a}^2} g_0 \quad \text{on } \overline{M} \setminus S$$

constructs a punctured AES with precisely that declared puncture set.

Proof. The equality follows from the theorem with its conformal hypothesis. For the converse, the factor is positive off S and the eikonal identity follows by inverting the factor. At each point of S a non-degenerate regular extension would give $0 = \mu^2 + \lambda^2 \tilde{a}^2$, which is impossible. This proves essentiality for the constructed metric, not a classification of arbitrary prescribed metrics. $\square$

Remark 11.8 (A construction rather than a classification). The corollary supplies models from assignments with finitely many critical points. It does not make an arbitrary AES metric extendible and does not identify all punctures with critical points. Section 12 constructs compactified carriers with every positive finite puncture count; the point at infinity in that family is essential because the assignment diverges, so it is not an additional critical point of a smooth extension.

11.4 Tearing across a puncture

The tearing of Definition 8.10 is a defect between two add-scale histories. On a punctured space a further question arises: a comparison of two motions may have to be performed around a puncture rather than through it, and one may ask whether the tearing is then measured by an ACS area, by a holonomy of a loop around the puncture, or by neither. The answer proved below is that it is measured by the ACS area in every case, that no canonical holonomy exists in the AES data, and that what the puncture changes is the admissibility of a *comparison on the surface* rather than the value of the tearing.

Proposition 11.9 (The tearing does not see the puncture). *Let γ, δ be scale-compatible add-scale histories over a field K , with translation coordinates ξ_γ, ξ_δ as in Section 7. Then*

$$\tau(\gamma, \delta) = \xi_\gamma - \xi_\delta$$

depends only on the two words. Consequently $\tau(\gamma, \delta)$ is unchanged by puncturing the space in which the histories are realized, by changing that space, or by choosing a different realization of the same words; and the area formula of Theorem 8.9 applies verbatim on a punctured model, with the filling taken in the charge plane rather than on the surface.

Proof. By Proposition 7.1 the target-frame translation ξ is a function of the chronological word alone, and Definition 8.7 defines τ as the difference of the two ξ 's. Nothing in either statement mentions a surface, a metric, or a puncture. The area formula is Theorem 8.9, whose data are the two charge chains. $\square$

The next statement explains why the charge plane is the canonical place for the comparison, and it has nothing to do with punctures: on a surface the two orders of a rectangle never close up.

Proposition 11.10 (The surface never closes the comparison). *Let $\mathfrak{E}$ be a regular AES with canonical frame (X_u, X_v) , let $p \in M$, and let $h, k > 0$ be such that the flows Φ_u^h and Φ_v^k of X_u and X_v are defined at p and along both compositions. Then the two chronological compositions end at points whose assignments differ by*

$$a(\Phi_v^k \Phi_u^h(p)) - a(\Phi_u^h \Phi_v^k(p)) = \mu h(e^{\lambda k} - 1) \neq 0, \quad (105)$$

whenever $\mu h \neq 0$ and $\lambda k \neq 0$. Hence the two flow paths alone bound no region of M : closing them requires a further path along which the assignment changes by that amount.

Consequently the ACS comparison of Section 8 is not a device for avoiding punctures. It is the canonical comparison for every regular AES, punctured or not, and the puncture changes nothing about it.

Proof. By Theorem 3.16, the flows satisfy $\frac{d}{ds}(a \circ \Phi_u^s) = \mu$ and $\frac{d}{ds}(a \circ \Phi_v^s) = \lambda(a \circ \Phi_v^s)$; hence $a \circ \Phi_u^h = a + \mu h$ and $a \circ \Phi_v^k = e^{\lambda k} a$. Composing in the two orders,

$$a \circ \Phi_v^k \circ \Phi_u^h = e^{\lambda k}(a + \mu h), \quad a \circ \Phi_u^h \circ \Phi_v^k = e^{\lambda k}a + \mu h,$$

whose difference is $\mu h(e^{\lambda k} - 1)$. This is nonzero under the stated hypotheses, so the two endpoints are distinct; two paths that do not share both endpoints bound no region. $\square$

Proposition 11.11 (No canonical holonomy around a puncture). *Let $\mathfrak{E}$ be a regular AES.*

1. *The horizontal structure determined by the assignment is the foliation by the level sets of a , whose tangent distribution $\ker da$ is integrable; it is therefore flat, and it has no holonomy.*
2. *The AES data determine no further connection on M . A connection with curvature requires a choice of splitting beyond $(M, g, a; \mu, \lambda)$; the contact model of Section 9 is one such choice, and relative to its added charge projection the curvature form ω of Proposition 9.6 is canonical.*
3. *Since the assignment is a globally single-valued function on M , no loop around a puncture can change the value of a : there is no monodromy of the assignment.*

Hence no tearing is measured by a holonomy of a loop around a puncture. The natural holonomy in this theory lives in the contact model, over a loop in the charge plane, where it is the closed horizontal drift $T_{\text{closed}} = e^{-\lambda k} T_{\text{open}}$ of Section 9 — a holonomy of the added structure, not of the puncture.

Proof. (1) The level sets of a smooth function foliate the open set where its differential does not vanish, and by Lemma 11.3 that is all of M ; the tangent distribution of a foliation is integrable by definition. (2) The canonical frame supplies two global directions, but a splitting of the tangent bundle is not part of the data; §9 fixes one by declaring the coordinates (u, v, a) and the charge projection. (3) A single-valued function has no monodromy along any loop. $\square$

What the puncture does change. It changes which comparisons can be performed *on the surface*. In the model of Theorem 11.6 the local picture is computed as follows.

Proposition 11.12 (The zero-set germ and the metric at the puncture). *In the model of Theorem 11.6, with $0 < \varepsilon < \rho$:*

1. the circle $S_2 = \varepsilon^2$ meets the zero locus in exactly eight points, four on the pair of circles tangent to the x -axis and four on the pair tangent to the y -axis;
2. the assignment changes sign at each of them, and the punctured disc is divided into eight sectors, four with $a > 0$ and four with $a < 0$; the negative sectors are the cusps $|y| < x^2/(2\rho)$ and $|x| < y^2/(2\rho)$ along the coordinate axes;
3. g is conformal to the Euclidean metric with conformal factor vanishing at the puncture, so the conformal class of g extends across the puncture although the metric does not; angles at the puncture are therefore well defined and equal those of the tangent cone of Proposition 11.2;
4. the puncture lies at finite g -distance from every interior point, so a path that approaches it does so with finite g -length, and M is incomplete.

Proof. (1) On the circle $S_2 = \varepsilon^2$ the union polynomial restricts to

$$P = \varepsilon^4(\varepsilon^2 - 4\rho^2 y^2)(\varepsilon^2 - 4\rho^2 x^2),$$

whose zeros are $y = \pm\varepsilon^2/(2\rho)$ and $x = \pm\varepsilon^2/(2\rho)$, each giving two points of the circle since $\varepsilon < \rho$ implies $\varepsilon^2/(2\rho) < \varepsilon$. (2) The sign of P on a small punctured neighbourhood is the sign of the product of the two factors $S_2^2 - 4\rho^2 y^2$ and $S_2^2 - 4\rho^2 x^2$, and near the origin $P = S_2^3(S_2 - 4\rho^2) + 16\rho^4 x^2 y^2$ is dominated by $-4\rho^2 S_2^3$ exactly where $|y| < x^2/(2\rho)$ and by $-4\rho^2 S_2^3$ with the roles of x and y exchanged exactly where $|x| < y^2/(2\rho)$. (3) and (4) are parts (1) and (5) of Theorem 11.6, the conformal factor being $|dP|^2/(\mu^2 + \lambda^2 P^2)$. $\square$

So the answer to the question of this subsection is the following. A tearing is always the ACS area of Theorem 8.9, and its value is independent of the model and of its punctures; the admissible fillings are the oriented two-chains of the charge plane with the given boundary, and by that theorem the value does not depend on which filling is chosen. A comparison that would have to pass through a puncture is simply not available on the surface, and in the computed model the puncture is exactly where the surface stops being metric (though not conformal) and where the zero-set germ has eight sign-changing branches. These scalar statements do not determine whether an assignment-preserving word moves the full surface state; Section 12 retains that distinction explicitly.

11.5 Six computed witnesses

The exploratory work recorded in [23] produced six separately computed statements about punctured configurations. They are reproduced here in the register’s own order, each with its exact content and its status. None of them is promoted by appearing in this paper: the register carries no authoritative status, and the items below are classified according to the labels of this repository.

1. *A topological witness.* Cut a disc carrying four right-angle sectors along the two diagonal seams and the four coordinate rays; eight triangular pieces result, each with a left radial edge, an outer arc, and a right radial edge. Regluing is a permutation σ of the eight left edges onto the eight right edges. If C is the number of cycles of σ , then the centre vertex has C orbits, the outer vertices form eight orbits, and $V = C + 8$, $E = 16$, $F = 8$, whence

$$\chi = V - E + F = C = b,$$

the number of boundary components. Consequently every gluing of this family produces either one disc or a disjoint union of C discs, and the compact three-holed sphere ($\chi = -1$ with three boundary components) is unreachable in this family. A surface of Euler characteristic -1 is obtained instead by removing two points from the disc: it has one boundary component, two punctures, and hence three ends, so its Euler characteristic agrees with that of the compact pair of pants while its boundary structure does not. An exhaustive check of all 40320 permutations confirms the distribution 5040, 13068, 13132, 6769, 1960, 322, 28, 1 for $C = 1, \dots, 8$. *Status:* the identity $\chi = C$ is elementary and provable as above; the enumeration is a computationally verified example. The topological conclusion holds for the stated gluing rule, not for arbitrary gluings of the eight pieces.

2. *A combinatorial witness.* If three components interact pairwise, the interaction sites are the three unordered pairs, and each site carries two directions of exchange, giving six directed transitions in total. *Status:* counting, given the definition of a hole as an interaction site. The definition is a choice, not a theorem.
3. *A frame witness.* Let the state of a three-component frame be (c, σ, φ) with component label $c \in \mathbb{Z}_3$, direction $\sigma \in \{\pm 1\}$, and phase $\varphi \in \mathbb{Z}_4$, and let one step act by

$$(c, \sigma, \varphi) \mapsto (c + \sigma, \sigma, \varphi + \sigma).$$

Since $\gcd(3, 4) = 1$, the Chinese remainder theorem identifies (c, φ) with a single counter k modulo 12, and the orbit of each direction has length 12. Within one such orbit the component label advances in the direction σ , so the transitions realized are the three pairs of that direction, each occurring four times — once per phase class. The six directed transitions of the combinatorial witness therefore appear four times each only over the two orbits together, that is, over 24 steps; a single 12-step orbit cannot contain 24 transitions. Since one full phase period is four steps and $4 \equiv 1 \pmod{3}$, one period of the phase returns the phase to its initial value while advancing the component label by

$$c \mapsto c + \sigma \pmod{3};$$

for the fixed direction $\sigma = +1$ this is the advance by one used in the register, and for $\sigma = -1$ it is the advance by two. In either case the phase loop is not closed in the total state: its monodromy is the generator $-\sigma$ of $\mathbb{Z}_3$. *Status:* exact arithmetic (Chinese remainder theorem), verified exhaustively on the two 12-state orbits and by replay.

4. *An observer witness.* Let a frame be described by an angle θ and three joint offsets $\delta_1, \delta_2, \delta_3$, and let the residuals available to an observer be

$$r_i = |2 \sin \delta_i| \quad (i = 1, 2, 3), \quad r_4 = |2 \sin(2\theta + \tfrac{1}{2}(\delta_1 + \delta_2 + \delta_3))|.$$

All four residuals vanish exactly at the ideal frame $(\theta, \delta) = (\pi/2, 0, 0, 0)$. Suppose an observer sees only residuals of magnitude at least ε , and that an iterative procedure stops as soon as every visible residual is below that threshold. At the halt one then has the following bound. From $r_i < \varepsilon$ we get $|\delta_i| < \arcsin(\varepsilon/2) = \varepsilon/2 + O(\varepsilon^3)$, and from $r_4 < \varepsilon$ together with $r_4 = |4(\theta - \pi/2) + (\delta_1 + \delta_2 + \delta_3)| + O(\|(\theta - \pi/2, \delta)\|^2)$ we get

$$|\theta - \pi/2| < \frac{5}{8} \varepsilon + O(\varepsilon^2).$$

The halting offset is therefore of order ε and decreases linearly in ε ; it is not zero, and it is the best that a procedure obeying this stopping rule can report. *Status:* the residual functions, their exact common zero at the ideal frame, and the bound are proved above. The particular iteration counts and offsets recorded in the register depend on the implementation of the update step, which the register does not fix, and they are accordingly *not* reproduced here. The witness is *not* an obstruction: nothing in the computation shows that no procedure can do better, and the exact zero of the residuals at the ideal frame shows the opposite.

5. *A semantic witness.* Model the three interaction sites by vocabularies built from four terms common to all three components, four further control terms distributed over the pairs as two, one, and one, and three terms private to the individual components (which therefore enter no site vocabulary). The three site vocabularies then have sizes

$$6 = 4 + 2, \quad 5 = 4 + 1, \quad 5 = 4 + 1.$$

Filling the three sites independently — one word chosen from each site's own vocabulary — gives $6 \cdot 5 \cdot 5 = 150$ possibilities; filling them so that the same core word is used at all three sites gives $4^3 = 64$. The difference 86 is carried by the adapters at the sites. On the control layer the pairwise intersections are non-empty while the common intersection is empty; on the full vocabularies the common intersection is the four-term core, so the emptiness statement is layer-dependent. *Status:* the arithmetic is verified given the model; the model is a modelling choice, and the "Borromean" description is an analogy with no link-theoretic content.

6. *A phase-geometry witness.* With components at phases $\{0, +1, -1\}$ modulo n , a forward motion of k steps and a reverse motion of m steps meet exactly when $k + m \equiv 0 \pmod{n}$. A meeting phase different from the three component phases exists precisely for $n \geq 4$; at $n = 4$ it is the first case in which the meeting is simultaneously balanced ($k = m$) and antipodal ($\varphi = n/2$), namely $k = m = 2$, $\varphi = 2$. *Status:* computationally verified example over the stated finite model, exhaustive for $n = 3, \dots, 8$. The value $n = 4$ is minimal, not unique: $n = 6$ and $n = 8$ also admit balanced antipodal meetings.

Remark 11.13 (Register bookkeeping). The exploration register introduces these items as five classes while tabulating six rows; the sixth row is the phase-geometry witness listed last above. This paper follows the table and counts six, recording the discrepancy rather than resolving it silently [23].

11.6 What the witnesses do not establish

The register's own summary sentence is that a hole "is not a defect to be removed but the position that carries exchange". That sentence is the motivation for this chapter, and it is not proved by any of the six witnesses. The following list of non-claims is part of the content of this section.

- No witness shows that punctures are unavoidable in AES. The witnesses are about chosen finite models: a chosen gluing family, a chosen frame, a chosen vocabulary, a chosen stopping rule. Changing the model changes the count, and the register says so for its own semantic witness.
- No witness establishes an obstruction of AEG. The monodromy of the frame witness is a monodromy of a stipulated finite flow; the calibration witness is a property of a stipulated halting rule; the gluing witness is a statement about one gluing family.
- The $\mathbb{Z}_3$ monodromy of the frame witness is not identified with any $\mathbb{Z}_4$ twist of sheet labels appearing elsewhere in the register. Those are different groups, and no computation in the register relates them; the identification asserted there is recorded as unproved.
- The word "hole" in this chapter means a puncture in the sense of Definition 11.1, or an interaction site in the sense of the combinatorial witness. It is not the "one-hole context" of Section 2, which is a placeholder in an expression, and the two must not be conflated.
- The engineering study that accompanies the register is explicitly recorded by its author as an analogy and not as a mathematical assertion; one of its three language prototypes was never compiled. It contributes motivation only.
- Nothing here changes the status of Paper I's regular-zero theorems or of Paper III's singular theory. A punctured AES is a specialization of the singular notion, and its classification is not attempted.

11.7 Open problems

The register closes with the author's own verdict that this route is not yet free, for three reviewable reasons: the family of cuts depends on a chosen permutation, the covering sheets are defined by the cut, and the graded description depends on an externally supplied frame. It registers three directions that have not been attempted, and they are the natural open problems of this chapter.

1. Replace the tangent cone of Proposition 11.2 by the formal neighbourhood of the puncture: compute the successive jets at p and test whether the description ceases to depend on the choice of tangent directions.
2. Blow up the puncture and rewrite the cut-and-glue construction as a canonical construction on the exceptional divisor, so that the seam is replaced by a construction the data itself determines.
3. Replace the cut-and-glue calculus by an operadic composition, in which the gluing of punctured surfaces is intrinsic and contains no chosen seam.

Each of these is a programme, not a result. Four further problems arise directly from the discussion above; the first of them is the natural generalization of the construction now proved, and the second is the comparison question raised in Subsection 11.4.

1. *Generalize the model of Theorem 11.6.* The construction there produces one puncture and a zero locus with four branches from a prescribed polynomial assignment and the metric that the eikonal identity forces. Two questions follow. Which curve configurations arise as zero loci of punctured AES in this way? The existence of some compactifiable model for each $k \geq 1$ is settled in Theorem 12.3; prescribed zero configurations, puncture locations, and conformal moduli remain separate questions. And does the construction survive if the assignment is required to come from the arithmetic motions rather than being prescribed as a polynomial? A negative answer to the last question would mean that the configuration is realizable as a zero locus but not as an expression.
2. *Classify full-state residue across punctured models.* The scalar affine defect is word-dependent and hence puncture-independent, while Proposition 12.7 exhibits a level-set displacement invisible to that scalar observation. Determine which residual data and admissible motion domains suffice for continuation across recorded collars, especially when the assignment does not extend to a puncture. The scalar statement alone does not answer this full-state question.
3. *Canonicalize the punctures when the assignment does not extend.* For smoothly extending assignments with an extending conformal background, Theorem 11.4 gives the critical set. Outside that class, metric singularities may occur where $da \neq 0$. When the assignment is not differentiable at the puncture, as in the disc model of Paper I, the critical set is not even defined there. Determine useful intrinsic end data in both cases. This is the first unfinished item of the exploration register's own programme.
4. *Carry ripple geometry across a puncture.* Decide whether a punctured model carries a pullback structure, in the sense of Section 4, that survives at the puncture; in the model of Theorem 11.6 the pole would have to be placed at the critical point, where the conformal factor of the metric vanishes.

A solution of any of them would turn the language of this chapter from a description into a theory.

12 Finite-Type Compactification and Three-Ended Reconstruction

The punctured models of Section 11 can be organized by their ends without identifying a singular point, a boundary circle, and a computational input. We first specify which structure is compactified, then construct a model for every finite positive puncture count, and finally state the collar data needed to cut and reconstruct the original AES.

The results concern existence and reconstruction under explicit hypotheses. They neither classify all AES metrics nor identify the complete hyperbolic metric of an auxiliary uniformization with the arithmetic metric. The port binding and execution layer is developed in Paper IV [22]; its inputs do not alter the geometry defined here.

12.1 End data and three meanings of completion

Definition 12.1 (Finite-type carrier and conformal point compactification). A finite-type carrier of genus γ with k punctures is a connected oriented surface M diffeomorphic to $\overline{M} \setminus S$, where $\overline{M}$ is a closed oriented surface of genus γ and $|S| = k$. It is a punctured AES carrier when $(g, a; \mu, \lambda)$ also satisfies Definition 11.1, including essentiality at every point of S . Its oriented conformal structure is the one determined by g . A *conformal point compactification* adds one point at each end and extends that conformal structure; it does not assert extension of g or a .

Proposition 12.2 (The local condition for conformal filling). *A finite-type conformal surface admits such a point compactification if and only if every end has a neighbourhood conformally equivalent to a punctured disc $0 < |z| < \epsilon$. A topological puncture alone does not impose this condition.*

Proof. If a conformal compactification exists, a disc coordinate at each added point restricts to the required end coordinate. Conversely, attach $z = 0$ in each of finitely many disjoint end charts. Their overlap maps are already holomorphic away from the added points, and no other chart contains those points; hence these charts extend the conformal atlas. The finite-type compact core together with the finitely many closed smaller end discs is compact. A finite-modulus annular end, for example, is not conformally a punctured-disc end although it is topologically annular. $\square$

For a complete hyperbolic cusp, the quotient of $\{\text{Im } z > Y\}$ by $z \mapsto z + 1$ has coordinate $q = e^{2\pi iz}$. Direct substitution gives

$$h = \frac{|dq|^2}{|q|^2(\log |q|)^2}.$$

Thus $q = 0$ fills the conformal puncture, while the cusp remains at infinite h -distance. Topological point filling, conformal point filling, and metric completion are different operations. Removing an open disc instead leaves a boundary circle; an oriented real blow-up can retain directions at a puncture, but does not automatically extend the arithmetic data. Markings, end germs, and any chosen angular or port data must be retained explicitly.

For a genus-zero complete finite-area hyperbolic surface with $k \geq 3$ cusps, the conformal compactification is a sphere with k distinct marked points. Normalizing three points to $0, 1, \infty$ leaves $k - 3$ complex parameters, modulo any explicitly chosen label symmetries. This is a statement about the carrier's conformal moduli, not a count of AES assignments or metrics.

12.2 An explicit family with every puncture count

Theorem 12.3 (Compactifiable AES with any positive finite puncture count). *Let $k \geq 1$. For $k > 1$ choose distinct real numbers $r_1, \dots, r_{k-1}$; for $k = 1$ use the empty product. Choose a polynomial P_k with*

$$P'_k(x) = \prod_{j=1}^{k-1} (x - r_j), \quad a(x, y) = P_k(x) + y^2,$$

and set $S = \{r_1, \dots, r_{k-1}, \infty\} \subset \mathbb{P}^1(\mathbb{C})$, $M = \mathbb{P}^1(\mathbb{C}) \setminus S$. For $\mu \neq 0$ and real λ , put

$$g = \frac{(P'_k(x))^2 + 4y^2}{\mu^2 + \lambda^2 a^2} (dx^2 + dy^2). \quad (106)$$

Then $(\mathbb{P}^1(\mathbb{C}), S, g, a; \mu, \lambda)$ is a punctured AES with exactly k essential punctures and conformal compactification $\mathbb{P}^1(\mathbb{C})$.

Proof. The Euclidean differential of a is $(P'_k(x), 2y)$, whose zeros are precisely $(r_j, 0)$. The numerator in (106) is consequently positive on M and the denominator is at least $\mu^2 > 0$. Inverting its positive conformal factor gives $|da|_g^2 = \mu^2 + \lambda^2 a^2$. At a finite point of S the polynomial assignment has vanishing differential; any smooth non-degenerate extension of the given AES would therefore contradict that identity. At infinity, along $x = 0$ and $|y| \rightarrow \infty$, the assignment $P_k(0) + y^2$ diverges, so it admits no continuous real-valued extension. These prove essentiality directly. The conformal class on M is the Euclidean class, which extends in the standard coordinate $1/z$ at infinity and at all finite punctures. For $k = 1$, $P'_1 = 1$ and the only puncture is infinity. No critical-set theorem at infinity is used. $\square$

This proves existence for every k , including the low-complexity cases $k = 1, 2$; it does not supply every marked conformal structure or a prescribed zero set. The notation $\mathfrak{E}_k$ remains descriptive, not a unique model determined by k . In the four-puncture calibration we take

$$a = \frac{x^4}{4} - \frac{x^2}{2} + y^2, \quad S = \{-1, 0, 1, \infty\}, \quad \mu = \lambda = 1. \quad (107)$$

The finite puncture values are $-1/4, 0, -1/4$. They are fixed by this assignment; arbitrary later computational inputs cannot be interpreted as arbitrary end limits without constructing a different parameterized AES family.

12.3 Three-ended blocks and exact geometric reconstruction

For $2\gamma - 2 + k > 0$, a finite-type carrier has a pants decomposition into three-ended blocks. An external end can remain a puncture, while an internal end of a cut block is a boundary circle with a collar. If N is the number of blocks and E the number of internal seams, Euler characteristic and end counting give

$$N = 2\gamma - 2 + k, \quad 3N = k + 2E, \quad E = 3\gamma - 3 + k. \quad (108)$$

Indeed each block has Euler characteristic -1 , and gluing circles changes no Euler characteristic. Existence of the decomposition and its hyperbolic coordinate interpretation are standard; see [15, §2]. In genus zero the adjacency graph is a tree with $k - 2$ vertices and $k - 3$ internal edges. If instead k counts extra operational ends in addition to three permanent ends, substitute $k + 3$ in these formulas. The exceptional sphere, plane, and cylinder cases are not assigned negative numbers of blocks.

Definition 12.4 (AES collar contract). A collar contract specifies the paired seam sides, their parametrizations and markings, and smooth two-sided collar charts for the proposed quotient. On overlapping collar charts, transition diffeomorphisms ϕ_e preserve the surface orientation and satisfy

$$\phi_e^* g_2 = g_1, \quad a_2 \circ \phi_e = a_1, \quad (109)$$

with identical signed constants μ, λ . Inverse and cocycle identities are required wherever transitions overlap. Boundary orientations of paired cut circles are opposite. The contract includes the complete seam inventory, external end labels and germs, and the requirement that the quotient atlas is a Hausdorff smooth oriented surface.

The conditions concern full collar germs, not just equal values on the seam. Equivalently, one may specify compatible smooth tensors in one signed collar coordinate and restrict them to its two sides. Arbitrary boundary identifications do not automatically satisfy the definition.

Proposition 12.5 (Cutting and descent of the original AES). *Cut a finite-type regular AES along a pants decomposition and retain its full collar contracts and external end data. Reassembly reconstructs the original AES up to the recorded marked isomorphism. More generally, AES blocks with contracts as in Definition 12.4 descend to a regular AES on the smooth quotient. Their canonical arithmetic frames agree across the seams. A finite motion trajectory crossing the seams transversely away from punctures is carried to the same trajectory in the quotient.*

Proof. The matching identities glue the metric tensors to a smooth positive-definite metric and the assignments to a smooth function on the quotient atlas. The eikonal identity is local and invariant

under change of chart, so it holds on the quotient. Orientation-preserving isometries preserving a preserve its gradient and the rotation J ; with the same signed μ, λ , the formula in Proposition 3.2 therefore matches both frame fields. Uniqueness for smooth local ordinary differential equations matches their flows on overlaps, hence each finite admissible transverse crossing. In the case obtained by cutting an existing AES, the quotient charts are its original charts and the seam identifications restore exactly the same points, tensors and function. Retained end germs restore the same essential punctures; no extension across them is asserted. $\square$

For (107), the circles $|z + 1/2| = 3/4$ and $|z - 1/2| = 3/4$ give two different single-seam decompositions. The first separates $\{-1, 0\}$ from $\{1, \infty\}$; the second separates $\{0, 1\}$ from $\{-1, \infty\}$. Neither seam meets S . Restricting the same a, g to either pair of blocks and retaining the original collar charts reconstructs exactly the same four-puncture AES by the proposition. This is a reconstruction result for the recorded atlas, not independence from erased seam data or a canonical choice of decomposition.

12.4 Auxiliary hyperbolic geometry and three-cusp compatibility

When the carrier admits a complete finite-area hyperbolic uniformization h , its pants carry internal geodesic boundary lengths $\ell_e > 0$ and twists τ_e . With fixed marking and fixed external cusp types, these are Fenchel–Nielsen coordinates on Teichmüller space. They reconstruct h , not the original AES tensors (g, a) . On pinching every internal seam of a chosen genus-zero decomposition, the limiting nodal curve consists of $k - 2$ thrice-punctured components, paired at $k - 3$ nodes. Local plumbing $uv = q_e$ smooths a node when $q_e \neq 0$; $q_e = 0$ is its nodal limit [15, §3]. Local coordinates and any marking or orbifold identifications must be specified. This standard geometric compatibility provides no AES deformation unless a compatible family $(g_{\mathbf{q}}, a_{\mathbf{q}})$ and its end data are also constructed.

Warning 12.6 (The arithmetic metric is not the complete cusp metric). A connected, complete, boundaryless regular AES with a globally single-valued assignment has underlying surface diffeomorphic to a plane or a cylinder. Indeed the rectified assignment $t(a) = \int_0^a (\mu^2 + \lambda^2 s^2)^{-1/2} ds$ has unit gradient; its complete flow gives $M \cong a^{-1}(0) \times \mathbb{R}$, and a connected one-dimensional level is a line or a circle. This is the complete-splitting theorem of Paper I [21]. Consequently a complete three-cusp hyperbolic metric cannot itself be the regular AES metric for such an assignment. Boundary, incompleteness, or a change of semantic category is an active hypothesis, not a removable inconvenience.

There is also an explicit special family covering the three-cusp sphere. For $k \geq 3$, put $d = k - 2$ and

$$X_k = \mathbb{P}^1(\mathbb{C}) \setminus (\{0, \infty\} \cup \{z : z^d = 1\}).$$

The map $z \mapsto z^d$ is an unramified degree- d cover from X_k to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$: any critical points lie in the removed set $\{0, \infty\}$, and each retained target has exactly d distinct preimages. It extends to a branched map between the compact spheres. This proves a compatible conformal model for each count, not a covering for every prescribed conformal structure. For a four-puncture unramified cover the degree must be 2 by Euler characteristic; its compact extension has two simple branch values among the three marked targets, and the four source punctures have cross-ratio orbit $\{-1, 2, 1/2\}$. Generic four-puncture moduli are excluded. This covering construction by itself specifies no arithmetic metric.

12.5 An assignment identity can retain motion residue

The scalar tearing of Section 8 compares induced affine operators. A word inducing the identity on the assignment need not be the identity on the full AES state. Here is an exact local certificate in (107), independent of a numerical trajectory plot.

Proposition 12.7 (A nonzero level-set residue in the four-puncture model). *Let A_s and M_t be the local flows of the canonical fields X_u, X_v in (107). For sufficiently small s, t , the chronological word $A_s, M_t, A_{-e^t s}, M_{-t}$ defines*

$$W_{s,t} = M_{-t} \circ A_{-e^t s} \circ M_t \circ A_s, \quad a \circ W_{s,t} = a.$$

At $p = (0, 1)$ it has nonzero mixed displacement:

$$\partial_s \partial_t W_{s,t}(p)|_{s=t=0} = ([X_u, X_v] - X_u)(p) = (-3/4, 0).$$

Thus for sufficiently small nonzero $s = t$ the full state moves while its assignment is unchanged.

Proof. The flow identities are $a \circ A_s = a + s$ and $a \circ M_t = e^t a$. Their displayed composition gives the identity on a exactly. Write $v = \nabla_{\text{Eucl}} a$, $Jv = (-a_y, a_x)$, and $G = |v|^2$. The canonical fields are

$$X_u = (v - aJv)/G, \quad X_v = (av + Jv)/G.$$

At p , $a = 1$, $v = (0, 2)$ and $G = 4$. Differentiating the polynomial and these rational fields gives $[X_u, X_v](p) = (-1/4, 1/2)$ and $X_u(p) = (1/2, 1/2)$. The usual two-parameter Taylor expansion of the four smooth local flows has mixed term $st[X_u, X_v]$; the extra factor e^t in the third flow subtracts stX_u . Since $W_{s,0} = W_{0,t} = \text{id}$, its remainder along $s = t$ is $O(s^3)$, proving the nonzero displacement. Finally $da([X_u, X_v] - X_u) = \mu\lambda - \mu = 1 - 1 = 0$, so the infinitesimal residue is tangent to a level set, as required by the exact assignment identity. $\square$

An interface that retains only the endpoint number can therefore discard a geometrically observable residue. Whether that residue is needed for a particular computation is a question about its declared future observations, answered by Paper IV's contextual residual criterion. Neither pants counts, circle packings, nor scalar assignment identities prove a faithful realization of Adva's triadic computation. The local circle surgery and sheet constructions that motivated this programme, with their precise non-equivalences, are recorded in Appendix E.

13 Interfaces to Papers I–IV

This paper was enlarged to become the geometric foundation of the series. The division of labour between Paper 0 and Paper I is fixed by the authoritative amendment `governance/00b-paper-0-geometric-foundation-amendment.md`, and this section records the resulting interface without restating either paper's theorems.

13.1 What Paper 0 establishes

The proved spine of this paper is

left and right comb grammars $\longrightarrow$ slot-(1)/slot-(2) one-hole maps,
distributive expansion $\longrightarrow$ same operator, different charges,
arithmetic motions on an AES $\longrightarrow$ point path in $\mathfrak{E}_0$,
projective operator $\longrightarrow$ pole, fixed points, ripple pencil,
affine cocycles $\longrightarrow$ relative defect $\longrightarrow$ ACS weighted area,
contact form $\longrightarrow$ non-integrable horizontal distribution $\longrightarrow$ curvature $\mu\lambda$,
 $\text{Aff}(1, K) = \text{Stab}(\infty) \subset \text{PGL}_2(K)$.

The paper also supplies four laboratories: the basic hyperbolic model $\mathfrak{E}_0$, the one-puncture model $\mathfrak{E}_1$, the pulled-back horocycle pencil, and the three-dimensional contact model. The finite geometric-series and continued-fraction examples are exact at every truncation; their infinite values are asserted only under the stated formal or analytic convergence hypotheses.

Two statements in the paper are not theorems about AES and are labelled as proposals where they occur: the descriptive name *ripple geometry* for the dual reading of Section 4, and the language of *tearing* of Section 8 together with the punctured reading of Section 11. Both rest on proved computations — the line-to-circle theorem, the weighted torsion–Stokes theorem, and the contact curvature — and neither claims a new isomorphism class of geometry.

13.2 What remains in Paper I

Paper I begins one level earlier and keeps the history layer. It retains responsibility for:

1. the intrinsic dependency-poset classification of all sequential binary trees, including mixed slot words;
2. marked planar sequential trees and bounded/free marked spinal histories, with operand-slot chirality, mirror, temporal reversal, and path inverse kept distinct;
3. projective evaluation of chronological marked histories, including ordinary-domain data and the affine sector;
4. the $q = 4$ Hecke sublanguage and its history-level relations;
5. the regular-zero theorems, the complete splitting theorem, the foundational definition of a singular arithmetic expression space, the verified isolated-zero model, and the smooth parameter-family zero-set lemma;
6. the interfaces to Papers II, III, and IV.

Paper I states the definition of a regular AES and its canonical frame as a concise imported interface, because its zero theorems are stated in those terms; everything else that moved here is referenced rather than repeated. The earlier version of that interface, which ran in the opposite direction, is superseded.

Two chapters of this paper are deliberately *not* offered as stable interfaces to the later papers. Section 10 states two proved boundaries — the boundary of finite-dimensional Lie-group closure and the boundary of observables of the charge endpoint — together with a vocabulary imported from a

companion programme; the boundaries are theorems, the vocabulary is a research stance, and the later papers depend on neither. Similarly, the *witness register* of Section 11.5 and the two registered problems of Subsection 11.7 are research reports. The new Section 12, in contrast, provides the stable finite-type construction and collar interface. Section 11 is not without proved content: the definition of a punctured space, the non-vanishing of the assignment gradient in a regular AES, and the explicit punctured model realizing the four-circle configuration (Theorem 11.6) are theorems, and the last of them may be imported wherever a concrete punctured instance is needed in place of the disc model of Paper I.

13.3 The dependency architecture

The series dependency is

$$\boxed{\text{Paper 0} \longrightarrow \text{Paper I} \longrightarrow \{\text{Papers II, III, IV}\}.$$

The first arrow exports the following interfaces:

Paper 0 object	use in later papers
comb and slot conventions	operand-slot chirality on sequential spines
distributive expansion	comparison of expansions with equal operator
regular AES and canonical frame	imported definition for zero theory
arithmetic motions and $\mathfrak{E}_0$	model for analysis and for the flow
one-puncture model $\mathfrak{E}_1$	first punctured instance
point path	bounded marked-history evaluation in the affine sector
ripple pencil and pole	motivation for bilateral projective continuation
affine cocycles	discrete order defect
history group and charge descent	what the ACS is, and what the motions keep
ACS and relative torsion	global torsion and its measures
contact form and horizontal curvature	horizontal calculus and function theory
the Reeb-quotient identification	the two models are one object
punctured models and their zero loci	verified finite-puncture instances
finite-type carrier and collar contracts	geometric input for Paper IV ports
tearing and the hole witnesses	research report; not yet an interface

Papers II and III import the regular-AES definition, the canonical frame, the basic hyperbolic model, the contact model, and the elementary bilateral operators from this paper; they do not redefine them. Paper IV imports the history layer from Paper I and the affine torsion, finite-type carriers and collar-reconstruction theorem from this paper. Its finite program substitution and numerical evaluation do not establish that every gate is an AES motion.

13.4 Open extensions

Several genuine problems remain, and they are recorded here rather than advertised as results.

1. Construct a projective analogue of the affine arithmetic expression space in which the pole and the ripple pencil are intrinsic rather than chosen through one upper-half-plane chart, and decide which path or ripple data descend through relations among history words.
2. Decide whether the contact structure of Section 9 admits a canonical compatible structure beyond the horizontal distribution, and if so which. Neither a complex structure nor a horizontal metric is determined by the contact form alone.
3. Determine whether the choice of punctures in Section 11 can be made canonical, and whether a punctured model carries a ripple structure that survives at the puncture; the three registered programmes of Subsection 11.7 are the concrete routes.
4. Extend the one-hole analysis to multi-wire expression networks while retaining ordinary-domain, pole, and fixed-point information, and develop a controlled theory of infinite arithmetic expressions that separates formal, analytic, projective, and computational convergence.
5. Determine whether the non-linear algebraic structures proposed in Section 10 can be defined intrinsically for AES, so that the two proved obstructions of that section are replaced by a positive description rather than a refusal.

None of these problems is solved merely by the matrix representation or by the existence of the contact form. The stable conclusion of the present paper is more elementary: the four arithmetic operations act as motions of one space, their order defect is measured in the commutativization of that space, and the contact structure of the same defect is what prevents the two motions from being calibrated independently.

Acknowledgements and provenance

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The manuscript was produced in a hybrid workflow that must be named explicitly rather than hidden. DeepSeek Harness, an artificial-intelligence research agent running the model `deepseek-v4-flash-vision-exp`, assisted with proof drafting, with the formalization and machine checking of finite statements under `paper-0/formal`, with the exact computations under `paper-0/scripts`, with red-team reading of the claim statuses, and with the preparation of L^AT_EX source. An agent is not a natural person: it cannot hold copyright, cannot sign a transfer, and cannot correspond, so it appears on the title page as assistance and not as a coauthor. Contribution-level provenance, including the commits and model identities where they are known, is maintained in `AUTHORSHIP_AND_PROVENANCE.md`; the same document records the standing rule that machine cross-checking is not independent human review. The mathematical statuses used throughout, and the boundary between proved statements and structural proposals, are those registered in `governance/05b-paper-0-status-register.md`.

A Elementary Calculations

This appendix records calculations used in the main text. They are included to keep the elementary claims reproducible and to fix chronology and sign conventions.

A.1 Chronological products and one-hole recursion

Let

$$F_k(z) = \frac{A^{(k)}z + B^{(k)}}{C^{(k)}z + D^{(k)}}$$

have matrix representative

$$M_k = \begin{pmatrix} A^{(k)} & B^{(k)} \\ C^{(k)} & D^{(k)} \end{pmatrix}.$$

If F_1 acts first, the two-step action is

$$F_2(F_1(z)),$$

and direct substitution gives the matrix M_2M_1 . Induction therefore gives

$$F_k \circ \cdots \circ F_1 \longleftrightarrow M_k \cdots M_1.$$

This is the convention used in (52).

For affine matrices

$$M_k = \begin{pmatrix} \alpha_k & \beta_k \\ 0 & 1 \end{pmatrix}, \quad P_{k-1} = \begin{pmatrix} A_{k-1} & B_{k-1} \\ 0 & 1 \end{pmatrix},$$

one has

$$M_k P_{k-1} = \begin{pmatrix} \alpha_k A_{k-1} & \alpha_k B_{k-1} + \beta_k \\ 0 & 1 \end{pmatrix},$$

which is the recursion (20).

A.2 The ripple circle calculation

For

$$F(z) = \frac{Az + B}{Cz + D}, \quad \Delta = AD - BC > 0,$$

one computes

$$\begin{aligned} F(z) - \overline{F(z)} &= \frac{(Az + B)(C\bar{z} + D) - (A\bar{z} + B)(Cz + D)}{|Cz + D|^2} \\ &= \frac{(AD - BC)(z - \bar{z})}{|Cz + D|^2}. \end{aligned}$$

Dividing by $2i$ gives

$$\operatorname{Im} F(z) = \frac{\Delta \operatorname{Im} z}{|Cz + D|^2}.$$

Writing $z = x + iy$ and imposing $\operatorname{Im} F(z) = t$ gives

$$\Delta y = t((Cx + D)^2 + C^2 y^2).$$

When $C \neq 0$, division by tC^2 yields

$$\left(x + \frac{D}{C}\right)^2 + y^2 - \frac{\Delta}{tC^2}y = 0,$$

and completing the square yields (38).

For $J(z) = -1/z$, one has $A = D = 0$, $B = -1$, $C = 1$, and $\Delta = 1$. Hence

$$x^2 + \left(y - \frac{1}{2t}\right)^2 = \left(\frac{1}{2t}\right)^2.$$

A.3 Powers of an affine map

Let $T(z) = az + b$ with $a \neq 0$. Matrix multiplication gives

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} a^n & b(1 + a + \cdots + a^{n-1}) \\ 0 & 1 \end{pmatrix}.$$

If $a \neq 1$, this is

$$\begin{pmatrix} a^n & b(1 - a^n)/(1 - a) \\ 0 & 1 \end{pmatrix}.$$

Writing $q = b/(1 - a)$, the induced map becomes

$$T^n(z) = a^n z + q(1 - a^n) = q + a^n(z - q).$$

If $a = 1$, the matrix power is

$$\begin{pmatrix} 1 & nb \\ 0 & 1 \end{pmatrix},$$

so $T^n(z) = z + nb$.

For the geometric-series update $T_r(s) = rs + 1$, apply the same formula with $a = r$, $b = 1$, and $s = 0$:

$$T_r^n(0) = \frac{1 - r^n}{1 - r}$$

when $r \neq 1$.

A.4 Continued-fraction matrices

Let

$$F_k(z) = a_k + \frac{b_k}{z}, \quad M_k = \begin{pmatrix} a_k & b_k \\ 1 & 0 \end{pmatrix}, \quad b_k \neq 0.$$

Write

$$P_k = M_k \cdots M_1 = \begin{pmatrix} p_k & r_k \\ q_k & s_k \end{pmatrix}.$$

Then

$$P_k = \begin{pmatrix} a_k & b_k \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{k-1} & r_{k-1} \\ q_{k-1} & s_{k-1} \end{pmatrix} = \begin{pmatrix} a_k p_{k-1} + b_k q_{k-1} & a_k r_{k-1} + b_k s_{k-1} \\ p_{k-1} & r_{k-1} \end{pmatrix}.$$

Thus

$$q_k = p_{k-1}, \quad p_k = a_k p_{k-1} + b_k p_{k-2},$$

with the analogous recurrence for the second column. The k th truncation at an initial value z_0 is

$$F_k \circ \cdots \circ F_1(z_0) = \frac{p_k z_0 + r_k}{q_k z_0 + s_k}.$$

An ordinary truncation is admissible precisely when each intermediate denominator is nonzero; the projective formula also records the stages at which the orbit reaches infinity.

In the stationary golden-ratio case $a_k = b_k = 1$, the first-column recurrence is

$$p_k = p_{k-1} + p_{k-2},$$

which is the Fibonacci recurrence. This gives the ratios displayed in Example 5.5.

A.5 The constructive projective factorization

For completeness, assume $C \neq 0$ and put

$$q = \frac{D}{C}, \quad s = \frac{AD - BC}{C^2}, \quad t = \frac{A}{C}.$$

Then

$$\begin{aligned} T_t \circ D_s \circ J \circ T_q(z) &= t - \frac{s}{z + q} \\ &= \frac{t(z + q) - s}{z + q} \\ &= \frac{(A/C)z + AD/C^2 - (AD - BC)/C^2}{z + D/C} \\ &= \frac{(A/C)z + B/C}{z + D/C} \\ &= \frac{Az + B}{Cz + D}. \end{aligned}$$

This verifies (54) with the signs used in the main text.

B Hyperbolic-Model Calculations

This appendix records the coordinate checks used for the basic hyperbolic arithmetic expression space. Throughout, $\mu, \lambda > 0$,

$$G = \{(x, y) : x \in \mathbb{R}, y > 0\}, \quad (x, y)(x', y') = (x + yx', yy'),$$

and

$$g_{\mu, \lambda} = \frac{1}{y^2} \left(\frac{dx^2}{\mu^2} + \frac{dy^2}{\lambda^2} \right), \quad a = -\frac{x}{y}.$$

The elementary grid-map and Baumslag-Solitar calculations formerly included here are now owned by Paper 0.

B.1 Group coordinates and the invariant frame

The identity of G is $(0, 1)$, and

$$(x, y)^{-1} = \left(-\frac{x}{y}, \frac{1}{y} \right).$$

For $g_0 = (x_0, y_0)$, left translation and its differential are

$$L_{g_0}(x, y) = (x_0 + y_0x, y_0y), \quad dL_{g_0}(\partial_x) = y_0\partial_x, \quad dL_{g_0}(\partial_y) = y_0\partial_y.$$

It follows that $y\partial_x$ and $y\partial_y$ are left-invariant. The normalized arithmetic frame

$$X_u = -\mu y \partial_x, \quad X_v = -\lambda y \partial_y$$

is therefore left-invariant, and its dual coframe is

$$-\frac{dx}{\mu y}, \quad -\frac{dy}{\lambda y}.$$

The sum of their squares is $g_{\mu,\lambda}$. The bracket is

$$\begin{aligned}[X_u, X_v] &= [-\mu y \partial_x, -\lambda y \partial_y] \\ &= -\mu \lambda y \partial_x = \lambda X_u,\end{aligned}$$

and hence $[X_u, X_v]a = \mu\lambda$, as required by the general arithmetic-frame identity.

The same invariance can be checked directly:

$$\begin{aligned}L_{(x_0, y_0)}^* g_{\mu,\lambda} &= \frac{1}{(y_0 y)^2} \left(\frac{(y_0 dx)^2}{\mu^2} + \frac{(y_0 dy)^2}{\lambda^2} \right) \\ &= g_{\mu,\lambda}.\end{aligned}$$

For a horocyclic chart, set

$$u = x, \quad y = e^{\lambda v}.$$

Then

$$(u, v)(u', v') = (u + e^{\lambda v} u', v + v'),$$

and

$$X_u = -\mu e^{\lambda v} \partial_u, \quad X_v = -\partial_v,$$

while

$$g_{\mu,\lambda} = e^{-2\lambda v} \frac{du^2}{\mu^2} + dv^2, \quad a = -u e^{-\lambda v}.$$

The letters u, v in this chart are genuine coordinates, but the arithmetic frame is not the coordinate frame.

B.2 Inverse metric, gradient, and the eikonal identity

In the (x, y) -chart,

$$(g_{ij}) = \begin{pmatrix} \frac{1}{\mu^2 y^2} & 0 \\ 0 & \frac{1}{\lambda^2 y^2} \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} \mu^2 y^2 & 0 \\ 0 & \lambda^2 y^2 \end{pmatrix},$$

and

$$\det(g_{ij}) = \frac{1}{\mu^2 \lambda^2 y^4}, \quad d\text{vol}_g = \frac{dx \wedge dy}{\mu \lambda y^2}.$$

Since

$$a_x = -\frac{1}{y}, \quad a_y = \frac{x}{y^2} = -\frac{a}{y},$$

the gradient is

$$\begin{aligned}\nabla a &= g^{xx} a_x \partial_x + g^{yy} a_y \partial_y \\ &= -\mu^2 y \partial_x + \lambda^2 x \partial_y.\end{aligned}$$

Its squared norm is

$$\begin{aligned}|\nabla a|_g^2 &= g^{xx} a_x^2 + g^{yy} a_y^2 \\ &= \mu^2 + \lambda^2 \frac{x^2}{y^2} \\ &= \mu^2 + \lambda^2 a^2.\end{aligned}$$

Alternatively,

$$X_u a = (-\mu y) \left(-\frac{1}{y} \right) = \mu, \quad X_v a = (-\lambda y) \left(\frac{x}{y^2} \right) = \lambda a,$$

so the framed and eikonal verifications agree.

B.3 Curvature and completeness

Two calculations give the curvature normalization. First set

$$X = \frac{\lambda}{\mu} x.$$

This is a global diffeomorphism of upper half-planes, and

$$g_{\mu,\lambda} = \frac{1}{\lambda^2} \frac{dX^2 + dy^2}{y^2}.$$

The standard metric $(dX^2 + dy^2)/y^2$ is complete with curvature -1 . Constant rescaling by λ^{-2} gives a complete metric of curvature $-\lambda^2$.

For a direct Cartan calculation, use the positively oriented orthonormal coframe

$$\omega^1 = \frac{dx}{\mu y}, \quad \omega^2 = \frac{dy}{\lambda y}.$$

Then

$$d\omega^1 = \lambda \omega^1 \wedge \omega^2, \quad d\omega^2 = 0.$$

The Levi-Civita connection form $\omega^1_2 = -\lambda \omega^1$ satisfies

$$d\omega^1 = -\omega^1_2 \wedge \omega^2, \quad d\omega^2 = \omega^1_2 \wedge \omega^1.$$

Its curvature form is

$$d\omega^1_2 = -\lambda d\omega^1 = -\lambda^2 \omega^1 \wedge \omega^2.$$

Thus $K = -\lambda^2$, consistently with the coordinate rescaling.

B.4 The Laplace–Beltrami operator

With the divergence-of-gradient convention,

$$\Delta_g f = \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} g^{ij} \partial_j f \right).$$

Here

$$\sqrt{\det g} = \frac{1}{\mu \lambda y^2}, \quad \sqrt{\det g} g^{xx} = \frac{\mu}{\lambda}, \quad \sqrt{\det g} g^{yy} = \frac{\lambda}{\mu}.$$

The last two coefficients are constant, so

$$\Delta_{g_{\mu,\lambda}} f = \mu^2 y^2 f_{xx} + \lambda^2 y^2 f_{yy}. \tag{110}$$

For $a = -x/y$,

$$a_{xx} = 0, \quad a_{yy} = -\frac{2x}{y^3} = \frac{2a}{y^2}.$$

Equation (110) therefore gives

$$\Delta_{g_{\mu,\lambda}} a = 2\lambda^2 a.$$

C Affine Cocycle Calculations

This appendix supplements Proposition 7.1 with expanded calculations. All discrete formulas hold over an arbitrary field K , and every multiplier is assumed nonzero.

C.1 Alternating expressions: the elementary arithmetic origin

Early drafts called these objects *alternating threadlike expressions* [18]. The elementary observation behind them is that an addition changes a value, whereas each subsequent multiplication transports that change to a new scale. We recover that calculation before using affine coordinates or a continuous model.

Definition C.1 (Alternating arithmetic expression). Fix $q_i \in K^\times$ and $\mu_i \in K$ for $i = 1, \dots, n$. An *alternating expression* α is the chronological slot-(1) word that, in each pair, first multiplies by q_i and then adds μ_i . Applied to the initial value μ_0 , its successive values are

$$w_0 = \mu_0, \quad w_i = q_i w_{i-1} + \mu_i. \quad (111)$$

The two operations in a pair remain distinct marked events, even when the pair is evaluated as one affine map.

A finite word consisting of additions and nonzero scalings can be padded to this alternating pattern by inserting $+0$ or $\times 1$. This preserves its operator, not its literal tree or full marked history, and supplies no unique normal form. Slot-(1) subtraction and division by a nonzero constant can first be interpreted as addition of a negative constant and multiplication by an inverse. A slot-(2) reciprocal c/z is not covered by this argument.

Proposition C.2 (Arithmetic expansion and exact perturbation). *With the empty product equal to 1, the recurrence (111) has value*

$$w_n = \mu_0 \prod_{j=1}^n q_j + \sum_{i=1}^n \mu_i \prod_{j=i+1}^n q_j. \quad (112)$$

Keep the same operations and constants, but replace the initial value by $\tilde{\mu}_0$, obtaining values $\tilde{w}_i$. Then, at every prefix,

$$\tilde{w}_i - w_i = \left(\prod_{j=1}^i q_j \right) (\tilde{\mu}_0 - \mu_0). \quad (113)$$

If $\tilde{\mu}_0 \neq \mu_0$, the exact difference quotient is

$$\frac{\tilde{w}_i - w_i}{\tilde{\mu}_0 - \mu_0} = \prod_{j=1}^i q_j. \quad (114)$$

Proof. The first two steps, using only distributivity, give

$$w_1 = q_1 \mu_0 + \mu_1, \quad w_2 = q_2 q_1 \mu_0 + q_2 \mu_1 + \mu_2.$$

At each next step multiplication by q_{n+1} extends every displayed product, and the final addition supplies the term μ_{n+1} with empty suffix product. Induction, starting at $n = 0$, proves (112). For the perturbation, subtract the two recurrences directly:

$$\tilde{w}_i - w_i = (q_i \tilde{w}_{i-1} + \mu_i) - (q_i w_{i-1} + \mu_i) = q_i (\tilde{w}_{i-1} - w_{i-1}).$$

Iterating from $\tilde{w}_0 - w_0 = \tilde{\mu}_0 - \mu_0$ proves (113); division by the nonzero initial difference proves (114). $\square$

No smallness hypothesis, derivative or limit is used. For example, the rational word “multiply by 2, add 1, multiply by 3, add 4” sends 5 to 37 and 6 to 43. The difference 6 is exactly $2 \cdot 3$ times the initial difference 1. It is an exact gain formula, not a truncation of a Taylor expansion. With $q_i = s_i$ and $\mu_i = t_i$, the same calculation is the elementary source of the general affine formulas below.

C.2 Prefix recursion and closed form

For the prefix history $(f_1, \dots, f_k)$, write

$$F_k = f_k \circ \dots \circ f_1, \quad F_k(x) = \Phi_k x + \xi_k,$$

with $F_0(x) = x$. If $f_{k+1}(x) = s_{k+1}x + t_{k+1}$, then

$$\Phi_{k+1} = s_{k+1}\Phi_k, \quad \xi_{k+1} = s_{k+1}\xi_k + t_{k+1}. \quad (115)$$

Iterating the first recursion gives

$$\Phi_k = s_k \cdots s_1.$$

Iterating the second one gives

$$\begin{aligned} \xi_1 &= t_1, \\ \xi_2 &= s_2 t_1 + t_2, \\ \xi_3 &= s_3 s_2 t_1 + s_3 t_2 + t_3, \\ \xi_4 &= s_4 s_3 s_2 t_1 + s_4 s_3 t_2 + s_4 t_3 + t_4. \end{aligned}$$

The pattern is

$$\xi_k = \sum_{i=1}^k t_i \prod_{j=i+1}^k s_j.$$

Substitution in (115) proves this formula inductively, including the final term through the empty-product convention.

The source-normalized prefix coordinate $\hat{\xi}_k = \Phi_k^{-1} \xi_k$ obeys a different recursion. From (115),

$$\hat{\xi}_{k+1} = \frac{s_{k+1}\xi_k + t_{k+1}}{s_{k+1}\Phi_k} = \hat{\xi}_k + \frac{t_{k+1}}{\Phi_{k+1}}. \quad (116)$$

Starting from $\hat{\xi}_0 = 0$ therefore gives

$$\hat{\xi}_k = \sum_{i=1}^k \frac{t_i}{\Phi_i} = \sum_{i=1}^k \frac{t_i}{\prod_{j=1}^i s_j}.$$

This second derivation makes the past normalization explicit at every prefix.

C.3 Suffix transport and the future-weight kernel

For $0 \leq i \leq n$, define the scale of the strict suffix after step i by

$$S_{i \rightarrow n} = \prod_{j=i+1}^n s_j, \quad S_{n \rightarrow n} = 1.$$

Then the target-frame translation is simply

$$\xi_n = \sum_{i=1}^n S_{i \rightarrow n} t_i. \quad (117)$$

To verify the interpretation, replace only t_i by $t_i + \epsilon$. All states before step i are unchanged. At step i the output gains ϵ , and every later affine map multiplies this change by its scale. Thus the final endpoint changes by

$$\epsilon s_{i+1} \cdots s_n = \epsilon S_{i \rightarrow n}.$$

The coefficient in (117) is therefore exactly the transport of an additive event to the target frame.

C.4 Alternating addition–scale histories

Now specialize the arithmetic multipliers to $q_i = e^{m_i} > 0$ over $\mathbb{R}$. This positivity concerns the multipliers; the initial value and additive increments may have either sign. The pairs become

$$x \xrightarrow{e^{m_i}} e^{m_i} x \xrightarrow{+\mu_i} e^{m_i} x + \mu_i, \quad i = 1, \dots, n,$$

applied from $i = 1$ to $i = n$. At the level of affine steps this has $s_i = e^{m_i}$ and $t_i = \mu_i$. Define prefix and suffix logarithmic scales by

$$M_i = \sum_{j=1}^i m_j, \quad M_{i+1:n} = \sum_{j=i+1}^n m_j.$$

Proposition 7.1 becomes

$$\Phi_n = e^{M_n}, \quad (118)$$

$$\xi_n = \sum_{i=1}^n \mu_i e^{M_{i+1:n}}, \quad (119)$$

$$\hat{\xi}_n = \sum_{i=1}^n \mu_i e^{-M_i}. \quad (120)$$

For $n = 3$, direct expansion checks the target formula:

$$\begin{aligned} x &\mapsto e^{m_1} x + \mu_1 \\ &\mapsto e^{m_2+m_1} x + e^{m_2} \mu_1 + \mu_2 \\ &\mapsto e^{m_3+m_2+m_1} x + e^{m_3+m_2} \mu_1 + e^{m_3} \mu_2 + \mu_3. \end{aligned}$$

Dividing the translation line by $e^{m_1+m_2+m_3}$ gives

$$\hat{\xi}_3 = \mu_1 e^{-m_1} + \mu_2 e^{-(m_1+m_2)} + \mu_3 e^{-(m_1+m_2+m_3)},$$

in agreement with (120).

The convention in this example is “scale, then add” within each pair. If a source instead groups the elementary operations as “add, then scale”, the translation attached to that pair is $e^{m_i} \mu_i$, not μ_i . Converting the pair to the common affine form $s_i x + t_i$ before using the cocycle formulas prevents this convention change from being hidden.

For the original initial perturbation $\tilde{\mu}_0 = e^{\eta_0} \mu_0 + \epsilon_0$, the arithmetic identity gives

$$\tilde{w}_n - w_n = e^{M_n} ((e^{\eta_0} - 1) \mu_0 + \epsilon_0).$$

The first recurrence starts at $w_0 = \mu_0$, and its companion starts at $\tilde{w}_0 = \tilde{\mu}_0$. Any ratio by their initial difference requires that difference to be nonzero.

C.5 From arithmetic weights to ACS and scale normalization

For this same word, draw the ACS path of Definition 8.1. The i th vertical edge adds m_i to M , and the following horizontal edge adds μ_i to A at height M_i . Thus direct integration along the horizontal edges gives

$$\int_{C_\alpha} e^{-M} dA = \sum_{i=1}^n \mu_i e^{-M_i} = \hat{\xi}_n, \quad w_n = e^{M_n}(\mu_0 + \hat{\xi}_n). \quad (121)$$

This recovers Proposition 8.2 for alternating words. The initial perturbation is multiplied by the total scale e^{M_n} ; an additive increment μ_i is multiplied only by its remaining suffix scale $e^{M_n - M_i}$. These are distinct quantities forced by the same expansion. The charge endpoint $(\sum_i \mu_i, M_n)$ forgets the heights at which the additions occurred, whereas the weighted integral retains their effect.

Grouping consecutive pairs into blocks uses the chronological cocycle law below: each block has an exact effective scale and translation. Retaining the original events distinguishes this operator summary from the full history. Accumulated scale and the normalization e^{-M_n} supplied an arithmetic starting point for the later renormalization discussion [20]. The finite identities proved here support that motivation; identifying a particular blocking and rescaling procedure with a renormalization transformation requires further definitions. The historical interpretation is recorded separately from those identities.

C.6 Concatenation in target and normalized coordinates

Let

$$\rho(\gamma) = g(\Phi_\gamma, \xi_\gamma), \quad \rho(\delta) = g(\Phi_\delta, \xi_\delta),$$

where $g(\Phi, \xi)(x) = \Phi x + \xi$. Since $\gamma\delta$ performs γ first,

$$\begin{aligned} \rho(\gamma\delta)(x) &= \rho(\delta)(\rho(\gamma)(x)) \\ &= \Phi_\delta(\Phi_\gamma x + \xi_\gamma) + \xi_\delta \\ &= \Phi_\delta \Phi_\gamma x + \Phi_\delta \xi_\gamma + \xi_\delta. \end{aligned}$$

This yields the target-coordinate pair

$$(\Phi_{\gamma\delta}, \xi_{\gamma\delta}) = (\Phi_\delta \Phi_\gamma, \Phi_\delta \xi_\gamma + \xi_\delta).$$

For normalized coordinates,

$$\begin{aligned} \hat{\xi}_{\gamma\delta} &= \frac{\Phi_\delta \xi_\gamma + \xi_\delta}{\Phi_\delta \Phi_\gamma} \\ &= \hat{\xi}_\gamma + \Phi_\gamma^{-1} \hat{\xi}_\delta. \end{aligned}$$

The first formula transports the earlier translation forward by the later scale. The second transports the later normalized translation back through the earlier scale.

For three histories, repeated use gives

$$\begin{aligned} \xi_{\gamma\delta\epsilon} &= \Phi_\epsilon \Phi_\delta \xi_\gamma + \Phi_\epsilon \xi_\delta + \xi_\epsilon, \\ \hat{\xi}_{\gamma\delta\epsilon} &= \hat{\xi}_\gamma + \Phi_\gamma^{-1} \hat{\xi}_\delta + (\Phi_\delta \Phi_\gamma)^{-1} \hat{\xi}_\epsilon. \end{aligned}$$

These are the block-history versions of future weighting and past normalization.

C.7 Inverse histories and relative comparison

The affine inverse is

$$g(\Phi, \xi)^{-1} = g(\Phi^{-1}, -\Phi^{-1}\xi). \quad (122)$$

Indeed,

$$g(\Phi^{-1}, -\Phi^{-1}\xi)(\Phi x + \xi) = x + \Phi^{-1}\xi - \Phi^{-1}\xi = x.$$

Using (122), the relative affine defect is

$$\rho(\delta)^{-1}\rho(\gamma) = g\left(\frac{\Phi_\gamma}{\Phi_\delta}, \frac{\xi_\gamma - \xi_\delta}{\Phi_\delta}\right). \quad (123)$$

In matrices, the same computation is

$$\begin{pmatrix} \Phi_\delta^{-1} & -\Phi_\delta^{-1}\xi_\delta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi_\gamma & \xi_\gamma \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Phi_\delta^{-1}\Phi_\gamma & \Phi_\delta^{-1}(\xi_\gamma - \xi_\delta) \\ 0 & 1 \end{pmatrix}.$$

If $\Phi_\gamma = \Phi_\delta = \Phi$, the relative element is a pure translation by $(\xi_\gamma - \xi_\delta)/\Phi$. The unnormalized endpoint difference is $\xi_\gamma - \xi_\delta$. These two scalar quantities differ by the common scale and should not be identified.

C.8 The elementary order square

Let

$$A_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \quad S_q = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}.$$

Under the chronological convention, first adding and then scaling has matrix

$$S_q A_\mu = \begin{pmatrix} q & q\mu \\ 0 & 1 \end{pmatrix},$$

whereas first scaling and then adding has matrix

$$A_\mu S_q = \begin{pmatrix} q & \mu \\ 0 & 1 \end{pmatrix}.$$

Their endpoint difference is

$$(qx + q\mu) - (qx + \mu) = \mu(q - 1).$$

Taking the second history as reference gives

$$(A_\mu S_q)^{-1}(S_q A_\mu) = \begin{pmatrix} 1 & \mu(1 - q^{-1}) \\ 0 & 1 \end{pmatrix}.$$

Thus the relative group translation is $\mu(1 - q^{-1})$, while the open endpoint defect is $\mu(q - 1)$. The latter is the former multiplied by the common target scale q .

Closing the comparison into a group commutator produces yet another ordered quantity. For example,

$$\begin{aligned} S_q A_\mu S_q^{-1} A_\mu^{-1} &= \begin{pmatrix} 1 & (q - 1)\mu \\ 0 & 1 \end{pmatrix}, \\ A_\mu S_q A_\mu^{-1} S_q^{-1} &= \begin{pmatrix} 1 & (1 - q)\mu \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The signs differ because the loop orientations differ. These closed-loop calculations are recorded only to audit conventions; the torsion theory and its orientation choices are introduced in Section 8.

C.9 Exact perturbation propagation at every prefix

Let x and $\tilde{x}$ be two initial values, and propagate both through the same history. Set

$$x_k = F_k(x), \quad \tilde{x}_k = F_k(\tilde{x}).$$

Taking $q_i = s_i$, $\mu_i = t_i$, $\mu_0 = x$ and $\tilde{\mu}_0 = \tilde{x}$ in the elementary identity (113) gives

$$\tilde{x}_k - x_k = \left(\prod_{i=1}^k s_i \right) (\tilde{x} - x) = \Phi_k(\tilde{x} - x). \quad (124)$$

The translations cancel at each step; only the scale cocycle propagates the initial difference. Over $\mathbb{R}$, if $s_i = e^{\lambda_i}$, then

$$\frac{\tilde{x}_k - x_k}{\tilde{x} - x} = e^{\lambda_1 + \dots + \lambda_k}.$$

No limiting process is involved in this identity. The displayed ratio requires $\tilde{x} \neq x$; the difference identity (124) also includes equal initial values.

C.10 Differential conversion check

Finally, on the positive real affine group let $\Phi = e^M$ and $\hat{\xi} = e^{-M}\xi$. The product rule gives

$$\begin{aligned} d\hat{\xi} &= d(e^{-M}\xi) \\ &= -e^{-M}\xi dM + e^{-M}d\xi \\ &= e^{-M}(d\xi - \xi dM). \end{aligned}$$

Therefore

$$d\xi - \xi dM = e^M d\hat{\xi}.$$

This verifies that the translation component of $dg g^{-1}$ is the target scale times the differential of the source-normalized coordinate. It does not identify $dg g^{-1}$ with $g^{-1}dg$, whose translation component is $e^{-M}d\xi$.

C.11 Left and right group data

The affine group is represented by

$$g(\Phi, \xi) = \begin{pmatrix} \Phi & \xi \\ 0 & 1 \end{pmatrix}, \quad \Phi \in K^\times.$$

With the column action, matrix multiplication gives

$$g(\Phi_2, \xi_2)g(\Phi_1, \xi_1) = g(\Phi_2\Phi_1, \Phi_2\xi_1 + \xi_2). \quad (125)$$

Thus the earlier translation is rescaled by the later affine map. Equation (125) is the two-factor origin of the target-frame cocycle.

Over the positive real component, write $\Phi = e^M$. The two Maurer–Cartan forms used in Section 7 have the following readings.

Form	Translation component	Frame
$g^{-1}dg$	$e^{-M}d\xi$	body/source
$dg g^{-1}$	$d\xi - \xi dM$	spatial/target

The words “left” and “right” here refer to invariant differential forms on the affine group. They do not refer to first and second operand slots. The operand-slot mirror is syntactic; the Maurer–Cartan distinction is group-theoretic.

C.12 Target and source-normalized translation coordinates

For $x \mapsto \Phi x + \xi$, the symbol ξ always denotes target-frame translation. The source-normalized coordinate is

$$\widehat{\xi} = \Phi^{-1}\xi.$$

Their discrete and differential conversions are

$$\xi = \Phi \widehat{\xi}, \quad \widehat{\xi} = \Phi^{-1}\xi,$$

and, when $\Phi = e^M$,

$$d\xi = \xi dM + e^M d\widehat{\xi}.$$

No formula in the main text changes from ξ to $\widehat{\xi}$ without this normalization being displayed.

D ACS and Contact Calculations

This appendix records sign checks and coordinate calculations used in Sections 8 and 9. The chronological convention is unchanged: entries in a history are performed from left to right. The ACS is oriented by $dA \wedge dM$, and the contact state space is oriented by $du \wedge da \wedge dv$.

D.1 The oriented ACS rectangle

Let

$$\gamma = (A_p, M_q), \quad \delta = (M_q, A_p).$$

Their common terminal multiplicative charge is $M_* = q$. The chain $C_\gamma - C_\delta$ traverses the bottom edge from $(0, 0)$ to $(p, 0)$, the right edge to (p, q) , the top edge back to $(0, q)$, and the left edge back to the origin. For $p, q > 0$ this is the positive boundary convention for $dA \wedge dM$. Since vertical edges do not contribute to $\eta_* = e^{q-M} dA$,

$$\begin{aligned} \int_{C_\gamma - C_\delta} \eta_* &= \int_0^p e^q dA + \int_p^0 1 dA \\ &= p(e^q - 1). \end{aligned}$$

On the other hand,

$$\int_{[0,p] \times [0,q]} e^{q-M} dA \wedge dM = p \int_0^q e^{q-M} dM = p(e^q - 1).$$

This verifies both the sign and the target-frame weight.

The same formula holds for signed p and q . More precisely, regard the rectangle as the oriented singular chain parametrized by

$$R_{p,q}(s, t) = (sp, tq), \quad 0 \leq s, t \leq 1.$$

Its pulled-back orientation form is $pq ds \wedge dt$, so orientation reverses automatically when exactly one side is negative. Direct integration still gives

$$\int_{R_{p,q}} e^{q-M} dA \wedge dM = p(e^q - 1).$$

Thus no separate “unsigned area” convention is needed.

D.2 A closed charge path with nonzero evaluation defect

Consider the chronological loop

$$\ell = (A_p, M_q, A_{-p}, M_{-q}).$$

Its total ACS charge is $(0, 0)$, the same as the empty history $\emptyset$, but

$$\begin{aligned}\nu_x(\ell) &= e^{-q}(e^q(x+p) - p) \\ &= x + p(1 - e^{-q}).\end{aligned}$$

Hence

$$\tau(\ell, \emptyset) = p(1 - e^{-q}).$$

Here $M_* = 0$, so $\eta_* = e^{-M}dA$. The ACS path of ℓ is the oriented rectangle boundary itself, and

$$\int_{C_\ell} e^{-M} dA = p - pe^{-q} = p(1 - e^{-q}).$$

This example simultaneously checks the closed-loop sign and shows why zero total charge is weaker than operator neutrality.

For a self-intersecting charge path, the same computation is performed on a singular two-chain with integer multiplicities. Decomposing a filling into oriented rectangles adds their signed weighted integrals; overlap is counted with the sum of multiplicities. This is the precise replacement for speaking of “the region enclosed” by a self-intersecting loop.

D.3 Finite contact compositions

Write $p = \mu h$ and $q = \lambda k$. On the assignment coordinate, the two horizontal flows act by

$$A_h(a) = a + p, \quad M_k(a) = e^q a.$$

The open comparison is

$$\begin{aligned}(M_k \circ A_h)(a) &= e^q(a + p), \\ (A_h \circ M_k)(a) &= e^q a + p,\end{aligned}$$

and therefore

$$(M_k \circ A_h)(a) - (A_h \circ M_k)(a) = p(e^q - 1).$$

For the closed chronological word A_h, M_k, A_{-h}, M_{-k} , the induced composition is

$$M_{-k} \circ A_{-h} \circ M_k \circ A_h,$$

and

$$\begin{aligned}(M_{-k} \circ A_{-h} \circ M_k \circ A_h)(a) &= e^{-q}(e^q(a + p) - p) \\ &= a + p(1 - e^{-q}).\end{aligned}$$

Consequently

$$T_{\text{closed}} = p(1 - e^{-q}) = e^{-q}p(e^q - 1) = e^{-q}T_{\text{open}}.$$

The equality after multiplication by e^q is a change to the common target frame, not an equality of the two raw finite defects.

D.4 Darboux coordinates and the Reeb field

Assume $\mu\lambda \neq 0$ and introduce natural coordinates

$$\tilde{u} = \mu u, \quad \tilde{v} = \lambda v.$$

Then

$$\alpha = da - d\tilde{u} - a d\tilde{v}.$$

Set

$$x = \tilde{v}, \quad y = a, \quad z = a - \tilde{u}.$$

This is a global affine change of coordinates on $\mathbb{R}^3$, and

$$dz - y dx = da - d\tilde{u} - a d\tilde{v} = \alpha.$$

Thus the contact form is in the standard Darboux form. The arithmetic structure is not a new local contact-isomorphism class; it is the distinguished frame (D_u, D_v) and its interpretation by addition and scaling.

For completeness, the Reeb field R is determined by

$$\alpha(R) = 1, \quad \iota_R d\alpha = 0.$$

Since $d\alpha = -\lambda da \wedge dv$, its kernel is spanned by ∂_u . Hence

$$R = -\frac{1}{\mu} \partial_u.$$

In the Darboux coordinates above this is $R = \partial_z$, as expected.

D.5 Expanded scalar-curvature calculation

Let $F = F(u, v, a)$. Expanding the two ordered derivatives gives

$$\begin{aligned} D_u D_v F &= F_{uv} + \lambda a F_{ua} + \mu F_{av} + \mu \lambda F_a + \mu \lambda a F_{aa}, \\ D_v D_u F &= F_{vu} + \mu F_{va} + \lambda a F_{au} + \mu \lambda a F_{aa}. \end{aligned}$$

Equality of the mixed coordinate derivatives leaves

$$D_u D_v F - D_v D_u F = \mu \lambda F_a.$$

Multiplying by $du \wedge dv$ yields

$$\mathcal{R}_H(F) = \mu \lambda (\partial_a F) du \wedge dv.$$

This is an antisymmetrized calculation on scalar fields. It neither defines δ_H on one-forms nor claims that a graded differential has been constructed.

E Circle Surgery and Branched-Sheet Models

The circle and sheet pictures motivating finite-type AES compactification are useful local models, but they preserve different structures. This appendix records the exact statements needed to keep those pictures from becoming unproved global identifications.

E.1 Triangular gaps and four initial branches

Proposition E.1 (Möbius equivalence of non-degenerate triangular gaps). *Let G be a specified curvilinear triangular gap bounded by three mutually tangent circles on the Riemann sphere, with three distinct tangency points and disjoint chosen circle interiors. Any two such marked gaps, with matching cyclic order, are Möbius equivalent. This equivalence carries every finite generation of their standard Apollonian inscribed-circle recursion to the corresponding generation.*

Proof. Send one tangency point to infinity. The two circles through it become parallel lines, and the third becomes a circle tangent to both lines. A Euclidean similarity makes their separation 2 and the circle radius 1; translation along the strip fixes its centre. Selecting the matching gap and cyclic marking fixes the remaining choice. This gives a common normal form. In that strip each of the two chosen cusp gaps has a unique circle inside it tangent to its three bounding circles: tangency to both parallel lines forces radius 1 and the centre onto their midline; tangency to the existing unit circle then places the centre at distance 2 along that midline, on the chosen side. Möbius transformations preserve circles, incidence, tangency and containment of the chosen spherical discs, so uniqueness identifies the inserted circle and then its three child gaps. Induction proves the assertion for every finite generation. The standard recursive packing is described in [9]. $\square$

The three tangencies lie on a dual circle orthogonal to the bounding circles. In its Poincaré disc, their arcs bound an ideal hyperbolic triangle; doubling that triangle along its geodesic sides produces a three-cusp sphere. Each child gap has its own dual disc. The newly inscribed circle is not thereby a horocycle for the original disc metric, and ordinary three-cusp point compactification does not generate an Apollonian packing.

If a central circle splits a four-sided initial gap into four triangular gaps, subsequent standard Apollonian filling produces $4 \cdot 3^n$ gaps at depth $n \geq 0$. Four initial entries do not define four-way recursion at every stage. A four-marked parent gap also retains marking data: the real modulus of a marked planar quadrilateral should not be confused with the complex cross-ratio modulus of four punctures on a sphere.

E.2 The AES four-circle configuration is not the tangent ring

Proposition E.2 (Incidence obstruction and an explicit deformation). *The four circles of Proposition 11.2 are not Möbius equivalent to four equal circles in a cyclic tangent ring with adjacent tangencies distinct. They are connected as circle configurations by the continuous family*

$$C_j(t) : |z - t\rho i^j| = \rho, \quad j = 0, 1, 2, 3, \quad 1 \leq t \leq \sqrt{2}.$$

At $t = \sqrt{2}$, adding $|z| = (\sqrt{2} - 1)\rho$ leaves four triangular gaps of the preceding proposition.

Proof. At $t = 1$ all four circles pass through 0; adjacent ones intersect at a right angle and opposite ones are tangent at 0. In the tangent ring there is no common point, adjacent circles are tangent, and opposite circles are disjoint. Möbius transformations preserve concurrence and intersection angles, so cannot identify the configurations. At $t = \sqrt{2}$, adjacent centre distance is $\sqrt{2}t\rho = 2\rho$, while opposite centre distance is $2t\rho > 2\rho$. The distance of any centre from 0 is $\sqrt{2}\rho$, equal to the sum of the large radius and $(\sqrt{2} - 1)\rho$; hence the new central circle is tangent to all four. Each adjacent pair and the central circle bound a non-degenerate triangular gap. $\square$

This is a deformation of circle incidence, not a deformation theorem for AES. The four-circle AES of Theorem 11.6 uses only local arcs inside D_r , $r < \rho$. Moving the entire circles does not prove that its assignment, metric, critical set or puncture count continues along the family. Those data must be constructed and checked separately.

E.3 A three-sheet model requires cuts and an extra base sheet

Let D be a disc about O , and write N, W, S, E for its northern, western, southern and eastern half-discs. Every open quadrant belongs to exactly two of these four sets. Three sheets per quadrant therefore require an independent copy of D as well. Assemble temporary sheets $A = D$, $B = N \cup S$, $C = W \cup E$, with the half-disc joins specified. Cut along the positive coordinate rays and identify banks by the transpositions $(A\ B)$ and $(A\ C)$. The monodromy around O is a three-cycle, with its direction depending on the declared crossing convention.

Proposition E.3 (The resulting local branched cover). *The transitive monodromy just described gives a connected three-sheet cover of $D \setminus \{O\}$. With the pulled-back complex structure its one-point extension has local form $\pi(w) = w^3$. The metric completion of the pulled-back Euclidean length metric has cone angle 6π at that point.*

Proof. The fundamental group of the punctured disc is $\mathbb{Z}$; the action of its generator is a three-cycle and its stabilizer subgroup is $3\mathbb{Z}$. The classification of covers identifies this connected cover with the punctured-disc power map [10, §1.3]. That map extends holomorphically by $w = 0$. If $w = re^{i\theta}$, its metric pullback is

$$9r^4 dr^2 + 9r^6 d\theta^2 = dR^2 + 9R^2 d\theta^2, \quad R = r^3, \quad \theta \in \mathbb{R}/(2\pi\mathbb{Z}).$$

A circle of radius R therefore has length $6\pi R$. This is a cone length metric; its tensor degenerates at $w = 0$, even though the conformal carrier is smooth there. $\square$

Identifying only the centres of independent discs does not give this surface: its punctured neighbourhood has several components. If instead each half-disc carries its own complete hyperbolic metric, O lies on an infinite-distance boundary and is not a finite seam point. Finally the pullback of $|z - \rho| = \rho$ is $|w^3 - \rho| = \rho$, generally not a circle, while its projection remains the original circle. The branched cover does not convert the AES four-circle configuration into the tangent ring. A faithful circle-domain uniformization carrying the arithmetic data remains an open problem.

E.4 Other completions and the genus-two comparison

The visual compactification $\mathbb{H}^2 \cup S_\infty^1$ is a closed disc; adding a direction circle by the unit tangent bundle is a different, three-dimensional construction. A finite-word tree has its own end space, and identifying it with a hyperbolic visual boundary requires a specified coding and convergence theorem. None of these operations is the conformal point filling of Section 12.

A companion algebraic example makes the distinction concrete [24]. After adjoining independent sheets

$$Y^2 = 2(E - U)(1 - U^2), \quad Z_m^2 = c + U^2,$$

the product-sign quotient $W = YZ_m$ obeys

$$W^2 = 2(E - U)(1 - U^2)(c + U^2).$$

At $E = 0, c = 1$, setting $w = W/\sqrt{2}$ yields $w^2 = U^5 - U$. Its five finite branch points and infinity are distinct, so the degree-two cover of the sphere has genus 2 by Riemann–Hurwitz; this is the Bolza model. The compactified object is the newly constructed algebraic curve, and the product quotient forgets the separate sheet signs. No canonical selection of c or faithful reconstruction of those signs follows from the equation.

Likewise, pairing all three boundary circles of two compact oriented pants with orientation-reversing boundary maps produces a closed genus-two surface: $\chi = -1 - 1 = -2$. Equality of genus

supplies only a topological comparison; matching the Bolza conformal structure, marked seams and process residues requires further data and a separate proof. These observations locate the cross-programme research question without asserting an equivalence between its distinct geometric constructions.

F Cuts and the Foundations of Arithmetic Expressions

This appendix collects background for a future formal theory of arithmetic expressions, continuing Subsection 2.6. The constructions of Bauer–Taylor and Conway are cited external mathematics; their proposed use in AEG is a research direction. No general convergence theory, new expression calculus, or additional mechanized theorem is asserted here.

F.1 Bauer–Taylor: cuts, observable predicates, and interval information

Bauer and Taylor construct the real line in Abstract Stone Duality (ASD) using two-sided Dedekind cuts [3, Sections 2 and 6]. In the familiar rational presentation, a real x has lower and upper predicates

$$\delta(q) \equiv q < x, \quad v(q) \equiv x < q, \quad q \in \mathbb{Q}.$$

Conversely, inhabited, rounded lower and upper predicates that are disjoint and satisfy locatedness specify a real. Roundedness means

$$\delta(q) \iff \exists r \in \mathbb{Q} (q < r \wedge \delta(r)), \quad v(q) \iff \exists r \in \mathbb{Q} (v(r) \wedge r < q),$$

and locatedness is the overlapping alternative

$$p < q \implies \delta(p) \vee v(q).$$

This is not a Boolean decision of $q < x$ versus $x < q$ at the same rational q . In particular, both strict predicates fail when $q = x$ is rational.

Taylor’s computational reading uses predicates valued in the Sierpiński space Σ , rather than Boolean tests: a positive observation may be witnessed by termination, while absence of a witness does not decide its negation [14, Section 2]. Dropping locatedness allows the two sides to leave an interval of candidates. Additional lower and upper evidence narrows that interval: inclusion of evidence corresponds to reverse inclusion of candidate intervals. This *information order* differs from numerical order.

Taylor develops a calculus for exact real analysis and translates it to interval computation [14, Sections 5–6]. Thus an exact denotation and its finite enclosure need not be the same object. The order-theoretic construction also permits other countable dense bases, including dyadics and real algebraic numbers; an order isomorphism alone does not preserve arithmetic. The interpolation discussion explicitly mentions Conway’s simplest interpolant as an alternative to an arithmetic midpoint [3, Section 6].

F.2 Conway: recursive options and simplicity

Conway forms a number $\{L \mid R\}$ from sets of previously constructed numbers with $\ell < r$ for every $\ell \in L, r \in R$. Either set may be empty. The simplicity theorem identifies its value as the unique simplest surreal satisfying those strict inequalities, where simplicity belongs to the recursive construction, not to numerical magnitude [6, Chapters 0–2]. For example,

$$\{|\} = 0, \quad \{0 \mid\} = 1, \quad \{0 \mid 1\} = \tfrac{1}{2}, \quad \{0 \mid 3\} = 1.$$

The last form selects 1; it does not retain an unknown target somewhere between 0 and 3, and does not select the arithmetic midpoint $3/2$.

Arithmetic is itself recursively defined. For example, using x^L, x^R and y^L, y^R for typical options,

$$x + y = \{x^L + y, x + y^L \mid x^R + y, x + y^R\}.$$

All available options of each indicated kind are included. Compatibility with equality and the field laws require the accompanying proofs; they do not follow from bracket notation alone. Transfinite construction yields the proper class **No**, a real-closed ordered field containing the reals, infinite numbers, and nonzero infinitesimals. For instance,

$$\omega = \{0, 1, 2, \dots \mid\}, \quad \omega^{-1} = \{0 \mid 1, \tfrac{1}{2}, \tfrac{1}{4}, \dots\}.$$

An infinitesimal here is a definite nonzero number, not an unresolved approximation error [7].

F.3 Finite names, constructions, and observations

Finite well-founded option trees in Conway’s primitive cut language yield exactly the dyadic rationals $\mathbb{Z}[1/2]$. Nevertheless, in the expanded field language $1/3$ has a finite expression, although its primitive cut construction is infinite [6, Chapter 2]. Finite syntax, finite unfolding, and terminating computation are therefore separate requirements. A finite rule for an infinite cut must state its semantics and any effective approximation guarantees. Over a fixed countable alphabet, finite names cannot cover every real without additional parameters; neither construction removes this limitation.

The ambient carrier also matters. In $\mathbb{R}$, all rational lower and upper comparisons determine a unique point. In **No**, an embedded *irrational* real r and $r + \omega^{-1}$ have identical strict comparisons with every rational: the absolute value of each such real difference exceeds the positive infinitesimal. Rational comparisons alone consequently do not distinguish them; Conway’s simplest choice supplies additional structure. Moreover, cuts require an ordered setting, whereas the ordinary expression syntax over a general field does not. They cannot be silently imposed on complex or positive-characteristic fields.

F.4 The Adva through interface and the next formal questions

Adva’s finite research adapter gives the middle bar a different, explicitly typed interpretation [1]. For maps to a declared middle object N , the through relation is the fibre product

$$\langle L \mid_N R \rangle = L \times_N R = \{(\ell, r) : q_L(\ell) = q_R(r)\}.$$

In the implemented adapter the specializations may instead be relations; a pair is retained when the two middle-image sets intersect. The middle object is a declared finite quotient by exact upper-cut wire indices, derived from checked program incidences. Copy may leave several matches; the complete program slice and occurrence distinctions remain attached. This is bounded research evidence, not a stable native cut operation, a Conway simplicity selector, or an implementation of ASD.

There is already a useful obstruction to identifying these interpretations [2]. Write **Obj** for Conway value selection and let $g(z) = 3z$, acting on each presented option. Then

$$g(\text{Obj}\{0 \mid 1\}) = \tfrac{3}{2}, \quad \text{Obj}\{0 \mid 3\} = 1.$$

Scaling the options is not the recursively defined multiplication of the number. A value projection therefore cannot be assumed to commute with arbitrary transformations of presentations.

For AEG the proposed next step is to specify, separately:

1. a finite expression signature, constant names, and partial ordinary interpretation with every division guard retained;
2. constructors for exact denotations, with explicit root, branch, embedding, or cut conditions when required;
3. observations that produce justified bounds while retaining shared dependencies and any symbolically unexpanded remainder;
4. distinct judgments for presentation equality, denotational equality, information refinement, and typed through compatibility.

These are open design and proof obligations. The comparison proposes no identification of Conway simplicity, AEG history, and computational resource cost, and no extension of the formal certificates already recorded for this manuscript.

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