

Local independent sets and colourings of triangle-free graphs

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Abstract

This AI-assisted draft constructs, for every sufficiently large D , a local rule of radius $O((\ln D)^2)$ that produces, on any triangle-free graph G of maximum degree at most D , a random independent set that includes each vertex with probability at least $(1 - o_D(1)) \ln D/D$. Further properties of the rule yield, for every fixed $\varepsilon > 0$ and all sufficiently large D , a randomised LOCAL algorithm that colours any such finite graph G with at most $(1 + \varepsilon)D/\ln D$ colours, with high probability, in $O_\varepsilon(\ln D) + (\ln \ln |V(G)|)^{O(1)}$ communication rounds. Applying known transference principles for Borel graphs, we obtain the same asymptotic bound on the number of colours for measurable and Baire-measurable colourings, and for Borel colourings of graphs with finite asymptotic separation index.

Contents

1	Introduction	2
2	Notation and preliminaries	5
3	The capped bucket process	5
4	The marginal estimate	8
5	Concentration for uncovered neighbours	12
6	Local constraints	14
7	The colouring algorithm	18
8	Borel and measurable colourings	22
9	Optimality of the constants	24

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1 Introduction

The study of independent sets in triangle-free graphs is closely related to the Ramsey number $R(3, k)$, the least integer n such that every triangle-free graph on n vertices has an independent set of size k . Following the work of Ajtai, Komlós and Szemerédi [AKS80], Shearer [She83, Theorem 1] proved that every triangle-free graph G on n vertices with average degree d satisfies, with $\alpha(G)$ denoting its independence number,

$$\alpha(G) \geq (1 - o(1)) \frac{n \ln d}{d} \quad \text{as } d \rightarrow \infty. \quad (1.1)$$

Since each neighbourhood is independent, this also gives $R(3, k) \leq (1 + o(1))k^2 / \ln k$. Kim’s construction [Kim95, Theorem 1.1] established a matching lower bound up to a constant factor. The lower coefficient was later improved to $1/4$ by Bohman and Keevash [BK21, Theorem 1.3] and, independently, Fiz Pontiveros, Griffiths and Morris [FGM20, Theorem 1.2], then to $1/3$ by Campos, Jenssen, Michelen and Sahasrabudhe [CJMS25, Theorem 1.1], and to $1/2$ by Hefty, Horn, King and Pfender [HHKP25, Theorem 1.2]. Thus the current bounds are

$$\left(\frac{1}{2} - o(1)\right) \frac{k^2}{\ln k} \leq R(3, k) \leq (1 + o(1)) \frac{k^2}{\ln k}. \quad (1.2)$$

For a short proof of the order of magnitude, see Harris, Hefty, Horn, King and Pfender [HHH⁺26, Theorem 1.2 and Corollary 5.2].

For colouring, Johansson proved that every triangle-free graph of maximum degree at most D has chromatic number $O(D / \ln D)$; see Molloy and Reed [MR02, Theorem 13.1] for a published proof. Molloy [Mol19, Theorem 1] strengthened this to $(1 + o(1))D / \ln D$, even for list colouring. Related lower bounds on the number of proper colourings at this scale were obtained by Bernshteyn, Brazelton, Cao and Kang [BBCK23, Theorem 1.2]. On the other hand, a theorem of Bollobás [Bol81, Theorem 1 and Corollary 3] gives finite triangle-free graphs of maximum degree D with chromatic number at least $(1/2 - o(1))D / \ln D$. The optimal leading coefficient for finite graphs is therefore still unknown. We obtain the coefficient 1 under additional requirements of locality or definability, and show that it is optimal in several of these settings.

The independent-set bound also has refinements concerning probability distributions. Davies, Jenssen, Perkins and Roberts [DJPR18, Theorem 1] showed that a uniformly random independent set in an n -vertex triangle-free graph of maximum degree at most D has expected size at least $(1 - o(1))n \ln D / D$. Lower bounds on the number of independent sets in terms of the average degree were obtained by Buys, van den Heuvel and Kang [BvdHK25, Theorem 1]. More recently, Martinsson and Steiner [MS25, Theorem 1.2] proved that every finite triangle-free graph has a distribution on independent sets for which each vertex v of positive degree is included with probability at least $(1 - o(1)) \ln d(v) / d(v)$, where the error tends to zero with $d(v)$. This resolves the local fractional-colouring conjecture of Kelly and Postle [KP24, Conjecture 2.2]. For a fixed maximum-degree bound D , we seek the uniform marginal $(1 - o(1)) \ln D / D$ under the additional requirement that each decision be determined by independent labels in a ball whose radius depends only on D .

Our first result produces such an independent set by a local rule. We give each vertex an independent uniform label from a finite alphabet Λ . Write $B_G(v, R)$ for the vertices at distance at most R from v . A rule has *determining radius* R if its decision at v is determined by the labelled graph induced on $B_G(v, R)$, with v distinguished as the root. The rule must be invariant under isomorphisms of labelled rooted neighbourhoods. In particular, it uses no vertex ordering or identifiers. The construction is based on the ordered independent-set process of Martinsson and Steiner [MS25, Section 4]. We adapt their forced-decline comparison and preservation of weights

in expectation [MS25, Claims 4.2 and 4.3], replacing the vertex ordering by finitely many random priority classes and capping the attempt probabilities. These changes give a bounded determining radius and pointwise domination by independent Bernoulli variables.

Theorem 1.1 (Local independent-set sampler). *For every sufficiently large integer D , put*

$$\ell := \lfloor \ln D \rfloor, \quad w := \frac{2\ell}{D}, \quad \eta := \ell^{-6}, \quad \text{and} \quad T := \ell^2.$$

There is a finite uniform label alphabet and a radius- $(T + 2)$ rule, depending only on D , such that the following hold for every triangle-free graph of maximum degree at most D .

1. *For every label assignment, the output I is independent.*
2. *Under independent uniform labels, every vertex v satisfies*

$$\mathbb{P}(v \in I) \geq \frac{\ln D - 7 \ln \ln D - 6}{D}. \quad (1.3)$$

3. *There are independent Bernoulli variables $K(v)$ of parameter η , each determined by the label at v , for which $\mathbf{1}_{\{v \in I\}} \leq K(v)$ for every label assignment.*

Each probability on an infinite graph is the probability of an event in a finite product of labels in a determining neighbourhood.

The independent cap in Part (3) is a pointwise coupling with a product measure. The output indicators themselves need not be independent. This additional control will be used in the colouring argument.

The second result concerns the *randomised LOCAL model*. In this model there is a processor at each vertex of a finite graph. In one synchronous round, every processor may send messages to each neighbour, receive its neighbours' messages, perform local computation, and generate independent private random bits. Message length and local computation are unrestricted. The cost is the number of communication rounds, until all vertices output their colours. The processors have distinct identifiers consisting of $O(\ln n)$ bits. They know n , the degree bound, and the requested number of colours. A radius- R rule can be executed in $O(R)$ rounds.

Theorem 1.2. *For every $\varepsilon \in (0, 1)$ there is Δ_0 such that the following holds. Let G be an n -vertex triangle-free graph of maximum degree $\Delta \geq \Delta_0$, and let $1 \leq k \leq (1 - \varepsilon) \ln \Delta$. There is a randomised LOCAL algorithm which, with probability at least $1 - n^{-1}$, properly colours G with at most $\lfloor \Delta/k \rfloor$ colours in*

$$O_\varepsilon(k) + \ln^{O(1)} \ln n \quad (1.4)$$

rounds. The exponent in the second term is absolute.

The earlier distributed work of Pettie and Su [PS15, Theorem 1] and Chung, Pettie and Su [CPS17, Theorem 9] was developed further by Davies-Peck and Zhang [DPZ26, Theorem 1]. The latter result, like Theorem 1.2, constructs a (Δ/k) -colouring using $O_\varepsilon(k) + \ln^{O(1)} \ln n$ rounds; however, their range of k is bounded above by $(1/4 - \varepsilon) \ln \Delta$. Davies-Peck and Zhang [DPZ26, Section 6.3] asked if $1/4$ can be increased to 1 here; thus Theorem 1.2 answers this question.

Theorem 1.2 also has consequences for definable colourings of infinite graphs. A graph is *Borel* if its vertex set is a standard Borel space and its adjacency relation is Borel. We write $\chi_B(G)$ for its Borel chromatic number, $\chi_B^f(G)$ for its fractional Borel chromatic number, and $\chi_\mu(G)$ for its

chromatic number with colour classes measurable with respect to a Borel probability measure μ . Given a Polish topology compatible with the Borel structure, we also consider *Baire-measurable* colourings, whose colour classes have the Baire property. The asymptotic separation index $\text{asi}(G)$ measures whether, at each finite distance scale, the vertices can be partitioned into a fixed finite number of Borel sets with finite components. Precise definitions are given in Section 8.

Theorem 1.3. *For every $\delta > 0$ there is D_0 such that the following holds for every integer $D \geq D_0$. Put*

$$q := \left\lfloor (1 + \delta) \frac{D}{\ln D} \right\rfloor. \quad (1.5)$$

If G is a triangle-free Borel graph of maximum degree at most D , then

1. $\chi_B^f(G) \leq q$;
2. $\chi_\mu(G) \leq q$ for every Borel probability measure μ ;
3. $\chi_B(G) \leq q$ if $\text{asi}(G) < \infty$;
4. for every Polish topology compatible with the Borel structure of $V(G)$, the graph G has a Baire-measurable proper q -colouring.

The threshold D_0 depends only on δ ; in particular it is independent of G , μ , the chosen topology and the finite value of $\text{asi}(G)$.

These bounds all have the form $(1 + o(1))D/\ln D$ as $D \rightarrow \infty$. For the fractional Borel chromatic number, the sampler gives the stronger explicit estimate

$$\chi_B^f(G) \leq \frac{D}{\ln D - 7 \ln \ln D - 6};$$

see Corollary 8.2. The measurable assertion also holds for every σ -finite Borel measure. The probability or σ -finiteness convention is necessary; Remark 8.3 explains the obstruction for arbitrary Borel measures.

The measurable and Baire-measurable assertions follow from Bernshteyn’s transfer theorem [Ber23, Theorem 2.14], and the Borel assertion under finite ASI follows from the transfer theorem of Bernshteyn and Weilacher [BW25, Theorem 1.42]. The measurable assertion answers Bernshteyn’s [Ber23, Open Problem 3.11]. Bernshteyn and Weilacher [BW25, Theorem 1.18] proved $\chi_B(G) \leq (4 + o(1))D/\ln D$ for triangle-free graphs with finite ASI and then asked whether the coefficient 4 could be replaced by 1. Theorem 1.3(3) answers this question affirmatively.

Section 9 proves that the leading coefficient 1 is optimal for the local independent-set marginal, for the number of colours of randomised LOCAL algorithms using $o(\ln n)$ rounds at fixed degree, and for the fractional Borel and measurable bounds on arbitrary triangle-free Borel graphs. These conclusions use the factor-of-i.i.d. obstruction of Rahman–Virág [RV17] and its Borel formulation due to Bernshteyn [Ber22] and in fact show that these bounds cannot be improved even for acyclic graphs. For the finite-ASI and Baire classes, the optimal leading coefficient remains undetermined; see Section 9.

Our distributed application uses the Local Lemma of Flin, Ghaffari, Halldórsson, Maus and Nolin [FGH⁺26, Theorem 1.1], together with fast degree-plus-one colouring [CLP20, RG20]. We use the corrected colouring analysis of Ghaffari, Halldórsson, Maus and Nolin [GHMN26]. The precise statements are given in Section 6. Davies’ resilient Local Lemma [Dav23] also suffices here, with a repair to its shattering argument; see Remark 6.2. Further algorithms for avoiding local bad events,

and for replacing random local choices by deterministic ones, are developed in [MT10, FG17, GHK18]. For each fixed Δ , one can also obtain an $o(\ln n)$ round bound for Theorem 1.2, as needed for the deduction of Theorem 1.3, using the earlier distributed Local Lemma of Fischer and Ghaffari [FG17], with the corrected analysis in [GHMN26].

2 Notation and preliminaries

All graphs are simple and undirected. For a graph G , write $N_G(v)$ for the neighbourhood of v , $B_G(v, R)$ for its ball of radius R , and $G[W]$ for the subgraph induced by $W \subseteq V(G)$. For $v \in V(G)$, we denote by $G - v$ the graph obtained by removing the vertex v from G . We may omit the subscript G when the graph is clear. The *power graph* G^r has the same vertex set $V(G)$ and joins distinct vertices at distance at most r in G .

We write $[q] := \{1, \dots, q\}$, $\ln$ for the natural logarithm, and $\mathbf{1}_E$ for the indicator of an event E . Probability and expectation are denoted by $\mathbb{P}$ and $\mathbb{E}$, respectively.

The parameter D is a degree bound for the sampler; in the colouring algorithm, Δ is the initial maximum degree and D_j is the degree bound at phase j . A subscript on O specifies the permitted dependence of its implicit constant. When we write $\ln^{O(1)} n$, the exponent is absolute; for small n this notation means $O((1 + \ln \ln(\max\{n, 3\}))^c)$ for some absolute c .

All samplers used below have rational parameters. The exact encoding in Lemma 3.1 therefore allows their analysis on finite probability spaces. We use the usual notation for filtrations, conditional expectations and martingales. In particular, a process (M_i) adapted to $(\mathcal{F}_i)$ is a supermartingale if $\mathbb{E}[M_{i+1} \mid \mathcal{F}_i] \leq M_i$. We use the tower property and the fact that a factor measurable with respect to the conditioning sigma-algebra can be taken outside a conditional expectation. We will also use the elementary inequality

$$\mathbb{E}[\min\{X, \eta\} \mid \mathcal{F}] \leq \min\{\mathbb{E}[X \mid \mathcal{F}], \eta\} \quad (X \geq 0), \quad (2.1)$$

which follows by bounding the conditional expectation separately by $\mathbb{E}[X \mid \mathcal{F}]$ and by η .

3 The capped bucket process

3.1 Definition of the process

Fix a positive integer T , a degree bound D , and parameters $0 < w < \eta < 1$. Initially the parameters may be real; rationality will be imposed when a finite uniform alphabet is required. On a finite triangle-free graph G , independently at each vertex u , choose a bucket $b(u)$ uniformly from $[T]$ and an independent uniform random variable $U(u)$ on $[0, 1]$. These continuous variables are only a convenient description of threshold trials. For rational parameters Lemma 3.1 replaces them exactly by finite uniform coordinates, which can be used throughout all subsequent probability calculations. A vertex is *pending after Stage i* if $b(u) > i$. Set $W_0(u) := w$ at every vertex.

At *Stage i* , for $i = 1, \dots, T$, every vertex u in bucket i makes an *attempt* if

$$U(u) < p_{i-1}(u), \quad \text{where} \quad p_{i-1}(u) := \min\{W_{i-1}(u), \eta\}.$$

All these decisions use the weights from before the stage. In particular, vertices in the same bucket i do not change one another's attempt probabilities during Stage i . Conditional on all buckets and on earlier trials, the current trials are independent Bernoulli trials with the displayed parameters.

Write $a(u)$ for the indicator that u attempts at Stage $b(u)$. Update the weight of each vertex z that remains pending by

$$W_i(z) := W_{i-1}(z) \prod_{\substack{u \in N(z) \\ b(u)=i}} \frac{1 - a(u)}{1 - p_{i-1}(u)}. \quad (3.1)$$

Weights of vertices already processed are frozen; they will not be used for further attempts. The denominator in every factor is positive because $p_{i-1}(u) \leq \eta < 1$.

For a vertex z pending after Stage i , iterating the update gives

$$W_i(z) = w \prod_{\substack{u \in N_G(z) \\ b(u) \leq i}} \frac{1 - a(u)}{1 - p_{b(u)-1}(u)}. \quad (3.2)$$

The factor associated with u occurs once, at Stage $b(u)$. Multiplication of the successive stage products therefore gives exactly the displayed product. If a numerator is zero, the product is zero; otherwise all numerators are one. A neighbour in a future bucket has not yet contributed a factor.

Let $\mathcal{A} := \{u : a(u) = 1\}$ be the set of all attempts, and output

$$I := \{u \in \mathcal{A} : \text{there is no } z \in N(u) \cap \mathcal{A} \text{ with } b(z) = b(u)\}. \quad (3.3)$$

Thus both ends of an edge with simultaneous attempts are removed from the output. An attempt that is removed from the output *still* contributed its factor zero in (3.1) to later neighbours. Forgetting this distinction would define a different process, for which the proof below does not apply.

Lemma 3.1 (Independence, locality, and exact encoding). *The set I is independent for every label assignment. Its membership rule has determining radius at most $T + 2$. Throughout the process,*

$$0 \leq W_i(u) \leq w(1 - \eta)^{-D}. \quad (3.4)$$

If w and η are rational, the process has an exact finite uniform label encoding. In that encoding there are independent Bernoulli variables $K(u)$ of parameter η such that

$$\mathbf{1}_{\{u \in I\}} \leq a(u) \leq K(u) \quad \text{for every assignment of labels.} \quad (3.5)$$

Proof. If uz is an edge with $b(u) < b(z)$ and u attempts, then (3.1) sets the pending weight of z to zero. This weight remains zero, so z cannot attempt. If $b(u) = b(z)$ and both vertices attempt, (3.3) removes both. Thus I is independent.

The state at a vertex after Stage i is determined by its label and the states and labels of its neighbours before that stage. Induction therefore bounds its determining radius by i . The final deletion step inspects neighbouring attempt decisions, so radius $T + 1$ suffices. In particular, the answer at v is unchanged if the process is run on $G[B_G(v, T + 2)]$. This also justifies the rule on infinite graphs. Each neighbour contributes to a pending weight at most once, and every nonzero multiplier is at most $(1 - \eta)^{-1}$, proving (3.4).

For rational w, η , define finite sets of nonnegative rational numbers by

$$F_0 := \{0, w\},$$

$$F_{i+1} := F_i \cup \{0\} \cup \left\{ x \prod_{r=1}^k \frac{1}{1 - \min\{y_r, \eta\}} : x, y_1, \dots, y_k \in F_i, 0 \leq k \leq D \right\}.$$

All denominators are positive. Induction using (3.1) shows that every weight after Stage i belongs to F_i ; the terms F_i and 0 account for frozen and extinguished weights. Thus the possible attempt probabilities belong to the finite rational set

$$P := \{\min\{x, \eta\} : x \in F_0 \cup \dots \cup F_T\},$$

which depends only on D, T, w, η . Choose an integer N divisible by the denominators of the elements of $P \cup \{\eta\}$. Independently at each vertex, choose $J(u)$ uniformly from $[N]$ and replace the threshold trial by $J(u) \leq Np_{b(u)-1}(u)$. Since $J(u)$ is first consulted in u 's bucket, this trial has exactly the prescribed conditional probability. Hence the uniform label $(b(u), J(u)) \in [T] \times [N]$ gives the same law as the original process. Finally, $K(u) := \mathbf{1}_{\{J(u) \leq N\eta\}}$ are independent Bernoulli variables of parameter η , and every attempt satisfies $a(u) \leq K(u)$. Together with $I \subseteq \mathcal{A}$, this proves (3.5). $\square$

The bounds above also explain precisely how the rule is interpreted on an infinite graph. The output at one vertex can be defined from a finite labelled ball. Definitions made from larger balls agree by locality, and independence is checked edge by edge by the same deterministic argument. The probability of membership at a root is therefore computed from only finitely many independent labels.

3.2 The cost of simultaneous attempts

Lemma 3.2 (Collision loss). *For every vertex v of a triangle-free graph of maximum degree at most D ,*

$$\mathbb{P}(v \in I) \geq \mathbb{P}(v \in \mathcal{A}) - \frac{Dw^2}{T}. \quad (3.6)$$

Proof. Work first on a finite graph and condition on all buckets. Let $\mathcal{H}_i$ record these buckets and the trial coordinates through Stage i . Every multiplier in (3.1) has conditional mean one, since

$$\mathbb{E} \left[\frac{1 - a(u)}{1 - p_{i-1}(u)} \mid \mathcal{H}_{i-1} \right] = 1 \quad \text{if } b(u) = i.$$

Multipliers belonging to distinct current vertices are conditionally independent.

Consider an edge uv with $b(u) = b(v) = j$. Triangle-freeness gives $N(u) \cap N(v) = \emptyset$. For each $i < j$, the multipliers updating u and v therefore use disjoint sets of fresh trial coordinates. Consequently

$$\mathbb{E}[W_i(u)W_i(v) \mid \mathcal{H}_{i-1}] = W_{i-1}(u)W_{i-1}(v).$$

Iterating from $W_0(u)W_0(v) = w^2$ gives

$$\mathbb{E}[W_{j-1}(u)W_{j-1}(v) \mid \text{buckets}] = w^2.$$

The two trials in Bucket j are independent conditional on $\mathcal{H}_{j-1}$, so

$$\begin{aligned} \mathbb{P}(u, v \in \mathcal{A} \mid \text{buckets}) &= \mathbb{E}[p_{j-1}(u)p_{j-1}(v) \mid \text{buckets}] \\ &\leq \mathbb{E}[W_{j-1}(u)W_{j-1}(v) \mid \text{buckets}] = w^2. \end{aligned}$$

Here the weights themselves need not be independent. Since two independent uniform buckets coincide with probability $1/T$, averaging gives

$$\mathbb{P}(u, v \in \mathcal{A}, b(u) = b(v)) \leq w^2/T.$$

An attempted vertex v is deleted only if this event occurs for one of its at most D neighbours. The union bound proves (3.6). Locality extends the conclusion to infinite graphs. $\square$

4 The marginal estimate

4.1 A forced-decline law and its likelihood factor

Fix a root v in a finite triangle-free graph, and write $S := N(v)$ and $n := |S| \leq D$. The set S is independent: an edge inside S , together with v , would be a triangle. Define another experiment on $G - v$, with the same independent bucket distribution. Every vertex in S is *forced to decline* when its bucket is processed, regardless of its own trial coordinate. It still applies the usual decline multiplier to its later neighbours. All vertices outside S make honest trials with the probabilities computed from this modified history. Denote this law by $\mathbb{Q}$, its weights by $\tilde{W}_i(u)$, and its capped weights by

$$\tilde{p}_i(u) := \min\{\tilde{W}_i(u), \eta\}.$$

The tilde is essential: these probabilities are computed in the forced experiment and need not equal the probabilities in an unrelated realisation of the original experiment. They do agree on matched histories in the comparison below. An additional bucket $b(v)$ is sampled independently of the entire forced experiment.

Recall that every vertex starts with weight w . At a forced vertex $u \in S$, the number $\tilde{p}_{b(u)-1}(u)$ is still used in the denominator of the update at later neighbours, but the attempt indicator is assigned zero. Thus the transmitted multiplier is $1/(1 - \tilde{p}_{b(u)-1}(u))$. An *honest trial* means a decision using its stated probability and its own previously unused trial coordinate. The bound (3.4) remains valid in the forced experiment: each numerator is zero or one and each denominator is at least $1 - \eta$. Define

$$X_i := \sum_{\substack{u \in S \\ b(u) \leq i}} -\ln(1 - \tilde{p}_{b(u)-1}(u)), \quad 0 \leq i \leq T. \quad (4.1)$$

In particular $X_0 = 0$, and X_i is nondecreasing.

Lemma 4.1 (Change of measure). *The attempt probability in the original experiment satisfies*

$$\mathbb{P}(v \in \mathcal{A}) = \frac{1}{T} \sum_{i=0}^{T-1} \mathbb{E}_{\mathbb{Q}} \min\{w, \eta e^{-X_i}\}. \quad (4.2)$$

Proof. Condition on $b(v) = i + 1$ and on the buckets of all other vertices. Before Stage $i + 1$, the vertex v makes no trial or update, so deleting it has no effect. Likewise, forcing a decline in a bucket after i has no effect on the history through Stage i .

Consider a history h of attempt outcomes through Stage i in which every processed vertex of S declines. We compare these outcome histories, without including the unused raw coordinates at forced vertices. On matching histories, the weights and all honest trial probabilities agree. A forced decline at u has probability one under $\mathbb{Q}$ and probability $1 - \tilde{p}_{b(u)-1}(u)$ under the original law. Therefore

$$\mathbb{P}(h \mid \text{buckets}, b(v) = i + 1) = \mathbb{Q}(h \mid \text{buckets}) \prod_{\substack{u \in S \\ b(u) \leq i}} (1 - \tilde{p}_{b(u)-1}(u)) = \mathbb{Q}(h \mid \text{buckets}) e^{-X_i(h)}.$$

All factors are positive because $\tilde{p} \leq \eta < 1$. On this history, (3.2) gives $W_i(v) = w e^{X_i(h)}$. Multiplying the conditional attempt probability by the likelihood factor yields

$$e^{-X_i(h)} \min\{w e^{X_i(h)}, \eta\} = \min\{w, \eta e^{-X_i(h)}\}.$$

Every other original history contains an earlier neighbouring attempt and has $W_i(v) = 0$, so contributes nothing. Summing over histories and averaging over the buckets of $G - v$ proves the corresponding term of (4.2). Finally average over the independent uniform choice of $b(v)$. $\square$

4.2 Deferred bucket exposure and normalised pending mass

Until the end of the marginal proof, expectations are under $\mathbb{Q}$ unless another law is specified. Instead of revealing all buckets initially, reveal the first i bucket classes and the trial coordinates of the vertices in those classes. Let $\mathcal{F}_i$ be the sigma-algebra generated by this information. Coordinates at forced vertices may be exposed too; the process does not consult them. All weights right after Stage i are $\mathcal{F}_i$ -measurable.

Conditional on $\mathcal{F}_i$, the buckets of the pending vertices are independent and uniform on $\{i+1, \dots, T\}$, and their trial coordinates remain independent and unused.

For $i < T$ define the intermediate sigma-algebra

$$\mathcal{G}_{i+1} := \mathcal{F}_i \vee \sigma(\mathbf{1}_{\{b(x)=i+1\}} : x \in V(G-v)). \quad (4.3)$$

Thus $\mathcal{G}_{i+1}$ reveals the next bucket, but not its trial coordinates, and $\mathcal{F}_i \subseteq \mathcal{G}_{i+1} \subseteq \mathcal{F}_{i+1}$.

Let

$$S_i := \{u \in S : b(u) > i\}, \quad B_i := \sum_{u \in S_i} \tilde{p}_i(u), \quad \text{and} \quad M_i := \frac{B_i}{1-i/T} \quad (0 \leq i < T), \quad (4.4)$$

and set $B_T := 0$ and $M_T := 0$. The denominator normalises for the expected fraction $1-i/T$ of buckets still unprocessed.

Lemma 4.2 (Capped pending mass and its noise). *The process $(M_i)_{i=0}^T$ is a nonnegative supermartingale with $M_0 = nw$. Define*

$$Y_i := \sum_{\substack{u \in S \\ b(u) \leq i}} \tilde{p}_{b(u)-1}(u), \quad A_i := \frac{1}{T} \sum_{r=0}^{i-1} M_r, \quad \text{and} \quad Z_i := Y_i - A_i. \quad (4.5)$$

Then (Z_i) is a martingale with $Z_0 = 0$, and

$$\mathbb{E}_{\mathbb{Q}} Z_T^2 \leq \eta nw \leq \eta Dw, \quad (4.6)$$

$$\mathbb{Q} \left(\max_{0 \leq i \leq T} |Z_i| > 1 \right) \leq \eta Dw, \quad (4.7)$$

$$X_i \leq \frac{Y_i}{1-\eta}. \quad (4.8)$$

Proof. Fix $i < T$ and put $q_i := 1/(T-i)$. Conditional on $\mathcal{G}_{i+1}$, consider a vertex $u \in S_i$ with $b(u) > i+1$. Since S is independent, every current neighbour of u makes an honest trial. Their update multipliers are conditionally independent and have mean one, whence

$$\mathbb{E}[\widetilde{W}_{i+1}(u) \mid \mathcal{G}_{i+1}] = \widetilde{W}_i(u).$$

Applying (2.1), multiplying by the $\mathcal{G}_{i+1}$ -measurable indicator of $b(u) > i+1$, and summing, we obtain

$$\mathbb{E}[B_{i+1} \mid \mathcal{F}_i] \leq \mathbb{E} \left[\sum_{u \in S_i} \tilde{p}_i(u) \mathbf{1}_{\{b(u) > i+1\}} \mid \mathcal{F}_i \right] = (1-q_i)B_i.$$

For $i < T-1$, division by $1-(i+1)/T$ gives $\mathbb{E}[M_{i+1} \mid \mathcal{F}_i] \leq M_i$. The last step follows from $M_T = 0$. Thus (M_i) is a nonnegative supermartingale; it is bounded by $Tn\eta$, and $M_0 = nw$.

Write $\Delta Y_{i+1} := Y_{i+1} - Y_i$, and similarly for Z . Conditional on $\mathcal{F}_i$, the indicators $C_u := \mathbf{1}_{\{b(u)=i+1\}}$, $u \in S_i$, are independent Bernoulli variables of parameter q_i . Hence

$$\Delta Y_{i+1} = \sum_{u \in S_i} \tilde{p}_i(u) C_u, \quad \text{and} \quad \mathbb{E}[\Delta Y_{i+1} \mid \mathcal{F}_i] = q_i B_i = M_i/T = A_{i+1} - A_i.$$

This proves that (Z_i) is a martingale. Conditional independence and $\tilde{p}_i(u)^2 \leq \eta \tilde{p}_i(u)$ also give

$$\mathbb{E}[(\Delta Z_{i+1})^2 \mid \mathcal{F}_i] = q_i(1 - q_i) \sum_{u \in S_i} \tilde{p}_i(u)^2 \leq \eta q_i B_i = \eta M_i/T.$$

Distinct martingale differences have zero expected product. Consequently

$$\mathbb{E}Z_T^2 = \sum_{i=0}^{T-1} \mathbb{E}(\Delta Z_{i+1})^2 \leq \frac{\eta}{T} \sum_{i=0}^{T-1} \mathbb{E}M_i \leq \eta n w,$$

which proves (4.6).

Let σ be the first i with $|Z_i| > 1$, or T if there is none. Since (Z_i^2) is a submartingale and $\{\sigma \leq i\} \in \mathcal{F}_i$,

$$\mathbb{E}(Z_T^2 - Z_\sigma^2) = \sum_{i=0}^{T-1} \mathbb{E}[\mathbf{1}_{\{\sigma \leq i\}}(Z_{i+1}^2 - Z_i^2)] \geq 0.$$

On the event $\max_i |Z_i| > 1$, including a first crossing at T , we have $Z_\sigma^2 > 1$. Thus

$$\mathbb{Q}(\max_i |Z_i| > 1) \leq \mathbb{E}Z_\sigma^2 \leq \mathbb{E}Z_T^2 \leq \eta D w,$$

proving (4.7). Finally,

$$-\ln(1 - p) = \int_0^p \frac{ds}{1 - s} \leq \frac{p}{1 - \eta} \quad (0 \leq p \leq \eta),$$

and summing this inequality gives (4.8). □

4.3 The time-budget inequality

The next lemma gives a lower bound on the expected value of the time τ needed to spend a fixed budget at a nonnegative supermartingale rate. We include a self-contained proof, with the convention $\tau := 1$ if the budget is not exhausted.

Lemma 4.3 (Time budget). *Let $\mathcal{F}_0 \subseteq \dots \subseteq \mathcal{F}_T$ be a filtration on a finite probability space, and let $(M_i)_{i=0}^T$ be a nonnegative supermartingale with respect to this filtration. Thus M_i is $\mathcal{F}_i$ -measurable and*

$$\mathbb{E}[M_{i+1} \mid \mathcal{F}_i] \leq M_i \quad (0 \leq i < T). \quad (4.9)$$

Suppose that $a, b > 0$, $\mathbb{E}M_0 \leq b$, and $b \geq 2a$. Put

$$t_i := i/T, \quad A_i := \frac{1}{T} \sum_{j=0}^{i-1} M_j, \quad \text{and} \quad A(t) := A_i + (t - t_i)M_i \quad (t_i \leq t \leq t_{i+1}). \quad (4.10)$$

Define τ to be the first time that $A(t)$ reaches a , or 1 if it does not reach a by time 1. Then

$$\frac{1}{T} \sum_{i=0}^{T-1} \mathbb{P}(A_i < a) \geq \mathbb{E}\tau \geq \frac{a}{b}. \quad (4.11)$$

Proof. Since $M_i \geq 0$, the function A is continuous and nondecreasing. Set

$$r_i := \max\{a - A_i, 0\}, \quad \text{and} \quad S := \sum_{i=0}^{T-1} M_i(r_i - r_{i+1}).$$

Then $0 \leq r_{i+1} \leq r_i \leq a$ and $r_0 = a$. Moreover, r_{i+1} is $\mathcal{F}_i$ -measurable, because $A_{i+1} = T^{-1} \sum_{j=0}^i M_j$. It follows that

$$\mathbb{E}[r_{i+1}(M_{i+1} - M_i) \mid \mathcal{F}_i] = r_{i+1}(\mathbb{E}[M_{i+1} \mid \mathcal{F}_i] - M_i) \leq 0. \quad (4.12)$$

This is the predictable-transform calculation of [Dur19, Theorem 4.2.8]. Discrete integration by parts gives

$$S + r_T M_T = a M_0 + \sum_{i=0}^{T-1} r_{i+1}(M_{i+1} - M_i).$$

Taking expectations, using (4.12) and $r_T M_T \geq 0$, yields

$$\mathbb{E}S \leq a \mathbb{E}M_0 \leq ab. \quad (4.13)$$

Write $M(t) := M_i$ on $[t_i, t_{i+1})$, and put $\lambda_i := \min\{\tau, t_{i+1}\} - \min\{\tau, t_i\}$. The amount spent in this interval before stopping is $r_i - r_{i+1} = M_i \lambda_i$, also when $M_i = 0$. Therefore

$$S = \sum_{i=0}^{T-1} M_i^2 \lambda_i = \int_0^\tau M(t)^2 dt. \quad (4.14)$$

Let $C := \{A_T \geq a\}$. We have $\tau > 0$, and on C the Cauchy–Schwarz inequality gives

$$a^2 = \left(\int_0^\tau M(t) dt \right)^2 \leq \tau \int_0^\tau M(t)^2 dt = \tau S.$$

Outside C we still have $S \geq 0$. Hence $a^2 \mathbf{1}_C / \tau \leq S$, and (4.13) implies

$$\mathbb{E}[\mathbf{1}_C / \tau] \leq b/a. \quad (4.15)$$

Put $s := a/b \leq 1/2$. Pointwise,

$$\tau \geq 2s - \frac{s^2}{\tau} \mathbf{1}_C :$$

on C the difference equals $(\tau - s)^2 / \tau$, while outside C we have $\tau = 1 \geq 2s$. Taking expectations and using (4.15) gives $\mathbb{E}\tau \geq 2s - s^2 b/a = a/b$. Finally, if K is the first index with $A_K \geq a$, then exactly K of $A_0, \dots, A_{T-1}$ are below a and $\tau \leq K/T$. If there is no such index, all T are below a and $\tau = 1$. Thus $\tau \leq T^{-1} \sum_{i < T} \mathbf{1}_{\{A_i < a\}}$, which proves (4.11) after taking expectations. $\square$

Remark 4.4. The bound $\mathbb{E}\tau \geq a/b$ in Lemma 4.3 is sharp: take $M_i = b$ deterministically, so that $\tau = a/b$.

4.4 Combining the likelihood, budget, and collision bounds

Proposition 4.5 (Quantitative marginal). *Suppose that D, T are positive integers and $0 < w < \eta < 1$ are rational. If $a > 0$ satisfies*

$$a \leq (1 - \eta) \ln(\eta/w) - 1, \quad \text{and} \quad Dw \geq 2a, \quad (4.16)$$

then the output of the bucket process satisfies, at every vertex $v \in V(G)$,

$$\mathbb{P}(v \in I) \geq \frac{a}{D} - \eta Dw^2 - \frac{Dw^2}{T}. \quad (4.17)$$

Proof. Let $E := \{\max_{0 \leq i \leq T} |Z_i| > 1\}$. Outside E , whenever $A_i < a$,

$$Y_i = A_i + Z_i < a + 1 \leq (1 - \eta) \ln(\eta/w).$$

Thus $X_i < \ln(\eta/w)$ by (4.8), so the minimum in (4.2) equals w . It follows that

$$\mathbb{P}(v \in \mathcal{A}) \geq \frac{w}{T} \sum_{i=0}^{T-1} \mathbb{Q}(E^c \cap \{A_i < a\}) \geq w \left(\frac{1}{T} \sum_{i=0}^{T-1} \mathbb{Q}(A_i < a) - \mathbb{Q}(E) \right).$$

Lemma 4.3 applies with $b := Dw$, since $M_0 = nw \leq Dw$ and $Dw \geq 2a$. Therefore

$$\mathbb{P}(v \in \mathcal{A}) \geq \frac{a}{D} - w\mathbb{Q}(E) \geq \frac{a}{D} - \eta Dw^2, \quad (4.18)$$

where the last inequality uses (4.7). Subtracting the collision loss from Lemma 3.2 proves (4.17). Locality extends the estimate from finite graphs to arbitrary graphs of the specified maximum degree. $\square$

Proof of Theorem 1.1. Put $\ell := \lfloor \ln D \rfloor$, $w := 2\ell/D$, $\eta := \ell^{-6}$ and $T := \ell^2$. For sufficiently large D we have $\ell \geq 2$ and $2\ell^7 < D$, so $0 < w < \eta < 1$. All parameters required for the exact encoding are rational. Set

$$a := (1 - \ell^{-6})(\ln D - 7 \ln \ell - \ln 2) - 1 = (1 - \eta) \ln(\eta/w) - 1.$$

Then $a > 0$ for sufficiently large D . Since $\ln D < \ell + 1$, we also have

$$a \leq \ln D - 7 \ln \ell - \ln 2 - 1 < \ell.$$

Thus $Dw = 2\ell \geq 2a$, and Proposition 4.5 applies. The two error terms satisfy

$$\eta D^2 w^2 = 4\ell^{-4}, \quad \text{and} \quad D^2 w^2 / T = 4.$$

Consequently

$$\begin{aligned} D\mathbb{P}(v \in I) &\geq (1 - \ell^{-6})(\ln D - 7 \ln \ell - \ln 2) - 1 - 4\ell^{-4} - 4 \\ &= \ln D - 7 \ln \ell - \ln 2 - 5 - \ell^{-6}(\ln D - 7 \ln \ell - \ln 2) - 4\ell^{-4}. \end{aligned}$$

Since $0 < \ln D - 7 \ln \ell - \ln 2 < \ell + 1$, the last two loss terms are at most

$$\ell^{-5} + \ell^{-6} + 4\ell^{-4} \longrightarrow 0.$$

For sufficiently large D this sum is at most $1 - \ln 2$. Using also $\ln \ell \leq \ln \ln D$, we obtain

$$D\mathbb{P}(v \in I) \geq \ln D - 7 \ln \ell - 6 \geq \ln D - 7 \ln \ln D - 6.$$

The remaining assertions follow from Lemma 3.1. $\square$

5 Concentration for uncovered neighbours

We will also need the following concentration result that follows from Boucheron, Lugosi and Massart [BLM09, Theorem 1]. For completeness, we include a self-contained proof using the exposure martingale.

Lemma 5.1 (Uncovered-set tail). *Let $B > 0$ and let $S_1, \dots, S_m$ be independent random subsets of a finite set V_0 of size n , and suppose that $|S_j| \leq B$ almost surely for every $j \in [m]$. Put*

$$U := \left| V_0 \setminus \left(\bigcup_{j=1}^m S_j \right) \right|, \quad \text{and} \quad q := n - \mathbb{E}U = \mathbb{E} \left| \bigcup_{j=1}^m S_j \right|.$$

If $q = 0$, then $U = n$ almost surely. If $q > 0$, then, for every $t > 0$,

$$\mathbb{P}(U - \mathbb{E}U \geq t) \leq \exp \left(-\frac{t^2}{2Bq} \right) \leq \exp \left(-\frac{t^2}{2Bn} \right). \quad (5.1)$$

Proof. If $q = 0$, the nonnegative random variable $n - U$ has expectation zero and hence vanishes almost surely. Suppose that $q > 0$. For $u \in V_0$, write

$$p_j(u) := \mathbb{P}(u \in S_j) \quad (j \in [m]), \quad \text{and} \quad r_j(u) := \prod_{k=j+1}^m (1 - p_k(u)) \quad (0 \leq j \leq m). \quad (5.2)$$

Thus $r_j(u)$ is the probability that all sets after the j th miss u , with $r_m(u) = 1$. Expose the sets in order and put $\mathcal{F}_j := \sigma(S_1, \dots, S_j)$. Independence between the sets gives the conditional expectation martingale

$$H_j := \mathbb{E}[U \mid \mathcal{F}_j] = \sum_{u \in V_0 \setminus (\bigcup_{k=1}^j S_k)} r_j(u), \quad j \in \{0, \dots, m\}. \quad (5.3)$$

In particular, $H_0 = \mathbb{E}U$ and $H_m = U$.

For $j \in [m]$, define

$$C_j := \sum_{u \in V_0 \setminus (\bigcup_{k=1}^{j-1} S_k)} r_j(u) \mathbf{1}_{\{u \in S_j\}}, \quad \text{and} \quad \mu_j := \mathbb{E}[C_j \mid \mathcal{F}_{j-1}]. \quad (5.4)$$

Since $0 \leq r_j(u) \leq 1$, we have $0 \leq C_j \leq |S_j| \leq B$. Using (5.3) and the identity $r_{j-1}(u) = (1 - p_j(u))r_j(u)$ gives

$$H_j - H_{j-1} = \mu_j - C_j, \quad \text{and} \quad \mu_j = \sum_{u \in V_0 \setminus (\bigcup_{k=1}^{j-1} S_k)} p_j(u) r_j(u) \leq \sum_{u \in V_0} p_j(u) r_j(u). \quad (5.5)$$

The last bound is deterministic. Its sum telescopes:

$$\sum_{j=1}^m \sum_{u \in V_0} p_j(u) r_j(u) = \sum_{u \in V_0} \sum_{j=1}^m (r_j(u) - r_{j-1}(u)) = \sum_{u \in V_0} (1 - r_0(u)) = q. \quad (5.6)$$

The final equality holds because $r_0(u)$ is the probability that u remains uncovered.

Fix $\lambda > 0$. Convexity gives the chord bound

$$e^{-\lambda x} \leq 1 + \frac{x}{B} (e^{-\lambda B} - 1) \quad (0 \leq x \leq B).$$

Also, for $y \geq 0$,

$$e^{-y} - 1 + y = \int_0^y (1 - e^{-s}) ds \leq \frac{y^2}{2}.$$

Applying these inequalities to C_j , and using $1 + x \leq e^x$, we obtain

$$\begin{aligned}
\mathbb{E}[e^{\lambda(H_j - H_{j-1})} \mid \mathcal{F}_{j-1}] &= e^{\lambda\mu_j} \mathbb{E}[e^{-\lambda C_j} \mid \mathcal{F}_{j-1}] \\
&\leq e^{\lambda\mu_j} \left(1 + \frac{\mu_j}{B}(e^{-\lambda B} - 1)\right) \\
&\leq \exp\left(\frac{\mu_j}{B}(e^{-\lambda B} - 1 + \lambda B)\right) \\
&\leq \exp\left(\frac{\lambda^2 B \mu_j}{2}\right) \leq \exp\left(\frac{\lambda^2 B}{2} \sum_{u \in V_0} p_j(u) r_j(u)\right). \tag{5.7}
\end{aligned}$$

The last expression is deterministic. For each $j \in [m]$, the tower property and the $\mathcal{F}_{j-1}$ -measurability of H_{j-1} give

$$\begin{aligned}
\mathbb{E}[e^{\lambda(H_j - H_0)}] &= \mathbb{E}\left[e^{\lambda(H_{j-1} - H_0)} \mathbb{E}[e^{\lambda(H_j - H_{j-1})} \mid \mathcal{F}_{j-1}]\right] \\
&\leq \exp\left(\frac{\lambda^2 B}{2} \sum_{u \in V_0} p_j(u) r_j(u)\right) \mathbb{E}[e^{\lambda(H_{j-1} - H_0)}].
\end{aligned}$$

Iterating this inequality for $j = 1, \dots, m$, starting with $\mathbb{E}[e^{\lambda(H_0 - H_0)}] = 1$, and using (5.6), we obtain

$$\mathbb{E}[e^{\lambda(U - \mathbb{E}U)}] \leq \exp(\lambda^2 B q / 2). \tag{5.8}$$

Markov's inequality with $\lambda := t/(Bq)$ now gives the first inequality in (5.1). The second follows from $q \leq n$. $\square$

6 Local constraints

We collect the local tools used in the colouring argument: a distributed Local Lemma, a fast degree-plus-one colouring theorem, and two lemmas that turn independent sampler runs into a degree reduction.

6.1 A distributed Local Lemma

Let $\mathcal{B}$ be a finite indexed family of *bad events*, that is, constraint failures to be avoided simultaneously. They are determined by a finite set $\mathcal{V}$ of independent finite-valued variables. For each $A \in \mathcal{B}$, specify a *determining set* $\mathcal{V}(A) \subseteq \mathcal{V}$: its coordinates suffice to decide A , but may include variables not consulted by its predicate. The *dependency graph* has vertex set $\mathcal{B}$, with two distinct events adjacent when their determining sets intersect. We use the following result of Flin, Ghaffari, Halldórsson, Maus and Nolin [FGH⁺26, Theorem 1.1].

Theorem 6.1 (Distributed Local Lemma). *Let $N := |\mathcal{B}| \geq 2$. Suppose that the independent variables have finite alphabets with given rational probability tables, that the dependency graph has maximum degree at most an integer $d \geq 2$, and that $\mathbb{P}(A) \leq p$ for every $A \in \mathcal{B}$. If*

$$pd^{1000} < 1, \tag{6.1}$$

then a randomised LOCAL algorithm avoids all bad events with probability at least $1 - N^{-10}$ in $O((1 + \ln \ln(\max\{N, 3\}))^c)$ rounds on the dependency graph, where c is an absolute constant. Each event processor is given its predicate, determining variables and their probability tables, and the bounds N, d, p . The processors have distinct identifiers of $O(\ln N)$ bits.

Remark 6.2. Davies’ resilient Local Lemma [Dav23, Theorem 5.1] also suffices for our application. However, it relies on [GHK18, Theorem V.1] whose statement was identified in [GHMN26, Section 3.2] as false. Although a version of [Dav23, Theorem 5.1] that suffices for our purposes can be recovered with some extra work, we will use Theorem 6.1 instead, in order to avoid this auxiliary argument.

6.2 Degree-plus-one colouring

Theorem 6.3 (Fast degree-plus-one colouring). *There is an absolute constant c and a randomised LOCAL algorithm which, on an N -vertex graph, $N \geq 2$, of maximum degree at most an integer $D \geq 0$, finds a proper colouring with $D + 1$ colours in $O((1 + \ln \ln(\max\{N, 3\}))^c)$ rounds, with failure probability at most N^{-c_0} for some absolute constant $c_0 > 0$. The processors have distinct $O(\ln N)$ -bit identifiers and know N and the degree bound D .*

This follows from the Chang–Li–Pettie algorithm [CLP20], with the corrected shattering analysis in [GHMN26, Theorem 5], and the deterministic network-decomposition result of Rozhoň and Ghaffari [RG20].

The implicit exponents are absolute. We will use the fast colouring theorem with failure probability at most n^{-6} . This follows from its stated guarantee by polynomial padding. More generally, for any fixed $A > 0$, choose a positive integer p with $pc_0 \geq A$ and, for $n \geq 2$, pad an n -vertex instance to $N := n^p$ vertices with isolated vertices. Each original processor simulates n^{p-1} processors, one representing the original vertex and the rest isolated, using distinct $O(\ln N)$ -bit identifiers $(\text{ID}(v), j)$ and independent randomness at each virtual processor. Thus a virtual round costs one physical round, and the restricted output fails with probability at most

$$N^{-c_0} \leq n^{-A}, \quad \text{and} \quad \ln \ln N = \ln \ln n + \ln p. \quad (6.2)$$

The asymptotic round bound is unchanged. The same padding argument works for Local Lemma instances by adding isolated identically false events, although Theorem 6.1 already supplies more than the failure probability needed below.

6.3 Solving local degree constraints

Lemma 6.4. *Let $D \geq 2$ be an integer, let H be a finite graph of maximum degree at most D , and let R be a positive integer. Suppose that $I_1, \dots, I_m$ are independent runs of a radius- $(R - 1)$ rule, using mutually independent uniform labels $X_{u,j}$ from known finite alphabets, where $u \in V(H)$ and $j \in [m]$. Fix $\rho \in (0, 1)$ and suppose that, for every $v \in V(H)$,*

$$\mathbb{P}\left(\left|N_H(v) \setminus (\cup_{j=1}^m I_j)\right| > \rho D\right) \leq p, \quad pd^{1000} < 1, \quad \text{and} \quad d := 2D^{2R}. \quad (6.3)$$

Then a LOCAL algorithm finds labels for which

$$\left|N_H(v) \setminus (\cup_{j=1}^m I_j)\right| \leq \rho D \quad \text{for every } v, \quad (6.4)$$

with failure probability at most $(\max\{n, 2\})^{-10}$, in $O(R(1 + \ln \ln(\max\{n, 3\}))^c)$ rounds on H , where $n := |V(H)|$ and c is an absolute constant.

If H is the active induced subgraph of an original communication graph with $\bar{n} \geq n$ processors, the same conclusion holds with n replaced by $\bar{n}$ in both bounds.

Proof. If $|V(H)| \leq 1$, then every neighbourhood is empty and any labels satisfy the conclusion. Otherwise, introduce one bad event at each vertex:

$$A_v := \left\{\left|N_H(v) \setminus (\cup_{j=1}^m I_j)\right| > \rho D\right\}, \quad \text{and} \quad \mathcal{V}(A_v) := \{X_{u,j} : u \in B_H(v, R), j \in [m]\}. \quad (6.5)$$

This is a determining set: membership of each neighbour of v in a run is determined within distance R of v . If two determining sets intersect, their centres have distance at most $2R$. Hence the number of events meeting any one determining set, including itself, is at most

$$\sum_{t=0}^{2R} D^t \leq 2D^{2R} = d. \quad (6.6)$$

Thus Theorem 6.1 applies.

The simulation costs $O(R)$ physical rounds per virtual round. The physical vertex v simulates its event processor and first gathers the required radius- $2R$ neighbourhood. Every edge of the dependency graph joins centres within distance $2R$, so flooding delivers every virtual message in at most $2R$ physical rounds. All such messages are sent concurrently; their sizes are unrestricted in LOCAL. After the solver has chosen consistent values for the labels, one further execution of the local rule takes $O(R)$ rounds. This gives the claimed bounds.

For the last assertion, retain one event processor at each inactive vertex, with an identically false predicate and empty determining set. These processors are isolated in the dependency graph, even if their physical hosts have neighbours in the original graph. The solver now runs on $\bar{n}$ event processors, so its size parameter and failure bound refer to $\bar{n}$. The same simulation gives the round bound. $\square$

6.4 A degree reduction with a polynomially small cap

Here is the concentration estimate in the form required below. It is convenient to retain a cap chosen from the initial degree, rather than change it at every phase.

Lemma 6.5. *Fix $b, C, \tau > 0$, $\gamma, \theta \in (0, 1)$, and $\rho \in (0, 1)$ such that*

$$e^{-\gamma C} < \rho. \quad (6.7)$$

There are $c > 0$ and Δ_0 , depending only on the above fixed constants, such that the following holds for every $\Delta \geq \Delta_0$. Suppose that

$$1 \leq k \leq \ln \Delta, \quad \frac{\tau \Delta}{k} < D \leq \Delta, \quad 1 \leq h \leq bk + 1, \quad \text{and} \quad \eta := \frac{1}{\lceil \Delta^\theta \rceil},$$

where D, h are integers. Suppose that a sampler on a graph H of maximum degree at most D has marginal at least $\gamma h/D$ and pointwise independent cap η : there are independent variables $K(u) \sim \text{Bernoulli}(\eta)$, each determined by the label at u , with $\mathbf{1}_{\{u \in I\}} \leq K(u)$ for every label assignment. Take

$$m := \left\lceil \frac{CD}{h} \right\rceil \quad (6.8)$$

independent runs. Then, for every $v \in V(H)$,

$$\mathbb{P} \left(\left| N_H(v) \setminus (\cup_{j=1}^m I_j) \right| > \rho D \right) \leq \exp(-c\Delta^\theta). \quad (6.9)$$

Proof. The assertion is immediate when $N_H(v) = \emptyset$. Fix a vertex v with a non-empty neighbourhood. We have

$$m \frac{\gamma h}{D} \geq \gamma C. \quad (6.10)$$

For a run j , put $Z_j := \sum_{u \in N_H(v)} K_j(u)$ and $B := \lceil 2D\eta \rceil$, where the $K_j(u)$ are the independent caps. The caps inside this run are mutually independent. Hence

$$\mathbb{E}[2^{Z_j}] = \prod_{u \in N_H(v)} \mathbb{E}[2^{K_j(u)}] = (1 + \eta)^{|N_H(v)|} \leq e^{D\eta}.$$

Put $\xi := e^{-D\eta/3}$. Since $B \geq 2D\eta$, Markov's inequality gives

$$\mathbb{P}(Z_j > B) \leq e^{D\eta} 2^{-B} \leq e^{-(2\ln 2 - 1)D\eta} \leq e^{-D\eta/3} = \xi. \quad (6.11)$$

Here we used $2\ln 2 - 1 > 1/3$.

Set $S_j := I_j \cap N_H(v)$ if $Z_j \leq B$, and $S_j := \emptyset$ otherwise. These sets are independent between runs and have size at most B . For every $u \in N_H(v)$,

$$\mathbb{P}(u \in S_j) \geq \frac{\gamma h}{D} - \xi.$$

Define the uncovered counts for the clipped sets and for the actual sampler outputs by

$$U := \left| N_H(v) \setminus (\cup_{j=1}^m S_j) \right|, \quad \text{and} \quad U_{\text{actual}} := \left| N_H(v) \setminus (\cup_{j=1}^m I_j) \right|. \quad (6.12)$$

The clipped sets are used only in the analysis; the algorithm retains the original outputs I_j . Since $S_j \subseteq I_j \cap N_H(v)$ for every outcome,

$$N_H(v) \setminus (\cup_{j=1}^m I_j) \subseteq N_H(v) \setminus (\cup_{j=1}^m S_j), \quad \text{and} \quad U_{\text{actual}} \leq U. \quad (6.13)$$

The error $m\xi = o(1)$ uniformly in the stated range. Indeed, $m \leq C\Delta + 1$, since $h \geq 1$, while

$$D\eta \geq \frac{\tau \Delta^{1-\theta}}{2k} \geq \frac{\tau \Delta^{1-\theta}}{2 \ln \Delta}.$$

In particular,

$$m\xi \leq (C\Delta + 1) \exp\left(-\frac{\tau \Delta^{1-\theta}}{6 \ln \Delta}\right) \rightarrow 0.$$

The same estimate gives $\Delta\xi \rightarrow 0$, and hence $\gamma h/D - \xi \geq \gamma/\Delta - \xi > 0$ eventually.

For each neighbour u , the events $\{u \notin S_j\}$ are independent over the runs. Its probability of being missed is therefore at most $(1 - \gamma h/D + \xi)^m \leq \exp(-m(\gamma h/D - \xi))$. Summing over at most D neighbours gives

$$\mathbb{E}U \leq D \exp\left(-m\left(\frac{\gamma h}{D} - \xi\right)\right). \quad (6.14)$$

No independence between different neighbours is needed.

Choose a fixed ρ_0 with $e^{-\gamma C} < \rho_0 < \rho$ and put $\delta := \rho - \rho_0$. By (6.10) and (6.14), eventually $\mathbb{E}U \leq \rho_0 D$.

By (6.13), the degree-failure event $\{U_{\text{actual}} > \rho D\}$ is contained in $\{U > \rho D\}$. Also $D\eta \geq 1$ eventually, so $B \leq 3D\eta$. Lemma 5.1, with ground-set size $|N_H(v)| \leq D$ and $t := \delta D$, gives

$$\mathbb{P}(U_{\text{actual}} > \rho D) \leq \mathbb{P}(U > \rho D) \leq \exp\left(-\frac{\delta^2 D^2}{2BD}\right) \leq \exp\left(-\frac{\delta^2}{6\eta}\right).$$

Since $\eta^{-1} \geq \Delta^\theta$, this is (6.9) with $c := \delta^2/6$. No cap-failure probability is added to this bound: (6.13) holds for every outcome, and the loss ξ has already been included in the clipped marginals. All lower bounds imposed on Δ depend only on the fixed constants, so a single Δ_0 works uniformly in k, D, h and H . $\square$

7 The colouring algorithm

Proof of Theorem 1.2. We first prove the assertion when k is rational and is supplied to the processors; we remove this convention at the end. Fix $\varepsilon \in (0, 1)$. All constants chosen in the next paragraph depend only on ε . Choose b in the non-empty interval $(1, 1/(1 - \varepsilon))$. Then choose $0 < \theta < \min\{1/2, (1 - b(1 - \varepsilon))/2\}$, so that

$$b(1 - \varepsilon) < 1 - 2\theta. \quad (7.1)$$

Choose γ with $1/b < \gamma < 1$. Then $b\gamma > 1$. Since

$$\lim_{C \downarrow 0} \frac{C}{b(1 - e^{-\gamma C})} = \frac{1}{b\gamma} < 1,$$

we can choose $C > 0$ and then ρ, σ with

$$e^{-\gamma C} < \rho < \sigma < 1, \quad \text{and} \quad Q := \frac{C}{b(1 - \rho/\sigma)} < 1. \quad (7.2)$$

Let us show that this is possible. We have $1 - e^{-\gamma C} = \gamma C + o(C)$ as $C \downarrow 0$. Choose C so that $C/[b(1 - e^{-\gamma C})] < 1$. By continuity, the same strict inequality still holds if $e^{-\gamma C}$ in the denominator is replaced by a slightly larger $\rho < 1$. The function $\sigma \mapsto C/[b(1 - \rho/\sigma)]$ tends to $C/[b(1 - \rho)] < 1$ as $\sigma \uparrow 1$. Therefore some $\rho < \sigma < 1$ gives the required $Q < 1$. Finally choose $\tau \in (0, 1)$ so that

$$Q + 3\tau < 1. \quad (7.3)$$

All these constants can be chosen rational, because the required inequalities are strict.

For a given Δ, k , define the integer degree bounds, stopping index, and sampler parameters by

$$D_0 := \Delta, \quad D_{j+1} := \lfloor \rho D_j \rfloor, \quad s := \min\{j \geq 0 : D_j \leq \tau \Delta/k\}, \quad (7.4)$$

$$h_j := \lceil bk\sigma^j \rceil, \quad \eta := \frac{1}{\lceil \Delta^\theta \rceil}, \quad T_j := \left\lceil \frac{8h_j}{1 - \gamma} \right\rceil \quad (0 \leq j < s). \quad (7.5)$$

Set $w_j := 2h_j/D_j$. The same cap η is used throughout. Since $D_j \leq \Delta\rho^j$ and $k \geq 1 > \tau$, we have

$$1 \leq s \leq \left\lceil \frac{\ln(k/\tau)}{-\ln \rho} \right\rceil = O_\varepsilon(1 + \ln k). \quad (7.6)$$

At every executed phase,

$$\frac{\tau \Delta}{k} < D_j \leq \Delta, \quad \text{and} \quad 1 \leq h_j \leq bk + 1 \leq b \ln \Delta + 1. \quad (7.7)$$

In particular D_j tends to infinity uniformly in all these phases and in the permitted choices of k .

We check the sampler hypotheses uniformly. Since $\eta \geq \Delta^{-\theta}/2$, (7.7) gives

$$\begin{aligned} \ln \left(\frac{\eta D_j}{2h_j} \right) &\geq (1 - \theta) \ln \Delta - \ln k - \ln(bk + 1) + \ln(\tau/4) \\ &\geq (1 - \theta) \ln \Delta - 2 \ln \ln \Delta - O_\varepsilon(1). \end{aligned} \quad (7.8)$$

Also $\ln(\eta D_j/(2h_j)) \leq \ln \Delta$, because $\eta \leq 1$, $D_j \leq \Delta$ and $h_j \geq 1$. By (7.1),

$$h_j + 1 \leq b(1 - \varepsilon) \ln \Delta + 2 < (1 - 2\theta) \ln \Delta + 2.$$

It follows from (7.8) that, eventually,

$$\begin{aligned} (1 - \eta) \ln \left(\frac{\eta D_j}{2h_j} \right) - (h_j + 1) \\ \geq \theta \ln \Delta - 2 \ln \ln \Delta - O_\varepsilon(1) - \eta \ln \Delta > 0. \end{aligned}$$

Finally, $\eta h_j \leq \Delta^{-\theta}(b \ln \Delta + 1) \rightarrow 0$. Thus, after increasing Δ_0 , we have uniformly

$$h_j \leq (1 - \eta) \ln(\eta/w_j) - 1, \quad \text{and} \quad 8\eta h_j \leq 1 - \gamma. \quad (7.9)$$

In particular $0 < w_j < \eta < 1$ and $D_j w_j = 2h_j$, so Proposition 4.5, with $a := h_j$, gives

$$D_j \mathbb{P}(v \in I) \geq h_j - 4\eta h_j^2 - \frac{4h_j^2}{T_j} \geq \gamma h_j. \quad (7.10)$$

Indeed, (7.9) and the choice of T_j bound each loss term by $(1 - \gamma)h_j/2$. Since w_j and η are rational, Lemma 3.1 supplies a finite uniform label alphabet and guarantees an independent set for every assignment, pointwise domination by independent Bernoulli variables of parameter η read from individual vertex labels, and determining radius $T_j + 2$. Allowing this fixed relative loss in the marginal makes the radius linear in h_j .

In phase j the current graph H_j is induced by the uncoloured vertices. Take

$$m_j := \left\lceil \frac{CD_j}{h_j} \right\rceil \quad (7.11)$$

runs of this sampler. Choose their labels by one of the two procedures specified below, so that all vertices of H_j have at most ρD_j neighbours uncovered by the runs. Give every covered vertex the colour of its first covering run, using m_j new colour names in this phase. Each colour class is contained in an independent set. The graph left uncoloured has maximum degree at most $\lfloor \rho D_j \rfloor = D_{j+1}$. At the end colour H_s with $D_s + 1$ further colours using the fast degree-plus-one algorithm stated above.

The number of colours fits the claimed bound even with all rounding errors. By (7.4), (7.5), and (7.11),

$$\sum_{j=0}^{s-1} m_j \leq \sum_{j=0}^{s-1} \left(\frac{CD_j}{h_j} + 1 \right) \leq \frac{C\Delta}{bk} \sum_{j=0}^{s-1} (\rho/\sigma)^j + s \leq Q \frac{\Delta}{k} + s. \quad (7.12)$$

Moreover, $D_s + 1 \leq \tau\Delta/k + 1$. Uniformly for $k \leq \ln \Delta$,

$$\frac{k(s+1)}{\Delta} = O_\varepsilon \left(\frac{\ln \Delta (1 + \ln \ln \Delta)}{\Delta} \right) \rightarrow 0.$$

Increase Δ_0 so that $s+1 \leq \tau\Delta/k$. Equations (7.3) and (7.12) give

$$\sum_{j=0}^{s-1} m_j + D_s + 1 \leq (Q + 2\tau) \frac{\Delta}{k} < \frac{\Delta}{k}. \quad (7.13)$$

The left-hand side is an integer, hence is at most $\lfloor \Delta/k \rfloor$.

Write $R_j := T_j + 3$ for the radius of a degree constraint. The total radius satisfies

$$\sum_{j=0}^{s-1} R_j \leq \frac{8}{1-\gamma} \sum_{j=0}^{s-1} h_j + 4s \leq \frac{8}{1-\gamma} \left(\frac{bk}{1-\sigma} + s \right) + 4s = O_\varepsilon(k). \quad (7.14)$$

Here (7.6) and $1 + \ln k = O(k)$ for $k \geq 1$ were used in the last step. This is the reason for decreasing h_j : using h_j of order k in all s phases would introduce a factor of order $\ln k$.

We next justify the label choices and their probability of success. Condition on the complete history before phase j , assuming all previous reductions succeeded. Then H_j is a fixed triangle-free graph with maximum degree at most D_j , so the sampler estimates and Lemma 6.5 apply to fresh independent labels. They apply uniformly over every such history. To combine such bounds, consider the event that phase j is the first unsuccessful phase. Its probability is at most the conditional failure bound for that phase. These first-failure events are disjoint, so summing their bounds requires no independence between phases. Let $c > 0$ be the constant in that lemma, fixed for the parameters already chosen. For each vertex, the degree-failure probability is at most $e^{-c\Delta^\theta}$.

The corresponding dependency bound is $d_j := 2D_j^{2R_j}$. Since $h_j \leq bk + 1$ and $k \leq \ln \Delta$,

$$\ln d_j \leq \ln 2 + 2R_j \ln \Delta = O_\varepsilon((\ln \Delta)^2). \quad (7.15)$$

As $\theta > 0$ is fixed, $\Delta^\theta/(\ln \Delta)^2 \rightarrow \infty$. Consequently, uniformly in j ,

$$p_j := d_j^{-1001}, \quad e^{-c\Delta^\theta} \leq p_j, \quad \text{and} \quad p_j d_j^{1000} < 1 \quad (7.16)$$

for all sufficiently large Δ . Lemma 6.4 therefore applies. Notice that d_j need not be a fixed power of Δ : the stronger probability estimate absorbs the radius in the exponent of the dependency bound.

There are two regimes. Increase Δ_0 once more so that $s \leq n$, which follows from (7.6) and $\Delta \leq n - 1$.

If $\Delta^\theta \geq (8/c) \ln n$, choose all labels independently and directly execute the samplers in every phase. Conditional on every preceding phase succeeding, the chance that phase j fails at any vertex is at most $ne^{-c\Delta^\theta} \leq n^{-7}$. Summing over $s \leq n$ phases gives failure probability at most n^{-6} . The communication time for these phases is $O_\varepsilon(k)$ by (7.14).

If $\Delta^\theta < (8/c) \ln n$, use Theorem 6.1 in every phase, via Lemma 6.4. Keep one event processor at every original vertex. At a vertex that has already been coloured, its event is identically false and has an empty determining set; labels are needed only on H_j . Thus the virtual instance has n processors, even if few original vertices remain uncoloured. Each call then fails with probability at most $n^{-10} \leq n^{-6}$. Equations (7.15) and (7.16) are unaffected by these isolated events. A union bound over the phases gives failure probability at most n^{-5} . In this regime,

$$\ln \Delta < \theta^{-1}(\ln \ln n + \ln(8/c)) = O_\varepsilon(1 + \ln \ln n). \quad (7.17)$$

For some absolute exponent a_1 , Lemma 6.4 and (7.14) therefore bound the total phase time by

$$O_\varepsilon \left(\sum_{j=0}^{s-1} R_j (1 + \ln \ln n)^{a_1} \right) = O_\varepsilon(k(1 + \ln \ln n)^{a_1}). \quad (7.18)$$

Since $k \leq \ln \Delta = O_\varepsilon(1 + \ln \ln n)$ by (7.17), this is $O_\varepsilon((1 + \ln \ln n)^{a_1+1})$. After increasing Δ_0 , we can absorb its ε -dependent prefactor into one further power of $1 + \ln \ln n$, since $n \geq \Delta + 1$. Thus the exponent in the second term of (1.4) is absolute.

For the final degree-plus-one colouring, keep all original vertices, making the previously coloured ones isolated, and use only the edges of H_s . Ignore the new colours returned at previously coloured vertices. This gives an n -processor invocation. Polynomial padding by isolated vertices makes its failure probability at most n^{-6} , and its time is $(1 + \ln \ln n)^{O(1)}$. Palettes of different phases and of this final step are disjoint, so their union is a proper colouring whenever all steps succeed. The sum of the failure probabilities is less than n^{-1} .

All instructions can be defined on unsuccessful histories as well. For the exact finite alphabet, form the catalogue in Lemma 3.1 using the original degree bound Δ , while retaining $w_j = 2h_j/D_j$, η and T_j . This still gives the same process on every graph of degree at most D_j , and makes every trial well-defined even if a previous phase failed to attain its advertised degree bound. Subroutines are run for their prescribed number of rounds, and outputs on invalid inputs can be assigned arbitrarily within the colour set. The success analysis uses only the conditional guarantees on valid histories.

Finally, the two regimes and (7.14) give (1.4).

For general k , put $q := \lfloor \Delta/k \rfloor$ and $\kappa := \Delta/q$. This is a rational input computable from q, Δ . For sufficiently large Δ , uniformly in the stated range of k ,

$$k \leq \kappa \leq \frac{k}{1 - k/\Delta} \leq 2k, \quad \text{and} \quad \kappa - k \leq \frac{k^2}{\Delta - k} = o(1).$$

Indeed, $q \geq \Delta/k - 1 > 0$ implies $\kappa \leq k/(1 - k/\Delta)$; subtracting k gives $\kappa - k \leq k^2/(\Delta - k)$. This is at most $(\ln \Delta)^2/(\Delta - \ln \Delta) \rightarrow 0$. The gap $(\varepsilon/2) \ln \Delta$ eventually exceeds this error, so $\kappa \leq (1 - \varepsilon/2) \ln \Delta$. Apply the rational case, already proved for every fixed positive error parameter, with $\varepsilon/2$ and κ . It uses at most $\lfloor \Delta/\kappa \rfloor = q$ colours, and its $O_{\varepsilon/2}(\kappa)$ term is $O_\varepsilon(k)$. Increasing Δ_0 to the threshold of that application completes the proof for the stated finite input. $\square$

7.1 Implementation and comparison

The finite uniform labels can be sampled with private fair bits, using rejection sampling as follows. For $M = 1$ the only symbol is chosen deterministically. For $M \geq 2$, put $a := \lceil \log_2 M \rceil$ and use a fair bits to draw a uniform integer J from $\{0, \dots, 2^a - 1\}$. If $J < M$, output $J + 1$; otherwise discard these bits and try again independently. Each accepted value is equally likely, and the acceptance probability $M/2^a$ is at least $1/2$. The same construction samples any finite rational distribution by using a common denominator. Use implementations of the samplers and auxiliary algorithms with a prescribed finite number of communication rounds and a bounded number of draws from finite rational probability tables per round. For fixed n and the input parameters, there are only finitely many inputs and histories, so there is a finite deterministic upper bound $F \geq 1$ on the total number of such draws. No polynomial bound on F is required. To obtain fixed finite random labels, stop each rejection sampler after $\lceil \log_2 F + 12 \log_2 n \rceil$ trials and use a fixed default value upon rejection throughout. Coupling with the untruncated procedure, the probability of any discrepancy is at most $F 2^{-\lceil \log_2 F + 12 \log_2 n \rceil} \leq n^{-12}$. Thus this change adds at most n^{-12} to the failure probability and no communication rounds. The slack in the preceding proof still gives the stated $1 - n^{-1}$ guarantee. Increasing the constant $8/c$ in the regime cutoff and the rejection-sampling cutoff, and applying polynomial padding to both auxiliary algorithms as in (6.2), gives $1 - n^{-A}$ for any prescribed fixed $A > 0$, with constants depending also on A .

All event predicates and their probability tables are finite, so the local computations required by the Local Lemma can be performed by finite enumeration. No polynomial bound is claimed for this local computation, for the alphabet size, or for message lengths. These resources are unrestricted in LOCAL. The communication bound is obtained on the original graph, with the factor R_j in the virtual simulation explicitly included.

Taking $k := (1 - \varepsilon) \ln \Delta$ gives, in particular, a colouring with at most $\lfloor \Delta / ((1 - \varepsilon) \ln \Delta) \rfloor$ colours in $O_\varepsilon(\ln \Delta) + \ln^{O(1)} n$ rounds. Together with Theorem 1.2, this gives the full constant-one range. The capped sampler supplies the marginal and independent cap; Lemma 6.5 supplies the small failure probabilities required by the distributed Local Lemma.

8 Borel and measurable colourings

A *standard Borel space* is a set equipped with the Borel σ -algebra of some Polish topology. A graph G on a standard Borel space X is *Borel* if its adjacency relation is a Borel subset of $X \times X$. A colouring with finitely many colours is Borel if each colour class is Borel. The smallest possible number of colours is the *Borel chromatic number* $\chi_B(G)$. Given a Borel probability measure μ on X , we define the *measurable chromatic number* $\chi_\mu(G)$ in the same way, requiring the colour classes to belong to the completion of the Borel σ -algebra with respect to μ .

Let τ be a Polish topology on X whose Borel σ -algebra is the specified one. A subset of X has the *Baire property* in τ if its symmetric difference with some open set is meagre, that is, a countable union of nowhere dense sets. A finite colouring is *Baire-measurable* with respect to τ if each colour class has the Baire property. All colourings considered here are proper everywhere, including on any exceptional null or meagre set.

For positive integers a, b , a *Borel b -fold colouring with a colours* is a Borel map $f : X \rightarrow \binom{[a]}{b}$ such that $f(u) \cap f(v) = \emptyset$ whenever $uv \in E(G)$. The *fractional Borel chromatic number* is

$$\chi_B^f(G) := \inf \left\{ \frac{a}{b} : G \text{ has a Borel } b\text{-fold colouring with } a \text{ colours} \right\}. \quad (8.1)$$

Equivalently, one may use a Borel independent sets covering each vertex at least b times: a fixed ordering of the sets allows one to keep precisely the first b memberships at each vertex. See [Ber22] for fractional Borel colourings of Schreier graphs and [Wei26, Section 2.2] for their relation to homomorphisms to Kneser graphs.

For a positive integer r , the graph G^r joins distinct vertices at distance at most r in G . The *asymptotic separation index* $\text{asi}(G)$ is the least nonnegative integer s such that, for every r , there is a Borel partition $X = X_0 \sqcup \dots \sqcup X_s$ for which every component of each $G^r[X_i]$ is finite. The partition may depend on r , and no uniform bound on the sizes of its components is required. If there is no such s , put $\text{asi}(G) := \infty$. This is the convention of [BW25, Definition 1.11].

8.1 Finite averaging and fractional colourings

The following elementary observation turns a finite local random construction into a fractional Borel colouring. A *partial q -colouring* takes values in $\{0\} \cup [q]$, where 0 means uncoloured; it is proper if adjacent coloured vertices have different colours.

Lemma 8.1 (Finite averaging). *Let G be a Borel graph of bounded degree and let $R \geq 1$. Suppose that a radius- R rule with a finite uniform label alphabet Λ produces a proper partial q -colouring for every label assignment. Suppose also that, under independent uniform labels, every vertex is coloured with probability at least $p > 0$. Then*

$$\chi_B^f(G) \leq \frac{q}{p}. \quad (8.2)$$

Proof. The graph G^{2R} is Borel and has bounded degree. Hence it has a Borel proper colouring $h : X \rightarrow [L]$ for some finite L , by [KST99, Proposition 4.6]. In particular, h is injective on every ball $B_G(v, R)$.

Put $\Sigma := \Lambda^{[L]}$ and $M := |\Sigma|$. For each $\sigma \in \Sigma$, give v the label $\sigma(h(v))$, apply the rule, and let $I_{\sigma,j}$ be the set assigned colour j , for $j \in [q]$. Each $I_{\sigma,j}$ is independent. It is also Borel: the labels are Borel, and a bounded-radius rule on a bounded-degree Borel graph with a finite alphabet is Borel. One can verify the latter assertion by partitioning vertices according to the finitely many possible labelled rooted neighbourhood types; these partitions are Borel by the Lusin–Novikov theorem, since all neighbourhoods are finite [Kec95, Theorem 18.10].

Choose σ uniformly from Σ . The labels on $B_G(v, R)$ are then independent and uniform, since the values of h on that ball are distinct. Consequently, if

$$m(v) := |\{\sigma \in \Sigma : v \text{ is coloured under } \sigma\}|,$$

then $m(v) \geq pM$. If X is empty there is nothing to prove. Otherwise, put $b := \lceil pM \rceil$. Since $m(v)$ is integral, $m(v) \geq b$ for every v . Every vertex belongs to exactly $m(v)$ of the qM sets $I_{\sigma,j}$. Keeping its first b memberships gives a Borel b -fold colouring with qM colours. Thus $\chi_B^f(G) \leq qM/b \leq q/p$. $\square$

Corollary 8.2. *For every sufficiently large integer D , every triangle-free Borel graph G of maximum degree at most D satisfies*

$$\chi_B^f(G) \leq \frac{D}{\ln D - 7 \ln \ln D - 6} = (1 + o(1)) \frac{D}{\ln D}. \quad (8.3)$$

Proof. Regard the independent set in Theorem 1.1 as a proper partial 1-colouring, and apply Lemma 8.1 with $p := (\ln D - 7 \ln \ln D - 6)/D$. $\square$

8.2 Transferring the LOCAL algorithm

A colouring problem is *locally checkable* if a fixed-radius neighbourhood determines whether the constraints at a vertex are satisfied. In particular, proper q -colouring is locally checkable with radius 1. We use two transfer theorems in the following form. Let $\mathcal{C}$ be a class of finite graphs of bounded degree, closed under adding isolated vertices, and suppose that a locally checkable problem on $\mathcal{C}$ has a randomised LOCAL algorithm taking $o(\ln n)$ rounds on n -vertex graphs, with failure probability at most $1/n$. If every finite induced subgraph of a bounded-degree Borel graph G belongs to $\mathcal{C}$, then:

1. the problem has a μ -measurable solution on G for every Borel probability measure μ , by [Ber23, Theorem 2.14(i)];
2. the problem has a Baire-measurable solution on G for every Polish topology compatible with its Borel structure, by [Ber23, Theorem 2.14(ii)];
3. if $\text{asi}(G) < \infty$, the problem has a Borel solution on G , by [BW25, Theorem 1.42].

The little- o condition concerns $n \rightarrow \infty$ with the degree bound and the colouring problem fixed. The solution may depend on the measure or the chosen topology.¹

The randomised model in these transfer statements uses independent finite random labels in place of preassigned distinct identifiers. We make the normalisation explicit. Fix a degree bound

¹The fixed-parameter distributed Local Lemma input in [Ber23, Theorem 5.10] also follows from the corrected analysis. If $p(d+1)^8 \leq 2^{-15}$ with fixed $d \geq 2$, take $q_0 := 1/\lceil 4(d+1) \rceil$ and $\zeta := 1/\ln d$ in the technical form of [GHMN26, Theorem 1, Section 5.4]. Then $q_0 < 1/\lceil e(d+1) \rceil$ and $p 2(d+1)d^{6+\zeta}/q_0 \leq 8e 2^{-15} < 1$. Together with their Lemma 9(1), whose slack condition holds since $eq_0 d(1+1/4) < 5e/16 < 1$, this gives $o(\ln n)$ rounds for fixed d , as required by the transfer argument. The cases $d \leq 1$ are elementary.

D and a palette supplied by Theorem 1.2. Given an n -vertex triangle-free graph H of maximum degree at most D , where $n \geq 1$, advertise $N := 4n^3 + D + 1$ vertices. For the analysis, pad H with a disjoint D -star and isolated vertices to obtain a graph on N vertices whose maximum degree is exactly D . Each actual processor chooses an independent uniform identifier in $[N]$ and runs the algorithm with advertised size N and degree bound D , using the finite private random strings from Section 7.1. The added components need not be simulated, since they have no edges to H . Conditional on the actual identifiers being distinct, they extend to distinct identifiers on the padded graph, so the conditional failure probability on H is at most $1/N$. The union bound for identifier collisions therefore gives

$$\mathbb{P}(\text{failure on } H) \leq \frac{1}{N} + \frac{\binom{n}{2}}{N} \leq \frac{1}{n}. \quad (8.4)$$

This produces a rule with independent finite uniform labels. Since $\ln N = O_D(\ln n)$ as $n \rightarrow \infty$, the running time remains $o(\ln n)$ for fixed D .

Proof of Theorem 1.3. Set $\varepsilon := \delta/(1 + \delta)$ and $k := \ln D/(1 + \delta) = (1 - \varepsilon) \ln D$. For sufficiently large D , we have $k \geq 1$. Theorem 1.2, together with the normalisation above, gives a randomised algorithm for proper q -colouring on the class $\mathcal{C}_D$ of all finite triangle-free graphs of maximum degree at most D . For fixed D its running time satisfies

$$t_D(n) = O_\delta(\ln D) + \ln^{O(1)} \ln(n + D) = o(\ln n). \quad (8.5)$$

Every finite induced subgraph of G belongs to $\mathcal{C}_D$. Thus [Ber23, Theorem 2.14] proves Parts (2) and (4), and [BW25, Theorem 1.42] proves Part (3).

For Part (1), increase D_0 so that

$$\frac{D}{\ln D - 7 \ln \ln D - 6} \leq \left(1 + \frac{\delta}{2}\right) \frac{D}{\ln D} \leq \left\lfloor (1 + \delta) \frac{D}{\ln D} \right\rfloor = q.$$

Then Corollary 8.2 applies. $\square$

Remark 8.3 (The measure convention). Part (2) holds also for every σ -finite Borel measure μ . Indeed, if μ is nonzero, choose a Borel partition $X = \bigsqcup_{j \geq 1} A_j$ with $\mu(A_j) < \infty$, and put

$$\nu_0(B) := \sum_{j \geq 1} \frac{2^{-j} \mu(B \cap A_j)}{1 + \mu(A_j)}, \quad \text{and} \quad \nu(B) := \frac{\nu_0(B)}{\nu_0(X)}.$$

Then ν is a Borel probability measure with the same null Borel sets as μ , so their completed Borel σ -algebras agree. A ν -measurable colouring is therefore μ -measurable. For the zero measure, every subset is measurable, and an ordinary q -colouring exists by the finite result and compactness [dBE51, Theorem 1].

One cannot interpret “every Borel measure” as including arbitrary non- σ -finite measures. For counting measure on the Borel sets of an uncountable standard Borel space, the only null set is the empty set, so measurable colourings are exactly Borel colourings. Marks constructed, for every positive integer D , a Borel forest of maximum degree D with Borel chromatic number $D + 1$ [Mar16, Theorem 1.3]. These graphs exclude the asserted asymptotic bound for that interpretation.

9 Optimality of the constants

Let us discuss the optimality of the leading constants in our results.

The coefficient 1 in Theorem 1.1 is optimal. Indeed, Rahman and Virág [RV17, Theorem 2.1] proved that every factor of i.i.d. independent set on the infinite D -regular tree has density at most $(1 + o(1)) \ln D / D$ as $D \rightarrow \infty$; see also [Ber22, Theorem 3.1]. Applying any isomorphism-invariant finite-radius sampler to this tree gives such a factor. A finite uniform label can be sampled from a uniform label in $[0, 1]$, so the conclusion applies to the label convention used here. There is no restriction on how the radius depends on D . This does not assert optimality of the additive $\ln \ln D$ term, the determining radius, or the independent cap.

The fractional Borel and measurable lower bounds follow directly from Bernshteyn [Ber22, Corollaries 1.2 and 1.4]. For $m \geq 2$, let F_m be the free group of rank m , and let G_m be the Schreier graph of the free part of the Bernoulli shift $F_m \curvearrowright \{0, 1\}^{F_m}$, with respect to the standard free generators. Let β_m be the fair-coin product probability measure restricted to the free part. Every component of G_m is a $2m$ -regular tree, and the cited corollaries give

$$\chi_{\beta_m}(G_m) \geq \chi_B^f(G_m) = (2 + o(1)) \frac{m}{\ln m}. \quad (9.1)$$

The comparison between the measurable and fractional Borel chromatic numbers in (9.1) is specific to these shift graphs. Taking $m := \lfloor D/2 \rfloor$ shows that both parameters can be at least $(1 - o(1))D / \ln D$ for forests of maximum degree at most D . Thus the coefficient 1 in Theorem 1.3 Items 1 and 2 is best possible, even for acyclic graphs.

The same examples give a lower bound for randomised algorithms with sublogarithmic running time. Here the degree is fixed before the number of vertices tends to infinity.

Proposition 9.1. *Suppose that, for each sufficiently large D , an integer q_D and a randomised LOCAL algorithm are given such that every n -vertex triangle-free graph of maximum degree at most D is properly q_D -coloured with probability at least $1 - 1/n$ in $t_D(n)$ rounds. If $t_D(n) = o(\ln n)$ for every fixed D , then*

$$q_D \geq (1 - o(1)) \frac{D}{\ln D} \quad \text{as } D \rightarrow \infty. \quad (9.2)$$

Proof. Fix D and put $m := \lfloor D/2 \rfloor$. All finite induced subgraphs of G_m are triangle-free and have maximum degree at most D . We first put the algorithm in the finite-random-label model used by the transfer theorem. For an n -vertex input, advertise $N := 4n^3 + D + 1$ vertices, padding with a disjoint D -star and isolated vertices, and give each actual processor an independent uniform identifier in $[N]$. For each fixed N , there are only finitely many input graphs and assignments of distinct identifiers from $[N]$. On every successful execution only finitely many private random bits and local computation steps are used. Hence common finite cutoffs for this finite collection of inputs add at most $1/N$ to the global failure probability. If a cutoff is exceeded, use fixed default messages and output a fixed colour. The cutoffs need not be efficient or uniform in N , since the transfer theorem only requires a finite-alphabet radius rule for each chosen input size. Distinct identifiers on the actual input extend to distinct identifiers on the padded graph. Thus the total failure probability on the actual input is at most

$$\frac{2}{N} + \frac{\binom{n}{2}}{N} \leq \frac{1}{n}.$$

Since $\ln N = O_D(\ln n)$, the running time remains $o(\ln n)$ for fixed D .

The measurable transfer theorem [Ber23, Theorem 2.14], in the form stated in Section 8, now gives a β_m -measurable proper q_D -colouring of G_m . Consequently,

$$q_D \geq \chi_{\beta_m}(G_m) \geq \chi_B^f(G_m) = (1 + o(1)) \frac{D}{\ln D},$$

where the last equality follows from (9.1). □

For every fixed D and $\varepsilon > 0$, the running time in Theorem 1.2 is $o(\ln n)$. Taking $k := (1 - \varepsilon) \ln D$ and then letting ε tend to zero shows that its leading coefficient is optimal in this class of running times. This does not establish optimality of the precise round complexity in (1.4) or of the number of colours for finite graphs without a locality restriction.

The shift graphs above do not give the same lower bounds for Baire-measurable colourings. Conley and Miller [CM16] proved that every locally finite Borel graph G with $\chi(G) < \infty$ has a Baire-measurable colouring with at most $2\chi(G) - 1$ colours, for every compatible Polish topology. In particular, every locally finite Borel forest has such a 3-colouring.

For Borel colourings, Conley, Jackson, Marks, Seward and Tucker-Drob [CJM⁺23, Theorem 8.1] proved that every locally finite Borel graph G with $\chi(G) < \infty$ satisfies

$$\chi_B(G) \leq (\text{asi}(G) + 1)(\chi(G) - 1) + 1 \quad \text{if } \text{asi}(G) < \infty.$$

In particular, this gives $2\chi(G) - 1$ when $\text{asi}(G) \leq 1$, and $\text{asi}(G) + 2$ for forests. However, the above shift graphs G_m have infinite ASI.

Finite graphs have asymptotic separation index zero. A classical result of Bollobás [Bol81, Theorem 1 and Corollary 3] (see also [Bol78]) implies that, for every sufficiently large D , there is a finite triangle-free graph F_D of maximum degree D with

$$\chi(F_D) > \frac{D}{2 \ln D}. \tag{9.3}$$

Such graphs can be chosen D -regular and with any prescribed lower bound on their girth. Equipped with the discrete topology, finite graphs give the same lower bound for Baire-measurable colourings. Hence the optimal leading coefficient in each of the finite-ASI and Baire-measurable bounds lies between $1/2$ and 1 . Determining these coefficients remains open.

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AI disclosure

AI tools (ChatGPT 6) were used in various ways in this project, including generation of candidate proofs and writing of this manuscript. Although almost all key mathematical ideas of this manuscript were suggested by AI, iterative guidance was required in order to obtain these. The current draft was primarily written by AI, with the authors doing some further editing to improve the presentation. The authors plan to produce a properly written manuscript in the near future.

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