

Counterexamples to log-concavity of cluster monomials and theta structure constants

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Abstract

We disprove Conjectures 1.1 and 5.5 of Chen–Huang–Sun and the log-concavity assertions in Chen’s Conjecture 5.11 and Chen–Sun’s Conjecture 8.2 by constructing counterexamples for cluster monomials and theta structure constants in coefficient-free type A_5 .

AI use and scholarly credit

The results of this paper were generated through two conversations with ChatGPT-6-Pro, totaling approximately two hours. The manuscript was subsequently polished and revised using Codex and underwent AI-assisted review with Rethlas-OSV [13, 16].

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1 Introduction

Log-concavity expresses a restriction on how neighboring coefficients can vary. A nonnegative sequence $(u_k)_{k \in \mathbb{Z}}$ is log-concave if $u_k^2 \geq u_{k-1}u_{k+1}$ for every integer k . On an interval of positive terms, these inequalities say that the successive ratios u_k/u_{k-1} are nonincreasing. A finite log-concave sequence whose positive terms form an interval is therefore unimodal: its terms first increase weakly and then decrease weakly. Newton’s inequalities give a classical source of log-concavity, through the coefficients of polynomials with nonnegative coefficients and only real roots. The connections with counting problems, Hilbert functions and convex geometry are developed in Stanley’s survey [20].

Representation-theoretic multiplicities give a related class of questions. Okounkov [17] connected their logarithms with entropy and proposed log-concavity phenomena for multiplicities. Chindris, Derksen and Weyman [5] constructed counterexamples to his conjecture for Littlewood–Richardson coefficients. Positive results hold in other settings. Huh, Mathorne, Mészáros and St. Dizier [9] proved logarithmic concavity results for normalized Schur polynomials, obtained by dividing each monomial coefficient by the product of the factorials of its exponents. Their work applies the theory of Lorentzian polynomials developed by Brändén and Huh [1].

Fomin and Zelevinsky introduced cluster algebras in connection with total positivity and canonical bases [6], and classified their finite types [7]. A cluster is a distinguished

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transcendence basis of the ambient rational function field. New clusters are obtained by exchange relations, and a cluster monomial is a product of nonnegative integer powers of variables from one cluster. The Laurent phenomenon states that every cluster variable is a Laurent polynomial in the variables of any chosen cluster. The positivity problem then asks whether its coefficients are nonnegative. In coefficient-free type A_n , clusters correspond to triangulations of a convex $(n + 3)$ -gon; diagonals give variables, and flips give Ptolemy relations.

Combinatorial expansion formulas made positivity accessible in this model. Schiffler [18] gave a formula for type A using paths in a triangulated polygon. Musiker, Schiffler and Williams [15] obtained Laurent expansion formulas for cluster algebras from surfaces, including principal coefficients and punctures. Lee and Schiffler [12] subsequently proved positivity for all skew-symmetric cluster algebras. Gross, Hacking, Keel and Kontsevich [8] constructed theta functions with nonnegative multiplication constants; their construction includes cluster monomials and also proves Laurent positivity. These results make it natural to ask how the positive coefficients vary with their exponent vectors.

The geometry of the support gives another part of the picture. Shen [19] showed that supports of products of canonical basis elements for cluster X -varieties of type A_n are Stasheff polytopes in tropical coordinates. This concerns the set of nonzero coefficients. Log-concavity additionally compares their sizes, and in the present paper the coefficients come from the commutative cluster $\mathcal{A}$ -algebra.

Chen, Huang and Sun [3] formulated the cluster log-concavity problem in a preprint of 2024, published in 2026. For a Laurent polynomial, they fix all but one exponent and require log-concavity of the resulting coefficient sequence. They proved this property for cluster variables of type A_n and cluster monomials of type A_2 . Their Conjecture 5.5 extends the assertion to cluster monomials of type A_n , $n \geq 3$. Their Conjecture 1.1 also asks for log-concavity of theta multiplication constants in every cluster chart, with the two input labels fixed and the output label varying; it includes all finite types.

Chen [2, Theorem 5.1] proved log-concavity and unimodality for every cluster monomial of type A_3 . The argument uses coefficient formulas and real-rootedness through hypergeometric and Jacobi polynomials. Conjecture 5.11 of the same paper has two parts: the corresponding assertion in type A_n for $n \geq 4$, and independence of these properties from the initial seed.

Chen and Sun [4] proved log-concavity and unimodality in every rank when the initial triangulation is a fan, meaning that all its diagonals meet one vertex. Their coefficient formula uses products of interval factors whose intervals are nested or disjoint. The proof treats the parity of the rank separately, using inequalities along integer lines in even rank and convolution in odd rank. Their Conjecture 8.2 restates Chen's seed-independence conjecture. Together, these results establish the desired inequalities in all seeds in rank three and in fan seeds in arbitrary rank, leaving the behavior of products in other seeds to be determined.

A different specialization leads to the work of Kang, Lee and Lim [10]. The earlier study of q -deformed rational numbers by Morier-Genoud and Ovsienko [14] led to unimodality questions for rank polynomials of fence posets. These polynomials count order ideals by cardinality; an order ideal is a subset containing every element below any of its members. Kantarcı Oğuz and Ravichandran [11] proved their unimodality. Kang, Lee and Lim extended the analysis to loop fence posets arising from notched arcs on surfaces. In cluster language, these rank polynomials are obtained by setting all coefficient variables of an F -polynomial equal to one variable q ; an F -polynomial itself is obtained from a cluster variable with principal coefficients by setting the initial cluster variables to 1. They also proved unimodality for the analogous specialization associated with a lamination consisting of a single curve on an unpunctured surface [10, Theorem 14], and conjectured log-concavity in that setting. This specialization sums coefficients over exponent vectors; coordinate log-concavity

fixes the other exponents.

We give a family of counterexamples in coefficient-free type A_5 . In an octagon with vertices $0, \dots, 7$, let p_{ij} denote the variable associated to the diagonal joining i and j . Relative to the initial cluster of algebraically independent variables

$$(p_{04}, p_{05}, p_{14}, p_{24}, p_{57}) = (a, b, c, d, e),$$

the compatible product $p_{27}^{20}p_{36}$ has, when the exponents of a, c, d, e are fixed at $-14, -4, 17, 3$, respectively, the complete coefficient sequence in the remaining variable b

$$(145350, 53010, 19380) = 570(255, 93, 34).$$

The inequality fails because

$$93^2 - 255 \cdot 34 = -21.$$

Theorem 2.1 gives the family $p_{27}^r p_{36}$ for every integer $r \geq 20$. It disproves Conjecture 5.5 of [3] and the log-concavity assertion in Conjecture 5.11(1) of [2]. In a cluster containing p_{27} and p_{36} the same element is a monomial. This comparison disproves the log-concavity assertion in Conjecture 5.11(2) of [2], restated as Conjecture 8.2 of [4]. Clearing denominators turns the three coefficients into theta multiplication constants for one input pair and three collinear output labels, disproving Conjecture 1.1 of [3].

The calculation uses Ptolemy relations followed by the binomial theorem. Two contributions have different polynomial growth in r , producing the failed coefficient inequality. The passage to theta structure constants uses the independence statement for cluster-monomial labels in [8, Theorem 7.20].

2 An octagon and its Laurent coefficients

We work over $\mathbb{Q}$ in the coefficient-free cluster algebra of type A_5 . The octagon vertices are labeled $0, 1, \dots, 7$ in cyclic order. For $0 \leq i < j \leq 7$, write p_{ij} for the variable of an interior diagonal. For boundary sides set $p_{i,i+1} = 1$ for $0 \leq i \leq 6$ and $p_{07} = 1$. A triangulation determines a cluster; noncrossing diagonals are compatible. For $0 \leq i < j < k < l \leq 7$, the exchange relation is

$$p_{ik}p_{jl} = p_{ij}p_{kl} + p_{il}p_{jk}. \quad (1)$$

Fix the ordered cluster

$$T = (p_{04}, p_{05}, p_{14}, p_{24}, p_{57}) = (a, b, c, d, e). \quad (2)$$

Here ijk denotes the triangle with vertices i, j, k . The triangles of T are 014, 124, 234, 045, 057, 567. The diagonals 27 and 36 belong to the ordered triangulation

$$U = (p_{02}, p_{26}, p_{27}, p_{35}, p_{36}). \quad (3)$$

Its triangles are 012, 027, 236, 267, 345, 356, which exhibit the compatibility of its diagonals. Thus $M_r = p_{27}^r p_{36}$ is a cluster monomial for every integer $r \geq 0$.

For a finite Laurent polynomial F in a, b, c, d, e and integers u, v, w, z , the notation $[a^u c^v d^w e^z]F$ denotes the Laurent polynomial in b obtained by retaining exactly those terms with the indicated four exponents and removing the factors $a^u c^v d^w e^z$. Missing coefficients are zero. For example,

$$[a^{-1} c^0 d^0 e^0](2a^{-1}b + 3a^{-1}b^2 + a) = 2b + 3b^2.$$

Coefficient extraction in other subsets of variables has the same meaning. A Laurent polynomial with nonnegative coefficients is coordinate log-concave if each sequence obtained by fixing all but one exponent satisfies the adjacent log-concavity inequalities, with zero coefficients outside its finite support. This is the notion in [3, Definition 2.17]. For integers $r \geq 0$ and $0 \leq k \leq r$, we use $\binom{r}{k} = r!/(k!(r-k)!)$, with $0! = 1$; the binomial coefficient is zero when $k < 0$ or $k > r$.

Theorem 2.1. *For every integer $r \geq 4$,*

$$[a^{6-r}c^{-4}d^{r-3}e^3]M_r = 30\binom{r}{4}b^{-5} + \left(10\binom{r}{4} + 4\binom{r}{3}\right)b^{-4} + 4\binom{r}{4}b^{-3}. \quad (4)$$

The square of the middle coefficient minus the product of its neighbors is

$$\left(10\binom{r}{4} + 4\binom{r}{3}\right)^2 - 120\binom{r}{4}^2 = \frac{\binom{r}{3}^2}{4}(-5r^2 + 110r - 221). \quad (5)$$

Consequently M_r is not coordinate log-concave in T for every integer $r \geq 20$.

Proof. The quadrilaterals 0124, 0457, and 0247 give, in this order,

$$cp_{02} = a + d, \quad bp_{47} = 1 + ae, \quad ap_{27} = p_{02}p_{47} + d.$$

Solving these equations yields

$$p_{02} = \frac{a+d}{c}, \quad p_{47} = \frac{1+ae}{b}, \quad p_{27} = \frac{(a+d)(1+ae) + bcd}{abc}.$$

The quadrilaterals 0234, 4567, 0567, and 0346 similarly give the four equations

$$dp_{03} = p_{02} + a, \quad ep_{46} = 1 + p_{47}, \quad ep_{06} = 1 + b, \quad ap_{36} = p_{03}p_{46} + p_{06}.$$

Substitution gives

$$p_{03} = \frac{d+a+ac}{cd}, \quad p_{46} = \frac{1+b+ae}{be}, \quad p_{06} = \frac{1+b}{e},$$

and

$$p_{36} = \frac{(d+a+ac)(1+b+ae) + bcd(1+b)}{abcde}.$$

For the coefficient calculation, denote the two numerator polynomials by

$$\begin{aligned} P &= (a+d)(1+ae) + bcd, \\ S &= (a+d)(1+b+ae) + c(a(1+b+ae) + bd(1+b)). \end{aligned} \quad (6)$$

Then

$$M_r = \frac{P^r S}{a^{r+1}b^{r+1}c^{r+1}de}. \quad (7)$$

The denominator shifts the four requested exponents to

$$(6-r) + (r+1) = 7, \quad -4 + (r+1) = r-3, \quad (r-3) + 1 = r-2, \quad 3+1 = 4.$$

Thus we extract $a^7c^{r-3}d^{r-2}e^4$ from $P^r S$. Each factor of P contributes either bcd or $(a+d)(1+ae)$, and each term of S contains either zero or one factor of c . In the first case, exactly

$r - 3$ factors of P contribute bcd ; in the second, exactly $r - 4$ do. The numbers of choices are $\binom{r}{r-3} = \binom{r}{3}$ and $\binom{r}{r-4} = \binom{r}{4}$, respectively. Hence

$$\begin{aligned} [c^{r-3}]P^r S &= \binom{r}{3} (bd)^{r-3} (a+d)^4 (1+ae)^3 (1+b+ae) \\ &\quad + \binom{r}{4} (bd)^{r-4} (a+d)^4 (1+ae)^4 (a(1+b+ae) + bd(1+b)). \end{aligned}$$

In the first summand, extracting e^4 from $(1+ae)^3(1+b+ae)$ gives a^4 . Extracting the remaining a^3d from $(a+d)^4$ gives 4. Its contribution to $[a^7c^{r-3}d^{r-2}e^4]P^r S$ is therefore $4\binom{r}{3}b^{r-3}$.

For the second summand, extract $a^7d^2e^4$ from

$$(a+d)^4(1+ae)^4(a(1+b) + a^2e + bd(1+b)).$$

For $a(1+b)$, the e^4 term of $(1+ae)^4$ is a^4e^4 , leaving the coefficient of a^2d^2 in $(a+d)^4$. For a^2e , the required term is $4a^3e^3$, again leaving a^2d^2 . For $bd(1+b)$, the required term is a^4e^4 , leaving a^3d . Consequently the three contributions are

$$\begin{aligned} [a^7d^2e^4](a+d)^4(1+ae)^4a(1+b) &= (1+b)[a^2d^2](a+d)^4 = 6(1+b), \\ [a^7d^2e^4](a+d)^4(1+ae)^4a^2e &= 4[a^2d^2](a+d)^4 = 24, \\ [a^7d^2e^4](a+d)^4(1+ae)^4bd(1+b) &= b(1+b)[a^3d](a+d)^4 = 4b(1+b). \end{aligned}$$

Their sum is $30 + 10b + 4b^2$. Thus

$$[a^7c^{r-3}d^{r-2}e^4]P^r S = 4\binom{r}{3}b^{r-3} + \binom{r}{4}b^{r-4}(30 + 10b + 4b^2).$$

Dividing by b^{r+1} sends the exponents $r-4, r-3, r-2$ to $-5, -4, -3$ and proves (4). Since $\binom{r}{4} = (r-3)\binom{r}{3}/4$, the coefficient defect is

$$\begin{aligned} \left(10\binom{r}{4} + 4\binom{r}{3}\right)^2 - 120\binom{r}{4}^2 &= -20\binom{r}{4}^2 + 80\binom{r}{4}\binom{r}{3} + 16\binom{r}{3}^2 \\ &= \frac{\binom{r}{3}^2}{4}(-5(r-3)^2 + 80(r-3) + 64) \\ &= \frac{\binom{r}{3}^2}{4}(-5r^2 + 110r - 221). \end{aligned}$$

This proves (5). Finally, for $r \geq 20$,

$$5r^2 - 110r + 221 = 5(r-20)^2 + 90(r-20) + 21 > 0,$$

so the defect is negative. □

At $r = 20$, formula (4) reads

$$[a^{-14}c^{-4}d^{17}e^3]M_{20} = 570b^{-5}(255 + 93b + 34b^2). \quad (8)$$

Corollary 2.2. *Conjecture 5.5 of Chen–Huang–Sun is false, as is the log-concavity assertion in Chen’s Conjecture 5.11(1). Coordinate log-concavity of a cluster monomial depends on the initial seed, disproving the log-concavity assertion in Chen’s Conjecture 5.11(2), also stated as Conjecture 8.2 by Chen–Sun.*

Proof. The type A_5 cluster monomial M_{20} violates coordinate log-concavity in T by (8). In U it is a monomial, and every nonempty coordinate coefficient sequence has just one term. Thus it is coordinate log-concave in U . □

3 Theta multiplication constants

Let $\mathcal{A}$ be the coefficient-free cluster variety associated to our type A_5 algebra, obtained by gluing the cluster tori through their exchange maps. We use the following results of [8]. Its theta labels are integral tropical points of the dual cluster variety: each seed gives integer coordinates for a label, and the coordinate changes are the tropicalizations of the dual cluster transformations. Write Θ for the subset of labels indexing the middle algebra in [8, Theorem 0.3]. Relative to T , the label coordinates lie in $\mathbb{Z}^5$, ordered by (a, b, c, d, e) .

The middle algebra $\text{mid}(\mathcal{A})$ is the $\mathbb{Q}$ -algebra with basis $\{\vartheta_q : q \in \Theta\}$ constructed in that theorem. It has a homomorphism

$$\nu : \text{mid}(\mathcal{A}) \longrightarrow \text{up}(\mathcal{A}),$$

where $\text{up}(\mathcal{A})$ is the intersection, inside the ambient rational function field, of the Laurent polynomial rings over $\mathbb{Q}$ of all clusters. The multiplication constants $\alpha(g, h, q)$ are defined by the finite expansion

$$\vartheta_g \vartheta_h = \sum_{q \in \Theta} \alpha(g, h, q) \vartheta_q \quad (g, h \in \Theta). \quad (9)$$

They are nonnegative integers. Each cluster monomial has a label in Θ whose coordinates in its own cluster are its exponents; ν sends that theta element to the cluster monomial. Put $x = (a, b, c, d, e)$. For $\beta = (\beta_1, \dots, \beta_5) \in \mathbb{Z}_{\geq 0}^5$, write $x^\beta = a^{\beta_1} b^{\beta_2} c^{\beta_3} d^{\beta_4} e^{\beta_5}$. Then $\nu(\vartheta_\beta) = x^\beta$; see [8, Theorem 0.3(2), (3), (5)]. Here and below a nonnegative vector also denotes its theta label in the T chart. We take the coefficient-free normalization obtained by setting all principal coefficient variables to 1. This specialization gives coefficient 1 on the corresponding cluster monomial, as in the proof of [8, Theorem 7.16].

Lemma 3.1. *Let $g \in \Theta$, let $D \in \mathbb{Z}_{\geq 0}^5$, and suppose*

$$x^D \nu(\vartheta_g) = \sum_{\beta \in \mathbb{Z}_{\geq 0}^5} p_\beta x^\beta$$

is a polynomial, where $p_\beta \in \mathbb{Q}$ and all but finitely many p_β vanish. Then $\alpha(g, D, \beta) = p_\beta$ for every $\beta \in \mathbb{Z}_{\geq 0}^5$.

Proof. Since D is nonnegative, $\nu(\vartheta_D) = x^D$. Apply the homomorphism ν to (9) with $h = D$ and subtract the polynomial expansion. This gives the finite relation

$$\sum_{q \in \Theta} \alpha(g, D, q) \nu(\vartheta_q) - \sum_{\beta \in \mathbb{Z}_{\geq 0}^5} p_\beta \nu(\vartheta_\beta) = 0.$$

Theorem 7.20 of [8] states that, in a finite linear relation among the images $\nu(\vartheta_q)$, the coefficient at every cluster-monomial label is zero. For a fixed nonnegative β , the coefficient of that label in the displayed relation is exactly $\alpha(g, D, \beta) - p_\beta$. It therefore vanishes. $\square$

For an integer $r \geq 4$, let $g_r \in \Theta$ be the theta label whose coordinates in the ordered cluster U are $(0, 0, r, 0, 1)$. Thus $\nu(\vartheta_{g_r}) = M_r$. In the T chart define the denominator exponent vector D_r and the three output labels β_j by

$$D_r = (r+1, r+1, r+1, 1, 1), \quad \beta_j = (7, r-4+j, r-3, r-2, 4) \quad (j = 0, 1, 2).$$

All D_r and β_j lie in the initial nonnegative cone.

Theorem 3.2. *For each integer $r \geq 20$, the fixed input pair (g_r, D_r) has output constants*

$$\begin{aligned}\alpha(g_r, D_r, \beta_0) &= 30 \binom{r}{4}, \\ \alpha(g_r, D_r, \beta_1) &= 10 \binom{r}{4} + 4 \binom{r}{3}, \\ \alpha(g_r, D_r, \beta_2) &= 4 \binom{r}{4}.\end{aligned}$$

They violate the midpoint log-concavity inequality. Consequently Conjecture 1.1 of Chen–Huang–Sun is false already in coefficient-free type A_5 .

Proof. The Ptolemy calculation in the proof of Theorem 2.1 gives the polynomial identity

$$\begin{aligned}x^{D_r} M_r &= ((a+d)(1+ae) + bcd)^r \\ &\quad \cdot ((a+d)(1+b+ae) + c(a(1+b+ae) + bd(1+b))).\end{aligned}$$

Lemma 3.1 identifies $\alpha(g_r, D_r, \beta_j)$ with the coefficient of $a^7 b^{r-4+j} c^{r-3} d^{r-2} e^4$ in this polynomial. After division by x^{D_r} , these exponents become

$$(6-r, -5+j, -4, r-3, 3).$$

Thus the constants are exactly the three coefficients in (4).

In the chosen chart, $2\beta_1 = \beta_0 + \beta_2$. The definition preceding [3, Conjecture 1.1] requires, for fixed inputs, a log-concave function interpolating the nonzero output constants in every cluster chart. Here log-concavity of a positive function f on a convex set means

$$f((1-t)u + tv) \geq f(u)^{1-t} f(v)^t \quad (0 \leq t \leq 1)$$

for any points u, v in its domain. Taking $u = \beta_0$, $v = \beta_2$ and $t = 1/2$ gives $f(\beta_1)^2 \geq f(\beta_0)f(\beta_2)$. All three constants are positive, and (5) contradicts this inequality. $\square$

For $r = 20$, the inputs are $(g_{20}, (21, 21, 21, 1, 1))$ and the three outputs are

$$(7, 16, 17, 18, 4), \quad (7, 17, 17, 18, 4), \quad (7, 18, 17, 18, 4).$$

Their constants are 145350, 53010, 19380, with defect -6822900 .

References

- [1] P. Brändén and J. Huh, *Lorentzian polynomials*, Ann. of Math. **192** (2020), 821–891. [doi:10.4007/annals.2020.192.3.4](https://doi.org/10.4007/annals.2020.192.3.4).
- [2] Z. Chen, *Log-concavity and unimodality of cluster monomials of type A_3* , [arXiv:2601.21496v4](https://arxiv.org/abs/2601.21496) (2026).
- [3] Z. Chen, G. Huang and Z. Sun, *Log-concavity of cluster algebras of type A_n* , Nagoya Math. J. **261** (2026), e28, 1–19. [doi:10.1017/nmj.2026.10110](https://doi.org/10.1017/nmj.2026.10110).
- [4] Z. Chen and Z. Sun, *Log-concavity and unimodality of cluster monomials in fan coordinates of type A_n* , [arXiv:2610.09312v1](https://arxiv.org/abs/2610.09312) (2026).
- [5] C. Chindris, H. Derksen and J. Weyman, *Counterexamples to Okounkov’s log-concavity conjecture*, Compos. Math. **143** (2007), 1545–1557. [doi:10.1112/S0010437X07003090](https://doi.org/10.1112/S0010437X07003090).

- [6] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529. [arXiv:math/0104151](#).
- [7] S. Fomin and A. Zelevinsky, *Cluster algebras II: Finite type classification*, Invent. Math. **154** (2003), 63–121. [doi:10.1007/s00222-003-0302-y](#).
- [8] M. Gross, P. Hacking, S. Keel and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31** (2018), 497–608. [arXiv:1411.1394v2](#).
- [9] J. Huh, J. P. Matherne, K. Mészáros and A. St. Dizier, *Logarithmic concavity of Schur and related polynomials*, Trans. Amer. Math. Soc. **375** (2022), 4411–4427. [doi:10.1090/tran/8606](#).
- [10] W. Kang, K. Lee and E. Lim, *Unimodality and cluster algebras from surfaces*, European J. Combin. **136** (2026), 104392. [doi:10.1016/j.ejc.2026.104392](#); [arXiv:2508.04396v3](#).
- [11] E. Kantarcı Oğuz and M. Ravichandran, *Rank polynomials of fence posets are unimodal*, Discrete Math. **346** (2023), 113218. [doi:10.1016/j.disc.2022.113218](#).
- [12] K. Lee and R. Schiffler, *Positivity for cluster algebras*, Ann. of Math. **182** (2015), 73–125. [doi:10.4007/annals.2015.182.1.2](#).
- [13] S. Liu, *Rethlas-OSV: Desktop Math Agents for Mathematicians*, 29 September 2026, updated 20 October 2026. <https://srliu.com/blog/rethlas-osv-guide/>.
- [14] S. Morier-Genoud and V. Ovsienko, *q -deformed rationals and q -continued fractions*, Forum Math. Sigma **8** (2020), e13. [doi:10.1017/fms.2020.9](#).
- [15] G. Musiker, R. Schiffler and L. Williams, *Positivity for cluster algebras from surfaces*, Adv. Math. **227** (2011), 2241–2308. [doi:10.1016/j.aim.2011.04.018](#).
- [16] F. Nie, H. Ye, S. Liu, Z. Wang, S. Ermon and J. Zou, *One-Sided Verification Under Limited Expert Review*, [arXiv:2610.18089v2](#) (2026).
- [17] A. Okounkov, *Why would multiplicities be log-concave?*, in The Orbit Method in Geometry and Physics, Progr. Math. **213**, Birkhäuser, Boston, 2003, 329–347. [arXiv:math/0002085](#).
- [18] R. Schiffler, *A cluster expansion formula (A_n case)*, Electron. J. Combin. **15** (2008), R64. [doi:10.37236/788](#).
- [19] L. Shen, *Stasheff polytopes and the coordinate ring of the cluster X -variety of type A_n* , Selecta Math. (N.S.) **20** (2014), 929–959. [arXiv:1104.3528v2](#).
- [20] R. P. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry*, Ann. New York Acad. Sci. **576** (1989), 500–535. [doi:10.1111/j.1749-6632.1989.tb16434.x](#).

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