

First-order phase transition in Onsager's model of hard rods in $\mathbb{R}^3$

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Abstract

We prove a first-order transition for hard rods in $\mathbb{R}^3$ whose orientations vary continuously over the sphere. For every sufficiently small positive ratio of diameter to length, the grand-canonical pressure has a nondifferentiable point, with a jump in number density at the intermediate Onsager scale. Following the phase-pressure strategy of Lebowitz, Mazel, and Presutti, we connect coexistence in the homogeneous Onsager functional to the microscopic pressure through an exact auxiliary-field representation, exponential decay of correlations in the rod gas at fixed auxiliary field, and estimates of the free-energy cost of interfaces between connected families of minimizing angular profiles.

1 The pressure transition and the proof strategy

Let $d\omega$ be probability area measure on $\mathbb{S}^2$. A rod of axial length one and diameter $\varepsilon > 0$, with centre x and axis u , occupies

$$\mathcal{R}_\varepsilon(x, u) = x + [-1/2, 1/2]u + B(0, \varepsilon/2).$$

The orientation varies continuously over $\mathbb{S}^2$, with u and $-u$ describing the same rod. For a bounded region Λ , activity $z > 0$, and admissible exterior configuration ξ , define

$$Z_{\Lambda, \varepsilon}(z \mid \xi) = \sum_{N \geq 0} \frac{z^N}{N!} \int_{(\Lambda \times \mathbb{S}^2)^N} \mathbf{1}_{\text{hc}}(X_1, \dots, X_N; \xi) \prod_{j=1}^N dx_j d\omega(u_j). \quad (1.1)$$

The indicator requires disjoint rod interiors. Centres in Λ are integrated; the exterior rods are fixed. For $\Lambda_s = [-s/2, s/2]^3$, write

$$P_\varepsilon(a) = \lim_{s \rightarrow \infty} \frac{1}{|\Lambda_s|} \log Z_{\Lambda_s, \varepsilon}(\varepsilon^{-1} e^a \mid \emptyset). \quad (1.2)$$

Theorem 1.1 (Nondifferentiability of the pressure). *The pressure P_ε exists, is finite, and is convex in a . There are $a_t \in \mathbb{R}$, $\varepsilon_0 > 0$, and constants $\Delta, c, C > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there is a_ε , with $a_\varepsilon \rightarrow a_t$ as $\varepsilon \downarrow 0$, satisfying*

$$P'_{\varepsilon, +}(a_\varepsilon) - P'_{\varepsilon, -}(a_\varepsilon) \geq \frac{\Delta}{2\varepsilon}, \quad \frac{c}{\varepsilon} \leq P'_{\varepsilon, \pm}(a_\varepsilon) \leq \frac{C}{\varepsilon}. \quad (1.3)$$

The one-sided derivatives are the limiting number densities from the corresponding sides of the transition.

Thus the transition occurs at number density of order ε^{-1} and occupied volume fraction of order ε . For axial length ℓ and diameter d , these scales become $(\ell^2 d)^{-1}$ and d/ℓ , respectively.

The strategy and its relation to LMP

Lebowitz, Mazel, and Presutti (LMP) [1, 2] transferred mean-field coexistence to a continuum system with a large but finite interaction range. Their method combines a variational description of the reference phases, estimates within each phase, control of interfaces, and an adjustment of the chemical potential that makes the corrected phase pressures equal. We follow this sequence. The model-specific work replaces their Kac interactions by hard exclusion and their scalar density minima by families of angular profiles. The contour step uses Pirogov–Sinai theory [3, 4], particularly Zahradník’s truncation and stability argument [5].

The conclusion here concerns the pressure. Once the physical pressure is identified with the maximum of phase pressures whose density responses are separated, their crossing proves the theorem. This avoids the further construction and analysis of distinct physical Gibbs measures carried out by LMP. The argument still needs locality of the reference gas and boundary estimates for restricted free energies; auxiliary Gibbs states are used to establish those estimates.

The following eleven steps are the subjects of the next eleven sections, in this order.

- 1. Identify the homogeneous phases.** The Onsager functional has an isolated isotropic minimizer and nonisotropic minimizers at one chemical potential. Their masses are separated. Analytic coordinates divide the complete minimizing set into finitely many connected components.
- 2. Introduce an exact auxiliary field.** Attach a normalized Gaussian variable to each spatial and angular cell. Integrating these variables restores the original partition function exactly. Conditional on the field, a quadratic count penalty provides the reference system used in the subsequent estimates.
- 3. Derive the variational limit.** The thin-rod overlap operator and a random graph representation give the finite-volume Onsager functional for the reference gas. This supplies the microscopic approximation that plays the role of coarse graining in LMP.
- 4. Prove locality of the reference gas.** The quadratic penalty makes the insertion generator coercive. Its inverse yields covariance decay uniformly in the field, the volume, and fixed exterior rods. This replaces LMP’s restricted reference-system uniqueness estimates.
- 5. Construct a positive field action.** Directional averaging splits the leading free energy into nonnegative terms. A transport rigidity theorem identifies their common zeros. Minimization over density profiles then gives a local field action and quantitative separation of its minimizing components.
- 6. Restore all particle counts.** A projection empties cells exceeding the reference count bound. A convergent expansion reconstructs their contribution and propagates the reference estimates to the exact rod integral.
- 7. Recover the exact field interaction.** Integrate the field response to express the exact interaction as the positive leading action plus a small summable error. A common additive normalization allows pressures with different reference profiles to be compared.
- 8. Control boundary costs.** Interpolate density profiles within a connected minimizing component and compare restricted field integrals. This treats the continuous families of angular minima. It gives surface bounds and a comparison of partition functions across a separating layer.
- 9. Assign contours and their costs.** Blocks where a phase cannot be continued form contours. Rigidity and isoperimetry supply their positive cost. The exact regional expansion retains the boundary terms needed to combine neighboring regions.

10. **Identify the physical pressure.** Following LMP's use of auxiliary phase pressures, truncate large contour weights and prove uniform surface estimates. An induction then shows that the physical pressure is the maximum of these phase pressures.
11. **Locate the crossing.** Differentiate the original Gaussian representation. The homogeneous mass gap gives separated increments of the complete phase pressures, while their leading values agree. Adjusting the chemical potential forces a crossing and hence the asserted density jump.

2 Homogeneous reference phases

Let $d\omega$ be probability area measure on S^2 and define

$$(Kf)(u) = \int_{S^2} |u \times v| f(v) d\omega(v), \quad \Omega_a(f) = \int_{S^2} f(\log f - 1 - a) d\omega + \langle f, Kf \rangle.$$

This is the grand-canonical form of Onsager's second-virial functional [6]. Profiles are nonnegative integrable functions; the entropy may be $+\infty$. Write $m_a = \inf \Omega_a$ and $\mathcal{W}_a = \{f : \Omega_a(f) = m_a\}$.

For bounds on fixed-mass Onsager minimizers, see Vollmer [7, Section 7.1]. The following lemma gives the bounds needed with variable mass.

Lemma 2.1 (Bounds and compactness of the minimizers). *The set $\mathcal{W}_a$ is nonempty. Every minimizer is even and satisfies*

$$\log f + 2Kf = a, \quad e^{a-2e^a} \leq f \leq e^a. \quad (2.1)$$

The minimizers are uniformly Lipschitz on the sphere. Their union over a compact parameter interval is relatively compact in the uniform norm, and limits at a convergent sequence of parameters are minimizers at the limiting parameter.

Proof. If $\rho = \int f d\omega$, Jensen's inequality and positivity of the kernel give

$$\Omega_a(f) \geq \rho \log \rho - (1 + a)\rho.$$

A minimizing sequence consequently has bounded mass and bounded positive entropy. It is uniformly integrable and has a weakly convergent subsequence in L^1 . The entropy is lower semicontinuous. The interaction is continuous under this convergence: approximate its continuous kernel uniformly on $S^2 \times S^2$ by finite sums of products of continuous functions, and use the uniform mass bound. This proves existence. Replacing $f(u)$ by $[f(u) + f(-u)]/2$ preserves its interaction and mass and strictly lowers its entropy unless f is even.

For positivity and the Euler equation, fix a minimizer f , set $V = 2Kf$, and consider $L(g) = \int g(\log g - 1 - a + V) d\omega$. Its unique minimizer is $g = e^{a-V}$. If $L(g) < L(f)$, convexity and the quadratic expansion give, for $f_t = (1 - t)f + tg$,

$$\Omega_a(f_t) - \Omega_a(f) \leq t[L(g) - L(f)] + t^2 \langle g - f, K(g - f) \rangle < 0$$

for small $t > 0$, a contradiction. Thus $f = e^{a-2Kf}$. Since $0 \leq Kf \leq \int f \leq e^a$, the bounds follow. The kernel is Lipschitz in its first argument, uniformly in the second. The Euler equation gives a uniform Lipschitz bound for f . Arzelà–Ascoli and continuity of the functional on the resulting bounded positive profiles prove the compactness assertions. $\square$

The spectrum below is the Onsager-kernel spectrum computed in [7, Corollary 10], divided by 4π because $d\omega$ is probability measure.

Lemma 2.2 (Angular spectrum). *On spherical harmonics of degree l , the eigenvalue of K is*

$$\kappa_0 = \frac{\pi}{4}, \quad \kappa_{2n} = -\frac{\pi}{2(2n-1)(2n+2)} \left(\frac{\binom{2n}{n}}{4^n} \right)^2 \quad (n \geq 1), \quad \kappa_{2n+1} = 0. \quad (2.2)$$

In particular $\inf_l \kappa_l = \kappa_2 = -\pi/32$, consecutive even eigenvalues are distinct, and $|\kappa_{2n}| = O(n^{-3})$.

Proof. The Funk–Hecke formula gives $\kappa_l = \frac{1}{2} \int_{-1}^1 \sqrt{1-t^2} P_l(t) dt$, where P_l is the Legendre polynomial. Put $w(t) = \sqrt{1-t^2}$ and denote the integrals of wP_l and P_l/w by I_l and J_l . Two integrations by parts in Legendre’s equation give $l(l+1)I_l = 2I_l - J_l$; the boundary terms vanish and the remaining endpoint singularities are integrable. Odd integrals vanish. The generating function gives

$$\sum_{l \geq 0} J_l z^l = \int_0^\pi (1 - ze^{i\theta})^{-1/2} (1 - ze^{-i\theta})^{-1/2} d\theta = \pi \sum_{n \geq 0} \left(\frac{\binom{2n}{n}}{4^n} \right)^2 z^{2n}.$$

Here conjugate off-diagonal terms combine into cosines with zero integral. This proves (2.2). Finally,

$$\frac{|\kappa_{2n+2}|}{|\kappa_{2n}|} = \frac{4n^2 - 1}{4(n+1)(n+2)} < 1,$$

and the central binomial coefficient bound gives the stated decay. $\square$

The isotropic critical point is the constant $r(a) > 0$ determined by $\log r + 2\kappa_0 r = a$; it increases continuously with a .

Onsager’s degree-two trial profiles show how global minimality can be lost before local stability [6, pp. 645–646]; see also Kayser and Raveché [8]. The following proof uses an explicit finite-amplitude comparison in the grand-canonical functional.

Theorem 2.3 (Homogeneous coexistence). *There is a_t for which $\mathcal{W} := \mathcal{W}_{a_t}$ contains an isolated isotropic profile $r_I = r(a_t) < 5$ and at least one nonisotropic profile.*

Proof. On the convex order interval $0 < f \leq M$, the Hessian obeys

$$D^2\Omega_a(f)[g, g] \geq \left(M^{-1} - \frac{\pi}{16} \right) \|g\|_2^2.$$

If $e^a < 16/\pi$, all minimizers lie in a strictly convex interval, so the minimizer is unique and rotation invariant. More generally, $r(a)$ is isolated among nearby critical points whenever $r(a) < 16/\pi$: choose $r(a) < M < 16/\pi$ and apply the same inequality in a small uniform neighborhood, uniformly for nearby a .

To find loss of global minimality before this threshold, take $g(u) = 1 + P_2(u \cdot n)$. It has mass one, lies between $1/2$ and 2 , and $\int P_2^2 d\omega = 1/5$. At density five,

$$\Omega_a(5g) - \Omega_a(5) = 5 \left(H - \frac{\pi}{32} \right), \quad H = \int_0^1 \frac{1+3t^2}{2} \log \frac{1+3t^2}{2} dt = \log 2 - 1 + \frac{2\pi}{9\sqrt{3}}.$$

This is strictly negative. Indeed, the series $\log 2 = 2 \sum_{n \geq 0} 3^{-2n-1}/(2n+1)$ gives $\log 2 < 1733/2500$. Together with $\sqrt{3} > 19/11$ and $\pi < 22/7$, this yields

$$H - \frac{\pi}{32} < \frac{1733}{2500} - 1 + \frac{22}{7} \left(\frac{22}{171} - \frac{1}{32} \right) = -\frac{8021}{11970000} < 0.$$

Let $a_5 = \log 5 + 10\kappa_0$. The same strict trial comparison holds near a_5 , while $r(a_5) = 5 < 16/\pi$.

Let $S = \{a : r(a) \in \mathcal{W}_a\}$ and set $a_t = \sup\{a : (-\infty, a] \subset S\}$. Compactness in Lemma 2.1 implies continuity of m_a and closedness of S . Thus $a_t \in S$ and $a_t < a_5$. There are $a_n \downarrow a_t$ with $a_n \notin S$. Choose minimizers f_n and take a uniformly convergent subsequence. Its limit f_* lies in $\mathcal{W}$ and cannot equal r_I : otherwise local strict convexity would force $f_n = r(a_n)$ eventually. Hence a nonisotropic well coexists with the isolated isotropic one. $\square$

Lemma 2.4 (Density separation). *There is $\Delta > 0$ such that*

$$\int_{S^2} f d\omega \geq r_I + \Delta \quad (f \in \mathcal{W} \setminus \{r_I\}). \quad (2.3)$$

Proof. We eliminate the interaction from Onsager's equal-pressure and equal-chemical-potential equations [6, Eq. (95)]. For $f \in \mathcal{W}$, write $f = \rho g$, $\int g = 1$, and $t = \rho/r_I$. The integrated Euler equation and equality of the minimum values give

$$\log \rho + \int g \log g + 2\rho \langle g, Kg \rangle = a_t, \quad \rho + \rho^2 \langle g, Kg \rangle = r_I + \kappa_0 r_I^2.$$

Eliminating the interaction yields

$$\int g \log g = -\log t + 2(1 + \kappa_0 r_I)(1 - t^{-1}).$$

The right side is strictly increasing on $(0, 1]$ and vanishes at one, whereas the left side is nonnegative. Thus $t \geq 1$, and equality forces $g = 1$. Finally $\mathcal{W} \setminus \{r_I\}$ is compact because r_I is isolated. Minimizing its positive mass difference proves (2.3). $\square$

2.1 The geometry of the well set

Let $\mathcal{P}_N$ project onto the even harmonic degrees at most $2N$, and put $\mathcal{Q}_N = I - \mathcal{P}_N$. The uniformly convergent zonal expansion and Lemma 2.2 show that

$$\|\mathcal{Q}_N K\|_{C \rightarrow C} \leq \sum_{n > N} (4n + 1) |\kappa_{2n}| \longrightarrow 0. \quad (2.4)$$

The reduction below is a global finite-mode version of the Lyapunov–Schmidt construction used for Onsager equilibria in [7, Section 3]: a uniform contraction on the harmonic tail covers the entire minimizing set.

Proposition 2.5 (Global analytic well coordinates). *For fixed a , there are a finite-dimensional space E , an open set $U \subset E$, a compact real analytic zero set $Z \Subset U$, and an analytic injective map $\Psi : U \rightarrow C_{\text{even}}(S^2)$ such that $\Psi(Z) = \mathcal{W}_a$.*

Proof. Set $j_- = a - 2e^a - 1$, $j_+ = a + 1$, and let $T_J(t) = \exp(\max(j_-, \min(t, j_+)))$. Choose N so large that $q_* = 2e^{j_+} \|\mathcal{Q}_N K\|_{C \rightarrow C} < 1$. For every $v \in E = \mathcal{P}_N C_{\text{even}}(S^2)$ the contraction

$$w \longmapsto -2\mathcal{Q}_N K T_J(v + w)$$

on $\mathcal{Q}_N C_{\text{even}}(S^2)$ has a unique fixed point $w(v)$. The fixed point depends globally Lipschitz continuously on v . Let U consist of the v for which $j_- < v + w(v) < j_+$ everywhere. On U the clipping is inactive. The derivative in w of $w + 2\mathcal{Q}_N K e^{v+w}$ is within q_* of the identity, so the analytic implicit function theorem makes w analytic on U .

For every well, (2.1) and the one-unit margin in J give $w(\mathcal{P}_N \log f) = \mathcal{Q}_N \log f$. Define

$$F(v) = \mathcal{P}_N[v + w(v) + 2Ke^{v+w(v)} - a], \quad G(v) = \Omega_a(e^{v+w(v)}) - m_a.$$

Both are real analytic. Their complete common zero set in U is exactly $Z = \mathcal{P}_N \log \mathcal{W}_a$: a common zero produces a positive profile of energy m_a , and every well satisfies both equations. Compactness and the positive lower bound in Lemma 2.1 show $Z \subseteq U$. The map $\Psi(v) = e^{v+w(v)}$ is injective because $\mathcal{P}_N \log \Psi(v) = v$. □

Corollary 2.6 (Finitely many well components). *The compact set $\mathcal{W}_a$ has finitely many connected components.*

Proof. The analytic set Z has finitely many connected components [9, Corollary 2.7]. Its continuous injection Ψ is a homeomorphism onto $\mathcal{W}_a$. □

Lemma 2.7 (Paths within a well component). *Any two wells in one component can be joined by a Lipschitz path with values in that component, in the uniform norm.*

Proof. A connected compact semianalytic set is rectifiably connected [9, Theorem 6.10]. The derivative of Ψ is bounded near Z , so its images of these curves are rectifiable in the uniform norm. Parametrize each image by arc length. □

3 An exact auxiliary field representation

Set $\beta = \varepsilon^{-1}$ and $\zeta = e^{a_t}$, where a_t is provided by Theorem 2.3. The parameter β measures rod thinness. For the source $h = a - a_t$, write

$$p_\varepsilon(h) = P_\varepsilon(a_t + h), \quad z = \beta\zeta e^h. \tag{3.1}$$

3.1 Cells, volumes, and reference measure

Partition space into congruent fine cubes B_i of volume v_s and partition the sphere into finitely many shape-regular cells U_a , of probability areas $\omega_a > 0$. A marked cell $C = (i, a)$ has measure $v_C = v_s \omega_a$. Let P denote orthogonal averaging on these marked cells. A coarse cube contains L^3 fine spatial cubes and has physical volume

$$V_L = v_s L^3.$$

Volumes in pressure exponents are Euclidean volumes. Surface estimates may absorb the fixed mesh factors in their constants.

Choose a preliminary source interval $|h| \leq H$, a pin strength α , and numbers

$$\rho > e\zeta e^H, \quad R > H + 2\alpha\rho + 2.$$

The choices of α and the mesh are specified by the operator estimates below. The count threshold in C is $K_C = \lceil \rho\beta v_C \rceil$. It is imposed only on a reference system. The original rod model in (1.1) remains unrestricted.

3.2 Normalized Gaussian augmentation

The completion of the square is the Gaussian auxiliary-field method of Stratonovich and Hubbard [10, 11]. We use normalized conditional densities on a bounded field interval, with the particle count reset above the threshold.

For a physical configuration η , let $n_C = N_C(\eta)$ and $\hat{n}_C = n_C \mathbf{1}_{\{n_C \leq K_C\}}$. Attach to each marked cell an independent field $\phi_C \in [-R, R]$, conditionally on η , with density

$$\frac{1}{c_{C,h}(\hat{n}_C)} \sqrt{\frac{\beta v_C}{4\pi\alpha}} \exp \left\{ -\frac{\beta v_C}{4\alpha} \left(\phi_C - h - \frac{2\alpha \hat{n}_C}{\beta v_C} \right)^2 \right\}. \quad (3.2)$$

The normalizer $c_{C,h}(n)$ is the Gaussian probability of $[-R, R]$. For sufficiently large β , all the means in (3.2) stay a fixed positive distance from the endpoints. Thus, uniformly in $0 \leq n \leq K_C$,

$$c_{C,h}(n) = 1 + O(e^{-c\beta}),$$

with the same conclusion for any fixed number of h -derivatives, after reducing $c > 0$.

Put

$$g_{C,h}(\phi) = \sqrt{\frac{\beta v_C}{4\pi\alpha}} e^{-\beta v_C(\phi-h)^2/(4\alpha)}.$$

Expansion of the square shows that, after extracting $g_{C,h}$, the exact cell weight is

$$w_{C,h,\phi}(n) = \begin{cases} (\beta\zeta)^n \exp\{\phi n - \alpha n^2/(\beta v_C)\} / c_{C,h}(n), & 0 \leq n \leq K_C, \\ (\beta\zeta e^h)^n / c_{C,h}(0), & n > K_C. \end{cases} \quad (3.3)$$

In particular, the second branch is independent of ϕ . All geometric integrals and factorials remain those of (1.1).

Use the probability reference $d\lambda_C = d\phi_C/(2R)$ for the fields. If $R_{\text{rod},\Lambda}(\phi \mid \xi)$ denotes the rod integral with weights (3.3), its full field density is

$$F_\Lambda(\phi \mid \xi) = (2R)^{|I(\Lambda)|} \prod_{C \in I(\Lambda)} g_{C,h}(\phi_C) R_{\text{rod},\Lambda}(\phi \mid \xi). \quad (3.4)$$

Here $I(\Lambda)$ is the set of marked cells over Λ . Throughout the field construction, a region is a union of complete fine spatial cubes and contains every angular cell above each of them. Since every conditional law (3.2) integrates to one, Tonelli's theorem gives the exact identity

$$Z_{\Lambda,\varepsilon}(\beta\zeta e^h \mid \xi) = \int F_\Lambda(\phi \mid \xi) d\lambda_\Lambda(\phi). \quad (3.5)$$

This equality holds for every finite volume and every admissible exterior configuration.

The explicit source contribution is

$$\log \frac{g_{C,h}(\phi_C)}{g_{C,0}(\phi_C)} = \frac{\beta v_C h \phi_C}{2\alpha} - \frac{\beta v_C h^2}{4\alpha}. \quad (3.6)$$

Its quadratic term sums to $-\beta h^2 |\Lambda|/(4\alpha)$, because $\sum_a \omega_a = 1$.

4 The finite-volume variational limit

We identify the free energy of the quadratically pinned gas. The spatial region and marked-cell partition remain fixed throughout the limit of vanishing diameter.

Let $D \subset \mathbb{R}^3$ be a bounded box and set $\Omega = D \times S^2$, with measure $d\nu = dx d\omega(u)$, where $\omega(S^2) = 1$. Functions on Ω are extended by zero in the spatial variable. Write $b_\varepsilon(X, Y)$ for the overlap indicator of the rods with marks $X = (x, u)$ and $Y = (y, v)$, and define

$$(Af)(x, u) = \int_{-1/2}^{1/2} f(x + tu, u) dt, \quad (Kf)(x, u) = \int_{S^2} |u \times v| f(x, v) d\omega(v), \quad T = 2A^*KA.$$

The operator T preserves positivity, is self-adjoint, has spatial range one, and has row integral $\pi/2$ on the full space. Fix a finite partition $\{C_i\}_{i=1}^m$ of Ω , put $v_i = \nu(C_i) > 0$, and let P be orthogonal averaging on these cells. For a measurable activity profile $0 < a \leq a_{\max} < \infty$, define

$$\mathcal{F}_a(q) = \int_{\Omega} q \left(\log \frac{q}{a} - 1 \right) d\nu + \frac{1}{2} \langle q, Tq \rangle + \alpha \|Pq\|_2^2. \quad (4.1)$$

At diameter ε , the reference partition function has activity a/ε , hard exclusion, the factor $\exp[-\alpha\varepsilon \sum_i N_i^2/v_i]$, and caps $N_i \leq \lceil \rho v_i/\varepsilon \rceil$.

Theorem 4.1 (Pinned finite-volume limit). *Suppose $T + 2\alpha P \geq -\kappa I$ and $\kappa a_{\max} < 1$. Then*

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log Z_{\varepsilon, \text{cap}} = - \min_{\substack{q \geq 0 \\ \int_{C_i} q d\nu \leq \rho v_i}} \mathcal{F}_a(q). \quad (4.2)$$

4.1 The overlap operator and common neighbors

The leading excluded volume is Onsager's angular interaction [6]. The following estimate strengthens this geometric limit to convergence in operator norm.

Lemma 4.2. *If B_ε is the integral operator with kernel b_ε , then*

$$\|B_\varepsilon/\varepsilon - T\|_{2 \rightarrow 2} \leq C\varepsilon, \quad \sup_X \int b_\varepsilon(X, Y) d\nu(Y) \leq C\varepsilon.$$

Proof. For fixed orientations u, v , the excluded displacement set is

$$[-1/2, 1/2]u + [-1/2, 1/2]v + \overline{B}(0, \varepsilon).$$

It contains the normal prism over the parallelogram, of thickness 2ε . Its volume is

$$2\varepsilon|u \times v| + 2\pi\varepsilon^2 + \frac{4\pi}{3}\varepsilon^3.$$

This follows by integrating the planar parallel-body formula over normal heights; the parallelogram has perimeter four. Parallel orientations follow by continuity. The remainder outside the prism has volume $O(\varepsilon^2)$ uniformly in u, v , so Schur's test bounds its operator norm by $C\varepsilon^2$.

For $k \in \mathbb{R}^3$ write $s_k(u) = \text{sinc}(k \cdot u/2)$, where $\text{sinc}(t) = \sin(t)/t$. The Fourier symbol of the prism divided by ε is the angular kernel

$$2|u \times v| s_k(u) s_k(v) \text{sinc}(\varepsilon k \cdot n(u, v)), \quad n(u, v) = \frac{u \times v}{|u \times v|}.$$

The symbol of T omits the last factor. Set

$$J(r) = \int_{S^2} s_k(u)^2 d\omega(u) \leq \frac{C}{1+r}, \quad r = |k|.$$

The angular Hilbert–Schmidt norm of the difference is at most $C \min(\varepsilon^2 r^2, 1) J(r) \leq C\varepsilon$. Plancherel and the remainder estimate prove the norm convergence. $\square$

Lemma 4.3 (Compactness of the limiting interaction). *The compression of T to a bounded spatial region is compact.*

Proof. Cut the symbol of T off to $|k| \leq R$. Its localization on the left to D has Hilbert–Schmidt norm squared

$$(2\pi)^{-3} |D| \int_{|k| \leq R} \|\widehat{T}(k)\|_{\text{HS}(L^2(S^2))}^2 dk < \infty.$$

Right localization preserves this property, and the Fourier tail has norm at most $C/(1+R)$, proving compactness. $\square$

For $d\mu = w d\nu$ with bounded $w \geq 0$, the limiting operator T_μ on $L^2(\mu)$ is defined by conjugation: the isometry $\mathcal{U}_\mu f = \sqrt{w}f$ maps $L^2(\mu)$ onto the subspace of $L^2(\nu)$ supported on $\{w > 0\}$, and

$$\mathcal{U}_\mu T_\mu \mathcal{U}_\mu^{-1} = M_{\sqrt{w}} T M_{\sqrt{w}}.$$

The same conjugation applies to $B_\varepsilon/\varepsilon$, proving norm convergence to the compact operator T_μ .

If the partitions refine with $P \rightarrow I$ strongly, compactness gives $\|T - PTP\| \rightarrow 0$. Since $PTP + 2\alpha P \geq (2\alpha - \|T\|)P$, the coercivity hypothesis of Theorem 4.1 holds on a sufficiently fine partition whenever $2\alpha > \|T\|$.

Lemma 4.4 (Common-neighbor bound). *Let μ have bounded density on Ω . For $X = (x, u)$ and $Y = (y, v)$ put $s = |u \times v|$, $d = |(y - x) \times u|$, and*

$$c_\varepsilon(X, Y) = \int b_\varepsilon(X, Z) b_\varepsilon(Y, Z) d\mu(Z).$$

Then

$$c_\varepsilon(X, Y) \leq \begin{cases} C\varepsilon, & s \leq \varepsilon, \\ C\varepsilon^2 s^{-1} (1 + \log(s/\varepsilon)), & s > \varepsilon, \end{cases} \quad (4.3)$$

$$c_\varepsilon(X, Y) \leq C\varepsilon \min\{1, (s + \varepsilon)/d\}. \quad (4.4)$$

The last quotient is interpreted as infinity when $d = 0$.

Proof. Fix the direction w of the common neighbor. Its possible centers lie in two slabs of thickness 2ε inside a fixed bounded ball. The sine of the angle between their normals is

$$\frac{|\det(u, v, w)|}{|u \times w| |v \times w|} \geq |\det(u, v, w)|.$$

Their intersection has volume at most $C \min\{\varepsilon, \varepsilon^2/|\det(u, v, w)|\}$. Integration over w , using $\det(u, v, w) = sn(u, v) \cdot w$, gives (4.3).

If both intersections occur, subtraction of the two intersection relations gives $y - x = tu - t'v + \tau w + e$, with $|t|, |t'| \leq 1/2$ and $|e| \leq 2\varepsilon$. Dotting with $u \times w$ yields $|((y - x) \times u) \cdot w| \leq Cs + C\varepsilon$. The allowed directions form a spherical belt of measure at most $C \min\{1, (s + \varepsilon)/d\}$. For each of them the center volume is at most $C\varepsilon$, proving (4.4). $\square$

For $r > \varepsilon$, splitting at $s = \sqrt{\varepsilon r}$ gives

$$\sup_{\max(s,d) \geq r} \frac{c_\varepsilon(X,Y)}{\varepsilon} \leq C \sqrt{\frac{\varepsilon}{r}} (1 + \log(1/\varepsilon)). \quad (4.5)$$

For independent marks of bounded density, the probability that both $s < r$ and $d < r$ is at most Cr^4 : the angular cap and the transverse spatial cylinder each contribute two powers of r .

4.2 Sampling the interaction uniformly over trial profiles

Operator norm control makes the interaction approximation uniform over weights chosen after sampling the point cloud.

Lemma 4.5 (Covariance sampling). *Let $v_1, \dots, v_m$ be independent copies of a random vector v in a separable Hilbert space with $\|v\|^2 \leq L$. Set $Q = m^{-1} \sum_j v_j \otimes v_j$, $A = \mathbb{E}(v \otimes v)$, and $k_m = (1 + 2\lceil \log m \rceil)^{1/2}$. Then, with $b_m = 2k_m \sqrt{L/m}$,*

$$\mathbb{E}\|Q - A\| \leq b_m \|A\|^{1/2} + b_m^2, \quad \mathbb{E}\|Q\| \leq (\|A\|^{1/2} + b_m)^2.$$

Proof. Let σ_j be independent signs taking values ± 1 with equal probability, independently of the vectors. Symmetrization and Tropp's matrix Rademacher inequality [12, Theorem 4.1] give

$$\begin{aligned} d := \mathbb{E}\|Q - A\| &\leq \frac{2}{m} \mathbb{E} \mathbb{E}_\sigma \left\| \sum_j \sigma_j (v_j \otimes v_j) \right\| \\ &\leq 2k_m \sqrt{\frac{L}{m}} \mathbb{E}\|Q\|^{1/2} \leq b_m \sqrt{\|A\| + d}. \end{aligned}$$

The matrices act on the sample span, of dimension at most m , and $(v \otimes v)^2 \leq L(v \otimes v)$. Solving the quadratic gives both bounds. Symmetrization follows by adjoining an independent sample and applying Jensen's inequality to these Bochner integrable trace-class operators. $\square$

Lemma 4.6 (Empirical operator approximation). *Suppose a real symmetric kernel R_ε on a probability space (Ω, μ) satisfies*

$$\|R_\varepsilon\|_\infty \leq C_0/\varepsilon, \quad \sup_y \int |R_\varepsilon(x, y)|^2 d\mu(x) \leq C_1/\varepsilon.$$

For independent $Z_1, \dots, Z_n$ of law μ , let B_n have entries $R_\varepsilon(Z_j, Z_k)/n$ off the diagonal and zero on the diagonal. Then, for even n ,

$$\mathbb{E}\|B_n\| \leq \|R_\varepsilon\|_{2 \rightarrow 2} + C \sqrt{\frac{\log(2n)}{n\varepsilon}} + \frac{C}{\varepsilon} e^{-cn\varepsilon}. \quad (4.6)$$

Proof. Write $n = 2m$ and first take independent samples $X_1, \dots, X_m$ and $Y_1, \dots, Y_m$. For $v_j = R_\varepsilon(\cdot, Y_j)$, the covariance operator $\Sigma_Y = m^{-1} \sum_j v_j \otimes v_j$ has expectation $R_\varepsilon R_\varepsilon^*$ and $\|v_j\|^2 \leq L = C_1/\varepsilon$. Lemma 4.5 gives

$$\mathbb{E}\|\Sigma_Y\| \leq (r_\varepsilon + b_1)^2, \quad r_\varepsilon = \|R_\varepsilon\|, \quad b_1 = 2k_m \sqrt{L/m}.$$

Define $r_Y(x) = m^{-1/2}(R_\varepsilon(x, Y_1), \dots, R_\varepsilon(x, Y_m))$. Its conditional covariance G_Y has the same nonzero eigenvalues as Σ_Y , by the identities $G_Y = J_Y^* J_Y$ and $\Sigma_Y = J_Y J_Y^*$ for $J_Y c = m^{-1/2} \sum_j c_j v_j$.

Put $S_Y(x) = \|r_Y(x)\|^2$. For each fixed x , its summands have mean at most L , bound $M = C_0^2/\varepsilon^2$, and variance at most ML . Scalar Bernstein therefore gives

$$\mathbb{P}_Y\{S_Y(x) > 2L\} \leq e^{-cm\varepsilon}, \quad \mathbb{E}_Y \mu\{x : S_Y(x) > 2L\} \leq e^{-cm\varepsilon}.$$

Truncate r_Y on this exceptional set. The truncated vectors have squared norm at most $2L$, and their covariance is bounded above by G_Y . Applying Lemma 4.5 conditionally on Y to the vectors at $X_1, \dots, X_m$ shows that the matrix $M_{ij} = R_\varepsilon(X_i, Y_j)/m$, with exceptional rows set to zero, has expected norm at most $r_\varepsilon + b_1 + b_2$, where $b_2 = 2k_m\sqrt{2L/m}$. The deleted rows have expected squared Frobenius norm at most $Me^{-cm\varepsilon}$. Thus

$$\mathbb{E}\|M\| \leq r_\varepsilon + C\sqrt{\frac{\log(2m)}{m\varepsilon}} + \frac{C}{\varepsilon}e^{-cm\varepsilon}.$$

For a fixed full cloud, choose uniformly a set S of m indices, and retain in B_n only entries crossing from S to S^c . Call the resulting symmetric matrix C_S . Each off-diagonal entry is retained with probability $n/[2(n-1)]$, so

$$B_n = \frac{2(n-1)}{n} \mathbb{E}_S C_S.$$

The norm of C_S is the norm of its off-diagonal block, whose entries are half those of M . Jensen's inequality gives $\mathbb{E}\|B_n\| \leq (n-1)\mathbb{E}\|M\|/n$, proving (4.6). $\square$

Apply this lemma to $R_\varepsilon = b_\varepsilon/\varepsilon - V$, where V is a fixed bounded symmetric finite rank kernel approximating T_μ on $L^2(\mu)$. Such kernels are dense among compact self-adjoint operators, because bounded functions are dense in $L^2(\mu)$. Lemma 4.2 supplies the required row-square bound. If $n\varepsilon/\log n \rightarrow \infty$, then

$$\limsup_{\varepsilon \downarrow 0} \mathbb{E}\|U_n - V_n\| \leq \|T_\mu - V\|, \quad (U_n)_{jk} = \frac{b_\varepsilon(Z_j, Z_k)}{n\varepsilon} \ (j \neq k), \quad (V_n)_{jk} = \frac{V(Z_j, Z_k)}{n}.$$

Here the diagonal correction is at most $\|V\|_\infty/n$. Consequently the quadratic form error is uniform over all $f \in [0, 1]^n$, since $n^{-1}\|f\|_2^2 \leq 1$.

4.3 The graph representation

For a finite graph G , let $Z_G(\lambda) = \sum_{I \subset V(G): I \text{ independent}} \lambda^{|I|}$. We use the graph case of Jenssen–Perkins–Potukuchi–Simkin [13, Theorem 1.4]. If the maximum degree Δ tends to infinity, the maximum common neighbor count is $o(\Delta)$, and $p\Delta$ stays in a compact subinterval of $(0, \infty)$, then, for $\lambda = p/(1-p)$,

$$\log Z_G(\lambda) = \max_{0 \leq x_j \leq p} \left\{ \sum_j \left(x_j - x_j \log \frac{x_j}{p} \right) - \sum_{\{j,k\} \in E(G)} x_j x_k \right\} + o(np) + O(np^2). \quad (4.7)$$

This follows by multiplying the independent-set probability by $(1+\lambda)^n$. Decreasing a coordinate above p improves both terms of the objective, justifying the displayed restriction.

Fix a vector of cell tilts $t = (t_i)$, set $a^{(t)} = ae^{-t_i}$ on C_i , and include the self-pin in

$$a_{\varepsilon,t} = a^{(t)}e^{-\alpha\varepsilon/v_i} \quad \text{on } C_i, \quad A_{\varepsilon,t} = \int a_{\varepsilon,t} d\nu, \quad d\mu_{\varepsilon,t} = a_{\varepsilon,t} d\nu / A_{\varepsilon,t}.$$

Take an even sample size $n \asymp \varepsilon^{-1-b}$, where $1/2 < b < 1$, and sample the marks with this probability law. In addition to every physical overlap edge, insert independently a pin edge for each same-cell pair with probability $p_i = 1 - e^{-2\alpha\varepsilon/v_i}$. Set $\lambda = A_{\varepsilon,t}/(n\varepsilon)$. Averaging the pin-edge avoidance weight gives $\exp[-\alpha\varepsilon \sum_i N_i(N_i - 1)/v_i]$; the self-pin supplies the missing diagonal part of N_i^2 .

Choose $\gamma \in ((1+b)/2, 1)$ and put $r = \varepsilon^\gamma$. By Lemma 4.4, the probability of any pair with both $s < r$ and $d < r$ is at most $Cn^2r^4 = o(1)$. On the complement, all common-neighbor probabilities are $o(\varepsilon)$ uniformly. Conditional Bernstein bounds and a union bound give physical maximum common-neighbor count $o(n\varepsilon)$ and maximum degree $O(n\varepsilon)$. Conditioning on the physical graph, mixed physical–pin and pin–pin common-neighbor counts are $O(n\varepsilon^2 + \log n) = o(n\varepsilon)$. Pin degrees are $O(n\varepsilon)$, and a cell of positive sampling mass supplies vertices of degree at least $cn\varepsilon$ with probability tending to one. Hence $cn\varepsilon \leq \Delta \leq Cn\varepsilon$, and $p\Delta$ is bounded above and below. These are precisely the hypotheses of (4.7) for graphs.

Matrix Bernstein [14, Theorem 1.4], conditional on the marks, gives

$$\left\| \frac{\text{pin adjacency} - \text{conditional mean}}{n\varepsilon} \right\| = O_{\mathbb{P}} \left(\sqrt{\frac{\log n}{n\varepsilon}} + \frac{\log n}{n\varepsilon} \right) = o_{\mathbb{P}}(1).$$

Indeed each centered edge matrix has norm at most one and the variance parameter is $O(n\varepsilon)$. The conditional mean, before physical overlaps are removed, converges to the finite rank kernel $2\alpha \sum_i \mathbf{1}_{C_i} \otimes \mathbf{1}_{C_i}/v_i$. Removing physical-overlap entries changes its normalized row norm by $O(\varepsilon)$, because the pin probabilities are $O(\varepsilon)$ and physical degrees are $O(n\varepsilon)$.

Set $x_j = pf_j$ in (4.7). The preceding operator estimates and finite rank approximation give

$$\varepsilon \log Z_G(\lambda) \longrightarrow \sup_{0 \leq f \leq 1} \left\{ A_t \int (f - f \log f) d\mu_t - \frac{1}{2} \langle a^{(t)} f, T(a^{(t)} f) \rangle_{L^2(\nu)} - \alpha \|P(a^{(t)} f)\|_{L^2(\nu)}^2 \right\}, \quad (4.8)$$

where $A_t = \int a^{(t)} d\nu$ and $d\mu_t = a^{(t)} d\nu / A_t$. Here $np\varepsilon \rightarrow A_t$.

To justify the variational maximum, express a bounded finite rank kernel in finitely many bounded features. The empirical joint laws of these features and weights in $[0, 1]$ have subsequential weak limits with the prescribed feature marginal. Conditional averaging of the weight given the features preserves the interaction moments and increases the concave entropy, giving the upper bound. Evaluating any fixed measurable continuum weight on the independent cloud gives the lower bound by the law of large numbers. The residual interaction is uniformly small by Lemma 4.6. Finally the sampling density $\mu_{\varepsilon,t}$ differs relatively from μ_t by $O(\varepsilon)$ at the fixed partition, which does not alter these limits.

Putting $q = a^{(t)}f$ in (4.8) gives $-\inf_{q \geq 0} \{\mathcal{F}_a(q) + \sum_i t_i \int_{C_i} q\}$. Clipping to $a^{(t)}$ decreases the entropy on $\{q > a^{(t)}\}$ and both nonnegative interactions, so the constraint $q \leq a^{(t)}$ does not change the minimum.

4.4 Annealing and the cap constraint

Let $Z_{\varepsilon,t}$ and $\mathbb{P}_{\varepsilon,t}$ be the partition function and Gibbs law of the uncapped pinned gas with activity $a^{(t)}/\varepsilon$. Write $(n)_N = n(n-1) \cdots (n-N+1)$. Averaging Z_G over positions and pin edges multiplies its N -particle integral by $(n)_N/n^N$ and discards terms above n . Thus $\mathbb{E}Z_G \leq Z_{\varepsilon,t}$. Repulsiveness bounds the factorial moments of the particle count by those of a Poisson variable of mean A_t/ε : insert prescribed particles and omit their repulsive factors. For a sufficiently large fixed K ,

$$\mathbb{P}_{\varepsilon,t}\{N > K/\varepsilon\} \leq e^{-c/\varepsilon}.$$

On $N \leq K/\varepsilon$, the binomial correction is bounded below by $\exp[-C/(n\varepsilon^2)]$. Consequently

$$0 \leq \log Z_{\varepsilon,t} - \log \mathbb{E} Z_G \leq C/(n\varepsilon^2) + o(1) = o(\varepsilon^{-1}).$$

Changing one sampled point changes only edges incident to its vertex. Deleting that vertex bounds the oscillation of $\log Z_G$ by $\log(1 + \lambda)$. Group the independent pin uniforms by their first endpoint; each group has the same oscillation bound. The exponential bounded-differences inequality [15] for these $2n$ independent blocks gives

$$0 \leq \log \mathbb{E} Z_G - \mathbb{E} \log Z_G \leq Cn\lambda^2 = O((n\varepsilon^2)^{-1}) = o(\varepsilon^{-1}).$$

The variables $\varepsilon \log Z_G$ are uniformly bounded, so their convergence in probability also implies convergence of expectations. This proves the following limit, without the coercivity hypothesis.

Proposition 4.7 (Uncapped variational limit). *For every fixed bounded tilt vector,*

$$\phi(t) := \lim_{\varepsilon \downarrow 0} \varepsilon \log Z_{\varepsilon,t} = - \inf_{q \geq 0} \left\{ \mathcal{F}_a(q) + \sum_i t_i \int_{C_i} q \, d\nu \right\}. \quad (4.9)$$

Lemma 4.8 (Uniqueness of the constrained minimizer). *Under the coercivity hypothesis of Theorem 4.1, $\mathcal{F}_a(q) + \sum_i t_i \int_{C_i} q$ has a unique minimizer for each $t_i \geq 0$, with or without the cell-mass constraints.*

Proof. Clipping restricts the minimization to $0 \leq q \leq a$. On this convex set the entropy curvature is at least $1/a_{\max}$, so (4.1) is strongly convex under the stated coercivity margin. The order interval is weakly compact in L^2 ; relative entropy is weakly lower semicontinuous, and compactness of T makes its quadratic form weakly continuous on bounded sets. This proves existence and uniqueness. $\square$

Lemma 4.9 (Multipliers for finitely many caps). *Let F be convex on a convex class $\mathcal{C}$, and let $m : \mathcal{C} \rightarrow [0, \infty)^k$ be affine. Suppose*

$$V(b) = \inf \{ F(q) : q \in \mathcal{C}, m(q) \leq b \}$$

is finite for every $b \in (0, \infty)^k$. For such a b , if q_b attains $V(b)$, there is $t \in [0, \infty)^k$ such that q_b minimizes $F(q) + t \cdot m(q)$ on $\mathcal{C}$ and $t_i(m_i(q_b) - b_i) = 0$ for every i .

Proof. The finite convex nonincreasing function V has a supporting subgradient $p \leq 0$ at b [16]. For $q \in \mathcal{C}$ and $\eta > 0$, use $V(m(q) + \eta \mathbf{1}) \leq F(q)$ in its supporting inequality and let $\eta \downarrow 0$. With $t = -p$ this gives

$$F(q) + t \cdot (m(q) - b) \geq V(b).$$

At q_b , every term in $t \cdot (m(q_b) - b)$ is nonpositive, so the inequality forces each to vanish. Equality there proves the tilted minimality. $\square$

Proof of Theorem 4.1. Apply Lemma 4.9 on $0 \leq q \leq a$, with $F = \mathcal{F}_a$, $m_i(q) = \int_{C_i} q$, and $b_i = \rho v_i$. The zero competitor and entropy lower bound make its value function finite. It gives nonnegative multipliers t_i and complementary slackness. Since $t_i \geq 0$, clipping extends tilted minimality to all $q \geq 0$. Its uniqueness remains true in a small open neighborhood of the tilt, because clipping gives $q \leq ae^s$ there and the coercivity margin is strict. Thus ϕ is differentiable there, with $\partial_{t_i} \phi = - \int_{C_i} q$. Applying exponential Markov bounds using (4.9) at $t \pm se_i$ proves exponential concentration of εN_i about this mass.

The cap upper bound follows by multiplying by $e^{-\sum_i t_i N_i}$ and using $N_i \leq \lceil \rho v_i / \varepsilon \rceil$; the ceiling contributes only $O(1)$ to the logarithm. For the lower bound solve the constrained problem at $\rho' < \rho$. Under its multiplier tilt, the concentration event lies strictly below all target caps. Undo the tilt on that event and use complementary slackness to obtain the lower bound $-\mathcal{F}_a(q_{\rho'})$. As $\rho' \uparrow \rho$, the comparison function $(\rho'/\rho)q_{\rho'}$ and bounded convergence give $\mathcal{F}_a(q_{\rho'}) \rightarrow \mathcal{F}_a(q_{\rho})$. $\square$

5 Locality of the reference gas

We now control covariances uniformly in the volume and exterior rods. This provides the locality supplied by the restricted-ensemble Dobrushin analysis in LMP [2, Sections 5–6]. For the gas conditioned on the auxiliary field, we use a coercive insertion operator. The method is the continuum Glauber coercivity argument, in particular the Mayer-kernel criterion of Kondratiev, Kuna, and Ohlerich [17, Theorem 4.2 and Corollary 4.3]. Here the cell pin makes the Mayer operator almost positive, and we estimate separately the terms produced by active count caps.

Use shape-regular marked cells with spatial side ℓ , angular diameter comparable with ℓ , and measures $v_c \asymp \ell^5$. The cells and α are fixed before $\varepsilon \downarrow 0$. Let $\beta = \varepsilon^{-1}$ and allow any measurable one-particle activity $0 \leq z(X) \leq M\beta$, including arbitrary fixed accessibility multipliers. Pair interactions are

$$U_\varepsilon(X, Y) = \begin{cases} +\infty, & b_\varepsilon(X, Y) = 1, \\ 2\alpha\varepsilon/v_c, & X, Y \in C_c \text{ and } b_\varepsilon(X, Y) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The self-pin $e^{-\alpha\varepsilon/v_c}$ is included in z . Impose caps $N_c \leq K_c$. A bounded measurable perturbation $J = \sum_A J_A$ is permitted, with supports consisting of complete cells, and norm

$$\kappa_{m,1} = \sup_i \sum_{A \ni i} |A| e^{m \text{diam}(A)} \|J_A\|_\infty. \quad (5.1)$$

Distances are measured on the spatial cell lattice; angular cells over the same spatial cube have distance zero. Constants may depend on the fixed mesh, α , and M . Write μ for the finite-volume Gibbs law, $D_X F(\eta) = F(\eta + X) - F(\eta)$ for insertion difference, and

$$r_X(\eta) = z(X) \mathbf{1}_{\{N_{c(X)}(\eta) < K_{c(X)}\}} \exp \left[- \sum_{Y \in \eta} U_\varepsilon(X, Y) - D_X J(\eta) \right]$$

for the insertion rate. Here $c(X)$ is the marked cell containing X . Forbidden insertions have rate zero.

Theorem 5.1 (Uniform covariance). *Choose $\alpha > \|T\| + 1$. For a sufficiently fine fixed mesh, all sufficiently small ε , and $\beta\kappa_{m,1}$ sufficiently small, there are $C, m > 0$, independent of the domain and accessibility mask, such that*

$$|\text{Cov}_\mu(F, G)| \leq C \sum_{i,j} e^{-md(i,j)} \mathcal{D}_i(F) \mathcal{D}_j(G), \quad \mathcal{D}_i(F)^2 = \mathbb{E}_\mu \int_{C_i} r_X(\eta) |D_X F(\eta)|^2 d\nu(X). \quad (5.2)$$

Here C_i comprises all angular cells over spatial cell i . The observables are square integrable and have finite displayed norms.

5.1 A lower bound for the Mayer operator

The limiting operator maps L^2 boundedly into spatial and angular H^1 . For the spatial derivative this follows from $\|\hat{T}(k)\| \leq C/(1 + |k|)$. For angular derivatives,

$$\int_{S^2} |\nabla_u s_k(u)|^2 d\omega(u) \leq \begin{cases} Cr, & r \geq 1, \\ Cr^4, & r \leq 1, \end{cases}$$

by the substitution $t = ru \cdot \widehat{k}/2$. Combining this estimate with $J(r) \leq C/(1+r)$ for the other sinc factor bounds the angular Hilbert–Schmidt norm of the differentiated symbol uniformly. The weak derivative of $|u \times v|$ is bounded. Cell Poincare inequalities therefore imply

$$\|(I - P_\ell)T\| \leq C\ell, \quad \|T - P_\ell T P_\ell\| \leq 2C\ell. \quad (5.3)$$

For $X \in C_c$ and $\varepsilon \leq \ell$,

$$\int_{C_c} b_\varepsilon(X, Y) d\nu(Y) \leq C\varepsilon\ell^4. \quad (5.4)$$

Indeed, the center displacement has the form $tu - sv + e$. In a single spatial cell $t - s = O(\ell)$, while a single angular cell gives $|u - v| = O(\ell)$. The relevant planar area is $O(\ell^2)$ and its thickening has volume $O(\varepsilon\ell^2 + \varepsilon^2\ell + \varepsilon^3)$. Integrating over the angular cell gives (5.4).

The exact pair Mayer kernel is

$$B_\varepsilon^{\text{pin}} = b_\varepsilon + (1 - b_\varepsilon) \sum_c \vartheta_c \mathbf{1}_{C_c} \otimes \mathbf{1}_{C_c}, \quad \vartheta_c = 1 - e^{-2\alpha\varepsilon/v_c}.$$

For $\varepsilon \leq cv_{\min}/\alpha$,

$$\sum_c v_c \vartheta_c P_c \geq \alpha\varepsilon P_\ell.$$

By (5.3), $T + \alpha P_\ell \geq -C\ell I$. The pin-overlap subtraction has row norm at most $C\alpha\varepsilon^2/\ell$, by (5.4). Lemma 4.2 now gives

$$B_\varepsilon^{\text{pin}} \geq -\varepsilon\delta_\varepsilon I, \quad \delta_\varepsilon \leq C(\ell + \varepsilon + \alpha\varepsilon/\ell). \quad (5.5)$$

Its row integral is at most $C(1 + \alpha)\varepsilon$, and its spatial range is fixed. Compression to a finite union of complete cells preserves these estimates.

5.2 The insertion commutator and active caps

Since $r_X \leq M\beta e^{2\kappa_{0,1}}$, the reversible birth–death generator

$$(LF)(\eta) = \int r_Y(\eta) D_Y F(\eta) d\nu(Y) + \sum_{Y \in \eta} [F(\eta - Y) - F(\eta)]$$

has bounded birth rates. Canceling factorials in the grand-canonical density gives the Georgii–Nguyen–Zessin identity [18, Theorem 2] and hence

$$\mathcal{E}(F, G) = \mathbb{E}_\mu \int r_X D_X F D_X G d\nu(X) = \langle F, -LG \rangle_\mu.$$

One-forms are measured in $\mu(d\eta)r_X(\eta)d\nu(X)$; their norm is denoted by $\|\cdot\|_1$.

Hold an inserted particle X fixed and let L^X evolve the background, with birth rates $r_Y(\eta + X)$ and deaths of particles already in η . Direct subtraction yields

$$\begin{aligned} D_X(-L)F &= (-L^X + I)D_X F + \int [r_Y(\eta) - r_Y(\eta + X)] D_Y F(\eta) d\nu(Y), \\ D(-L) &= \mathcal{H}D, \quad \mathcal{H} = -L^X + I + \mathcal{K}. \end{aligned} \quad (5.6)$$

The identity

$$r_X(\eta)r_Y(\eta + X) = r_Y(\eta)r_X(\eta + Y)$$

makes L^X reversible for the one-form measure and $\mathcal{K}$ self-adjoint. In particular, $-L^X \geq 0$.

First take $J = 0$. Away from an active cap, the curvature kernel in $\mathcal{K}$ is $B_\varepsilon^{\text{pin}}$. Applying (5.5) to the function $r_X g_X$, and using $r_X^2 \leq M\beta r_X$, bounds its quadratic form below by $-M\delta_\varepsilon \|g\|_1^2$. If $N_c(\eta) = K_c - 1$, the additional same-cell block is

$$e^{-2\alpha\varepsilon/v_c} \mathbf{1}_{C_c}(X) \mathbf{1}_{C_c}(Y) (1 - b_\varepsilon(X, Y)) r_Y(\eta).$$

In its quadratic form the term 1 is a positive rank-one operator. The subtracted overlap term has norm at most $CM\ell^4$, by (5.4). Retaining the positive block gives

$$\langle g, \mathcal{H}g \rangle_1 \geq (1 - M\delta_\varepsilon - CM\ell^4) \|g\|_1^2. \quad (5.7)$$

Choose the mesh first so that the ℓ terms are small, and then choose ε small at that mesh. The lower bound is a positive constant independent of the volume and mask.

For a bounded perturbation, an allowed double birth introduces the factor $e^{-D_X D_Y J}$. Its difference from one is bounded by

$$C \sum_{A \ni c(X), c(Y)} \|J_A\|_\infty, \quad |D_X D_Y J| \leq 4 \sum_{A \ni c(X), c(Y)} \|J_A\|_\infty.$$

The other insertion-ratio factors lie in $[0, 1]$. Weighted Schur summation bounds the resulting one-form operator by $C\beta\kappa_{m,1}$. The previous pair and cap bounds apply with the factor $e^{2\kappa_{0,1}}$ in the rate bound. Requiring $C\beta\kappa_{m,1}$ smaller than a fixed reserve in (5.7) therefore gives $\mathcal{H} \geq cI$.

5.3 Weighted inverse and covariance

We obtain spatial decay by the exponential conjugation method of Combes and Thomas [19]. For a spatial-cell set A , multiply one-forms by $w_X = e^{md(c_{\text{spatial}}(X), A)}$. This weight commutes with L^X and with each positive active-cap block, because it is constant on the whole spatial cell. The pair part has row mass $O(\varepsilon)$, rate $O(\beta)$ and finite spatial range. Its weighted commutator is therefore $O(m)$ at the fixed mesh. The weak part is controlled by (5.1). For a sufficiently small $m > 0$, both conjugations by w and w^{-1} have real part bounded below by $c' > 0$. In finite volume the weights are bounded and invertible, so the conjugated operators are invertible. Their accretivity gives the uniform inverse bound $1/c'$. If P_i selects insertions in spatial cell i , this proves

$$\|P_i \mathcal{H}^{-1} P_j\|_{1 \rightarrow 1} \leq C e^{-md(i,j)}. \quad (5.8)$$

The caps bound the number of rods in every finite volume. Together with bounded total birth rates and reversibility, this makes the generators bounded on their respective L^2 spaces; the identities therefore extend from bounded functions by density. Integration by parts and (5.6) give

$$\|LF\|_2^2 = \langle DF, \mathcal{H}DF \rangle_1 \geq c\mathcal{E}(F, F).$$

Successive deletions reach the empty configuration. The Georgii–Nguyen–Zessin identity therefore propagates a zero insertion gradient through every particle-number sector, so the Dirichlet kernel consists of constants. The spectral theorem gives a gap at least c .

For centered G , solve $(-L)u = G$. The commutator gives $Du = \mathcal{H}^{-1}DG$, and hence

$$\text{Cov}_\mu(F, G) = \langle DF, \mathcal{H}^{-1}DG \rangle_1.$$

This is the birth–death counterpart of the Helffer–Sjöstrand covariance representation [20]. Inserting (5.8) proves Theorem 5.1.

In particular, marked-cell counts satisfy

$$|\text{Cov}_\mu(N_i, N_j)| \leq C\beta e^{-md(i,j)}. \quad (5.9)$$

If the activity on any measurable set of marks is multiplied by e^{-t} , its counting observable has insertion norm in a cell at most $C\sqrt{\beta}e^{-t/2}$ times the square root of the mark measure in that cell. This bound remains valid with a fixed bounded perturbation, because the insertion rate retains the factor e^{-t} . It is integrable over $t \geq 0$ and will be used for boundary deletion. For a bounded local observable V supported on A ,

$$|\text{Cov}_\mu(N_i, V)| \leq C\beta\|V\|_\infty \sum_{j \in A} e^{-md(i,j)}, \quad (5.10)$$

since $|D_X V| \leq 2\|V\|_\infty$ on its support.

6 Spatial rigidity and a positive field action

Write $D = u \cdot \nabla_x$ for the transport derivative.

The next lemma is the polynomiality property of Killing tensors in Euclidean space [21, Section IV], stated here for distributional coefficients.

Lemma 6.1 (Polynomial transport coefficients). *Let $B \subset \mathbb{R}^3$ be connected and open. Suppose $H(x, z)$ is a homogeneous polynomial of degree l in $z \in \mathbb{R}^3$, with distributional coefficients on B , and $(z \cdot \nabla_x)H = 0$. Every coefficient is a polynomial in x of degree at most l .*

Proof. For $D_z = z \cdot \nabla_x$ one has $[D_z, \partial_{z_j}] = -\partial_{x_j}$. Induction gives

$$D_z^j \partial_z^\gamma H = (-1)^j j! \sum_{\substack{\xi \leq \gamma \\ |\xi|=j}} \binom{\gamma}{\xi} \partial_x^\xi \partial_z^{\gamma-\xi} H \quad (0 \leq j \leq |\gamma|).$$

Taking $|\gamma| = j = l+1$ gives $\partial_x^\gamma H = 0$. Mollifying any coefficient on a relatively compact ball therefore produces polynomials of degree at most l on smaller balls. This finite-dimensional polynomial space is closed in distributions, so the coefficient itself is locally such a polynomial. Polynomials agreeing on an open overlap agree identically. Connectedness of B then gives one polynomial on the whole domain. $\square$

Theorem 6.2 (Local transport rigidity). *If r is measurable on $B \times S^2$, $r(x, \cdot) \in \mathcal{W}_a$ for almost every x in a connected open set B , and $Dr = 0$ in distributions, then $r(x, u) = f(u)$ almost everywhere for a single $f \in \mathcal{W}_a$.*

Proof. The well bounds give $0 < m \leq r \leq M$. Spatial mollification preserves the transport equation and these bounds. The ordinary chain rule for the mollified functions, followed by local L^2 convergence and Lipschitz continuity of the logarithm on $[m, M]$, proves $D \log r = 0$. The Euler equation then gives $DKr = 0$.

Let r_l be the angular harmonic component of degree l ; odd components vanish. Multiplication by u_j maps degree l into degrees $l-1, l+1$. At any odd degree j , the equations $Dr = DKr = 0$ therefore read

$$A_j + B_j = 0, \quad \kappa_{j-1}A_j + \kappa_{j+1}B_j = 0,$$

where A_j and B_j are the degree- j projections of Dr_{j-1} and Dr_{j+1} . Distinct consecutive even eigenvalues imply $A_j = B_j = 0$. Thus $Dr_l = 0$ for every even l .

Represent r_l by its homogeneous harmonic polynomial $H_l(x, z)$. The transport identity on the sphere extends by homogeneity to all z . Lemma 6.1 makes its spatial coefficients polynomials of degree at most l . In the coordinates of Proposition 2.5,

$$v(x) = \mathcal{P}_N \log r(x) = a - 2\mathcal{P}_N K r(x)$$

is consequently the restriction of a polynomial map $\mathbb{R}^3 \rightarrow E$. It belongs to Z almost everywhere on B , and hence everywhere there by continuity.

Fix $x_0 \in B$ and a direction e . The polynomial curve $v_e(t) = v(x_0 + te)$ lies in Z near zero. On the component I containing zero of $\{t : v_e(t) \in U\}$, the analytic functions $F(v_e(t))$ and $G(v_e(t))$ vanish identically. Thus $v_e(I) \subset Z$. A finite endpoint of I would have its image in compact $Z \Subset U$, contradicting maximality of I . Hence $I = \mathbb{R}$. The polynomial curve is bounded and therefore constant. Since e was arbitrary, v is constant, and the identity $r = \Psi(v)$ proves the claim. $\square$

Following LMP's averaged-density representation [2, Section 3], we separate the action into an entropy loss under averaging and a homogeneous excess. Rods require averaging along their axes, for which Theorem 6.2 gives the equality case.

Put $h(t) = t \log t - t$. With the longitudinal averaging operator A , define

$$D_q(x) = \int_{S^2} \left[\int_{-1/2}^{1/2} h(q(x + tu, u)) dt - h(Aq(x, u)) \right] d\omega(u),$$

$$V_a(f) = \Omega_a(f) - m_a.$$

Jensen's inequality gives $D_q \geq 0$; $V_a \geq 0$ for all nonnegative profiles, without an evenness restriction.

Corollary 6.3 (Zero local action). *If a bounded nonnegative profile q satisfies $\int_B [D_q + V_a(Aq)] dx = 0$ on a connected open set B , then $q(x, u) = f(u)$ almost everywhere on $B \times S^2$ for one $f \in \mathcal{W}_a$.*

Proof. Set $r = Aq$. The zero potential gives $r(x, \cdot) \in \mathcal{W}_a$ almost everywhere. Strict convexity in the equality case of Jensen's inequality gives $q(x + tu, u) = r(x, u)$ for almost every (x, u, t) with $x \in B$. Choose $\chi \in C_c^\infty((-1/2, 1/2))$ of integral one. In distributions,

$$r = \int \chi(t) q(x + tu, u) dt, \quad Dr = - \int \chi'(t) q(x + tu, u) dt = 0.$$

Theorem 6.2 gives $r = f$. For $B' \Subset B$ and $0 < b \leq 1/2$ with $B' + B(0, b) \subset B$, a change of variables gives

$$2b \int_{B' \times S^2} |q - f|^2 \leq \int_{B \times S^2} \int_{-b}^b |q(x + tu, u) - f(u)|^2 dt dx d\omega = 0.$$

An exhaustion proves the assertion on B . $\square$

6.1 Minimizing the density profile at a fixed field

Fix $a = a_t$ and $\zeta = e^{at}$. Retain the marked cells $C_{i,b} = B_i \times U_b$, their measures $v_{i,b} = v_s \omega_b$, and the cell projection P . Fields take values in $[-R, R]$.

Set $S = T + 2\alpha P$. Both summands have nonnegative kernels and bounded row sums; T has fixed finite spatial range. The operator estimate required below is

$$S \geq -\delta I, \quad M = \zeta e^{R+1}, \quad c = M^{-1} - \delta > 0. \quad (6.1)$$

Choose $\alpha \geq \|T\|$. The mesh estimate (5.3) gives $\|T - PTP\| \leq C\ell$, where ℓ is the fine marked-cell mesh. Since $PTP \geq -\|T\|P$,

$$S \geq (2\alpha - \|T\|)P - C\ell I \geq -C\ell I.$$

After fixing α and R , choose the mesh so that $\delta = C\ell < M^{-1}$. Choose

$$\rho > \sup_{f \in \mathcal{W}} \|f\|_\infty, \quad R > 2\alpha \sup_{f \in \mathcal{W}} \|f\|_\infty. \quad (6.2)$$

Admissible density profiles satisfy $q \geq 0$ and $\int_C q \leq \rho v_C$ on every marked cell.

Proposition 6.4 (The capped minimizing profile). *Let D be a finite union of complete spatial cells. For bounded measurable sources $-R_- \leq \phi \leq R + 1$, where $R_- < \infty$ is fixed, the functional on profiles extended by zero outside D ,*

$$G_{D,\phi}(q) = \int_{D \times S^2} q(\log q - 1 - a - \phi) + \frac{1}{2} \langle q, S_D q \rangle$$

has a unique capped minimizer $q_{D,\phi}$. It satisfies uniform bounds $0 < m_ \leq q_{D,\phi} < M$ on D , and cell multipliers $\lambda_C \geq 0$ satisfy*

$$\log(q/\zeta) - \phi + Sq + \lambda_C = 0 \quad \text{on } C, \quad \lambda_C \left(\int_C q - \rho v_C \right) = 0. \quad (6.3)$$

Proof. Replacing q by $q \wedge M$ decreases both terms: the derivative $\log(q/\zeta) - \phi$ is positive above M , and S has a nonnegative kernel. On $[0, M]$, write the entropy as a convex function plus $q^2/(2M)$ and combine the latter with $S \geq -\delta I$. This proves strong convexity with constant c , weak lower semicontinuity, existence by weak compactness in L^2 , and uniqueness.

Apply Lemma 4.9 on $[0, M]$ to G and the cell masses. The zero competitor and the entropy lower bound make the value function finite. This gives nonnegative multipliers λ_C and complementary slackness. For that minimizer q_* , set $V = -a - \phi + Sq_* + \lambda$. The pointwise minimizer of $\int [h(q) + Vq]$ on $[0, M]$ is $g = \min(M, e^{-V}) > 0$. If $g \neq q_*$, its negative first-order improvement dominates the quadratic remainder along the segment joining the two profiles, contradicting minimality. Thus $q_* = g$. Positivity of $q_* = g$ gives $Pq_* > 0$ on every marked cell. Hence

$$\log M + V = R + 1 - \phi + Sq_* + \lambda \geq 2\alpha Pq_* > 0.$$

Thus $q_* < M$ and (6.3) holds. If $\lambda_C > 0$, saturation and $q \leq \zeta e^{R+1-\lambda_C}$ give

$$0 \leq \lambda_C \leq (\log \zeta + R + 1 - \log \rho)_+.$$

The bounded row sum of S now gives the uniform positive lower bound. □

We apply the exponential conjugation method of Combes and Thomas [19] to the constrained Euler equation.

Lemma 6.5 (Weighted response). *If W is a positive spatial weight constant on complete spatial cells and $|\log W_i - \log W_j| \leq m d(i, j)$ for sufficiently small $m > 0$, then minimizers at two sources obey*

$$\|W(q_{D,\phi_1} - q_{D,\phi_2})\|_2 \leq \frac{2}{c} \|W(\phi_1 - \phi_2)\|_2. \quad (6.4)$$

Proof. Subtract (6.3) for two sources and test with $W^2(q_1 - q_2)$. The cap term is nonnegative because a positive multiplier occurs only at a saturated cap:

$$\sum_C W_C^2(\lambda_C^1 - \lambda_C^2) \left(\int_C q_1 - \int_C q_2 \right) \geq 0.$$

The entropy difference quotient is at least M^{-1} . Because W commutes with P , only T changes under conjugation. Finite range and bounded row sums imply $\|WTW^{-1} - T\| \leq C(e^{Cm} - 1) \leq c/2$ for small enough rate m . Coercivity and Cauchy–Schwarz give (6.4). $\square$

Corollary 6.6 (The profile in the whole space). *For every field in the compact product box there is a unique bounded positive solution in the whole space q_ϕ of the capped Euler equations. It commutes with lattice translations and is locally Lipschitz under changes in finitely many field coordinates. Agreement of fields in a radius- d neighborhood of a target cell implies*

$$\|q_\phi - q_\psi\|_{L^2(C)} \leq Ce^{-md}.$$

Coordinatewise convergence of fields implies local L^2 convergence of the profiles.

Proof. For $D \subset D'$, the restriction of $q_{D',\phi}$ to D solves the capped problem at the effective source $\phi - T_{D,D' \setminus D} q_{D',\phi}$. This perturbation is nonpositive, bounded, and supported in a fixed boundary strip. Apply (6.4) with a weight decreasing away from a target cell. The squared weighted source norm is an exponentially decaying sum over the strip, bounded by Ce^{-2md} after decreasing m to absorb lattice shell counts. Thus nested free-box minimizers are Cauchy locally. The local Euler equations and multipliers pass to the limit.

For two bounded solutions in the whole space use the same decaying weight on their difference. All terms are integrable, the cap contribution is nonnegative, and coercivity forces equality. This also proves lattice translation covariance. Weighted response proves locality and finite-coordinate continuity. To obtain product continuity, first make the exterior contribution small using locality, then use coordinate convergence in the resulting fixed finite neighborhood. $\square$

6.2 Local charges and an exact positive interaction

Define

$$e_i(\phi) = \int_{B_i} [D_{q_\phi} + V_a(Aq_\phi)] dx + \frac{1}{4\alpha} \int_{B_i \times S^2} |\phi - 2\alpha P q_\phi|^2. \quad (6.5)$$

The charges are nonnegative, bounded and continuous in the product topology. Their response to a field change decays exponentially with distance, by the profile locality and the Lipschitz continuity of entropy on the uniform positive profile interval. For a homogeneous well f and $r = 2\alpha P f$, the Euler equation shows $q_r = f$ by Corollary 6.6; the caps are inactive by (6.2). Consequently $e_i(r) = 0$.

Proposition 6.7 (Positive interactions on lattice balls). *There are nonnegative continuous interactions $\Phi_{i,n}^0$, with declared support the entire fine lattice ball $B_n(i)$, such that*

$$e_i = \sum_{n \geq 0} \Phi_{i,n}^0, \quad \|\Phi_{i,n}^0\|_\infty \leq Ce^{-m(n-1)} \quad (n \geq 1).$$

Every atom vanishes at a homogeneous well field. For every fixed p and $0 < a' < m/2$,

$$\sum_{i,n: 0 \in B_n(i)} |B_n(i)|^p e^{a' \text{diam } B_n(i)} \|\Phi_{i,n}^0\|_\infty < \infty.$$

Proof. Set $e_i^{[n]}(\phi) = \inf\{e_i(\psi) : \psi = \phi \text{ on } B_n(i)\}$. Compactness gives attainment, and exponential locality gives $0 \leq e_i - e_i^{[n]} \leq Ce^{-mn}$. Continuity follows by uniform finite-range approximation of e_i . The localizations increase with n . Take $\Phi_{i,0}^0 = e_i^{[0]}$ and $\Phi_{i,n}^0 = e_i^{[n]} - e_i^{[n-1]}$ for $n \geq 1$. Their nonnegativity, bounds, and root values follow immediately. There are $O((n+1)^3)$ possible centers with $0 \in B_n(i)$; $|B_n(i)| = O((n+1)^3)$ and its diameter is $2n$. The asserted norm is therefore finite. $\square$

6.3 Alignment with the well set

Write $\|\cdot\|_i$ for the norm of $L^2(B_i \times S^2)$. If $f \in \mathcal{W}$, the Euler equation cancels the linear terms and yields

$$\begin{aligned} D_Q(x) + V_a(AQ(x)) &= \int_{S^2} \int_{-1/2}^{1/2} [h(Q(x+tu, u)) - h(f(u)) \\ &\quad - \log f(u)(Q(x+tu, u) - f(u))] dt d\omega(u) \\ &\quad + \langle AQ(x) - f, K(AQ(x) - f) \rangle. \end{aligned}$$

For $z \geq 0$, $f > 0$, the entropy bracket is at most $(z - f)^2/f$, by $\log t \leq t - 1$. With $m_{\mathcal{W}} = \inf_{f \in \mathcal{W}, u} f(u) > 0$,

$$0 \leq \int_{B_i} [D_Q + V_a(AQ)] \leq C \sum_{|j-i| \leq r_*} \|Q_j - f\|_{L^2(B_j \times S^2)}^2, \quad (6.6)$$

where r_* is a fixed fine-lattice range.

In the field coordinates

$$s_{i,b} = \sqrt{\omega_b} \phi_{i,b}, \quad (\Pi Q_i)_b = \frac{2\alpha\sqrt{\omega_b}}{v_s \omega_b} \int_{B_i \times U_b} Q_i,$$

one has $\|\Pi\| \leq 2\alpha/\sqrt{v_s}$ and the exact identity

$$\frac{1}{4\alpha} \int_{B_i \times S^2} |\phi - 2\alpha PQ|^2 = \frac{v_s}{4\alpha} |s_i - \Pi Q_i|^2. \quad (6.7)$$

Copying whole spatial cells preserves every marked-cell cap. The normalized Lebesgue field measure is pushed forward under this fixed diagonal coordinate change. Projected wells lie in its interior by (6.2), giving uniform polynomial lower bounds for small-ball probabilities.

Lemma 6.8 (Alignment on a fixed patch). *For a fixed connected inner patch I , choose a connected outer patch O containing a neighborhood of I . There is a bounded, nondecreasing concave modulus ω , with $\omega(t) \rightarrow 0$ as $t \downarrow 0$, such that, writing $E_O = \sum_{i \in O} e_i(\phi)$,*

$$\min_{f \in \mathcal{W}} \sum_{i \in I} \|q_{\phi,i} - f\|_i^2 + \sum_{\substack{i,j \in I \\ i \sim j}} \|f_i - f_j\|_{L^2(S^2)}^2 + \sum_{i \in I} |s_i - \Pi f_i|^2 \leq \omega(E_O).$$

Here $i \sim j$ means neighboring spatial cells and f_i is any nearest well to $q_{\phi,i}$ in $\|\cdot\|_i$.

Proof. If the outer charge sum is zero, Corollary 6.3 makes q_ϕ one common well on the inner patch; the pinning square then gives $\phi = 2\alpha P f$ there. If the asserted vanishing modulus failed, choose fields whose outer energy tends to zero and whose inner error stays bounded below. Compactness of the countable product field box gives a coordinatewise convergent subsequence. Corollary 6.6 and

continuity of the charges pass zero energy and local profile convergence to the limit, a contradiction. For any nearest well f_i , $\|f_i - f\|_i \leq 2\|q_{\phi,i} - f\|_i$, so the same modulus controls all such choices. A measurable nearest-well selection can be used: the minimum is over a compact metric set, the distance is continuous, and the Kuratowski–Ryll–Nardzewski selection theorem applies [22].

Let $\psi(t)$ be the nondecreasing supremum of the bounded total inner error at energy at most t , and choose $C \geq \sup \psi$. The function $\omega(t) = \inf_{s>0}[\psi(s) + Ct/s]$ is a bounded, nondecreasing concave majorant of ψ and tends to zero at zero. This proves the quantitative form used under averaging. $\square$

Lemma 6.9 (Separation after cell projection). *The restriction of P to $\mathcal{W}$ is injective. The images of distinct well components are compact sets at positive mutual distance.*

Proof. If $Pf = Pg$ for two wells, both solve the same homogeneous pinned Euler equation. The angular operator $2K + 2\alpha P_{\text{ang}}$ obeys the same lower bound as S ; this follows by testing the spatial inequality on profiles constant in larger and larger boxes and dividing by volume. Subtracting the two Euler equations and testing with $f - g$ gives at least $c\|f - g\|_2^2$, but the expression is zero. Thus $f = g$. The cell projection is therefore a continuous injection on compact $\mathcal{W}$, and is a homeomorphism onto its image. Its finitely many compact components have positive mutual distance. $\square$

6.4 Changes of field and comparison with trial profiles

For an arbitrary trial Q , let $\ell_i(Q, \phi)$ be the expression in (6.5) with Q replacing q_ϕ .

Proposition 6.10 (Exact change of action). *If $\phi - \psi$ is supported on finitely many marked cells, then*

$$\begin{aligned} H^0(\phi; \psi) &:= \sum_i [e_i(\phi) - e_i(\psi)] \\ &= \int_0^1 \sum_C (\phi_C - \psi_C) \left[\frac{v_C \{\psi_C + t(\phi_C - \psi_C)\}}{2\alpha} - \int_C q_{\psi+t(\phi-\psi)} \right] dt. \end{aligned} \quad (6.8)$$

The sum is absolutely convergent. It equals the sum of changes of the interactions $\Phi_{i,n}^0$ from Proposition 6.7.

Proof. Use the same finite free box D for both profiles and extend both profiles and fields by zero outside it. The cancellation

$$D_q + V_a(Aq) = \int_{S^2} \int_{-1/2}^{1/2} h(q(x + tu, u)) dt d\omega - a \int_{S^2} Aq d\omega + \langle Aq, K Aq \rangle - m_a$$

removes the $h(Aq)$ terms. Integration over all spatial centers, translation invariance of the longitudinal average, and expansion of the pinning square give the exact identity

$$\begin{aligned} &\sum_i \{\ell_i(q_{D,\phi}, \phi_D) - \ell_i(q_{D,\psi}, \psi_D)\} \\ &= G_{D,\phi}(q_{D,\phi}) + \frac{1}{4\alpha} \int_D \phi^2 - G_{D,\psi}(q_{D,\psi}) - \frac{1}{4\alpha} \int_D \psi^2. \end{aligned}$$

The common constant baseline outside D cancels. Weighted response bounds each centerwise difference by an exponentially summable function of its distance from the changed field set, uniformly

in D . To estimate entropy use the common positive interval inside D ; outside it both profiles vanish. Dominated convergence gives the all-space charge sum.

The minimizing inequalities at two sources and the continuity of the minimizer give the envelope formula of Danskin [23]: the minimized full action $\min_q G_{D,\phi}(q) + (4\alpha)^{-1} \int_D \phi^2$ is differentiable, with coordinate derivative $v_C \phi_C / (2\alpha) - \int_C q_{D,\phi}$. Integrate along the straight segment of fields and pass to the limit to obtain (6.8). For the ball interaction, atoms on balls missing the changed set have zero difference. At distant centers, only radii reaching the set contribute; their exponential norms are summable. This justifies the double sum and proves equality of the two action differences. $\square$

Proposition 6.11 (Comparison with a trial profile). *Suppose $\phi' - \phi$ is finitely supported, and Q' is capped, $0 \leq Q' \leq M$, and agrees with q_ϕ outside finitely many complete cells. Then*

$$\sum_i [e_i(\phi') - e_i(\phi)] \leq \sum_i [\ell_i(Q', \phi') - \ell_i(q_\phi, \phi)]. \quad (6.9)$$

Proof. Put $q' = q_{\phi'}$ and $\Delta = Q' - q'$. Locality gives $\Delta \in L^1 \cap L^2$. Subtract the Euler equation for q' from the action difference (all integrals below converge) to obtain

$$\begin{aligned} \sum_i [\ell_i(Q', \phi') - \ell_i(q', \phi')] &= \int [h(Q') - h(q') - \log q'(Q' - q')] + \frac{1}{2} \langle \Delta, S\Delta \rangle - \sum_C \lambda'_C \int_C \Delta \\ &\geq \frac{c}{2} \|\Delta\|_2^2 \geq 0. \end{aligned}$$

The cap term is nonnegative, because a positive multiplier means q' saturates its cap, while Q' is admissible. Bounded multipliers and L^1 decay justify the sum. Combining this with Proposition 6.10 gives (6.9). $\square$

Corollary 6.12 (A homogeneous trial profile). *For a homogeneous root $r = 2\alpha P f$ and a finite region D ,*

$$0 \leq H_D^0(\phi_D \mid r) \leq \frac{1}{4\alpha} \sum_{C \subset D} v_C |\phi_C - r_C|^2, \quad H_D^0(\phi_D \mid r) = \sum_i e_i(\phi_D r_{D^c}). \quad (6.10)$$

Proof. Use $Q' = f$ in Proposition 6.11. Its local Onsager action is zero, and only the displayed pinning square remains. Nonnegativity gives the lower bound. $\square$

7 Restoring all particle counts

We use the fixed spatial and angular mesh chosen above. Its marked cells are denoted by C_i , their one-particle measures by ν_i , and their particle counts by N_i . Constants may depend on this mesh. In particular $0 < v_- \leq \nu_i \leq v_+ < \infty$. The graph G joins any two cells in which rods can overlap. It has bounded degree, finite spatial range, and polynomial volume growth of degree three. All neighborhoods and distances below refer to this full graph, including when the integration domain has holes.

Write $z_h = \beta \zeta e^h$, $K_i = \lceil \rho \beta \nu_i \rceil$, and use the parameters and Gaussian normalizers of (3.2)–(3.3). In particular,

$$0 \leq -\log c_{i,h}(n) \leq C e^{-c\beta}, \quad 0 \leq n \leq K_i. \quad (7.1)$$

With ν_i denoting one-particle measure on C_i , the two cell weights relative to $d\mathfrak{L}_i = \sum_{n \geq 0} d\nu_i^{\otimes n} / n!$ are

$$a_i^\phi = \frac{(\beta \zeta)^{N_i}}{c_{i,h}(N_i)} e^{\phi_i N_i - \alpha N_i^2 / (\beta \nu_i)} \quad (N_i \leq K_i), \quad b_i = \frac{z_h^{N_i}}{c_{i,h}(0)} \quad (N_i > K_i). \quad (7.2)$$

The empty-cell weight is $E_i = c_{i,h}(0)^{-1}$. All estimates below allow additional one-particle accessibility multipliers in $[0, 1]$.

7.1 The empty-cell projection

For a finite set D of marked cells and an admissible exterior rod configuration ξ , let $\mathcal{R}_D^{\text{full}}$ be the rod partition function with the weights (7.2), and let $\mathcal{R}_D^{\text{cap}}$ be its restriction to $N_i \leq K_i$ in every cell. The exterior ξ is not restricted by a cap. Define $\pi\omega$ by emptying every cell whose count exceeds its cap. Thus

$$N_i(\pi\omega) = \hat{N}_i(\omega), \quad \hat{N}_i = N_i \mathbf{1}_{\{N_i \leq K_i\}}. \quad (7.3)$$

A nonempty G^2 -connected set $S \subset D$ has dependency support $U_S = N_G[S]$. Two such sets are compatible when their dependency supports are disjoint, equivalently when their G -distance is at least three. For a capped configuration σ set

$$\begin{aligned} B_S(\sigma; \xi) &= \int_{N_i > K_i, i \in S} \prod_{i \in S} b_i(\omega_i) \chi_S(\omega_S; \sigma_{D \setminus S}, \xi) d\mathfrak{L}_S(\omega_S), \\ w_S(\sigma; \xi) &= \frac{\mathbf{1}_{\{\sigma_S = \emptyset\}}}{\prod_{i \in S} E_i} B_S(\sigma; \xi), \\ \Xi_D^e(\sigma; \xi) &= \sum_{\mathcal{F} \text{ compatible}} \prod_{S \in \mathcal{F}} w_S, \quad K_D^e = -\log \Xi_D^e. \end{aligned} \quad (7.4)$$

Here χ_S is the product of hard-core factors with at least one particle in S ; the empty polymer family contributes one.

Proposition 7.1 (Exact projection). *For every D, ξ, ϕ ,*

$$\mathcal{R}_D^{\text{full}}(\phi \mid \xi) = \mathcal{R}_D^{\text{cap}}(\phi \mid \xi) \mathbb{E}_{D, \phi, \xi}^{\text{cap}} e^{-K_D^e}. \quad (7.5)$$

The pushforward of the full law by π is the capped law tilted by $e^{-K_D^e}$. The correction K_D^e is independent of ϕ .

Proof. Decompose the full integral according to its actual bad set $B = \{i : N_i > K_i\}$. The G^2 -components of B form a unique compatible family. Their closed G -neighborhoods are disjoint, so the bad integrals factor conditional on the retained rods. In the corresponding term on the right of (7.5), the indicators force the cells in B to be empty. Integrating them contributes exactly $\prod_{i \in B} E_i$, canceling the denominator in (7.4). All remaining factors agree with the original sector. This proves the identity sector by sector. Inserting any bounded function of $\pi\omega$ proves the pushforward assertion. Finally $b_i/E_i = z_h^{N_i}$ and the hard factors have no field dependence. $\square$

Lemma 7.2 (Exceptional-count tails). *There is $q_\beta = e^{-c\beta}$, after reducing $c > 0$ if necessary, such that*

$$0 \leq w_S \leq q_\beta^{|S|}, \quad \sum_{n > K_i} n^m \frac{(\beta \zeta e^H v_i)^n}{n!} \leq C_m \beta^m q_\beta \quad (7.6)$$

for every fixed integer $m \geq 0$. These estimates are uniform in the exterior rods and all accessibility multipliers.

Proof. Drop the hard constraints and accessibility multipliers. The empty weights cancel the factors $c_{i,h}(0)^{-1}$. Writing $r = e\zeta e^H/\rho < 1$, the inequality $n! \geq (n/e)^n$ gives

$$\sum_{n > K_i} \frac{(\beta \zeta e^H v_i)^n}{n!} \leq \frac{r^{K_i+1}}{1-r}.$$

Since $v_i \geq v_-$, this is exponentially small in β . The ratio of successive summands is at most $\zeta e^H/\rho < 1$ beyond the cap. Summing the geometric tail with the extra polynomial n^m gives the second estimate. $\square$

7.2 The expansion with a weak interaction

We shall allow a bounded measurable interaction $J = \sum_A J_A$ on the actual rod configuration and the field. Every declared support contains all rod and field arguments. Supports are connected in a fixed finite-range enlargement of G . For $a > 0$ write

$$\|J\|_a = \sup_x \sum_{A \ni x} e^{a|A|} \|J_A\|_\infty = \kappa. \quad (7.7)$$

The exponent a is chosen with a fixed reserve for polynomial factors and spatial decay.

Expand $e^{-J_A} = 1 + f_A$, with $\eta_A = e^{\|J_A\|_\infty} - 1$. A bad-site atom at i has support $i^+ = N_G[i]$ and majorant q_β ; a weak atom A has support A and majorant η_A . An object γ is a nonempty set of distinct atoms connected by overlap of their supports. Write

$$U_\gamma = \bigcup_{a \in \gamma} B_a, \quad \ell_\gamma = \sum_{a \in \gamma} |B_a|, \quad z_\gamma^\# = \prod_{a \in \gamma} w_a. \quad (7.8)$$

For $\gamma = (S, \mathcal{F})$ its exact weight is

$$z_\gamma(\sigma; \xi) = \frac{\mathbf{1}_{\{\sigma_S = \emptyset\}}}{\prod_{i \in S} E_i} \int_{N_i > K_i, i \in S} \prod_{i \in S} b_i(\omega_i) \chi_S(\omega_S; \sigma_{D \setminus S}, \xi) \\ \times \prod_{A \in \mathcal{F}} f_A(\omega_S \sigma_{D \setminus S} \xi, \phi) d\mathfrak{L}_S(\omega_S). \quad (7.9)$$

When S is empty this is simply the product of the selected weak factors on the retained configuration. The factors in (7.9) are evaluated on the reconstructed actual configuration. The bound

$$|z_\gamma| \leq z_\gamma^\# = q_\beta^{|S|} \prod_{A \in \mathcal{F}} \eta_A \quad (7.10)$$

follows from Lemma 7.2.

Lemma 7.3 (Projection with a weak interaction). *Let $K_D(J)$ be minus the logarithm of the gas of objects (7.9), with disjoint dependency supports. Then*

$$\mathcal{R}_D^{\text{full}, J} = \mathcal{R}_D^{\text{cap}} \mathbb{E}_D^{\text{cap}} e^{-K_D(J)}. \quad (7.11)$$

The full law projects to the capped law tilted by $e^{-K_D(J)}$.

Proof. Expand the weak factors and partition each selected atom set into its overlap components. Distinct components have disjoint dependence and no hard factor connecting their bad rods, so their integrals factor. The empty weights cancel as in Proposition 7.1. Inserting a bounded function of the projected configuration proves the assertion about its law. $\square$

The logarithmic expansion uses Penrose's partition identity [24] and the Kotecký–Preiss convergence criterion [25], in the formulation by iteration over trees in Fernández and Procacci [26, Section 4].

Lemma 7.4 (Mixed cluster expansion). *Suppose that, for the required fixed $s \geq 0$,*

$$\rho_s := \sup_x \sum_{a:x \in B_a} w_a e^{(s+2)|B_a|} \leq \frac{1}{4}. \quad (7.12)$$

The gas has an absolutely convergent logarithm. Grouping the logarithm by connected union supports gives

$$K_D(J) = \sum_U K_{D,U}(J), \quad \sup_x \sum_{U \ni x} e^{b|U|} (1 + |U|)^p \|K_{D,U}(J)\|_\infty \leq C_{b,p}(q_\beta + \kappa) \quad (7.13)$$

whenever (7.12) holds with sufficient reserve beyond b . A cluster marked by one prescribed bad site retains a factor q_β ; a cluster marked by a weak atom retains a factor κ ; the two marks together retain $q_\beta \kappa$.

Proof. For an ordered tuple $C = (\gamma_1, \dots, \gamma_n)$, let $\varphi^T(C)$ be the alternating sum over connected spanning subgraphs of its incompatibility graph. Set $\ell(C) = \sum_j \ell_{\gamma_j}$ and $U(C) = \bigcup_j U_{\gamma_j}$. The tree bound is

$$|\varphi^T(C)| \leq \sum_{T \text{ tree}} \prod_{\{i,j\} \in E(T)} \mathbf{1}_{\{U_{\gamma_i} \cap U_{\gamma_j} \neq \emptyset\}}. \quad (7.14)$$

To prove it, order edges and group connected graphs by their minimum spanning tree. Each group is a Boolean interval, whose alternating sum is zero or has absolute value one.

For atom trees, iteration from zero in

$$T_a = w_a e^{u|B_a|} \exp \left\{ \sum_{b: B_b \cap B_a \neq \emptyset} T_b \right\}$$

is bounded by $w_a e^{(u+1)|B_a|}$ whenever $\sup_x \sum_{a:x \in B_a} w_a e^{(u+1)|B_a|} \leq 1$. Every connected atom set has a spanning tree. Allowing repeated species in the tree sum therefore gives

$$\sum_{\gamma: a \in \gamma} z_\gamma^\# e^{u\ell_\gamma} \leq w_a e^{(u+1)|B_a|}, \quad \sum_{\gamma: x \in U_\gamma} z_\gamma^\# e^{u\ell_\gamma} \leq \sum_{a: x \in B_a} w_a e^{(u+1)|B_a|}. \quad (7.15)$$

Apply the same recursion to object trees. Its supersolution is $z_\gamma^\# e^{s\ell_\gamma + |U_\gamma|}$, since

$$\sum_{\gamma': U_{\gamma'} \cap U_\gamma \neq \emptyset} z_{\gamma'}^\# e^{s\ell_{\gamma'} + |U_{\gamma'}|} \leq |U_\gamma| \rho_s.$$

Rooting at an object containing x and using (7.14) yields

$$\sum_{n \geq 1} \frac{1}{n!} \sum_C |\varphi^T(C)| \prod_j z_{\gamma_j}^\# e^{s\ell(C)} \mathbf{1}_{\{x \in U(C)\}} \leq \rho_s. \quad (7.16)$$

If $n_i(C)$ counts bad-atom occurrences at i , rooting at such an occurrence instead gives

$$\sum_{n \geq 1} \frac{1}{n!} \sum_C |\varphi^T(C)| n_i(C) \prod_j z_{\gamma_j}^\# e^{s\ell(C)} \leq q_\beta e^{(s+2)|B_i|}. \quad (7.17)$$

Extra occurrence marks cost powers of $\ell(C)$, absorbed by the reserved exponent. For a weak mark, let κ_s be the weak-atom part of ρ_s and multiply weak majorants by $\lambda_0 = 1 + (4\kappa_s)^{-1}$. Their sum and the bad-atom sum remain below $3/4$. The inequality $k \leq (\lambda_0^k - 1)/(\lambda_0 - 1)$ applied to the

nonnegative majorant series gives $C\kappa_s$ for one weak mark, and, using (7.17), $Cq_\beta\kappa_s$ for both marks. The case $\kappa_s = 0$ is immediate.

Because $|U(C)| \leq \ell(C)$, these bounds prove (7.13). A cluster joining two sites has occurrence size at least a fixed multiple of their distance, so the reserved exponent also gives spatial decay. For finitely many atoms the series is the formal logarithm of the gas polynomial. The same bounds hold in a complex activity disk larger than the unit disk; analytic continuation identifies its exponential with that polynomial. Positivity of the original fiber integral selects the real logarithm. Absolute convergence then permits countable atom families and marked derivatives. $\square$

For fixed norm exponents, (7.12) follows from $q_\beta \rightarrow 0$ and sufficiently small κ in the stronger norm (7.7). For $J = 0$, $K_D(0) = K_D^e$ is field independent. The same bad-atom formula defines its coefficients on the full lattice: selected cells are empty in the background, and inserted rods interact with all retained rods. This definition also allows uncapped background counts and has the same uniform majorants.

Evaluate both expansions at $\eta = \sigma_D \xi$, retaining the original exterior rods. The finite expansion consists of exactly those clusters whose bad-atom sites lie in D . Its difference from the ordinary full-lattice conditional sum consists of clusters with an exterior bad-atom site and dependency union meeting D . A term in this difference with nonzero oscillation at i connects i to D^c . The marked tree bound with an exponential distance reserve therefore gives, for the differences Δ_U of the conditional coefficients,

$$\sum_{U \ni i} \text{osc}_i \Delta_U \leq Cq_\beta e^{-md(i, D^c)}. \quad (7.18)$$

7.3 Uniform response and reconstruction

We use Theorem 5.1 in the following form. If a capped reference is perturbed by Q with $\beta\|Q\|_a$ sufficiently small, then

$$|\text{Cov}_{\text{cap}, Q}(F, G)| \leq C \sum_{i, j} e^{-md(i, j)} \mathcal{D}_i(F) \mathcal{D}_j(G), \quad \mathcal{D}_i(F)^2 = \mathbb{E} \int_{C_i} r_X |D_X F|^2 dX. \quad (7.19)$$

Insertion rates satisfy $r_X \leq C\beta$. If particle activity in a measurable set is multiplied by e^{-t} , its restricted insertion quadratic form retains that factor. All these assertions are uniform under hard accessibility masks and admissible exterior rods. By (7.1) and (7.13), they apply to the exact projected law whenever

$$\beta(q_\beta + \kappa) \leq c_0 \quad (7.20)$$

with the fixed constant c_0 chosen small enough.

Lemma 7.5 (Reset-count covariance). *In the bare full reference,*

$$|\text{Cov}(\hat{N}_i, \hat{N}_j)| \leq C\beta e^{-md(i, j)}. \quad (7.21)$$

Proof. The projected reset counts are the capped counts. Their insertion gradients equal one in their own cells and zero elsewhere, so (7.19) applies. $\square$

Lemma 7.6 (Response to deleting particle states). *Let A be any measurable set of one-particle states. Multiply both branches of (7.2) by e^{-tN_A} , keeping h , the Gaussian attachment and its normalizers fixed. If $I_A = \{j : \nu_j(A \cap C_j) > 0\}$, then*

$$\left| \frac{d}{dt} \beta^{-1} \mathbb{E}_t \hat{N}_i \right| \leq C(1 + \beta q_\beta) e^{-t/2} \sum_{j \in I_A} e^{-md(i, j)}. \quad (7.22)$$

Proof. Under deletion, the same projection formula holds with correction K_t . Differentiation of a bad integral inserts $-N_A$. Whenever $N_A \geq 1$, $N_A e^{-tN_A} \leq e^{-t/2} N_A$. Lemma 7.2 and the marked cluster estimate therefore give

$$\partial_t K_t = \sum_{j \in I_A} \sum_{U \ni j} D_{t,U}^{(j)}, \quad \sum_{U \ni j} e^{b|U|} (1 + |U|)^p \|D_{t,U}^{(j)}\|_\infty \leq C_{b,p} \beta q_\beta e^{-t/2}. \quad (7.23)$$

If T_A counts retained particles in A , differentiation of the normalized capped law yields

$$\frac{d}{dt} \mathbb{E}_t \hat{N}_i = -\text{Cov}_{\text{cap}, K_t}(N_i, T_A + \partial_t K_t).$$

The insertion norm of a deleted count in cell j is at most $C\sqrt{\beta}e^{-t/2}$. For a bounded local V_U , its insertion difference is at most $2\|V_U\|_\infty$, giving

$$|\text{Cov}(N_i, V_U)| \leq C\beta\|V_U\|_\infty \sum_{x \in U} e^{-md(i,x)}.$$

Apply this to (7.23); connected support and its exponential reserve move x to the anchor j . The two terms cost $C\beta e^{-t/2}$ and $C\beta^2 q_\beta e^{-t/2}$, respectively, which proves (7.22). $\square$

For an exterior configuration ξ , the interior one-particle states forbidden by ξ lie in a fixed boundary strip. Deleting those states and integrating (7.22) from zero to infinity shows that

$$\left| \beta^{-1} \mathbb{E}_D^\xi \hat{N}_i - \beta^{-1} \mathbb{E}_D^{\text{free}} \hat{N}_i \right| \leq C e^{-md(i, D^c)}. \quad (7.24)$$

The same argument works for every bounded local observable of the projected configuration: its insertion differences are bounded by twice its supremum norm. Its unscaled boundary error has a constant finite for each fixed β . Conditioning larger boxes on the original exterior rods then gives convergence of all projected local marginals in total variation.

To recover the actual configuration, condition on the projected configuration σ . A nonempty retained cell is fixed. An empty retained cell contains either vacuum or an entire bad configuration, whose configuration space is $\Omega_i^{\text{bad}} = \bigsqcup_{n > K_i} C_i^n / \mathfrak{S}_n$ and whose mark measure relative to the vacuum is

$$d\nu_i^{\text{bad}}(\omega_i) = \sum_{n > K_i} \frac{z_h^n}{n!} d\nu_i^{\otimes n}(\omega_i), \quad \nu_i^{\text{bad}}(\Omega_i^{\text{bad}}) \leq q_\beta. \quad (7.25)$$

The hard constraints are imposed separately and only reduce this measure. Under a weak interaction B of norm at most κ , the energy change relative to the vacuum in one cell is at most 2κ , so its nonvacuum conditional mass is at most $q_\beta e^{2\kappa}$.

The following is Dobrushin's comparison argument [27, Theorem 3] applied to the dilute conditional system.

Lemma 7.7 (Fiber comparison). *The conditional bad-cell system has a unique infinite-volume law. If c_{ij} is its total-variation influence matrix, then*

$$c_{ij} \leq Cq_\beta \left[\mathbf{1}_{\{i \sim_G j\}} + \sum_{A \ni i, j} \|B_A\|_\infty \right], \quad \sup_i \sum_j e^{md(i,j)} c_{ij} < \frac{1}{2}. \quad (7.26)$$

For bounded F supported on a finite set U , changing the retained configuration only at distance at least r from U changes its conditional expectation by at most $C|U|\|F\|_\infty e^{-mr}$, uniformly in exterior rods.

Proof. Changing a hard neighbor changes only the nonvacuum part of the one-cell conditional law, of mass $O(q_\beta)$. For a weak term meeting both i and j , the change in its energy relative to the vacuum is at most $4\|B_A\|_\infty$. The total-variation change is therefore at most that energy oscillation times Cq_β . This proves the first estimate. Connected supports and the exponential reserve in (7.7) give the second for large β .

For the comparison statement, couple single-cell heat-bath updates by maximal coupling. If two specifications differ by conditional discrepancies b_i , their time-averaged disagreement probabilities obey $d_i \leq b_i + \sum_j c_{ij}d_j$ in the limit along a subsequence. Such a subsequence exists since these are finitely many numbers in $[0, 1]$; the marginal laws remain their prescribed finite-volume Gibbs laws throughout the coupling. Iteration gives $d \leq (I - C)^{-1}b$. The weighted row bound in (7.26) makes the resolvent converge and gives exponential decay of its entries. A boundary change, or a change of retained rods outside a large ball, supplies discrepancies only near that change, together with the summable weak tails. Thus local marginals are uniformly Cauchy in total variation as boundaries recede. Their consistent limits give the asserted law and quasilocality. $\square$

Lemma 7.8 (The bare reference law). *For every field with $|\phi_i| \leq R$, the full bare reference has a unique infinite-volume Gibbs law ν_ϕ . Finite-volume expectations of bounded local observables converge uniformly over admissible exterior rods, with an error decaying exponentially in the distance to the boundary; the prefactor may depend on β and the observable.*

Proof. The deletion estimate gives convergence in total variation of the projected local marginals. Approximate a conditional fiber expectation by retaining its arguments in a finite neighborhood of the observable. Lemma 7.7 bounds the approximation error exponentially; projected convergence then constructs the actual local limits. Consistency gives a law on configurations. Local finite-range conditional distributions pass to the limit in total variation, so this law satisfies the DLR equations. Conditioning any Gibbs law on larger boxes and using the uniform boundary estimate proves uniqueness. $\square$

7.4 Covariance for actual rods

Proposition 7.9. *Let B be a weak interaction satisfying (7.20), including any coefficientwise partial sum or interpolation of such an interaction. For bounded actual-rod observables F_U, G_V supported on finite sets U, V ,*

$$|\text{Cov}_{\text{full}, B}(F_U, G_V)| \leq C\beta \|F_U\|_\infty \|G_V\|_\infty \sum_{u \in U, v \in V} e^{-md(u, v)}, \quad (7.27)$$

$$|\text{Cov}_{\text{full}, B}(F_U, \hat{N}_j)| \leq C\beta \|F_U\|_\infty \sum_{u \in U} e^{-md(u, j)}. \quad (7.28)$$

The estimates hold in finite volumes uniformly in frozen admissible exteriors and accessibility masks.

Proof. Let $K(s, t)$ be minus the logarithm of the positive fiber integral with the added energy $sF_U + tG_V$. Differentiating the projected partition function twice gives

$$\text{Cov}_{\text{full}, B}(F_U, G_V) = \text{Cov}_{\text{cap}, K}(K_s, K_t) - \mathbb{E}_{\text{cap}, K} K_{st}. \quad (7.29)$$

Conditional differentiation of the positive fiber integral yields

$$K_s = \mathbb{E}_{\text{fiber}, B} F_U, \quad K_t = \mathbb{E}_{\text{fiber}, B} G_V, \quad K_{st} = -\text{Cov}_{\text{fiber}, B}(F_U, G_V).$$

Perturbing the fiber by tG_V changes a one-cell conditional law only in V , by at most $Cq_\beta|t|\|G_V\|_\infty$. Divide the comparison estimate of Lemma 7.7 by $|t|$ and let $t \rightarrow 0$ to get

$$|K_{st}| \leq Cq_\beta\|F_U\|_\infty\|G_V\|_\infty \sum_{u \in U, v \in V} e^{-md(u,v)}. \quad (7.30)$$

An allowed retained insertion in cell x changes F_U directly only if $x \in U$. It also changes bad-cell eligibility at x , hard accessibility nearby, and weak terms meeting x . All changes of fiber conditional laws have size $O(q_\beta)$ with the same exponential profile. Hence

$$|D_X K_s| \leq C\|F_U\|_\infty \left[\mathbf{1}_{\{x \in U\}} + q_\beta \sum_{u \in U} e^{-md(x,u)} \right]. \quad (7.31)$$

Applying (7.19) to the two gradients in (7.31), and convolving exponential kernels, bounds the first term of (7.29) by the right side of (7.27), with a smaller exponent. Equation (7.30) bounds its second term. If $G = \hat{N}_j$, that source is constant on every fiber: $K_t = N_j(\sigma)$ and $K_{st} = 0$. Its capped insertion gradient is one in cell j , which proves (7.28). $\square$

For an infinite weak background B , define $B^{(R)} = \sum_{A \subset B_R(0)} B_A$ and tilt the bare reference ν_ϕ by $e^{-B^{(R)}}$. These are bounded scalar tilts. For $S > R$, interpolation of the additional terms, followed by (7.27), gives for local F_U

$$\left| \mathbb{E}_{\nu_{\phi}^{B(S)}} F_U - \mathbb{E}_{\nu_{\phi}^{B(R)}} F_U \right| \leq C_{\beta,U} \kappa \|F_U\|_\infty e^{-c \text{dist}(U, B_R(0)^c)}. \quad (7.32)$$

Taking the supremum over $\|F_U\|_\infty \leq 1$ constructs the local limits in total variation. Their projective consistency defines ν_ϕ^B , independently of the exhaustion. Terms whose support meets a fixed region have uniformly summable tails, and the bare specification has finite range. Approximation by functions of finitely many cells therefore passes the DLR identity to this limit. This construction also covers lexicographic partial backgrounds and preserves (7.27)–(7.28).

7.5 Weak free energies at a frozen boundary

Lemma 7.10 (Deletion response for actual observables). *Let F_U be bounded and supported on a finite set U , and let N_B count particles in a finite set B of marked cells. Multiply their activities by e^{-t} . For $t \geq 0$ and $0 \leq s \leq 1$, under the weak background sJ_D ,*

$$|\text{Cov}_{\text{full},t,sJ_D}(F_U, N_B)| \leq C\beta(1 + \beta q_\beta) e^{-t/2} \|F_U\|_\infty \sum_{x \in U, y \in B} e^{-md(x,y)}. \quad (7.33)$$

Proof. Use a scalar source uF_U and write $S = \partial_u K_t(0) = \mathbb{E}_{\text{fiber}} F_U$. Let T_B count retained particles in B and N_B^{bad} the reconstructed particles there. Differentiating the fiber and capped expectations gives the exact identity

$$\text{Cov}_{\text{full}}(F_U, N_B) = \text{Cov}_{\text{cap}}(S, T_B + \partial_t K_t) + \mathbb{E}_{\text{cap}} \text{Cov}_{\text{fiber}}(F_U, N_B^{\text{bad}}). \quad (7.34)$$

The insertion norm of T_B retains $e^{-t/2}$. For $\partial_t K_t$, differentiate a bad atom in the mixed expansion. Weak-only objects have zero derivative, and (7.6) supplies one count factor $C\beta$. The cluster estimate with a bad-site mark gives (7.23) with the same anchors in B , uniformly under the weak background. Thus (7.19) and (7.31) bound the first term of (7.34) by the right side of (7.33). For the fiber term, the derivative of a deleted cell's conditional law has total mass at most $C\beta q_\beta e^{-t/2}$ by the marked tail. The influence resolvent then gives the same bound with coefficient $C\beta q_\beta$. $\square$

Lemma 7.11 (Response of a logarithmic tilt). *Let μ be a probability measure and A, B real measurable functions, with A bounded. Write $\mu_{t,s}$ for the normalized tilt of μ by e^{-tB-sA} . If $\mathbb{E}_\mu e^{-(t\pm\eta)B} < \infty$ for some $\eta > 0$, then*

$$\frac{d}{dt} \left[-\log \mathbb{E}_{\mu_{t,0}} e^{-A} \right] = - \int_0^1 \text{Cov}_{\mu_{t,s}}(A, B) ds. \quad (7.35)$$

Proof. Differentiating the numerator and denominator gives $\mathbb{E}_{\mu_{t,1}} B - \mathbb{E}_{\mu_{t,0}} B$. Since $\partial_s \mathbb{E}_{\mu_{t,s}} B = -\text{Cov}_{\mu_{t,s}}(A, B)$, integration in s proves the identity. The exponential moments justify differentiation under the integrals. $\square$

Proposition 7.12. *Let $J_D = \sum_{A \subset D} J_A$, and let μ_D^ξ be the bare full reference at fixed fields and exterior rods ξ . Extend the fields outside D in any bounded way and let ν_ϕ be the infinite bare reference. Then*

$$\left| -\log \mathbb{E}_{\mu_D^\xi} e^{-J_D} + \log \mathbb{E}_{\nu_\phi} e^{-J_D} \right| \leq C\beta(1 + \beta q_\beta) \kappa A(D), \quad (7.36)$$

where $A(D)$ is its lattice boundary area. The estimate is uniform for finite cubical unions, including regions with holes.

Proof. Choose a strip $B \subset D$ along every boundary face, of fixed physical width greater than the hard interaction range. Then $|B| \leq CA(D)$. Delete all particle activities in B by e^{-t} , holding h , the normalizers and J_D fixed, and put $V_\xi(t) = -\log \mathbb{E}_{\mu_{D,t}^\xi} e^{-J_D}$. Apply Lemma 7.11 with $A = J_D$ and $B = N_B$; finite-volume count tails supply its exponential moments. Summing (7.33) over the terms of J_D gives $|V'_\xi(t)| \leq C\beta(1 + \beta q_\beta) \kappa e^{-t/2} |B|$. At $t = \infty$ the strip is empty. The remaining rods are beyond hard interaction range of ξ , and J_D has no arguments outside D ; consequently $V_\xi(\infty)$ is independent of ξ . Integrating proves that all $V_\xi(0)$ lie in an interval of width at most the right side of (7.36). Finally the bare DLR identity gives

$$\mathbb{E}_{\nu_\phi} e^{-J_D} = \mathbb{E}_{\nu_\phi|D^c} \mathbb{E}_{\mu_D^\xi} e^{-J_D}.$$

The negative logarithm of this average lies between the extreme values of $V_\xi(0)$, completing the proof. $\square$

7.6 A common field interaction for all regional weak gases

Let J be a weak interaction covariant under a fixed sublattice of translations, with regional restrictions

$$J_D = \sum_{U \subset D} J_U. \quad (7.37)$$

If J is obtained by expanding a gas and grouping each logarithmic cluster by the union of its supports, restricting the original gas to supports contained in D gives exactly (7.37).

Choose a translation-covariant lexicographic order and anchor each support at its least site. Define

$$J_i = \sum_{\text{anchor}(U)=i} J_U, \quad J_{<i} = \sum_{\text{anchor}(U)<i} J_U, \quad v_i(\phi) = -\log \mathbb{E}_{\nu_\phi^{J_{<i}}} e^{-J_i}. \quad (7.38)$$

The state for the interaction $J_{<i}$ is defined by the preceding construction by finite tilts. Since $\|J_i\|_\infty \leq \kappa$, we have $|v_i| \leq \kappa$. For a finite D , use $J_{i,D}$ and $J_{<i,D}$ with the additional support restriction $U \subset D$, and define $v_{i,D}$ in the same way. Finite telescoping of normalized tilts gives

$$V_D(\phi) := -\log \mathbb{E}_{\nu_\phi} e^{-J_D} = \sum_{i \in D} v_{i,D}(\phi). \quad (7.39)$$

Put $t_\beta = C\beta\kappa$, with C large enough for the estimates below.

Lemma 7.13 (Comparison of regional increments). *For $d_i = \text{dist}(i, D^c)$,*

$$|v_{i,D} - v_i| \leq C(\kappa + \beta\kappa^2)e^{-cd_i} \leq t_\beta e^{-cd_i}. \quad (7.40)$$

Proof. For background changes, use Lemma 7.11 with $\mu = \nu_\phi^{B_0}$ and tilt observable $B = \Delta B$. Apply it to finite sums first, then pass to summable interactions by (7.27). If $d_i = \text{dist}(i, D^c)$, the strong norm and connected support imply

$$\|J_i - J_{i,D}\|_\infty \leq C\kappa e^{-cd_i}, \quad \sum_{U \ni x, U \not\subset D} e^{b|U|} \|J_U\|_\infty \leq C\kappa e^{-c \text{dist}(x, D^c)} \quad (x \in D),$$

with bound $C\kappa$ outside D . The profile of the anchor source J_i is at most $C\kappa e^{-cd(x,i)}$. Applying (7.35) and convolving the exponential profiles yields (7.40). For every finite cubical union, $\sum_{i \in D} e^{-cd_i} \leq CA(D)$: assign each site to a nearest boundary face and sum the polynomial number of sites at each distance from that face. Equations (7.39) and (7.40) therefore show

$$|V_D - \sum_{i \in D} v_i| \leq Ct_\beta A(D). \quad (7.41)$$

□

Lemma 7.14 (Local dependence of the increments). *If ϕ and ϕ' agree on $B_r(i)$, then*

$$|v_i(\phi) - v_i(\phi')| \leq Ct_\beta e^{-cr}. \quad (7.42)$$

Proof. First hold the weak functions fixed. Changing ϕ_j by t tilts the bare law by $e^{t\hat{N}_j}$: this finite tilt satisfies the new conditional specification and Lemma 7.8 identifies its law. The identity passes through finite weak tilts and their local limits. Lemma 7.11, with $A = J_i$ and $B = -\hat{N}_j$, therefore gives

$$\partial_{\phi_j} v_i = \int_0^1 \text{Cov}_{\nu_\phi^{J_{<i+tJ_i}}} (J_i, \hat{N}_j) dt.$$

The mixed covariance estimate bounds the sum over changed fields by $Ct_\beta e^{-cr}$. Then change the weak source functions to their ϕ' values. Terms that change have support meeting the complement of $B_r(i)$. Their contribution to J_i is bounded directly by $C\kappa e^{-cr}$, using $|\log \mathbb{E}e^{-A} - \log \mathbb{E}e^{-A'}| \leq \|A - A'\|_\infty$. Changes in the past background are bounded by (7.35) with the same profiles as above. This proves (7.42).

□

We now convert the local increments into an interaction by shell telescoping, following Kozlov's reconstruction method [28, Sections 2 and 4].

Proposition 7.15 (Regional interaction). *There are bounded measurable field potentials $\Psi_{i,r}^J$, covariant under the same translations and supported on full radius- r balls $B_r(i)$, such that*

$$\sup_x \sum_{i,r:x \in B_r(i)} e^{c'r} (1+r)^p \|\Psi_{i,r}^J\|_\infty \leq C_{p,c'} t_\beta, \quad t_\beta = C\beta\kappa, \quad (7.43)$$

for fixed p and sufficiently small $c' > 0$. Their ordinary conditional Hamiltonian, the sum of all terms whose full support meets D , satisfies

$$|V_D(\phi_D r_{D^c}) - H_D^J(\phi_D | r)| \leq Ct_\beta A(D). \quad (7.44)$$

Proof. Fix a reference field r_* for telescoping and put

$$v_i^{[r]}(\phi) = v_i(\phi_{B_r(i)r_*, B_r(i)^c}), \quad \Psi_{i,0}^J = v_i^{[0]}, \quad \Psi_{i,r}^J = v_i^{[r]} - v_i^{[r-1]} \quad (r \geq 1).$$

Their exact sum is v_i , and (7.42) bounds the r th shell by $Ct_\beta e^{-cr}$. Polynomial volume growth proves (7.43). In an ordinary conditional sum, every term anchored in D is included; terms anchored outside D that meet D have radius at least their distance from D . Summing their exponential tails costs at most $Ct_\beta A(D)$. Combine this with (7.41) to obtain (7.44). $\square$

The weak potentials above are contributions to the exponent. If the field action is written βH , their contribution to its energy interaction is Ψ^J/β .

8 The effective field interaction

The auxiliary field lives on the fine spatial lattice. A site carries the angular-cell vector $\phi_i \in K = [-R, R]^{n_\omega}$, with normalized Lebesgue reference measure λ ; n_ω is the number of angular cells. Write v_s for the physical volume of a spatial cell and v_a for its marked-cell volumes, so that $\sum_a v_a = v_s$. All meshes, caps, and field bounds in this section are fixed before $\beta = \varepsilon^{-1}$ tends to infinity. The positive interaction Φ^0 is expressed in these original coordinates by $s_{i,a} = \sqrt{\omega_a} \phi_{i,a}$.

8.1 From finite boxes to an exact field interaction

We follow LMP in separating a leading variational functional from a local remainder [2, Sections 3–5].

Let $\mathcal{R}_{\beta,D}^h(\phi \mid \xi)$ be the rod-only fixed-field partition function of the reset model. The exact reset gives

$$\mathcal{R}_{\beta,D}^h(\phi \mid \xi) = \int w_{\beta,h,D}(\eta \mid \xi) \exp \left\{ \sum_{i \in D,a} \phi_{i,a} \hat{N}_{i,a}(\eta) \right\} d\eta, \quad \hat{N}_{i,a} = N_{i,a} \mathbf{1}_{\{N_{i,a} \leq K_a\}}, \quad (8.1)$$

where w is nonnegative and independent of ϕ . Thus $\log \mathcal{R}$ is convex. This formula also defines it on the enlarged box $K^+ = [-R-1, R+1]^{n_\omega}$. The enlargement is used for difference quotients; the physical field law is still supported on K .

The finite-box variational limit, the microscopic locality estimate, and the response estimate for the minimizing profile give the following facts. First, in each fixed free box,

$$\begin{aligned} P_{\beta,D}^h(\phi) &:= \beta^{-1} \log \mathcal{R}_{\beta,D}^h(\phi) \longrightarrow P_{0,D}(\phi), \\ P_{0,D}(\phi) &= - \min_q \left\{ \int_D q(\log(q/\zeta) - 1) + \frac{1}{2} \langle q, T_D q \rangle + \alpha \|P_D q\|_2^2 - \int_D \phi q \right\}, \end{aligned} \quad (8.2)$$

with the marked-cell mass caps imposed on q . The limiting minimizer is unique and continuous in ϕ . The normalizer and empty-cell estimates make the convergence uniform in the fixed source interval. Second, for the full reset reference,

$$\begin{aligned} \left| \beta^{-1} \mathbb{E}_{D,\phi,\xi} \hat{N}_{i,a} - \beta^{-1} \mathbb{E}_\phi \hat{N}_{i,a} \right| &\leq C e^{-m_0 d(i,D^c)}, \\ \beta^{-1} \left| \text{Cov}_\phi(\hat{N}_{i,a}, \hat{N}_{j,b}) \right| &\leq C e^{-m_0 d(i,j)}. \end{aligned} \quad (8.3)$$

These bounds are uniform in the original admissible exterior rods. Third, the density minimizers in free boxes converge exponentially locally to the unique bounded all-space minimizer q_ϕ .

The expectations in (8.3) use the unique whole-space bare reference of Lemma 7.8.

Lemma 8.1 (Uniform force convergence). *Put*

$$\begin{aligned} g_{\beta,h;i,a}(\phi) &= \frac{v_a}{2\alpha}(\phi_{i,a} - h) - \beta^{-1}\mathbb{E}_\phi \hat{N}_{i,a}, \\ g_{0;i,a}(\phi) &= \frac{v_a}{2\alpha}\phi_{i,a} - \int_{C_{i,a}} q_\phi. \end{aligned}$$

Uniformly over all fields in $K^{\mathbb{Z}^3}$ and the fixed source interval,

$$d_\beta := \sup_{\phi,h,i,a} \left| g_{\beta,h;i,a}(\phi) - g_{0;i,a}(\phi) + \frac{hv_a}{2\alpha} \right| \longrightarrow 0. \quad (8.4)$$

Proof. In a fixed box, we use the convergence theorem for gradients of convex functions [16, Theorem 25.7], with the following uniform estimate. The capped counts make the functions in (8.2) equi-Lipschitz on K^+ . A finite net upgrades their pointwise convergence to uniform convergence. If its error in a fixed box is $\eta_{\beta,D}$ and the limiting gradient has modulus ω_D , convexity gives, for $0 < t < 1$ and $\phi \in K$,

$$|\partial_{i,a} P_{\beta,D}^h(\phi) - \partial_{i,a} P_{0,D}(\phi)| \leq 2\eta_{\beta,D}/t + \omega_D(t).$$

The two derivative bounds are the forward and backward difference quotients. Let $\beta \rightarrow \infty$ first and then $t \downarrow 0$. Use a free box whose center is distance ℓ from its boundary. The locality estimates for the microscopic gas and minimizing density bound the difference between the global forces by the fixed-box derivative error plus $2Ce^{-m\ell}$. Choose ℓ first and β second. This proves (8.4). $\square$

Lemma 8.2 (Locality of the force error). *The error force $E_{\beta,h}$ in this formula is quasilocal: if $\phi = \psi$ on $B_r(i)$, then*

$$|E_{\beta,h;i}(\phi) - E_{\beta,h;i}(\psi)| \leq Ce^{-mr}. \quad (8.5)$$

Proof. Integrating the covariance bounds and variational response bounds along successive field coordinates proves (8.5) for finitely supported changes. Summability and exhaustion extend the bound to arbitrary exterior changes. $\square$

For finite-volume fields the force is the gradient of minus β^{-1} times the logarithm of the full fixed-field density. Boundary convergence passes the vanishing of every finite-coordinate loop integral to the global force. The variational force has the same property. Consequently $E_{\beta,h}$ is closed and defines the finite-change cocycle

$$\mathcal{C}_E(\phi; \psi) = \int_0^1 \sum_{i \in A} E_i(\psi + t(\phi - \psi)) \cdot (\phi_i - \psi_i) dt, \quad A = \{i : \phi_i \neq \psi_i\}, \quad (8.6)$$

whenever A is finite.

Reconstructing potentials from quasilocal conditional energies goes back to Sullivan [29] and Kozlov [28]. The construction below adapts Kozlov's lexicographic regrouping [28, Section 4]; exponential locality gives the weighted norm.

Proposition 8.3 (Small interaction representing the error). *Fix a homogeneous well field r_* . There is a translation-covariant interaction $\Delta_{\beta,h}$ on complete lattice balls whose cocycle is (8.6) and for which, whenever $a < m$ and p is fixed,*

$$\|\Delta_{\beta,h}\|_{a,p} := \sup_x \sum_{i,r: x \in B_r(i)} e^{ar}(1+r)^p \|\Delta_{i,r}\|_\infty \longrightarrow 0. \quad (8.7)$$

Every term vanishes on r_ .*

Proof. Order the lattice lexicographically and set

$$U_i(\phi) = \int_0^1 E_i(\phi_{j \prec i}, r_* + t(\phi_i - r_*), r_{*, j \succ i}) \cdot (\phi_i - r_*) dt.$$

This is well defined on the infinite past by quasilocality. Let $U_i^{(r)}$ replace every field outside $B_r(i)$ by r_* and put $\Delta_{i,0} = U_i^{(0)}$ and $\Delta_{i,r} = U_i^{(r)} - U_i^{(r-1)}$ for $r \geq 1$. Declare the support to be the entire ball. The force bound and locality separately give

$$\|\Delta_{i,r}\|_\infty \leq C \min\{d_\beta, e^{-mr}\}.$$

Finite lexicographic telescoping proves the cocycle identity for fields equal to r_* outside a finite box. Exhaustion then proves

$$\mathcal{C}_E(\phi; \psi) = \sum_{i,r} [\Delta_{i,r}(\phi) - \Delta_{i,r}(\psi)]$$

for arbitrary finite changes, since centers far from the changed set have an exponentially summable contribution. Finally,

$$\|\Delta_{\beta,h}\|_{a,p} \leq C \sum_{r \geq 0} (1+r)^{p+3} e^{ar} \min\{d_\beta, e^{-mr}\} \longrightarrow 0.$$

Dominated convergence applies because $a < m$. Splitting the sum at $m^{-1} \log(1/d_\beta)$ also gives the bound $C d_\beta^{1-a/m} (1 + |\log d_\beta|)^{p+4}$. $\square$

Retain the positive leading interaction Φ^0 and define

$$\Psi_{\beta,h} = \Phi^0 + \Delta_{\beta,h} - \sum_i \frac{h}{2\alpha} \sum_a v_a(\phi_{i,a} - r_{*,a}). \quad (8.8)$$

Its force is exactly $g_{\beta,h}$. The complete signed perturbation $\mathcal{E}_{\beta,h} = \Psi_{\beta,h} - \Phi^0$ satisfies

$$\|\mathcal{E}_{\beta,h}\|_{a,p} \leq C|h| + \delta_{\text{micro},\beta}, \quad \delta_{\text{micro},\beta} \longrightarrow 0. \quad (8.9)$$

Let F_D denote the full fixed-field density, including its Gaussian factor relative to λ . For a coarse union D , let $A(D)$ be its number of boundary faces multiplied by L^2 .

Lemma 8.4 (Pressure at a fixed homogeneous field). *For every homogeneous field $r \in K$, the limit*

$$b_r = \lim_D (v_s \# D)^{-1} \log F_D(r \mid \text{free})$$

exists along aligned boxes. Changing the admissible exterior rods changes the logarithm by at most $C_\beta A(D)$, uniformly in r .

Proof. We use the argument with separated boxes for the thermodynamic limit [30, Section 2]. Divide each cell weight by its empty weight and remove the Gaussian prefactor. The normalized empty weight is one. Dropping exclusions, (3.3) bounds the one-cell sum by $2 \exp[C\beta v_a]$, uniformly in r and $|h| \leq H$. Emptying a complete-cell strip therefore costs at most a constant per cell at fixed β . A strip wider than the rod range removes all exterior exclusions. Dropping exclusions between boxes gives the product upper bound; separating boxes by empty strips gives the product lower bound. Tiling a large aligned box and letting the tile side increase proves existence. The removed cell prefactors have an exact volume density and can then be restored. $\square$

For a finite region D , let H_D^Ψ be the sum of terms whose declared supports meet D , evaluated with the specified exterior. Define its homogeneous energy density by equal allocation to support sites: $e(r) = v_s^{-1} \sum_{A \ni 0} \Psi_A(r)/|A|$.

Lemma 8.5 (Common pressure normalization). *With $b_* = b_{r_*}$,*

$$b_r + \beta e(r) = b_* =: b_0. \quad (8.10)$$

Proof. Integrate the difference between the finite-box and global forces along $r_* + t(r - r_*)$ on D . The boundary decay in (8.3) sums to $CA(D)$, so

$$\log \frac{F_D(r \mid \text{free})}{F_D(r_* \mid \text{free})} = -\beta [H_D^\Psi(r_D \mid r_*) - H_D^\Psi(r_* \mid r_*)] + O_\beta(A(D)).$$

The first spatial moment of the interaction is summable, so the energy difference equals $(e(r) - e(r_*))v_s \# D + O(A(D))$. Every interaction term vanishes at r_* , hence $e(r_*) = 0$. Divide by volume and apply Lemma 8.4. $\square$

Define

$$\sigma_D(r) = H_D^\Psi(r \mid r) - e(r)v_s \# D. \quad (8.11)$$

We retain $\sigma_D(r)$ when changing the root.

For frozen admissible rods ξ , subtract the canonical action from $-\beta^{-1} \log F_D(\phi \mid \xi)$ and subtract its value at the root. The resulting boundary action satisfies

$$|\partial_{i,a} B_D^\xi(\phi)| \leq C e^{-md(i, D^c)}.$$

If O contains a boundary band of width cL , summing this tail on a coarse cubical union gives

$$\sup_\phi |B_D^\xi(\phi) - B_D^\xi(\phi \text{ on } D \setminus O)| \leq C e^{-m'L} A(D). \quad (8.12)$$

The retained function of the outer fields is part of the boundary weight.

9 Surface estimates for restricted fields

9.1 Local interpolation

Write $Q(\phi) = q_\phi$ for the minimizing profile and use the trial action ℓ_i of (6.5). In the original angular field coordinates, set $|z|_\omega^2 = \sum_a \omega_a z_a^2$ and $(\Pi_0 Q_i)_a = 2\alpha(v_s \omega_a)^{-1} \int_{B_i \times U_a} Q_i$. Its pinning term is $(v_s/4\alpha)|\phi_i - \Pi_0 Q_i|_\omega^2$. We use the quadratic estimate (6.6) and the global comparison (6.9); write r_0 for their fixed local profile range. Copying complete cells and inserting wells preserve all marked-cell caps.

Fix a component $\mathcal{W}_q$, put $\mathcal{M}_q = \Pi_0 \mathcal{W}_q$, and choose a root f_q with field $r_q = \Pi_0 f_q$. All field neighborhoods use the norm $|\cdot|_\omega$. Lemma 6.8 gives measurable nearest wells f_i and a bounded concave modulus ω_0 controlling squared local profile errors and neighboring-well differences. Set

$$\omega(t) = C\{\omega_0(t) + \sqrt{\omega_0(t)}\}.$$

This bounded concave modulus controls both the squared errors and the linear neighboring-well differences used in randomized rounding below. The results 2.6, 2.7, and 6.9 give finitely many separated projected components and rectifiable paths within each component.

Let E be a signed perturbation with exponentially summable norm at most δ , in energy units before multiplication by β . Coarse blocks have side L . Assume that the declared supports of E are connected and that E , its supports, and the reference measure are covariant under translations by $L\mathbb{Z}^3$. This holds for the canonical perturbation and for the regional interactions of Proposition 7.15. The phase restriction allows at most θL^3 fine spins outside a fixed neighborhood of $\mathcal{M}_q$ in each block. A radius- s ball about any projected well has λ -mass at least cs^{n_ω} and lies inside the field box for sufficiently small s .

For a finite vector X with probability product reference λ_X , write $\mathcal{H}(X | W) = -\mathbb{E}D(\mathcal{L}(X | W) || \lambda_X)$; omit W for unconditional entropy.

Lemma 9.1 (Entropy after replacing coordinates). *Let X be a finite vector determined by source randomness S , and let U retain a deterministic subset of its coordinates, possibly permuted in a way that preserves their reference measures. If fresh coordinates V satisfy $\mathcal{H}(V | S, W) \geq -a$, then*

$$\mathcal{H}(U, V | W) \geq \mathcal{H}(X | W) - a.$$

Proof. The chain rule and $\mathcal{H}(\cdot | \cdot) \leq 0$ give $\mathcal{H}(U | W) \geq \mathcal{H}(X | W)$. Since U is determined by S , conditioning gives $\mathcal{H}(V | U, W) \geq \mathcal{H}(V | S, W)$. Adding these two bounds proves the claim; these are the entropy inequalities of [31, Proposition 1]. $\square$

For m fresh coordinates sampled independently, conditional on S, W , from λ restricted and normalized to radius- β^{-2} balls about projected wells, the ball-mass bound above gives $a = Cm \log \beta$, for $\beta \geq 2$.

The elementary cubical bound used repeatedly is

$$\#\{i : d_\infty(i, \partial D) \leq t\} \leq C(t+1)A(D), \quad 1 \leq t \leq cL. \quad (9.1)$$

For one coarse face the relevant prism, edge, and corner volumes are bounded by $C(tL^2 + t^2L + t^3) \leq CtL^2$; summing over faces proves the assertion for arbitrary coarse unions.

Lemma 9.2 (Buffer energy). *Let $A > 0$ be an area bound in fine-lattice units. Let B be a buffer of width comparable to L in a restricted Gibbs field with arbitrary fixed boundary fields. Assume that B and its radius- r_0 neighborhood lie in a union J of freely integrated complete coarse blocks, with $\#J \leq CLA$ and at most CA fine sites in its fixed-width boundary strip. Then*

$$\mathbb{E} \sum_{i \in B} e_i \leq CA + CLA \left(\delta + \frac{\log \beta}{\beta} + \beta^{-4} \right). \quad (9.2)$$

Consequently a fine moving cut through a distance interval of length cL samples mean patch energy at most

$$\eta_L = C \left(L^{-1} + \delta + \frac{\log \beta}{\beta} + \beta^{-4} \right). \quad (9.3)$$

Proof. Replace the fields in J by independent uniform fields in balls of radius β^{-2} about r_q . Use a profile trial equal to f_q inside J and to the old minimizing profile outside. Only a fixed-width interface has a bounded new density-action cost, of order A . Keep the negative old terms centered in B in (6.9) and drop the other negative terms. The weak change costs at most $C\delta LA$. Lemma 9.1 bounds the conditional entropy loss by $CLA \log \beta$. Conditional Gibbs optimality yields (9.2). Whole-block replacement preserves the phase restriction. A fixed-width patch belongs to only a bounded number of the cL cut positions. Averaging and (9.1) give (9.3). $\square$

Lemma 9.3 (Interpolation through a well component). *For source laws satisfying Lemma 9.2, choose a ν -net in $\mathcal{W}_q$ of cardinality N_ν and paths to f_q of length at most D_ν . A moving interface can be joined to a root plateau through R_1 fine layers with expected energy cost at most $A\kappa_E$, where $\kappa_E \leq \kappa$ and*

$$\begin{aligned} \kappa \leq C \Big[& \theta + \nu^2 + \frac{D_\nu^2}{R_1} + \left(1 + \frac{R_1 N_\nu}{\nu}\right) \omega(\eta_L) \\ & + L\delta + \frac{L \log \beta}{\beta} + L\beta^{-4} \Big] \end{aligned} \quad (9.4)$$

also includes its entropy cost divided by βA . The coefficient of θ is independent of the subsequent net, paths, and R_1 .

Proof. Fix a separator made of coarse-block faces. Assign the blocks on each side to the source that will be retained there. On either side choose a cut at fine distance between $L/4$ and $L/3$ from the separator, and put its interpolation strip on that same side, between the cut and the separator. Require $R_1 + r_0 < L/6$. Copy that source beyond the strip and put the root profile between the strips. The root plateau therefore crosses the fixed separator, and every coarse block retains coordinates from only its assigned source. The hypothesis of Lemma 9.2 is imposed on the available bands on both sides. For a single source the other side is already at the root.

First map a nearest well f_i outside $\mathcal{W}_q$ to f_q , leaving wells in $\mathcal{W}_q$ fixed. This map is Lipschitz on $\mathcal{W}$: different components have a positive separation and the set has finite diameter. Thus the transformed wells h_i retain the neighboring rigidity bound.

Let the net be $v_1, \dots, v_{N_\nu}$. For the assignment step we use a fixed-order variant of the random-radius partitions of Călinescu, Karloff, and Rabani [32]; see also the construction in [33, Lemma 3.1]. Choose u uniformly in $[\nu, 2\nu]$ and assign h to the first v_j with $\|h - v_j\| \leq u$. For two wells,

$$\mathbb{P}_u(n(h) \neq n(h')) \leq \frac{CN_\nu}{\nu} \|h - h'\|.$$

Indeed, a different first successful index requires u to lie between two distances to at least one net point; the total interval length is at most $N_\nu \|h - h'\|$. Parametrize each chosen path on $[0, 1]$ with Lipschitz constant at most D_ν . Across the strip use $Q'_i = \gamma_{n(h_i)}(s_i)$, where s_i increases from zero to one in R_1 layers. Equal assignments have neighboring discrepancy at most CD_ν/R_1 . Unequal assignments cost a bounded amount and have the preceding probability. The quadratic bound (6.6), patch rigidity, and concavity of ω give

$$CA \left[\frac{D_\nu^2}{R_1} + \frac{R_1 N_\nu}{\nu} \omega(\eta_L) \right].$$

The initial interface costs $CA[\nu^2 + \omega(\eta_L) + \theta]$. Write E_i^{patch} for the sum of charges on the fixed enlarged patch around i . Projected component separation gives

$$\mathbf{1}_{\{f_i \notin \mathcal{W}_q\}} \leq \mathbf{1}_{\{\phi_i \notin \mathcal{N}(\mathcal{M}_q)\}} + C\omega(E_i^{\text{patch}}).$$

Only the fixed-width initial interface pays this cost. Each minority site is charged $O(L^{-1})$ after averaging over cuts, and majority in a buffer of volume $O(LA)$ gives $C\theta A$. Inside the strip all h_i already belong to $\mathcal{W}_q$, so no factor $R_1\theta$ arises.

Conditional on the source fields, sample the fresh spins independently from λ restricted and normalized to radius- β^{-2} balls about their trial projected wells. There are $O(LA)$ such spins. Their

pinning cost is $O(LA\beta^{-4})$, their conditional entropy is at least $-CLA\log\beta$, and the weak cost is $O(\delta LA)$. The fixed source assignment makes each output block retain a subset of one old block. All fresh spins lie in the phase neighborhood, so the majority constraint is preserved. Extra erased cells lie on the root plateau and have only the pinning cost just estimated. Average over cuts and u at the level of the expected energy, then choose deterministic parameters attaining the bound. The retained coordinate sets are consequently deterministic for the entropy arguments below. $\square$

One first fixes θ , then ν , the finite paths and R_1 , then $L \gg R_1 + r_0$, and finally small δ and large β . The right side of (9.4) can be made arbitrarily small.

9.2 Rooted restricted partition functions

Let Z_D^{q,r_q} be the restricted field partition on a coarse union D with root field r_q outside. Its leading rooted energy is

$$H_D^0(\phi_D \mid r_q) = \sum_i e_i(\phi_D r_{q,D^c}) \geq 0.$$

The weak part is the ordinary conditional Hamiltonian $H_D^E = \sum_{A:A \cap D \neq \emptyset} E_A$, including its root value.

Lemma 9.4 (Partition bound near a homogeneous well). *For sufficiently large β , uniformly over finite coarse unions D ,*

$$-C(\log\beta + \beta\delta)\#D \leq \log Z_D^{q,r_q} \leq C\beta\delta\#D.$$

The same bounds hold after omitting nonnegative leading terms or weak interaction terms.

Proof. The leading energy is nonnegative, the weak energy has absolute value at most $C\delta\#D$, and the reference is a probability measure; these give the upper bound. For the lower bound, integrate each field in a radius- β^{-2} ball about r_q . The resulting product set satisfies the phase restriction and has reference mass at least $\exp[-C\#D\log\beta]$. Its leading energy is at most $C\beta^{-4}\#D$ by (6.10). Omitting nonnegative leading terms preserves both energy bounds; omitting weak terms cannot increase their absolute norm. $\square$

For a signed interaction, allocate every term E_A equally among its $|A|$ declared sites. If B_L is one coarse block and μ is coarse-translation invariant, define

$$e_0 = \frac{1}{L^3} \sum_{i \in B_L} \mathbb{E}_\mu e_i, \quad e_E = \frac{1}{L^3} \sum_{i \in B_L} \sum_{A \ni i} \frac{\mathbb{E}_\mu E_A}{|A|}.$$

Absolute summability gives $|e_E| \leq C\delta$.

Lemma 9.5 (Restricted pressure). *Along complete-block cubes D , the limit*

$$\pi_q = \lim_D \frac{\log Z_D^{q,r_q}}{\#D}$$

exists and is independent of the bounded exterior field.

Proof. We again use the proof of the thermodynamic limit by division into boxes [30, Section 2]. Normalize the reference on the allowed configurations of each block. For any coarse union D , the norm of interactions meeting both D and D^c is

$$b(D) = \sum_{A:A \cap D \neq \emptyset, A \cap D^c \neq \emptyset} \|\Phi_A^0 + E_A\|_\infty = O(A(D)).$$

Deleting those terms changes $\log Z_D$ by at most $\beta b(D)$. Tiling a large cube by fixed smaller cubes gives matching upper and lower limits as the smaller side tends to infinity. The same bound removes the exterior field. Restore the block-reference normalization. $\square$

Lemma 9.6 (Restricted Gibbs state). *There is a restricted Gibbs state μ invariant under translations by coarse blocks.*

Proof. This is the finite-volume compactness construction of Gibbs states, adapted to the allowed-block reference; compare [34, Appendix A]. Relative to the normalized allowed-block reference, summability bounds each fixed marginal density in L^∞ , uniformly in the containing volume. Diagonal weak-* compactness passes bounded measurable cylinder tests. Uniform interaction tails pass the DLR kernels. Averaging translates and repeating the limit gives an invariant Gibbs state. $\square$

Lemma 9.7 (Entropy bounds). *The state μ has an entropy density s . For every union C of complete blocks,*

$$\mathcal{H}_\mu(X_C) \geq s\#C, \quad \mathcal{H}_\mu(X_C \mid X_{C^c}) \leq s\#C. \quad (9.5)$$

Proof. We use the entropy inequalities and lexicographic construction of Robinson and Ruelle [31, Propositions 1–2]. In a lexicographic block order, conditioning on larger finite pasts converges to conditioning on the infinite past. The chain rule on large cubes gives its entropy per site as s . The conditioning sets for a marginal of C are contained in that past; those for the conditional law given C^c contain it. $\square$

The next identity is the restricted-block form of the Gibbs variational principle [35, Theorem 3]; the passage from Gibbs conditional laws to this identity follows the boundary comparison in [34, Appendix A].

Lemma 9.8 (Pressure identity). *The entropy and mean energies of μ satisfy*

$$\pi_q = s - \beta(e_0 + e_E). \quad (9.6)$$

Proof. For a complete-block box D , conditional Gibbs densities with two exterior fields have ratios between $e^{-4\beta b(D)}$ and $e^{4\beta b(D)}$. Comparing with their mixture bounds the mutual information between X_D and X_{D^c} by $4\beta b(D)$, so their conditional and marginal entropy densities agree. The identity

$$\mathcal{H}_\mu(X_D \mid X_{D^c}) = \beta \mathbb{E}_\mu H_D(X_D \mid X_{D^c}) + \mathbb{E}_\mu \log Z_D(X_{D^c})$$

then gives (9.6) after division by $\#D$. The equal allocation of interaction terms changes the mean energy only by the boundary contribution $O(b(D))$. $\square$

Lemma 9.4 gives the pressure bounds below; (9.6), $s \leq 0$, and $|e_E| \leq C\delta$ then bound the mean leading energy:

$$-C(\log \beta + \beta\delta) \leq \pi_q \leq C\beta\delta, \quad 0 \leq e_0 \leq C\left(\delta + \frac{\log \beta}{\beta}\right). \quad (9.7)$$

For a finite fine-site set J containing a complete-block union C , nonpositivity and the chain rule also give

$$\mathcal{H}_\mu(X_J \mid X_{J^c}) = \mathcal{H}_\mu(X_{J \setminus C} \mid X_{J^c}) + \mathcal{H}_\mu(X_C \mid X_{C^c}) \leq s\#C. \quad (9.8)$$

Gibbs optimality on J follows by disintegrating the kernel on an enclosing complete-block region, keeping each partial-block constraint.

The next bound has the role of LMP's surface error in comparing restricted partition functions [2, Statement 3.5 and Section 6]. Here the boundary modification interpolates within an entire connected component of minimizing profiles.

Proposition 9.9 (Rooted surface bound). *Under the preceding hypotheses and scale choices,*

$$|\log Z_D^{q,r_q} - \pi_q \#D| \leq \beta \kappa A(D) \quad (9.9)$$

uniformly over finite unions of complete coarse blocks, with κ of the form (9.4).

Proof. For the lower bound take $X \sim \mu$. Copy its deep part into D , interpolate to a root plateau in the boundary band, and use root fields outside D . Apply (6.9) relative to the global root, using a trial density profile equal to f_q outside D . The plateau makes all outside-centered trial costs vanish. Copied local terms are a subset of nonnegative old terms centered in D , so

$$\mathbb{E}H_D^0(Z_D \mid r_q) \leq e_0 \#D + \kappa_E A(D).$$

The equal site allocation of signed weak terms gives $e_E \#D + C\delta LA(D)$: terms wholly in the copied core agree, and terms meeting its boundary band have total norm $O(\delta LA(D))$. Apply Lemma 9.1 to X_D , using the complete field X as source randomness, and then (9.5):

$$\mathcal{H}(Z_D) \geq s \#D - CLA(D) \log \beta.$$

The finite-volume variational principle proves the lower estimate.

For the upper bound let Y have the rooted restricted Gibbs law on D , independently of $X \sim \mu$. Insert its deep core into X , using the interpolation above. Let J be the actual finite set of changed fine sites and choose a complete-block core $C \subset J$ such that every range- r_0 stencil centered in C lies among the changed cells and

$$|\#D - \#C| \leq CLA(D).$$

Every cell of J is copied from Y or is fresh. A partially cut block retains X only outside J ; the whole-block source assignment prevents it from also retaining Y . Apply (6.9) relative to X , keep the negative old terms centered in C , and drop the other negative terms. The copied Y costs are bounded by its entire nonnegative rooted energy. Thus

$$\begin{aligned} \mathbb{E}H^0(Z; X) &\leq \mathbb{E}H_D^0(Y_D \mid r_q) - e_0 \#C + \kappa_E A(D), \\ \mathbb{E}H^E(Z; X) &\leq \mathbb{E}H_D^E(Y_D \mid r_q) - e_E \#C + C\delta LA(D). \end{aligned}$$

Apply Lemma 9.1 to Y_D , with source randomness (X, Y) and conditioning X_{J^c} . Independence gives

$$\mathcal{H}(Z_J \mid X_{J^c}) \geq \mathcal{H}(Y_D) - CLA(D) \log \beta.$$

Conditional Gibbs optimality of μ and (9.8) now imply

$$\log Z_D^{q,r_q} = \mathcal{H}(Y_D) - \beta \mathbb{E}H_D(Y_D \mid r_q) \leq \pi_q \#C + \beta \kappa A(D).$$

Replacing $\#C$ by $\#D$ costs at most $C\beta LA(D)(\delta + \log \beta/\beta)$ by (9.7), already included in (9.4). If an eroded core is empty, the affected region has at most $CLA(D)$ sites; Lemma 9.4 gives both bounds directly. $\square$

For the chosen perturbation E , the pressure in physical units is

$$p_q(E) = b_0 + \pi_q/v_s. \quad (9.10)$$

The root boundary normalization is (8.11).

For a regional weak gas, add the potentials of Proposition 7.15 divided by β to the weak energy interaction. Their additional norm is at most Ct_β/β . Equation (7.44) compares the actual regional correction with its ordinary conditional Hamiltonian at a logarithmic cost $Ct_\beta A(D)$. Applying Proposition 9.9 with this enlarged weak interaction therefore gives the same rooted surface estimate for the actual auxiliary field integral, on every coarse union, including punctured regions. The homogeneous value of the correction is included in its branch pressure.

9.3 Collar kernels and arbitrary caps

Let S be a finite union of coarse blocks. Partition its complement into a bounded inner region and an outer region, called the two caps, so that every connected set joining them meets S . Prescribe fields a, b on these caps and impose only the phase- q majority constraint in S . Let T be a union of complete coarse blocks containing the inner cap and disjoint from the outer cap.

If P is the set of coarse cubes incident to faces of ∂T , assume its three-neighborhood lies in S . Its two-neighborhood is then a complete-block reset region for Lemma 9.2. Writing $A = L^2$ times the number of separator faces, the intervening bands have volume $O(LA)$. Cuts at distance between $L/4$ and $L/3$ have area $O(A)$, and a site belongs to an R_1 -thick moving strip for at most CR_1 cut positions. These bounds follow by counting prisms over separator faces. Assume that all declared interaction supports are connected.

Proposition 9.10 (Collar cross ratio). *Let $K(a, b) > 0$ be the restricted collar kernel with arbitrary bounded inner and outer cap fields. Under this geometry,*

$$|\log K(a, b) + \log K(a', b') - \log K(a, b') - \log K(a', b)| \leq \beta A \kappa, \quad (9.11)$$

with κ as in (9.4), uniformly in the caps and in the collar's size and topology.

Proof. Take independent source fields X, Y from the kernels with caps (a, b) and (a', b') . Swap the complete bounded side T . On each side of its boundary place a moving interpolation strip, with a root plateau between them wider than r_0 . Outside T an output retains its original source; inside it retains the opposite source. Fresh fields occupy the intervening erased band. The source assignment is made by whole coarse blocks, so each output block retains at most one source and satisfies the majority condition.

For the first output use a global trial density equal to $Q(X)$ outside the finite modified set, to $Q(Y)$ in its copied interior, and to the interpolation profiles in between. Reverse the roles for the second output. The trials preserve all cell caps. Add the two inequalities (6.9). Each trial agrees with its own source profile outside a finite set, and every stencil copied entirely from a source cancels across the pair. The plateau has only the small field-pinning cost. Dropping the remaining negative old terms and applying Lemmas 9.2–9.3 bounds the leading change by $A\kappa_E$.

The weak changes are also added before they are bounded. Terms wholly in a copied side cancel; every remaining connected support meets the erased band. Their total norm is at most $C\delta LA$, independent of the volume of the swapped inner cap. Choose one deterministic pair of cuts and one net radius after averaging this expected energy estimate.

The retained collar coordinates are a deterministic permutation of a marginal of (X_S, Y_S) . Lemma 9.1, with the complete source pair as randomness, bounds the joint output entropy below by $\mathcal{H}(X_S) + \mathcal{H}(Y_S) - CLA \log \beta$. Their joint law is a trial for the product target law.

An interaction missing S lies on one cap side, by connectedness of its declared support. Its contribution cancels because the complete bounded side was swapped. The sum of the two full cocycles therefore equals the difference of the target and source collar Hamiltonians. The Gibbs variational principle gives

$$\log K(a', b) + \log K(a, b') \geq \log K(a, b) + \log K(a', b') - \beta A \kappa.$$

Interchanging the source and target cap pairings gives the reverse inequality. Positivity follows from bounded action and the positive-mass small-ball trial. $\square$

10 Contours and interface costs

We use fine spatial lattice units in this section. A fine cell has physical volume v_s , and a coarse cube contains L^3 fine cells. Thus

$$V_L = v_s L^3, \quad \text{Vol}(D) = V_L n(D), \quad (10.1)$$

where $n(D)$ counts coarse cubes. Surface areas $\mathcal{A}(D)$ are measured in fine-grid units. Constants may absorb the fixed mesh conversion in surface estimates; the factor V_L is retained in every bulk identity.

Let $Q_v = L(v + [0, 1]^3)$, $v \in \mathbb{Z}^3$, be closed coarse cubes. Faces are assigned measurably to one adjacent cube when integrating variables. Topology always refers to closed cubes. Two indices are star adjacent if their sup-norm distance is one. Thus a union of cubes is connected exactly when its indices are star connected. Define

$$N_k(B) = \{v : \text{dist}_\infty(v, B) \leq k\}.$$

Every spatial variable includes the entire angular fiber and all rod and field variables assigned to that spatial cell. We choose the physical coarse side longer than the microscopic interaction range.

Use the compact projected sets $\mathcal{M}_1, \dots, \mathcal{M}_m$ and the norm $|\cdot|_\omega$ defined above. Choose $\rho_* > 0$ smaller than one quarter of their minimum separation. A fine site has label q if its field lies within ρ_* of $\mathcal{M}_q$, and label zero otherwise. A coarse cube has $\tau_v = q$ when at least $(1 - \theta)L^3$ fine sites have label q , where $0 < \theta < 1/2$.

Set $\tau_v = 0$ when no phase has this majority, and define

$$\Theta_v = q \iff \tau_w = q \text{ for every } \|w - v\|_\infty \leq 1; \quad \Theta_v = 0 \text{ otherwise.}$$

Two star-adjacent nonzero Θ labels coincide. Write $B = \{v : \Theta_v = 0\}$. A contour support is a star component of the full, unclipped set $N_k(B)$. Its bad set B_Γ consists of the bad cubes whose k -neighborhood lies in that component, and its support is $A_\Gamma = N_k(B_\Gamma)$.

10.1 Complementary components and regional domains

Lemma 10.1 (Connected complementary boundaries). *For a finite connected union A of closed coarse cubes, the open complementary components correspond to face-connected components of unused indices. Each complementary boundary, and its set of touching cubes on either side, is connected in the respective Euclidean or star sense.*

Proof. A path in the open complement can be perturbed away from grid edges and vertices, so it crosses only open faces; the converse follows by joining face neighbors. Compactify space to S^3 . For a complementary component D , the sets $F = S^3 \setminus D$ and $G = \overline{D}$ are connected finite subcomplexes of a common triangulation: F consists of A and the other components, whose closures meet A . If vertices x, y belonged to different components of $F \cap G$, choose chains $a \subset F$, $b \subset G$ with boundary $y - x$. The following is the reduced Mayer–Vietoris argument [36, Section 2.2, pp. 149–150]. Since $H_1(S^3) = 0$, write $a - b = \partial(z_F + z_G)$ by assigning each triangle of a filling chain to F or G . Then $a - \partial z_F = b + \partial z_G$ is a chain in $F \cap G$ joining x to y , a contradiction. Thus $\partial D = F \cap G$ is connected. Its closed boundary faces have a connected intersection graph, which connects the touching cubes by star steps. An edge or vertex contact also belongs to this graph: among incident cubes, a face path from a D cube to an A cube first leaves D across a boundary face. $\square$

Lemma 10.2 (Containment of inherited regions). *If D is a bounded complementary component of a connected closed support and $A' \subseteq D$ is another such support, every bounded complementary component of A' lies in D .*

Proof. The connected set $S^3 \setminus D$ avoids A' and contains infinity, so it lies in the unbounded component of $S^3 \setminus A'$. $\square$

A *physical recursion domain* is a box or a bounded complementary component of a connected closed support.

Lemma 10.3. *Every complementary boundary of a contour support has a single nonzero facing label. Components with different facing labels have index distance at least $2k + 2$.*

Proof. A cube outside the support and touching it has distance at least $k + 1$ from every bad cube. Lemma 10.1 connects these facing cubes by star steps, so their nonzero labels coincide. A star path between differently labeled facing cubes contains a bad cube, and hence has length at least $2k + 2$. For arbitrary cubes in differently labeled components, take a coordinate path in their bounding box. Its first exit and last entry give facing cubes no farther apart. $\square$

Lemma 10.4 (Regional gap). *If D is a complementary face component of A , then for $v \in D$,*

$$\text{dist}_\infty(v, D^c) = \text{dist}_\infty(v, A).$$

For a child support $A' = N_k(B') \subset D$, the regional gap is exactly

$$\text{dist}_\infty(A', D^c) \geq 2 \iff \text{dist}_\infty(B', D^c) \geq k + 2. \quad (10.2)$$

Proof. For the distance identity, choose a nearest cube of D^c . If it is unused, a coordinate path inside the corresponding sup-norm box must first leave D through A : face-adjacent unused cubes remain in one component. The reverse inequality follows from $A \subset D^c$. Finally, for any set C ,

$$\text{dist}_\infty(N_k(B'), C) = \max\{\text{dist}_\infty(B', C) - k, 0\},$$

by the triangle inequality and a shortest star path. This proves (10.2). $\square$

Write $U_q(D)$ for the restriction $\tau = q$ throughout D . The physical regional restriction $R_q(D)$ is $\Theta = q$ in the first $k + 1$ inside layers. For contour indexing, extend the outside labels by q ; the actual adjacent outside stencil agrees with this extension, while exterior rods remain arbitrary. On a physical recursion domain, (10.2) is equivalent to regional admissibility. Indeed, the connected complement of D lies in the child's unbounded component. The gap excludes bad cubes from the first $k + 1$ layers. A coordinate path from a cube in this band to a nearest outside cube stays in the band until its first exit. Its nonzero labels propagate the outside value q back to the starting cube. The converse follows immediately from the absence of bad cubes in the band.

On arbitrary finite unions, including punctured sets, we will also use an auxiliary gas indexed by support inclusion.

10.2 Removed regions, shielding, and the contour identity

A contour datum records B_Γ , A_Γ , the τ pattern on the support, and the facing label of each complementary component. Filling every complementary component with its facing label must

produce exactly the recorded bad set B_Γ . Fix an exterior phase q . Let D_j denote its bounded components with facing labels $p_j \neq q$. Define

$$V_\Gamma = A_\Gamma \cup \bigcup_j D_j, \quad K_\Gamma = \{\text{support cubes touching a } q \text{ component}\}, \quad U_\Gamma = V_\Gamma \setminus K_\Gamma.$$

The set K_Γ is the full star-facing layer, including boundaries of q holes. Two contours are strongly compatible when their closed supports do not touch.

Lemma 10.5 (Nesting of removed regions). *The removed regions of strongly compatible q contours are nested or disjoint. Distinct maximal regions have star distance at least two.*

Proof. A connected support avoiding another lies in one complementary component. If this component is bounded, Lemma 10.2 keeps all its interiors there: the removed regions are nested in a non- q component and disjoint in a q hole. If bounded interiors H_1, H_2 overlap while the second support avoids H_1 , connectedness gives $H_1 \subset H_2$; its boundary and then the first support lie in H_2 , again giving nesting. Finally the boundary of each removed region lies in its support, so disjoint removed regions cannot touch unless their supports touch. $\square$

Lemma 10.6 (Microscopic separation). *Every microscopic factor meeting U_Γ depends on exterior variables only through K_Γ .*

Proof. Lemma 10.3 separates the inherited non- q components from all q components. The layer K_Γ covers the remaining contacts, and a coarse side exceeds the microscopic interaction range. $\square$

Let x_v contain the rods and fields of a coarse cube. Write ρ_U for the nonnegative conditional microscopic density, including every hard exclusion. On compatible q data x_K define the joint integrals

$$N_\Gamma(x_K) = \int \rho_U(x_U \mid x_K) \mathbf{1}_{\{\text{recorded pattern on } A\}} \prod_j \mathbf{1}_{R_{p_j}(D_j)} dx_U, \quad (10.3)$$

$$D_\Gamma(x_K) = \int \rho_U(x_U \mid x_K) \mathbf{1}_{U_q(A \setminus K)} \prod_j \mathbf{1}_{R_q(D_j)} dx_U, \quad W_\Gamma = N_\Gamma / D_\Gamma. \quad (10.4)$$

The empty rod configuration and fields near the phase root give $D_\Gamma > 0$. Shielding makes W_Γ a function of x_K alone and makes it independent of the ambient regional domain. Extend it by zero on incompatible collar fields. The extension set is source independent.

Lemma 10.7 (Interior contour indices). *After a maximal q contour has been selected, its residual virtual contours in each D_j are precisely the regional q contours there. The indices for distinct D_j form a Cartesian product.*

Proof. Compatibility with the parent is equivalent to (10.2). This gap also ensures compatibility with siblings and exterior contours and admissibility in the parent domain. The denominator in (10.4) prescribes phase q in every D_j . $\square$

The cancellation of interior partition functions is the contour reweighting used by Disertori–Giuliani [37, Section 4.3, Lemma 1] and Disertori–Giuliani–Jauslin [38, Section 4.2, Lemma 2]. Here the quotients remain conditional on the collar variables, and the identity is established for each exterior configuration.

Proposition 10.8 (Exact regional identity). *Let $\mathcal{C}_q(D)$ be the admissible q contour data in a physical regional domain. Set*

$$G_q(D; x) = \sum_{\mathcal{F} \subset \mathcal{C}_q(D)} \prod_{\text{compatible } \Gamma \in \mathcal{F}} W_\Gamma(x_{K_\Gamma}),$$

allowing nested removed regions. For every compatible exterior configuration ξ ,

$$\int_{U_q(D)} \rho_D(x_D \mid \xi) G_q(D; x_D) dx_D = \int_{R_q(D)} \rho_D(x_D \mid \xi) dx_D. \quad (10.5)$$

Proof. Physical configurations have a unique outside decomposition. Extract the components of the full thickened bad set, start in the exterior q sea, and retain all q contours reached through that sea and its q holes, stopping at each non- q inherited component. To justify enclosure, a contour with exterior-facing phase $p \neq q$ faces a finite p sea component. Its connected outer boundary lies in the disjoint union of supports and hence in one support. This enclosing support is different from the original one, since the chosen facing cube was in its unbounded component. Repeating outward terminates by strict containment and finiteness. The retained removed regions are disjoint; everything outside them has $\tau = q$. Conversely, the recorded support patterns, regional interior restrictions, and q exterior reconstruct exactly this decomposition. The full regional bands prevent new bad stencils or merged k -neighborhoods at the interfaces.

Induct on the number of cubes of D . Group a virtual family by its maximal removed regions. By Lemma 10.7, the residual family indices are a product of regional q -family indices in the inherited components. Freeze every variable outside the first such component, including rods in touching siblings, and apply the conditional induction identity there. Repeat for all components. Each transformed component retains its q boundary band, so the next conditional identity has the required exterior stencil. All selected weights are measurable outside the changed component. The resulting integral over a selected U is exactly D_Γ ; multiplication by W_Γ replaces it by N_Γ . Shielding permits successive cancellation for the disjoint maximal regions. The resulting sum is the physical outside decomposition. Every inherited domain is strictly smaller. $\square$

Following Zahradník's truncation construction [5], used in the continuum setting by LMP [2, Section 3], we clip contour weights to define controlled phase pressures before proving stability.

For fixed thresholds t_Γ , classify a contour as active at a reference configuration when $W_\Gamma > t_\Gamma$. In each virtual family select active contours maximal under inclusion of removed regions. Their removed regions are disjoint. All descendants inside selected inherited components are unrestricted; every remaining contour is stable, including ancestors surrounding selected regions. Put $E = D \setminus \bigcup_{\Gamma \in \mathcal{F}} V_\Gamma$. The selected contribution is exactly

$$\int_{U_q(D)} \rho_D \prod_{\Gamma \in \mathcal{F}} (W_\Gamma \mathbf{1}_{\{W_\Gamma > t_\Gamma\}}) G_{q,E}^{\text{stable}, \mathcal{F}} \prod_{\Gamma \in \mathcal{F}, j} G_q(D_j) dx_D. \quad (10.6)$$

The stable gas retains the original parent admissibility and compatibility conditions. Its actual supports lie in E , even when a stable contour's filled hull surrounds a hole of E . The active indicators depend on K_Γ , and the stable gas depends on E . The same conditional cancellations therefore apply to (10.6). Dropping the active indicators and enlarging the stable gas gives the positive bound

$$\text{selected contribution} \leq \int \rho_D \mathbf{1}_{R_{\mathcal{F}}(\phi)} G_{q,E}^{\text{aux}}(x_E) dx_D, \quad (10.7)$$

where the auxiliary gas contains all compatible q contours with $A_\Gamma \subset E$ and activities $w_\Gamma = \min\{W_\Gamma, t_\Gamma\}$.

10.3 A positive charge on each contour

The contour cost supplies the energy bound in Peierls' energy–entropy argument [39]; here it follows from local rigidity and lattice isoperimetry.

Call a fine site defective when its label is zero or it has a nearest neighbor with a different nonzero label. The fixed-patch rigidity estimate supplies a finite patch P_i and $e_* > 0$ such that

$$i \text{ defective} \implies \sum_{x \in P_i} e_x(\phi) \geq e_*. \quad (10.8)$$

Here e_x are the nonnegative leading charges. To obtain a uniform e_* , take the finite union of closed defect events with distances $\geq \rho_*$ for zero labels and $\leq \rho_*$ for prescribed nonzero labels. If the infimum of the patch charge vanished, product compactness and exponential locality would give a limiting zero-charge patch. Local rigidity makes all marked cells one common well, contradicting the separation of those closed events.

Use the positive ball atoms of Proposition 6.7, writing $\Phi_{x,r} = \Phi_{x,r}^0$. Their localization error is Ce^{-cr} . Fix R_0 so that $|P_i|Ce^{-cR_0} \leq e_*/2$. Then

$$F_i = \sum_{x \in P_i} \sum_{r=0}^{R_0} \Phi_{x,r} \geq e_*/2$$

at each defect, and its support lies in a fixed ball $B_S(i)$.

Lemma 10.9 (Relative lattice isoperimetry). *For $H \subset \{1, \dots, M\}^3$ and its internal nearest-neighbor edge boundary $\partial_Q H$,*

$$|\partial_Q H| \geq (1 - 2^{-1/3}) \min\{|H|, |Q \setminus H|\}^{2/3}.$$

Proof. Replace H by its complement so $a = |H| \leq M^3/2$. If P_j counts its nonempty coordinate- j fibers, the Loomis–Whitney projection inequality [40, Theorem 2] $a^2 \leq P_1 P_2 P_3$ implies some $P_j \geq a^{2/3}$. The projection inequality follows by applying Cauchy–Schwarz twice to $\sum_{x,y,z} a_{xy} b_{yz} c_{zx}$, with the three projection indicators; its square is at most $(\sum a_{xy})(\sum b_{yz}^2)(\sum c_{zx}^2)$. At most a/M nonempty fibers are full. Each remaining fiber has an internal crossing edge, whence

$$|\partial_Q H| \geq a^{2/3} - a/M \geq (1 - 2^{-1/3})a^{2/3}.$$

□

Proposition 10.10 (Protected Peierls charge). *There is $c_0 > 0$, independent of L, θ, β and the contour shape, such that for each contour of size $n = |A_\Gamma|$ there is a fixed set of leading atom labels, depending only on B_Γ , whose sum P_{B_Γ} satisfies*

$$\text{supp } P_{B_\Gamma} \subset N_2(B_\Gamma), \quad P_{B_\Gamma} \geq \tau L^2 n, \quad \tau = \frac{c_0 \theta^{2/3}}{(2k+1)^3}. \quad (10.9)$$

For strongly compatible contours these protected atom sets are disjoint.

Proof. First suppose $\tau_v = 0$ and write $M = L^3$. If the number a_0 of zero-labeled sites is at least $\theta M/2$, it is a positive integer and

$$a_0 \geq \max\{1, \theta M/2\} \geq 2^{-2/3}(\theta M)^{2/3}.$$

Otherwise at least $3M/4$ sites have nonzero labels. A most frequent phase has population $a_q \geq 3M/(4m)$ and complement larger than θM . Hence $\min(a_q, M - a_q) \geq c_m \theta M$. Relative isoperimetry

applied to that phase gives at least $c\theta^{2/3}L^2$ crossing edges. Each crossing has a defective endpoint, and a site is incident to at most six edges.

If $\Theta_b = 0$, either one of its 27 neighboring coarse cubes has $\tau = 0$, or two have different nonzero labels. In the latter case the fine cube $N_1(b)$ contains at least $L^3/2$ sites of one phase and at least that many of its complement. Relative isoperimetry again gives $c\theta^{2/3}L^2$ defects. Summing over B_Γ and using the multiplicity 27 gives

$$|\{\text{defects in } N_1(B_\Gamma)\}| \geq c\theta^{2/3}L^2|B_\Gamma|/27.$$

Color fine sites by coordinate residues modulo $2S + 1$. One color has at least $(2S + 1)^{-3}$ of these defects, and their $B_S(i)$ are disjoint. Summing the corresponding F_i uses no atom twice. Since $n \leq (2k + 1)^3|B_\Gamma|$, this proves the lower bound.

To make the protected Hamiltonian independent of the field, include, once per label, every $\Phi_{x,r}$ with $r \leq R_0$ and center within the patch radius of $N_1(B_\Gamma)$. This fixed set contains every possible color selection and preserves the lower bound by positivity. For $L \geq S + 2$ its full supports lie in $N_2(B_\Gamma)$. Distinct full k -neighborhood supports have bad-set distance at least $2k + 2$; their N_2 neighborhoods are disjoint for $k \geq 24$. $\square$

10.4 Seams, restoration, and separated collars

Let D_j be inherited components of a support or of a family with total support size N . Every boundary face has a support cube on the other side. Therefore

$$\mathcal{A}_{\text{in}} := \sum_j \mathcal{A}(D_j) \leq 6L^2N. \quad (10.10)$$

Suppose $\|\Phi_{i,r}\|_\infty \leq C_0e^{-c_0r}$ and let $m(i,r)$ be the number of inherited components met by its declared ball.

Lemma 10.11 (Seam estimate). *For every fixed $p \geq 0$,*

$$\sum_{m(i,r) \geq 2} (1 + m(i,r))^p \|\Phi_{i,r}\|_\infty \leq C_p \mathcal{A}_{\text{in}}/L. \quad (10.11)$$

If a signed interaction ψ has connected declared supports and local norm at most δ , then

$$\sum_{X: m(X) \geq 2} m(X) \|\psi_X\|_\infty \leq C\delta \mathcal{A}_{\text{in}}.$$

Proof. For $r < L/4$, a ball meets at most eight coarse cubes. Two of these in different complementary components cannot share a face; if both meet the ball, they meet each other only at an edge or vertex. In the sup norm, the intersection of their r -neighborhoods lies in the r -neighborhood of that edge or vertex. It belongs to the skeleton of an inherited boundary face. The number of possible fine centers is bounded by

$$C \sum_j |\partial D_j| \{L(r+1)^2 + (r+1)^3\} \leq C \frac{\mathcal{A}_{\text{in}}}{L} (r+1)^2.$$

Here $|\partial D_j|$ counts coarse boundary faces. Summation against e^{-c_0r} proves the short-radius estimate, with bounded $m \leq 8$. For $r \geq L/4$, a ball meeting two components crosses a boundary face. One

face has at most $CL^2(r+1)(1+(r+1)/L)^2$ possible centers, and $m \leq C(1+r/L)^3$. The remaining sum is at most

$$C_p \mathcal{A}_{\text{in}} \sum_{r \geq L/4} (r+1)(1+(r+1)/L)^{2+3p} e^{-c_0 r} \leq C_p e^{-cL} \mathcal{A}_{\text{in}}.$$

A connected weak support crossing ∂D_j contains the endpoints of a fine boundary edge. Charge it to one such edge and sum the local norm. Summation over j counts a term meeting m interiors at most m times. A fixed finite-range connected-support convention can be enlarged to a nearest-neighbor connected hull at a fixed norm cost. $\square$

Deleting or restoring terms of total norm M changes a log integral by at most βM . A term meeting m interiors is restored in at most m interior factors, m root divisors, and a bounded number of core or exterior factors. Its total multiplicity is $C(1+m)$, covered by the first two estimates. Thus the total logarithmic cost is at most $C\beta(L^{-1} + \delta)\mathcal{A}_{\text{in}}$.

Take $k \geq 24$. Inside D_j put

$$\ell_j(v) = \text{dist}_\infty(v, D_j^c), \quad S_j = \{1 \leq \ell_j \leq 8\}, \quad I_j = \{\ell_j \geq 9\}.$$

Use the level-four/five faces as separator Σ_j . The regional restriction makes S_j p_j -correct. The one-Lipschitz property of ℓ_j separates I_j from D_j^c . Each separator face lies a fixed number of coarse steps from an original boundary face, so

$$|\Sigma_j| \leq C\mathcal{A}(D_j), \quad |S_j| + |N_t(\Sigma_j)| \leq CL\mathcal{A}(D_j), \quad (10.12)$$

for $L/4 \leq t \leq L/3$. Moreover

$$|N_{t+R}(\Sigma_j) \setminus N_t(\Sigma_j)| \leq C(R+1)\mathcal{A}(D_j).$$

Indeed a single coarse face contributes $O(L^3)$ to the tube volume and $O((R+1)L^2)$ to the enlargement; a union's increment lies in the union of the individual increments. Any fixed alignment margin fits inside the collar for large L , with distance cL from both caps.

For a selected family let V be the union of removed regions and let E include its complement and all q holes. Define

$$S_0 = \{v \in A : 1 \leq \text{dist}_\infty(v, E) \leq 8\}.$$

The same level-four/five construction gives its outer separator. Every inherited non- q component lies beyond it. Propagation from the facing q layer makes S_0 q -correct, and

$$\text{dist}_\infty(S_0, D_j) \geq 2k + 2 - 8.$$

Leading balls meeting both outer and inner collars therefore have radius at least cL and total norm $Ce^{-cL}\mathcal{A}_{\text{in}}$; weak terms cost $C\delta\mathcal{A}_{\text{in}}$. Delete these along with the multi-interior terms. Every remaining declared support meets at most one collar. Connectedness of supports assigns every other term to exactly one cap. Conditional collar integrations now factor.

The collar switch exchanges the complete bounded sides $\{\ell_j \geq 5\}$ and $\{v \in V : \text{dist}_\infty(v, E) \geq 5\}$ of the inner and outer separators. These contain the inherited interiors and exclude q holes. Omitted cap coordinates may be extended arbitrarily when comparing cut and full kernels, since all terms reaching them were deleted. If $I_j = \emptyset$, the cross ratio is one.

Collar kernels impose $\tau = q$. Once these labels are fixed, the remaining Θ restrictions concern individual caps: a radius-one stencil cannot cross a width-eight collar, and touching sibling collars have equal labels by Lemma 10.3.

For a conditional single-contour quotient, freeze $K = K_\Gamma$ and work in $U = V_\Gamma \setminus K$. Set $\ell(v) = \text{dist}_\infty(v, K)$ and use

$$O = \{\ell \leq 4\}, \quad S = \{5 \leq \ell \leq 12\}, \quad V' = U \setminus (O \cup S).$$

The level-eight/nine separator has the same tube bounds. The collar lies in the q support buffer, every D_j remains in V' , and every protected atom lies outside all these collars. For each separator let P be its incident coarse cubes. Their distance levels are 4, 5 for inner and outer collars, and 8, 9 for the shifted collar. Since the distance function is one-Lipschitz, $N_3(P) \subset S$: the possible levels are respectively $[1, 8]$ and $[5, 12]$. In the outer case these cubes remain in A , because positive distance excludes E and the component gap excludes every D_j . The region $N_2(P)$ is a union of complete coarse blocks inside the collar. The two interpolation strips lie on their respective sides of the separator, at fine distances between $L/4$ and $L/3$. Their alignment patches remain in the collar once $R_1 + r_0 < L/6$. Thus all three collars satisfy the buffer and block conditions used in the surface estimates.

The boundary-cocycle estimate $|\partial_i B^\xi| \leq C e^{-c \text{dist}(i, U^c)}$ gives

$$\|B^\xi(\phi) - B^\xi(\phi_{Or_{O^c}})\|_\infty \leq C e^{-c'L} \mathcal{A}(U). \quad (10.13)$$

To sum the gradient bound, assign each fine site to a nearest boundary face; at distance d there are at most $C \mathcal{A}(U)(d+1)^2$ such sites. The sum starts at $d \geq cL$. Before detaching inner collars in $V_{\text{aug}} = S \cup V'$, also delete terms meeting both S_j and V_{aug}^c . Their leading norm is $C e^{-cL} \sum_j \mathcal{A}(D_j)$ and their weak norm is $C \delta \sum_j \mathcal{A}(D_j)$. This ensures that an inner root divisor has its own phase on its whole exterior. Every deleted term meets a collar or an inherited component, whereas the protected atoms lie in $N_2(B_\Gamma)$. The Peierls charge survives all cuts.

10.5 Removed volume and contour counting

Lemma 10.12. *For a contour of support size n ,*

$$n(V_\Gamma) \leq C n^{3/2}.$$

The number of size- n contour data whose removed region contains a specified coarse site is at most C_1^n .

Proof. By the same Loomis–Whitney inequality, a finite lattice set d has projection sizes satisfying $|d|^2 \leq P_1 P_2 P_3$. Each occupied coordinate line contributes two boundary faces, giving $|\partial d| \geq 2(P_1 + P_2 + P_3) \geq 6|d|^{2/3}$. Every inherited boundary face meets a support cube, so

$$\sum_j |D_j| \leq C \sum_j |\partial D_j|^{3/2} \leq C \left(\sum_j |\partial D_j| \right)^{3/2} \leq C n^{3/2}.$$

Add the support. Every bounded complementary component lies in its support's coordinate bounding box, since a ray beyond an extreme coordinate reaches infinity without meeting the support. A star-connected support of size n has coordinate span at most $n - 1$. A removed region containing the specified site therefore has only $O(n^3)$ possible support anchors. A deterministic spanning-tree search encodes each anchored support by a depth-first walk of length $2(n - 1)$ with 26 choices per step. Finite support labels and at most $6n$ complementary-component labels add only exponential factors. Absorb the anchor polynomial. $\square$

For a contour put $n_\Gamma = |A_\Gamma|$ and let its declared support S_Γ be all marked cells above A_Γ . Thus $|S_\Gamma| \leq d_L n_\Gamma$. For a tuple $C = (\Gamma_1, \dots, \Gamma_m)$ set $s(C) = \sum_j n_{\Gamma_j}$ and $S(C) = \bigcup_j S_{\Gamma_j}$. Let $\varphi^T(C)$ be its connected incompatibility coefficient, as in (7.14).

Lemma 10.13 (Anchored contour clusters). *For every fixed $b \geq 0$, there is $\mathcal{T}_0(L, b)$ such that*

$$\sup_x \sum_{m \geq 1} \frac{1}{m!} \sum_C |\varphi^T(C)| e^{bs(C)} \prod_{j=1}^m e^{-\mathcal{T} n_{\Gamma_j}} \mathbf{1}_{\{x \in S(C)\}} \leq C_L e^{-\mathcal{T}/2} \quad (10.14)$$

for $\mathcal{T} \geq \mathcal{T}_0(L, b)$, uniformly over regional contour index sets.

Proof. The counting bound gives at most C_1^n contours of size n containing a prescribed coarse site. A contour has at most $27n_\Gamma$ incompatibility anchors. For majorants $a_\Gamma = e^{-(\mathcal{T}-b)n_\Gamma}$,

$$\sum_{\Gamma' \neq \Gamma} a_{\Gamma'} e^{n_{\Gamma'}} \leq 27n_\Gamma \sum_{n \geq 1} C_1^n e^{-(\mathcal{T}-b-1)n} \leq n_\Gamma$$

when $\mathcal{T}$ is large. This is the Kotecký–Preiss criterion [25, Section 2, Theorem]. The rooted tree bound from Lemma 7.4 bounds the left side of (10.14) by $C_L \sum_{n \geq 1} C_1^n e^{-(\mathcal{T}-b-1)n}$. Increasing $\mathcal{T}_0(L, b)$ bounds this sum by $C_L e^{-\mathcal{T}/2}$. $\square$

For $|w_\Gamma| \leq e^{-\mathcal{T} n_\Gamma}$, group the convergent logarithm by connected union supports. Since $|S(C)| \leq d_L s(C)$ and $\text{diam } S(C) \leq C_L s(C)$, increasing b controls the required exponential support weights and polynomial factors. The absolute majorants, uniformly in the rods and fields, give interaction norm at most $\kappa = C_L e^{-\mathcal{T}/2}$.

Lemma 10.14 (Localization of additional contour species). *Let D_j be disjoint physical recursion domains in a coarse union E , and put $C = E \setminus \bigcup_j D_j$. An auxiliary q contour on E absent from the union of the regional q contour sets on D_j has support meeting*

$$S = C \cup \bigcup_j \{v \in D_j : \text{dist}_\infty(v, D_j^c) < 2\}.$$

Proof. A connected support avoiding S lies in one D_j and has distance at least two from D_j^c . The equivalence (10.2) makes it regionally admissible there; Lemma 10.2 keeps all its inherited regions inside D_j . Conversely every regional index is an auxiliary index. $\square$

Lemma 10.15 (Cost of adding species). *Suppose two clipped contour gases have nested index sets, identical activities on common indices, and every additional species has support meeting a coarse set S . Then*

$$\|\log G_2 - \log G_1\|_\infty \leq C_L \kappa \#S.$$

Proof. This is the local cluster comparison underlying [25, Section 2, estimate (4)]. Their logarithmic cluster series differ only in clusters containing an additional species. Mark one such species and a site of its support in S . The anchored estimate in Lemma 10.13, summed over these sites, gives the bound. It is pointwise in the rods and fields and hence also bounds the ratio of the corresponding joint partition functions. $\square$

11 The physical pressure as a maximum of phase pressures

Following LMP [2, Sections 3 and 6], we first define auxiliary pressures using clipped contours and then bound the physical partition functions in terms of those pressures. Roots, phase neighborhoods, regional restrictions, and the threshold $\mathcal{T}$ are independent of the source.

11.1 Normalization and quantitative inputs

Fix the source h . Let ν_r be the unique whole-space bare rod state at constant field r (Lemma 7.8), and $\nu_{r,\text{out}}$ its marginal outside D . Write $\nu_{r,\phi;D}$ for the bare state at the complete field $\phi_D r_{D^c}$. The fixed-field density $F_D(r \mid \xi)$ includes the Gaussian prefactor; denote its pressure per physical volume by b_r . For a field restriction R and a bounded joint perturbation J , define

$$C_D^{R,J}(r) = e^{b_r \text{Vol}(D)} \mathbb{E}_{\nu_{r,\text{out}}} \frac{Z_D^{R,J}(\xi)}{F_D(r \mid \xi)}, \quad V_D^J(\phi) = -\log \mathbb{E}_{\nu_{r,\phi;D}} e^{-J}.$$

The conditional Hamiltonian H_D^Ψ sums every interaction term whose declared support meets D , with exterior field r . Recall

$$b_r + \beta e(r) = b_0, \quad \sigma_D(r) = H_D^\Psi(r \mid r) - e(r) \text{Vol}(D).$$

Lemma 11.1 (Exact normalized field integral). *With the probability reference measure λ_D ,*

$$\mathfrak{Z}_D^{R,J}(r) := e^{-\beta \sigma_D(r)} C_D^{R,J}(r) = e^{b_0 \text{Vol}(D)} \int_R e^{-\beta H_D^\Psi(\phi \mid r) - V_D^J(\phi)} d\lambda_D. \quad (11.1)$$

Proof. Let $\mathcal{G}_D$ be the Gaussian prefactor in (3.4) and set $\mathcal{L}_D = \sum_{i \in I(D)} (\phi_i - r_i) \hat{N}_i$. At fixed h , the normalizers in (3.3) cancel, giving

$$\frac{F_D(\phi \mid \xi)}{F_D(r \mid \xi)} = \frac{\mathcal{G}_D(\phi)}{\mathcal{G}_D(r)} \mathbb{E}_{\nu_r(\cdot \mid \xi)} e^{\mathcal{L}_D}.$$

Average over $\nu_{r,\text{out}}$. The normalized finite tilt $e^{\mathcal{L}_D} \nu_r / \mathbb{E}_{\nu_r} e^{\mathcal{L}_D}$ satisfies the bare specification at $\phi_D r_{D^c}$; uniqueness identifies it with $\nu_{r,\phi;D}$. Thus the perturbation contributes the factor $e^{-V_D^J(\phi)}$. The logarithmic derivative of $\mathcal{G}_D(\phi) \mathbb{E}_{\nu_r} e^{\mathcal{L}_D} / \mathcal{G}_D(r)$ is $-\beta g_{\beta,h}$, which is also the derivative of $-\beta H_D^\Psi$. Integrating from r to ϕ gives

$$\frac{\mathcal{G}_D(\phi)}{\mathcal{G}_D(r)} \mathbb{E}_{\nu_r} e^{\mathcal{L}_D} = e^{-\beta [H_D^\Psi(\phi \mid r) - H_D^\Psi(r \mid r)]}.$$

Insert this identity into $C_D^{R,J}$ and multiply by $e^{-\beta \sigma_D(r)}$. The remaining constant is $e^{(b_r + \beta e(r)) \text{Vol}(D)}$. $\square$

Set $\mathfrak{Z}_q^{\text{phys}}(D) = \mathfrak{Z}_D^{R_q(D),0}(r_q)$. The auxiliary partition uses $R = U_q(D)$ and $J = -\log G_{q,D}^{\text{aux}}$. Write I for the corresponding field integrals, so $\mathfrak{Z} = e^{b_0 \text{Vol}} I$. For a selected-family term $Z_{D,\mathcal{F}}(\xi)$, define

$$\mathfrak{J}_{\mathcal{F}} = e^{b_{r_q} \text{Vol}(D) - \beta \sigma_D(r_q)} \mathbb{E}_{\nu_{r_q,\text{out}}} \frac{Z_{D,\mathcal{F}}(\xi)}{F_D(r_q \mid \xi)}.$$

The interaction $\Psi = \Phi^0 + \psi$ has nonnegative leading atoms vanishing at constant wells and $\|\psi\| \leq \delta_\beta$. Its boundary cocycle satisfies (10.13); its homogeneous trial bound is (6.10).

Apply the restricted-pressure construction to the field interaction obtained from the clipped contour gas, and denote its physical pressure by p_q . Lemma 10.13 and the rooted estimate 9.9 give, on every finite coarse union G ,

$$\left| \log I_q^{\text{aux}}(G) - (p_q - b_0) \text{Vol}(G) \right| \leq \sigma_0 \mathcal{A}(G), \quad |p_q - b_0| \leq r_\beta. \quad (11.2)$$

Here $r_\beta = O(\log \beta + \beta \delta_\beta + t_\beta)$. If the field pressure is measured per fine site as π_q , then $p_q = b_0 + \pi_q / v_s$.

The positive bare collar kernels satisfy, by 9.10, for arbitrary bounded caps,

$$K_S(v, w) = \frac{K_S(v, r_q)K_S(r_q, w)}{k_S} e^{U_S(v, w)}, \quad k_S = K_S(r_q, r_q), \quad |U_S| \leq \beta \epsilon_K \mathcal{A}(S). \quad (11.3)$$

The marked estimates 7.12 and 7.15 bound changes outside the source region E in $V_D^{J_E}$ by $Ct_\beta \mathcal{A}(E)$. Adding species meeting H costs $C\kappa$ times its fixed-neighborhood volume. Frozen-rod comparisons add this weak boundary cost to the bare cocycle. Moreover,

$$V_D^{J_E}(\phi_E r_{D \setminus E}) = V_E^{J_E}(\phi_E). \quad (11.4)$$

At fixed L , t_β is a fixed L -dependent constant times a polynomial in β times κ .

Put $P = \max_q p_q$ and fix $\sigma = C_\sigma(\sigma_0 + t_\beta + \beta\delta_\beta)$, where C_σ covers the fixed constants for area conversion, boundary terms on parent domains, and addition of contour species. Define

$$\eta_{\text{cut}} = C/L + C\delta_\beta + Ce^{-cL}, \quad (11.5)$$

$$\mathcal{E}_\beta = C[\sigma + \beta\epsilon_K + \beta\eta_{\text{cut}} + \beta e^{-cL} + t_\beta]L^2 + C(r_\beta + t_\beta)V_L, \quad (11.6)$$

$$B_* = \beta\tau L^2 - \mathcal{E}_\beta. \quad (11.7)$$

The induction requires $B_* \geq 3\mathcal{T}$ and B_* sufficiently large compared with $\log C_1$. The sufficient conditions $\mathcal{T} = \beta\tau L^2/16$ and $\mathcal{E}_\beta \leq \beta\tau L^2/2$ give $B_* \geq 8\mathcal{T}$.

The root divisors satisfy

$$\log k_S \geq -C(\log \beta + \beta\delta_\beta) \text{Vol}(S). \quad (11.8)$$

Apply Lemma 9.4 on S with both caps at r_q ; the bound also allows the omitted nonnegative leading terms.

11.2 Single-contour and simultaneous-family estimates

Lemma 11.2 (Allocation of interactions to collars). *Partition the integrated coordinates into an exterior cap E , a core cap C , inner caps I_j , and collars S_0, S_j . The collar S_0 separates E from the other caps, and S_j separates I_j from the other caps. All components of each cap are treated together. Delete the interaction terms specified in Section 10.3, so that a remaining declared support meets at most one collar. A support missing all collars is then assigned to its unique cap; a support meeting a collar is assigned to that collar. Assign by full declared supports, even when a term is constant at the fields in question. With the fixed exterior coordinates included in the appropriate cap, this gives the exact decomposition*

$$H^{\text{cut}} = H_E + H_C + \sum_j H_{I_j} + h_0(S_0; E, C) + \sum_j h_j(S_j; C, I_j).$$

Suppose the restrictions, after fixing the prescribed majority label on every collar, are products of collar-majority indicators and cap-only indicators. Allow any additional nonnegative cap weights, and absorb these and $e^{-\beta H_\alpha}$ into α_α . Write $K_0^{\text{cut}}(c, e)$ and $K_j^{\text{cut}}(i, c)$ for the collar integrals of $e^{-\beta h_0}$ and $e^{-\beta h_j}$.

Proof. A connected support meeting distinct caps crosses a collar. If it meets only S_j , its caps are C, I_j ; if it meets only S_0 , they are C, E . This proves the energy decomposition. A radius-one correctness stencil cannot cross a width-eight collar, so after fixing the collar labels its indicator depends on one cap. Touching inner collars have equal labels by Lemma 10.3. $\square$

Lemma 11.3 (Factorization through collar kernels). *With the preceding allocation, let I^{cut} be its joint field integral, including any nonnegative weights on the separate caps. For each collar let r_e be its phase root, and let M_e bound the total energy norm of terms restored to compare its cut kernel with the full kernel, uniformly in every cap extension. If the full kernel has cross-ratio bound ϵ_e , then*

$$K_e^{\text{cut}}(v, w) \leq e^{\epsilon_e + 4\beta M_e} \frac{K_e^{\text{cut}}(v, r_e) K_e^{\text{cut}}(r_e, w)}{k_e^{\text{cut}}}, \quad k_e^{\text{cut}} = K_e^{\text{cut}}(r_e, r_e).$$

Consequently the cut integral obeys

$$I^{\text{cut}} \leq e^{\sum_e (\epsilon_e + 4\beta M_e)} \frac{F_E^{\text{cut}} F_C^{\text{cut}} \prod_j F_j^{\text{cut}}}{k_0^{\text{cut}} \prod_j k_j^{\text{cut}}}, \quad (11.9)$$

where

$$\begin{aligned} F_E^{\text{cut}} &= \int a_E(e) K_0^{\text{cut}}(r_q, e) d\lambda_E(e), \\ F_C^{\text{cut}} &= \int a_C(c) K_0^{\text{cut}}(c, r_q) \prod_j K_j^{\text{cut}}(r_{p_j}, c) d\lambda_C(c), \\ F_j^{\text{cut}} &= \int a_j(i) K_j^{\text{cut}}(i, r_{p_j}) d\lambda_{I_j}(i). \end{aligned}$$

Proof. The energy comparison gives $|\log K_e^{\text{cut}} - \log K_e| \leq \beta M_e$ at each cap pairing. The four pairings in a cross ratio thus contribute at most $4\beta M_e$. Tonelli's theorem first integrates the collars and writes the cut integral as

$$\int a_E(e) a_C(c) \prod_j a_j(i_j) K_0^{\text{cut}}(c, e) \prod_j K_j^{\text{cut}}(i_j, c) d\lambda_E d\lambda_C \prod_j d\lambda_{I_j}.$$

Apply the cross-ratio inequality pointwise. The remaining factors depend on separate caps, and Tonelli gives (11.9). The cap weights may include boundary cocycles. For an empty inner cap the root factor cancels its divisor. $\square$

In each factor, terms assigned to an incident collar are integrated with that collar's copied variables. Its root divisor retains the same terms at the root. A term meeting m inherited domains is restored in at most $C(1+m)$ factors; Lemma 10.11 bounds the total cost.

Lemma 11.4. *Suppose, in every strictly smaller physical inherited domain H and for every phase p ,*

$$\mathfrak{Z}_p^{\text{phys}}(H) \leq e^{P \text{Vol}(H) + \sigma \mathcal{A}(H)}.$$

Then every q contour admissible in D satisfies uniformly in its compatible frozen collar variables

$$\log W_\Gamma \leq -B_* n_\Gamma + (P - p_q) \text{Vol}(V_\Gamma). \quad (11.10)$$

Proof. Keep numerator and denominator of (10.4) as joint microscopic integrals on the same $U = (A \setminus K) \cup \bigcup_j D_j$. In the denominator apply the exact regional identity successively and conditionally inside the D_j , then clip the resulting activities. Positivity gives a lower bound with the original regional cutoff species. Their full regional gaps make cross-sibling compatibility automatic. Their union is one compatible subgas. By Lemma 10.14, every added auxiliary species meets the core or the first inherited boundary layer. The core and these layers contain at most $C_L n_\Gamma$ sites, by the boundary bounds above. Lemma 10.15 therefore bounds the comparison cost by $C t_\beta V_L n_\Gamma$.

Use the shifted collar O, S, V' constructed above and retain only auxiliary species with actual supports in V' , again a positive denominator restriction. Integrating all rods on U gives, for every restriction R and source J , the common normalization

$$Z_U^{R,J}(\xi) = \underbrace{F_U(r_q | \xi) e^{\beta H_U^\Psi(r_q | r_q)}}_{\mathcal{A}_U(\xi, r_q)} \int_R e^{-\beta H_U^\Psi(\phi | r_q) - \beta B_U^{\xi, r_q}(\phi) - V_U^{J, \xi}(\phi)} d\lambda_U.$$

The prefactor is the same for numerator and denominator and cancels. The frozen-boundary weak comparison replaces $V_U^{J, \xi}$ by the canonical correction with a small surface error and leaves the bare B_U^{ξ, r_q} unchanged. Truncate that cocycle to O , with error $C\beta e^{-cL} L^2 n_\Gamma$. Set the O, S arguments in the denominator weak correction to the root, paying $Ct_\beta \mathcal{A}(V')$. Equation (11.4) leaves exactly $V_{V'}^{J, V'}$, including its scalar. Both sides now have the same bare collar kernel.

Let $a_\xi(w)$ denote the entire positive outer-cap amplitude, including the possibly large retained cocycle. Equation (11.3) gives

$$\int a_\xi(w) K_S(v, w) d\lambda_O(w) = A_\xi K_S(v, r_q) e^{\varepsilon(v)}, \quad A_\xi = \frac{\int a_\xi(w) K_S(r_q, w) d\lambda_O(w)}{k_S}, \quad |\varepsilon| \leq \beta \epsilon_K \mathcal{A}(S).$$

The same A_ξ cancels. On $V_{\text{aug}} = S \cup V'$, restore the S reference fields and add auxiliary species meeting S , using the marked comparison. The denominator is bounded below by the auxiliary field integral on V_{aug} times a small exponential loss. Equation (11.2) applies also when this domain is punctured.

Choose the inner collars $S_j \subset D_j$. Delete the wrong-root long tails and all multi-collar interactions. Lemma 10.11 counts every restoration; the protected charge is untouched. Allocate terms by full declared supports and apply Lemma 11.3. Keep both cap factors and each root divisor, then restore the deleted terms. If $A_0 = V_{\text{aug}} \setminus \bigcup_j D_j$, this yields

$$I_N(V_{\text{aug}}) \leq e^{E_{\text{in}}} \frac{I_{\text{core}^+} \prod_j I_{p_j}^{\text{phys}}(D_j)}{\prod_j k_j}, \quad \text{Vol}(\text{core}^+) = \text{Vol}(A_0) + \sum_j \text{Vol}(S_j).$$

The core integral is at most $\exp[-\beta \tau L^2 n_\Gamma + C r_\beta \text{Vol}(\text{core}^+)]$ by positivity and Proposition 10.10. Apply the induction hypothesis to the unchanged D_j , (11.8) to the divisors, and (11.2) to the denominator. The only nonlocal bulk difference is exactly

$$\begin{aligned} & (P - b_0) \sum_j \text{Vol}(D_j) - (p_q - b_0) \text{Vol}(V_{\text{aug}}) \\ &= (P - p_q) \text{Vol}(V_{\text{aug}}) - (P - b_0) \text{Vol}(A_0). \end{aligned}$$

The last term is a core error, and the first is at most $(P - p_q) \text{Vol}(V_\Gamma)$. All surfaces are at most $CL^2 n_\Gamma$ and all core and copied volumes at most $CV_L n_\Gamma$. The budget (11.7) absorbs these errors and proves (11.10). $\square$

Lemma 11.5. *Under the same smaller-domain hypothesis, a selected maximal active q -family $\mathcal{F}$, with $N = \sum_\Gamma n_\Gamma$ and exterior $E = D \setminus \bigcup_\Gamma V_\Gamma$, has canonical contribution*

$$\mathfrak{I}_{\mathcal{F}} \leq \exp\{P \text{Vol}(D) + \sigma \mathcal{A}(D) - B_* N - (P - p_q) \text{Vol}(E)\}. \quad (11.11)$$

Proof. Apply (10.7), integrate all rods together, and use the common canonical normalization once. The result is bounded by

$$e^{b_0 \text{Vol}(D)} \int_{R_{\mathcal{F}}} e^{-\beta H_D^\Psi(\phi | r_q) - V_D^{JE}(\phi)} d\lambda_D.$$

Set fields outside E to the root in the weak correction, paying $Ct_\beta \mathcal{A}(E)$, and use (11.4). Let $S_0 \subset A = \bigcup_\Gamma A_\Gamma$ be the joint outer collar, and $S_j \subset D_j$ the inherited inner collars. Put $C = A \setminus S_0$ and $I_j = D_j \setminus S_j$. Allocate the restrictions to product-majority collars and cap-only indicators. Delete every term meeting multiple collars and the required long tails. Conditional on the caps, Lemma 11.3 factors the collar integrations and applies the cross ratios with complete root divisors. Restore all terms with their multiplicities. The exterior acquires a copy of S_0 , the core copies of all S_j , and each inherited domain remains unchanged.

Write $E^+ = E \cup S_0$. In the exterior factor first restore the S_0 reference fields while keeping J_E , then add species meeting S_0 . The errors are $Ct_\beta \mathcal{A}(E) + C\kappa \text{Vol}(S_0)$; the full weak scalar is preserved. We obtain

$$e^{-b_0 \text{Vol}(D)} \mathfrak{J}_{\mathcal{F}} \leq e^{E_{\mathcal{F}}} \frac{I_q^{\text{aux}}(E^+) (\prod_j I_{p_j}^{\text{phys}}(D_j)) I_{\text{core}^+}}{k_0 \prod_j k_j},$$

where

$$\text{Vol}(E^+) = \text{Vol}(E) + \text{Vol}(S_0), \quad \text{Vol}(\text{core}^+) = \text{Vol}(A) + \sum_j \text{Vol}(S_j).$$

The copied field volumes obey

$$\text{Vol}(E^+) + \sum_j \text{Vol}(D_j) + \text{Vol}(\text{core}^+) - \text{Vol}(S_0) - \sum_j \text{Vol}(S_j) = \text{Vol}(D).$$

Apply the auxiliary surface estimate, the physical induction, the root-divisor trial, and the protected charge. The bulk expression is

$$\begin{aligned} & b_0 \text{Vol}(D) + (p_q - b_0)[\text{Vol}(E) + \text{Vol}(S_0)] + (P - b_0) \sum_j \text{Vol}(D_j) \\ &= P \text{Vol}(D) - (P - p_q) \text{Vol}(E) + (p_q - b_0) \text{Vol}(S_0) - (P - b_0) \text{Vol}(A). \end{aligned}$$

The last two terms have magnitude at most $Cr_\beta V_L N$. Furthermore $\mathcal{A}(E^+) \leq \mathcal{A}(D) + CL^2 N$ and $\sum_j \mathcal{A}(D_j) \leq CL^2 N$. The constant σ covers the parent boundary terms; the other errors enter (11.7), leaving $-B_* N$. $\square$

11.3 The numerical tree and the physical pressure

The simultaneous induction below adapts Zahradník's stability argument [5, Theorem 1.7 and Section 1.8]: an active contour must be large relative to the pressure deficit, and summing such contours yields the physical pressure bound.

Proposition 11.6 (Finite-volume pressure bounds). *For the canonical model constructed above, let the error budget (11.7) satisfy $B_* \geq 3\mathcal{T}$, with B_* sufficiently large. Then for every physical regional domain D and phase q ,*

$$e^{p_q \text{Vol}(D) - \sigma \mathcal{A}(D)} \leq \mathfrak{Z}_q^{\text{phys}}(D) \leq e^{P \text{Vol}(D) + \sigma \mathcal{A}(D)}. \quad (11.12)$$

Proof. Induct simultaneously over all phases and physical domains, ordered by their number of coarse cubes. The cutoff pressures and P are already defined. The smaller-domain hypothesis gives both preceding lemmas. Put $a = (P - p_q)V_L \geq 0$. If a contour is active, then

$$-\mathcal{T}n < \log W_\Gamma \leq -B_* n + C a n^{3/2},$$

so for $a > 0$ its size is at least

$$N(a) = \max \left\{ 1, \left\lceil c \left(\frac{B_* - \mathcal{T}}{a} \right)^2 \right\rceil \right\}.$$

At $a = 0$ there are no active contours. Extracting the common factor $e^{P \text{Vol}(D) + \sigma \mathcal{A}(D)}$ from (11.11) leaves a sum of weights $e^{-B_* \sum n_\Gamma} e^{-an(E)}$ over disjoint removed regions.

Expose the first unexamined coarse site. Either it belongs to E , with weight e^{-a} , or to one removed region of size $n \geq N(a)$, with at most C_1^n choices by Lemma 10.12. Erase that entire removed region and continue; its q holes remain available. Induction on the number of remaining sites bounds this sum by one provided

$$e^{-a} + \sum_{n \geq N(a)} C_1^n e^{-B_* n} \leq 1.$$

For $a \geq 1$ the geometric tail is smaller than $1 - e^{-1}$ for large B_* . For $0 < a \leq 1$, the hypotheses give a tail at most $C \exp[-cB_*^3/a^2] \leq a/2 \leq 1 - e^{-a}$, uniformly in that interval. The case $a = 0$ has only the empty active family. This proves the upper bound.

For the lower bound, clip the exact regional gas before enlarging it. Positivity makes the physical partition at least this original regional cutoff partition. Apply Lemma 10.14 with one domain and empty core. Added species meet its first boundary layer, whose size is bounded by $C\mathcal{A}(D)/L^2$. Lemma 10.15 gives a surface cost, proving the lower bound by the choice of σ . □

Theorem 11.7 (Pressure maximum). *For the parameter range in Proposition 11.6,*

$$p_\varepsilon(h) = \max_q p_q. \tag{11.13}$$

Proof. At fixed β , emptying a fixed-width boundary strip costs at most activity times strip volume and removes the physical exterior. The fixed-field reference admits the same comparison by its ideal-gas cell bound and positive empty-cell weight. Each conditional Gaussian law assigns a root neighborhood probability at least $e^{-C\beta}$, uniformly in the reset count, so imposing the regional q band also costs a surface term. Thus $F_D(r \mid \xi)$ has bulk $e^{b_r \text{Vol}(D)}$, uniformly in ξ , which cancels in the canonical ratio; $\sigma_D(r)$ is a surface correction. Divide (11.12) by volume along boxes. The upper bound gives a pressure limsup at most P , and the lower bound for a maximizing phase gives a liminf at least P . This proves (11.13). □

12 The crossing and the density jump

We first record the elementary pressure facts, then compare the source responses of the phase pressures. All pressures are per Euclidean volume. The mesh, count caps, field interval, phase neighborhoods, and contour index sets remain fixed as h varies.

Lemma 12.1 (Thermodynamic pressure). *For each $\varepsilon > 0$, the limit defining $p_\varepsilon(h)$ exists, is finite, and is convex in h .*

Proof. We use the separated-box construction for the thermodynamic limit [30, Section 2]. Dropping exclusions between boxes gives a product upper bound. Boxes separated by an empty strip wider than $1 + \varepsilon$ give a product lower bound. Deleting a set S of allowed centres changes $\log Z_\Lambda$ by at most $z|S|$: integrate its particles with all exclusions dropped for the upper bound, and leave it empty for

the lower bound. Tile large cubes with boxes of fixed side, then let that side increase. The upper and lower bounds converge to the same limit; the strip estimate also removes exterior boundary effects. Finally, $0 \leq \log Z_\Lambda \leq z|\Lambda|$, and finite-volume log partition functions are convex in h . $\square$

Lemma 12.2 (Density response). *At every differentiability point of p_ε , the finite-volume number densities converge to p'_ε . The one-sided derivatives are the corresponding limits as h approaches through differentiability points from the left or right.*

Proof. The first two derivatives of $\log Z_\Lambda$ are $\mathbb{E}N_\Lambda$ and $\text{Var } N_\Lambda$. Convex difference-quotient bounds give convergence of the first derivatives wherever the limiting pressure is differentiable. Monotonicity of the derivative gives its one-sided limits; these are the convex convergence principles of [16, Theorems 24.1 and 24.5]. $\square$

12.1 Source response of the joint model

Let D be a union of complete fine spatial cells. For a field event E and a joint perturbation $J_h = \sum_A J_{A,h}$, let $Z_D^{E,J}(h \mid \xi)$ be the integral of the joint rod-field density (3.4) over E , multiplied by e^{-J_h} . The exterior rods ξ and the event E are independent of h . We assume

$$\sup_i \sum_{A \ni i} \|J_{A,h}\|_\infty \leq \kappa, \quad \sup_i \sum_{A \ni i} \text{Lip}_h(J_{A,h}) < \infty. \quad (12.1)$$

Fix $0 < h_* \leq H$. The constants below are uniform for $|h| \leq h_*$.

Lemma 12.3 (Source score). *For almost every h , the logarithmic derivative of the positive joint density is*

$$\mathcal{S}_h = \sum_{i \in I(D)} \left[\frac{\beta v_i}{2\alpha} (\phi_i - h) + N_i - \hat{N}_i - \partial_h \log c_{i,h}(\hat{N}_i) \right] - \partial_h J_h. \quad (12.2)$$

Proof. Differentiate the physical activity factor and the attached Gaussian in (3.2). Their count terms combine to $N_i - \hat{N}_i$. The reset count is independent of h . The Lipschitz assumption gives the derivative of J_h almost everywhere. $\square$

Lemma 12.4 (Exceptional counts). *If $\rho > e\zeta e^{h_*}$, then, uniformly over E and compatible exterior rods with $Z_D^{E,J} > 0$,*

$$\mathbb{E}_h^{E,J} \sum_{i \in I(D)} (N_i - \hat{N}_i) \leq C\beta|D|e^{-c\beta}. \quad (12.3)$$

Proof. Condition on the fields and the rods outside cell i . Emptying that cell changes J_h by at most 2κ . In (3.3), the empty term and every term with $n > K_i$ have the same Gaussian normalizer. Divide by the empty term of the conditional partition function and drop the hard exclusions. For $F \geq 0$,

$$\mathbb{E}[F(N_i)\mathbf{1}_{\{N_i > K_i\}} \mid \phi, \eta_{i^c}] \leq e^{2\kappa} \sum_{n > K_i} F(n) \frac{(\beta\zeta e^{h_*} v_i)^n}{n!}. \quad (12.4)$$

Apply the scalar tail bound in Lemma 7.2 with $m = 1$ and H replaced by h_* . It gives $C\beta e^{-c\beta}$ per marked cell. There are at most $|D|/v_-$ such cells, proving the claim. $\square$

The Gaussian means stay at distance at least $d_0 > 0$ from the endpoints. The endpoint densities therefore give

$$1 - c_{i,h}(m) \leq 2e^{-\beta v_i d_0^2/(4\alpha)}, \quad |\partial_h c_{i,h}(m)| \leq C\sqrt{\beta v_i} e^{-\beta v_i d_0^2/(4\alpha)}.$$

In particular, $c_{i,h}(m) \geq 1/2$ for large β , and

$$\sum_{i \in I(D)} |\partial_h \log c_{i,h}(\hat{N}_i)| \leq C\beta|D|e^{-c\beta}. \quad (12.5)$$

These bounds justify differentiation under the finite-volume integral, so $\partial_h \log Z_D^{E,J} = \mathbb{E}_h^{E,J} \mathcal{S}_h$ almost everywhere.

12.2 Differentiating the cutoff gas

The numerator and denominator of W_Γ are joint integrals on the same region with the same frozen collar. Their restrictions are independent of h . If n_Γ is the number of support cubes and V_Γ its removed region, the geometric bound gives $|V_\Gamma| \leq CV_L n_\Gamma^{3/2}$. Applying the preceding estimates with $J = 0$ to both integrals yields

$$|\partial_h \log W_\Gamma| \leq C\beta V_L n_\Gamma^{3/2}. \quad (12.6)$$

A zero numerator has an allowed set of zero reference measure and vanishes for every h . Extend the cutoff by zero off its fixed compatible collar set. For an h -independent threshold $\mathcal{T}$,

$$w_\Gamma = \min\{W_\Gamma, e^{-\mathcal{T}n_\Gamma}\}, \quad |\partial_h w_\Gamma| \leq C\beta V_L n_\Gamma^{3/2} e^{-\mathcal{T}n_\Gamma} \quad (12.7)$$

almost everywhere. Its declared marked-cell support is unchanged, with $|S_\Gamma| \leq d_L n_\Gamma$.

Lemma 12.5 (Differentiated cutoff expansion). *For every fixed $a > 0$, the interaction $J_h = -\log G_h$ of the cutoff gas, grouped by declared union supports, satisfies*

$$\sup_i \sum_{A \ni i} e^{a|A|} \|J_{A,h}\|_\infty \leq C_L e^{-\mathcal{T}/2}, \quad (12.8)$$

$$\sup_i \sum_{A \ni i} e^{a|A|} \text{Lip}_h(J_{A,h}) \leq C_L \beta e^{-\mathcal{T}/4}, \quad (12.9)$$

when $\mathcal{T}$ exceeds a constant depending on a , the mesh, and L .

Proof. Since $|S(C)| \leq d_L s(C)$, (12.8) follows from (10.14) with $b = ad_L$. Differentiating a cluster term and using (12.7) bounds its absolute majorant by $C\beta V_L \sum_j n_{\Gamma_j}^{3/2} \leq C\beta V_L s(C)^{3/2}$ times the original majorant. Absorb this factor with $s^{3/2} \leq C e^{\mathcal{T}s/4}$, and apply the same lemma with threshold $3\mathcal{T}/4$. Its bound $C_L \beta e^{-3\mathcal{T}/8} \leq C_L \beta e^{-\mathcal{T}/4}$ proves (12.9). This is the rooted derivative expansion of [26, Section 2, equations (2.6)–(2.7)]. Uniform absolute convergence permits integration between source values and gives the stated Lipschitz norm. $\square$

12.3 Increment bounds for the phase pressures

Let $[m_q^-, m_q^+]$ contain the masses of the wells in component q . On the q -majority event, the good-cell tolerance and minority fraction imply

$$m_q^- - \omega_0 \leq M_D(\phi) := \frac{1}{2\alpha|D|} \sum_{i \in I(D)} v_i \phi_i \leq m_q^+ + \omega_0. \quad (12.10)$$

Indeed, if a good spatial cell has mass error at most $\eta/(2\alpha)$ and at most a fraction θ of cells are exceptions, one may take $\omega_0 = \eta/(2\alpha) + R\theta/\alpha$. Apply this estimate block by block; boundary blocks have vanishing volume fraction.

Lemma 12.6 (Microscopic realization of a branch). *Let $Z_{D,q}^{\text{cut}}(h)$ be the complete joint partition function restricted to q -majority fields and perturbed by the regional cutoff interaction $J_{q,D}$. With a fixed compatible exterior configuration,*

$$\lim_{D \uparrow \mathbb{R}^3} |D|^{-1} \log Z_{D,q}^{\text{cut}}(h) = p_q(h).$$

Proof. The bare boundary cocycle, the frozen weak comparison (7.44), and the rooted surface estimate give

$$\log Z_{D,q}^{\text{cut}}(h) = (b_0(h) + \pi_q(h)/v_s)|D| + O_\beta(A(D)).$$

Divide by the volume of aligned boxes and use (9.10). $\square$

Lemma 12.7 (Branch increments). *Put $Q_q(h) = \beta^{-1}p_q(h) + h^2/(4\alpha)$. There is $\sigma_\beta = Ce^{-c\beta} + C_L e^{-\mathcal{T}/4} \rightarrow 0$ such that, for $h_1 < h_2$,*

$$(m_q^- - \omega_0 - \sigma_\beta)(h_2 - h_1) \leq Q_q(h_2) - Q_q(h_1) \leq (m_q^+ + \omega_0 + \sigma_\beta)(h_2 - h_1). \quad (12.11)$$

Proof. The source score, exceptional-count bound, and cutoff estimate imply

$$\frac{\partial_h \log Z_{D,q}^{\text{cut}}}{\beta|D|} = \mathbb{E}_{D,q,h} M_D - \frac{h}{2\alpha} + \mathcal{E}_{D,q}(h), \quad |\mathcal{E}_{D,q}(h)| \leq \sigma_\beta. \quad (12.12)$$

Conversion from the marked-cell norm uses the positive minimum of v_i . Integrate in finite volume, use (12.10), and pass to the pressure limit supplied by the preceding lemma. $\square$

12.4 Choosing the scales

Reduce the homogeneous mass gap so that $0 < \Delta \leq r_I$. Choose the parameters in the following order.

1. Fix $H > 0$, then choose $\alpha > \|T\| + 1$, $\rho > e\zeta e^H$, and $R > H + 2\alpha\rho + 2$. With $M = \zeta e^{R+1}$, choose the spatial and angular mesh to satisfy the coercivity and covariance estimates. Its marked-cell volumes are now fixed; the margin absorbs the ceiling in K_i for large β .
2. Choose disjoint phase neighborhoods preserving the mass gap, with good-cell mass error at most $\Delta/16$. Fix a rigidity patch, a radius retaining half its positive defect charge, and $k \geq 24$. The full leading interaction is retained.
3. The protected charge has coefficient $\tau = c_P \theta^{2/3}$. Choose θ so that both the minority mass error and $C_m \theta / \tau$ are sufficiently small. The constant C_m is independent of the subsequent net and interpolation choices.
4. Choose a well net of radius $\nu \leq 1$, its finite path family, and the interpolation width R_{int} , so that $C\nu$ and CD_ν^2/R_{int} are small fractions of τ . Choose $\eta_0 > 0$ such that

$$C \left(1 + \frac{R_{\text{int}} N_\nu}{\nu} \right) \omega(\eta_0) \leq c_0 \tau,$$

where c_0 is a sufficiently small fixed constant. Then choose L large enough for the geometric margins, $C/L \leq \eta_0/3$, and $C/L + Ce^{-cL} \ll \tau$.

5. The weak interaction obeys $\delta_\beta(h) \leq C|h| + \delta_{\text{micro},\beta}$, with $\delta_{\text{micro},\beta} \rightarrow 0$. Choose $0 < h_* \leq H$ so that CLh_* and Ch_* fit the preceding error bounds and $h_*/(2\alpha) \leq r_I/4$. Finally increase β to control the microscopic remainder, $L \log \beta/\beta$, and the Gaussian and exceptional-count errors.

All choices except β are now fixed. Set

$$\mathcal{T} = \frac{\beta\tau L^2}{16}. \quad (12.13)$$

This threshold satisfies Lemma 12.5 for large β . Equations (7.6) and (12.8) give $\beta(q_\beta + \kappa) \rightarrow 0$, verifying (7.20). For the error $\mathcal{E}_\beta$ in (11.7), $r_\beta/\beta \leq Ch_* + o(1)$ and $t_\beta/\beta \rightarrow 0$. After division by βL^2 , the volume term is $O(Lh_*) + o(1)$; the other terms are controlled by the successive choices above. Taking their finitely many allocated fractions sufficiently small gives

$$B_* = \beta\tau L^2 - \mathcal{E}_\beta \geq \frac{1}{2}\beta\tau L^2 = 8\mathcal{T}. \quad (12.14)$$

Lemma 12.8 (Equal limiting intercepts). *At zero source,*

$$\max_q \left| \frac{p_q(0) - b_0(0)}{\beta} \right| \leq C \left(\frac{\log \beta}{\beta} + \delta_{\text{micro},\beta} + \frac{t_\beta}{\beta} + \beta^{-4} \right) \rightarrow 0. \quad (12.15)$$

Proof. At $h = 0$, the canonical perturbation and the regional cutoff correction have combined energy norm at most $C(\delta_{\text{micro},\beta} + t_\beta/\beta)$, by (8.9) and Proposition 7.15. Apply (9.7) with this norm and use $p_q = b_0 + \pi_q/v_s$. Division by β gives the bound. $\square$

12.5 Proof of the main theorem

The source adjustment is the LMP coexistence mechanism [2, Section 6, proof of Statement 3.3]: separated density responses force a crossing of the corrected phase pressures.

Proof of Theorem 1.1. The reserve (12.14) satisfies the quantitative conditions of Proposition 11.6. Theorem 11.7 therefore gives

$$p_\varepsilon(h) = \max_q p_q(h), \quad |h| \leq h_*. \quad (12.16)$$

The isotropic component has mass r_I and the others have mass at least $r_I + \Delta$. Choose the tolerances above so that $\omega_0 \leq \Delta/8$, and take β large enough that $\sigma_\beta \leq \Delta/16$. Equation (12.11) gives

$$(r_I - 3\Delta/16)(h_2 - h_1) \leq Q_I(h_2) - Q_I(h_1) \leq (r_I + 3\Delta/16)(h_2 - h_1), \quad (12.17)$$

$$Q_q(h_2) - Q_q(h_1) \geq (r_I + 13\Delta/16)(h_2 - h_1), \quad q \neq I. \quad (12.18)$$

There are finitely many components, and their increment bounds imply continuity. Thus $D_\beta(h) = \max_{q \neq I} Q_q(h) - Q_I(h)$ is continuous. Choosing a maximizing component at h_1 shows that

$$D_\beta(h_2) - D_\beta(h_1) \geq \frac{5\Delta}{8}(h_2 - h_1). \quad (12.19)$$

By (12.15), $D_\beta(0) \rightarrow 0$. Consequently D_β has a unique zero $h_\varepsilon \in (-h_*, h_*)$ for large β , and $|h_\varepsilon| \leq 8|D_\beta(0)|/(5\Delta) \rightarrow 0$.

To the left of the zero the physical pressure equals the isotropic branch; to the right it equals the maximum of the other branches. Convexity gives one-sided derivatives. Their secant bounds are

$$\begin{aligned} p'_{\varepsilon,-}(h_\varepsilon) &\leq \beta \left(r_I + \frac{3\Delta}{16} - \frac{h_\varepsilon}{2\alpha} \right), \\ p'_{\varepsilon,+}(h_\varepsilon) &\geq \beta \left(r_I + \frac{13\Delta}{16} - \frac{h_\varepsilon}{2\alpha} \right). \end{aligned}$$

Therefore

$$p'_{\varepsilon,+}(h_\varepsilon) - p'_{\varepsilon,-}(h_\varepsilon) \geq \frac{5\Delta}{8\varepsilon} \geq \frac{\Delta}{2\varepsilon}. \quad (12.20)$$

The isotropic lower increment bound, $\Delta \leq r_I$, and $h_*/(2\alpha) \leq r_I/4$ give a positive lower bound proportional to β for both derivatives. The upper increment bounds give the matching upper bound. Setting $a_\varepsilon = a_t + h_\varepsilon$ proves the theorem for $P_\varepsilon(a) = p_\varepsilon(a - a_t)$. $\square$

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