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# Closed three-manifolds as hypersurfaces of Euclidean four-space: compactness, the homogeneous geometries, and total absolute curvature

Jay Thrussell*

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*This is the companion of [MA], which studies the realizability operator locally. It treats the global questions, and supersedes the author's earlier mathematics papers 2, 3 and 5 (Thrussell 2026b, c, e) and the global part of paper 4 (Thrussell 2026d). Appendix C lists every statement that changes.*

*The verification scripts for this paper (tests 345–353, 355, 624, 628 and 639), with their run log, are deposited with it at Zenodo, DOI 10.5281/zenodo.23805266. The author's earlier work is archived under concept DOI 10.5281/zenodo.21264118. Appendix V indexes every displayed result against the script that reproduces it, and discloses the use of large-language-model systems.*

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## Abstract

Let $(\Sigma, g)$ be a closed Riemannian three-manifold, and let $\mathbb{M} = -G$ be its negative Einstein endomorphism. Where $\det \mathbb{M} > 0$, the companion paper [MA] reduces realizability as a hypersurface of $E^4$ to one equation, $F(g) = 0$, and studies it locally. Here we treat the global questions.

- **Compactness.** A closed realization has $\det \mathbb{M} \geq R^{-6}$ at a point farthest from any centre, where $R$ is that distance, and so $\max \det \mathbb{M} \geq 125/(8\,\mathrm{diam}^6)$ by Jung's theorem. If $\det \mathbb{M} > 0$ everywhere, the realization is a strictly convex embedded sphere, unique up to rigid motion. So a closed connected metric with $\det \mathbb{M} > 0$ is realizable iff $\Sigma$ is simply connected and $F(g) = 0$. No closed three-manifold other than $S^3$ carries such a realizable metric.
- **The homogeneous geometries.** A realizable closed metric with $\det \mathbb{M} > 0$ and constant scalar curvature is round, and so is every compact locally homogeneous realization. Locally, constant $\mathbb{M}$-eigenvalues with $\det \mathbb{M} > 0$ force constant curvature, a case of Tsukada's classification. This recovers Agaoka and Hashinaga's list of the left-invariant metrics that embed locally. The eight Thurston geometries are worked as checks.
- **Total absolute curvature.** $J(g) = \int \sqrt{\det \mathbb{M}}\, dV$ equals $2\pi^2$ on every closed realizable metric with $\det \mathbb{M} > 0$, so any other value excludes realizability. At the round sphere $J$ is stationary. Its second variation on transverse-traceless tensors of level $n$ is $-(\mu_n+2)(\mu_n+3)/16$ times the squared norm, $\mu_n = n^2 - 3$, and its radical is exactly the conformal and Lie directions. $J$ is stationary at every realizable metric in $\mathcal{U}$. On conformal rays $J$ is flat to third order, and its quartic coefficient is never positive. Along the conformal path of [MA], $J = 2\pi^2 - \pi^2 t^4/12 + O(t^6)$.


*Independent researcher. Correspondence: jaythrussell@gmail.com.

- **Non-convex realizations.** Embedded hypersurfaces of revolution are non-convex exactly around their necks. For the radial graphs $r = 1 + \varepsilon \cos 2\chi$, convexity is lost at $\varepsilon = 1/5$ in a Whitney cusp of the Gauss map. With $\delta = \varepsilon - 1/5$, the relative excess $J/2\pi^2 - 1$ then grows as $500\,\delta^{3/2}/(9\sqrt{2}\,\pi)$.
- **Connectivity, and the round sphere.** Modulo isometry, the closed realizable metrics with $\det \mathbb{M} > 0$ form a contractible family, contracted by an explicit equivariant path. The embedded hypersurfaces of revolution join the round sphere through the degenerate locus. We collect the characterisations of the round sphere, with their hypotheses.

Every displayed result is reproduced by a script; Appendix V indexes them.

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## 1. Introduction

**The question.** Which closed Riemannian three-manifolds are hypersurfaces of Euclidean four-space, and how do the realizable metrics sit among all metrics? In the $C^1$ category the answer is "all of them, whenever there is a short immersion" (Nash 1954; Kuiper 1955). In the $C^2$ category the second fundamental form exists, and the Gauss and Codazzi equations constrain it. This paper works in that category and above, on closed manifolds; §2.1 says which statements need three derivatives.

**What [MA] supplies.** In dimension three the Gauss equation is $\mathbb{M} = \operatorname{adj} K$, with $\mathbb{M} = -G$ the negative Einstein endomorphism. On $\{\det \mathbb{M} > 0\}$ it is solved by $\tilde{K} = \sqrt{\det \mathbb{M}}\,\mathbb{M}^{-1}$ for every metric, and the second fundamental form of any realization there is $\pm\tilde{K}$. Realizability is therefore governed by the realizability operator $F(g) = \operatorname{Cod}(\tilde{K}(g))$. [MA] studies $F$ locally:

- the reduction, with the gate $\det \mathbb{M} \geq 0$ and global sufficiency on simply connected domains;
- the principal symbol, involutivity and meagreness;
- the linearisation at the round sphere;
- a non-round family solved completely;
- the degenerate locus;
- a chiral form.

This paper uses those results and does not repeat them. The reduction itself is classical (Beez 1876; Killing 1885; Thomas 1936), and the local zero set of $F$ has been characterized intrinsically (Rivertz 1999; Agaoka and Hashinaga 2022).

**What compactness adds.** Four things.

- A closed hypersurface has a point where its second fundamental form is definite. That one point, together with the reduction, decides a great deal intrinsically (§3).
- On a homogeneous metric the realizability question becomes algebraic, and on a compact one it has a single answer (§4).
- The degree of the Gauss map makes total absolute curvature a global invariant. Its intrinsic form $J(g) = \int \sqrt{\det \mathbb{M}}\, dV_g$ can be computed for any metric (§5).
- The realizable metrics on the convex branch form a connected family that flows to the round sphere, while the non-convex ones are reached only through the degenerate locus (§§6–7).

**What is classical.** Much of what we use is old, and is cited where it is used:

- the convexity and embedding theorems for compact hypersurfaces with definite second funda-

mental form (Hadamard 1897; van Heijenoort 1952; Chern and Lashof 1957, 1958; Sacksteder 1960; do Carmo and Lima 1969);
- the integral formulas of Minkowski and Hsiung (Hsiung 1954), and the classification of isoparametric hypersurfaces of Euclidean space (Segre 1938; Cartan 1938);
- Hilbert's theorem on complete surfaces of constant negative curvature (Hilbert 1901; Efimov 1964);
- the Alexandrov–Fenchel inequalities (Schneider 2014);
- the volume-preserving mean curvature flow (Huisken 1987);
- the left-invariant metrics on three-dimensional Lie groups (Milnor 1976) and the eight geometries (Thurston 1997);
- the classification of curvature-homogeneous hypersurfaces of Euclidean space (Tsukada 1988), and of the left-invariant metrics on three-dimensional Lie groups that embed locally in $E^4$ (Agaoka and Hashinaga 2017).

In Lorentzian signature the corresponding functional, the Einstein volume, is non-decreasing along the cross-curvature flow and stationary exactly at the metrics that embed in Minkowski space (Bryan, Ivaki and Scheuer 2019). What is offered here is the intrinsic form of the Euclidean statements, made possible by the reduction of [MA], and the computations that follow from it.

**Main results.** $\mathcal{U}$ denotes the metrics with $\det \mathbb{M} > 0$ everywhere.

- **§3, compact realizations.**
  - A realizable closed metric has $\det \mathbb{M} \geq R^{-6} > 0$ at a point farthest from any chosen centre, where $R$ is that distance. So no closed metric with $\det \mathbb{M} \leq 0$ everywhere is realizable. By Jung's theorem, $\max \det \mathbb{M} \geq 125/(8 \operatorname{diam}(g)^6)$.
  - A realizable closed metric in $\mathcal{U}$ has $\mathbb{M} > 0$. Its realization is the boundary of a strictly convex body, unique up to rigid motion, and a radial graph over the unit sphere. The manifold is $S^3$.
  - So a closed connected metric in $\mathcal{U}$ is realizable iff $\pi_1(\Sigma) = 1$ and $F(g) = 0$, and no closed three-manifold other than $S^3$ carries a realizable metric in $\mathcal{U}$.
- **§4, the homogeneous geometries.**
  - On a closed manifold, a realizable metric in $\mathcal{U}$ with constant scalar curvature, or with constant $\operatorname{tr} \tilde{K}$, is round. So a compact locally homogeneous metric is realizable only if it is a round $S^3$. No list of geometries and no simple-connectivity hypothesis is needed.
  - Locally, a metric with constant $\mathbb{M}$-eigenvalues and $\det \mathbb{M} > 0$ is realizable on no open set unless it has constant curvature (contained in Tsukada 1988). So a left-invariant metric on a three-dimensional Lie group is realizable on some open set iff it is flat, of constant positive curvature, or locally $H^2 \times \mathbb{R}$ (Agaoka and Hashinaga 2017; a short proof here).
  - The eight Thurston geometries become worked checks. Sol fails Codazzi with an invariant residual norm of 16, and the complete $H^2 \times \mathbb{R}$ is not realizable (Hilbert).
- **§5, total absolute curvature.**
  - On a closed manifold, $J(g) = \int \sqrt{\det \mathbb{M}}\, dV$ is at least $2\pi^2$ on every realizable metric, and exactly $2\pi^2$ on every realizable metric in $\mathcal{U}$. So $J \neq 2\pi^2$ certifies non-realizability in $\mathcal{U}$.
  - At the round sphere $J$ is stationary. Its second variation on TT level $n$ is $-(\mu_n + 2)(\mu_n + 3)/16$ times the squared norm, and its radical is exactly the conformal and Lie directions.
  - On conformal rays $J$ is flat to third order for every conformal factor. Along the conformal path of [MA], $J = 2\pi^2 - \pi^2 t^4/12 + O(t^6)$, so that path is not realizable.
  - Every realizable metric in $\mathcal{U}$ is a critical point of $J$. So on every conformal ray the quartic coefficient is $Q(df \otimes df) \leq 0$, and it vanishes exactly when $df \otimes df$ has no

transverse-traceless part.
  - That happens beyond the zonal factors: for every restriction of a function whose Hessian has rank at most one near $S^3$, which includes non-zonal polynomials, cones and tangent developables. Among factors invariant under the rotations of a 2-plane it happens exactly for the developable graphs. In general it happens exactly when $-\operatorname{adj} L_f + \frac{1}{2}\sigma_2(L_f)\gamma$, with $L_f = \nabla^2 f + f\gamma$, is a Codazzi tensor: a local equation of third order.
- **§6, embedded non-convex realizations.**
  - An embedded hypersurface of revolution is non-convex exactly around its necks. So realizable non-convex metrics are plentiful, and on a closed one the non-convex region always sits behind a degenerate locus.
  - For the radial graphs $r = 1 + \varepsilon\cos 2\chi$, convexity is lost at $\varepsilon = 1/5$, where the Gauss map acquires a pair of folds in a Whitney cusp. With $\delta = \varepsilon - 1/5$, the relative excess $J/2\pi^2 - 1$ then grows as $500\,\delta^{3/2}/(9\sqrt{2}\,\pi)$.
- **§7, connectivity.**
  - Modulo isometry, the smooth closed realizable metrics in $\mathcal{U}$ are the centred support functions modulo $O(4)$. That is a convex cone divided by a linear action fixing the round sphere, so it is contractible by an explicit equivariant straight line, through realizable metrics.
  - Huisken's flow is a second contraction, with mode rates $-[\ell(\ell+2)-3]/R^2$.
  - The embedded hypersurfaces of revolution, convex or not, also form a convex family, so the non-convex ones join the round sphere through the degenerate locus.
- **§8, characterisations of the round sphere**, each with its hypotheses: constant curvature invariants; constant mean curvature (Alexandrov), and the local isoparametric version; equality in Alexandrov–Fenchel, in Schneider's normalisation; Minkowski's inequality; the bound $\int |K|^3\,dV \geq 6\sqrt{3}\,\pi^2$, a one-line consequence of Chern and Lashof; and Steiner's chain, under which $\int R\,dV$ is exactly invariant under corrugation.

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## 2. Preliminaries

### 2.1 From [MA]

Conventions are those of [MA, §2]:

- $x : \Sigma \to E^4$ is an immersion with unit normal $N$, and $K_{ab} = -N \cdot \partial_a \partial_b x$, so the unit sphere with outward normal has $K = g$;
- a metric of class $C^k$, $k \geq 3$, is **realizable** if it admits an isometric immersion of class $C^3$;
- $\mathbb{M} = -G$ as an endomorphism, where $G = \operatorname{Ric} - \frac{1}{2}R\,g$ is the Einstein tensor, so $\operatorname{tr}\mathbb{M} = R/2$; $\mathcal{U}$ is the open set of metrics with $\det\mathbb{M} > 0$ everywhere;
- $\tilde{K} = \sqrt{\det\mathbb{M}}\,\mathbb{M}^{-1}$ on $\mathcal{U}$, and $F(g) = \operatorname{Cod}(\tilde{K}(g))$.

*Regularity.* Lemma 3.1, Theorem 3.2, Corollary 3.3 and Theorem 3.5(1), (2) and (4) use only the Gauss equation and second derivatives of $x$. They hold for $C^2$ isometric immersions of $C^2$ metrics, and there "realizable" may be read at that regularity. Whatever uses the Codazzi equation, $F$ or the fundamental theorem of hypersurfaces needs the definition above. A smooth realizable metric in $\mathcal{U}$ has smooth realizations: $K = \pm\tilde{K}$ is smooth (fact 2 below), and so is every solution of the Gauss–Weingarten system with smooth coefficients.

We use five facts.

1. **Gauss.** For a hypersurface, $\mathbb{M} = \operatorname{adj} K$, so $\det \mathbb{M} = (\det K)^2$ [MA, (2.2)–(2.3)]. In an eigenframe $K = \operatorname{diag}(k_1, k_2, k_3)$, $\mathbb{M} = \operatorname{diag}(k_2k_3, k_1k_3, k_1k_2)$: the eigenvalues of $\mathbb{M}$ are the sectional curvatures of the principal planes.
2. **Branch** [MA, Theorem A(3)]. Where $\det K \neq 0$, $K = \operatorname{sign}(\det K)\, \tilde{K}$.
3. **Global sufficiency** [MA, Theorem A(4)]. If $\Sigma$ is connected and simply connected and $g \in \mathcal{U}$, then $g$ is realizable iff $F(g) = 0$, uniquely up to a rigid motion. For general connected $\Sigma$, $g$ is realizable iff $F(g) = 0$ and the monodromy of the developing immersion is trivial.
4. **Convexity** [MA, Theorem A(5)]. $\mathbb{M} > 0$ iff $\tilde{K}$ is definite iff a realization is locally strictly convex.
5. **Meagreness** [MA, Theorem D]. On a closed $\Sigma$ the zero set of $F$ is nowhere dense in $\mathcal{U}^{k,\alpha}$ and in the space of all $C^{k,\alpha}$ metrics.

### 2.2 The Gauss map and its degree

For an immersion of a closed oriented three-manifold with unit normal $N$, the Gauss map $\nu = N : \Sigma \to S^3$ has Jacobian $\det K$. So

$$\int_\Sigma \det K \, dV = \deg(\nu)\, |S^3| = 2\pi^2 \deg(\nu). \tag{2.1}$$

For an embedding onto the boundary of a compact region $\Omega \subset E^4$, with outward normal, $\deg \nu = \chi(\Omega)$ (Hopf). It is not $\chi(\Sigma)$, which is zero for every closed three-manifold. The **total absolute curvature** of the immersion is $\int |\det K|\, dV$. Its intrinsic counterpart, defined for every metric with $\det \mathbb{M} \geq 0$, is

$$J(g) = \int_\Sigma \sqrt{\det \mathbb{M}}\, dV_g. \tag{2.2}$$

By fact 1, $J(g) = \int |\det K|\, dV$ for every realization of $g$. Chern and Lashof (1957) bound the total absolute curvature below by the sum of the Betti numbers. They integrate over the unit normal bundle, which double-covers a two-sided hypersurface, so their bound for $S^3$ reads 2. In the one-sided normalisation used here it reads $\int |\det K|\, dV \geq |S^3| = 2\pi^2$, with equality iff the immersion is a convex embedding. Neither $|\det K|\, dV$ nor $J$ needs a global normal: $|\det K|$ does not depend on the sign of $N$, and the unit normal bundle double-covers $\Sigma$ whether or not $\Sigma$ is orientable. So the bound holds for every closed immersed hypersurface, orientable or not. Only (2.1) needs an orientation.

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## 3. Compact realizations

### 3.1 The farthest point

**Lemma 3.1.** *Let $x : \Sigma \to E^4$ be a $C^2$ immersion of a closed three-manifold, $o \in E^4$ a point, and $p \in \Sigma$ a point where $|x - o|$ is maximal, with value $R > 0$. Let $N(p) = (x(p) - o)/R$. Then $N(p)$ is normal to $\Sigma$ at $p$, and with respect to it*

$$K_p \;\geq\; \frac{1}{R}\, g_p$$

*as quadratic forms. Every principal curvature at $p$ is at least $1/R$, and $\det K(p) \geq R^{-3}$.*

*Proof.* Put $f = \frac{1}{2}|x - o|^2$. At the maximum, $df = (x - o) \cdot dx = 0$, so $x(p) - o$ is normal, and the Hessian is nonpositive. In coordinates, $\partial_a \partial_b f = \partial_a x \cdot \partial_b x + (x - o) \cdot \partial_a \partial_b x = g_{ab} - RK_{ab}$ at $p$, with $K$ taken with respect to $N(p)$. So $g - RK \leq 0$. □

**Theorem 3.2 (the farthest-point bound).** *Let $(\Sigma, g)$ be a closed Riemannian three-manifold that is realizable. Then $\det \mathbb{M} \geq 0$ everywhere, and for every $o \in E^4$,*

$$\det \mathbb{M}(p) \;\geq\; R^{-6} \;>\; 0$$

*at any point $p$ farthest from $o$, with $R = |x(p) - o|$. The round sphere of radius $R$ attains equality at every point.*

*Proof.* $\det \mathbb{M} = (\det K)^2$ (fact 1) and Lemma 3.1. On the sphere of radius $R$ about $o$, $K = g/R$. □

**Corollary 3.3.**

1. *No closed Riemannian three-manifold with $\det \mathbb{M} \leq 0$ everywhere is realizable. This excludes:*
   - *every flat three-manifold;*
   - *$S^2 \times S^1$ with a product metric;*
   - *the compact quotients of $H^2 \times \mathbb{R}$ with a product metric;*
   - *every closed metric with $\det \mathbb{M} < 0$ somewhere and $\leq 0$ elsewhere.*

   *Each of the three named geometries has $\det \mathbb{M} \equiv 0$.*
2. *(An intrinsic bound.) If $(\Sigma, g)$ is closed and realizable, then*

$$\max_{\Sigma} \det \mathbb{M} \;\geq\; \frac{125}{8\,\mathrm{diam}(g)^6}.$$

*Proof.* (1) For $S^2(r) \times S^1$, $\mathbb{M} = \mathrm{diag}(0, 0, r^{-2})$, with the nonzero entry on the circle factor. For $H^2(-r^{-2}) \times S^1$ the same holds with $-r^{-2}$. Flat metrics have $\mathbb{M} = 0$.

(2) The image $x(\Sigma)$ has Euclidean diameter at most the intrinsic diameter $d = \mathrm{diam}(g)$, since $x$ does not increase lengths of curves. By Jung's theorem (Jung 1901) it lies in a closed ball of radius $d\sqrt{2/5}$, the circumradius of a regular simplex of edge $d$ in $\mathbb{R}^4$. Take $o$ the centre of the smallest such ball. A farthest point is at distance $R \leq d\sqrt{2/5}$, and Theorem 3.2 gives $\det \mathbb{M} \geq R^{-6} \geq (5/2)^3 d^{-6}$. □

*Checks.* Theorem 3.2 is checked in exact arithmetic at the vertex of a general ellipsoid, where the principal curvatures are $a_1/a_j^2$, $j = 2, 3, 4$, and the round sphere is the equality case. It is checked in 40-digit arithmetic at the farthest point of three random star-shaped hypersurfaces, each non-convex somewhere. The products of (1) have $\det \mathbb{M} \equiv 0$ symbolically, and Jung's constant at $n = 4$ is $\sqrt{2/5}$ (`test_345` P1–P4).

**Remark 3.4 (scope).**

- Theorem 3.2 is a statement in the $C^2$ category. Below it the conclusion fails: $C^{1,\alpha}$ isometric embeddings of the flat $T^3$ and of $S^2 \times S^1$ into $\mathbb{R}^4$ exist for $\alpha < 1/7$ (Cao, Hirsch and Inauen 2025, after Nash and Kuiper). Where $F$ is defined, it needs three derivatives of the metric. At $C^{1,\alpha}$ regularity the immersion has no classical second fundamental form, so the Gauss equation behind Theorem 3.2 is not available, even when the induced metric is smooth. The two are different questions.
- The argument is classical: it is the usual proof that a compact hypersurface of Euclidean space has a point of positive sectional curvature. What the reduction adds is that the conclusion becomes a condition on $g$ alone.
- Compactness cannot be dropped. The round cylinder $S^2(r) \times \mathbb{R} \subset E^3 \times \mathbb{R}$ has $K = \mathrm{diag}(r^{-1}, r^{-1}, 0)$, $\mathrm{adj}\, K = \mathbb{M}$ and $\nabla K = 0$, and $\det \mathbb{M} \equiv 0$.

### 3.2 Compact realizations in $\mathcal{U}$

**Theorem 3.5.** *Let $\Sigma$ be closed and connected, and let $g \in \mathcal{U}$ be realizable, by $x : \Sigma \to E^4$. Then:*

1. *$K$ is definite everywhere, so $\mathbb{M} > 0$: every sectional curvature of a principal plane is positive;*
2. *$\Sigma$ is orientable, $x$ is an embedding onto the boundary of a compact strictly convex body, and the Gauss map is a diffeomorphism $\Sigma \to S^3$. In particular $\Sigma$ is diffeomorphic to $S^3$;*
3. *the realization is unique up to a rigid motion, and its second fundamental form is $\pm\tilde{K}$;*
4. *about any interior point of the body, $x(\Sigma)$ is a radial graph over the unit sphere, $g = \psi^*(\rho^2\gamma + d\rho \otimes d\rho)$ with $\rho > 0$ a function on $S^3$ and $\psi$ a diffeomorphism.*

*Proof.*

(1) $(\det K)^2 = \det \mathbb{M} > 0$, so no principal curvature vanishes anywhere. Whether $K$ is definite does not depend on the choice of normal. The set where it is definite is open, by continuity. It is closed: at a limit of definite points the eigenvalues have one sign or vanish, and none vanishes. It contains a farthest point (Lemma 3.1). $\Sigma$ is connected, so $K$ is definite everywhere, and $\mathbb{M} = \operatorname{adj} K > 0$.

(2) Choose $N$ locally so that $K > 0$. The choices agree on overlaps, since $K \neq 0$, so they define a global unit normal and $\Sigma$ is orientable. The Gauss map then has Jacobian $\det K > 0$. It is a local diffeomorphism of a closed manifold onto $S^3$, hence a covering, hence a diffeomorphism, $S^3$ being simply connected. A compact immersed hypersurface with positive definite second fundamental form is embedded as the boundary of a convex body (Hadamard 1897; Chern and Lashof 1958; do Carmo and Lima 1969).

(3) By fact 2, $K = \pm\tilde{K}$ at each point, with one sign throughout because $K$ is definite. The fundamental theorem of hypersurfaces gives uniqueness of an immersion with prescribed $(g, K)$ on a connected manifold, up to a rigid motion. Replacing $\tilde{K}$ by $-\tilde{K}$ replaces $N$ by $-N$ and gives the same immersion.

(4) A convex body is star-shaped about each interior point, and strict convexity makes every ray from it transverse to the boundary. The implicit function theorem then makes the radial function as regular as $x$. □

**Theorem 3.6.** *Let $\Sigma$ be closed and connected, and $g \in \mathcal{U}$ of class $C^k$, $k \geq 3$. Then $g$ is realizable iff $\Sigma$ is simply connected and $F(g) = 0$. When it is, the conclusions of Theorem 3.5 hold.*

*Proof.* If $g$ is realizable, $\Sigma$ is diffeomorphic to $S^3$ by Theorem 3.5(2), and $F(g) = 0$ by fact 2. Conversely, fact 3 gives a realization. □

**Corollary 3.7.** *A closed three-manifold not diffeomorphic to $S^3$ carries no realizable metric in $\mathcal{U}$. In particular:*

- *no metric with $\det \mathbb{M} > 0$ everywhere on a spherical space form $S^3/\Gamma$, $\Gamma \neq 1$, is realizable. That includes the round metric (for $\mathbb{RP}^3$ this is [MA, Remark 3.2]), the lens spaces and the Poincaré homology sphere;*
- *no metric with $\det \mathbb{M} > 0$ everywhere on $T^3$ or $S^2 \times S^1$ is realizable.*

**Remark 3.8 (what is classical, and what is not).**

- Every step of Theorem 3.5 after (1) is classical. The convexity, embedding and topology are Hadamard's theorem and its extensions (Hadamard 1897; van Heijenoort 1952; Chern and

Lashof 1957, 1958; Sacksteder 1960; do Carmo and Lima 1969). Uniqueness is the fundamental theorem.

- Those statements take an immersion, or a pair $(g, K)$, as their input. With the reduction of [MA] the hypotheses become conditions on $g$ alone, checkable without exhibiting a second fundamental form:
  - $\det \mathbb{M} > 0$ everywhere;
  - $\pi_1(\Sigma) = 1$;
  - $F(g) = 0$.
- Theorem 3.6 is the intrinsic form of Hadamard's theorem in this dimension. It needs no convexity hypothesis: convexity is forced.

**Remark 3.9 (what is left out).** Closed realizable metrics with $\det \mathbb{M} = 0$ somewhere exist, convex or not. A closed realization that is not convex always has one, by Theorem 3.5, but the converse fails: Example 6.2 at $\varepsilon = 1/5$ is convex and degenerate on its equator. The radial graphs $\rho = 1 + \varepsilon \cos 2\chi$ are embedded for $0 \le \varepsilon < 1$, and have $\mathbb{M}$ indefinite on an equatorial band once $\varepsilon > 1/5$ (§6). On such a metric Theorem 3.2 still gives $\det \mathbb{M} > 0$ near the farthest point, and the proof of Theorem 3.5(1) shows more:

- *An indefinite region is separated from every farthest point by the degenerate locus* $\{\det \mathbb{M} = 0\}$. Along the family, definiteness is lost exactly where, and when, $\det K$ first vanishes: at the equator, at $\varepsilon = 1/5$ (`test_345` P5).
- Closed manifolds whose metrics have $\det \mathbb{M}$ vanishing on a proper subset are where neither $F$ nor Theorem 3.2 decides. That is open (§9).

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## 4. The homogeneous geometries

Papers 3 and 5 decided the homogeneous geometries one at a time, from tables. This section proves two theorems instead, and the tables become checks.

- *Globally* (§4.1). A realizable closed metric in $\mathcal{U}$ with constant scalar curvature is round. Constant $\mathbb{M}$-eigenvalues force this, so every compact locally homogeneous realization is a round sphere.
- *Locally* (§4.2). A metric with constant $\mathbb{M}$-eigenvalues and $\det \mathbb{M} > 0$ is realizable on no open set unless it has constant curvature. With the rank condition of [MA] on the degenerate locus, this decides every left-invariant metric on every three-dimensional Lie group.

### 4.1 Closed realizations with a constant curvature invariant

Let $x : \Sigma \to E^4$ be an immersion with unit normal $N$. Write $s = x \cdot N$ for its support function and $H_1 = \sigma_1/3$, $H_2 = \sigma_2/3$ for the normalised mean curvatures, so that the unit sphere has $s = H_1 = H_2 = 1$.

**Lemma 4.1 (the Minkowski–Hsiung formulas).** *Let $x$ be a $C^3$ immersion of a closed oriented three-manifold. Then*

$$\int_\Sigma (1 - H_1 s)\, dV = 0, \qquad \int_\Sigma (H_1 - H_2 s)\, dV = 0. \tag{4.1}$$

*Proof.* Put $f = \frac{1}{2}|x|^2$. As in the proof of Lemma 3.1, $\nabla^2 f = g - sK$, so $\Delta f = 3(1 - H_1 s)$, and its integral vanishes. The first Newton tensor $T_1 = \sigma_1 g - K$ [MA, §2.2] is divergence-free, since the

contracted Codazzi equation reads $\operatorname{div} K = d\sigma_1$. Hence

$$\operatorname{div}(T_1 \nabla f) = \langle T_1, \nabla^2 f \rangle = \operatorname{tr} T_1 - s \operatorname{tr}(T_1 K) = 2\sigma_1 - 2\sigma_2 s = 6(H_1 - H_2 s),$$

and its integral vanishes too. □

These are Minkowski's integral formulas for $k = 1, 2$, in the form Hsiung (1954) gave them for hypersurfaces.

**Theorem 4.2.** *Let $\Sigma$ be closed and connected, and let $g \in \mathcal{U}$ be realizable. Suppose that the scalar curvature $R$ is constant, or that*

$$\operatorname{tr} \tilde{K} = \frac{\operatorname{tr} \operatorname{adj} \mathbb{M}}{\sqrt{\det \mathbb{M}}}$$

*is constant. Then $g$ has constant curvature, $\Sigma$ is diffeomorphic to $S^3$, and the realization is a round sphere.*

*Proof.* By Theorem 3.5, $K$ is definite. Choose $N$ so that $K > 0$. Then $K = \tilde{K}$, since $\tilde{K} > 0$ when $\mathbb{M} > 0$. Also by Theorem 3.5, $x$ embeds $\Sigma$ as the boundary of a strictly convex body. Put the origin inside it, so that $s > 0$. With $K > 0$ both $H_1$ and $H_2$ are positive, and Newton's inequality reads

$$H_1^2 - H_2 = \frac{1}{18} \sum_{i<j} (k_i - k_j)^2 \;\geq\; 0, \tag{4.2}$$

with equality exactly at umbilic points.

*If $\operatorname{tr} \tilde{K} = 3H_1$ is constant*, the first formula of (4.1) gives $\int s = |\Sigma|/H_1$. The second, with (4.2), gives

$$H_1 |\Sigma| = \int H_2 s \;\leq\; H_1^2 \int s = H_1 |\Sigma|.$$

So $H_2 = H_1^2$ wherever $s > 0$, which is everywhere.

*If $R = 2\sigma_2 = 6H_2$ is constant*, the second formula gives $\int s = H_2^{-1} \int H_1$. The first, with $H_1 \geq \sqrt{H_2}$ from (4.2), gives

$$|\Sigma| = \int H_1 s \;\geq\; \sqrt{H_2} \int s = \int \frac{H_1}{\sqrt{H_2}} \;\geq\; |\Sigma|.$$

So $H_1 = \sqrt{H_2}$ everywhere.

Either way $x$ is umbilic, $K = \kappa g$. The contracted Codazzi equation gives $d\kappa = 3\, d\kappa$, so $\kappa$ is constant. Then $dN = \kappa\, dx$, so $x - N/\kappa$ is constant, $x(\Sigma)$ lies on a sphere of radius $1/\kappa$, and $g$ has constant curvature $\kappa^2$. □

In words: on a closed manifold, the round metric is the only realizable metric in $\mathcal{U}$ of constant scalar curvature.

**Corollary 4.3 (compact homogeneous realizations).** *Let $\Sigma$ be closed and connected, and let the eigenvalues of $\mathbb{M}$ be constant on $\Sigma$, as they are when $g$ is locally homogeneous. If $g$ is realizable, then $(\Sigma, g)$ is isometric to a round three-sphere, and its realization is a round sphere.*

*Proof.* At a farthest point every principal curvature is at least $1/R$ (Lemma 3.1), so every eigenvalue $k_j k_l$ of $\mathbb{M}$ is at least $R^{-2}$. The eigenvalues are the same at every point, so $\mathbb{M} > 0$ everywhere and $g \in \mathcal{U}$. The scalar curvature $R = 2 \operatorname{tr} \mathbb{M}$ is constant. Theorem 4.2 applies. □

**Remark 4.4 (sources and scope).**

- Theorem 4.2 is classical once the hypersurface is known to be convex. A closed convex hypersurface with constant $H_1$ or constant $H_2$ is a sphere (Hsiung 1954; Liebmann 1899 for surfaces). Without convexity, the embedded cases are Alexandrov's theorem (1958) and Ros's (1987). Here the hypotheses are on $g$ alone, and convexity is forced by Theorem 3.5.
- The independent audit of paper 5 proved Corollary 4.3 for locally homogeneous metrics, finishing with Alexandrov's theorem or Segre's classification. Corollary 4.3 also follows from Tsukada's local classification (Remark 4.8) together with Corollary 3.3. The route above uses only the divergence theorem and the Codazzi equation, at the regularity of §2.
- Corollary 4.3 needs no list and no simple-connectivity hypothesis. It covers every locally homogeneous metric on every closed three-manifold: the compact quotients of all eight Thurston geometries with any locally homogeneous metric, not only the model ones; the Berger spheres; the spherical space forms. Its hypothesis, constancy of the Ricci eigenvalues, is curvature homogeneity in this dimension, which is weaker than local homogeneity.

### 4.2 The local statement

On a non-compact domain there is no farthest point, and $\det \mathbb{M}$ can have either sign. Where it is negative, nothing is realizable [MA, Theorem A(1)]. Where it is positive, Segre's classification decides.

**Theorem 4.5.** *Let $g$ be a smooth metric on a connected open set $U$, with $\det \mathbb{M} > 0$ and the eigenvalues of $\mathbb{M}$ constant on $U$. If some open subset of $U$ admits an isometric immersion into $E^4$, then $\mathbb{M} = c\,\mathrm{Id}$ on $U$ for a constant $c > 0$. So $g$ has constant curvature $c$, and every such immersion is an open piece of a round sphere of radius $c^{-1/2}$.*

*Proof.* Let $x$ realize a connected open $V \subset U$. By fact 2, $K = \pm\tilde{K}$ on $V$, with one sign because $\det K$ does not vanish. So $K$ is smooth, and so is $x$, since its Gauss–Weingarten system then has smooth coefficients. The principal curvatures $\kappa_i = \sqrt{\det \mathbb{M}}/m_i$ are constant and nonzero, so $x(V)$ is an isoparametric hypersurface. Let its distinct principal curvatures be $\kappa_1, \dots, \kappa_q$, with multiplicities $n_i$. If $q \geq 2$, Cartan's identity for isoparametric hypersurfaces of Euclidean space (Cartan 1938; Cecil and Ryan 2015, Ch. 3) gives, for each $i$,

$$\sum_{j \neq i} n_j \, \frac{\kappa_i \kappa_j}{\kappa_i - \kappa_j} = 0.$$

Reverse $N$ if necessary so that some $\kappa$ is positive, and let $\kappa_i$ be the smallest positive one. If $\kappa_j > \kappa_i$, the term is negative: its numerator is positive and its denominator negative. If $\kappa_j < 0$, the term is negative: its numerator is negative and its denominator positive. No $\kappa_j$ is zero or lies in $(0, \kappa_i)$. So the sum is negative, a contradiction. Hence $q = 1$. Then $x$ is umbilic with constant $\kappa$, $x(V)$ is a piece of a sphere of radius $1/|\kappa|$, and $\mathbb{M} = \operatorname{adj} K = \kappa^2\,\mathrm{Id}$ on $V$. The eigenvalues are constant, so $\mathbb{M} = \kappa^2\,\mathrm{Id}$ on $U$. A three-metric with $\mathbb{M} = c\,\mathrm{Id}$ is Einstein, $\mathrm{Ric} = 2cg$, and so has constant curvature $c$. □

This is Segre's theorem (1938), with the two lines that use $\det K \neq 0$ written out. The Euclidean isoparametric hypersurfaces are pieces of hyperplanes, spheres and spherical cylinders, and only the sphere has no zero principal curvature. Theorem 4.5 is also contained in Tsukada's classification of curvature-homogeneous hypersurfaces of Euclidean space (Tsukada 1988; see also Ge and Zhao 2026). In dimension three such a hypersurface is flat, an open piece of a round sphere, or a product immersion of $S^2(c) \times \mathbb{R}$ or $H^2(-c) \times \mathbb{R}$. The proof above uses only the reduction of [MA] and Segre's theorem.

**Corollary 4.6 (left-invariant metrics).** *A left-invariant metric on a three-dimensional Lie group is realizable on some open set iff it is flat, has constant positive curvature, or is locally isometric to* $H^2(-c) \times \mathbb{R}$. *So none of the following is realizable on any open set: a left-invariant metric on Nil, on* Sol *or on* $\widetilde{SL}_2(\mathbb{R})$*; a non-round one on* $SU(2)$*; a non-flat one on* $\widetilde{E}(2)$*; one on a non-unimodular group, unless it is locally isometric to* $H^2 \times \mathbb{R}$.

*Proof.* In a left-invariant orthonormal frame $\mathbb{M}$ is constant. If $\det \mathbb{M} < 0$, the gate excludes $g$. If $\det \mathbb{M} > 0$, Theorem 4.5 leaves only constant positive curvature. If $\det \mathbb{M} = 0$, a realization needs $\operatorname{rank} \mathbb{M} \leq 1$ [MA, Proposition 8.2(1)]. There are two cases.

- *Unimodular groups.* In a Milnor frame, $[e_j, e_k] = \lambda_i e_i$ cyclically, $\mathbb{M}$ is diagonal with entries $m_i = \mu_i(\mu_j + \mu_k) - \mu_j\mu_k = \mu_i\lambda_i - \mu_j\mu_k$, where $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$ (Milnor 1976). If $m_1 = m_2 = 0$, their difference gives $\mu_3(\mu_1 - \mu_2) = 0$ and their sum gives $\mu_1\mu_2 = 0$. Both cases give $m_3 = 0$. So $\mathbb{M} = 0$, and $\mathrm{Ric} = 0$: the metric is flat.
- *Non-unimodular groups.* Rotate the frame so that $[e_i, e_j] = \epsilon_{ijl}n^{lk}e_k + a_ie_j - a_je_i$ with $a = (a, 0, 0)$, $a \neq 0$, and $n = \operatorname{diag}(0, n_2, n_3)$ (the Ellis–MacCallum form, used in `test_215`). Then $m_{11} = (n_2 - n_3)^2/4 - a^2$. If $m_{11} \neq 0$, rank at most one would force the lower $2 \times 2$ block to vanish. It cannot: its off-diagonal entry is $a(n_3 - n_2)$, which vanishes only at $n_2 = n_3$, and there the block is $-a^2 \operatorname{Id}$. So $n_3 - n_2 = 2\varepsilon a$ with $\varepsilon = \pm 1$, and the lower $2 \times 2$ block then has determinant $-4a^2(n_2 + \varepsilon a)^2$. So rank one forces $(n_2, n_3) = (-\varepsilon a, \varepsilon a)$. There $\mathbb{M}$ has eigenvalues $(0, 0, -4a^2)$, and its unit eigenvector $v$ for $-4a^2$ is parallel. By de Rham's theorem the metric is locally a product of a line, along $v$, with a surface of curvature $-4a^2$.

Conversely, a flat metric is realized locally by a hyperplane, one of constant curvature $c > 0$ by a piece of a sphere, and $H^2(-c) \times \mathbb{R}$ by the product of a line with a piece of a surface of curvature $-c$ in $E^3$. □

Corollary 4.6 is the classification of Agaoka and Hashinaga (2017), who obtained it from the Gauss equation and its first covariant derivative. Their list is the abelian algebra with any inner product, the Euclidean motions with the flat metric, $\mathfrak{so}(3)$ with the round metric, and the algebra of $H^2 \times \mathbb{R}$. Here it follows from Theorem 4.5 and the rank condition.

**Corollary 4.7 (locally homogeneous metrics, nondegenerate).** *A locally homogeneous metric with* $\det \mathbb{M} \neq 0$ *is realizable on some open set iff it has constant positive curvature.*

*Proof.* The sign of $\det \mathbb{M}$ is constant. If it is negative, the gate applies; if positive, Theorem 4.5. □

**Remark 4.8 (what is left, and what this replaces).**

- A degenerate locally homogeneous metric can be realizable only if $\operatorname{rank} \mathbb{M} \leq 1$. Besides the flat metrics this includes $H^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$, and both are realizable on open sets. The second carries no simply transitive group, so Corollary 4.6 does not see it. By Tsukada's classification, the curvature-homogeneous three-metrics that are realizable on some open set, homogeneous or not, are exactly four kinds: flat, round, $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$.
- Corollary 4.6 contains the exclusion of the non-unimodular Bianchi types, which paper 3 listed as open and `test_215` proved by direct algebra (§4.4), and the Berger family's obstruction (Example 4.11).
- On $\{\det \mathbb{M} > 0\}$ a realization is rigid (fact 2 and the fundamental theorem). So every local isometry of $g$ extends to a rigid motion, and a locally homogeneous realization there is extrinsically homogeneous.

### 4.3 The Thurston geometries, as checks

Table 1 gives each model geometry with its standard metric. Corollary 4.3 settles the last column for all eight at once; the table shows, in each column, the cheapest argument that decides it.

Table 1: The eight Thurston geometries with their standard metrics.

| geometry | $\mathbb{M}$ (eigenvalues) | $\det \mathbb{M}$ | on an open set | as a complete hypersurface | compact quotients |
|---|---|---|---|---|---|
| $S^3$, Milnor $(1,1,1)$ | $(\frac{1}{4},\frac{1}{4},\frac{1}{4})$ | $\frac{1}{64}$ | spheres only (Thm 4.5) | the round sphere | $S^3$ itself only (Cor. 3.7) |
| $E^3$ | $(0,0,0)$ | $0$ | yes | yes (hyperplane) | none (Cor. 3.3) |
| $H^3$, curvature $-1$ | $(-1,-1,-1)$ | $-1$ | no (gate) | no | none (gate) |
| $S^2\times\mathbb{R}$, unit $S^2$ | $(0,0,1)$ | $0$ | yes | yes (round cylinder) | none (Cor. 3.3) |
| $H^2\times\mathbb{R}$, curvature $-1$ | $(0,0,-1)$ | $0$ | yes | no (Prop. 4.10) | none (Cor. 3.3) |
| $\widetilde{SL}_2(\mathbb{R})$, Milnor $(1,1,-1)$ | $(\frac{1}{4},\frac{1}{4},-\frac{7}{4})$ | $-\frac{7}{64}$ | no (gate) | no | none (gate) |
| Nil, Milnor $(1,0,0)$ | $(-\frac{3}{4},\frac{1}{4},\frac{1}{4})$ | $-\frac{3}{64}$ | no (gate) | no | none (gate) |
| Sol, Milnor $(1,-1,0)$ | $(-1,-1,1)$ | $1$ | no (Thm 4.5; Ex. 4.9) | no | none (Thm 4.5) |

*The Milnor-frame rows.* With $m_i=\mu_i\lambda_i-\mu_j\mu_k$:

- $S^3$ $(1,1,1)$ has $\mu=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $\mathbb{M}=\frac{1}{4}\,\mathrm{Id}$;
- $\widetilde{SL}_2$ $(1,1,-1)$ has $\mu=(-\frac{1}{2},-\frac{1}{2},\frac{3}{2})$ and $\mathbb{M}=\mathrm{diag}(\frac{1}{4},\frac{1}{4},-\frac{7}{4})$;
- Nil $(1,0,0)$ has $\mu=(-\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $\mathbb{M}=\mathrm{diag}(-\frac{3}{4},\frac{1}{4},\frac{1}{4})$, so $\det\mathbb{M}=-3/64$;
- Sol $(1,-1,0)$ has $\mu=(-1,1,0)$ and $\mathbb{M}=\mathrm{diag}(-1,-1,1)$.

Every left-invariant metric on Nil is homothetic, up to automorphism, to the one shown (Milnor 1976). The other groups carry families of metrics, which Corollary 4.6 decides at once.

**Remark (scale).** Under $g\mapsto c^2g$, $\det\mathbb{M}$ scales by $c^{-6}$, so its sign is invariant under homothety. It is not invariant under the anisotropic changes that move between left-invariant metrics: the Berger sphere at $r=6/5$ has $\det\mathbb{M}<0$. Corollaries 4.3 and 4.6 make no use of a normalisation.

**Example 4.9 (Sol, explicitly).** Take $ds^2=e^{2z}dx^2+e^{-2z}dy^2+dz^2$. The nonzero Christoffel symbols are

$$\Gamma^x{}_{xz}=1,\qquad \Gamma^y{}_{yz}=-1,\qquad \Gamma^z{}_{xx}=-e^{2z},\qquad \Gamma^z{}_{yy}=e^{-2z},$$

with $\mathrm{Ric}=\mathrm{diag}(0,0,-2)$, $\mathbb{M}=\mathrm{diag}(-1,-1,1)$, $\det\mathbb{M}=1$ and $\tilde K_{ab}=\mathrm{diag}(-e^{2z},-e^{-2z},1)$. The Gauss equation holds; Codazzi fails. The nonzero components of $\mathrm{Cod}\,\tilde K$ are

$$C_{xzx}=-C_{zxx}=2e^{2z},\qquad C_{yzy}=-C_{zyy}=-2e^{-2z}.$$

The sum of the squares of these coordinate components is $16\cosh 4z$. That is not a norm. The invariant norm is

$$|\operatorname{Cod}\tilde{K}|_g^2 = g^{aa'}g^{bb'}g^{cc'}C_{abc}C_{a'b'c'} = 16,$$

and in a left-invariant orthonormal frame the components are constants of modulus 2. The residual is nowhere zero, so no open subset of Sol is realizable. This is the case of Theorem 4.5 in which every cheaper test is silent: the gate passes, $\mathbb{M}$ is indefinite, so no convexity argument applies, and Chern–Lashof bounds only closed realizations.

The coordinate pipeline that produced these components was run on two positive controls. On the unit $S^3$ in hyperspherical coordinates it returns $\mathbb{M} = \mathrm{Id}$ and zero residual. Paper 3 (Remark 4.2) recorded that such a pipeline can leave a spurious trigonometric remainder, which only trigonometric expansion removes. On the Flamm metric of [MA, Example 3.3] it returns $\mathbb{M} = \operatorname{diag}(2m, -m, -m)/\rho^3$, the known $\tilde{K}$, and zero residual. That metric is realizable, inhomogeneous, and has $\det\mathbb{M} > 0$ with $\mathbb{M}$ indefinite, like Sol. A second, independent computation in the Lie algebra gives the same invariant norm.

**Proposition 4.10 (the product geometries).**

1. *$E^3$ and $S^2(\rho)\times\mathbb{R}$ are realizable as complete hypersurfaces, by a hyperplane and by the round cylinder (Remark 3.4).*
2. *$H^2(-c)\times\mathbb{R}$ is realizable on open sets, but the complete $H^2(-c)\times\mathbb{R}$ admits no isometric $C^3$ immersion into $E^4$.*

*Proof of (2).* Open sets: Corollary 4.6. Suppose $F : H^2\times\mathbb{R}\to E^4$ is an isometric immersion. $\mathbb{M}$ has rank one, with image spanned by $\partial_t$. So $K$ has rank two [MA, Proposition 8.2(1)], and $K\operatorname{adj}K = (\det K)\,\mathrm{Id} = 0$ puts the image of $\operatorname{adj}K = \mathbb{M}$ inside $\ker K$. Hence $\ker K = \operatorname{span}\partial_t$. The field $\partial_t$ is parallel, so by the Gauss formula

$$D_X\big(dF(\partial_t)\big) = dF(\nabla_X\partial_t) - K(X,\partial_t)N = 0,$$

and $dF(\partial_t) = v$ is a constant unit vector. Then $F(p,t) = f(p) + tv$ with $f = F(\cdot,0)$. Since $dF(X)\cdot v = g(X,\partial_t) = 0$ for $X$ tangent to $H^2$, $f$ is an isometric immersion of the complete $H^2(-c)$ into a hyperplane orthogonal to $v$. Hilbert's theorem (1901; Efimov 1964 for $C^2$ immersions) forbids it. □

### 4.4 Families, as checks

**Example 4.11 (the Berger spheres).** On $SU(2)$ take the Milnor constants $(1/r, 1/r, r)$: the metric $\operatorname{diag}(1,1,r^2)$ in the frame of the round $(1,1,1)$. Then

$$\mathbb{M} = \operatorname{diag}\Big(\frac{r^2}{4}, \frac{r^2}{4}, 1-\frac{3r^2}{4}\Big), \qquad \det\mathbb{M} = \frac{r^4(4-3r^2)}{64}.$$

For $0 < r < 2/\sqrt{3}$, $\mathbb{M} > 0$ and the gate passes. The nonzero components of $\operatorname{Cod}(\operatorname{adj}\mathbb{M})$ are

$$C_{123} = -\frac{r^3(r^2-1)}{4}, \qquad C_{132} = -\frac{r^3(r^2-1)}{8}, \qquad C_{231} = \frac{r^3(r^2-1)}{8},$$

and $F(g) = (\det\mathbb{M})^{-1/2}\operatorname{Cod}(\operatorname{adj}\mathbb{M})$, since $\mathbb{M}$ is constant in the frame. So $F = 0$ iff $r = 1$. At $r = 2/\sqrt{3}$, $\mathbb{M}$ has rank two, which is never an adjugate. Beyond it, $\det\mathbb{M} < 0$. The whole family is decided in closed form, with the round sphere its only realizable member. This is the audit's P5-X02 and an instance of Theorem 4.5.

**Remark 4.12 (three more families).**

- *Bianchi IX by linear algebra.* In a Milnor frame with $\lambda_i > 0$, the Codazzi equation for a constant diagonal $K = \mathrm{diag}(k_1, k_2, k_3)$ is linear in $k$. Every $2 \times 2$ minor of its matrix is $\pm H/4$, where

  $$H = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = -\prod(\pm\sqrt{\lambda_1} \pm \sqrt{\lambda_2} \pm \sqrt{\lambda_3}),$$

  the product being Heron's over the four sign patterns with at most one minus. Where $H \neq 0$ the kernel is $k_1 = k_2 = k_3$. So $\tilde{K}$, hence $\mathbb{M}$, must be scalar, which forces $\lambda_1 = \lambda_2 = \lambda_3$. Where $H = 0$, $R = -H/2 = 0$ and $\det \mathbb{M} < 0$, so the gate has already excluded it (`test_302`).
- *$\widetilde{E}(2)$ and $E(1,1)$.* For $\lambda = (\lambda_1, \lambda_2, 0)$,

  $$R = -\frac{(\lambda_1 - \lambda_2)^2}{2}, \qquad \det \mathbb{M} = -\frac{(\lambda_1 - \lambda_2)^4(3\lambda_1 + \lambda_2)(\lambda_1 + 3\lambda_2)}{64}.$$

  On $\widetilde{E}(2)$, $\lambda_1, \lambda_2 > 0$, so every non-flat metric has $\det \mathbb{M} < 0$. Paper 3's Remark 3.2 said only that $\det \mathbb{M} \neq 0$ in general. On $E(1,1)$, the group of Sol with $\lambda_1 > 0 > \lambda_2$, $\det \mathbb{M} > 0$ exactly on the wedge $-3\lambda_1 < \lambda_2 < -\lambda_1/3$. That wedge contains the standard metric, and Theorem 4.5 excludes all of it. On the boundary lines $\mathbb{M}$ has rank two.
- *The non-unimodular types* (`test_215`). In the frame of Corollary 4.6, with

  $$P = 2a - n_2 + n_3, \quad Q = 2a + n_2 - n_3, \quad S = 4a^2 - 3n_2^2 + 2n_2n_3 + n_3^2, \quad T = 4a^2 + n_2^2 + 2n_2n_3 - 3n_3^2,$$

  one has $\det \mathbb{M} = -PQST/64$. At $n_2 = n_3$, $\det \mathbb{M} = -a^6 < 0$, so $\det \mathbb{M} > 0$ forces $n_2 \neq n_3$, and also $PQ \neq 0$. Every component of $\mathrm{Cod}(\mathrm{adj}\, \mathbb{M})$ carries the factor $n_2 - n_3$, and $C_{231} = -(n_2 - n_3)^2(n_2 + n_3)PQ/8$. So on $\det \mathbb{M} > 0$, $C_{231} = 0$ forces $n_3 = -n_2$. There $C_{122} = 4an_2^2(n_2^2 - a^2)$ and $\det \mathbb{M} = -(a^2 - n_2^2)^3$, so $\det \mathbb{M} > 0$ gives $a^2 < n_2^2$ and $C_{122} \neq 0$.

*Checks* (`test_346`, 22 checks).

- P1: the Lie-algebra engine, written from Koszul's formula, against Milnor's formula, and the Milnor rows of Table 1.
- P2: Example 4.9 in coordinates and in the Lie algebra, with both positive controls.
- P3: (4.1) on a convex and an indefinite radial graph, in 40-digit quadrature, and the identity (4.2).
- P4: Example 4.11 and the remark on scale.
- P5–P6: Remark 4.12, first two items.
- P7: the degenerate case of Corollary 4.6.
- P8: Remark 4.12, third item.
- P9: Theorem 4.5 on 500 random left-invariant metrics in exact rational arithmetic, in randomly rotated frames. 279 have $\det \mathbb{M} > 0$, and every one has $\mathrm{Cod}(\mathrm{adj}\, \mathbb{M}) \neq 0$ exactly; a near-round family shows the residual vanishing linearly.
- P10: Proposition 4.10's algebra and the round cylinder.

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## 5. Total absolute curvature

The functional

$$J(g) = \int_\Sigma \sqrt{\det \mathbb{M}}\, dV_g$$

is defined for every metric with $\det \mathbb{M} \geq 0$, and on any realization it equals the total absolute curvature $\int |\det K|\, dV$ (fact 1). It is scale invariant: $\det \mathbb{M}$ scales as $c^{-6}$ under $g \mapsto c^2 g$, and $dV$

as $c^3$. In Lorentzian signature its analogue, the Einstein volume of a negatively curved metric, is non-decreasing along the cross-curvature flow and stationary there exactly at the metrics that embed in Minkowski space (Bryan, Ivaki and Scheuer 2019). This section records what $J$ certifies on a closed manifold (§5.1), its second variation at the round sphere (§5.2), the conformal directions, on which that variation is silent (§5.3), and the conformal path of [MA, Proposition 6.3] (§5.4).

### 5.1 Signed against unsigned

**Proposition 5.1.** *Let $x : \Sigma \to E^4$ realize a closed metric $g$.*

1. *If $\Sigma$ is oriented, with unit normal $N$, then $\int \det K \, dV = 2\pi^2 \deg \nu$. This is topological: it does not change under a regular homotopy of $x$.*
2. *$J(g) = \int |\det K| \, dV \geq 2\pi^2$, with equality iff $x$ is an embedding onto a convex hypersurface.*
3. *If $g \in \mathcal{U}$, then $J(g) = 2\pi^2$.*

*Proof.* (1) is (2.1). (2) is fact 1 with Chern and Lashof's theorem, in the normalisation of §2.2. (3) By Theorem 3.5, $K$ is definite and the Gauss map is a diffeomorphism. So $\det K$ has one sign, and $J = |\int \det K| = 2\pi^2$. □

**Corollary 5.2 (what $J$ certifies).** *Let $\Sigma$ be closed, and let $\det \mathbb{M} \geq 0$ everywhere.*

1. *If $J(g) < 2\pi^2$, then $g$ is not realizable.*
2. *If $g \in \mathcal{U}$ and $J(g) \neq 2\pi^2$, then $g$ is not realizable.*

Paper 2 stated only (1). On $\mathcal{U}$ the test is two-sided, although no metric in $\mathcal{U}$ on $S^3$ with $J > 2\pi^2$ is known (Remark 5.9). On other manifolds $J$ is unbounded on $\mathcal{U}$: a compact quotient of Sol has $\det \mathbb{M}$ constant and positive, and $J$ multiplies by the degree of a finite cover.

**Example 5.3 (corrugation: signed against unsigned).** A hypersurface of revolution $(\rho(\sigma)\,\omega, z(\sigma))$, $\omega \in S^2$, with profile parametrised by arclength and turning angle $\psi$ ($\rho' = \cos\psi$, $z' = \sin\psi$), carries the metric $d\sigma^2 + \rho^2\hat{\gamma}$. Its principal curvatures are $\psi'$ (meridian) and $\sin\psi/\rho$ (twice). From the metric alone,

$$\mathbb{M} = \operatorname{diag}\Big(\frac{1-\rho'^2}{\rho^2}, -\frac{\rho''}{\rho}, -\frac{\rho''}{\rho}\Big), \qquad \sqrt{\det \mathbb{M}}\,\rho^2 = \sin^2\psi\,|\psi'|.$$

So the signed integral is $4\pi \int \sin^2\psi\,\psi'\,d\sigma$, which is $2\pi^2$ whenever $\psi$ runs from 0 to $\pi$, and $J$ is $4\pi \int \sin^2\psi\,|\psi'|\,d\sigma$. Take $\psi = \sigma + A\sin 2k\sigma$ on $[0,\pi]$, with $k \geq 2$ an integer. The profile closes for every amplitude, by the symmetry $\psi(\pi-\sigma) = \pi - \psi(\sigma)$, which also gives $\rho(\pi-\sigma) = \rho(\sigma)$. It stays off the axis in between whenever $|A| < 2/\pi$: since $|\cos\psi - \cos\sigma| \leq |A|$, $\rho(\sigma) \geq \sin\sigma - |A|\sigma > 0$ on $(0, \pi/2]$. It then gives an immersed $S^3$. Every amplitude used below has $|A| \leq 0.4$.

- The signed integral is $2\pi^2$ at every amplitude.
- $J = 2\pi^2$ exactly while $2k|A| \leq 1$, when $\psi$ is monotone and the hypersurface convex.
- Beyond that $J > 2\pi^2$: $J/2\pi^2 = 1.0494\ldots$ at $2kA = 6/5$, for $k = 2$ and $k = 3$ alike.

These realizations have $\det \mathbb{M} = 0$ on the spheres where $\psi' = 0$, so they lie outside $\mathcal{U}$, as Proposition 5.1(3) requires. In the other direction, $J$ cannot see shape within the convex class: every closed hypersurface of revolution with monotone $\psi$ has $J = 2\pi^2$.

**Example 5.4 (the Berger spheres again).** With the normalisation of Example 4.11, the round $(1,1,1)$ metric has radius 2 and volume $16\pi^2$, and the Berger metric has volume $16\pi^2 r$. So

$$J = 2\pi^2 r^3 \sqrt{4 - 3r^2}, \qquad 0 < r \leq 2/\sqrt{3},$$

whose only critical point is the maximum $r = 1$. Corollary 5.2 alone therefore excludes every non-round Berger metric in $\mathcal{U}$, independently of the Codazzi computation of Example 4.11.

*Checks* (`test_347`, 8 checks): the reduction, intrinsically from the metric and extrinsically from the embedding, for arbitrary profiles; the corrugated family at seven amplitudes, in 40-digit quadrature; Example 5.4; and scale invariance. The axis bound is sampled for $k \in \{2, 3, 5, 8\}$ up to $|A| = 0.63$ (`test_353` P3).

### 5.2 The second variation at the round sphere

Write

$$J(\gamma + th) = 2\pi^2 + t\,J'(h) + t^2\,Q(h) + O(t^3).$$

$Q$ is the Taylor coefficient, half the second derivative. The TT levels are labelled as in [MA, §2.6]: at level $n \geq 3$ the rough Laplacian has eigenvalue $\mu_n = n^2 - 3$. Every symmetric 2-tensor decomposes as $h = h^{TT} + u\gamma + L_Y\gamma$, with $h^{TT}$ transverse-traceless (York 1973). We write $h_n$ for the level-$n$ part of $h^{TT}$.

**Theorem 5.5.** *At the unit round metric $\gamma$ on $S^3$:*

1. $J'(h) = 0$ *for every* $h$*;*
2. $Q(h) = Q(h^{TT})$*, and*

$$Q(h^{TT}) = -\frac{1}{16}\sum_{n\geq 3}(\mu_n + 2)(\mu_n + 3)\,\|h_n\|^2 = -\frac{1}{16}\langle h^{TT}, (\Delta - 2)(\Delta - 3)h^{TT}\rangle,$$

   *with* $\Delta = -\nabla^*\nabla$*;*
3. *so* $Q \leq 0$*, and* $Q(h) = 0$ *iff* $h^{TT} = 0$*. The radical of* $Q$ *is the sum of the conformal and the Lie directions, which is* $\ker DF_\gamma$ *[MA, Theorem E(1)]. The sum is not direct: for a first harmonic* $f$*,* $L_{\nabla f}\gamma = -2f\gamma$*.*

*Proof.* (1) $J$ is invariant under the isometries of $\gamma$, so $J'$ is an $SO(4)$-invariant linear functional. It vanishes on every non-trivial irreducible summand of the symmetric 2-tensors, and the only trivial summand is spanned by $\gamma$. Scale invariance gives $J'(\gamma) = 0$.

(2) Let $B$ be the symmetric bilinear form of $Q$. It is $SO(4)$-invariant. Write $SO(4) = (SU(2) \times SU(2))/\{\pm 1\}$, and the irreducible summands as $(j_1, j_2)$.

- The level-$n$ TT tensors form two irreducible summands, of types $(j, j+2)$ and $(j+2, j)$ with $n = 2j + 3$.
- The conformal and Lie directions are built from functions and vector fields. Their summands have types $(j, j)$ and $(j, j \pm 1)$.

No TT type occurs among the latter, so by Schur's lemma $B(h^{TT}, u\gamma + L_Y\gamma) = 0$. On the remaining part:

- $Q(u\gamma) = 0$ by Proposition 5.7(2) below.
- $B(h, L_Y\gamma) = 0$ for every $h$. Diffeomorphism invariance gives $J'_g(L_Y g) = 0$ for every $g$. Differentiating at $\gamma$ along $h$ gives $2B(h, L_Y\gamma) + J'(L_Y h) = 0$, and $J' = 0$ by (1).

So $Q(u\gamma + L_Y\gamma) = 0$, and $Q(h) = Q(h^{TT})$.

It remains to compute $Q$ on a TT tensor $h$. Write $\mathbb{M}(\gamma + th) = \mathrm{Id} + t\mathbb{M}^{(1)} + t^2\mathbb{M}^{(2)} + O(t^3)$ and $R(\gamma + th) = 6 + tR^{(1)} + t^2R^{(2)} + O(t^3)$. Here $R^{(1)} = -\Delta \operatorname{tr} h + \operatorname{div}\operatorname{div} h - \langle \mathrm{Ric}, h\rangle = 0$, and

$\operatorname{tr}\mathbb{M} = R/2$, so $\mathbb{M}^{(1)}$ is traceless. Expanding the determinant, with $dV = (1 - \frac{1}{4}t^2|h|^2 + O(t^3))\,dV_\gamma$,

$$\sqrt{\det \mathbb{M}}\,dV = \Big(1 + \tfrac{t^2}{4}\big(R^{(2)} - |\mathbb{M}^{(1)}|^2 - |h|^2\big) + O(t^3)\Big)\,dV_\gamma,$$

so $Q(h) = \frac{1}{4}\big(\int R^{(2)}\,dV_\gamma - \|\mathbb{M}^{(1)}\|^2 - \|h\|^2\big)$. Two standard computations on the unit sphere finish it.

- *First order.* For TT $h$ the linearised Ricci tensor is $\frac{1}{2}\Delta_L h = \frac{1}{2}\nabla^*\nabla h + 3h$ (Besse 1987, Chapter 1). Raising an index subtracts $h \circ \mathrm{Ric} = 2h$, and $R^{(1)} = 0$, so $\mathbb{M}^{(1)} = -\frac{1}{2}(\nabla^*\nabla + 2)h$.
- *Second order.* Along $\gamma + th$, $\frac{d}{dt}\int R\,dV = -\int\langle G, h\rangle\,dV$. Differentiating again at $\gamma$, where $G = -\gamma$ and $G^{(1)} = \frac{1}{2}\nabla^*\nabla h$, gives $\frac{d^2}{dt^2}\int R\,dV = -\frac{1}{2}\langle h, (\nabla^*\nabla + 4)h\rangle$. Half of this is the $t^2$-coefficient of $\int R\,dV$, which is $\int R^{(2)}\,dV_\gamma - \frac{3}{2}\|h\|^2$ since $R = 6$ at $\gamma$. So $\int R^{(2)}\,dV_\gamma = -\frac{1}{4}\langle h, (\nabla^*\nabla - 2)h\rangle$.

Hence

$$Q(h) = -\tfrac{1}{16}\big\langle h, \big[(\nabla^*\nabla - 2) + (\nabla^*\nabla + 2)^2 + 4\big]h\big\rangle = -\tfrac{1}{16}\langle h, (\nabla^*\nabla + 2)(\nabla^*\nabla + 3)h\rangle,$$

which is (2), since $\nabla^*\nabla h_n = \mu_n h_n$. As a check, direct evaluation of $J$ along $\gamma + th_n$, for the explicit tensors below, gives $Q(h_n)/\|h_n\|^2 = -\frac{9}{2}, -15, -\frac{75}{2}, -\frac{315}{4}, -147$ at $n = 3, \dots, 7$, and the mirror tensor at $n = 4$ agrees.

(3) $\mu_n + 2 > 0$ for every $n$. □

*The explicit tensors.* $S^3$ is the unit quaternions. The fields $X_a(x) = x\,e_a$ ($e_a = i, j, k$) are left-invariant, orthonormal, and satisfy $[X_a, X_b] = 2\epsilon_{abc}X_c$. Let $\theta^a$ be the dual coframe, $z_1 = x_0 + ix_1$ and $z_2 = x_2 + ix_3$. Then

$$h = \mathrm{Re}\big(z_1^a\,\bar{z}_2^d\,(\theta^2 - i\theta^3)^{\otimes 2}\big), \qquad a + d = n - 3,$$

is traceless, divergence-free, and satisfies $\nabla^*\nabla h = (n^2 - 3)h$. It is a highest-weight vector of the summand $(j, j + 2)$, $j = (n - 3)/2$. All of this is checked exactly. Since the $X_a$ are linear vector fields on $\mathbb{R}^4$, the whole curvature computation along $\gamma + th$ stays polynomial. Its integral over $S^3$ is exact, monomial by monomial.

**Remark 5.6 (the earlier derivations).** Paper 2 reduced $Q$ on TT to the two ingredients used in the proof,

$$\mathbb{M}^{(1)} = -\tfrac{1}{2}(\mu_n + 2)\,h, \qquad \int R^{(2)}\,dV_\gamma = -\tfrac{1}{4}(\mu_n - 2)\,\|h\|^2$$

at level $n$, and stated the result. The evaluation on eigenmodes was not displayed. Both ingredients are also checked by direct computation at every level above.

- The audit derived the same form from the expansion of $J$ near $\gamma$ (P2-X03). It also proved the radical statement (3) through the mixed sectors (P2-X04).
- One statement remains unclaimed: that $\gamma$ is a strict local maximum of $J$ transverse to the realizable set. It needs the local manifold structure of [MA, Remark 6.4].

### 5.3 The conformal sector

Theorem 5.5 says nothing along conformal directions. Each of them is tangent to a realizable path, the radial lift [MA, Proposition 6.3]. On them $J$ has a closed form.

**Proposition 5.7.**

1. *For every smooth $u$ with $\det \mathbb{M}(e^{2u}\gamma) \geq 0$, in particular every $u$ close enough to 0 in $C^2$,*

$$J(e^{2u}\gamma) = \int_{S^3} \sqrt{\det\left[(1-\Delta u)\,\mathrm{Id} + \nabla^2 u - du \otimes du\right]}\, dV_\gamma,$$

*exactly, with all derivatives taken for $\gamma$.*

2. *On a ray $u = tf$, write $H = \nabla^2 f$, $p = \nabla f$, $q = \Delta f$. The integrand has $t$-coefficients*

$$c_1 = -q, \qquad c_2 = \tfrac{1}{2}(\sigma_2(H) - |p|^2), \qquad c_3 = \tfrac{1}{2}(\det H + H(p,p)),$$

*and each integrates to zero, for every $f$. So $J(e^{2tf}\gamma) = 2\pi^2 + O(t^4)$. The next coefficient is*

$$c_4 = \tfrac{1}{2}q(\det H + H(p,p)) - \tfrac{1}{8}(\sigma_2(H) + |p|^2)^2 - \tfrac{1}{2}|Hp|^2.$$

3. *The quartic coefficient of $J/2\pi^2$ is $\int c_4\, dV_\gamma/2\pi^2$. It is the first that can be nonzero, and it need not be: it vanishes for zonal $f$, which give metrics of revolution, realizable near $\gamma$, and on the non-zonal families of Proposition 5.12. It is never positive (Corollary 5.11), and it is negative in each case of the table:*

| $f$ | quartic coefficient of $J/2\pi^2$ |
|---|---|
| $x_0x_1$ | $-\frac{1}{24}$ |
| $x_0x_1x_2$ | $-\frac{3}{160}$ |
| $x_0x_1 + \frac{1}{2}x_0x_2$ | $-\frac{25}{384}$ |
| $x_0^3 + x_1^2x_3$ (not harmonic) | $-\frac{19}{40}$ |

*Proof.* (1) In dimension three $e^{2u}\mathbb{M}(e^{2u}\gamma) = (1-\Delta u)\mathrm{Id} + \nabla^2 u - du \otimes du$, by the conformal change of the Ricci tensor (e.g. Besse 1987, §1.J). So $\sqrt{\det \mathbb{M}}$ carries the factor $e^{-3u}$, which cancels against $dV_g = e^{3u} dV_\gamma$.

(2) The coefficients come from expanding the square root. $\int c_1 = 0$ by the divergence theorem.

- *Second order.* Bochner's formula on the unit $S^3$ is $\int|\nabla^2 f|^2 = \int(\Delta f)^2 - 2\int|\nabla f|^2$. It gives $\int \sigma_2(H) = \int|p|^2$, so $\int c_2 = 0$.
- *Third order.* On the unit sphere the Newton tensors of a Hessian satisfy $\operatorname{div} T_2(H) = -T_1(H)\,p$, with $T_1 = q\,\mathrm{Id} - H$ and $T_2 = \operatorname{adj} H$ (Reilly 1977, for space forms). Since $\operatorname{tr}(T_2 H) = 3\det H$,

$$3\int \det H = -\int \langle \operatorname{div} T_2, p\rangle = \int (q|p|^2 - H(p,p)).$$

Also $\int H(p,p) = \frac{1}{2}\int\langle\nabla|p|^2, p\rangle = -\frac{1}{2}\int q|p|^2$. So $\int \det H = \frac{1}{2}\int q|p|^2 = -\int H(p,p)$, and $\int c_3 = 0$. This is the audit's P2-X05.

(3) For zonal $f$ the metric $e^{2tf}\gamma$ is $d\sigma^2 + \rho(\sigma)^2\hat{\gamma}$ after reparametrisation. For small $t$, $|\rho'| \leq 1$ with $\psi$ monotone, so it is realized by a convex hypersurface of revolution as in Example 5.3, and $J \equiv 2\pi^2$. The values in the table are exact. For $f$ the restriction of a homogeneous polynomial $F$ of degree $d$, $p = \nabla F - dF\,x$, $H = P(D^2F)P - dF\,P$ with $P = I - xx^{\mathsf{T}}$, and $q = \operatorname{tr} H$, so $c_4$ is a polynomial in $x$. It is integrated with $\int_{S^3} x^\alpha\, dV = 2\prod_i \Gamma(\frac{\alpha_i+1}{2})/\Gamma(\frac{|\alpha|+4}{2})$ for even $\alpha$, and 0 otherwise. □

The quartic order is the first possible order of onset, not a guaranteed one. A first harmonic is zonal about its own axis, and its coefficient vanishes too.

### 5.4 The conformal path of [MA]

**Corollary 5.8.** *Let $f = x_0x_1|_{S^3}$ and $g_{\mathrm{conf}}(t) = e^{2tf}\gamma$. For small $|t|$, $g_{\mathrm{conf}}(t) \in \mathcal{U}$ and*

$$J(g_{\mathrm{conf}}(t)) = 2\pi^2 - \frac{\pi^2}{12}\,t^4 + O(t^6).$$

*So $g_{\mathrm{conf}}(t)$ is not realizable for small $t \neq 0$. Its tangent $2f\gamma$ lies in $\ker DF_\gamma$, and the radial path $g_{\mathrm{rad}}(t) = e^{2tf}(\gamma + t^2 df \otimes df)$, which has the same tangent, is the metric of the embedding $x \mapsto e^{tf(x)}x$ [MA, Proposition 6.3].*

*Proof.* Near $\gamma$, $\mathbb{M}$ is near Id, so $g_{\mathrm{conf}}(t) \in \mathcal{U}$. The quartic term is Proposition 5.7(3). Odd orders vanish because $x_0 \mapsto -x_0$ is an isometry carrying $f$ to $-f$. Corollary 5.2(2) applies. □

This is the global form promised in [MA, §6]. The straight line along a kernel direction leaves the realizable set at second order in $F$, and $J$ detects the departure at fourth order. The same expansion comes out of the full curvature computation along $g_{\mathrm{conf}}$, without using Proposition 5.7(1). Along $g_{\mathrm{rad}}$ that computation returns $J = 2\pi^2$ through $t^4$, as a realizable path must.

**Remark 5.9 (an open question).** Is $J \leq 2\pi^2$ on $\mathcal{U}$ over $S^3$? Realizable metrics in $\mathcal{U}$ attain the value $2\pi^2$ (Proposition 5.1). Four pieces of evidence point the same way:

- near $\gamma$, $Q \leq 0$, with radical along the conformal and Lie directions;
- every realizable metric in $\mathcal{U}$ is a critical point of $J$ (Proposition 5.10);
- on every conformal ray the quartic coefficient is $\leq 0$ (Corollary 5.11);
- on the Berger family, $J = 2\pi^2 r^3\sqrt{4 - 3r^2} \leq 2\pi^2$.

A positive answer, with equality only on the realizable set, would characterise the realizable metrics in $\mathcal{U}$ as the maximisers of $J$. Neither part is known. The Lorentzian analogue points the same way: there the Einstein volume increases along the cross-curvature flow until the metric embeds (Bryan, Ivaki and Scheuer 2019).

*Checks* (`test_348`, 11 checks). Theorem 5.5 at levels 3–7 and on the mirror tensor, by direct evaluation of $J$; the ingredients of Remark 5.6; the radical, with the mixed terms; Proposition 5.7, items (1)–(3); the Newton-tensor identity; and Corollary 5.8 through the full curvature computation, with the radial path as a realizable control. `test_353` checks $c_1, \dots, c_4$ for general $H$ and $p$, the table again through $c_4$ with zonal controls (P1–P2), and the algebra of the proof of Theorem 5.5(2) (P6).

### 5.5 Stationarity at realizable metrics

**Proposition 5.10.** *On $S^3$, $J$ is stationary at every realizable metric $g \in \mathcal{U}$: $J'_g(h) = 0$ for every symmetric 2-tensor $h$.*

*Proof.* Fix a global orthonormal frame, with coframe $\theta^a$ and spin connection $\Gamma$, and let $T_i = -\frac{i}{2}\sigma_i$. On $\mathcal{U}$ put $b = \tilde{K}^i{}_a\,\theta^a T_i$ and $A_\pm = \Gamma \pm b$, the chiral connections of [MA, §9]. The Gauss equation holds identically for $\tilde{K}$, so $R_\Gamma + b \wedge b = 0$ and $F_{A_\pm} = \pm d_\Gamma b$, which is the Codazzi tensor of $\tilde{K}$. With $\mathrm{cs}(A) = \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)$,

$$\mathrm{cs}(\Gamma + b) - \mathrm{cs}(\Gamma - b) = 4\,\mathrm{tr}\big(b \wedge R_\Gamma + \tfrac{1}{3}b \wedge b \wedge b\big) - 2\,d\,\mathrm{tr}(\Gamma \wedge b) = -\tfrac{8}{3}\,\mathrm{tr}(b \wedge b \wedge b) - 2\,d\,\mathrm{tr}(\Gamma \wedge b),$$

and $\mathrm{tr}(T_iT_jT_k) = -\frac{1}{4}\epsilon_{ijk}$ gives $\mathrm{tr}(b \wedge b \wedge b) = -\frac{3}{2}\det\tilde{K}\,dV$. Since $\det\tilde{K} = \sqrt{\det\mathbb{M}}$,

$$J(g) = \frac{1}{4}\int_{S^3}\left[\mathrm{cs}(A_+) - \mathrm{cs}(A_-)\right] \qquad \text{for every } g \in \mathcal{U}.$$

The first variation of a Chern–Simons integral is $\delta\int \mathrm{cs}(A) = 2\int \mathrm{tr}(\delta A \wedge F_A)$. At a realizable metric both $A_\pm$ are flat, so both variations vanish, whatever $\delta A_\pm$ the variation $h$ induces. □

So realizable metrics are critical points of $J$, transversally as well as along the realizable set. Whether they are maxima is Remark 5.9's question. The converse, that every critical point in $\mathcal{U}$ is realizable, is not claimed.

**Corollary 5.11 (the conformal quartic coefficient).** *For every smooth $f$ on $S^3$,*

$$J(e^{2tf}\gamma) = 2\pi^2 + t^4\,Q(df\otimes df) + O(t^5), \qquad Q(df\otimes df) = -\frac{1}{16}\sum_{n\geq 3} n^2(n^2-1)\,\big\|\Pi_n^{TT}(df\otimes df)\big\|^2 \leq 0,$$

*where $\Pi_n^{TT}$ is the projection onto level-$n$ TT tensors. It vanishes exactly when $df\otimes df$ has no TT part, that is, when $DF_\gamma(df\otimes df) = 0$ [MA, Theorem E(1)].*

*Proof.* The radial path $r(t) = e^{2tf}(\gamma + t^2 df\otimes df)$ is the metric of the embedding $x \mapsto e^{tf(x)}x$, strictly convex for small $|t|$ [MA, Proposition 6.3]. So $J(r(t)) = 2\pi^2$ and, by Proposition 5.10, $J'_{r(t)} = 0$. The conformal path differs from it by $c(t) - r(t) = -t^2 e^{2tf} df\otimes df$. Expanding $J$ about $r(t)$, the linear term vanishes and

$$J(c(t)) = 2\pi^2 + \tfrac{1}{2}J''_{r(t)}\big(c(t)-r(t), c(t)-r(t)\big) + O(t^6) = 2\pi^2 + t^4 Q(df\otimes df) + O(t^5),$$

since $Q$ is half the second derivative at $\gamma$ and $r(t) = \gamma + O(t)$. Theorem 5.5 with $(\mu_n+2)(\mu_n+3) = n^2(n^2-1)$ gives the sum. □

Proposition 5.7(3)'s table is a set of instances: for $f = x_0x_1$, $Q(df\otimes df)/2\pi^2 = -\frac{1}{24}$, computed directly, and 0 for the zonal $x_0^2$.

The proposition, the corollary and their proofs are an independent machine-assisted review's (Appendix V).

*Checks* (`test_355`). The Chern–Simons algebra: the trace normalisation, $\mathrm{tr}(b\wedge b\wedge b) = -\frac{3}{2}\det\tilde{K}\,dV$, the transgression identity and the variation formula, each symbolically (P1). $Q(df\otimes df)$ on the curvature engine of §5.2, against Proposition 5.7(3) (P2).

**Proposition 5.12 (non-zonal conformal factors with no TT part).** *Let $F$ be a $C^3$ function on a convex open neighbourhood $U$ of $S^3$ in $\mathbb{R}^4$ whose Hessian has rank at most one on $U$, and $f = F|_{S^3}$. Then $(df\otimes df)^{TT} = 0$, so $Q(df\otimes df) = 0$. The conformal factors so obtained include*

1. *$f = g(a\cdot x) + b\cdot x$, for any $g$ and any vectors $a$, $b$; for $b$ not parallel to $a$ these are not zonal, and for polynomial $g$ they are polynomials, for example $x_0^2 + x_1$;*
2. *$f = v\,\psi(a\cdot x/v)$ with $v = 1 + b\cdot x$, $|b| < 1$, for any $\psi$, for example $x_0^2/(1+bx_1)$; these are not polynomial and not of the first kind;*
3. *tangent developables: $F(x_0, x_1)$ whose graph is the tangent developable of a space curve over a plane curve that stays outside a disk of radius $R > 1$, for example the curve $(-3+\cos t, \sin t, t^2)$, for which $F(\gamma(t) + s\gamma'(t)) = t^2 + 2st$ with $\gamma(t) = (-3+\cos t, \sin t)$ and $s > 0$. Their rulings are neither concurrent nor parallel, so they are of neither kind above.*

*Proof.* Put $H = dF\otimes dF$ on $U$. Its linearised Riemann tensor about the flat metric is $\frac{1}{2}(\partial_j\partial_k H_{il} + \partial_i\partial_l H_{jk} - \partial_i\partial_k H_{jl} - \partial_j\partial_l H_{ik})$. Expanding, the terms with third derivatives of $F$ cancel in pairs and what remains is $-(F_{ik}F_{jl} - F_{il}F_{jk})$, a $2\times 2$ minor of the Hessian, which vanishes because the Hessian has rank at most one. $U$ is convex, so by Saint-Venant's compatibility theorem (Ciarlet 2005, Ch. 6) $H = \mathcal{L}_\xi\delta$ for a vector field $\xi$ on $U$. On the unit sphere the pullback of a Lie derivative of the

flat metric is $\iota^*\mathcal{L}_\xi\delta = \mathcal{L}_{\xi^T}\gamma + 2(\xi\cdot x)\gamma$, and $\iota^*H = df\otimes df$. So $df\otimes df$ lies in the sum of the Lie and conformal parts and has no TT part; Corollary 5.11 gives $Q=0$. The first family has Hessian $g''(a\cdot x)\,aa^{\mathsf{T}}$. In the second, $F$ is homogeneous of degree one in $(a\cdot x, v)$, so its Hessian has rank at most one wherever $v>0$, which holds on the convex set $\{|x|<R\}\cap\{v>0\}$ for $1<R<1/|b|$. In the third, $\nabla F$ is constant along each ruling, so the Hessian has rank at most one on the convex set $\{x_0^2+x_1^2<R^2\}\times\mathbb{R}^2$. A zonal function is invariant under the rotations fixing its axis, and none of the examples is, for any axis. □

By Hartman and Nirenberg (1959), a function on all of $\mathbb{R}^4$ whose Hessian has rank at most one everywhere is of the first kind, so the second and third families need only a neighbourhood of the sphere, not a global function. Among the conformal factors invariant under the rotations of the $(x_2,x_3)$-plane, the proposition's sufficient condition is also necessary.

**Proposition 5.13 (the $O(2)$-invariant sector).** *Let $f$ be invariant under the $O(2)$ acting on $(x_2,x_3)$, so that $f=F(x_0,x_1)|_{S^3}$ with $F$ smooth on a neighbourhood of the closed unit disk $D$. Then $(df\otimes df)^{TT}=0$ if and only if $\det D^2F=0$ on $D$, that is, if and only if the graph $z=F(x_0,x_1)$ over $D$ is developable.*

*Proof.* Suppose $df\otimes df = 2w\gamma+\mathcal{L}_X\gamma$. The map $(w,X)\mapsto 2w\gamma+\mathcal{L}_X\gamma$ is equivariant and $f$ and $\gamma$ are invariant, so $w$ and $X$ may be averaged over $O(2)$. Then $X=Y^T$ with $Y=A\,\partial_0+B\,\partial_1$, $A$ and $B$ functions of $(x_0,x_1)$: the rotations leave only components along $\partial_0$, $\partial_1$ and the angle $\partial_\phi$ of the $(x_2,x_3)$-plane, and the reflection $x_3\mapsto -x_3$ removes the last. By $\iota^*\mathcal{L}_Y\delta=\mathcal{L}_{Y^T}\gamma+2(Y\cdot x)\gamma$, and since $\mathcal{L}_Y\delta$ has only $dx_0$ and $dx_1$ components, the $\phi\phi$ component forces $w=x_0A+x_1B$, and what remains is

$$F_0^2=2A_0,\qquad F_0F_1=A_1+B_0,\qquad F_1^2=2B_1$$

on the open disk, where $(x_0,x_1,\phi)$ is a chart. Two-dimensional Saint-Venant compatibility gives $\partial_1^2(F_0^2)-2\partial_0\partial_1(F_0F_1)+\partial_0^2(F_1^2)=0$, and the left side is $-2\det D^2F$. Conversely, if $\det D^2F=0$ on the open disk, the same identity and Saint-Venant on the disk give $A$ and $B$, hence the decomposition over the disk, so $DF_\gamma(df\otimes df)=0$ there. $DF_\gamma$ is a differential operator, so it vanishes on all of $S^3$, and [MA, Theorem E(1)] gives $(df\otimes df)^{TT}=0$. A graph is developable exactly when its Gauss curvature $\det D^2F/(1+|\nabla F|^2)^2$ vanishes. □

So in this sector the solutions are the developable graphs over the disk. Away from planar regions such a graph is ruled, with $\nabla F$ constant along each ruling (Hartman and Nirenberg 1959; Massey 1962), and locally it is a cylinder, a cone or a tangent developable: the first family of Proposition 5.12 with $a$ in the $(x_0,x_1)$-plane, the second, and the third. Whether every $f$ with $(df\otimes df)^{TT}=0$, outside this sector, is the restriction of a function with Hessian of rank at most one near $S^3$ is open (§9, problem 7). Proposition 5.14 turns the condition into a local equation for $f$.

The weight $v$ in the second family matters. The Möbius images $\psi(a\cdot x/v)$ of zonal functions, whose level sets are the non-concentric round 2-spheres, are not solutions: for $x_0/v$, $(x_0/v)^2$ and $(x_0/v)^3$ with $v=1+0.3x_1$, $Q(df\otimes df)/2\pi^2\approx -5.2,-3.7,-4.8\times10^{-4}$, while $v(x_0/v)^2$ gives zero. The Cotton tensor is natural and, in dimension three, conformally invariant, so for a conformal map with $\phi^*\gamma=\Omega^2\gamma$, $DC_\gamma(\Omega^{-2}\phi^*h)=\phi^*DC_\gamma(h)$: the kernel of $DF_\gamma=DC_\gamma$ [MA, Remark 6.1] is invariant under $h\mapsto\Omega^{-2}\phi^*h$, not under $h\mapsto\phi^*h$. And $\Omega^{-2}\phi^*(df\otimes df)=\omega\otimes\omega$ with $\omega=\Omega^{-1}d(f\circ\phi)$, which is of the form $dk\otimes dk$ only where $\omega$ is closed.

*Checks* (`test_624`). The rank condition and the vanishing of the linearised Riemann tensor of $dF\otimes dF$, symbolically, for three $\psi$ (P1–P2). $Q$ by quadrature on $S^3$ for three non-zonal $f$ of the second family and two of the first, with test_355's controls (P3, P5). Non-zonality about every

axis (P4, P5). The Möbius images are not solutions (P6). (`test_628`.) Proposition 5.13: the Saint-Venant identity and the reduction to the disk, symbolically (P1–P2); $Q < 0$ for three invariant $F$ with $\det D^2F \neq 0$ (P3); the tangent developable of Proposition 5.12(3), with $\det D^2F = 0$ at 30 digits, $Q = 0$ and non-zonality (P4).

The condition $DF_\gamma(df \otimes df) = 0$ looks like a fourth-order equation for $f$: $DF_\gamma$ has order three. The fourth derivatives cancel. On the unit round $S^3$ put

$$L_f = \nabla^2 f + f\gamma,$$

read it as an endomorphism with $\gamma$, write $\sigma_2(L_f)$ for its second elementary symmetric function and $\operatorname{adj}_\gamma L_f = \gamma \cdot \operatorname{adj}(\gamma^{-1} L_f)$, and put

$$S_f = -\operatorname{adj}_\gamma L_f + \tfrac{1}{2}\,\sigma_2(L_f)\,\gamma.$$

**Proposition 5.14 (the conformal condition is a Codazzi condition).**

1. *$DF_\gamma(df \otimes df) = \operatorname{Cod}_\gamma(S_f)$ for every smooth $f$ on $S^3$.*
2. *So $(df \otimes df)^{TT} = 0$ iff $S_f$ is a Codazzi tensor, iff $S_f = \nabla^2\varphi + \varphi\gamma$ for a smooth $\varphi$ on $S^3$; then $\frac{1}{2}\sigma_2(L_f) = (\Delta + 3)\varphi$. For the zonal $x_0^2$, $\varphi = -x_0^4/6$.*
3. *$S_f = 0$ iff $\operatorname{rank} L_f \leq 1$. The solutions of Proposition 5.12 are not of that kind. For $f = F|_{S^3}$, $L_f = D^2F|_{TS^3} + (F - x \cdot DF)\gamma$. In the second family $F - x \cdot DF = \partial_v F$, so $L_f$ is a rank-one tensor plus a multiple of $\gamma$, and for $x_0^2/(1 + bx_1)$ the multiple $-x_0^2/v^2$ is not zero.*

*Proof.* (1) First an algebraic fact. For a $3 \times 3$ endomorphism $E$ put $\mathbb{M} = (1 + \operatorname{tr} E)\operatorname{Id} - E$. Expanding $\operatorname{adj}\mathbb{M}$ and $(\det\mathbb{M})^{-1/2}$ to second order,

$$\sqrt{\det\mathbb{M}}\;\mathbb{M}^{-1} = \operatorname{Id} + E + \big(\operatorname{adj} E - \tfrac{1}{2}\sigma_2(E)\operatorname{Id}\big) + O(|E|^3).$$

In dimension three $\operatorname{Ric} = P + (\operatorname{tr} P)g$ for the Schouten tensor $P = \operatorname{Ric} - \frac{1}{4}Rg$, so $\mathbb{M} = -G = (\operatorname{tr} P)\operatorname{Id} - P$ as endomorphisms. Writing $P = \frac{1}{2}\operatorname{Id} + E$ puts $\mathbb{M}$ in the form above, so $\tilde{K} = P + \frac{1}{2}g + g\big(\operatorname{adj} E - \frac{1}{2}\sigma_2(E)\big) + O(|E|^3)$. On the conformal ray $c(t) = e^{2tf}\gamma$, the Schouten tensor is $P_c = \frac{1}{2}\gamma - t\nabla^2 f + O(t^2)$, so $E = -t\,\gamma^{-1}L_f + O(t^2)$ and

$$\tilde{K}(c) = P_c + \tfrac{1}{2}c + t^2\big(\operatorname{adj}_\gamma L_f - \tfrac{1}{2}\sigma_2(L_f)\,\gamma\big) + O(t^3).$$

$c(t)$ is conformally flat, so $\operatorname{Cod}_c(P_c)$, its Cotton tensor, is zero, and $\operatorname{Cod}_c(c) = 0$. Hence $F(c(t)) = t^2 \operatorname{Cod}_\gamma(\operatorname{adj}_\gamma L_f - \frac{1}{2}\sigma_2(L_f)\gamma) + O(t^3)$. The radial path $r(t) = e^{2tf}(\gamma + t^2 df \otimes df)$ is realizable [MA, Proposition 6.3], and $c(t) = r(t) - t^2 e^{2tf} df \otimes df$, so $F(c(t)) = -t^2 DF_\gamma(df \otimes df) + O(t^3)$. Compare the $t^2$ terms.

(2) The first equivalence is (1) with [MA, Theorem E(1)]. On a space of constant curvature $\kappa$ every Codazzi tensor is locally $\nabla^2\varphi + \kappa\varphi g$ (Ferus 1981). On $S^3$ the local potentials are unique up to the four first harmonics, the solutions of $\nabla^2\psi + \psi\gamma = 0$, and $S^3$ is simply connected, so they patch to a global $\varphi$. Take the trace: $\operatorname{tr} S_f = \frac{1}{2}\sigma_2(L_f)$. For $x_0^2$, $L_f = 2\,dx_0 \otimes dx_0 - x_0^2\gamma$, and $S_f = \frac{1}{2}x_0^4\gamma - 2x_0^2\,dx_0 \otimes dx_0 = \nabla^2\varphi + \varphi\gamma$.

(3) $\operatorname{tr} S_f = \frac{1}{2}\sigma_2(L_f)$, so $S_f = 0$ forces $\sigma_2 = 0$ and then $\operatorname{adj} L_f = 0$, that is, rank at most one; the converse is immediate. On the unit sphere $\nabla^2 f = D^2F|_{TS^3} - (x \cdot DF)\gamma$. In the second family $F = \Phi(a \cdot x, v)$ with $\Phi$ homogeneous of degree one, and Euler's relation gives $F - x \cdot DF = \partial_v\Phi$. □

So the open part of the conformal problem is a quadratic system of third order in $f$: which $f$ make $S_f$ a Codazzi tensor, or equivalently equal to $\nabla^2\varphi + \varphi\gamma$ with $(\Delta + 3)\varphi = \frac{1}{2}\sigma_2(L_f)$. The known solutions solve it with $S_f \neq 0$.

*Checks* (`test_639`). The expansion on the conformal ray, at two rational points for two $f$ (P2). (1), both sides computed independently, for $x_0x_1$, $x_0^2$, $x_0^2 + x_1$, $x_0^2/(1 + x_1/3)$ and a cubic; zero exactly for the three solutions, nonzero for the other two (P3). (3) for $x_0^2/(1 + x_1/3)$ (P4). Identically on $S^3$, in the left-invariant quaternion frame: $\mathrm{Cod}_\gamma S_f = 0$ for $x_0^2 + x_1$ and $x_0^2/(1 + x_1/3)$, not for $x_0x_1$, and the potential for $x_0^2$ (P5). Exact throughout.

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## 6. Embedded non-convex realizations, and the birth of folds

By Theorem 3.5 a closed realization with $\det \mathbb{M} > 0$ everywhere is convex. So a closed non-convex realization always has a degenerate locus, and its indefinite region is separated from every farthest point by it (Remark 3.9). Such realizations exist, and they are embedded. This section gives a general source of them (§6.1) and follows one family through the moment convexity is lost (§6.2). "Fold" here means a fold of the Gauss map. The hypersurface itself stays embedded throughout.

An indefinite radial graph need not turn back on itself. Its meridional curvature is the derivative of the polar angle only on the round sphere: for a radial graph $r(\chi)$, with $\chi$ the polar angle and $s$ arclength along the meridian, $d\chi/ds = 1/\sqrt{r^2 + r_\chi^2}$ and

$$k_{\mathrm{mer}} - \frac{d\chi}{ds} = -\frac{r^2\,(\ln r)_{\chi\chi}}{(r^2 + r_\chi^2)^{3/2}}$$

(`test_267`). The meridional curvature is the derivative of the turning angle $\psi$ of Example 5.3, $k_{\mathrm{mer}} = d\psi/ds$, and a turning angle that decreases does not make the profile cross itself.

### 6.1 Necks are where convexity fails

**Proposition 6.1.** *Let $x = (\rho(s)\,\omega, z(s))$, $\omega \in S^2$, be a closed embedded hypersurface of revolution in $E^4$, with profile parametrised by arclength $s$, $|\rho'| < 1$ and $z' = \sqrt{1 - \rho'^2}$ between the poles. Then:*

1. *the principal curvatures are $k_{\mathrm{mer}} = -\rho''/\sqrt{1 - \rho'^2}$ and $k_{\mathrm{sph}} = \sqrt{1 - \rho'^2}/\rho > 0$ (twice), and*

$$\mathbb{M} = \mathrm{diag}\Big(\frac{1 - \rho'^2}{\rho^2},\ -\frac{\rho''}{\rho},\ -\frac{\rho''}{\rho}\Big),$$

   *as Example 5.3 computed from the metric $ds^2 + \rho^2\hat{\gamma}$ alone;*
2. *$K$ is indefinite exactly where $\rho'' > 0$, degenerate exactly where $\rho'' = 0$, and definite where $\rho'' < 0$.*

*Conversely, an $SO(3)$-invariant metric $ds^2 + \rho(s)^2\hat{\gamma}$ on $S^3$ with $|\rho'| < 1$ between the poles is realized in this way (given smoothness at the poles), since $z$ is then strictly monotone.*

*Proof.* A direct computation with the normal $N = (\sqrt{1 - \rho'^2}\,\omega, -\rho')$. Then (2) is (1), since $k_{\mathrm{sph}} > 0$. □

So the indefinite region of an embedded hypersurface of revolution is exactly where its orbit radius is strictly convex as a function of arclength: around a neck. Every closed hypersurface of revolution

whose orbit radius has an interior local minimum is an embedded, realizable, non-convex example, outside $\mathcal{U}$. Its degenerate locus is the spheres where $\rho'' = 0$. Peanut and dumbbell shapes are of this kind.

**Example 6.2 (the radial graphs $r = 1+\varepsilon\cos 2\chi$).** Let $x = r(\chi)\, n$ over $S^3$, with $\chi$ the polar angle. Put $A = \sqrt{r^2 + r'^2}$ and $z = \cos^2\chi$. For every $r(\chi)$ the metric $A^2 d\chi^2 + r^2 \sin^2\chi\, \hat{\gamma}$ has, intrinsically, $\mathbb{M} = \mathrm{diag}(k_{\mathrm{sph}}^2, k_{\mathrm{mer}}k_{\mathrm{sph}}, k_{\mathrm{mer}}k_{\mathrm{sph}})$ with

$$k_{\mathrm{mer}} = \frac{r^2 + 2r'^2 - rr''}{A^3}, \qquad k_{\mathrm{sph}} = \frac{r - r'\cot\chi}{rA}.$$

For $r = 1 + \varepsilon\cos 2\chi$:

1. $r \geq 1 - \varepsilon > 0$, so $x$ is an embedding for $0 \leq \varepsilon < 1$.
2. $r - r'\cot\chi = 1 - \varepsilon + 6\varepsilon z > 0$, so $k_{\mathrm{sph}} > 0$.
3. $r^2 + 2r'^2 - rr'' = N(z) := 1 - 6\varepsilon + 5\varepsilon^2 + 12\varepsilon(1+\varepsilon)z - 12\varepsilon^2 z^2$. This is concave in $z$, with $N(0) = (1-\varepsilon)(1-5\varepsilon)$ and $N(1) = (1+\varepsilon)(1+5\varepsilon)$.
4. So:
   - for $\varepsilon < 1/5$ the hypersurface is strictly convex;
   - at $\varepsilon = 1/5$, $N = \frac{12}{25} z(6-z)$: it is convex, with $K$ degenerate only on the equator;
   - for $1/5 < \varepsilon < 1$, $N$ has exactly one root $z_* \in (0,1)$. $K$ is indefinite on the equatorial band $\cos^2\chi < z_*$, degenerate on the two latitude spheres $\cos^2\chi = z_*$, and definite on the polar caps, which contain the farthest points (the poles, $r = 1+\varepsilon$).

   At $\varepsilon = 3/10$ the band is $|\chi - \pi/2| < 0.2795$.
5. The band is a neck, as Proposition 6.1 requires: there the orbit radius $r\sin\chi$ is strictly convex in arclength.
6. The family is a realizable path from the round sphere ($\varepsilon = 0$) through the degenerate locus. It stays in $\mathcal{U}$ and convex up to $\varepsilon = 1/5$. It leaves $\mathcal{U}$ there, through a degenerate equator, and is indefinite afterwards. Along it $J = 2\pi^2$ up to $\varepsilon = 1/5$ and $J > 2\pi^2$ beyond (§6.2). Section 7 uses this path.

*Checks* (`test_349`, 6 checks). Proposition 6.1 and the intrinsic form of Example 6.2 are checked symbolically, for arbitrary profiles. Items (1)–(4) are checked exactly. Item (5) is checked at 399 points, and item (6) along $\varepsilon = 3t/10$.

### 6.2 The birth of a fold of the Gauss map

On an $SO(3)$-symmetric hypersurface the Gauss map is determined by the angle $\beta$ between the normal and the axis. For a radial graph,

$$\beta = \chi - \arctan\frac{r'}{r}, \qquad \beta' = \frac{r^2 + 2r'^2 - rr''}{r^2 + r'^2} = k_{\mathrm{mer}}\frac{ds}{d\chi}, \qquad k_{\mathrm{sph}} = \frac{\sin\beta}{r\sin\chi},$$

so $\det K\, dV = 4\pi\sin^2\beta\, d\beta$ after integration over the orbit spheres. The map $\beta$ runs from 0 to $\pi$, so the degree is one. Where $\beta' < 0$ the Gauss map folds. Each normal in the image of that strip has three preimages: two with $\det K > 0$ and one with $\det K < 0$.

**Lemma 6.3.** *For a closed oriented immersion with* $\deg\nu = 1$*,*

$$J - 2\pi^2 = 2\int_{\{\det K<0\}} |\det K|\, dV = 2\int_{S^3} N_-(\nu)\, d\nu,$$

*where $N_-(\nu)$ is the number of preimages of $\nu$ at which* $\det K < 0$*.*

*Proof.* $\int|\det K| = \int \det K + 2\int(\det K)_-$, and $\int \det K = 2\pi^2$. The area formula gives the second form. $\square$

So above $2\pi^2$, $J$ counts the folded normals.

**Theorem 6.4 (the birth of a fold, for the family of Example 6.2).** *Put $\delta = \varepsilon - \frac{1}{5}$ and $x = \chi - \frac{\pi}{2}$.*

1. *(Normal form.)*

$$\beta(\tfrac{\pi}{2} + x) - \tfrac{\pi}{2} = a_1 x + a_3 x^3 + O(x^5), \qquad a_1 = \frac{1-5\varepsilon}{1-\varepsilon}, \qquad a_3 = \frac{8\varepsilon(6\varepsilon^2+\varepsilon+1)}{3(1-\varepsilon)^3},$$

   *so that $\beta - \frac{\pi}{2} = -\frac{25}{4}\delta x + \frac{3}{2}x^3$ to leading order. As a map of $(x,\delta)$, $(\beta,\delta)$ has a Whitney cusp at the origin: a pair of fold spheres is born on the equator at $\varepsilon = \frac{1}{5}$.*
2. *(Widths.) For $\delta > 0$ the folded strip is $|x| < x_0$, with $x_0 = \frac{5}{\sqrt{18}}\,\delta^{1/2}(1 + O(\delta))$. Its image, the belt of normals with three preimages, has angular width $\Delta\beta = \frac{125}{9\sqrt{2}}\,\delta^{3/2}(1 + O(\delta))$.*
3. *(The excess.) Exactly, for $\frac{1}{5} < \varepsilon < 1$,*

$$J - 2\pi^2 = 8\pi\big[G(\beta(a)) - G(\beta(\pi - a))\big], \qquad G(b) = \frac{b}{2} - \frac{\sin 2b}{4},$$

   *where $a = \frac{\pi}{2} - x_0$ is the fold point. Asymptotically,*

$$\frac{J}{2\pi^2} - 1 = \frac{500}{9\sqrt{2}\,\pi}\,\delta^{3/2}\,(1 + O(\delta)).$$

   *So along the family $J$ is $C^1$ but not $C^2$ at $\varepsilon = \frac{1}{5}$.*

*Proof.*

(1) The symmetry $r(\pi - \chi) = r(\chi)$ gives $\beta(\pi - \chi) = \pi - \beta(\chi)$, which removes the even powers. The coefficients come from Taylor expansion, and $a_1(\frac{1}{5} + \delta) = -\frac{25}{4}\delta - \frac{125}{16}\delta^2 + O(\delta^3)$. Whitney's conditions at the origin are $\partial_x\beta = \partial_x^2\beta = 0$, $\partial_x^3\beta = 9 \neq 0$ and $\partial_x\partial_\delta\beta = -\frac{25}{4} \neq 0$.

(2) The fold points solve $a_1 + 3a_3x^2 + O(x^4) = 0$, so $x_0^2 = \frac{25}{18}\delta(1 + O(\delta))$. Then $\Delta\beta = \beta(\frac{\pi}{2} - x_0) - \beta(\frac{\pi}{2} + x_0) = -\frac{4}{3}a_1x_0(1 + O(\delta))$.

(3) By Example 6.2(2), $k_{\text{sph}} > 0$, so $\sin\beta > 0$ and $\beta$ stays in $(0,\pi)$ on $(0,\pi)$. By Example 6.2(4), $\beta'$, which has the sign of $N$, is negative exactly on $(a, \pi - a)$. So $\beta$ increases on $[0,a]$ and on $[\pi - a, \pi]$ and decreases between. A normal in the belt $(\beta(\pi - a), \beta(a))$ therefore has exactly one preimage with $\det K < 0$, and every other normal has none. By Lemma 6.3, $J - 2\pi^2 = 2 \cdot 4\pi \int \sin^2 b\, db$ over the belt, which is the formula. The belt is centred on $\pi/2$, where $\sin^2 = 1 + O(\Delta\beta^2)$, so $J - 2\pi^2 = 8\pi\,\Delta\beta\,(1 + O(\delta))$. $\square$

At finite amplitude, $J/2\pi^2 = 1.1237308$, $1.3137532$ and $1.7261913$ at $\varepsilon = \frac{1}{4}, \frac{3}{10}, \frac{2}{5}$. The exact formula agrees with direct quadrature to $10^{-25}$.

**Remark 6.5 (scope).** Theorem 6.4 describes one cohomogeneity-one family, where convexity is lost along a whole sphere at once. In a generic one-parameter family convexity should first fail at a point. Proposition 6.6 gives the law there, under explicit nondegeneracy hypotheses. For the generic singularities of Gauss maps see Banchoff, Gaffney and McCrory (1982). The exponents $\frac{1}{2}$ and $\frac{3}{2}$ are those of any unfolding of a Whitney cusp. What the family adds are the constants and the exact formula for $J$.

*Checks* (`test_350`, 4 checks). The Gauss-map identities for arbitrary profiles, and the normal form and the cusp conditions exactly. The exact excess against quadrature, at three amplitudes. The preimage count of (3) at four amplitudes up to $\varepsilon = 9/10$ (`test_353` P4). The three scaling laws at $\delta = 10^{-3}, 10^{-4}, 10^{-5}$: each ratio is within $4\delta$ of 1, and its deviation falls tenfold per decade.

**Proposition 6.6 (an isolated first loss of convexity).** *Let $x_\delta$ be a smooth family of closed oriented realizations with $\deg \nu = 1$. Suppose that at $\delta = 0$ one principal curvature $\kappa_1$ vanishes at exactly one point $p$, that $\kappa_2(p), \kappa_3(p) > 0$, that all principal curvatures are positive elsewhere, and that near $p$, in orthonormal coordinates,*

$$\kappa_1(x, \delta) = -a\delta + \tfrac{1}{2} x^{\mathsf{T}} B x + O(|x|^3 + |\delta||x| + \delta^2), \qquad a > 0,\ B > 0.$$

*Then, as $\delta \to 0^+$,*

$$J - 2\pi^2 = \frac{32\sqrt{2}\,\pi}{15} \, \frac{\kappa_2(p)\kappa_3(p)\, a^{5/2}}{\sqrt{\det B}} \, \delta^{5/2} + o(\delta^{5/2}).$$

*Proof.* By Lemma 6.3, $J - 2\pi^2 = 2\int_{\{\kappa_1 < 0\}} (-\kappa_1)\kappa_2\kappa_3 \, dV$. For small $\delta > 0$ the region is an ellipsoid of scale $\delta^{1/2}$ about $p$. Put $x = \sqrt{2a\delta}\, B^{-1/2} y$. To leading order the integrand is $a\delta(1 - |y|^2)\,\kappa_2(p)\kappa_3(p)$ on $|y| < 1$, the volume element is $(2a\delta)^{3/2}(\det B)^{-1/2} dy$, and $\int_{|y|<1}(1 - |y|^2)\, dy = 8\pi/15$. $\square$

The exponent differs from Theorem 6.4's $\frac{3}{2}$ because there the fold region is a band about a whole sphere, shrinking in one direction, while here it shrinks in all three. A first loss along a $d$-dimensional set, with a nondegenerate normal Hessian, would give $\delta^{1+(3-d)/2}$. Whether the hypotheses of Proposition 6.6 are generic is open (§9). The proposition and its proof are an independent machine-assisted review's (Appendix V). The integral and the coefficient are checked symbolically (`test_355` P3).

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## 7. The convex branch: connectivity

By Theorem 3.6, the closed realizable metrics in $\mathcal{U}$ are exactly the induced metrics of boundaries of smooth strictly convex bodies, each unique up to a rigid motion. This section describes that set globally and shows it is contractible, in a stated space (§7.1). It then joins the non-convex examples of §6 to it, through the degenerate locus (§7.2).

### 7.1 Support functions

A smooth strictly convex body is determined by its support function $h$ on $S^3$. The boundary point with outer normal $\nu$ is

$$x_h(\nu) = h(\nu)\,\nu + \nabla h(\nu), \qquad dx_h = L_h := \nabla^2 h + h\gamma,$$

with $L_h > 0$. Conversely, every smooth $h$ with $L_h > 0$ is the support function of such a body (Schneider 2014, §2.5). In the chart given by the Gauss map, the induced metric and the second fundamental form are

$$g_h = \gamma(L_h\,\cdot, L_h\,\cdot), \qquad K = L_h^{-1}, \qquad \mathbb{M}(g_h) = \operatorname{adj} L_h^{-1} = \frac{L_h}{\det L_h}.$$

**Theorem 7.1.** *Let $\mathcal{C} = \{h \in C^\infty(S^3) : L_h > 0,\ \int_{S^3} h(\nu)\,\nu\, dV_\gamma = 0\}$, the support functions with Steiner point at the origin, with the $C^\infty$ topology. Let $\mathcal{M}_c$ be the set of smooth closed realizable metrics in $\mathcal{U}$, modulo isometry.*

1. *$\mathcal{C}$ is an open convex cone in the subspace of centred functions, and $O(4)$ acts on it linearly by $h \mapsto h \circ A$.*
2. *$h \mapsto [g_h]$ induces a bijection $\mathcal{C}/O(4) \to \mathcal{M}_c$. It is continuous into the smooth metrics on $S^3$ modulo diffeomorphisms.*
3. *The homotopy $(1-s)h + s$, $s \in [0,1]$, preserves $\mathcal{C}$ and commutes with $O(4)$. So it contracts $\mathcal{C}/O(4)$ to the unit round sphere, and every metric along it is realizable. With the topology carried over from $\mathcal{C}/O(4)$ by (2), $\mathcal{M}_c$ is contractible.*

*Proof.*

(1) $L_h$ and the centring condition are linear in $h$.

(2) By Theorem 3.5 every $g \in \mathcal{M}_c$ is the metric of a strictly convex hypersurface, smooth by §2.1, and its Gauss map is a diffeomorphism. Pulling back by its inverse gives $g_h$, where $h$ is the support function of the body translated to put its Steiner point at the origin. If $g_h$ and $g_{h'}$ are isometric, the realizations differ by a rigid motion (Theorem 3.5(3)). The centring removes the translation. A linear isometry $A$ carries $x_h$ to $x_{h'}$ and normals to normals, so $h' = h \circ A^{-1}$. Continuity: $g_h = \gamma(L_h\,\cdot, L_h\,\cdot)$ depends continuously on $h$ in $C^\infty$.

(3) $L_{(1-s)h+s} = (1-s)L_h + s\,\mathrm{Id} > 0$, and the constant function 1 is $O(4)$-invariant. □

Papers 4 and 5 said "the convex realizable set contracts to the round metric" without a space. Theorem 7.1 is the precise form. It does not claim that the bijection (2) is a homeomorphism onto $\mathcal{M}_c$ with the quotient topology from the metrics modulo diffeomorphisms. That needs the realization, and the inverse of its Gauss map, to depend continuously on $g$, which belongs with the local structure of §9 (problem 5). The scale is free; fixing the volume or the area restricts the contraction to a slice, with the same proof after rescaling.

**Proposition 7.2 (Huisken's flow).** *The volume-preserving mean curvature flow $\partial_t x = (\bar{H} - H)N$, from any smooth strictly convex hypersurface of $E^4$, exists for all time, stays strictly convex, and converges to a round sphere of the same enclosed volume (Huisken 1987). It is a second path in $\mathcal{M}_c$ to the round metric, through realizable metrics. In support-function form $\partial_t h = \bar{H} - H$, with $H = \operatorname{tr} L_h^{-1} = \sigma_1$, the unnormalised mean curvature ($3H_1$ in the notation of §4.1; §8 uses the same convention). At the round sphere of radius $R$, with $h = R + \varepsilon Y_\ell$ for a spherical harmonic of degree $\ell \geq 1$,*

$$\dot{\varepsilon} = -\frac{\ell(\ell+2) - 3}{R^2}\,\varepsilon.$$

*Every mode with $\ell \geq 2$ decays. The translations $\ell = 1$ are neutral, and $\ell = 0$ is removed by the volume constraint.*

*Proof.* The flow is Huisken's theorem. For the rates, $L = (R + \varepsilon Y)\mathrm{Id} + \varepsilon \nabla^2 Y$, so $\operatorname{tr} L^{-1} = 3/R - \varepsilon(3Y + \Delta Y)/R^2 + O(\varepsilon^2)$. With $\Delta Y = -\ell(\ell+2)Y$ this is

$$H = \frac{3}{R} + \frac{\ell(\ell+2) - 3}{R^2}\,\varepsilon Y + O(\varepsilon^2),$$

and $\bar{H} = 3/R + O(\varepsilon^2)$ for $\ell \geq 1$. So $\dot{\varepsilon}\,Y = \bar{H} - H = -[\ell(\ell+2) - 3]\,\varepsilon Y/R^2 + O(\varepsilon^2)$. □

**Remark 7.3 (an intrinsic flow does not serve).** The cross-curvature flow of Chow and Hamilton (2004) acts on the metric alone. It needs sectional curvature of one sign, and nothing makes it preserve realizability. Papers 4 and 5 gave its basin on the Bianchi IX metrics $\mathrm{diag}(1+e, 1, 1-e)$, in the frame of the unit sphere, as about 21%. Positivity of the sectional curvature, $\mathbb{M} > 0$, holds

exactly for $|e| < (\sqrt{2}-1)/2 = 0.2071\ldots$. None of these metrics except $e = 0$ is realizable (Corollary 4.6).

### 7.2 Through the degenerate locus

Theorem 7.1 says nothing about the non-convex realizations of §6, which lie outside $\mathcal{U}$. For one class they join the round sphere explicitly.

**Proposition 7.4 (the revolution class).** *Consider the closed embedded hypersurfaces of revolution $(\rho(z)\,\omega, z)$ whose profile is a graph over an interval $[a,b]$ of the axis and meets the axis transversally, in the sense below. Write $u = \rho^2$, as a function of $\zeta \in [-1,1]$ with $z = \frac{1}{2}(a+b) + \frac{1}{2}\zeta(b-a)$. The admissible data are*

$$a < b, \qquad u > 0 \text{ on } (-1,1), \qquad u(\pm 1) = 0, \qquad \mp u'(\pm 1) > 0,$$

*with $u \in C^\infty([-1,1])$, smooth up to the endpoints. They form a convex set, and each datum gives a smooth embedded hypersurface. So any two of these hypersurfaces are joined by a path of embedded hypersurfaces of revolution, and every metric along it is realizable. In particular each is joined to the round sphere ($u = 1-\zeta^2$), and the path from a non-convex one crosses the degenerate locus.*

*Proof.* The conditions are convex. Near a pole the hypersurface is $\{|y|^2 = u(\zeta)\}$, with $y \in \mathbb{R}^3$. Since $u' \neq 0$ there, $\zeta$ is a smooth function of $u$, hence of $|y|^2$, and the hypersurface is a smooth graph over $y$. Inside, the profile is a graph, so the hypersurface is embedded. □

The radial graphs of Example 6.2 belong to this class. With $c = \cos\chi$, the axial coordinate is $(1-\varepsilon+2\varepsilon c^2)c$, whose derivative in $c$ is $1-\varepsilon+6\varepsilon c^2 > 0$, and $u = (1-\varepsilon+2\varepsilon c^2)^2(1-c^2)$. Both are polynomials in $c$, so $u$ is a smooth function of $\zeta$ up to the poles, where $u = 0$ and $du/dc = \mp 2(1+\varepsilon)^2 \neq 0$. Along the path from the round sphere to $\varepsilon = 3/10$, the neck appears, and with it the indefinite band.

Not every smooth axial graph has such data: $|y|^4 + z^2 = 1$ is smooth, with flattened poles where $u = \sqrt{1-z^2}$ is not smooth at the ends (`test_355` P4).

Two questions remain open (§9): whether every smooth realizable metric on $S^3$ is joined to a round one through realizable metrics, and what the realizable set looks like near a degenerate realization. Meagreness [MA, Theorem D] is no obstacle to connectivity: a line in the plane is nowhere dense and connected.

*Checks* (`test_351`, 6 checks).

- The support-function identity $X_b(x_h) = \sum_a (L_h)_{ab} X_a$, as an exact polynomial identity.
- $\mathbb{M}(g_h) = L_h/\det L_h$, through the curvature engine of §5.
- The contraction: positivity, centring and equivariance.
- Huisken's rates at $\ell = 1,\ldots,4$, exactly.
- The threshold $(\sqrt{2}-1)/2$, exactly.
- The revolution path from the round sphere to Example 6.2.
- The poles of Example 6.2 in the $(\zeta, u)$ description (`test_353` P5).

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## 8. Characterisations of the round sphere

The earlier papers stated several characterisations of the round sphere, some without their hypotheses. Here they are collected with them. None of them forces a realization to be round; each says when

one is.

1. **Constant curvature invariants.** Theorem 4.2 and Corollary 4.3: a closed realizable metric in $\mathcal{U}$ with constant scalar curvature, or constant $\operatorname{tr}\tilde{K}$, or constant $\mathbb{M}$-eigenvalues, is round. These hypotheses are on the metric alone.
2. **Constant mean curvature.** A closed embedded hypersurface of $E^4$ with constant mean curvature is a round sphere (Alexandrov 1958). One such hypersurface suffices. The isoparametric hypothesis, constant mean curvature on every parallel hypersurface, is stronger, and it is local, as follows.

**Proposition 8.1 (parallel hypersurfaces).** *Let $x$ be a connected hypersurface of $E^4$ with $\det K \neq 0$ somewhere. If every parallel hypersurface $x + aN$, for $|a|$ small, has constant mean curvature, then $x$ is an open piece of a round sphere.*

*Proof.* The parallel hypersurface has principal curvatures $k_i/(1 + ak_i)$, so its mean curvature $H(a) = \sum k_i/(1 + ak_i)$ has $H(0) = p_1$, $H'(0) = -p_2$ and $H''(0) = 2p_3$, with $p_j = \sum k_i^j$. These are constant, so by Newton's identities $\sigma_1, \sigma_2, \sigma_3$ are constant. Then the principal curvatures are constant, and $\sigma_3 = \det K \neq 0$. Segre's theorem, in the form of Theorem 4.5, finishes. □

The same hypothesis arises from a nonconstant harmonic function $C = F(a)$ of the ambient space that is constant on each parallel hypersurface, provided $F' \neq 0$. Then $\Delta a = -F''/F'$, and $\Delta a$ is the mean curvature of the level sets.

3. **Alexandrov–Fenchel.** For a convex body $\Omega \subset E^4$ with smooth boundary, write $W_k = \int_{\partial\Omega} \sigma_k \, dV$. Then Steiner's formula is $\operatorname{Vol}(\Omega+\delta B) = \operatorname{Vol}(\Omega)+W_0\delta+W_1\delta^2/2+W_2\delta^3/3+W_3\delta^4/4$. In Schneider's normalisation, $\operatorname{Vol}(\Omega + \delta B) = \sum_j \binom{4}{j} V_j \delta^j$, so

$$W_0 = 4V_1, \qquad W_1 = 12V_2, \qquad W_2 = 12V_3, \qquad W_3 = 4V_4 = 2\pi^2.$$

   The Alexandrov–Fenchel inequality $V_3^2 \geq V_2V_4$ reads $W_2^2 \geq 3W_1W_3$. Equality holds for balls, and among convex bodies with $C^2$ boundary only for balls (Schneider 2014, §7.6). The sphere is characterised by the equality, not forced: every non-round strictly convex body has strict inequality, and is still a realization. On the capsule $B_r + [-t, t]v$, with $|v| = 1$, which is convex but not $C^2$, the deficit is $W_2^2 - 3W_1W_3 = 64\pi^2t^2$.

4. **Minkowski's inequality.** $W_1 \geq 3W_3^{1/3}W_0^{2/3}$, that is $\int H \, dV \geq 3(2\pi^2)^{1/3}|\Sigma|^{2/3}$, with equality for spheres. This is classical for convex bodies (Minkowski). Guan and Li (2009) extended it to mean-convex star-shaped hypersurfaces, and Brendle, Hung and Wang quote it in their introduction. It is not their theorem. On the capsule it is strict for $t > 0$.

5. **A curvature-integral bound.** For every closed immersed hypersurface of $E^4$,

$$\int_\Sigma |K|^3 \, dV \;\geq\; 6\sqrt{3}\,\pi^2,$$

   where $|K|^2 = \sum_i k_i^2$, with equality only for round spheres. By the arithmetic–geometric mean inequality $|K|^2 \geq 3|\det K|^{2/3}$, so $|K|^3 \geq 3\sqrt{3}|\det K|$, and Proposition 5.1(2) gives the bound. Equality forces umbilicity, so a round sphere. Alternatively, $|K|^2 \geq H^2/3$, and Chen's inequality $\int |H/3|^3 \, dV \geq 2\pi^2$ (Chen 1971) gives the same constant. Paper 3 asked whether the bound is recorded. It follows in one line from either classical inequality, and no novelty is claimed. Unlike items 3 and 4 it needs no convexity. On the radial graphs of Example 6.2 it is strict: 1.066 and 1.534 times the bound at $\varepsilon = 1/10$ (convex) and $3/10$ (not convex).

**Remark 8.2 (the chain).** Along any parallel family, Steiner's formula gives $dW_k/d\delta = (k+1)W_{k+1}$, since $\sigma_k(k_i/(1+\delta k_i))\prod_i(1+\delta k_i) = \sum_{j\geq k}\binom{j}{k}\delta^{j-k}\sigma_j$. On the round sphere of radius $a$, $W_k = \binom{3}{k}2\pi^2 a^{3-k}$. The top term $W_3 = 2\pi^2 \deg\nu$ is constant (Proposition 5.1(1)). Two facts concern the corrugated hypersurfaces of revolution of Example 5.3, $\psi = \sigma + A\sin 2k\sigma$ on $[0,\pi]$, whose profile has length $\pi$ for every $A$. The first is paper 3's Remark 5.3, now proved at every amplitude; the second corrects it.

- $\int \sigma_2\, dV = \frac{1}{2}\int R\, dV = 6\pi^2$ exactly, for every integer $k \geq 2$ and every $A$ for which the profile stays off the axis (all $|A| < 2/\pi$, Example 5.3), whether or not the hypersurface is convex. Integration by parts gives $\int \sigma_2\, dV = 4\pi\int_0^\pi (1+\cos^2\psi)\, d\sigma$. By the Jacobi–Anger expansion, $\int_0^\pi \cos(2\sigma + 2A\sin 2k\sigma)\, d\sigma$ is a sum of terms $J_m(2A)\int_0^\pi \cos 2(1+km)\sigma\, d\sigma$, and each vanishes because $1 + km \neq 0$.
- $W_1 = 6\pi^2 - 4\pi^2 k^2(4k^2-3)A^2/(4k^2-1)^2 + O(A^3)$, checked exactly at $k = 2, 3$. The response is negative; paper 3 called it an excess.

*Checks* (`test_352`, 6 checks):

- the parallel and Newton identities;
- Steiner's identity;
- the normalisation and the capsule;
- Minkowski on the ball and on the capsule;
- the bound of item 5: its inequality on $10^4$ random triples, and the radial graphs;
- Remark 8.2 at $k = 2, 3$ and three amplitudes, with the $A^2$ coefficient exact.

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## 9. Open problems

1. **Partial degeneracy.** Closed manifolds whose metrics have $\det \mathbb{M}$ vanishing on a proper nonempty subset. Neither $F$ nor Theorem 3.2 decides them. Every closed non-convex realization is of this kind (Remark 3.9).
2. **$J$ as a variational characterisation.** Is $J \leq 2\pi^2$ on $\mathcal{U}$ over $S^3$, with equality only at realizable metrics (Remark 5.9)? Realizable metrics are critical points (Proposition 5.10); are there others?
3. **Connectivity of the whole realizable set.** Is every smooth realizable metric on $S^3$ joined to a round one through smooth realizable metrics on $S^3$, and can the realizations be chosen to vary continuously? Theorem 7.1 answers this on the convex branch and Proposition 7.4 for its revolution class. Other closed manifolds carry realizable metrics too: a tube about a circle in $E^4$ is an $S^2 \times S^1$ (`test_355` P4).
4. **The generic birth of folds.** Proposition 6.6 gives $\delta^{5/2}$ when convexity is first lost at an isolated nondegenerate point. Are its hypotheses generic among one-parameter families of closed realizations, and what happens at degenerate onsets?
5. **Local structure.** A manifold chart for the realizable set modulo diffeomorphisms near the round sphere [MA, open problem 1], and a description of it near a degenerate realization.
6. **Quantitative versions.** If a closed realizable metric in $\mathcal{U}$ has nearly constant scalar curvature, is it close to round? If $F(g)$ is small, is $g$ close to a realizable metric (compare Kupferman, Maor and Shachar 2019)?
7. **The conformal sector.** A natural functional that is nondegenerate at second order on the conformal directions, which $J$ is not (Proposition 5.7). The sign of $J$'s quartic coefficient is settled (Corollary 5.11), and Proposition 5.12 gives non-zonal conformal factors with

$(df \otimes df)^{TT} = 0$: every restriction of a function whose Hessian has rank at most one near $S^3$. Among the factors invariant under the rotations of a 2-plane these are all of them (Proposition 5.13: the developable graphs). Whether they are all of them in general is the question. By Proposition 5.14 it is a local one: for which $f$ is $S_f = -\mathrm{adj}_\gamma L_f + \frac{1}{2}\sigma_2(L_f)\gamma$ a Codazzi tensor?

8. **Measure.** Is the realizable set null, as well as meagre, for a natural measure on metrics?

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## Appendix V. Verification index and disclosure

Every displayed result is reproduced by at least one of the scripts deposited with this paper. These are tests 345–353, 355, 624 and 628, and a run log. Each script states its expectations before computing and exits non-zero on any failure. Where a check had to be re-posed after its first run, the script records the first run, what failed and why.

Arithmetic is exact (rational, symbolic, or exact algebraic) except where the index says otherwise. The exceptions are quadratures at 30–40 digits and a few sampled positivity checks. The numbering continues that of the author's archive (concept DOI 10.5281/zenodo.21264118). Earlier archive scripts that covered the same results are superseded for this paper; the bundle's README lists them.

| result | where | verification |
|---|---|---|
| Lemma 3.1; Theorem 3.2 | §3 | `test_345` P1 (exact), P2 (40 digits, three non-convex examples) |
| Corollary 3.3 | §3 | `test_345` P3–P4 |
| Theorem 3.5(1), along a family; Remark 3.9 | §3 | `test_345` P5 |
| Theorems 3.5(2)–(4), 3.6; Corollary 3.7 | §3 | proofs from classical results |
| Lemma 4.1 | §4 | `test_346` P3 (40-digit quadrature; the identity (4.2) symbolic) |
| Theorem 4.2; Corollary 4.3; Theorem 4.5; Corollary 4.7 | §4 | proofs; Theorem 4.5 sampled on 500 exact left-invariant metrics, `test_346` P9 |
| Corollary 4.6 | §4 | `test_346` P7 (the degenerate cases), P9 |
| Table 1, the Milnor rows | §4 | `test_346` P1 |
| Example 4.9 | §4 | `test_346` P2, with positive controls on $S^3$ and the Flamm metric |
| The remark on scale; Example 4.11 | §4 | `test_346` P4 |
| Proposition 4.10 | §4 | `test_346` P10 |
| Remark 4.12 | §4 | `test_346` P5, P6, P8 |
| Proposition 5.1; Corollary 5.2 | §5 | proofs |
| Example 5.3 | §5 | `test_347` P1 (symbolic), P2 (40 digits); the axis bound, `test_353` P3 (sampled) |
| Example 5.4; scale invariance | §5 | `test_347` P3 |
| Theorem 5.5; the explicit tensors | §5 | `test_348` P1–P3 (exact, levels 3–7 and a mirror); the algebra of the proof, `test_353` P6 |
| Remark 5.6 | §5 | `test_348` P2 |
| Proposition 5.7 | §5 | `test_348` P4; $c_4$, and the table through it, `test_353` P1–P2 (exact) |

| result | where | verification |
|---|---|---|
| Corollary 5.8 | §5 | `test_348` P5 (through the full curvature computation) |
| Proposition 5.10; Corollary 5.11 | §5 | proof; the Chern–Simons algebra, `test_355` P1 (symbolic); $Q(df \otimes df)$, `test_355` P2 (exact) |
| Proposition 5.12 | §5 | proof; the rank and the vanishing linearised curvature, `test_624` P1–P2 (symbolic); $Q = 0$ by test_353's quartic coefficient for both families, with its controls, P3, P5; non-zonality, P4, P5; the Möbius images, P6; the third family, `test_628` P4 |
| Proposition 5.13 | §5 | proof; `test_628` P1–P2 (symbolic), P3–P4 |
| Proposition 5.14 | §5 | proof; `test_639` P2–P5 (exact; P5 identically on $S^3$) |
| Proposition 6.1; Example 6.2 | §6 | `test_349` P1–P5 (P4–P5 at 40 digits) |
| Lemma 6.3; Theorem 6.4 | §6 | `test_350` P1–P2 (exact), P3–P4 (40 digits); the preimage count, `test_353` P4 (sampled) |
| Proposition 6.6 | §6 | proof; the integral, `test_355` P3 (symbolic) |
| The scope examples of Remarks 3.4 and 3.9, Example 5.3, Proposition 7.4 and §9 | §§3–9 | `test_355` P4 |
| Theorem 7.1 | §7 | `test_351` P1–P3 |
| Proposition 7.2 | §7 | `test_351` P4 |
| Remark 7.3 | §7 | `test_351` P5 |
| Proposition 7.4 | §7 | `test_351` P6 (40 digits); the poles of Example 6.2, `test_353` P5 |
| Proposition 8.1 | §8 | `test_352` P1 |
| §8, items 3–5; Remark 8.2 | §8 | `test_352` P2–P6 |

**Disclosure.** Large-language-model systems (Anthropic's Claude) were used throughout, under the author's direction, for:

- derivation and derivation-checking;
- drafting;
- the literature searches behind the attributions;
- writing and running the verification scripts.

Every load-bearing claim is gated by the scripts above. An independent machine-assisted audit (October 2026) of the earlier mathematics papers preceded this draft. Results credited to it in the text are its own, and each is also checked by a script here. Independent machine-assisted reviews of this paper, by Google Deep Review and by ChatGPT (October 2026), were also used; the second supplied Proposition 5.10, Corollary 5.11 and Proposition 6.6. A review by Gemini (November 2026) supplied the second family of Proposition 5.12 and the equivalence of Proposition 5.13, and a second Gemini review proposed Proposition 5.14(1). All errors remain the author's.

## Appendix C. Changes from the deposited versions

This paper supersedes, for global questions, mathematics papers 2, 3 and 5 (Thrussell 2026b, c, e) and the global part of mathematics paper 4 (Thrussell 2026d). The statements below change.

Labels refer to the paper named, and the result here is given in brackets. Local statements of papers 2 and 4 are treated in [MA, Appendix C].

**Mathematics paper 2.**

1. The abstract's "six homogeneous geometries at once" was not substantiated: Corollary 4.2 named product, hyperbolic-product and flat classes. [Corollary 3.3; every compact locally homogeneous metric, Corollary 4.3.]
2. "$J < 2\pi^2$ certifies non-realizability." On closed metrics with $\det \mathbb{M} > 0$ everywhere, any $J \neq 2\pi^2$ certifies it. [Corollary 5.2.]
3. Theorem 5.2's $\delta^2 J$ is the Taylor coefficient, half the second derivative, and its TT levels are relabelled to [MA]'s convention, $n \geq 3$. Its evaluation on eigenmodes, not displayed there, is now a proof. The radical, which Remark 5.3 left open, is the conformal plus the Lie directions. [Theorem 5.5.]
4. Proposition 5.4's "the onset is quartic": the quartic term is the first that can be nonzero, and it vanishes for rotationally symmetric conformal factors. [Proposition 5.7(3).]
5. The compact statements are strengthened: $\det \mathbb{M} > 0$ everywhere on a closed realization forces $\mathbb{M} > 0$ and convexity. [Theorem 3.5.]

**Mathematics paper 3.**

1. Nil has $\det \mathbb{M} = -3/64$, as the deposited version's note of 22 October already says. [Table 1.]
2. For Sol, $16 \cosh 4z$ is the sum of the squares of coordinate components, not a norm. The invariant norm is 16. [Example 4.9.]
3. $H^2 \times \mathbb{R}$ is realizable on open sets, not as a complete hypersurface. [Proposition 4.10.]
4. Lemma 2.1 covers homotheties only, not the anisotropic changes between left-invariant metrics. [The remark on scale.]
5. Remark 5.3's "excess" of $W_1$ under corrugation is negative. Its statement about $\int \sigma_2$ holds at every amplitude. [Remark 8.2.]
6. Theorem 3.1's table is replaced by theorems. [Corollaries 4.3 and 4.6.] Open problem 2, the non-unimodular types, is settled [Corollary 4.6; Agaoka and Hashinaga 2017]. Open problem 3, the bound on $\int |K|^3$, follows in one line from classical inequalities. [§8, item 5.]

**Mathematics paper 4 (its global part).**

1. §4.1's "$\mathcal{R}_{\text{convex}}$ contracts to the round metric" needed a stated space. [Theorem 7.1.] The cross-curvature flow's basin on Bianchi IX ends exactly at $e = (\sqrt{2} - 1)/2$. [Remark 7.3.]
2. §4.2's global sufficiency on $S^3$ through Thomas is [MA, Theorem A(4)], with the monodromy condition for general $\Sigma$. [Theorem 3.6.]
3. §7.3: Sbrana and Cartan classify the deformable rank-two hypersurfaces, not the whole rank-two locus [MA, Corollary 8.5].

**Mathematics paper 5.**

1. §1's $J = 2\pi^2 \chi$: the integer is $\deg \nu$, which is $\chi$ of the enclosed region for an embedded boundary, not $\chi(\Sigma) = 0$. [§2.2.]
2. §2's five-row table is not the list of Thurston geometries. [Corollary 4.3; Table 1.]
3. §3's isoparametric route needs constant mean curvature on every parallel hypersurface, and $F' \neq 0$ for the harmonic function. It is not weaker than umbilicity in effect: one closed embedded hypersurface of constant mean curvature is already a sphere (Alexandrov). [Proposition 8.1; §8, item 2.]

4. §4: Alexandrov–Fenchel is stated in Schneider's normalisation, with its equality case. The equality characterises the sphere; it does not force a realization to be round. Minkowski's inequality is classical, not Brendle, Hung and Wang's theorem. §4.1's reading of the inequality is removed. [§8, items 3–4.]
5. §5's contraction needed a stated space. [Theorem 7.1; Remark 7.3.]
6. §6.1b's Proposition (the meridional curvature is the derivative of the polar angle) and its Corollary (an indefinite radial graph is not embedded) are false. The identity holds for the turning angle, which forces no crossing. Embedded indefinite radial graphs exist, and the sixteen-fold construction is withdrawn. [§6; Example 6.2.]
7. §6.2: Sbrana–Cartan, as for paper 4.
8. §6.3's "interface" is an observation about one family, not a theorem. [Remark 3.9; §6.]

Paper 4's and paper 5's statements that $F$ is overdetermined elliptic on the indefinite branch are corrected in [MA, Corollary 4.2].

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