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Nonpathological extensions of F_σ ideals and fractional coloring

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Abstract

The random graph ideal is pathological, answering a question of López-Abad et al. We prove this by constructing a positive hereditarily pathological restriction using Kneser graphs. For an F_σ ideal $\mathcal{I}$ and any lower semicontinuous submeasure φ with $\mathsf{FIN}(\varphi) = \mathcal{I}$, we show that the nonpathological envelope $\widehat{\varphi}$ generates an ideal that depends only on $\mathcal{I}$ and, when proper, is the least nonpathological F_σ extension of $\mathcal{I}$. Thus, $\mathcal{I}$ is nonpathological if and only if $\mathsf{FIN}(\varphi) = \mathsf{FIN}(\widehat{\varphi})$. We also characterize nonpathological F_σ ideals both as intersections of summable ideals and as intersections of matrix ideals, answering both converses asked by Filipów and Tryba. The least nonpathological F_σ extension of the random graph ideal is proper and is described by the fractional cochromatic numbers of its finite induced subgraphs.

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1 Introduction

Submeasure representations are a fundamental tool in the study of definable ideals on countable sets. Classical theorems of Mazur [27] and Solecki [30] provide representations by lower semicontinuous submeasures for F_σ ideals and analytic P-ideals, respectively. These representations lead to the distinction between pathology and nonpathology, studied in particular by Farah [12]. A submeasure is nonpathological if it is a supremum of measures; an F_σ ideal is nonpathological if it admits such a representation by a nonpathological submeasure, and pathological otherwise. We recall the precise definitions in Section 2.

For analytic P-ideals, Hrušák's measure dichotomy yields a characterization of nonpathology through the Katětov order [20, Corollary 5.6]. More recent work has focused on pathology for F_σ ideals and its connections with representations in Banach spaces [26, 24], following earlier work on representations of analytic P-ideals [5, 4]. These results motivate the search for combinatorial criteria that distinguish pathological from nonpathological F_σ ideals. A related question is whether nonpathology of an ideal can be decided from any fixed submeasure representation. We answer this question affirmatively and obtain, for graphs with no infinite clique, a criterion in terms of chromatic and fractional chromatic numbers.

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The random graph ideal $\mathcal{R}$, generated by the cliques and independent sets of the countable random graph, is a natural example in this setting. It is a tall F_σ ideal closely connected to Ramsey properties and the Katětov order [18, 20]; see also [8] for recent work on Ramsey properties of ideals. A theorem of Hrušák, Meza-Alcántara, Thümmel and Uzcátegui shows that every tall F_σ ideal has a positive restriction above $\mathcal{R}$ in the Katětov order [18].

López-Abad, Olmos-Prieto and Uzcátegui-Aylwin asked whether $\mathcal{R}$ is pathological [24, Question 5.19]. We prove that $\mathcal{R}$ is pathological by constructing a positive hereditarily pathological restriction: every further positive restriction of this witness is pathological. Together with a positive restriction isomorphic to the nonpathological ideal $\mathcal{ED}_{\mathrm{fin}}$, this gives the two local behaviors of Corollary 3.9.

The pathological witness is an induced disjoint union of finite Kneser graphs. Their unbounded chromatic numbers and independent star covers of uniformly positive density satisfy the block criterion of Martínez, Meza-Alcántara and Uzcátegui [26, Theorem 3.8], which also applies on every positive restriction of the witness. For the nonpathological restriction, universality gives an induced copy of a disjoint union of finite complete graphs of increasing sizes, whose homogeneous-set ideal is isomorphic to $\mathcal{ED}_{\mathrm{fin}}$ [26, Examples 3.5 and 4.5]. Proposition 3.10 also shows that a positive hereditarily pathological restriction of $\mathcal{R}$ lies above Solecki's ideal in the Katětov order, connecting the Kneser construction with the ideal appearing in Hrušák's measure dichotomy.

For the general criterion, let $\mathcal{I}$ be a proper F_σ ideal on V, and choose any lower semicontinuous submeasure φ with $\mathcal{I} = \mathsf{FIN}(\varphi)$. Its nonpathological envelope $\widehat{\varphi}$ is the supremum of all measures dominated by φ. Theorem 4.4 shows that $\mathcal{J}_\mathcal{I} = \mathsf{FIN}(\widehat{\varphi})$ depends only on $\mathcal{I}$, not on the choice of φ. If $\mathcal{J}_\mathcal{I}$ is proper, it is the least nonpathological F_σ extension of $\mathcal{I}$; otherwise, no proper such extension exists.

In particular,

$$\mathcal{I} \text{ is nonpathological} \quad \Longleftrightarrow \quad \mathsf{FIN}(\varphi) = \mathsf{FIN}(\widehat{\varphi}).$$

Thus, to prove pathology of $\mathcal{I}$, it suffices to find a set A with $\varphi(A) = \infty$ and $\widehat{\varphi}(A) < \infty$ for the chosen representation.

The main step is a comparison between arbitrary submeasure representations. If $\widehat{\varphi}(A) < \infty$, Hrušák's quantitative form of Christensen's lemma [20, Lemma 5.4] and compactness give a probability on a compact family of subsets of A with uniformly bounded φ-values, such that every vertex of A is covered with a uniformly positive probability. This covering bound shows that every nonpathological lower semicontinuous submeasure ψ with $\mathsf{FIN}(\varphi) \subseteq \mathsf{FIN}(\psi)$ is also finite on A. Hence, $\mathcal{J}_\mathcal{I} \subseteq \mathsf{FIN}(\psi)$, which gives the minimality of $\mathcal{J}_\mathcal{I}$.

Filipów and Tryba [14, Theorem 11.2] and, independently, Figueroa-Sierra and Hrušák [13, Corollary 6.1.2] proved that every nonpathological F_σ ideal is an intersection of summable ideals. Combining this representation with the least-extension theorem, we identify $\mathcal{J}_\mathcal{I}$ with the intersection of all proper summable extensions of $\mathcal{I}$. Corollary 4.8 also identifies it with the intersection of all matrix ideals containing $\mathcal{I}$. Consequently, for an F_σ ideal, being nonpathological, being an intersection of summable ideals, and being an intersection of matrix ideals are equivalent conditions. This answers both converses in [14, Question 11.6].

Corollary 4.9 shows that every nonpathological analytic P-ideal containing $\mathcal{I}$ also contains $\mathcal{J}_\mathcal{I}$. In particular, every F_σ ideal representable by unconditional convergence in a Banach space is also representable by bounded finite subsums, answering [24, Question 2.4].

For a countable graph H, put $\varphi_\chi(A) = \chi(H[A])$, the chromatic number of $H[A]$, i.e., the

least number of independent sets covering A. Its envelope can be computed using fractional colorings. Define the fractional submeasure by

$$\varphi_{\chi_f}(A) = \sup\{\chi_{\mathrm{f}}(H[F]) \mid F \subseteq A \text{ finite}\}.$$

Finite covering duality gives $\widehat{\varphi_\chi} = \varphi_{\chi_f}$, as we show in Theorem 5.3.[1]

If H has no infinite clique, independent sets generate its homogeneous-set ideal $\mathcal{I}_H$, so $\mathcal{I}_H = \mathsf{FIN}(\varphi_\chi)$. Assuming that $\mathcal{I}_H$ is proper, Theorem 5.5 expresses the general criterion in terms of finite subgraphs: $\mathcal{I}_H$ is nonpathological if and only if, for every $C > 0$, the chromatic numbers $\chi(H[F])$ are bounded as F ranges over the finite vertex sets satisfying $\chi_{\mathrm{f}}(H[F]) \leq C$. For a disjoint union of finite graphs, this bound must hold uniformly over the induced subgraphs of all blocks. The criterion gives partial answers to [24, Questions 5.21 and 5.22] for graphs with no infinite clique.

If the condition fails, we adapt the extraction argument in [17, proof of Theorem 17] to obtain a positive set whose induced graph is a disjoint union of finite graphs with unbounded chromatic numbers and uniformly bounded fractional chromatic numbers. The restriction to this set is hereditarily pathological.

The least-extension theorem also describes super-pathology, the property of admitting no proper nonpathological F_σ extension. For any representation $\mathcal{I} = \mathsf{FIN}(\varphi)$, Theorem 8.4 shows that a positive restriction $\mathcal{I} \restriction A$ is super-pathological if and only if $\widehat{\varphi}(A) < \infty$. Thus, $\mathcal{J}_\mathcal{I} \setminus \mathcal{I}$ consists exactly of the positive sets on which the restriction is super-pathological. In particular, every pathological F_σ ideal has a positive super-pathological restriction. For a graph H with no infinite clique and $A \in \mathcal{I}_H^+$, the envelope condition becomes $\varphi_{\chi_f}(A) < \infty$.

The random graph ideal $\mathcal{R}$ is represented by the cochromatic submeasure, which assigns to each set the least number of cliques and independent sets covering it. Theorem 6.5 shows that its least nonpathological F_σ extension is proper and is computed by the fractional cochromatic numbers of finite induced subgraphs of R. Thus, $\mathcal{R}$ is not super-pathological, whereas its Kneser restriction is.

For $d \geq 3$, pathology of the random d-uniform hypergraph ideal $\mathcal{R}_d$ and the existence of positive hereditarily pathological restrictions already follow from the constructions in [24, Theorem 5.14 and Corollary 5.15]. We recover these conclusions from the graph case: in Theorem 7.4, we use a standard extension of graph colorings to obtain a positive restriction of $\mathcal{R}_d$ isomorphic to $\mathcal{R}$ for every $d \geq 2$. This also transfers the positive nonpathological restriction isomorphic to $\mathcal{ED}_{\mathrm{fin}}$. Comparing the homogeneous-cover submeasures and their envelopes, we show that the least nonpathological F_σ extension of $\mathcal{R}_d$ is proper.

Finally, for every $d \geq 3$ and $n \geq d+1$, the homogeneous-set ideal of the universal $K_n^{(d)}$-free hypergraph is pathological but not super-pathological. Remark 8.13 explains why the super-pathology assertion in [24, Corollary 5.18] does not follow from the positive restrictions constructed there.

Section 2 fixes the notation and recalls the submeasure framework. Section 3 proves pathology of $\mathcal{R}$ and describes its two local behaviors. Section 4 proves the least-extension theorem for arbitrary submeasure representations and studies summable, matrix and nonpathological analytic P-ideal extensions. Section 5 proves the chromatic criterion for graphs with no infinite clique and derives the corresponding block criteria. Section 6 describes the

[1] Here the supremum is taken over finite induced subgraphs: assigning weights to all independent sets of an infinite graph can give a strictly larger value [15, Theorem 6 and the following remark].

least extension of $\mathcal{R}$, applies Prömel–Rödl universality to Erdős sequences, and relates the envelope to Kelley covering densities. Section 7 transfers the random-graph results to uniform hypergraphs. Section 8 describes super-pathological restrictions, gives the block classification, and treats universal hypergraphs with a finite complete hypergraph forbidden.

2 Preliminaries

Let $\omega = \{0, 1, \ldots\}$, with each natural number identified with the set of its predecessors. For a set X and $k \in \omega$, write $[X]^k$ for the family of subsets of X of cardinality k. All underlying sets are countably infinite unless stated otherwise. An ideal $\mathcal{I}$ on X is closed under subsets and finite unions, contains all finite subsets of X, and is *proper*, i.e., $X \notin \mathcal{I}$. For an ideal $\mathcal{I}$ on X, put $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$ and $\mathcal{I} \restriction Y = \mathcal{I} \cap \mathcal{P}(Y)$. This restriction is proper precisely when $Y \in \mathcal{I}^+$. We identify $\mathcal{P}(X)$ with 2^X via characteristic functions and equip it with the product topology. An ideal on X is F_σ if it is a countable union of closed sets in this topology, and is tall if every infinite subset of X contains an infinite member of the ideal. All notions are invariant under bijections of the underlying sets.

A *lower semicontinuous submeasure* (lscsm) on X is a map $\varphi : \mathcal{P}(X) \to [0, \infty]$ satisfying $\varphi(\emptyset) = 0$, monotonicity, subadditivity, $\varphi(\{x\}) < \infty$ for $x \in X$, and

$$\varphi(A) = \sup\{\varphi(F) \mid F \subseteq A \text{ finite}\}.$$

Write $\mathsf{FIN}(\varphi) = \{A \subseteq X \mid \varphi(A) < \infty\}$. The proper ideals of this form are exactly the F_σ ideals, by Mazur's theorem [27]; see also [19, Theorem 1.5].

A *measure* means a countably additive function $\mu(A) = \sum_{x \in A} m(x)$, where $m : X \to [0, \infty)$ is finite at each point. For submeasures write $\psi \leq \varphi$ for pointwise domination and let $\mathcal{M}(X, \varphi)$ be the measures dominated by φ. The submeasure φ is *nonpathological* if

$$\varphi(A) = \sup_{\mu \in \mathcal{M}(X,\varphi)} \mu(A) \quad (A \subseteq X).$$

This follows the countably additive convention of [26, §3]; for lsc submeasures it is equivalent to Farah's notion [12, p. 21], by [14, Proposition 4.1(7a)]. For F_σ ideals we use [26, Definition 3.1]: $\mathcal{I}$ is *nonpathological* if $\mathcal{I} = \mathsf{FIN}(\varphi)$ for some nonpathological lscsm, and *pathological* otherwise.

A pathological lscsm φ may generate a nonpathological ideal, since another lscsm ψ can be nonpathological and satisfy $\mathsf{FIN}(\psi) = \mathsf{FIN}(\varphi)$. Theorem 4.4 will show that the existence of such a ψ can be decided from φ, by comparing the ideal generated by φ with the one generated by its nonpathological envelope $\widehat{\varphi}$. An F_σ ideal is *hereditarily pathological* if $\mathcal{I} \restriction Y$ is pathological for every $Y \in \mathcal{I}^+$.

Fact 2.1 (Restriction)**.** *Restrictions of nonpathological submeasures are nonpathological. Consequently, every positive restriction of a nonpathological F_σ ideal is nonpathological. Equivalently, an F_σ ideal with a pathological positive restriction is pathological.*

Indeed, if $\mathcal{I} = \mathsf{FIN}(\varphi)$ with φ nonpathological, then $\mathcal{I} \restriction Y = \mathsf{FIN}(\varphi \restriction \mathcal{P}(Y))$, and positivity makes this ideal proper. The last assertion follows by contraposition.

For a graph $H = (V, E)$ and $A \subseteq V$, write $H[A] = (A, E \cap [A]^2)$ for the subgraph induced by A. A set is *homogeneous* if it is a clique or an independent set. Write $\mathcal{H}(H)$ for this family and $\langle \mathcal{H}(H) \rangle$ for the sets covered by finitely many members of it.

The chromatic number $\chi(H)$ is the least number of independent sets covering V, with $\chi(\emptyset) = 0$ and value ∞ when no finite independent cover exists. A proper coloring with m colors is a map $c : V \to m$ whose fibers are independent. Since every finite independent cover can be refined to a partition, $\chi(H)$ is equivalently the least number of colors in a proper coloring.

3 The random graph ideal: pathology and local extremes

The random graph $R = (\omega, E)$ is the unique countable graph with the extension property: for disjoint finite $a, b \subseteq \omega$, some vertex outside $a \cup b$ is adjacent to every point of a and none of b. Every countable graph occurs as an induced subgraph [7, Proposition 4.1]. Its homogeneous-set ideal $\mathcal{R} = \langle \mathcal{H}(R) \rangle$ is proper, tall and F_σ [20, p. 834]. For an induced copy $H = R[X]$, hereditary closure of homogeneity gives

$$\mathcal{R} \restriction X = \langle \mathcal{H}(H) \rangle : \tag{1}$$

one restricts a finite homogeneous cover to X, and conversely every homogeneous subset of H is homogeneous in R.

Lemma 3.1 (Block representation)**.** *Let $G = \bigsqcup_{n\in\omega} G_n$ be a disjoint union of nonempty finite graphs, with $V_n = V(G_n)$ and $V = \bigsqcup_n V_n$. For $A \subseteq V$, put*

$$\varphi_\chi(A) = \sup_{n\in\omega} \chi(G_n[A \cap V_n]).$$

Then φ_χ is an lscsm and $\varphi_\chi(A) = \chi(G[A])$ for every $A \subseteq V$. Moreover, $\varphi_\chi(\{x\}) = 1$ for every $x \in V$ and $\varphi_\chi(V_n) = \chi(G_n)$ for every $n \in \omega$. If H is a countable graph with $V \subseteq V(H)$ and $H[V] = G$, then

$$\langle \mathcal{H}(H) \rangle \restriction V = \langle \mathcal{H}(G) \rangle = \mathsf{FIN}(\varphi_\chi).$$

This is a proper F_σ ideal exactly when $\sup_{n\in\omega} \chi(G_n) = \infty$; in that case V is positive for $\langle \mathcal{H}(H) \rangle$.

Proof. Fix $A \subseteq V$. Restricting independent covers to each block gives $\varphi_\chi(A) \leq \chi(G[A])$. Conversely, if $m = \varphi_\chi(A) < \infty$, for each $n \in \omega$ choose independent subsets $(I_i^n)_{i<m}$ of $A \cap V_n$ covering it, allowing empty sets. Since there are no edges between distinct blocks, each $I_i = \bigcup_{n\in\omega} I_i^n$ is independent. These m sets cover A, so $\chi(G[A]) \leq m$, proving the equality.

Monotonicity, the value at the empty set, and the stated values on singletons and blocks follow directly from the definition. For subadditivity, it suffices to consider $A, B \subseteq V$ with finite φ_χ-values. Combining independent covers of A and B with $\varphi_\chi(A)$ and $\varphi_\chi(B)$ terms, respectively, gives

$$\varphi_\chi(A \cup B) \leq \varphi_\chi(A) + \varphi_\chi(B).$$

For lower semicontinuity, each trace $A \cap V_n$ is finite, so

$$\varphi_\chi(A) = \sup_{n\in\omega} \varphi_\chi(A \cap V_n) \leq \sup\{\varphi_\chi(F) \mid F \subseteq A \text{ finite}\} \leq \varphi_\chi(A).$$

Every clique of G is contained in a finite block. A finite homogeneous cover can therefore be replaced by a finite independent cover, by splitting each clique into singletons. Conversely, every independent set is homogeneous. Hence,

$$\langle \mathcal{H}(G) \rangle = \{A \subseteq V \mid \chi(G[A]) < \infty\} = \mathsf{FIN}(\varphi_\chi).$$

The identity for H follows by restricting homogeneous covers to V, as in (1).

Finally, $\mathsf{FIN}(\varphi_\chi)$ is F_σ by lower semicontinuity, and it is proper exactly when $\varphi_\chi(V) = \sup_{n\in\omega} \chi(G_n) = \infty$. The identity for H then gives the positivity of V. Compare [24, Propositions 5.11–5.12] for the corresponding properness criteria. □

We recall a block criterion for pathology.

For a nonempty finite set K and a covering sequence $\mathcal{S} = (S_i)_{i<m}$, put

$$\delta(K, \mathcal{S}) = \min_{x\in K} \frac{|\{i < m \mid x \in S_i\}|}{m}.$$

We call $\delta(K, \mathcal{S})$ the density of $\mathcal{S}$ on K; thus $\mathcal{S}$ covers K with density at least ε if $\delta(K, \mathcal{S}) \geq \varepsilon$. Kelley's covering number of a family is the supremum of these densities over its nonempty finite sequences [23, Corollary 6]; see also [20, §5, p. 841].

Lemma 3.2 (Block criterion; [26, Theorem 3.8]). *Let $X = \bigsqcup_n X_n$ be a partition into nonempty finite sets and let φ be an lscsm with*

$$\varphi(A) = \sup_{n\in\omega} \varphi(A \cap X_n), \qquad \sup_{n\in\omega} \varphi(X_n) = \infty.$$

Suppose each X_n has a finite covering sequence $\mathcal{S}_n$ whose terms have φ-value at most M, where $M < \infty$ is independent of n, and $\inf_{n\in\omega} \delta(X_n, \mathcal{S}_n) > 0$. Then $\mathsf{FIN}(\varphi)$ is pathological.

Remark 3.3 (The multiplicity adaptation). The paragraph before [26, Theorem 3.8] explicitly permits arbitrary finite partitions. Our only change is to allow repeated terms. The measure estimate in [26, Lemma 3.7] still holds, since counting each occurrence separately gives

$$\sum_{i<m} \mu(S_i) = \sum_{x\in K} \mu(\{x\})|\{i < m \mid x \in S_i\}| \geq m\delta(K, \mathcal{S})\mu(K).$$

Hence, some term has measure at least $\delta(K, \mathcal{S})\mu(K)$. This is the only use of the covering family in the cited proof, so the theorem applies to sequences as well. Empty terms may be discarded; doing so only increases the density.

For integers $1 \leq k \leq m$, the Kneser graph $\mathrm{KG}(m, k)$ has vertex set $[m]^k$, i.e., the subsets of m of cardinality k. Two distinct vertices are adjacent exactly when they are disjoint.

Theorem 3.4 (Lovász [25, Theorem 1′, p. 319]). *For integers m, r with $m \geq 2r \geq 2$,*

$$\chi(\mathrm{KG}(m, r)) = m - 2r + 2.$$

Fact 3.5. *For integers $1 \leq k \leq m$, the stars*

$$S_i(m, k) = \{a \in [m]^k \mid i \in a\}, \qquad i < m,$$

are independent and cover $V(\mathrm{KG}(m, k))$, each vertex belonging to exactly k stars. Thus, their covering density is k/m.

Proof. All members of $S_i(m, k)$ contain i, so this set is independent. A vertex $a \in [m]^k$ belongs to $S_i(m, k)$ exactly when $i \in a$, hence belongs to exactly k of the m stars. □

Earlier applications of Kneser-type constructions to ideals include the use of stable Kneser hypergraphs by Alon, Drewnowski and Łuczak to construct an ideal with the Nikodým property [1, Theorem 2.3 and its proof]. Their averaging argument over stars is an antecedent of the measure estimate used here [1, Lemma 3.1]. The particular family $\mathrm{KG}(3k-1,k)$ and its bounded fractional cochromatic numbers already appear in [16, Example 2, p. 6]; we use these graphs to obtain positive restrictions of $\mathcal{R}$.

Lovász's formula and the star covers provide the unbounded chromatic numbers and the uniformly positive covering densities needed for the block criterion.

Theorem 3.6 (Kneser restrictions)**.** *Let $(m_n,k_n)_{n\in\omega}$ be a sequence of pairs of positive integers with $2k_n \le m_n$ and*

$$\sup_{n\in\omega}(m_n-2k_n)=\infty, \qquad \inf_{n\in\omega}\frac{k_n}{m_n}>0.$$

If $R[X]\cong\bigsqcup_{n\in\omega}\mathrm{KG}(m_n,k_n)$, then $X\in\mathcal{R}^+$ and $\mathcal{R}\restriction X$ is hereditarily pathological.

Proof. We apply Lemma 3.2 on every positive subset of X. Write $X=\bigsqcup_{n\in\omega}X_n$ for the blocks of the induced copy, and let $S_i^n\subseteq X_n$ be the image of the star $S_i(m_n,k_n)$. Put

$$\varphi(A)=\sup_{n\in\omega}\chi(R[A\cap X_n]) \qquad (A\subseteq X).$$

By Lemma 3.1, φ is an lscsm and $\mathcal{R}\restriction X=\mathsf{FIN}(\varphi)$. Lovász's formula gives

$$\varphi(X)=\sup_{n\in\omega}(m_n-2k_n+2)=\infty,$$

so X is positive.

Let $A\subseteq X$ be positive. The nonempty traces $A\cap X_n$ partition A into finite blocks, and for every $B\subseteq A$,

$$\varphi(B)=\sup_{n\in\omega}\varphi\big(B\cap(A\cap X_n)\big).$$

Moreover, $\sup_{n\in\omega}\varphi(A\cap X_n)=\varphi(A)=\infty$. For each nonempty trace, the sequence $(S_i^n\cap A)_{i<m_n}$ consists of independent sets, each of φ-value at most 1. Every vertex belongs to exactly k_n terms, so this sequence covers the trace with density k_n/m_n. These densities have a common positive lower bound. Lemma 3.2, applied to $\varphi\restriction\mathcal{P}(A)$ with $M=1$, shows that $\mathcal{R}\restriction A$ is pathological. Thus, $\mathcal{R}\restriction X$ is hereditarily pathological. □

The following corollary answers the question of whether $\mathcal{R}$ is pathological [24, Question 5.19]. Put $G_k=\mathrm{KG}(3k-1,k)$ for $k\ge 2$.

Corollary 3.7 (The pathological local extreme)**.** *There is an $\mathcal{R}$-positive set X inducing $\bigsqcup_{k\ge2}G_k$ such that $\mathcal{R}\restriction X$ is hereditarily pathological. Consequently, $\mathcal{R}$ is pathological.*

Proof. By universality, choose such an induced copy, with blocks X_k. Identify each G_k with its image on X_k, and write $S_i^k\subseteq X_k$ for the image of $S_i(3k-1,k)$. Since

$$(3k-1)-2k=k-1\longrightarrow\infty, \qquad \frac{k}{3k-1}>\frac{1}{3},$$

Theorem 3.6 shows that X is positive and $\mathcal{R}\restriction X$ is hereditarily pathological. Fact 2.1 then gives pathology of $\mathcal{R}$. □

For completeness, we recall the following standard observation, obtained by embedding a disjoint union of finite complete graphs into R and using the nonpathology of $\mathcal{ED}_{\mathrm{fin}}$.

Fact 3.8. *There is an $\mathcal{R}$-positive set Y with*

$$\mathcal{R} \restriction Y \cong \mathcal{ED}_{\mathrm{fin}} = \left\{ A \subseteq \bigsqcup_{n\geq 1} V_n \mid \sup_{n\geq 1} |A \cap V_n| < \infty \right\}, \qquad |V_n| = n.$$

This restriction is nonpathological, so $\mathcal{R}$ is not hereditarily pathological.

Proof. The homogeneous-set ideal of $\bigsqcup_{n\geq 1} K_n$ is $\mathcal{ED}_{\mathrm{fin}}$, and it is nonpathological, by [26, Examples 3.5 and 4.5]; see also [24, Example 5.26]. Universality realizes this disjoint sum as $R[Y]$. Equation (1) identifies its ideal with $\mathcal{R} \restriction Y$. The block sizes are unbounded, so $Y \notin \mathcal{ED}_{\mathrm{fin}}$ under the identification and $Y \in \mathcal{R}^+$. Fact 2.1 also shows that every positive restriction inside Y is nonpathological. □

Corollary 3.7 and Fact 3.8 together give the two local behaviors of $\mathcal{R}$.

Corollary 3.9. *The random graph ideal $\mathcal{R}$ is pathological and has positive sets X, Y such that $\mathcal{R} \restriction X$ is hereditarily pathological and $\mathcal{R} \restriction Y \cong \mathcal{ED}_{\mathrm{fin}}$ is nonpathological.*

The Kneser construction also connects the pathological local behavior with Solecki's ideal, which appears in Hrušák's measure dichotomy for analytic P-ideals [20, Corollary 5.6]. We now give an explicit comparison in the Katětov order.

Let m be the usual product probability measure on 2^ω, and put

$$\Omega = \left\{ U \in \mathrm{Clop}(2^\omega) \mid m(U) = \frac{1}{2} \right\}.$$

Solecki's ideal $\mathcal{S}$ on Ω is generated by the sets $I_x = \{U \in \Omega \mid x \in U\}$, for $x \in 2^\omega$. Recall that, for ideals $\mathcal{I}$ on U and $\mathcal{J}$ on V, the relation $\mathcal{I} \leq_K \mathcal{J}$ means that there is a map $f : V \to U$ such that $f^{-1}(A) \in \mathcal{J}$ for every $A \in \mathcal{I}$.

Let $\mathcal{S}_{1/4}$ denote the analogous ideal on the clopen subsets of 2^ω of measure $1/4$. The argument in the proof of [21, Theorem 3.1] gives $\mathcal{S} \leq_K \mathcal{S}_{1/4}$.

Proposition 3.10. *There is an $\mathcal{R}$-positive set X such that $\mathcal{R} \restriction X$ is hereditarily pathological and*

$$\mathcal{S} \leq_K \mathcal{R} \restriction X.$$

Proof. We will code the vertices of the Kneser blocks by clopen sets of measure $1/4$ to witness $\mathcal{S}_{1/4} \leq_K \mathcal{R} \restriction X$.

By universality, choose an induced copy of

$$G = \bigsqcup_{n\in\omega} \mathrm{KG}(2^{n+2}, 2^n)$$

on a set X. Since $2^n/2^{n+2} = 1/4$ and $2^{n+2} - 2 \cdot 2^n = 2^{n+1} \to \infty$, Theorem 3.6 shows that $X \in \mathcal{R}^+$ and $\mathcal{R} \restriction X$ is hereditarily pathological. Write $X = \bigsqcup_{n\in\omega} X_n$, where X_n is the vertex set of the nth block. Label the points of X_n by pairs (n, a), with $a \in [2^{n+2}]^{2^n}$, so that two distinct points (n, a) and (n, b) are adjacent exactly when $a \cap b = \emptyset$.

For a finite binary sequence s, write $[s] = \{x \in 2^\omega \mid s \subseteq x\}$. Enumerate the binary sequences of length $n+2$ as $(s_i^n)_{i<2^{n+2}}$. For $v = (n, a) \in X$, put

$$f(v) = \bigcup_{i \in a} [s_i^n].$$

The cylinders in this union are pairwise disjoint and each has measure $2^{-(n+2)}$, so $m(f(v)) = 1/4$.

Fix $x \in 2^\omega$. For each $n \in \omega$, let $i_x(n)$ be the unique index such that $s_{i_x(n)}^n \subseteq x$. The inverse image under f of the generator $\{U \in \mathrm{Clop}(2^\omega) \mid m(U) = 1/4,\ x \in U\}$ is

$$\bigcup_{n \in \omega} \{(n, a) \in X_n \mid i_x(n) \in a\}.$$

This set is independent: its trace on each block is a star, and there are no edges between distinct blocks. Thus, f witnesses $\mathcal{S}_{1/4} \leq_K \mathcal{R} \restriction X$. Together with $\mathcal{S} \leq_K \mathcal{S}_{1/4}$, this proves the assertion. □

Together with Corollary 3.9, this comparison relates the contrasting local behavior of $\mathcal{R}$ to two familiar ideals: $\mathcal{ED}_{\mathrm{fin}}$ on the nonpathological side and Solecki's ideal on the pathological side. The former occurs as a positive restriction of $\mathcal{R}$, up to isomorphism, while the latter lies below a hereditarily pathological positive restriction in the Katětov order.

4 Nonpathological envelopes and least extensions

For an lscsm φ on V, its *nonpathological envelope* is[2]

$$\widehat{\varphi}(A) = \sup_{\mu \in \mathcal{M}(V,\varphi)} \mu(A).$$

Since $\widehat{\varphi} \leq \varphi$, we have $\mathsf{FIN}(\varphi) \subseteq \mathsf{FIN}(\widehat{\varphi})$ and $\mathsf{FIN}(\widehat{\varphi})$ is proper exactly when $\widehat{\varphi}$ is unbounded.

An F_σ ideal $\mathcal{I}$ is nonpathological if $\mathcal{I} = \mathsf{FIN}(\psi)$ for some nonpathological lscsm ψ. We will show that this existence question can be settled using any fixed representation $\mathcal{I} = \mathsf{FIN}(\varphi)$: the ideal is nonpathological if and only if $\mathsf{FIN}(\varphi) = \mathsf{FIN}(\widehat{\varphi})$.

We begin by recalling some basic facts about the envelope.

Proposition 4.1 (Envelope facts; [26, §3.1])**.** *Let φ be an lscsm on V.*

(i) The envelope $\widehat{\varphi}$ is the largest nonpathological lscsm dominated by φ. In particular, φ is nonpathological exactly when $\widehat{\varphi} = \varphi$.

(ii) The supremum defining $\widehat{\varphi}$ is unchanged if only finitely supported measures are used.

(iii) For every finite $F \subseteq V$,

$$\widehat{\varphi}(F) = \max \left\{ \nu(F) \;\middle|\; \begin{array}{l} \nu \text{ is a measure on } F, \\ \nu(B) \leq \varphi(B) \text{ for every } B \subseteq F \end{array} \right\}. \tag{2}$$

Every measure ν on F satisfying $\nu(B) \leq \varphi(B)$ for every $B \subseteq F$ extends by zero to a measure on V dominated by φ.

[2]Recall that $\mathcal{M}(V, \varphi)$ is the family of measures μ on V such that $\mu(B) \leq \varphi(B)$ for every $B \subseteq V$; see Section 2.

Proof. Assertion (i) follows from [26, §3.1], originating in [12, p. 21].

For (ii), let μ be a measure dominated by φ and, for each finite $F \subseteq V$, define $\mu_F(B) = \mu(B \cap F)$. Then μ_F has finite support and

$$\mu_F(B) = \mu(B \cap F) \leq \varphi(B \cap F) \leq \varphi(B),$$

so $\mu_F \leq \varphi$. Moreover, for every $A \subseteq V$,

$$\mu(A) = \sup_{F \in [A]^{<\omega}} \mu(F) = \sup_{F \in [A]^{<\omega}} \mu_F(A).$$

Hence, restricting the supremum defining $\widehat{\varphi}$ to finitely supported measures does not change its value.

For (iii), every measure dominated by φ restricts to a measure ν on F satisfying $\nu(B) \leq \varphi(B)$ for every $B \subseteq F$. Conversely, if ν satisfies these conditions, its extension by zero to V satisfies

$$\widetilde{\nu}(A) = \nu(A \cap F) \leq \varphi(A \cap F) \leq \varphi(A) \qquad (A \subseteq V).$$

Thus, $\widetilde{\nu} \leq \varphi$. A measure ν on F is determined by the values $\nu(\{x\})$, for $x \in F$. The conditions $\nu(B) \leq \varphi(B)$ define a closed subset of the compact finite product $\prod_{x \in F}[0, \varphi(\{x\})]$. Since $\nu(F) = \sum_{x \in F} \nu(\{x\})$ is continuous in these coordinates, the maximum is attained. □

If $\widehat{\varphi}(A) < \infty$, the next lemma constructs a probability on subsets of A with uniformly bounded φ-values, so that each vertex is covered with probability at least $1/4$. The finite covering estimate comes from Hrušák's quantitative form of Christensen's lemma [20, Lemma 5.4].

Lemma 4.2. *Let φ be an lscsm on V, let $A \subseteq V$, and suppose that $\widehat{\varphi}(A) \leq c < \infty$, where $c \geq 1$. Put*

$$\mathcal{K} = \{K \subseteq A \mid \varphi(K) \leq 2c\}, \qquad E_v = \{K \in \mathcal{K} \mid v \in K\} \quad (v \in A).$$

There is a probability measure λ defined on the Borel subsets of $\mathcal{K}$ such that[3]

$$\lambda(E_v) \geq \frac{1}{4} \qquad (v \in A).$$

Proof. The family $\mathcal{K}$ is compact by lower semicontinuity of φ. For $K \in \mathcal{K}$, let δ_K be the probability concentrated at K. If $A = \emptyset$, take $\lambda = \delta_\emptyset$; the required inequalities are vacuous. Now assume that $A \neq \emptyset$.

We first construct a probability satisfying the required inequalities on any given nonempty finite set $F \subseteq A$. If $\varphi(F) \leq 2c$, then $F \in \mathcal{K}$, and $\lambda_F = \delta_F$ satisfies $\lambda_F(E_v) = 1$ for every $v \in F$.

Otherwise, apply [20, Lemma 5.4] to the normalized submeasure $B \mapsto \varphi(B)/\varphi(F)$ on F, with $\varepsilon = 2c/\varphi(F)$. It shows that Kelley's covering number[4] of

$$\{B \subseteq F \mid \varphi(B) < 2c\}$$

[3] Thus, λ measures families of members of $\mathcal{K}$, whereas the measures dominated by a submeasure are defined on subsets of V.

[4] As in Section 3, the covering sequences may have repeated terms.

is at least

$$1 - \frac{\widehat{\varphi}(F)}{2c} \geq \frac{1}{2}.$$

Choose a finite sequence $(K_i)_{i<m}$ from this family with density greater than $1/4$, and put

$$\lambda_F = \frac{1}{m} \sum_{i<m} \delta_{K_i}.$$

Every K_i belongs to $\mathcal{K}$, so λ_F is a probability on $\mathcal{K}$. Moreover,

$$\lambda_F(E_v) = \frac{|\{i < m \mid v \in K_i\}|}{m} > \frac{1}{4} \qquad (v \in F).$$

We have therefore constructed λ_F for each nonempty finite $F \subseteq A$. To obtain one probability satisfying the inequalities for all $v \in A$, we use compactness.

The space of Borel probability measures on the compact metrizable space $\mathcal{K}$ is compact in the topology of weak convergence [22, Theorem 17.22]. Each E_v is clopen, so its characteristic function is continuous. By the definition of weak convergence, $\lambda \mapsto \lambda(E_v)$ is therefore continuous, and the condition $\lambda(E_v) \geq 1/4$ defines a closed set of probabilities.

These closed sets have the finite intersection property: for every nonempty finite $F \subseteq A$, the probability λ_F satisfies all the conditions indexed by $v \in F$. Compactness supplies a probability λ satisfying them for all $v \in A$. □

The uniform covering bound allows us to discard the members of $\mathcal{K}$ on which another submeasure is large while preserving a positive covering bound at every vertex. This gives the comparison needed for the least extension.

Lemma 4.3. *Let φ and ψ be lscsms on V, with ψ nonpathological. If*

$$\mathsf{FIN}(\varphi) \subseteq \mathsf{FIN}(\psi),$$

then

$$\mathsf{FIN}(\widehat{\varphi}) \subseteq \mathsf{FIN}(\psi).$$

Proof. Fix $A \subseteq V$ with $\widehat{\varphi}(A) < \infty$. We must show that $\psi(A) < \infty$. Choose $c \geq \max\{1, \widehat{\varphi}(A)\}$ and let $\mathcal{K}$, E_v and λ be as in Lemma 4.2. For $m \in \omega$, put

$$D_m = \{K \in \mathcal{K} \mid \psi(K) \leq m\}.$$

As $\mathcal{K} \subseteq \mathsf{FIN}(\varphi) \subseteq \mathsf{FIN}(\psi)$, the increasing sequence $(D_m \mid m \in \omega)$ covers $\mathcal{K}$. Each D_m is closed because ψ is lower semicontinuous. By continuity from below, $\lim_{m\to\infty} \lambda(D_m) = 1$. Choose m such that $\lambda(\mathcal{K} \setminus D_m) < 1/8$.

For every $v \in A$, we have

$$\lambda(D_m \cap E_v) = \lambda(E_v) - \lambda(E_v \setminus D_m) \geq \frac{1}{4} - \lambda(\mathcal{K} \setminus D_m) \geq \frac{1}{8}. \tag{3}$$

Let μ be a measure on V dominated by ψ, with finite support F. We will show that $\mu(A) \leq 8m$.

Note that $\mu(K) = \sum_{v \in F \cap A} \mu(\{v\}) \mathbf{1}_{E_v}(K)$ for $K \in \mathcal{K}$. Also, $\mu(K) \leq \psi(K) \leq m$ for $K \in D_m$.

Multiplying (3) by $\mu(\{v\})$ and summing over $v \in F \cap A$, we obtain

$$\frac{1}{8}\mu(A) \le \sum_{v\in F\cap A} \mu(\{v\})\lambda(D_m \cap E_v) = \int_{D_m} \mu(K)\, d\lambda(K) \le m\lambda(D_m) \le m.$$

The equality follows from linearity of the integral, since the sum is over the finite set $F \cap A$.

Thus, $\mu(A) \le 8m$ for every finitely supported measure $\mu \le \psi$. Taking the supremum and using Proposition 4.1(ii), we obtain $\widehat{\psi}(A) \le 8m$. Since ψ is nonpathological, $\psi(A) = \widehat{\psi}(A) < \infty$. □

Theorem 4.4 (Least nonpathological extension)**.** *Let $\mathcal{I}$ be a proper F_σ ideal on V, and let φ be any lscsm with $\mathcal{I} = \mathsf{FIN}(\varphi)$. Put $\mathcal{J}_\mathcal{I} = \mathsf{FIN}(\widehat{\varphi})$. Then:*

(i) If $\mathcal{J}_\mathcal{I}$ is proper, it is the least nonpathological F_σ ideal containing $\mathcal{I}$.

(ii) If $\mathcal{J}_\mathcal{I} = \mathcal{P}(V)$, no proper nonpathological F_σ ideal contains $\mathcal{I}$.

Moreover, $\mathcal{J}_\mathcal{I}$ does not depend on the choice of φ. In particular,

$$\mathcal{I} \text{ is nonpathological} \iff \mathsf{FIN}(\varphi) = \mathsf{FIN}(\widehat{\varphi}).$$

Proof. Let $\mathcal{L} = \mathsf{FIN}(\psi)$ be a proper nonpathological F_σ ideal containing $\mathcal{I}$, with ψ nonpathological. Since $\mathsf{FIN}(\varphi) \subseteq \mathsf{FIN}(\psi)$, Lemma 4.3 gives $\mathcal{J}_\mathcal{I} \subseteq \mathcal{L}$.

By Proposition 4.1(i), $\widehat{\varphi}$ is a nonpathological lscsm. Since $\widehat{\varphi} \le \varphi$, we have $\mathcal{I} \subseteq \mathcal{J}_\mathcal{I}$. If $\mathcal{J}_\mathcal{I}$ is proper, it is therefore a nonpathological F_σ extension of $\mathcal{I}$, and the preceding paragraph shows that it is the least one. If $\mathcal{J}_\mathcal{I} = \mathcal{P}(V)$, the inclusion $\mathcal{J}_\mathcal{I} \subseteq \mathcal{L}$ forces $\mathcal{L} = \mathcal{P}(V)$, so no such proper extension exists.

Thus, $\mathcal{J}_\mathcal{I}$ is the least such extension when one exists, and is $\mathcal{P}(V)$ otherwise. This description depends only on $\mathcal{I}$, not on the choice of φ.

For the last assertion, if $\mathcal{I}$ is nonpathological, take $\mathcal{L} = \mathcal{I}$ to obtain $\mathcal{J}_\mathcal{I} = \mathcal{I}$. Conversely, if $\mathsf{FIN}(\varphi) = \mathsf{FIN}(\widehat{\varphi})$, then $\widehat{\varphi}$ is a nonpathological lscsm representing $\mathcal{I}$. □

The theorem shows that one positive set can rule out every nonpathological representation of an F_σ ideal. Indeed, if $\mathcal{I} = \mathsf{FIN}(\varphi)$ and $\varphi(A) = \infty$ while $\widehat{\varphi}(A) < \infty$, then $\mathcal{I}$ is pathological, regardless of the representation chosen. The same envelope determines the least nonpathological F_σ extension whenever a proper one exists.

The second alternative is precisely super-pathology of $\mathcal{I}$, which we study in Section 8.

For $g : V \to (0, \infty)$ with $\sum_{v\in V} g(v) = \infty$, write $\mathcal{I}_g = \{A \subseteq V \mid \sum_{v\in A} g(v) < \infty\}$ for the associated proper summable ideal.

Filipów and Tryba [14, Theorem 11.2] and, independently, Figueroa-Sierra and Hrušák [13, Corollary 6.1.2] proved that every nonpathological F_σ ideal is an intersection of summable ideals. Using the least-extension theorem, we identify the intersection of all proper summable ideals containing an arbitrary F_σ ideal and obtain the converse.

Corollary 4.5. *For every proper F_σ ideal $\mathcal{I}$ on V,*

$$\mathcal{J}_\mathcal{I} = \bigcap \{\mathcal{I}_g \mid \mathcal{I} \subseteq \mathcal{I}_g,\ \mathcal{I}_g \text{ is a proper summable ideal on } V\},$$

where the intersection of the empty family is $\mathcal{P}(V)$. In particular, an F_σ ideal is nonpathological if and only if it is an intersection of summable ideals.

Proof. Let $\mathcal{L}$ denote the intersection on the right-hand side. Every proper summable ideal is nonpathological and F_σ, since it is represented by the measure $\mu_g(A) = \sum_{v\in A} g(v)$. Thus, Theorem 4.4 gives $\mathcal{J}_\mathcal{I} \subseteq \mathcal{L}$.

We prove the reverse inclusion. If $\mathcal{J}_\mathcal{I} = \mathcal{P}(V)$, it is immediate. Otherwise, $\mathcal{J}_\mathcal{I}$ is a proper nonpathological F_σ ideal, so the theorem of Filipów–Tryba and Figueroa-Sierra–Hrušák represents it as an intersection of proper summable ideals. Each ideal in this representation contains $\mathcal{I}$ and therefore occurs among those defining $\mathcal{L}$. Thus, $\mathcal{L}$ is contained in each of them, and hence in their intersection $\mathcal{J}_\mathcal{I}$.

Finally, if $\mathcal{I}$ is an intersection of summable ideals, it equals the intersection of all proper summable ideals containing it. Hence, $\mathcal{I} = \mathcal{L} = \mathcal{J}_\mathcal{I}$ is nonpathological. □

Combining Corollary 4.5 with the characterization of Filipów and Tryba [14, Proposition 11.4] gives the following formulation in the Katětov order.

Corollary 4.6. *An F_σ ideal $\mathcal{I}$ is nonpathological if and only if every positive restriction $\mathcal{I}\restriction A$ is Katětov below some summable ideal.*

We next consider matrix extensions. A nonnegative matrix $B = (b_{n,v} \mid n \in \omega, v \in V)$ with finite row sums is *regular* if

$$\lim_{n\to\infty} \sum_{v\in V} b_{n,v} = 1, \qquad \lim_{n\to\infty} b_{n,v} = 0 \quad (v \in V).$$

Thus, the row sums converge to 1, while each column converges to 0. In particular, the row sums are uniformly bounded. For $A \subseteq V$, put $\mu_n(A) = \sum_{v\in A} b_{n,v}$. The associated *matrix ideal* is

$$\mathcal{M}_B = \left\{ A \subseteq V \mid \lim_{n\to\infty} \mu_n(A) = 0 \right\};$$

see [14, §8]. Convergence to zero in each column gives $\lim_{n\to\infty} \mu_n(F) = 0$ for every finite $F \subseteq V$, so $\mathcal{M}_B$ contains all finite sets. Since $\lim_{n\to\infty} \mu_n(V) = 1$, we have $V \notin \mathcal{M}_B$, so the ideal is proper.

Proposition 4.7. *Let $\mathcal{I}$ be a proper F_σ ideal on V. Every matrix ideal containing $\mathcal{I}$ also contains $\mathcal{J}_\mathcal{I}$.*

Proof. Suppose $\mathcal{I} \subseteq \mathcal{M}_B$, and write $\mathcal{I} = \mathsf{FIN}(\varphi)$ for an lscsm φ. Fix $A \in \mathcal{J}_\mathcal{I}$ and choose $c \geq \max\{1, \widehat{\varphi}(A)\}$. By Lemma 4.2, the family

$$\mathcal{K} = \{K \subseteq A \mid \varphi(K) \leq 2c\}$$

admits a Borel probability λ such that $\lambda(E_v) \geq 1/4$ for every $v \in A$, where $E_v = \{K \in \mathcal{K} \mid v \in K\}$.

For each $K \in \mathcal{K}$, we have $\varphi(K) \leq 2c$, so $K \in \mathcal{I} \subseteq \mathcal{M}_B$. Hence, $\lim_{n\to\infty} \mu_n(K) = 0$. The functions

$$K \longmapsto \mu_n(K) = \sum_{v\in A} b_{n,v} \mathbf{1}_{E_v}(K)$$

are Borel, since each E_v is clopen and the partial sums are continuous. They are bounded by $L = \sup_{n\in\omega} \mu_n(V) < \infty$. Since λ is a probability, dominated convergence gives

$$\lim_{n\to\infty} \int_\mathcal{K} \mu_n(K)\, d\lambda(K) = 0.$$

Moreover,

$$\frac{1}{4}\mu_n(A) \leq \sum_{v\in A} b_{n,v}\lambda(E_v) = \int_{\mathcal{K}} \mu_n(K)\, d\lambda(K).$$

The inequality uses $\lambda(E_v) \geq 1/4$, and the equality follows from Tonelli's theorem, since all summands are nonnegative. Thus, $\lim_{n\to\infty} \mu_n(A) = 0$, so $A \in \mathcal{M}_B$. This proves $\mathcal{J}_\mathcal{I} \subseteq \mathcal{M}_B$. □

Every nonpathological F_σ ideal is an intersection of matrix ideals [14, Theorem 8.5]. Together with Proposition 4.7, this identifies the intersection of all matrix ideals containing a given F_σ ideal.

Corollary 4.8. *For every proper F_σ ideal $\mathcal{I}$ on V,*

$$\mathcal{J}_\mathcal{I} = \bigcap\{\mathcal{M} \mid \mathcal{I} \subseteq \mathcal{M},\ \mathcal{M} \text{ is a matrix ideal on } V\},$$

where the intersection of the empty family is $\mathcal{P}(V)$. In particular, the following are equivalent:

(i) $\mathcal{I}$ is nonpathological.

(ii) $\mathcal{I}$ is an intersection of summable ideals.

(iii) $\mathcal{I}$ is an intersection of matrix ideals.

Proof. Let $\mathcal{L}$ denote the intersection on the right-hand side. Proposition 4.7 gives $\mathcal{J}_\mathcal{I} \subseteq \mathcal{L}$.

We prove the reverse inclusion. If $\mathcal{J}_\mathcal{I} = \mathcal{P}(V)$, it is immediate. Otherwise, $\mathcal{J}_\mathcal{I}$ is a proper nonpathological F_σ ideal, so it is an intersection of matrix ideals. Each ideal in this representation contains $\mathcal{I}$ and therefore occurs among those defining $\mathcal{L}$. Thus, $\mathcal{L}$ is contained in each of them, and hence in their intersection $\mathcal{J}_\mathcal{I}$.

The equivalence of (i) and (ii) is Corollary 4.5, and the representation recalled above gives (i)⇒(iii). For (iii)⇒(i), suppose that $\mathcal{I}$ is an intersection of matrix ideals. It then equals the intersection of all matrix ideals containing it, so the identity just proved gives $\mathcal{I} = \mathcal{J}_\mathcal{I}$. Thus, $\mathcal{I}$ is nonpathological. □

The equivalence of (i)–(iii) answers both converses asked in [14, Question 11.6].

We now turn to nonpathological analytic P-ideal extensions. For these ideals, the definition of nonpathology uses a different submeasure representation.

Recall that an ideal $\mathcal{L}$ is a P-ideal if, for every sequence $(A_n)_{n\in\omega}$ in $\mathcal{L}$, there is $A \in \mathcal{L}$ such that $A_n \setminus A$ is finite for every n. Solecki's representation theorem [30] provides the counterpart of Mazur's theorem for analytic P-ideals. On ω, these are precisely the proper ideals of the form

$$\mathsf{Exh}(\psi) = \left\{A \subseteq \omega \mid \lim_{n\to\infty} \psi(A \setminus n) = 0\right\},$$

where ψ is an lscsm. Such an ideal is called nonpathological if it admits this representation with ψ nonpathological [14, Definition 8.3].

For analytic P-ideals, Tryba [32, Corollary 3.12] proved that nonpathology is equivalent to being an intersection of matrix ideals, with nonpathology understood through Exh. Thus, for an F_σ P-ideal, the term nonpathological can refer either to a FIN representation or to an Exh representation. The definitions alone do not identify these properties. The following corollary shows that they coincide and gives the Banach-space conclusion asked for in [24, Question 2.4].

Corollary 4.9. *Let $\mathcal{I}$ be a proper F_σ ideal on ω. Every nonpathological analytic P-ideal containing $\mathcal{I}$ also contains $\mathcal{J}_\mathcal{I}$. If $\mathcal{I}$ is a P-ideal, it is nonpathological as an analytic P-ideal if and only if it is nonpathological as an F_σ ideal. Consequently, every F_σ ideal representable by unconditional convergence in a Banach space is also representable by bounded finite subsums.*

Proof. Let $\mathcal{L}$ be a nonpathological analytic P-ideal containing $\mathcal{I}$. By [14, Theorem 8.5], $\mathcal{L}$ is an intersection of matrix ideals. Each ideal in this representation contains $\mathcal{I}$, so Proposition 4.7 shows that it contains $\mathcal{J}_\mathcal{I}$. Taking their intersection gives $\mathcal{J}_\mathcal{I} \subseteq \mathcal{L}$.

Now suppose that $\mathcal{I}$ is a P-ideal. If it is nonpathological as an analytic P-ideal, take $\mathcal{L} = \mathcal{I}$ to obtain $\mathcal{I} = \mathcal{J}_\mathcal{I}$. Thus, $\mathcal{I}$ is nonpathological as an F_σ ideal. Conversely, if it is nonpathological as an F_σ ideal, Corollary 4.8 represents it as an intersection of matrix ideals. By [32, Corollary 3.12], it is nonpathological as an analytic P-ideal.

For the last assertion, use the representation results recalled in [24, §2, after Definition 2.2]: nonpathological analytic P-ideals are precisely the ideals representable by unconditional convergence in a Banach space, while nonpathological F_σ ideals are precisely those representable by bounded finite subsums. □

We now turn to covering submeasures, whose envelopes can be computed by finite fractional covers. This will give the chromatic and cochromatic descriptions in the following sections. The next lemma is folklore, but we could not find a reference for its proof, so we include one here.

Lemma 4.10. *If $\mathcal{C} \subseteq 2^V$ is closed and hereditary and contains all singletons, then the function*

$$\varphi_\mathcal{C}(A) = \min \left\{ r \in \omega \;\middle|\; A \subseteq \bigcup_{j\in r} K_j \text{ for some } (K_j)_{j\in r} \in \mathcal{C}^r \right\}, \qquad \min \emptyset = \infty,$$

is an lscsm and $\mathsf{FIN}(\varphi_\mathcal{C}) = \langle \mathcal{C} \rangle$.

Proof. Monotonicity and the value at the empty set are immediate. If A and B can be covered by r and s members of $\mathcal{C}$, respectively, then $A \cup B$ can be covered by $r + s$ members, so $\varphi_\mathcal{C}$ is subadditive. Since $\mathcal{C}$ contains all singletons, $\varphi_\mathcal{C}$ is finite on finite sets. Moreover,

$$\mathsf{FIN}(\varphi_\mathcal{C}) = \langle \mathcal{C} \rangle$$

directly from the definition.

It remains to prove lower semicontinuity. For $r \in \omega$, let $E_r = \{A \subseteq V \mid \varphi_\mathcal{C}(A) \leq r\}$. We will show that each E_r is closed. Since $\mathcal{C}$ is closed in the compact space 2^V, the product $2^V \times \mathcal{C}^r$ is compact. Moreover,

$$\left\{ (A, (K_j)_{j\in r}) \in 2^V \times \mathcal{C}^r \mid A \subseteq \bigcup_{j\in r} K_j \right\}$$

is closed: failure of the covering condition is witnessed by some $x \in V$ with $x \in A$ and $x \notin K_j$ for every $j \in r$. Hence, this set is compact, and so its projection E_r is compact, therefore closed. It is also hereditary. Consequently,

$$A \in E_r \quad \iff \quad F \in E_r \text{ for every finite } F \subseteq A.$$

Thus,

$$\varphi_{\mathcal{C}}(A) = \sup\{\varphi_{\mathcal{C}}(F) \mid F \subseteq A \text{ finite}\},$$

which is lower semicontinuity. □

We call $\varphi_{\mathcal{C}}$ the *covering submeasure* associated with $\mathcal{C}$.

Every F_σ ideal is generated by a compact hereditary family containing all singletons; this fact is also recorded in [3, Remark 8.6(3)]. To see this directly, identify its underlying set with ω and write $\mathcal{I} = \mathsf{FIN}(\varphi)$ for an lscsm φ, by Mazur's theorem [27]. Put

$$\mathcal{C}_n = \{B \subseteq \omega \setminus n \mid \varphi(B) \leq n\}, \qquad \mathcal{C} = [\omega]^{\leq 1} \cup \bigcup_{n\in\omega} \mathcal{C}_n.$$

Each $\mathcal{C}_n$ is compact, and every neighborhood of $\emptyset$ contains $\mathcal{C}_n$ for all sufficiently large n. Since $[\omega]^{\leq 1}$ is also compact, $\mathcal{C}$ is compact. It is hereditary, contains all singletons and is contained in $\mathcal{I}$.

If $B \in \mathcal{I}$, choose $n \in \omega$ with $n \geq \varphi(B)$. Then $B \setminus n \in \mathcal{C}_n$, while $B \cap n$ is finite, so $B \in \langle\mathcal{C}\rangle$. Thus, $\mathcal{I} = \langle\mathcal{C}\rangle$.

Definition 4.11. Let F be a finite set and $\mathcal{A} \subseteq \mathcal{P}(F)$. A fractional cover of F by members of $\mathcal{A}$ is a function $w : \mathcal{A} \to [0, \infty)$ satisfying

$$\sum_{\substack{K\in\mathcal{A}\\ v\in K}} w(K) \geq 1 \qquad (v \in F).$$

We call $w(K)$ the weight of K and $\sum_{K\in\mathcal{A}} w(K)$ the total weight of the cover.

Fix a compact hereditary family $\mathcal{C} \subseteq 2^V$ containing all singletons. For $F \subseteq V$, put $\mathcal{C}_F = \mathcal{C} \cap \mathcal{P}(F) = \{K \cap F \mid K \in \mathcal{C}\}$, where the last equality holds as $\mathcal{C}$ is hereditary. The following proposition computes $\widehat{\varphi_{\mathcal{C}}}(F)$ for finite F as the minimum total weight of a fractional cover by members of $\mathcal{C}_F$.

Proposition 4.12 (Finite covering duality)**.** *For every $F \in [V]^{<\omega} \setminus \{\emptyset\}$,*

$$\widehat{\varphi_{\mathcal{C}}}(F) = \min\left\{ \sum_{K\in\mathcal{C}_F} w(K) \;\middle|\; w : \mathcal{C}_F \to [0,\infty),\quad \sum_{\substack{K\in\mathcal{C}_F\\ v\in K}} w(K) \geq 1 \ (v \in F) \right\}. \tag{4}$$

Moreover, there is a measure ν_F on F such that $\nu_F(F)$ equals this minimum and $\nu_F(K) \leq 1$ for every $K \in \mathcal{C}_F$. Its extension $\mu_F(B) = \nu_F(B \cap F)$ is dominated by $\varphi_{\mathcal{C}}$ on all of V.

Proof. Assigning weight 1 to each singleton and 0 to the other members of $\mathcal{C}_F$ gives a fractional cover of F, so the minimum in (4) is finite.

By finite covering duality [9, Theorem 2.1], this minimum equals the maximum of $\nu(F)$ over measures on F such that $\nu(K) \leq 1$ for every $K \in \mathcal{C}_F$.

We claim that a measure ν on F satisfies $\nu(K) \leq 1$ for every $K \in \mathcal{C}_F$ if and only if $\nu(B) \leq \varphi_{\mathcal{C}}(B)$ for every $B \subseteq F$. Indeed, if $\nu \leq \varphi_{\mathcal{C}}$ on $\mathcal{P}(F)$, then $\nu(K) \leq \varphi_{\mathcal{C}}(K) \leq 1$ for $K \in \mathcal{C}_F$. Conversely, suppose that $\nu(K) \leq 1$ for every $K \in \mathcal{C}_F$. For $B \subseteq F$, put $r = \varphi_{\mathcal{C}}(B)$,

which is finite since B is finite. Since $\mathcal{C}$ is hereditary, we can choose a cover of B by r members K_j of $\mathcal{C}_F$. Then

$$\nu(B) \le \sum_{j<r} \nu(K_j) \le r = \varphi_{\mathcal{C}}(B).$$

By (2), the dual maximum is $\widehat{\varphi_{\mathcal{C}}}(F)$. Choose a maximizing measure ν_F. By Proposition 4.1(iii), its extension μ_F by zero is dominated by $\varphi_{\mathcal{C}}$. □

For $\varphi_{\mathcal{C}}$, finite covering duality gives a more precise form of Lemma 4.2: the probability can be supported on the original generators in $\mathcal{C}_A$, with a bound determined by the envelope. Suppose that $\widehat{\varphi_{\mathcal{C}}}(A) \le c < \infty$, where $c \ge 1$. Then there is a Borel probability λ on $\mathcal{C}_A$ such that

$$\lambda(\{K \in \mathcal{C}_A \mid v \in K\}) \ge \frac{1}{c} \qquad (v \in A).$$

Indeed, for each nonempty finite $F \subseteq A$, Proposition 4.12 gives fractional-cover weights w_F of total weight at most c. As in the proof of Lemma 4.2, let δ_K denote the probability concentrated at K, and put

$$\lambda_F = \frac{1}{c} \sum_{K \in \mathcal{C}_F} w_F(K)\delta_K + \left(1 - \frac{1}{c} \sum_{K \in \mathcal{C}_F} w_F(K)\right) \delta_\emptyset.$$

This is a probability on $\mathcal{C}_A$ satisfying the required inequalities for $v \in F$. The same compactness argument gives a probability satisfying them for every $v \in A$. If $A = \emptyset$, take $\lambda = \delta_\emptyset$. This is a compact-family form of Kelley's intersection-number criterion [23, Theorem 2 and Corollary 3].

For $\mathcal{I} = \langle \mathcal{C} \rangle$ proper, Theorem 4.4 gives $\mathcal{J}_{\mathcal{I}} = \mathsf{FIN}(\widehat{\varphi_{\mathcal{C}}})$. Proposition 4.12 and lower semicontinuity therefore describe this extension by fractional covers of finite sets. In the next section we apply this description to covers by independent sets.

5 Fractional coloring and graphs with no infinite clique

We apply the results of Section 4 to the family of independent sets of a graph. The envelope identity holds for every countable graph. When the graph has no infinite clique, independent sets generate its homogeneous-set ideal, so the chromatic submeasure represents that ideal. We also use this hypothesis to extract pairwise disjoint finite subgraphs with no edges between them and obtain a criterion for nonpathology.

For a graph G, write $\mathrm{Ind}(G)$ for the family of its independent sets.

Definition 5.1. For a finite graph K, a fractional coloring of K is a fractional cover of $V(K)$ by members of $\mathrm{Ind}(K)$. Thus, its weights satisfy

$$\sum_{\substack{I \in \mathrm{Ind}(K) \\ v \in I}} w(I) \ge 1 \qquad (v \in V(K)).$$

The fractional chromatic number $\chi_{\mathrm{f}}(K)$ is the infimum of the total weights of these covers. Put $\chi_{\mathrm{f}}(\emptyset) = 0$.

Fact 5.2 (Fractional colorings and covering sequences)**.** *For every nonempty finite graph K, there is an optimal fractional coloring with rational weights. Consequently, some nonempty finite sequence of independent sets covers each vertex in at least a $1/\chi_{\mathrm{f}}(K)$ fraction of its terms.*

Proof. The existence of an optimal rational fractional coloring is a standard consequence of finite linear programming [29, §1.3].

Choose such a coloring w and a positive integer N such that $Nw(I)$ is an integer for every $I \in \mathrm{Ind}(K)$. Form a sequence $(I_j)_{j<m}$ by taking exactly $Nw(I)$ copies of each independent set I. Its length is

$$m = N \sum_{I \in \mathrm{Ind}(K)} w(I) = N\chi_{\mathrm{f}}(K) > 0.$$

For every $v \in V(K)$, the fractional coloring constraint gives

$$|\{j < m \mid v \in I_j\}| = N \sum_{\substack{I \in \mathrm{Ind}(K) \\ v \in I}} w(I) \geq N.$$

Hence, the fraction of terms containing each vertex is at least $N/m = 1/\chi_{\mathrm{f}}(K)$. □

For a countable graph $G = (V, E)$, define

$$\varphi^G_\chi(A) = \chi(G[A]), \qquad \varphi^G_{\chi_f}(A) = \sup_{F \in [A]^{<\omega}} \chi_{\mathrm{f}}(G[F]).$$

The compact hereditary family $\mathrm{Ind}(G)$ has covering submeasure φ^G_χ, so φ^G_χ is lower semicontinuous by Lemma 4.10. Equivalently, for each $r \in \omega$, the graph $G[A]$ is r-colorable if and only if every finite induced subgraph of $G[A]$ is r-colorable. This is the de Bruijn–Erdős compactness theorem [10, Theorem 1]. We suppress the superscript when the graph is fixed.

For fractional coloring, assigning weights to all independent sets of an infinite graph can give an infimum strictly larger than the supremum of the fractional chromatic numbers of its finite induced subgraphs; see [15, remark following Theorem 6]. Thus, the analogous compactness statement fails for this extension of fractional coloring. Here $\varphi^G_{\chi_f}(A)$ is defined directly as the supremum over finite induced subgraphs of $G[A]$. All linear programs used here are associated with finite graphs.

Theorem 5.3 (Fractional envelope)**.** *For every countable graph G and every $A \subseteq V(G)$,*

$$\widehat{\varphi^G_\chi}(A) = \varphi^G_{\chi_f}(A).$$

In particular, $\varphi^G_{\chi_f}$ is a nonpathological lscsm.

Proof. For a nonempty finite $F \subseteq V(G)$, Proposition 4.12, applied to $\mathrm{Ind}(G)$, gives $\widehat{\varphi^G_\chi}(F) = \chi_{\mathrm{f}}(G[F])$. The equality also holds for $F = \emptyset$, when both values are zero. By lower semicontinuity of the envelope,

$$\widehat{\varphi^G_\chi}(A) = \sup_{F \in [A]^{<\omega}} \widehat{\varphi^G_\chi}(F) = \sup_{F \in [A]^{<\omega}} \chi_{\mathrm{f}}(G[F]) = \varphi^G_{\chi_f}(A).$$

Since $\widehat{\varphi^G_\chi}$ is a nonpathological lscsm by Proposition 4.1(i), so is $\varphi^G_{\chi_f}$. □

For a disjoint union of finite graphs, we can compute $\varphi^G_{\chi_f}(A)$ from the intersections of A with the blocks. We use the standard identity for finite disjoint unions [29, §3.10, Exercise 13]:

$$\chi_{\mathrm{f}}(K_0 \sqcup \cdots \sqcup K_{r-1}) = \max_{j<r} \chi_{\mathrm{f}}(K_j). \tag{5}$$

Consequently, if $G = \bigsqcup_{n\in\omega} G_n$ and $V_n = V(G_n)$, then

$$\varphi^G_{\chi_f}(A) = \sup_{n\in\omega} \chi_{\mathrm{f}}(G_n[A \cap V_n]). \tag{6}$$

For every finite $F \subseteq A$, monotonicity and (5) give

$$\chi_{\mathrm{f}}(G[F]) \le \sup_{n\in\omega} \chi_{\mathrm{f}}(G_n[A \cap V_n]).$$

Taking the supremum over F proves the inequality $\le$ in (6). Conversely, each $A \cap V_n$ is a finite subset of A, so $\chi_{\mathrm{f}}(G_n[A \cap V_n]) \le \varphi^G_{\chi_f}(A)$ for every n.

The next lemma adapts the extraction argument in [17, proof of Theorem 17] to a hereditary family of finite sets and a monotone subadditive function. The absence of an infinite clique is used in the first step of the proof.

Lemma 5.4. *Let $H = (V, E)$ be a countable graph with no infinite clique, and let $\mathcal{A} \subseteq [V]^{<\omega}$ be hereditary. Suppose that $\varphi : [V]^{<\omega} \to [0, \infty)$ is monotone and subadditive, with $\varphi(\emptyset) = 0$. If*

$$\sup\{\varphi(F) \mid F \in \mathcal{A}\} = \infty,$$

then there is a sequence $(B_k \mid k \in \omega)$ of pairwise disjoint nonempty members of $\mathcal{A}$ such that $\varphi(B_k) > k$ for every k and there are no edges between distinct B_k.

Proof. In this proof, call $U \subseteq V$ unbounded (with respect to φ and $\mathcal{A}$) if

$$\sup\{\varphi(F) \mid F \in \mathcal{A},\ F \subseteq U\} = \infty.$$

For $v \in U$, write $N_U(v)$ for its neighborhood in $H[U]$. There is an unbounded set U such that $N_U(v)$ is not unbounded for any $v \in U$. Indeed, otherwise we could start with $U_0 = V$ and choose recursively $v_j \in U_j$ such that $U_{j+1} = N_{U_j}(v_j)$ is unbounded. The vertices v_j would be distinct and pairwise adjacent, contrary to the hypothesis on H.

Fix such a set U. For each $v \in U$, choose $C_v < \infty$ such that

$$\varphi(F) \le C_v \qquad (F \in \mathcal{A},\ F \subseteq N_U(v)).$$

Suppose that B_i has been chosen for $i < k$, and put $P = \bigcup_{i<k} B_i$. Since P is finite and φ is finite on finite sets, unboundedness of U gives $F \in \mathcal{A}$, $F \subseteq U$, with

$$\varphi(F) > k + \varphi(P) + \sum_{v\in P} C_v.$$

Set

$$B_k = F \setminus \left(P \cup \bigcup_{v\in P} N_U(v) \right).$$

Then $B_k \cap P = \emptyset$ and there are no edges between B_k and P; also, as $\mathcal{A}$ is hereditary, $B_k \in \mathcal{A}$. Each $F \cap N_U(v)$ also belongs to $\mathcal{A}$, so monotonicity and subadditivity give

$$\varphi(F) \le \varphi(B_k) + \varphi(P) + \sum_{v \in P} \varphi(F \cap N_U(v)) \le \varphi(B_k) + \varphi(P) + \sum_{v \in P} C_v.$$

Consequently, $\varphi(B_k) > k$. In particular, B_k is nonempty. □

For a countable graph H with no infinite clique, every clique is finite, so a finite homogeneous cover can be replaced by a finite independent cover. Consequently,

$$\mathcal{I}_H = \langle \mathcal{H}(H) \rangle = \mathsf{FIN}(\varphi_\chi^H).$$

We characterize nonpathology by asking whether bounded fractional chromatic number forces bounded chromatic number uniformly over the finite induced subgraphs of H.

Theorem 5.5. *Let $H = (V, E)$ be a countable graph with no infinite clique and with $\chi(H) = \infty$, and put $\mathcal{I}_H = \langle \mathcal{H}(H) \rangle$. Write φ_χ and φ_{χ_f} for its chromatic and fractional submeasures. The following are equivalent:*

(i) $\mathcal{I}_H$ is nonpathological;

(ii) $\mathsf{FIN}(\varphi_\chi) = \mathsf{FIN}(\varphi_{\chi_f})$;

(iii) for every $C > 0$ there is $M_C < \infty$ such that, for every finite $F \subseteq V$,

$$\chi_{\mathrm{f}}(H[F]) \le C \quad \Longrightarrow \quad \chi(H[F]) \le M_C. \tag{7}$$

If these conditions fail, there is $A \in \mathcal{I}_H^+$ such that $H[A]$ is a disjoint union of nonempty finite graphs with unbounded chromatic numbers and uniformly bounded fractional chromatic numbers. Moreover, $\mathcal{I}_H \restriction A$ is hereditarily pathological.

Proof. (i)⇔(ii). We have $\mathcal{I}_H = \mathsf{FIN}(\varphi_\chi)$ and, by Theorem 5.3, $\widehat{\varphi_\chi} = \varphi_{\chi_f}$. Thus, the equivalence follows from Theorem 4.4, applied to φ_χ.

(iii)⇒(ii). The inclusion $\mathsf{FIN}(\varphi_\chi) \subseteq \mathsf{FIN}(\varphi_{\chi_f})$ follows from $\varphi_{\chi_f} \le \varphi_\chi$. For a nonempty $A \subseteq V$ with $\varphi_{\chi_f}(A) < \infty$, put $C = \varphi_{\chi_f}(A) > 0$. Every finite $F \subseteq A$ then satisfies $\chi_{\mathrm{f}}(H[F]) \le C$, and hence $\chi(H[F]) \le M_C$. Lower semicontinuity of φ_χ gives $\varphi_\chi(A) \le M_C < \infty$.

To prove (ii)⇒(iii) by contraposition and establish the final assertion, suppose that (iii) fails for some $C > 0$. Put

$$\mathcal{A} = \{F \in [V]^{<\omega} \mid \chi_{\mathrm{f}}(H[F]) \le C\}, \qquad \varphi(F) = \chi(H[F]).$$

The family $\mathcal{A}$ is hereditary, as fractional chromatic number is monotone under induced restriction. The function $\varphi : [V]^{<\omega} \to [0, \infty)$ is monotone and subadditive, with $\varphi(\emptyset) = 0$. Since (iii) fails for C, for every $M < \infty$ there is $F \in \mathcal{A}$ with $\varphi(F) > M$.

Lemma 5.4 therefore gives pairwise disjoint sets B_k, with no edges between distinct sets, such that

$$\chi(H[B_k]) > k, \qquad \chi_{\mathrm{f}}(H[B_k]) \le C.$$

Put $A = \bigcup_{k\in\omega} B_k$. Since $\chi(H[A]) \geq \chi(H[B_k]) > k$ for every k, we have $\varphi_\chi(A) = \infty$. On the other hand, every nonempty finite $F \subseteq A$ meets only finitely many B_k. There are no edges between distinct B_k, so (5) gives

$$\chi_{\mathrm{f}}(H[F]) = \max_{\substack{k\in\omega \\ F\cap B_k\neq\emptyset}} \chi_{\mathrm{f}}(H[F\cap B_k]) \leq C.$$

Taking the supremum over finite $F \subseteq A$, we obtain $\varphi_{\chi_f}(A) \leq C$. Thus, $A \in \mathsf{FIN}(\varphi_{\chi_f}) \setminus \mathsf{FIN}(\varphi_\chi)$, so (ii) fails. Moreover, $A \in \mathcal{I}_H^+$.

It remains to prove hereditary pathology. Let $D \subseteq A$ with $D \notin \mathcal{I}_H$. Restricting homogeneous covers gives $\mathcal{I}_H \restriction D = \mathcal{I}_{H[D]}$. The graph $H[D]$ has no infinite clique, and its chromatic and fractional submeasures are the restrictions of φ_χ and φ_{χ_f} to $\mathcal{P}(D)$. Since

$$\varphi_\chi(D) = \infty \qquad \text{and} \qquad \varphi_{\chi_f}(D) \leq \varphi_{\chi_f}(A) \leq C,$$

condition (ii) fails for $H[D]$. By the equivalence (i)$\Leftrightarrow$(ii), already proved, $\mathcal{I}_{H[D]}$ is pathological. Thus, $\mathcal{I}_H \restriction A$ is hereditarily pathological. □

Let $H = (V, E)$ be a countable graph with no infinite clique, and let ψ be an lscsm on V that is finite on every independent set of H. Then there is $L > 0$ such that $\psi(I) \leq L$ for every independent set I.

Indeed, if these values were unbounded, lower semicontinuity would give unbounded values on finite independent sets. Lemma 5.4, applied to this family and to ψ on finite sets, would give independent sets B_k with $\psi(B_k) > k$ and no edges between distinct sets. Their union is independent and, by monotonicity, has infinite ψ-value, a contradiction.

A cover by independent sets and subadditivity give $\psi \leq L\varphi_\chi^H$. If ψ is nonpathological, taking envelopes and using Theorem 5.3 gives

$$\psi = \widehat{\psi} \leq L\widehat{\varphi_\chi^H} = L\varphi_{\chi_f}^H.$$

Thus, every nonpathological lscsm that is finite on the independent sets of H is dominated on all subsets of V by a single constant multiple of $\varphi_{\chi_f}^H$.

We now specialize these results to graph blocks. For the remainder of this section, let $G = \bigsqcup_{n\in\omega} G_n$, where the blocks are finite and nonempty, put $V = V(G)$ and $V_n = V(G_n)$, and assume $\sup_{n\in\omega} \chi(G_n) = \infty$. We use φ_χ and φ_{χ_f} for the submeasures associated with this graph. The graph G has no infinite clique. By Lemma 3.1 and equation (6),

$$\varphi_\chi(A) = \sup_{n\in\omega} \chi(G_n[A\cap V_n]), \qquad \varphi_{\chi_f}(A) = \sup_{n\in\omega} \chi_{\mathrm{f}}(G_n[A\cap V_n]).$$

Thus, $\mathcal{I}_G = \langle \mathcal{H}(G)\rangle = \mathsf{FIN}(\varphi_\chi)$. We call these proper ideals *graph block ideals*. By Theorem 5.5, $\mathcal{I}_G$ is nonpathological if and only if for every $C > 0$ there is $M_C < \infty$ such that

$$\chi_{\mathrm{f}}(G_n[B]) \leq C \quad \Longrightarrow \quad \chi(G_n[B]) \leq M_C \qquad (n \in \omega,\ B \subseteq V_n).$$

Indeed, it suffices to check condition (7) on subsets of individual blocks: if a finite $F \subseteq V$ satisfies $\chi_{\mathrm{f}}(G[F]) \leq C$, then each nonempty $F \cap V_n$ has fractional chromatic number at most C, and hence chromatic number at most M_C. Since there are no edges between distinct blocks, $\chi(G[F]) \leq M_C$.

The star covers of the Kneser blocks in Corollary 3.7 may now be read as fractional colorings: assigning weight $1/k$ to each star gives

$$\chi_{\mathrm{f}}(\mathrm{KG}(3k-1,k)) \leq \frac{3k-1}{k} < 3. \tag{8}$$

Thus, the concrete witness behind Corollary 3.7 is an instance of the general obstruction isolated in Theorem 5.5: uniformly bounded fractional chromatic number together with unbounded chromatic number.

Farah's degree of pathology is defined through ratios of a submeasure to its envelope [12, p. 21]. For lscsms it is enough to consider finite subsets, as in [26, §3.1]; see also [14, Proposition 7.2]. To distinguish this version from the normalized ratio on the whole space used in [20, §5], write

$$P_{\mathrm{fin}}(\varphi) = \sup\left\{ \frac{\varphi(F)}{\widehat{\varphi}(F)} \;\middle|\; F \text{ finite and } \widehat{\varphi}(F) > 0 \right\}.$$

For any countable graph H, Theorem 5.3 gives

$$P_{\mathrm{fin}}(\varphi_\chi^H) = \sup\left\{ \frac{\chi(H[F])}{\chi_{\mathrm{f}}(H[F])} \;\middle|\; F \in [V(H)]^{<\omega} \setminus \{\emptyset\} \right\}.$$

For graph blocks, it suffices to consider subsets of individual blocks.

Indeed, let $F \subseteq V$ be finite and nonempty. Since there are no edges between distinct blocks, choose n such that $\chi(G[F]) = \chi(G_n[F \cap V_n])$. By monotonicity of fractional chromatic number,

$$\frac{\chi(G[F])}{\chi_{\mathrm{f}}(G[F])} \leq \frac{\chi(G_n[F \cap V_n])}{\chi_{\mathrm{f}}(G_n[F \cap V_n])}.$$

Conversely, every nonempty subset of a block is a finite subset of V. Taking suprema therefore gives

$$P_{\mathrm{fin}}(\varphi_\chi) = \sup\left\{ \frac{\chi(G_n[B])}{\chi_{\mathrm{f}}(G_n[B])} \;\middle|\; \emptyset \neq B \subseteq V_n \text{ for some } n \in \omega \right\}. \tag{9}$$

Finiteness of $P_{\mathrm{fin}}(\varphi_\chi)$ means that a single constant bounds χ/χ_{f} on all nonempty induced subgraphs of the blocks. The nonpathology criterion in Theorem 5.5 allows this bound to depend on an upper bound C for χ_{f}. Indeed, for each fixed $C \geq 1$, bounding χ is equivalent to bounding χ/χ_{f} on the subgraphs with $1 \leq \chi_{\mathrm{f}} \leq C$. In particular, $P_{\mathrm{fin}}(\varphi_\chi) < \infty$ implies that $\mathcal{I}_G$ is nonpathological.

On a nonempty whole block, Hrušák's degree is $\chi(G_n)/\chi_{\mathrm{f}}(G_n)$;[5] its quantitative relation to Kelley coverings is given by [20, Lemma 5.4].

Remark 5.6 (Complexity in the block class). Question 5.21 of [24] asks, in particular, for the descriptive complexity of the colorings whose coloring ideal is nonpathological. Theorem 5.5 gives a partial answer for the graph block ideals considered here.

Code nonempty finite graphs by natural numbers and give the space of sequences of such codes the product topology. Restrict attention to sequences with unbounded chromatic numbers. Within this class, nonpathology is expressed by

$$(\forall C \in \omega \setminus \{0\})(\exists M \in \omega)(\forall n \in \omega)\ R(C, M, n),$$

[5] Hrušák's formulation is entirely in terms of submeasures and Kelley covering numbers. The appearance of χ_{f} here comes from our identification $\widehat{\varphi_\chi} = \varphi_{\chi_f}$ in Theorem 5.3.

where $R(C, M, n)$ says that every $B \subseteq V_n$ satisfying $\chi_{\mathrm{f}}(G_n[B]) \leq C$ also satisfies $\chi(G_n[B]) \leq M$. For fixed C, M, n, this condition depends only on the finite graph coded by the nth coordinate, so it defines a clopen set. These quantifiers therefore give a Π^0_3 definition of nonpathology relative to this class.

6 The canonical envelope and further consequences

We now describe the least nonpathological F_σ extension of the random graph ideal $\mathcal{R}$ using fractional cochromatic numbers of finite induced subgraphs. Theorem 4.4, applied to the cochromatic submeasure, gives this extension once we show that its envelope is unbounded. We prove this unboundedness and show that the Kneser set from Corollary 3.7 witnesses the strict inclusion of $\mathcal{R}$ in the extension.

We first describe the corresponding envelope for an arbitrary countable graph $G = (V, E)$. The family $\mathcal{H}(G)$ of homogeneous sets is closed, hereditary and contains all singletons. Write $\varphi^G_Z = \varphi_{\mathcal{H}(G)}$, so

$$\mathsf{FIN}(\varphi^G_Z) = \langle \mathcal{H}(G) \rangle = \mathcal{I}_G.$$

Definition 6.1. For a finite graph K, the cochromatic number $Z(K)$ is the least number of homogeneous sets covering $V(K)$. A fractional cocoloring of K is a fractional cover of $V(K)$ by members of $\mathcal{H}(K)$ (Definition 4.11). The fractional cochromatic number $Z_{\mathrm{f}}(K)$ is the infimum of the total weights of these covers. Put $Z(\emptyset) = Z_{\mathrm{f}}(\emptyset) = 0$.

As with fractional chromatic number, we extend the finite parameter by taking the supremum over finite induced subgraphs. For a countable graph G and $A \subseteq V(G)$, put

$$\varphi^G_{Z_f}(A) = \sup_{F \in [A]^{<\omega}} Z_{\mathrm{f}}(G[F]).$$

Applying Proposition 4.12 to $\mathcal{H}(G)$ gives the following envelope identity; see also [16, §2, p. 3] for the finite cochromatic duality.

Proposition 6.2 (Cochromatic envelope)**.** *For every countable graph G and every $A \subseteq V(G)$,*

$$\varphi^G_Z(A) = \sup_{F \in [A]^{<\omega}} Z(G[F]), \qquad \widehat{\varphi^G_Z}(A) = \varphi^G_{Z_f}(A).$$

Proof. For finite $F \subseteq V(G)$, the members of $\mathcal{H}(G)$ contained in F are exactly the homogeneous sets of $G[F]$, so $\varphi^G_Z(F) = Z(G[F])$. Since $\mathcal{H}(G)$ is closed, hereditary and contains all singletons, Lemma 4.10 makes φ^G_Z lower semicontinuous. This proves the first equality for arbitrary A.

Fractional covers of F by these homogeneous sets are precisely fractional cocolorings of $G[F]$. Proposition 4.12 therefore gives $\widehat{\varphi^G_Z}(F) = Z_{\mathrm{f}}(G[F])$ for nonempty finite F; the equality also holds for $F = \emptyset$, when both values are zero. By Proposition 4.1, the envelope is lower semicontinuous, so

$$\widehat{\varphi^G_Z}(A) = \sup_{F \in [A]^{<\omega}} Z_{\mathrm{f}}(G[F]) = \varphi^G_{Z_f}(A). \qquad \square$$

We next compare the cochromatic envelope with the chromatic envelope from Theorem 5.3.

Proposition 6.3. *Suppose $\mathcal{I}_G$ is proper and H is either G or its complement. If H has no infinite clique, then*

$$\mathsf{FIN}(\widehat{\varphi_Z^G}) = \mathsf{FIN}(\varphi_{\chi_f}^H).$$

Proof. The graphs G and H have the same homogeneous sets. Since all cliques of H are finite, its independent sets generate $\mathcal{I}_G$. Hence, $\mathsf{FIN}(\varphi_Z^G) = \mathcal{I}_G = \mathsf{FIN}(\varphi_\chi^H)$. By the independence of the representation in Theorem 4.4,

$$\mathsf{FIN}(\widehat{\varphi_Z^G}) = \mathsf{FIN}(\widehat{\varphi_\chi^H}).$$

Theorem 5.3 identifies $\widehat{\varphi_\chi^H}$ with $\varphi_{\chi_f}^H$. □

Under these hypotheses, allowing all homogeneous sets in place of independent sets of H may change the fractional covering values, but it does not change which sets have finite value. Thus, the chromatic and cochromatic constructions determine the same least nonpathological F_σ extension whenever it is proper.

For the random graph, the two constructions do differ at the level of ideals: an infinite clique C satisfies $\varphi_{Z_f}^R(C) = 1$, whereas $\varphi_{\chi_f}^R(C) = \infty$. We therefore use the homogeneous generators of $\mathcal{R}$. Write $\varphi_Z = \varphi_Z^R$ and $\varphi_{Z_f} = \varphi_{Z_f}^R$, so $\mathcal{R} = \mathsf{FIN}(\varphi_Z)$.

The fact that $\mathsf{FIN}(\varphi_{Z_f})$ is proper follows from the next proposition.

Proposition 6.4. *For every integer $n \geq 1$, there is a set $W_n \subseteq \omega$ with $|W_n| = n^2$ and $\varphi_{Z_f}(W_n) \geq n$. In particular, φ_{Z_f} is unbounded.*

Proof. Fix $n \geq 1$. By universality, choose $W_n \subseteq \omega$ such that $H = R[W_n]$ is a disjoint union of n copies of K_n. Every clique of H is contained in one copy, and every independent set contains at most one vertex from each copy. Thus, every homogeneous set has at most n vertices.

For any fractional cocoloring w of H, we have $\sum_{J \in \mathcal{H}(H),\, v \in J} w(J) \geq 1$ for every $v \in W_n$. Summing these inequalities over all vertices gives

$$n^2 \leq \sum_{J \in \mathcal{H}(H)} |J| w(J) \leq n \sum_{J \in \mathcal{H}(H)} w(J).$$

Hence, $\varphi_{Z_f}(W_n) = Z_{\mathrm{f}}(H) \geq n$.[6] □

Theorem 6.5. *The ideal $\mathsf{FIN}(\varphi_{Z_f})$ is the least nonpathological F_σ extension of $\mathcal{R}$, and*

$$\mathcal{R} \subsetneq \mathsf{FIN}(\varphi_{Z_f}) \subsetneq \mathcal{P}(\omega).$$

Proof. By Proposition 6.2, $\varphi_{Z_f} = \widehat{\varphi_Z}$, and Proposition 6.4 shows that this submeasure is unbounded. Hence, $\mathsf{FIN}(\varphi_{Z_f})$ is proper. Theorem 4.4, applied to φ_Z, makes it the least nonpathological F_σ extension of $\mathcal{R}$.

[6] In fact, $Z_{\mathrm{f}}(H) = n$: assigning weight 1 to each of the n cliques gives a fractional cocoloring of total weight n.

It remains to prove that $\mathcal{R} \subsetneq \mathsf{FIN}(\varphi_{Z_f})$. The Kneser set X from Corollary 3.7 satisfies $X \notin \mathcal{R}$.[7] On the other hand, for every finite $F \subseteq X$, the disjoint-union formula (5) and the star covers in (8) give

$$Z_{\mathrm{f}}(R[F]) \leq \chi_{\mathrm{f}}(R[F]) = \max_{k\geq 2} \chi_{\mathrm{f}}(R[F \cap X_k]) \leq 3.$$

The first inequality holds because every independent set is homogeneous. Taking the supremum over finite $F \subseteq X$ gives $\varphi_{Z_f}(X) \leq 3$,[8] so $X \in \mathsf{FIN}(\varphi_{Z_f}) \setminus \mathcal{R}$.[9] □

One might try to prove pathology of $\mathcal{R}$ by applying Lemma 3.2 directly to φ_Z on all of ω. The following fact rules out the uniform covering bounds required for this approach.

Fact 6.6 (The block criterion on the whole space)**.** *Let $(K_n)_{n\in\omega}$ be an increasing exhaustion of ω by nonempty finite sets, and let $\mathcal{S}_n = (S_i^n)_{i<m_n}$ be a finite covering sequence of K_n for each n. There are no constants $M < \infty$ and $\varepsilon > 0$ such that*

$$\delta(K_n, \mathcal{S}_n) \geq \varepsilon, \qquad \varphi_Z(S_i^n) \leq M \qquad (n \in \omega,\ i < m_n).$$

Proof. Suppose such constants exist, and let $\mu \leq \varphi_Z$ be a measure on ω. For each n, every vertex of K_n belongs to at least $m_n\varepsilon$ terms of $\mathcal{S}_n$. Hence,

$$m_n\varepsilon\,\mu(K_n) \leq \sum_{i<m_n} \mu(S_i^n) \leq \sum_{i<m_n} \varphi_Z(S_i^n) \leq m_n M.$$

Thus, $\mu(K_n) \leq M/\varepsilon$ for every n. Since $(K_n)_{n\in\omega}$ is increasing and $\bigcup_{n\in\omega} K_n = \omega$, continuity from below gives $\mu(\omega) = \sup_{n\in\omega} \mu(K_n) \leq M/\varepsilon$.[10] Taking the supremum over all measures $\mu \leq \varphi_Z$, we obtain

$$\varphi_{Z_f}(\omega) = \widehat{\varphi_Z}(\omega) \leq \frac{M}{\varepsilon},$$

contrary to Proposition 6.4. □

If φ_Z satisfied the hypotheses of Lemma 3.2 for a partition $\omega = \bigsqcup_n B_n$, put $K_n = \bigcup_{j\leq n} B_j$. On K_n, take the covering sequence of all unions obtained by choosing one term from each of the first $n+1$ local covering sequences, counting repetitions. Each vertex occurs in the same fraction of terms as in its local sequence, so the densities have the same positive lower bound. The identity $\varphi_Z(A) = \sup_j \varphi_Z(A \cap B_j)$ keeps the values of these unions uniformly bounded. The preceding fact therefore rules out applying the block criterion directly to φ_Z on ω. Our proof of pathology applies the criterion to the chromatic submeasure on the positive Kneser restriction.

We now turn to graph block ideals arising from Erdős sequences. We will prove their pathology by using a theorem of Prömel–Rödl to find induced copies of $\mathrm{KG}(3k-1, k)$, for $k \geq 2$, in distinct blocks.

[7] Recall that $R[X]$ is the disjoint union of the graphs $R[X_k] \cong \mathrm{KG}(3k-1, k)$, for $k \geq 2$.

[8] In fact, $\varphi_{Z_f}(X) = 3$: by [16, Example 2, p. 6], $Z_{\mathrm{f}}(R[X_k]) = 3 - 1/k$, so $\varphi_{Z_f}(X) \geq \sup_{k\geq 2}(3 - 1/k) = 3$.

[9] Since $\mathsf{FIN}(\varphi_{Z_f})$ is nonpathological, strictness also follows from the pathology of $\mathcal{R}$ in Corollary 3.7. The proof above gives an explicit witness.

[10] This is the bounded-envelope estimate from [26, Theorem 3.9], with repeated terms allowed as in Remark 3.3.

For a sequence $(G_n)_{n\in\omega}$ of nonempty finite graphs, put $V_n = V(G_n)$. Call it an *Erdős sequence* if $|V_n| \to \infty$ and, for all sufficiently large n, every homogeneous set in G_n has size at most

$$2\log_2 |V(G_n)|.$$

Such sequences exist by [11], and every infinite subsequence of an Erdős sequence is again an Erdős sequence.

Theorem 6.7 (Prömel–Rödl [28, Theorem 1])**.** *For every $a > 0$ there is $c > 0$ such that every graph on N vertices with no homogeneous set of cardinality at least $a\log_2 N$ contains every graph on at most $c\log_2 N$ vertices as an induced subgraph.*

Proposition 6.8. *Let $(G_n)_{n\in\omega}$ be an Erdős sequence and put $G = \bigsqcup_{n\in\omega} G_n$. Then $\mathcal{I}_G$ is pathological and has a positive hereditarily pathological restriction. In particular, if $R[X] \cong G$, then $X \in \mathcal{R}^+$ and $\mathcal{R} \restriction X$ has the same properties.*

Proof. We will find an induced copy of the disjoint union of Kneser graphs from Corollary 3.7.

Let $c > 0$ be given by Theorem 6.7 for $a = 4$. For all sufficiently large n, every homogeneous set in G_n has size at most $2\log_2 |V_n| < 4\log_2 |V_n|$, so the theorem applies to G_n. Since $|V_n| \to \infty$, we can choose strictly increasing indices n_k, for $k \geq 2$, such that

$$c\log_2 |V_{n_k}| \geq |V(\mathrm{KG}(3k-1,k))|.$$

Thus, G_{n_k} contains an induced copy of $\mathrm{KG}(3k-1,k)$ on a set $B_k \subseteq V_{n_k}$. Theorem 3.4 and equation (8) give

$$\chi(G[B_k]) = k+1, \qquad \chi_{\mathrm{f}}(G[B_k]) < 3.$$

Put $A = \bigcup_{k\geq 2} B_k$. Since there are no edges between distinct B_k, Lemma 3.1 shows that $\mathcal{I}_G \restriction A$ is isomorphic to the Kneser restriction in Corollary 3.7. Thus, $A \in \mathcal{I}_G^+$ and $\mathcal{I}_G \restriction A$ is hereditarily pathological. Fact 2.1 then gives pathology of $\mathcal{I}_G$.

Finally, if $X \subseteq \omega$ induces a copy of the disjoint union in the random graph, Lemma 3.1 identifies $\mathcal{R} \restriction X$ with the same graph block ideal. Since that ideal is proper, the lemma also gives $X \in \mathcal{R}^+$. □

The Erdős sequences show that condition (7) in Theorem 5.5 cannot be restricted to whole blocks. For a nonempty finite graph H, write $\alpha(H)$ for the maximum size of an independent set in H. Summing the fractional covering inequalities gives $\chi_{\mathrm{f}}(H) \geq |V(H)|/\alpha(H)$. Since (G_n) is an Erdős sequence, for all sufficiently large n we obtain

$$\chi_{\mathrm{f}}(G_n) \geq \frac{|V_n|}{\alpha(G_n)} \geq \frac{|V_n|}{2\log_2 |V_n|} \longrightarrow \infty.$$

Hence, for each fixed $C > 0$, only finitely many blocks satisfy $\chi_{\mathrm{f}}(G_n) \leq C$, and their chromatic numbers are bounded. The condition therefore holds when tested only on whole blocks, but fails on the induced subgraphs $G[B_k]$ constructed above, whose fractional chromatic numbers are less than 3 and whose chromatic numbers are unbounded.

We conclude by examining the role of the chosen covers in [26, Question 3.10]. Under the hypotheses of their Theorem 3.9, the authors ask whether boundedness of the envelope implies a positive uniform lower bound on the covering densities, and whether the absence of

such a bound implies nonpathology of the ideal. Both questions were answered negatively by Filipów and Tryba using singleton covers [14, Lemma 11.7 and the discussion following it].

The Kneser construction also illustrates why the choice of covers matters. Let G be the disjoint union of Kneser graphs used in Corollary 3.7, and put $\varphi = \varphi_\chi^G$. Then φ is unbounded, $\widehat{\varphi}(V(G)) \leq 3$, and $\mathsf{FIN}(\varphi)$ is pathological. Nevertheless, for any increasing exhaustion $(K_n)_{n\in\omega}$ of $V(G)$ by nonempty finite sets, the families $\mathcal{S}_n = \{\{v\} \mid v \in K_n\}$ consist of sets of φ-value 1 and satisfy

$$\delta(K_n, \mathcal{S}_n) = \frac{1}{|K_n|} \longrightarrow 0.$$

This suggests considering the best available covering densities. We optimize over finite covering sequences, allowing repetitions, rather than over families of distinct sets. Let $\mathcal{C} \subseteq 2^V$ be compact and hereditary and contain all singletons. For nonempty finite $F \subseteq V$, put[11]

$$\delta^*_{\mathcal{C}}(F) = \max\{\delta(F, \mathcal{S}) \mid \mathcal{S} \text{ is a finite covering sequence of } F \text{ with terms in } \mathcal{C}_F\}.$$

Finite covering duality gives

$$\delta^*_{\mathcal{C}}(F) = \frac{1}{\widehat{\varphi_{\mathcal{C}}}(F)}.$$

Indeed, an optimal fractional cover can be chosen with rational weights. Clearing denominators gives a covering sequence of density at least $1/\widehat{\varphi_{\mathcal{C}}}(F)$. Conversely, a sequence of length m and density $\delta > 0$ gives a fractional cover of total weight $1/\delta$: assign weight $1/(m\delta)$ to each term, adding weights when a set is repeated. Proposition 4.12 gives the reverse inequality and shows that the maximum is attained.

Consequently, for every increasing exhaustion $(K_n)_{n\in\omega}$ of V by nonempty finite sets,

$$\widehat{\varphi_{\mathcal{C}}}(V) < \infty \quad \Longleftrightarrow \quad \inf_{n\in\omega} \delta^*_{\mathcal{C}}(K_n) > 0.$$

Thus, for covering submeasures, boundedness of the envelope is equivalent to the existence of covering sequences whose densities have a positive uniform lower bound.[12]

The absence of a positive uniform lower bound for these optimal densities still does not imply nonpathology. Let $G = \bigsqcup_{n\in\omega} G_n$ arise from an Erdős sequence, and put $K_n = \bigcup_{j\leq n} V_j$. Taking $\mathcal{C} = \mathrm{Ind}(G)$, Theorem 5.3 gives

$$\delta^*_{\mathrm{Ind}(G)}(K_n) = \frac{1}{\chi_{\mathrm{f}}(G[K_n])} \leq \frac{1}{\chi_{\mathrm{f}}(G_n)} \longrightarrow 0,$$

whereas $\mathcal{I}_G$ is pathological by Proposition 6.8.

7 Uniform hypergraphs

For $d \geq 3$, pathology of the random d-uniform hypergraph ideal and the existence of positive hereditarily pathological restrictions already follow from the constructions of López-Abad et al. [24, Theorem 5.14 and Corollary 5.15]. Here we recover these conclusions from the

[11] Recall that $\mathcal{C}_F = \mathcal{C} \cap \mathcal{P}(F)$.

[12] More generally, let φ be an lscsm and let $\mathcal{S} = (S_i)_{i<m}$ cover a nonempty finite set F, with $S_i \subseteq F$ and $\varphi(S_i) \leq M < \infty$ for every $i < m$. Then $\delta(F, \mathcal{S})\widehat{\varphi}(F) \leq M$. This follows from the measure estimate in Remark 3.3 by taking the supremum over $\mu \leq \varphi$. For a graph G, taking $\varphi = \varphi_\chi^G$ gives $\delta(F, \mathcal{S}) \leq M/\chi_{\mathrm{f}}(G[F])$.

graph case by constructing a positive restriction isomorphic to $\mathcal{R}$ for every $d \geq 2$. This also transfers the positive nonpathological restriction isomorphic to $\mathcal{ED}_{\mathrm{fin}}$. Comparing the homogeneous-cover submeasures and their envelopes, we then show that the least nonpathological F_σ extension of the ambient ideal is proper.

For $d \geq 2$, a *d-uniform hypergraph* is a pair $H = (V, E)$ with $E \subseteq [V]^d$. For $A \subseteq V$, write $H[A] = (A, E \cap [A]^d)$ for the induced subhypergraph. A set A is complete if $[A]^d \subseteq E$, independent if $[A]^d \cap E = \emptyset$, and homogeneous if either holds. Every set of size below d is both complete and independent. Write $\mathcal{H}(H)$ for this closed hereditary family and, as for graphs, put $\varphi_Z^H = \varphi_{\mathcal{H}(H)}$. By Lemma 4.10, φ_Z^H is an lscsm and $\mathsf{FIN}(\varphi_Z^H) = \langle \mathcal{H}(H) \rangle$.

Let $R^{(d)}$ denote the countable universal homogeneous d-uniform hypergraph, equivalently the Fraïssé limit of the class of all finite d-uniform hypergraphs; see [31]. Put $\mathcal{R}_d = \langle \mathcal{H}(R^{(d)}) \rangle = \mathsf{FIN}(\varphi_Z^{R^{(d)}})$. Thus, $\mathcal{R}_2 = \mathcal{R}$.

Fact 7.1 (Induced restriction)**.** *For a countable d-uniform hypergraph $H = (V, E)$ and $X \subseteq V$,*

$$\langle \mathcal{H}(H) \rangle \restriction X = \langle \mathcal{H}(H[X]) \rangle.$$

Indeed, if A is homogeneous in H, then $A \cap X$ is homogeneous in $H[X]$. Conversely, if $B \subseteq X$ is homogeneous in $H[X]$, then, since $H[X]$ is induced, B is homogeneous in H as well.

We now pass from graphs to uniform hypergraphs by letting the color of an increasing d-tuple depend only on its first two vertices (see, for example, [6, proof of Proposition 4.1]). We record its effect on homogeneous covers and their envelopes.

Let $G = (V, E)$ be a graph with $V \subseteq \omega$, and fix $d \geq 2$. For vertices listed in increasing order, $x_0 < \cdots < x_{d-1}$, define a d-uniform hypergraph $G^{(d)}$ on V by

$$\{x_0, \dots, x_{d-1}\} \in E(G^{(d)}) \quad \iff \quad \{x_0, x_1\} \in E.$$

Fact 7.2. *Every homogeneous set of G is homogeneous in $G^{(d)}$, and every homogeneous set of $G^{(d)}$ can be partitioned into at most $d-1$ homogeneous sets of G. Moreover, if G has no triangles, then $G^{(d)}$ has no complete set of cardinality $d+1$.*

Proof. If a set is homogeneous in G, the first two vertices of each of its subsets of cardinality d form an edge when the set is a clique, and a nonedge when it is independent. Thus, the set is homogeneous in $G^{(d)}$.

Conversely, let J be homogeneous in $G^{(d)}$. If J is infinite, any pair $x < y$ from J can be extended to a subset of J of cardinality d by adding $d-2$ vertices larger than y. Since x and y are its first two vertices, all pairs from J have the same color in G. Thus, J itself is homogeneous in G.

Suppose now that J is finite. If $|J| < d$, its singletons give the required partition. Otherwise, let T consist of the $d-2$ greatest elements of J. We show that $J \setminus T$ is homogeneous in G. For every $x < y$ in $J \setminus T$, the set $\{x, y\} \cup T$ has cardinality d and its first two vertices are x and y. Homogeneity of J in $G^{(d)}$ therefore implies that all pairs from $J \setminus T$ have the same color in G. The set $J \setminus T$ and the singletons $\{t\}$, for $t \in T$, give the required partition into $d-1$ homogeneous sets.

Finally, suppose that $G^{(d)}$ has a complete set $\{x_0, \dots, x_d\}$, with $x_0 < \cdots < x_d$. Adjoining the $d-2$ vertices outside $\{x_0, x_1, x_2\}$ to each pair from this triple gives an edge of $G^{(d)}$ whose first two vertices are those of the pair. Thus, all three pairs are edges of G, so $\{x_0, x_1, x_2\}$ is a triangle. $\square$

Lemma 7.3 (Transfer to higher dimensions)**.** *Let G and $G^{(d)}$ be as above. For every $A \subseteq V$,*

$$\varphi_Z^{G^{(d)}}(A) \leq \varphi_Z^G(A) \leq (d-1)\varphi_Z^{G^{(d)}}(A), \tag{10}$$

and

$$\widehat{\varphi_Z^{G^{(d)}}}(A) \leq \widehat{\varphi_Z^G}(A) \leq (d-1)\widehat{\varphi_Z^{G^{(d)}}}(A). \tag{11}$$

In particular, $\langle \mathcal{H}(G^{(d)}) \rangle = \langle \mathcal{H}(G) \rangle$.

Proof. By Fact 7.2, every homogeneous set of G is homogeneous in $G^{(d)}$, which gives the first inequality in (10). For the second, replace each member of a finite homogeneous cover in $G^{(d)}$ by at most $d-1$ homogeneous sets of G. These inequalities also give the equality of the generated families.

Every measure dominated by $\varphi_Z^{G^{(d)}}$ is dominated by φ_Z^G. Taking the supremum over all such measures gives the first inequality in (11). For the second, let $\mu \leq \varphi_Z^G$. By (10), the measure $\mu/(d-1)$ is dominated by $\varphi_Z^{G^{(d)}}$, so

$$\frac{\mu(A)}{d-1} \leq \widehat{\varphi_Z^{G^{(d)}}}(A).$$

Taking the supremum over all measures $\mu \leq \varphi_Z^G$ gives the second inequality. □

The transfer identifies the random graph ideal with a positive restriction in every higher dimension. In particular, the two local behaviors require no new witnesses.

Theorem 7.4. *For every $d \geq 2$, $\mathcal{R}_d$ has a positive restriction isomorphic to $\mathcal{R}$. Consequently, $\mathcal{R}_d$ is pathological and has both a positive hereditarily pathological restriction and a positive nonpathological restriction isomorphic to $\mathcal{ED}_{\mathrm{fin}}$.*

Proof. Apply Lemma 7.3 to the random graph R. By universality of $R^{(d)}$, the resulting hypergraph has an induced copy in $R^{(d)}$ on a set Y. The lemma and Fact 7.1 give $\mathcal{R}_d \restriction Y \cong \mathcal{R}$. Since $\mathcal{R}$ is proper, Y is positive. The two positive restrictions of $\mathcal{R}$ from Corollary 3.9 transfer through this isomorphism, and Fact 2.1 gives pathology of $\mathcal{R}_d$. □

The comparison of envelopes shows that the least nonpathological extension is proper.

Proposition 7.5. *For every $d \geq 2$, $\widehat{\varphi_Z^{R^{(d)}}}$ is unbounded. In particular,*

$$\mathcal{R}_d \subsetneq \mathsf{FIN}(\widehat{\varphi_Z^{R^{(d)}}}) \subsetneq \mathcal{P}(V(R^{(d)})).$$

The middle ideal is the least proper nonpathological F_σ extension of $\mathcal{R}_d$.

Proof. Use the induced copy constructed in the proof of Theorem 7.4, and write $H = R^{(d)}[Y]$ for this hypergraph. Identify Y with the vertex set of R. Proposition 6.4 makes $\widehat{\varphi_Z^R}$ unbounded, so (11), applied to $G = R$, makes $\widehat{\varphi_Z^H}$ unbounded as well.

We show that every measure $\mu \leq \varphi_Z^H$ extends by zero to a measure dominated by $\varphi_Z^{R^{(d)}}$. Since the copy is induced, $\varphi_Z^H(B) = \varphi_Z^{R^{(d)}}(B)$ for every $B \subseteq Y$. Thus, for every $A \subseteq V(R^{(d)})$,

$$\widetilde{\mu}(A) = \mu(A \cap Y) \leq \varphi_Z^H(A \cap Y) = \varphi_Z^{R^{(d)}}(A \cap Y) \leq \varphi_Z^{R^{(d)}}(A).$$

It follows that $\widehat{\varphi_Z^{R^{(d)}}}(B) \geq \widehat{\varphi_Z^H}(B)$ for every $B \subseteq Y$, and hence $\widehat{\varphi_Z^{R^{(d)}}}$ is unbounded.

Proposition 4.1 now makes $\mathsf{FIN}(\widehat{\varphi_Z^{R^{(d)}}})$ a proper nonpathological F_σ ideal containing $\mathcal{R}_d$. Equality with $\mathcal{R}_d$ would contradict Theorem 7.4. By Theorem 4.4, applied to $\varphi_Z^{R^{(d)}}$, this is the least such extension. □

8 A strengthening: super-pathology

We now study F_σ ideals that have no proper nonpathological F_σ extension. Using the least-extension theorem, we show that $\mathcal{J}_\mathcal{I} \setminus \mathcal{I}$ consists exactly of the positive sets on which the restriction of $\mathcal{I}$ has this property[13]. In particular, every pathological F_σ ideal has such a positive restriction. For graphs with no infinite clique, we express the criterion in terms of fractional coloring. We also apply these results to the random ideals and to universal hypergraphs with a finite complete hypergraph forbidden.

Following [24, Definition 5.5], an ideal $\mathcal{I}$ on X is *super-pathological* if every proper F_σ ideal on X extending it is pathological. In particular, a super-pathological F_σ ideal is pathological. The definition concerns extensions of $\mathcal{I}$, but for F_σ ideals any fixed representation suffices to test super-pathology.

Fact 8.1. *Let $\mathcal{I}$ be a proper F_σ ideal on X, and let φ be an lscsm with $\mathcal{I} = \mathsf{FIN}(\varphi)$. Then $\mathcal{I}$ is super-pathological if and only if $\widehat{\varphi}(X) < \infty$.*

Proof. By Theorem 4.4, $\mathcal{I}$ has no proper nonpathological F_σ extension if and only if $\mathsf{FIN}(\widehat{\varphi}) = \mathcal{P}(X)$. By monotonicity of $\widehat{\varphi}$, this is equivalent to $\widehat{\varphi}(X) < \infty$. □

In the terminology just introduced, Corollary 4.5 says that a proper F_σ ideal is super-pathological if and only if it is not contained in any proper summable ideal. Corollary 4.8 gives another equivalent condition: the ideal is not contained in any matrix ideal.

Mazur constructed an F_σ ideal that is not contained in any proper summable ideal [27, Theorem 1.9]. Thus, his ideal is super-pathological; see also [26, §3.2] and [14, §8].

The same example is hereditarily pathological, because super-pathology passes to positive restrictions.

Fact 8.2 (Positive restrictions)**.** *Super-pathology passes to positive restrictions. Consequently, every super-pathological F_σ ideal is hereditarily pathological, and an F_σ ideal with a positive nonpathological restriction is not super-pathological.*

Proof. Let $\mathcal{I}$ be an F_σ ideal on X. We prove the contrapositive of the first assertion. Suppose $Y \in \mathcal{I}^+$ and $\mathcal{I} \restriction Y$ is not super-pathological. Choose a nonpathological lscsm ψ on Y with $\mathcal{I} \restriction Y \subseteq \mathsf{FIN}(\psi) \subsetneq \mathcal{P}(Y)$, and put $\psi'(A) = \psi(A \cap Y)$ for $A \subseteq X$. Extending each measure dominated by ψ by zero off Y shows that ψ' is nonpathological. It is lower semicontinuous, and $\psi'(X) = \psi(Y) = \infty$. Thus, $\mathsf{FIN}(\psi')$ is a proper nonpathological F_σ extension of $\mathcal{I}$, so $\mathcal{I}$ is not super-pathological. The remaining assertions follow from the definition. □

[13] Recall that $\mathcal{J}_\mathcal{I}$ is the least proper nonpathological F_σ extension of $\mathcal{I}$ when one exists, and the full power set otherwise; see Theorem 4.4.

It follows that Mazur's ideal and the super-pathological ideals in [24, Corollary 5.15] already provide examples for [24, Question 5.33], which asks for a hereditarily pathological F_σ ideal.

We now return to the block constructions. The criterion used to prove pathology also rules out proper nonpathological F_σ extensions.

Proposition 8.3. *Under the hypotheses of Lemma 3.2, $\mathsf{FIN}(\varphi)$ is super-pathological. In particular, the Kneser restriction of Corollary 3.7 and the restrictions extracted in Theorem 5.5 are super-pathological.*

Proof. The first assertion is [26, Theorem 3.8(iii)], with repeated terms allowed by Remark 3.3.

For the Kneser restriction, the block hypotheses were verified in the proof of Theorem 3.6. The restrictions extracted in Theorem 5.5 are induced by disjoint unions of finite graphs with unbounded chromatic numbers, so Lemma 3.1 gives the required block representation. Their uniformly bounded fractional chromatic numbers, $\chi_{\mathrm{f}} \le C$, give independent covering sequences of density at least $1/C$ by Fact 5.2. Every term has chromatic submeasure at most 1, so the block criterion applies. □

The least-extension theorem gives a description of super-pathology for every F_σ ideal, including its positive restrictions.

Theorem 8.4. *Let $\mathcal{I}$ be a proper F_σ ideal on V, and let φ be an lscsm with $\mathcal{I} = \mathsf{FIN}(\varphi)$. Put $\mathcal{J}_\mathcal{I} = \mathsf{FIN}(\widehat{\varphi})$. For every $A \in \mathcal{I}^+$,*

$$\mathcal{I} \restriction A \text{ is super-pathological} \iff \widehat{\varphi}(A) < \infty.$$

Consequently,

$$\mathcal{J}_\mathcal{I} = \mathcal{I} \cup \{A \in \mathcal{I}^+ \mid \mathcal{I} \restriction A \text{ is super-pathological}\}. \tag{12}$$

In particular, every pathological F_σ ideal has a super-pathological positive restriction.

Proof. We first prove the equivalence. Fix $A \in \mathcal{I}^+$ and put $\varphi_A = \varphi \restriction \mathcal{P}(A)$. Then φ_A is an lscsm on A and $\mathsf{FIN}(\varphi_A) = \mathcal{I} \restriction A$. This ideal is proper because $A \in \mathcal{I}^+$.

We also have $\widehat{\varphi_A}(B) = \widehat{\varphi}(B)$ for every $B \subseteq A$. Indeed, a measure on V dominated by φ restricts to one dominated by φ_A. Conversely, a measure $\nu \le \varphi_A$ on A extends by zero to a measure on V satisfying

$$\widetilde{\nu}(B) = \nu(B \cap A) \le \varphi_A(B \cap A) = \varphi(B \cap A) \le \varphi(B) \qquad (B \subseteq V).$$

Taking suprema gives the claimed equality of envelopes.

By Fact 8.1, $\mathcal{I} \restriction A$ is super-pathological if and only if $\widehat{\varphi_A}(A) < \infty$, equivalently when $\widehat{\varphi}(A) < \infty$. This proves the equivalence.

We now prove (12). By definition, $\mathcal{J}_\mathcal{I} \setminus \mathcal{I}$ consists of the sets $A \in \mathcal{I}^+$ with $\widehat{\varphi}(A) < \infty$. The equivalence just proved identifies these as exactly the sets for which $\mathcal{I} \restriction A$ is super-pathological. Since $\mathcal{I} \subseteq \mathcal{J}_\mathcal{I}$, taking the union with $\mathcal{I}$ gives (12).

Finally, suppose that $\mathcal{I}$ is pathological. Then $\mathcal{J}_\mathcal{I} \neq \mathcal{I}$, since equality would provide a nonpathological representation of $\mathcal{I}$. Choose $A \in \mathcal{J}_\mathcal{I} \setminus \mathcal{I}$. Then $A \in \mathcal{I}^+$ and, by the equivalence above, $\mathcal{I} \restriction A$ is super-pathological. □

For graphs without infinite cliques, Theorem 5.3 expresses this description in terms of fractional coloring.

Theorem 8.5. *Let $H = (V, E)$ have no infinite clique and satisfy $\chi(H) = \infty$. Then*

$$\mathcal{I}_H \text{ is super-pathological} \iff \varphi_{\chi_f}(V) < \infty.$$

More generally, $\mathcal{I}_H \restriction A$ is super-pathological for $A \in \mathcal{I}_H^+$ if and only if $\varphi_{\chi_f}(A) < \infty$. In particular, for a graph block ideal with $G = \bigsqcup_{n\in\omega} G_n$,

$$\mathcal{I}_G \text{ is super-pathological} \iff \sup_{n\in\omega} \chi_{\mathrm{f}}(G_n) < \infty.$$

Proof. Since H has no infinite clique, $\mathcal{I}_H = \mathsf{FIN}(\varphi_\chi)$. Apply Theorem 8.4 to φ_χ, using the identity $\widehat{\varphi_\chi} = \varphi_{\chi_f}$ from Theorem 5.3. The last assertion follows from equation (6). □

Combining this characterization with Theorem 5.5 separates the three possible behaviors of a proper graph block ideal. The distinction depends on fractional chromatic bounds for the whole blocks and on uniform chromatic bounds for their induced subgraphs.

Corollary 8.6 (Block classification)**.** *Let $G = \bigsqcup_{n\in\omega} G_n$ be a disjoint union of nonempty finite graphs with $\sup_{n\in\omega} \chi(G_n) = \infty$, and write $V_n = V(G_n)$. Exactly one of the following holds:*

(a) For every $C > 0$ there is $M_C < \infty$ such that, for every $n \in \omega$ and $B \subseteq V_n$,

$$\chi_{\mathrm{f}}(G_n[B]) \le C \implies \chi(G_n[B]) \le M_C.$$

In this case, $\mathcal{I}_G$ is nonpathological.

(b) The uniform bound in (a) fails and $\sup_{n\in\omega} \chi_{\mathrm{f}}(G_n) = \infty$. In this case, $\mathcal{I}_G$ is pathological but not super-pathological, and it has a super-pathological positive restriction.

(c) $\sup_{n\in\omega} \chi_{\mathrm{f}}(G_n) < \infty$. In this case, $\mathcal{I}_G$ is super-pathological.

Proof. We first check that the three alternatives are mutually exclusive and exhaustive. If (a) holds, the fractional chromatic numbers of the whole blocks must be unbounded: otherwise, taking $B = V_n$ would give a uniform bound on $\chi(G_n)$, contrary to the hypothesis. Thus, neither (b) nor (c) holds. If (a) fails, unbounded fractional chromatic numbers give case (b), while bounded fractional chromatic numbers give case (c).

For a disjoint union of finite graphs, condition (a) is equivalent to condition (iii) of Theorem 5.5, which gives nonpathology in that case. In case (b), Theorem 5.5 gives pathology, while Theorem 8.5 rules out super-pathology. The existence of a super-pathological positive restriction follows from Theorem 8.4. Finally, Theorem 8.5 gives super-pathology in case (c). □

Theorem 8.5 also gives a descriptive complexity bound for super-pathology. In the sequence coding of Remark 5.6, restrict attention to block sequences with unbounded chromatic numbers. Within this class, super-pathology is expressed by

$$(\exists m \in \omega)(\forall n \in \omega)\ \chi_{\mathrm{f}}(G_n) \le m.$$

For fixed m and n, the inequality depends only on the finite graph G_n, so it defines a clopen condition in the sequence space. Thus, super-pathology has a Σ^0_2 definition relative to this class. For nonpathology, Remark 5.6 gives a Π^0_3 upper bound.

For finite uniform hypergraphs, define χ and χ_{f} using independent covers and fractional independent covers, respectively, with both values zero on the empty hypergraph. The classification in Corollary 8.6 also holds for disjoint unions $G = \bigsqcup_{n\in\omega} G_n$ of nonempty finite d-uniform hypergraphs, where $d \geq 2$ and $\sup_{n\in\omega} \chi(G_n) = \infty$.

To see this, put $V_n = V(G_n)$ and $\mathcal{I}_G = \langle \mathcal{H}(G)\rangle$, and let φ^G_χ be the covering submeasure of the compact hereditary family $\mathrm{Ind}(G)$. Then $\mathcal{I}_G = \mathsf{FIN}(\varphi^G_\chi)$ and, for $A \subseteq V(G)$,

$$\varphi^G_\chi(A) = \sup_{n\in\omega} \chi(G_n[A \cap V_n]),$$
$$\widehat{\varphi^G_\chi}(A) = \sup_{n\in\omega} \chi_{\mathrm{f}}(G_n[A \cap V_n]).$$

Indeed, every complete set is finite: it either lies in one block or has fewer than d vertices. Replacing complete sets by singletons and combining independent sets from different blocks gives the representation of $\mathcal{I}_G$ and the formula for φ^G_χ. For the envelope, apply Proposition 4.12. The fractional chromatic number of a finite disjoint union is the maximum of the values of its components: clear a common denominator in optimal rational covers, extend each resulting sequence to the maximum of their lengths by adding empty terms, and unite corresponding terms. Restriction gives the reverse inequality, and lower semicontinuity gives the formula on arbitrary sets.

Since $\sup_{n\in\omega} \chi(G_n) = \infty$, the ideal $\mathcal{I}_G$ is proper. By Theorem 4.4, it is nonpathological exactly when $\mathsf{FIN}(\varphi^G_\chi) = \mathsf{FIN}(\widehat{\varphi^G_\chi})$. The uniform bound in Corollary 8.6(a) implies this equality. If the bound fails for some $C > 0$, choose $B_k \subseteq V_{n_k}$ with $\chi(G_{n_k}[B_k]) > k$ and $\chi_{\mathrm{f}}(G_{n_k}[B_k]) \leq C$, using distinct blocks. This is possible because the induced chromatic numbers of finitely many blocks are bounded. Then $A = \bigcup_{k\in\omega} B_k$ is positive and $\widehat{\varphi^G_\chi}(A) \leq C$. Theorem 8.4 shows that $\mathcal{I}_G \restriction A$ is super-pathological, hence hereditarily pathological. The same theorem, applied to $V(G)$, gives super-pathology of $\mathcal{I}_G$ exactly when $\sup_{n\in\omega} \chi_{\mathrm{f}}(G_n) < \infty$. These conclusions give the three cases of Corollary 8.6.

We now return to the random ideals $\mathcal{R}_d$, $d \geq 2$. The transferred Kneser restrictions are super-pathological, and the positive nonpathological restrictions prevent super-pathology of the ambient ideals.

Corollary 8.7. *For every $d \geq 2$, $\mathcal{R}_d$ is pathological but not super-pathological, and has both a super-pathological positive restriction and a nonpathological positive restriction isomorphic to $\mathcal{ED}_{\mathrm{fin}}$.*

Proof. Theorem 7.4 gives pathology and the nonpathological restriction. The latter rules out super-pathology by Fact 8.2. By Proposition 8.3, the Kneser restriction of $\mathcal{R}$ is super-pathological. The positive copy of $\mathcal{R}$ supplied by Theorem 7.4 transfers this restriction to $\mathcal{R}_d$. The canonical envelopes in Theorem 6.5 and Proposition 7.5 also give explicit proper nonpathological extensions of the ambient ideals. □

Corollary 8.7 shows that neither nonpathology nor super-pathology passes from a positive restriction to the ambient ideal. For super-pathology, positivity can be lost when the ideal

is enlarged: a set $Y \notin \mathcal{I}$ may belong to a proper extension $\mathcal{J} \supseteq \mathcal{I}$. Then every subset of Y belongs to $\mathcal{J}$, so $\mathcal{J} \restriction Y = \mathcal{P}(Y)$. Thus, restricting a proper nonpathological extension of $\mathcal{I}$ need not produce a proper extension of $\mathcal{I} \restriction Y$. Indeed, $\mathsf{FIN}(\widehat{\varphi_Z^R})$ contains the Kneser set X, by the proof of Theorem 6.5, although $\mathcal{R} \restriction X$ is super-pathological. Consequently,

$$\mathsf{FIN}(\widehat{\varphi_Z^R}) \restriction X = \mathcal{P}(X).$$

The extension therefore becomes improper on X, so it does not contradict super-pathology of $\mathcal{R} \restriction X$.

The random graph also realizes the intermediate case of the block classification.

Corollary 8.8. *All three cases of Corollary 8.6 occur among the restrictions of $\mathcal{R}$ to positive sets inducing disjoint unions of finite graphs. In particular, if $Y \subseteq \omega$ induces a copy of*

$$\bigsqcup_{k\geq 2} \big(K_k \sqcup \mathrm{KG}(3k-1,k)\big),$$

then $Y \in \mathcal{R}^+$ and $\mathcal{R} \restriction Y$ is pathological but not super-pathological.

Proof. Fact 3.8 gives the nonpathological case. The super-pathological Kneser restriction is supplied by Proposition 8.3. To obtain the intermediate case, use each K_k and each $\mathrm{KG}(3k-1,k)$ as a separate block. The two families supply the two requirements of Corollary 8.6(b).

The Kneser blocks have fractional chromatic numbers below 3 and chromatic numbers $k+1$, by Theorem 3.4 and equation (8). Thus, the uniform bound in Corollary 8.6(a) fails already on these whole blocks. On the other hand, $\chi_{\mathrm{f}}(K_k) = k$, so the sequence of fractional chromatic numbers is unbounded. Corollary 8.6(b) therefore gives the intermediate case. Universality supplies an induced copy on a set Y, and Lemma 3.1 gives its positivity and identifies its ideal with $\mathcal{R} \restriction Y$. □

The Erdős sequences from Section 6 provide another source of this intermediate behavior.

Corollary 8.9. *The graph block ideal of every Erdős sequence, and of each of its infinite subsequences, is pathological but not super-pathological and has a super-pathological positive restriction. The same holds for induced block copies in $\mathcal{R}$.*

Proof. Proposition 6.8 gives pathology and a positive restriction whose block covers satisfy Proposition 8.3. The estimate following Proposition 6.8 shows that the fractional chromatic numbers of the blocks tend to infinity, so Theorem 8.5 rules out super-pathology of the graph block ideal. Every infinite subsequence is again an Erdős sequence, and Lemma 3.1 gives the assertion about induced copies. □

These examples distinguish having a proper summable extension from being an intersection of summable ideals. The ideal $\mathcal{R}$ has a proper summable extension, by Corollaries 8.7 and 4.5. It is not an intersection of summable ideals, by its pathology and Corollary 4.5. The same holds for every ideal in case (b) of Corollary 8.6.

We finally consider the universal hypergraphs obtained by forbidding a finite complete hypergraph. For certain forbidden sizes, their homogeneous-set ideals occur alongside $\mathcal{R}_d$, $d \geq 3$, in the super-pathology assertion of [24, Corollary 5.18]. We show that these ideals are

pathological but not super-pathological for every forbidden size $n \geq d+1$: the transferred Kneser witness gives positive super-pathological restrictions, and a separate finite construction makes the canonical envelopes unbounded. Remark 8.13 explains the relation with [24, Corollary 5.18].

For $d \geq 3$ and $n \geq d+1$, let $K_n^{(d)}$ be the complete d-uniform hypergraph on n vertices, and let $R^d_{n\text{-free}}$ denote the countable universal homogeneous $K_n^{(d)}$-free hypergraph; see [24, §5.1]. Thus, $R^d_{n\text{-free}}$ contains no complete subset of cardinality n and contains an induced copy of every countable d-uniform hypergraph with this property.

Proposition 8.10. *For every $d \geq 3$ and $n \geq d+1$, the homogeneous-set ideal of $R^d_{n\text{-free}}$ has a positive super-pathological restriction. In particular, it is pathological.*

Proof. Let $G = \bigsqcup_{k\geq 2} \mathrm{KG}(3k-1, k)$. This graph has no triangles, since three pairwise disjoint k-subsets would require at least $3k$ points. By Fact 7.2, $G^{(d)}$ is $K_{d+1}^{(d)}$-free, and Lemma 7.3 shows that it has the same homogeneous-set ideal as G. That ideal is proper and super-pathological by Proposition 8.3.

Since $n \geq d+1$, the hypergraph $G^{(d)}$ is also $K_n^{(d)}$-free. Embed it as an induced subhypergraph of $R^d_{n\text{-free}}$. Fact 7.1 identifies its ideal with a positive restriction of the ambient ideal, and Fact 2.1 gives pathology. □

The following lemma gives the finite construction needed for Proposition 8.12. Its proof adapts the probabilistic argument of [2, Theorem 3.1.2] from graphs to d-uniform hypergraphs.

Lemma 8.11. *For every integer $d \geq 3$ and every integer $m \geq 1$, there is a nonempty finite $K_{d+1}^{(d)}$-free d-uniform hypergraph H such that every homogeneous set has cardinality less than $|V(H)|/m$.*

Proof. Fix d and m. For an integer $k \geq 2d$, choose each possible edge on mk vertices independently with probability $1/(mk)$. We seek a choice with no complete subset of cardinality $d+1$ and no independent subset of cardinality k.

Let Z count the complete subsets of cardinality $d+1$. Since each such subset requires $d+1$ edges, its expectation is

$$\mathbb{E}(Z) = \binom{mk}{d+1}\left(\frac{1}{mk}\right)^{d+1} < \frac{1}{(d+1)!}.$$

Let U count the independent subsets of cardinality k. Using $\binom{mk}{k} \leq 2^{mk}$ and $\binom{k}{d} \geq \frac{k^d}{2^d d!}$, we obtain

$$\begin{aligned}\mathbb{E}(U) &= \binom{mk}{k}\left(1-\frac{1}{mk}\right)^{\binom{k}{d}} \\ &\leq \exp\left(mk\ln 2 - \frac{1}{mk}\binom{k}{d}\right) \\ &\leq \exp\left(mk\ln 2 - \frac{k^{d-1}}{2^d m d!}\right).\end{aligned}$$

The last expression tends to zero as $k \to \infty$: since $d \geq 3$ and d, m are fixed, the negative term in the exponent grows at least quadratically, whereas the positive term grows linearly.

Thus, $\mathbb{E}(Z+U) < 1$ for all sufficiently large k. Since $Z+U$ is a nonnegative integer, some choice has $Z = U = 0$. The resulting hypergraph is $K^{(d)}_{d+1}$-free; its complete sets have cardinality at most $d < k$, and its independent sets have cardinality less than k. Hence, every homogeneous set has cardinality less than $k = |V(H)|/m$. □

Proposition 8.12. *Let $d \geq 3$ and $n \geq d+1$, and put $H = R^d_{n\text{-free}}$. Then $\widehat{\varphi^H_Z}$ is unbounded. Consequently, $\mathsf{FIN}(\widehat{\varphi^H_Z})$ is the least proper nonpathological F_σ extension of $\langle \mathcal{H}(H) \rangle$. In particular, $\langle \mathcal{H}(H) \rangle$ is not super-pathological.*

Proof. We will construct measures dominated by φ^H_Z whose values on $V(H)$ are arbitrarily large. For each integer $m \geq 1$, take a hypergraph supplied by Lemma 8.11. Since $n \geq d+1$, it is also $K^{(d)}_n$-free. By universality, realize it as the subhypergraph induced on a finite set $W_m \subseteq V(H)$, and define

$$\mu_m(A) = \frac{m|A \cap W_m|}{|W_m|} \qquad \big(A \subseteq V(H)\big).$$

Every homogeneous set meets W_m in fewer than $|W_m|/m$ vertices, so its μ_m-measure is at most 1. A cover of A by r homogeneous sets therefore gives $\mu_m(A) \leq r$, and hence $\mu_m \leq \varphi^H_Z$. Consequently,

$$\widehat{\varphi^H_Z}(W_m) \geq \mu_m(W_m) = m.$$

Since m is arbitrary, $\widehat{\varphi^H_Z}$ is unbounded. Theorem 4.4, applied to φ^H_Z, now identifies $\mathsf{FIN}(\widehat{\varphi^H_Z})$ as the least proper nonpathological F_σ extension of $\langle \mathcal{H}(H) \rangle$. □

Remark 8.13 (On [24, Corollary 5.18]). We use the numbering of arXiv:2501.15643v1 for references to [24]. For each $d \geq 3$, the construction in [24, Corollary 5.15] supplies a d-uniform hypergraph that generates a proper super-pathological ideal and forbids $K^{(d)}_{n(d)}$ for some $n(d)$. Universality and Fact 7.1 identify this ideal with a positive restriction of the homogeneous-set ideals of $R^{(d)}$ and $R^d_{n\text{-free}}$ for every $n \geq n(d)$. These restrictions are also hereditarily pathological by Fact 8.2.

The final step in the proof of [24, Corollary 5.18] passes from super-pathology of a positive restriction to super-pathology of the ambient ideal. This implication fails: a proper extension of the ambient ideal may contain the entire set supporting the restriction. Applying Fact 2.1 to the positive restrictions instead gives pathology of the ambient ideals.

Corollary 8.7 shows that $\mathcal{R}_d$ is pathological but not super-pathological. For the ideals of $R^d_{n\text{-free}}$, Propositions 8.10 and 8.12 give both conclusions for every $n \geq d+1$. Thus, the transferred Kneser witness extends the pathology conclusion beyond the range $n \geq n(d)$ supplied by the construction in [24].

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