

Nonunimodal Laurent coefficients of cluster monomials

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Abstract

We prove that cluster monomials of coefficient-free type A_5 can have Laurent coefficient sequences that are neither unimodal nor log-concave. This disproves the log-concavity and unimodality assertions in Chen’s Conjecture 5.11 and gives a further family of counterexamples to Chen–Sun’s Conjecture 8.2, including its unimodality assertion.

AI use and scholarly credit

The results of this paper were generated using ChatGPT-6-Pro in conversations totaling approximately thirty minutes. The manuscript was subsequently polished and revised using Codex and underwent AI-assisted review with Rethlas-OSV [13, 16].

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1 Introduction

Log-concavity expresses a restriction on how neighboring coefficients can vary. A nonnegative sequence $(u_k)_{k \in \mathbb{Z}}$ is log-concave if $u_k^2 \geq u_{k-1}u_{k+1}$ for every integer k . On an interval of positive terms, these inequalities say that the successive ratios u_k/u_{k-1} are nonincreasing. A finite log-concave sequence whose positive terms form an interval is therefore unimodal: its terms first increase weakly and then decrease weakly. Newton’s inequalities give a classical source of log-concavity, through the coefficients of polynomials with nonnegative coefficients and only real roots. The connections with counting problems, Hilbert functions and convex geometry are developed in Stanley’s survey [20].

Representation-theoretic multiplicities give a related class of questions. Okounkov [17] connected their logarithms with entropy and proposed log-concavity phenomena for multiplicities. Chindris, Derksen and Weyman [5] constructed counterexamples to his conjecture for Littlewood–Richardson coefficients. Positive results hold in other settings. Huh, Matherné, Mészáros and St. Dizier [9] proved logarithmic concavity results for normalized Schur polynomials, obtained by dividing each monomial coefficient by the product of the factorials of its exponents. Their work applies the theory of Lorentzian polynomials developed by Brändén and Huh [1].

Fomin and Zelevinsky introduced cluster algebras in connection with total positivity and canonical bases [6], and classified their finite types [7]. A cluster is a distinguished transcendence basis of the ambient rational function field. New clusters are obtained by exchange relations, and a cluster monomial is a product of nonnegative integer powers of

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variables from one cluster. The Laurent phenomenon states that every cluster variable is a Laurent polynomial in the variables of any chosen cluster. The positivity problem then asks whether its coefficients are nonnegative. In coefficient-free type A_n , clusters correspond to triangulations of a convex $(n + 3)$ -gon; diagonals give variables, and flips give Ptolemy relations.

Combinatorial expansion formulas made positivity accessible in this model. Schiffler [18] gave a formula for type A using paths in a triangulated polygon. Musiker, Schiffler and Williams [15] obtained Laurent expansion formulas for cluster algebras from surfaces, including principal coefficients and punctures. Lee and Schiffler [12] subsequently proved positivity for all skew-symmetric cluster algebras. Gross, Hacking, Keel and Kontsevich [8] constructed theta functions with nonnegative multiplication constants; their construction includes cluster monomials and also proves Laurent positivity. These results make it natural to ask how the positive coefficients vary with their exponent vectors.

The geometry of the support gives another part of the picture. Shen [19] showed that supports of products of canonical basis elements for cluster X -varieties of type A_n are Stasheff polytopes in tropical coordinates. This concerns the set of nonzero coefficients. Log-concavity additionally compares their sizes, and in the present paper the coefficients come from the commutative cluster $\mathcal{A}$ -algebra.

Chen, Huang and Sun [3] formulated the cluster log-concavity problem in a preprint of 2024, published in 2026. For a Laurent polynomial, they fix all but one exponent and require log-concavity of the resulting coefficient sequence. They proved this property for cluster variables of type A_n and cluster monomials of type A_2 . Their Conjecture 5.5 extends the assertion to cluster monomials of type A_n , $n \geq 3$. Their Conjecture 1.1 also asks for log-concavity of theta multiplication constants in every cluster chart, with the two input labels fixed and the output label varying; it includes all finite types.

In January 2026, Chen [2, Theorem 5.1] proved log-concavity and unimodality for every cluster monomial of type A_3 . The argument uses coefficient formulas and real-rootedness through hypergeometric and Jacobi polynomials. Conjecture 5.11 of the same paper has two parts. The first asserts that every cluster monomial of type A_n , $n \geq 4$, is log-concave and unimodal. The second asserts that these properties are independent of the initial seed. Chen also established unimodality for cluster variables of type A_n and cluster monomials of type A_2 [2, Proposition 3.8]. Thus the higher-rank question extends both the earlier coefficient inequalities and their associated unimodality statements.

In October 2026, Chen and Sun [4] proved log-concavity and unimodality in every rank when the initial triangulation is a fan, meaning that all its diagonals meet one vertex. Their coefficient formula uses products of interval factors whose intervals are nested or disjoint. The proof treats the parity of the rank separately, using inequalities along integer lines in even rank and convolution in odd rank. Their Conjecture 8.2 restates Chen's seed-independence conjecture for a fixed cluster monomial expanded in different initial clusters. The established cases are therefore all initial seeds in rank three and fan seeds in arbitrary rank. The extension to other initial seeds is the assertion addressed here.

A related line of work concerns single-variable specializations of cluster expansions. The earlier study of q -deformed rational numbers by Morier-Genoud and Ovsienko [14] led to unimodality questions for rank polynomials of fence posets. These polynomials count order ideals by cardinality; an order ideal is a subset containing every element below any of its members. Kantarcı Oğuz and Ravichandran [11] proved their unimodality. Kang, Lee and Lim [10], in work first posted in August 2025, extended the analysis to loop fence posets arising from notched arcs on surfaces. They obtained unimodality for the rank polynomials associated with tagged arcs, both plain and notched.

In cluster language, these rank polynomials are obtained by setting all coefficient variables

of an F -polynomial equal to one variable q ; an F -polynomial itself is obtained from a cluster variable with principal coefficients by setting the initial cluster variables to 1. They also proved unimodality for the analogous specialization associated with a lamination consisting of a single curve on an unpunctured surface [10, Theorem 14], and conjectured log-concavity in that setting. The single-lamination statement uses the coefficient pattern determined by the curve. These specializations sum coefficients over exponent vectors, while the coordinate fibers studied here fix the other exponents.

We construct cluster monomials with a nonunimodal coordinate coefficient sequence in type A_5 . In an octagon with vertices $0, \dots, 7$, let p_{ij} denote the variable of the diagonal joining i and j , and set all boundary variables to 1. In the initial cluster

$$(p_{04}, p_{05}, p_{14}, p_{24}, p_{57}) = (a, b, c, d, e),$$

the compatible product $p_{27}^{2693} p_{36}^2$ has, with the exponents of a, c, d, e fixed at $-2641, -49, 2688, 47$, a complete five-term coefficient sequence proportional to

$$(293250, 292400, 294911, 53782, 23805).$$

Its first three terms form a strict valley. More generally, Theorem 4.1 gives a nonunimodal fiber of $p_{27}^r p_{36}^2$ for every pair of integers $q \geq 46$ and $r \geq q(q-2)$. It follows that unimodality of cluster monomials fails in arbitrary initial clusters and depends on the initial seed. The same valley also violates log-concavity, giving counterexamples to both assertions in Chen's Conjecture 5.11(1) and to both seed-independence assertions in Conjecture 5.11(2), restated as Chen–Sun's Conjecture 8.2.

The earlier family $p_{27}^r p_{36}$, $r \geq 20$, already gives counterexamples to the log-concavity assertion of Conjecture 8.2 [21]. Its displayed three-term fibers are decreasing. The present family $p_{27}^r p_{36}^2$ gives further counterexamples with nonunimodal fibers. Squaring the second compatible factor produces three binomial contributions to a five-term sequence, whose coefficients decrease and then increase. The calculation uses Ptolemy relations and coefficient extraction in the original cluster variables.

2 The cluster monomials and their expansions

We work over $\mathbb{Q}$ in the coefficient-free cluster algebra of type A_5 . The vertices of the octagon are labeled $0, \dots, 7$ in cyclic order. For $0 \leq i < j \leq 7$, write p_{ij} for an interior-diagonal variable; for boundary sides set $p_{i,i+1} = 1$ for $0 \leq i \leq 6$ and $p_{07} = 1$. The Ptolemy relation is

$$p_{ik}p_{jl} = p_{ij}p_{kl} + p_{il}p_{jk} \quad (0 \leq i < j < k < l \leq 7). \quad (1)$$

Fix the ordered cluster

$$T = (p_{04}, p_{05}, p_{14}, p_{24}, p_{57}) = (a, b, c, d, e).$$

Writing ijk for the triangle with vertices i, j, k , its triangles are 014, 124, 234, 045, 057, 567. The ordered cluster

$$U = (p_{02}, p_{26}, p_{27}, p_{35}, p_{36})$$

has triangles 012, 027, 236, 267, 345, 356. In particular, p_{27} and p_{36} are compatible, and

$$M_r = p_{27}^r p_{36}^2 \quad (r \in \mathbb{Z}_{\geq 0})$$

is a cluster monomial.

For a finite Laurent polynomial F in a, b, c, d, e and integers u, v, w, z , the expression $[a^u c^v d^w e^z]F$ retains exactly the terms with these four exponents and removes the indicated monomial, giving a Laurent polynomial in b . For example,

$$[a^{-1} c^0 d^0 e^0](2a^{-1}b + 3a^{-1}b^2 + a) = 2b + 3b^2.$$

We use the same convention for other subsets of variables; absent coefficients are zero. A coordinate fiber fixes all but one Laurent exponent and reads the coefficients at consecutive integer exponents of the remaining variable. Coordinate unimodality and coordinate log-concavity mean that every such sequence has the corresponding property. For unimodality, the mode is chosen after the other exponents have been fixed; see [4, Definitions 3.6–3.7]. We regard the zero sequence as unimodal. For integers $n \geq 0$, set $\binom{n}{k} = n!/(k!(n-k)!)$ when $0 \leq k \leq n$, and $\binom{n}{k} = 0$ otherwise, with $0! = 1$.

The quadrilaterals 0124, 0457, 0247 give

$$cp_{02} = a + d, \quad bp_{47} = 1 + ae, \quad ap_{27} = p_{02}p_{47} + d.$$

The quadrilaterals 0234, 4567, 0567, 0346 give

$$dp_{03} = p_{02} + a, \quad ep_{46} = 1 + p_{47}, \quad ep_{06} = 1 + b, \quad ap_{36} = p_{03}p_{46} + p_{06}.$$

Hence

$$p_{03} = \frac{a + d + ac}{cd}, \quad p_{46} = \frac{1 + b + ae}{be}, \quad p_{06} = \frac{1 + b}{e}.$$

Define the two numerator polynomials

$$\begin{aligned} P &= (a + d)(1 + ae) + bcd, \\ S &= (a + d)(1 + b + ae) + c(a(1 + b + ae) + bd(1 + b)). \end{aligned}$$

Substitution into the preceding Ptolemy equations gives

$$p_{27} = \frac{P}{abc}, \quad p_{36} = \frac{S}{abcde}, \quad M_r = \frac{P^r S^2}{a^{r+2} b^{r+2} c^{r+2} d^2 e^2}. \quad (2)$$

3 The five-term coefficient formula

Theorem 3.1. *For integers $r \geq q \geq 5$ and $j = 0, \dots, 4$, let A_j be the coefficient at exponent $-q - 2 + j$ of b in the coordinate fiber below. Then the entire fiber is*

$$[a^{q+3-r} c^{-q} d^{r-5} e^{q-2}]M_r = b^{-q-2} \sum_{j=0}^4 A_j b^j. \quad (3)$$

The five coefficients are positive integers given by

$$A_0 = \binom{q}{3} \binom{q+2}{2} \binom{r}{q}, \quad (4)$$

$$A_1 = 2(q+1) \binom{q+1}{4} \binom{r}{q} + 2(q+1) \binom{q}{4} \binom{r}{q-1}, \quad (5)$$

$$\begin{aligned} A_2 &= \left(\binom{q}{5} + 2(q+2) \binom{q}{4} + \binom{q}{3} \right) \binom{r}{q} \\ &\quad + \left(4 \binom{q}{4} + 2 \binom{q}{5} \right) \binom{r}{q-1} + \binom{q}{5} \binom{r}{q-2}, \end{aligned} \quad (6)$$

$$A_3 = 2 \binom{q+1}{5} \binom{r}{q} + 2 \binom{q}{5} \binom{r}{q-1}, \quad (7)$$

$$A_4 = \binom{q}{5} \binom{r}{q}. \quad (8)$$

Proof. The denominator in (2) shifts the requested exponents to

$$(q + 3 - r) + (r + 2) = q + 5, \quad -q + (r + 2) = r + 2 - q,$$

$$(r - 5) + 2 = r - 3, \quad (q - 2) + 2 = q.$$

Thus we extract $a^{q+5}c^{r+2-q}d^{r-3}e^q$ from $P^r S^2$. The power of c contributed by S^2 is 2, 1 or 0. In these three cases respectively, q , $q - 1$ or $q - 2$ factors of P contribute $(a + d)(1 + ae)$, with the other factors contributing bcd . Expanding in c gives

$$\begin{aligned} & [c^{r+2-q}]P^r S^2 \\ &= \binom{r}{q}(bd)^{r-q}(a+d)^q(1+ae)^q(a(1+b+ae)+bd(1+b))^2 \\ &+ 2\binom{r}{q-1}(bd)^{r-q+1}(a+d)^q(1+ae)^{q-1} \\ &\quad \cdot (1+b+ae)(a(1+b+ae)+bd(1+b)) \\ &+ \binom{r}{q-2}(bd)^{r-q+2}(a+d)^q(1+ae)^{q-2}(1+b+ae)^2. \end{aligned}$$

We compute the remaining coefficients for each summand.

In the first summand, expansion of the squared factor gives

$$a^2(1+b+ae)^2 + 2abd(1+b)(1+b+ae) + b^2d^2(1+b)^2.$$

The required e coefficients are

$$\begin{aligned} [e^q](1+ae)^q(1+b+ae)^2 &= a^q \left((1+b)^2 + 2q(1+b) + \binom{q}{2} \right), \\ [e^q](1+ae)^q(1+b+ae) &= a^q(q+1+b), \\ [e^q](1+ae)^q &= a^q. \end{aligned}$$

For the first equality, the two factors in the square contribute e degree 0, 1 or 2, giving respectively $(1+b)^2$, $2q(1+b)$ or $\binom{q}{2}$. The other two equalities follow from e degrees 0, 1 and 0, respectively. After including the displayed powers of a and d , the remaining terms of $(a+d)^q$ are

$$a^3d^{q-3}, \quad a^4d^{q-4}, \quad a^5d^{q-5}.$$

Their coefficients are $\binom{q}{3}$, $\binom{q}{4}$, $\binom{q}{5}$. Define the residual polynomial

$$\begin{aligned} H_0(b) &= \binom{q}{3} \left((1+b)^2 + 2q(1+b) + \binom{q}{2} \right) \\ &+ 2\binom{q}{4}b(1+b)(q+1+b) + \binom{q}{5}b^2(1+b)^2. \end{aligned}$$

The first summand therefore contributes $\binom{r}{q}b^{r-q}H_0(b)$.

In the second summand, multiply the last two factors to get

$$a(1+b+ae)^2 + bd(1+b)(1+b+ae).$$

Here

$$\begin{aligned} [e^q](1+ae)^{q-1}(1+b+ae)^2 &= a^q(2(1+b) + (q-1)) = a^q(q+1+2b), \\ [e^q](1+ae)^{q-1}(1+b+ae) &= a^q. \end{aligned}$$

The square must contribute e degree 1 or 2 in the first line, and the final factor must contribute degree 1 in the second. The remaining powers in $(a + d)^q$ are now $a^4 d^{q-4}$ and $a^5 d^{q-5}$, because the prefactor already contains d^{r-q+1} . Define the residual polynomial

$$H_1(b) = 2b \left(\binom{q}{4} (q + 1 + 2b) + \binom{q}{5} b(1 + b) \right).$$

Including the factor $2b$ relative to b^{r-q} , the second contribution is $\binom{r}{q-1} b^{r-q} H_1(b)$.

In the third summand, the only way to obtain e^q is to take the top power from both factors:

$$[e^q](1 + ae)^{q-2}(1 + b + ae)^2 = a^q.$$

The required remaining term of $(a + d)^q$ is $a^5 d^{q-5}$. Hence the third contribution is

$$\binom{r}{q-2} \binom{q}{5} b^{r-q+2}.$$

Combining the three summands and dividing by b^{r+2} proves

$$[a^{q+3-r} c^{-q} d^{r-5} e^{q-2}] M_r = b^{-q-2} \left(\binom{r}{q} H_0(b) + \binom{r}{q-1} H_1(b) + \binom{r}{q-2} \binom{q}{5} b^2 \right).$$

To collect coefficients, expand

$$(1 + b)^2 + 2q(1 + b) + \binom{q}{2} = \binom{q+2}{2} + 2(q+1)b + b^2$$

and $b(1+b)(q+1+b) = (q+1)b + (q+2)b^2 + b^3$. Using $\binom{q}{3} + \binom{q}{4} = \binom{q+1}{4}$ and $\binom{q}{4} + \binom{q}{5} = \binom{q+1}{5}$ gives (4)–(8). The degrees of H_0 and H_1 are at most 4 and 3, respectively; the third contribution inside the parentheses has degree 2, and all five coefficients of H_0 are positive for $q \geq 5$. Since $\binom{r}{q} > 0$, the support is exactly the five exponents stated in (3). $\square$

4 A strict valley and seed dependence

Theorem 4.1. *For every pair of integers $q \geq 46$ and $r \geq q(q-2)$, the coefficients in (3) satisfy $A_0 > A_1 < A_2$. Consequently M_r is neither coordinate unimodal nor coordinate log-concave in T .*

Proof. The binomial ratios are

$$\frac{\binom{r}{q-1}}{\binom{r}{q}} = \frac{q}{r-q+1}, \quad \frac{\binom{r}{q-2}}{\binom{r}{q}} = \frac{q(q-1)}{(r-q+1)(r-q+2)}.$$

Also $\binom{q}{4}/\binom{q}{3} = (q-3)/4$ and $\binom{q}{5}/\binom{q}{3} = (q-3)(q-4)/20$. Dividing (4) and (5) by the positive number $\binom{r}{q}\binom{q}{3}$ gives

$$\frac{A_0}{\binom{r}{q}\binom{q}{3}} = \frac{(q+1)(q+2)}{2}, \quad \frac{A_1}{\binom{r}{q}\binom{q}{3}} = \frac{(q+1)^2}{2} + \frac{q(q+1)(q-3)}{2(r-q+1)}.$$

Their difference is therefore

$$\frac{A_0 - A_1}{\binom{r}{q}\binom{q}{3}} = \frac{q+1}{2} \left(1 - \frac{q(q-3)}{r-q+1} \right) > 0,$$

because $r - q + 1 \geq q(q - 3) + 1 > q(q - 3)$.

For the second difference, the same divisions in (5)–(6) give

$$\begin{aligned}
\frac{20(A_2 - A_1)}{\binom{r}{q}\binom{q}{3}} &= (q - 3)(q - 4) + 10(q + 2)(q - 3) + 20 - 10(q + 1)^2 \\
&\quad + (20(q - 3) + 2(q - 3)(q - 4) - 10(q + 1)(q - 3)) \frac{q}{r - q + 1} \\
&\quad + \frac{q(q - 1)(q - 3)(q - 4)}{(r - q + 1)(r - q + 2)} \\
&= (q + 1)(q - 38) - \frac{2q(q - 3)(4q - 1)}{r - q + 1} \\
&\quad + \frac{q(q - 1)(q - 3)(q - 4)}{(r - q + 1)(r - q + 2)}.
\end{aligned}$$

The last term is positive. Since $r - q + 1 > q(q - 3)$, this expression is strictly greater than

$$(q + 1)(q - 38) - 2(4q - 1) = q^2 - 45q - 36 = (q - 46)(q + 1) + 10 \geq 10.$$

Thus $A_2 > A_1$. A unimodal sequence with $A_0 > A_1$ must have its mode at or before the first term, which would require $A_1 \geq A_2$. The strict valley excludes this possibility. Moreover, positivity gives $A_1^2 < A_0 A_2$, so log-concavity fails as well. $\square$

For an explicit example, take $q = 49$ and $r = 2693$. Since $2693 \geq 49 \cdot 47$, Theorem 4.1 applies. The exact fiber is

$$\begin{aligned}
&[a^{-2641}c^{-49}d^{2688}e^{47}]M_{2693} \\
&= \frac{\binom{49}{3}\binom{2693}{49}}{230} b^{-51}(293250 + 292400b + 294911b^2 + 53782b^3 + 23805b^4). \quad (9)
\end{aligned}$$

Indeed, the ratios used to substitute into (4)–(8) are

$$\frac{\binom{2693}{48}}{\binom{2693}{49}} = \frac{49}{2645}, \quad \frac{\binom{2693}{47}}{\binom{2693}{49}} = \frac{8}{23805}, \quad \frac{\binom{49}{4}}{\binom{49}{3}} = \frac{23}{2}, \quad \frac{\binom{49}{5}}{\binom{49}{3}} = \frac{207}{2}.$$

Substitution gives the five normalized coefficients individually:

$$\begin{aligned}
\frac{A_0}{\binom{49}{3}\binom{2693}{49}} &= 1275 = \frac{293250}{230}, \\
\frac{A_1}{\binom{49}{3}\binom{2693}{49}} &= 1250 + 1150 \cdot \frac{49}{2645} = \frac{292400}{230}, \\
\frac{A_2}{\binom{49}{3}\binom{2693}{49}} &= \frac{2555}{2} + 253 \cdot \frac{49}{2645} + \frac{207}{2} \cdot \frac{8}{23805} = \frac{294911}{230}, \\
\frac{A_3}{\binom{49}{3}\binom{2693}{49}} &= 230 + 207 \cdot \frac{49}{2645} = \frac{53782}{230}, \\
\frac{A_4}{\binom{49}{3}\binom{2693}{49}} &= \frac{207}{2} = \frac{23805}{230}.
\end{aligned}$$

This proves (9). The first two differences in its integer vector are -850 and 2511 .

Corollary 4.2. *The log-concavity and unimodality assertions in Chen’s Conjecture 5.11(1) are false. In particular, Conjecture 5.5 of Chen–Huang–Sun is false. Both coordinate unimodality and coordinate log-concavity of cluster monomials depend on the initial seed, disproving the assertions in Chen’s Conjecture 5.11(2), restated as Chen–Sun’s Conjecture 8.2.*

Proof. The type A_5 example (9) lies in the range $n \geq 4$ of Chen’s conjecture. In T its first three coefficients form a strict valley. In U the same element is the monomial $p_{27}^{2693} p_{36}^2$; each nonzero coordinate fiber then consists of one coefficient and has both properties. $\square$

References

- [1] P. Brändén and J. Huh, *Lorentzian polynomials*, Ann. of Math. **192** (2020), 821–891. [doi:10.4007/annals.2020.192.3.4](https://doi.org/10.4007/annals.2020.192.3.4).
- [2] Z. Chen, *Log-concavity and unimodality of cluster monomials of type A_3* , [arXiv:2601.21496v4](https://arxiv.org/abs/2601.21496v4) (2026).
- [3] Z. Chen, G. Huang and Z. Sun, *Log-concavity of cluster algebras of type A_n* , Nagoya Math. J. **261** (2026), e28, 1–19. [doi:10.1017/nmj.2026.10110](https://doi.org/10.1017/nmj.2026.10110).
- [4] Z. Chen and Z. Sun, *Log-concavity and unimodality of cluster monomials in fan coordinates of type A_n* , [arXiv:2610.09312v1](https://arxiv.org/abs/2610.09312v1) (2026).
- [5] C. Chindris, H. Derksen and J. Weyman, *Counterexamples to Okounkov’s log-concavity conjecture*, Compos. Math. **143** (2007), 1545–1557. [doi:10.1112/S0010437X07003090](https://doi.org/10.1112/S0010437X07003090).
- [6] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529. [arXiv:math/0104151](https://arxiv.org/abs/math/0104151).
- [7] S. Fomin and A. Zelevinsky, *Cluster algebras II: Finite type classification*, Invent. Math. **154** (2003), 63–121. [doi:10.1007/s00222-003-0302-y](https://doi.org/10.1007/s00222-003-0302-y).
- [8] M. Gross, P. Hacking, S. Keel and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31** (2018), 497–608. [arXiv:1411.1394v2](https://arxiv.org/abs/1411.1394v2).
- [9] J. Huh, J. P. Matherne, K. Mészáros and A. St. Dizier, *Logarithmic concavity of Schur and related polynomials*, Trans. Amer. Math. Soc. **375** (2022), 4411–4427. [doi:10.1090/tran/8606](https://doi.org/10.1090/tran/8606).
- [10] W. Kang, K. Lee and E. Lim, *Unimodality and cluster algebras from surfaces*, European J. Combin. **136** (2026), 104392. [doi:10.1016/j.ejc.2026.104392](https://doi.org/10.1016/j.ejc.2026.104392); [arXiv:2508.04396v3](https://arxiv.org/abs/2508.04396v3).
- [11] E. Kantarcı Oğuz and M. Ravichandran, *Rank polynomials of fence posets are unimodal*, Discrete Math. **346** (2023), 113218. [doi:10.1016/j.disc.2022.113218](https://doi.org/10.1016/j.disc.2022.113218).
- [12] K. Lee and R. Schiffler, *Positivity for cluster algebras*, Ann. of Math. **182** (2015), 73–125. [doi:10.4007/annals.2015.182.1.2](https://doi.org/10.4007/annals.2015.182.1.2).
- [13] S. Liu, *Rethlas-OSV: Desktop Math Agents for Mathematicians*, 29 September 2026, updated 20 October 2026. <https://srliu.com/blog/rethlas-osv-guide/>.
- [14] S. Morier-Genoud and V. Ovsienko, *q -deformed rationals and q -continued fractions*, Forum Math. Sigma **8** (2020), e13. [doi:10.1017/fms.2020.9](https://doi.org/10.1017/fms.2020.9).
- [15] G. Musiker, R. Schiffler and L. Williams, *Positivity for cluster algebras from surfaces*, Adv. Math. **227** (2011), 2241–2308. [doi:10.1016/j.aim.2011.04.018](https://doi.org/10.1016/j.aim.2011.04.018).
- [16] F. Nie, H. Ye, S. Liu, Z. Wang, S. Ermon and J. Zou, *One-Sided Verification Under Limited Expert Review*, [arXiv:2610.18089v2](https://arxiv.org/abs/2610.18089v2) (2026).

- [17] A. Okounkov, *Why would multiplicities be log-concave?*, in The Orbit Method in Geometry and Physics, Progr. Math. **213**, Birkhäuser, Boston, 2003, 329–347. [arXiv:math/0002085](#).
- [18] R. Schiffler, *A cluster expansion formula (A_n case)*, Electron. J. Combin. **15** (2008), R64. [doi:10.37236/788](#).
- [19] L. Shen, *Stasheff polytopes and the coordinate ring of the cluster X -variety of type A_n* , Selecta Math. (N.S.) **20** (2014), 929–959. [arXiv:1104.3528v2](#).
- [20] R. P. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry*, Ann. New York Acad. Sci. **576** (1989), 500–535. [doi:10.1111/j.1749-6632.1989.tb16434.x](#).
- [21] Q. Tang, *Counterexamples to log-concavity of cluster monomials and theta structure constants*, unpublished manuscript (2026).

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