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Riemannian three-manifolds as hypersurfaces of Euclidean four-space: the realizability operator, its symbol, and its linearisation at the round sphere

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This paper supersedes, for local questions, the author's earlier mathematics papers 1, 2 and 4 (Thrussell 2026a, b, d). Appendix C lists every statement that changes. A companion paper treats the global questions.

The verification scripts for this paper (tests 325–327, 329–344, 354, 625–627, 629 and 640, their library and their run log) are deposited with it at Zenodo, DOI 10.5281/zenodo.23765514. The author's earlier work is archived under concept DOI 10.5281/zenodo.21264118. Appendix V indexes every displayed result against the script that reproduces it, and discloses the use of large-language-model systems.

Abstract

For a Riemannian three-manifold (Σ, g) with negative Einstein endomorphism $\mathbb{M} = -G$, the Gauss equation of a hypersurface of E^4 reads $\mathbb{M} = \operatorname{adj} K$. On $\{\det \mathbb{M} > 0\}$ it is solved identically by $\tilde{K} = \sqrt{\det \mathbb{M}}\, \mathbb{M}^{-1}$. Realizability with nondegenerate second fundamental form is therefore one equation, $F(g) = \operatorname{Cod}(\tilde{K}(g)) = 0$. This reduction is classical (Beez, Killing, Thomas), and the zero set of F has been characterized intrinsically (Rivertz; Agaoka and Hashinaga). We study F as a nonlinear operator on metrics.

- **The symbol.** At every nondegenerate point the principal symbol of F has rank two. Its kernel is the diffeomorphism directions plus the normal direction K: exactly the symbol of the linearised immersions, at every covector and on both branches. So F is injective modulo the linearised immersions everywhere, and is nowhere elliptic modulo diffeomorphisms alone.
- **Transverse-traceless tensors.** The symbol is injective on transverse-traceless tensors at ξ iff $\xi^{\mathsf{T}}(\sigma_1 g - K)\xi \neq 0$. So the transverse-traceless slice serves exactly where the first Newton tensor is definite, and a slice adapted to K serves everywhere.
- **The count.** The nine Codazzi components satisfy four identities, which leaves five conditions per point, at every nondegenerate shape operator. Cartan's test holds there with equality at order three, and the formal solutions of the symbol are exactly the linearised immersions.
- **Meagreness.** On a closed three-manifold the zero set of F, which contains the realizable metrics, is nowhere dense in the nondegenerate domain.

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- **The round sphere.** On transverse-traceless tensors the linearisation is $\mathrm{Cod} \circ \frac{1}{2}(-\Delta + 2)$, with singular values $n(n^2-1)/\sqrt{2}$ at level n, and the bound $\|DF\,h\|^2 \geq 288\|h\|^2$. At the round sphere DF is the linearised Cotton operator, and its dual on TT, with eigenvalues $\pm n(n^2-1)$, is Branson's third-order conformal intertwiner. The regularised count of transverse-traceless modes is $10 = \dim SO(4,1)$. The straight path along a kernel direction leaves the realizable set at second order, while the radial lift stays in it.
- **Shells.** For metrics foliated by round 2-spheres we solve in closed form the first-order problem for deformations of the radial coefficient by a shell harmonic. We write down the metrics of all hypersurfaces swept by round 2-spheres: the radial coefficient is $H^2 + (\alpha + \Phi\, b \cdot \hat{n})^2$, with b a free tilt of the spheres' 3-planes. So the $\ell = 1$ first-order direction is realized exactly by a tilt, with no in-plane drift of the centres. An in-plane drift realizes it together with a shear of the shell labels.
- **The rest.** We treat the degenerate locus through two polynomial forms of the equation. One of them is Rivertz's six equalities in tensor form: its six components span the same space of polynomials. We give Gauss–Codazzi as the flatness of two real $\mathfrak{su}(2)$ connections, the connection form of Lawn and Roth's two-spinor characterization; on the nondegenerate domain, realizability is equivalent to a single parallel spinor.

Every displayed result is reproduced by a script; Appendix V indexes them.

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1. Introduction

The question. Which Riemannian three-manifolds are hypersurfaces of Euclidean four-space? Isometric embedding has two classical regimes. In the C^1 category, codimension one already suffices locally for every metric, and globally whenever there is a short immersion (Nash 1954; Kuiper 1955). In the C^2 category the second fundamental form exists and is constrained by the intrinsic curvature through the Gauss and Codazzi equations. This paper is about the rigid regime, in the one dimension where the constraint takes a special form.

Why dimension three. For a hypersurface $\Sigma^n \subset E^{n+1}$ the Gauss equation $R_{abcd} = K_{ac}K_{bd} - K_{ad}K_{bc}$ has $n^2(n^2-1)/12$ components and the second fundamental form has $n(n+1)/2$. At $n = 2$ there is one equation, $\det K = \kappa$, for three unknowns. The problem is then a single Monge–Ampère equation for a height function, determined rather than overdetermined:

- analytic metrics always realize locally (Janet 1926; Cartan 1927);
- smooth metrics realize locally under sign conditions on the curvature (Lin 1985, 1986);
- local realization can fail, but only by delicate constructions (Pogorelov 1971 at finite regularity; Nadirashvili–Yuan 2002, smooth metrics with no C^3 realization).

At $n = 3$ the counts agree, six and six. The Gauss equation is then an algebraic system for K that can be solved in closed form, and the Codazzi equation becomes a condition on the metric alone. For $n \geq 4$ the Gauss equation overdetermines K. Codazzi then follows from Gauss when the second fundamental form has rank at least four (Thomas 1936), so the criterion carries no separate content (§10).

The reduction. In dimension three the Gauss equation contracts, with no loss, to a statement about the Einstein tensor. Writing $\mathbb{M} = -G$ as an endomorphism,

$$\mathbb{M} = \mathrm{adj}\, K, \tag{1.1}$$

the adjugate of the shape operator (§2). Where $\det \mathbb{M} > 0$ the tensor

$$\tilde{K} = \sqrt{\det \mathbb{M}}\, \mathbb{M}^{-1} \tag{1.2}$$

satisfies $\operatorname{adj} \tilde{K} = \mathbb{M}$ for every metric, realizable or not. When $\det K \neq 0$, the second fundamental form of any realization is $\pm\tilde{K}$. Realizability with nondegenerate second fundamental form is therefore the vanishing of the **realizability operator**

$$F(g) = \operatorname{Cod}(\tilde{K}(g)), \qquad \operatorname{Cod}(K)_{abc} = \nabla_a K_{bc} - \nabla_b K_{ac}. \tag{1.3}$$

F is a third-order nonlinear operator on metrics. Its values lie in the nine-dimensional bundle of tensors antisymmetric in their first two indices.

What is classical. The determination of K by the metric is the rigidity theorem of Beez (1876) and Killing (1885): a hypersurface of E^{n+1} whose second fundamental form has rank at least three is locally rigid. Thomas (1936) characterised Riemannian spaces of class one algebraically. His paper already contains, in other notation:

- the double-adjugate identity $\operatorname{adj}\operatorname{adj} K = (\det K)K$ (his (9.3));
- the clearing of the square root by a factor $(\det \mathbb{M})^{3/2}$ (his §10);
- the continuity of the sign field on simply connected domains (§9);
- the sufficiency of his conditions over simply connected spaces of type at least three (§11).

Further work:

- Allendoerfer (1939) extended rigidity to higher codimension, and Berger, Bryant and Griffiths (1983) sharpened the generic picture.
- Vilms (1977) identified the curvature operator on Λ^2 and its determinant condition.
- Fodor (2019) reached the same sign structure through the logarithm of the curvature operator.
- In the literature on metrics of embedding class one, the reduction to Gauss-then-Codazzi goes back to Eiesland (1925). Karmarkar (1948) gave the eliminant for spherically symmetric metrics, and Pandey and Sharma

  (1982) showed that it needs a nondegeneracy hypothesis, the analogue of $\det \mathbb{M} > 0$.
- Rivertz (1999) found, beyond the sign condition, six polynomial equalities in the curvature and its first derivative that a hypersurface metric must satisfy. Agaoka and Hashinaga (2022) proved that they suffice. A simply connected three-manifold whose curvature determinant $|\tilde{R}|$, which is $\det \mathbb{M}$ up to a positive factor, vanishes nowhere is locally a hypersurface of $\mathbb{R}^4$ iff $|\tilde{R}| > 0$ and the six equalities hold. They solve Gauss by 2×2 minors divided by $\sqrt{|\tilde{R}|}$, which is (1.2) in components. That is an intrinsic characterization of the zero set of F on $\mathcal{U}$. They do not study the operator: its symbol, involutivity, linearisation or degenerate locus. Proposition 8.1 gives a polynomial form of F with the bidegree of Rivertz's equalities, and Theorem 8.6 shows that its six components and Rivertz's six equalities span the same space.

In Lorentzian signature the tensor (1.2) appears in Bryan, Ivaki and Scheuer (2019), with a Minkowski ambient and negative curvature. Their result: $\sqrt{\det \mathcal{E}}\, \mathcal{E}^{-1}$ is Codazzi iff the universal cover embeds in $\mathbb{R}^{3,1}$ as a locally convex spacelike hypersurface. They also show that the Einstein volume is monotone along the cross curvature flow of Chow and Hamilton (2004).

The identity (1.1) is the case $k = 2$ of the classical fact that the Einstein tensor of a hypersurface of flat space is minus its second Newton transformation (Reilly 1973). Equivalently, it is the intrinsic character of the even-order elementary symmetric functions (Labbi 2007). Codazzi tensors on

space forms are classified locally by Ferus (1981); see also Derdziński and Shen (1983), Derdziński's appendix in Besse (1987), and Merton (2013).

What this paper does. We take the reduction as given and study the operator F on the space of metrics: its symbol and the slices on which it is injective, how overdetermined it is, the size of its zero set, its linearisation at the round sphere, its behaviour on a non-round family where it can be solved completely, its degenerate locus, and a chiral form. Along the way we write down the metrics of all hypersurfaces swept by round 2-spheres. This shows that a sphere in E^4 can realize an $\ell = 1$ change of its radial coefficient by tilting as well as by translating. Results credited in the text to *an independent machine-assisted audit (October 2026)* are that audit's own; Appendix V describes it.

Main results. Throughout, $\mathcal{U}$ denotes the open set of metrics with $\det \mathbb{M} > 0$.

- **Theorem A (§3, the reduction).**
  - The gate $\det \mathbb{M} = (\det K)^2 \geq 0$.
  - The Gauss equation is solved by construction, and the branch is $\tilde{K} = \operatorname{sign}(\det K)\, K$.
  - $\mathbb{M} > 0$ exactly on the locally convex branch.
  - Global sufficiency on simply connected Σ. The round $\mathbb{RP}^3$ shows that the hypothesis is needed.
  - A complete, two-ended, non-convex example on the indefinite branch.
- **Theorem B and Corollary 4.2 (§4, the symbol).**
  - At every nondegenerate point and every covector $\xi \neq 0$, $\sigma(\xi)$ has rank exactly two, with kernel the gauge directions plus $\operatorname{span}\{PKP\}$, P the projection off ξ.
  - That kernel is exactly the symbol of the linearised immersions $\mathcal{L}_X g - 2\varphi K$. So F is injective modulo the immersions at every covector, on a slice adapted to K, and is nowhere elliptic modulo diffeomorphisms alone.
  - The symbol is injective on transverse-traceless tensors iff $\xi^{\mathsf{T}} T_1 \xi \neq 0$, where $T_1 = \sigma_1 g - K$ is the first Newton tensor. So the TT slice serves where T_1 is definite: everywhere on the convex branch, and on the indefinite branch iff $k_1 + k_2$ and $k_2 + k_3$ share a sign.
- **Theorem C (§4, involutivity).**
  - The nine components of F satisfy four identities for every metric. They are independent, and $F = 0$ is exactly five conditions per point, at every nondegenerate shape operator.
  - The symbol passes Cartan's test at order three with equality, $78 = 55 + 19 + 4$, and the first prolongation adds no integrability condition. So $F = 0$ is involutive, at every nondegenerate shape operator.
  - The formal solutions of the symbol are exactly the linearised immersions. The general analytic solution depends on four functions of three variables: the immersion itself.
- **Theorem D (§5, meagreness).** On a closed three-manifold the zero set of F, which contains the realizable metrics, is closed with empty interior in $\mathcal{U}$, on both branches, and so nowhere dense.
- **Theorem E (§6, the round sphere).**
  - $\ker DF_\gamma$ is the conformal plus Lie directions, a sum that is not direct.
  - On transverse-traceless tensors, $DF_\gamma = \operatorname{Cod} \circ \frac{1}{2}(-\Delta + 2)$, and $\|DF_\gamma h\|^2 = \frac{1}{2}\langle h, (-\Delta + 3)(-\Delta + 2)^2 h\rangle \geq 288\|h\|^2$.
  - At level n the singular values are $n(n^2-1)/\sqrt{2}$. These are the multipliers of Branson's P_3 on functions, level by level, because DF_γ is the linearised Cotton operator at γ, conformally covariant.
- **Remark 6.1 and Proposition 6.3 (§6).** $2\star DF_\gamma$ is Branson's third-order intertwiner on trace-free tensors, with eigenvalues $\pm n(n^2-1)$ on the two chiral summands of TT, and

closes the exact sequence $0 \to \mathfrak{so}(4,1) \to \mathfrak{X} \to \mathrm{Sym}^2_0 \to TT \to 0$. The regularised count of transverse-traceless modes is $\zeta(0) = 10 = \dim SO(4,1)$, a theorem: the trace-free tensors count zero and the conformal Killing fields are the difference. Along the straight conformal path, F leaves its zero set at second order, while the radial lift stays in it.

- **Theorems F and G, Proposition 7.4 (§7, a worked non-round family).**
  - On shell metrics, for deformations of the radial coefficient by a shell harmonic, the first-order problem is solved completely: kernel $\ell \in \{0, 1\}$ for that family.
  - The metrics of hypersurfaces swept by round 2-spheres are written down exactly. Their radial coefficient is $H^2 + (\alpha + \Phi b \cdot \hat{n})^2$: an in-plane speed tied to the chart drift, plus a free tilt. The tilt realizes the $\ell = 1$ kernel exactly, with no drift. The drift realizes it together with an $\ell = 1$ shear of the shell labels.
  - Nested shells are a future-directed timelike curve in $\mathbb{R}^{1,3}$.
  - At second order on the round base, the $\ell = 1$ obstruction is pure $\ell = 2$. It is removed by the tilt or by the drift, on interior shells. On every base with $\Phi' \neq 0$ the drift's second-order residual is a multiple of $(1 - s)^2$: a double root (Theorem 7.7).
- **§8, the degenerate locus:** two polynomial forms of F, one of them Rivertz's equalities in tensor form, and what they see there.
- **§9, the chiral form:** Gauss and Codazzi as the flatness of two $\mathfrak{su}(2)$ connections, and realizability as a single parallel spinor.
- **§§10–12 (not load-bearing):** why $n = 3$, regularity, and higher dimensions.

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2. Preliminaries and conventions

2.1 Hypersurfaces

Let $x : \Sigma^3 \to E^4$ be an immersion with unit normal N, induced metric g and second fundamental form $K_{ab} = -N \cdot \partial_a \partial_b x$, signed so that the unit sphere with outward normal has $K = \gamma$. The structure equations of the flat ambient space are

$$\text{(Gauss)} \quad R_{abcd} = K_{ac}K_{bd} - K_{ad}K_{bc}, \qquad \text{(Codazzi)} \quad \mathrm{Cod}(K) = 0. \tag{2.1}$$

Conversely, on a simply connected Σ a pair (g, K) satisfying both is realized by an immersion, unique up to a rigid motion (the fundamental theorem of hypersurfaces; Mardare 2003, 2005 for low regularity).

We say that a metric g of class C^k, $k \geq 3$, is **realizable** if it admits an isometric immersion of class C^3 into E^4. Then K is C^1 and both equations hold classically. Below this regularity the questions change (§11).

2.2 Newton tensors

With σ_k the elementary symmetric functions of the eigenvalues of the shape operator, the Newton transformations are

$$T_0 = g, \qquad T_1 = \sigma_1 g - K, \qquad T_2 = \sigma_2 g - \sigma_1 K + K^2.$$

By Cayley–Hamilton, $T_2 = \mathrm{adj}\, K$ in dimension three. In an orthonormal eigenframe $K = \mathrm{diag}(k_1, k_2, k_3)$ the transformation T_1 is diagonal with entries the pairwise sums $k_2 + k_3$, $k_1 + k_3$, $k_1 + k_2$. Likewise T_2 has the pairwise products.

2.3 The Gauss equation in dimension three

Contracting (2.1) gives $\mathrm{Ric} = \sigma_1 K - K^2$ and $R = \sigma_1^2 - |K|^2 = 2\sigma_2$, so

$$G = \mathrm{Ric} - \tfrac{1}{2} R g = -(\sigma_2 g - \sigma_1 K + K^2) = -T_2.$$

Writing $\mathbb{M}$ for the endomorphism $-G$ (indices raised with g),

$$\mathbb{M} = T_2 = \operatorname{adj} K. \tag{2.2}$$

In dimension three the Riemann tensor is determined by the Ricci tensor, so (2.2) is equivalent to the full Gauss equation, not merely a contraction of it.

2.4 The adjugate

For a 3×3 symmetric K:

$$\det \operatorname{adj} K = (\det K)^2, \qquad \operatorname{adj} \operatorname{adj} K = (\det K)\, K. \tag{2.3}$$

For invertible K the derivative of the adjugate is

$$D \operatorname{adj}_K(X) = \det K \left[(\operatorname{tr} K^{-1} X)\, K^{-1} - K^{-1} X K^{-1} \right]. \tag{2.4}$$

In an eigenframe it acts on diagonal X by the matrix $\partial(k_j k_l)/\partial k_i$, with determinant $2k_1k_2k_3$, and on the off-diagonal pair (ij) by $-k_l$, $l \neq i, j$. Hence

$$\det D \operatorname{adj}_K = -2(\det K)^2,$$

and $D \operatorname{adj}_K$ is invertible iff $\det K \neq 0$, whatever the signs of the k_i. In particular adj is a local diffeomorphism of symmetric matrices near every invertible K.

2.5 The operator and its domain

For $k \geq 2$ and $\alpha \in (0,1)$ let

$$\mathcal{U}^{k,\alpha} = \{ g \in C^{k,\alpha} \text{ metrics on } \Sigma : \det \mathbb{M}(g) > 0 \text{ everywhere} \},$$

an open subset (on compact Σ). For $k \geq 3$, (1.2) and (1.3) define a smooth map

$$F : \mathcal{U}^{k,\alpha} \to C^{k-3,\alpha}(\Lambda^2 T^*\Sigma \otimes T^*\Sigma).$$

It is natural: $F(\phi^* g) = \phi^* F(g)$ for every diffeomorphism ϕ. On non-compact Σ the same holds locally. We write $\mathcal{U}$ when the regularity is immaterial.

2.6 Conventions

- Curvature: $[\nabla_a, \nabla_b] V^c = R^c{}_{dab} V^d$ and $R_{ab} = R^c{}_{acb}$. For a $(0,2)$-tensor, $[\nabla_a, \nabla_b] h_{cd} = -R^e{}_{cab} h_{ed} - R^e{}_{dab} h_{ce}$.
- The unit round metric γ on S^3 has $\mathrm{Ric} = 2\gamma$, $R = 6$, $G = -\gamma$, $\mathbb{M} = \mathrm{Id}$ and $\tilde{K} = \gamma$.
- **One labelling of the S^3 spectra.** Scalar harmonics at level $n \geq 1$ have $-\Delta$-eigenvalue $n^2 - 1$ and multiplicity n^2. Transverse-traceless (TT) tensor harmonics exist for $n \geq 3$, with rough-Laplacian eigenvalue $\mu_n = n^2 - 3$ and multiplicity $2(n^2 - 4)$, so $\mu = 6, 13, 22, \ldots$. The first scalar level above the constants, $n = 2$, is the four first harmonics (eigenvalue 3).
- The spectrum of the unit S^2, which governs the shells of §7, is written $\Lambda = \ell(\ell+1)$ and never shares a glyph with the above.
- A symmetric 2-tensor is **TT at** ξ if $h\xi = 0$ and $\operatorname{tr} h = 0$. This is the pointwise, symbol-level version, a two-dimensional space for each $\xi \neq 0$.

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3. The algebraic reduction

Theorem A. *Let (Σ^3, g) be a Riemannian three-manifold of class C^k, $k \geq 3$, and $\mathbb{M} = -G$.*

1. *(Gate.) If g is realizable then $\det \mathbb{M} = (\det K)^2 \geq 0$ everywhere. For constant curvature κ, $\mathbb{M} = \kappa \operatorname{Id}$ and $\det \mathbb{M} = \kappa^3$, so a hyperbolic metric admits no realization on any open set.*
2. *(Gauss by construction.) On $\{\det \mathbb{M} > 0\}$ the tensor $\tilde{K} = \sqrt{\det \mathbb{M}}\, \mathbb{M}^{-1}$ satisfies $\operatorname{adj} \tilde{K} = \mathbb{M}$, so $(g, \tilde{K})$ satisfies the Gauss equation, for every metric.*
3. *(Branch.) If (g, K) satisfies the Gauss equation at a point with $\det K \neq 0$, then $K = \operatorname{sign}(\det K)\, \tilde{K}$ there. Consequently, near a point where $\det \mathbb{M} > 0$, g is realizable with nondegenerate second fundamental form iff $F(g) = 0$ there.*
4. *(Global sufficiency.) Let Σ be connected and simply connected with $\det \mathbb{M} > 0$ everywhere. Then g is realizable iff $F(g) = 0$. The realization is unique up to a rigid motion, and for one choice of normal its second fundamental form is $\tilde{K}$. For general connected Σ with $\det \mathbb{M} > 0$ everywhere, $F(g) = 0$ gives an immersion of the universal cover that is equivariant under a homomorphism $\rho : \pi_1(\Sigma) \to \operatorname{Isom}(E^4)$. Then g is realizable iff ρ is trivial.*
5. *(Convexity.) $\mathbb{M}$ is positive definite iff $\tilde{K}$ is definite iff a realization is locally strictly convex. $\mathbb{M}$ is never negative definite on $\{\det \mathbb{M} > 0\}$. If $\det \mathbb{M} > 0$ and $\mathbb{M}$ is indefinite, then $\mathbb{M}$ has signature $(+,-,-)$, and $\tilde{K}$ has exactly one principal value of sign opposite to the other two.*

Proof. $\mathbb{M}$ is self-adjoint with respect to g, so it diagonalises in an orthonormal frame.

(1) This is (2.2) together with (2.3).

(2) Let $\mathbb{M}$ have eigenvalues m_1, m_2, m_3, real, nonzero, with $m_1 m_2 m_3 > 0$; no sign condition on the individual m_i is needed. Then $\tilde{K}$ has eigenvalues $\kappa_i = \sqrt{m_1 m_2 m_3}/m_i$, and

$$\kappa_j \kappa_l = \frac{m_1 m_2 m_3}{m_j m_l} = m_i \qquad (\{i,j,l\} = \{1,2,3\}),$$

which is $\operatorname{adj} \tilde{K} = \mathbb{M}$ in the common eigenframe.

(3) If $\mathbb{M} = \operatorname{adj} K$ with $K = \operatorname{diag}(k_i)$ and $\det K \neq 0$, then $m_i = k_j k_l$, $\det \mathbb{M} = (\det K)^2$, and $\kappa_i = |\det K|/k_j k_l = |\det K|\, k_i / \det K$. So $\tilde{K} = \operatorname{sign}(\det K)\, K$. If g is realized near the point with $\det K \neq 0$, then K is continuous with $\det K \neq 0$, so the sign is locally constant and $\operatorname{Cod} \tilde{K} = \pm \operatorname{Cod} K = 0$. Conversely, $F(g) = 0$ makes $(g, \tilde{K})$ a Gauss–Codazzi pair on a simply connected neighbourhood, which the fundamental theorem realizes.

(4) If $F(g) = 0$, then $(g, \tilde{K})$ is a Gauss–Codazzi pair on Σ, by (2). The fundamental theorem gives an immersion with second fundamental form $\tilde{K}$, unique up to rigid motion. Conversely, let x realize g. Then $(\det K)^2 = \det \mathbb{M} > 0$, so $\det K$ never vanishes. By (3), $K = \varepsilon \tilde{K}$ with $\varepsilon = \operatorname{sign} \det K$ continuous, hence constant on connected Σ. So $F(g) = 0$, and reversing the normal if $\varepsilon = -1$ gives second fundamental form $\tilde{K}$.

For general Σ, apply this on the universal cover. Deck transformations are isometries of the lifted metric, so by uniqueness each composes the immersion with a rigid motion, and this defines ρ. An immersion of Σ lifts to one with trivial ρ. Conversely, trivial ρ means the immersion descends.

(5) $m_i = k_j k_l > 0$ for all i iff the k_i share a sign. Definiteness of the second fundamental form is local strict convexity. If $\mathbb{M}$ were negative definite, $\det \mathbb{M} < 0$. If exactly one eigenvalue of K has the opposite sign, say $k_1 < 0 < k_2, k_3$, then $m_1 = k_2 k_3 > 0$ and $m_2, m_3 < 0$. □

Remark 3.1 (relation to Beez–Killing and Thomas). Part (3) is the Beez–Killing determination in closed form: in dimension three the system $k_j k_l = m_i$ has at most the two solutions $\pm K$. Part (2)

is a different kind of statement. $\tilde{K}$ is defined for every metric in $\mathcal{U}$ and satisfies Gauss whether or not the metric is realizable. Rigidity says a solution is unique if it exists; part (2) says a canonical candidate always exists. Part (4) is Thomas's §11 in this notation. His continuity argument for the sign field (§9) is not needed here, because (1.2) involves no choice of sign.

Remark 3.2 (simple connectivity is needed). The round metric on $\mathbb{RP}^3 = S^3/\{\pm 1\}$ has $\mathbb{M} = \mathrm{Id}$ and $F = 0$. The developing immersion of its universal cover is the unit sphere, and the deck transformation acts on it by $\rho = -\mathrm{Id} \in O(4)$, which is not trivial. So the round $\mathbb{RP}^3$ is not realizable, although it satisfies $F = 0$ and the gate at every point.

Example 3.3 (the indefinite branch: the Flamm hypersurface). On $(2m, \infty) \times S^2$ take

$$g = \Big(1 - \frac{2m}{\rho}\Big)^{-1} d\rho^2 + \rho^2 \hat{\gamma}, \qquad \hat{\gamma} \text{ the unit round metric on } S^2.$$

The hypersurface of revolution $\{(\rho\,\omega, z) : \omega \in S^2\} \subset \mathbb{R}^3 \times \mathbb{R}$ with profile $z^2 = 8m(\rho - 2m)$ realizes it. Its principal curvatures are

$$\kappa_{\mathrm{mer}} = -\frac{\sqrt{2m}}{2\rho^{3/2}}, \qquad \kappa_{\mathrm{sph}} = \frac{\sqrt{2m}}{\rho^{3/2}} \text{ (twice)},$$

so $\mathbb{M} = \operatorname{adj} K$ has eigenvalues $(2m/\rho^3, -m/\rho^3, -m/\rho^3)$. Thus $\det \mathbb{M} = 2m^3/\rho^9 > 0$ while $\mathbb{M}$ is indefinite.

- **Completeness.** The sheet $\rho > 2m$ is not complete: the distance to $\rho = 2m$ is finite. The complete hypersurface is the two-ended surface of revolution with $\rho(z) = 2m + z^2/8m$, $z \in \mathbb{R}$, diffeomorphic to $\mathbb{R} \times S^2$. It carries the metric above on each sheet. On the throat $z = 0$ it has $\det \mathbb{M} = 1/256m^6 > 0$, so $\det \mathbb{M} > 0$ everywhere.
- **What it shows.** It satisfies $F = 0$ and is not convex. So the convex branch is a proper part of the nondegenerate realizable set, and the gate (1) is necessary but not sufficient.
- **The TT slice.** Its first Newton tensor is $T_1 = \mathrm{diag}(2k, k/2, k/2)$ with $k = \kappa_{\mathrm{sph}} > 0$, which is definite. So the TT slice serves along it (§4).

Remark 3.4 (the degenerate branch). Where $\det \mathbb{M} = 0$ the equivalence in (3) fails. The Gauss equation with $\det K = 0$ has large solution families, with flat metrics the extreme case (Hartman and Nirenberg 1959). This is the regime of rank at most two, where Beez–Killing rigidity also fails and the deformable hypersurfaces of Sbrana (1909) and Cartan (1916) live. Section 8 treats it.

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4. The principal symbol

Throughout this section $g \in \mathcal{U}$, $p \in \Sigma$, and we work in a g-orthonormal frame at p. We write K for $\tilde{K}(g)(p)$, which is invertible, and σ_1, σ_2 for its elementary symmetric functions. We use the *formal* symbol, which replaces each ∂_a by ξ_a. It differs from the usual symbol by a power of i, which changes no rank and no kernel.

4.1 The symbol and its kernel

On $\mathcal{U}$ the map $\mathbb{M} \mapsto \tilde{K}$ is a local inverse of adj, so $D\tilde{K} = (D \operatorname{adj}_K)^{-1} \circ D\mathbb{M}$. $\mathbb{M}$ is of second order in g and Cod adds one derivative. Every other term of DF_g, including the variation of the connection

inside Cod, has order at most two. Hence the third-order symbol at $\xi \in T^*_p\Sigma \setminus 0$ is

$$\sigma(\xi) = \sigma(\mathrm{Cod})(\xi) \circ (D\,\mathrm{adj}_K)^{-1} \circ \sigma_2(D\mathbb{M})(\xi), \qquad \sigma(\mathrm{Cod})(\xi)L = \xi \wedge L, \quad (\xi \wedge L)_{abc} = \xi_a L_{bc} - \xi_b L_{ac}. \tag{4.1}$$

It maps $\mathrm{Sym}^2 T^*_p$, of dimension 6, to $\Lambda^2 T^*_p \otimes T^*_p$, of dimension 9.

Fix $|\xi| = 1$ and let $P = g - \xi \otimes \xi$ be the projection onto $\xi^\perp$. For Y symmetric and supported on $\xi^\perp$, write $\mathrm{adj}_2 Y = (\mathrm{tr}\, Y)P - Y$, the adjugate of Y as a 2×2 matrix on the plane. It is an involution of $\mathrm{Sym}^2(\xi^\perp)$. Put

$$Q_\xi = PKP,$$

the compression of K to $\xi^\perp$. Every $h \in \mathrm{Sym}^2 T^*_p$ splits uniquely as $h = PhP + \xi \odot v$, with $\xi \odot v = \frac{1}{2}(\xi \otimes v + v \otimes \xi)$.

Lemma 4.1 (the three factors).

1. $\sigma_2(D\mathbb{M})(\xi)h = -\frac{1}{2}\,\mathrm{adj}_2(PhP)$. It vanishes on $\{\xi \odot v\}$ and maps $\mathrm{Sym}^2(\xi^\perp)$ bijectively to itself.
2. For every symmetric K,

$$D\,\mathrm{adj}_K(X) = KX + XK - \sigma_1 X - (\mathrm{tr}\, X)K + (\sigma_1\,\mathrm{tr}\, X - \mathrm{tr}\, KX)\, g,$$

   and $D\,\mathrm{adj}_K(\xi \otimes \xi) = \mathrm{adj}_2(Q_\xi)$.
3. On symmetric L, $\xi \wedge L = 0$ iff $L \in \mathrm{span}\{\xi \otimes \xi\}$.

Proof. (1) The linearised Ricci tensor has symbol $\frac{1}{2}(\xi \otimes h\xi + h\xi \otimes \xi - h - (\mathrm{tr}\, h)\,\xi \otimes \xi)$, and the linearised scalar curvature has symbol $h(\xi,\xi) - \mathrm{tr}\, h$. Substituting $h = \xi \odot v$ gives zero for both. For $h = Y$ supported on $\xi^\perp$ we have $h\xi = 0$, and the symbol of G is $-\frac{1}{2}Y - \frac{1}{2}(\mathrm{tr}\, Y)\,\xi \otimes \xi + \frac{1}{2}(\mathrm{tr}\, Y)\, g = \frac{1}{2}\,\mathrm{adj}_2 Y$, and $\mathbb{M} = -G$ gives the sign. Bijectivity holds because adj_2 is an involution.

(2) Differentiate $\mathrm{adj}\, K = K^2 - \sigma_1 K + \sigma_2 g$. In a frame with $\xi = e_1$ and $K = \begin{pmatrix} k_{11} & k^{\mathsf{T}} \\ k & Q \end{pmatrix}$, the first row and column of $D\,\mathrm{adj}_K(e_1 \otimes e_1)$ vanish, and the lower block is $(\sigma_1 - k_{11})\mathrm{Id} - Q = \mathrm{adj}_2 Q$.

(3) Contracting $\xi_a L_{bc} = \xi_b L_{ac}$ with ξ^a gives $L = \xi \otimes w$, and symmetry forces $w \parallel \xi$. □

Theorem B (the symbol). *Let $g \in \mathcal{U}$, $p \in \Sigma$, $K = \tilde{K}(g)(p)$ and $\xi \neq 0$.*

1. *$\sigma(\xi)$ has rank exactly 2, and*

$$\ker \sigma(\xi) = \{\xi \odot v : v \in T^*_p\} \oplus \mathrm{span}\{Q_\xi\},$$

   a space of dimension 4.
2. *Let $T_1 = \sigma_1 g - K$ be the first Newton tensor. On $TT_\xi = \{h : h\xi = 0,\ \mathrm{tr}\, h = 0\}$, $\sigma(\xi)$ is injective iff $\xi^{\mathsf{T}} T_1 \xi \neq 0$.*
3. *Q_ξ is a multiple of P iff $K|_{\xi^\perp}$ is a multiple of the identity. This holds for every ξ iff $K \propto g$.*

Proof. (1) By Lemma 4.1(1) the first factor has kernel $\{\xi \odot v\}$ and image $\mathrm{Sym}^2(\xi^\perp)$, of dimension 3. The middle factor is invertible because $\det K \neq 0$ (§2.4). By Lemma 4.1(2) its inverse carries $\mathrm{adj}_2(Q_\xi) \in \mathrm{Sym}^2(\xi^\perp)$ to $\xi \otimes \xi$. By Lemma 4.1(3) that is the only direction the last factor kills. So the rank is $3 - 1 = 2$. The kernel is the set of h with $\mathrm{adj}_2(PhP) \in \mathrm{span}\{\mathrm{adj}_2 Q_\xi\}$, that is, $PhP \in \mathrm{span}\{Q_\xi\}$. Finally $Q_\xi \neq 0$, since $Q_\xi = 0$ would make $K = \xi \odot w$, of rank at most 2.

(2) A kernel element is $h = cQ_\xi + \xi \odot v$. Since $Q_\xi \xi = 0$, transversality gives $\frac{1}{2}((v \cdot \xi)\xi + v) = 0$, so $v = 0$. Then $h = cQ_\xi$ is trace-free iff $c\,\mathrm{tr}\, Q_\xi = 0$, and $\mathrm{tr}\, Q_\xi = \sigma_1 - K(\xi,\xi) = \xi^{\mathsf{T}} T_1 \xi$.

(3) If $K|_{\xi^\perp}$ is scalar for every ξ, then for every orthonormal pair u, w (take $\xi \perp u, w$) $K(u,u) = K(w,w)$ and $K(u,w) = 0$. $\square$

Attribution and checks. The proof of (1) through Lemma 4.1 is the one given by an independent machine-assisted audit (October 2026). The same audit gave the trace form of (2), with the exact counterexample below, and (3). Earlier versions had established (1) only at sample shape operators, and had claimed (2) without the condition. The formula for $D\,\mathrm{adj}_K$ shows that Lemma 4.1(2) is a polynomial identity, valid for every symmetric K. Lemma 4.1 is checked symbolically; rank and kernel exactly at twelve random rational K, definite and indefinite; the TT statement, in floating point, on random K and at null covectors of T_1 (`test_325` P1–P2, `test_326` P1). The kernel is checked again, exactly, in `test_332` P3. The formula (4.1) itself is checked against the full nonlinear operator: at a point of a metric with $g(0) = 1$, the linearisation of F on $h = ((\xi \cdot x)^3/6)\, h_0$ is $\sigma(\xi)h_0$ exactly, on both branches (`test_335` P5).

Corollary 4.2 (the kernel is the immersions'; slices). *Let $g \in \mathcal{U}$, $K = \tilde{K}(g)(p)$ and $\xi \neq 0$.*

1. $\ker \sigma(\xi) = \{\xi \odot v\} \oplus \mathrm{span}\{K\}$. *This is the symbol at ξ of the linearised immersions $h = \mathcal{L}_X g - 2\varphi K$, with X counted one order above φ. At a realizable metric these are the actual first-order deformations. So the symbol sequence $(X, \varphi) \mapsto h \mapsto \sigma(\xi)h$ is exact at h, at every ξ and every nondegenerate K, on both branches. In particular F is nowhere elliptic modulo diffeomorphisms alone: its symbol kills one direction more, the normal variation.*
2. *$\sigma(\xi)$ is injective on every complement of its kernel. The space $S_\xi = \{h : h\xi = 0,\ \langle h, Q_\xi\rangle = 0\}$ is one, at every ξ. When $h\xi = 0$, $\langle h, K\rangle = \langle h, Q_\xi\rangle$, so S_ξ is the symbol of the conditions* $\mathrm{div}\, h = 0$, $\langle h, \tilde{K}\rangle = 0$. *At an umbilic point it is TT_ξ.*
3. *TT_ξ is a complement iff $\xi^{\mathsf{T}} T_1 \xi \neq 0$ (Theorem B(2)). Call g **TT-admissible** at p if this holds for every $\xi \neq 0$, that is, if T_1 is definite at p. With principal values $k_1 \leq k_2 \leq k_3$, the eigenvalues of T_1 are the pairwise sums $k_i + k_j$, so:*
   - *a TT-characteristic covector exists iff $k_1 + k_2 \leq 0 \leq k_2 + k_3$;*
   - *the convex branch is TT-admissible everywhere;*
   - *on the indefinite branch, TT-admissibility holds exactly where $k_1 + k_2$ and $k_2 + k_3$ have the same sign.*

Proof. (1) $K - Q_\xi = \xi \odot w$ with $w = 2K\xi - K(\xi,\xi)\,\xi$, so the kernel of Theorem B(1) is $\{\xi \odot v\} + \mathrm{span}\{K\}$. The sum is direct because K has rank 3 and $\xi \odot v$ has rank at most 2. The symbol of $\mathcal{L}_X g$ at ξ is a multiple of $\xi \odot \hat{X}$, and $-2\varphi K$ is of order zero in φ.

(2) A kernel element in S_ξ is $h = cQ_\xi + \xi \odot v$ with $h\xi = 0$, so $v = 0$ as in the proof of Theorem B(2). Then $c|Q_\xi|^2 = 0$, and $Q_\xi \neq 0$. S_ξ has dimension $3 - 1 = 2$, the codimension of the kernel.

(3) is Theorem B(2) and the eigenvalues of T_1. $\square$

The existence criterion in (3) is the audit's. Examples:

- $K = \mathrm{diag}(2, 2, -1)$ has $T_1 = \mathrm{diag}(1, 1, 4)$, so it is TT-admissible.
- The Flamm hypersurface (Example 3.3) is TT-admissible everywhere.
- $K = \mathrm{diag}(2, 3, -3)$ has $T_1 = \mathrm{diag}(0, -1, 5)$, so it is not. At $\xi = e_1$ the TT tensor $h = \mathrm{diag}(0, 1, -1) = \frac{1}{3}Q_{e_1}$ satisfies $\sigma(e_1)h = 0$ exactly, while $\sigma(e_1)$ is injective on S_{e_1} (`test_332` P3).

Remark 4.3 (what a TT-characteristic covector is). $\xi^{\mathsf{T}} T_1 \xi = \mathrm{tr}(K|_{\xi^\perp})$. So ξ is TT-characteristic iff the second fundamental form restricted to the plane $\xi^\perp$ is trace-free: the normal

curvatures in any two orthonormal directions of that plane sum to zero. When T_1 is positive definite, the hypersurface is *2-convex* in the sense of Huisken and Sinestrari (2009). So TT-admissibility is 2-convexity for one of the two orientations. It is a property of the slice, not of F: at a TT-characteristic covector the kernel direction Q_ξ happens to be trace-free.

Remark 4.4 (the York reading). At each ξ, $\mathrm{Sym}^2 T_p^* = TT_\xi \oplus \{\xi \odot v\} \oplus \mathrm{span}\{P\}$, with dimensions $2+3+1$: York's decomposition (York 1973) at the level of symbols. The kernel of $\sigma(\xi)$ is the gauge part plus Q_ξ.

- At an umbilic point Q_ξ is a multiple of P. The kernel is then gauge plus conformal, and the image is $\sigma(TT_\xi)$. This is the situation at the round sphere (§6).
- Off umbilics the fourth kernel direction turns with K, and S_ξ turns with it. TT_ξ remains a complement exactly when ξ is not TT-characteristic.

Remark 4.5 (what the slices are used for). Meagreness (§5) uses only that $\sigma \neq 0$, and holds on both branches whatever T_1 does. The round-sphere analysis (§6) works on TT, which serves there. Neither TT_ξ nor S_ξ is a subbundle as ξ varies, so neither gives an estimate by itself; §6 repairs this at γ by augmenting the operator. Off umbilics the natural gauge is $\operatorname{div} h = 0$, $\langle h, \tilde{K} \rangle = 0$, whose symbol is S_ξ at every covector (§13, problem 4).

4.2 Four identities, five conditions, and Cartan's test

Lemma 4.6 (identities of the Codazzi tensor). *Let K be a symmetric C^2 tensor on a Riemannian three-manifold and $C = \operatorname{Cod} K$, $C_{abc} = \nabla_a K_{bc} - \nabla_b K_{ac}$.*

1. $C_{abc} + C_{bca} + C_{cab} = 0$.
2. $\nabla^a (\operatorname{adj} K)_{ab} = B_b(C)$, *where* $B_b(C) := K^c{}_b\, C^a{}_{ca} - \sigma_1\, C^a{}_{ba} + K^{ac} C_{abc}$.
3. $\sum_{\mathrm{cyc}(d,a,b)} \nabla_d C_{abc} = -\big(R^e{}_{cda} K_{be} + R^e{}_{cab} K_{de} + R^e{}_{cbd} K_{ae}\big)$. *The right-hand side vanishes when* $R_{ecda} = K_{ed}K_{ca} - K_{ea}K_{cd}$.

Proof. (1) The six terms cancel in pairs by the symmetry of K.

(2) Write $K_{ab;c} = \nabla_c K_{ab}$. From $\operatorname{adj} K = K^2 - \sigma_1 K + \sigma_2 g$ and $\nabla_b \sigma_2 = \sigma_1 \sigma_{1;b} - K^{cd} K_{cd;b}$,

$$\nabla^a (\operatorname{adj} K)_{ab} = \sigma_1 (\sigma_{1;b} - K^a{}_{b;a}) + K^c{}_b (K^a{}_{c;a} - \sigma_{1;c}) + K^{ac} (K_{cb;a} - K_{ac;b}).$$

The three brackets are $-C^a{}_{ba}$, $C^a{}_{ca}$ and C_{abc}.

(3) The cyclic sum is $[\nabla_d, \nabla_a] K_{bc} + [\nabla_a, \nabla_b] K_{dc} + [\nabla_b, \nabla_d] K_{ac}$. The Ricci identity expands each commutator into two curvature terms, and the terms of the form $R^e{}_{bda} K_{ec}$ cancel by the first Bianchi identity. With Gauss's curvature, the remaining sum is

$$(K^2)_{bd} K_{ca} - (K^2)_{ba} K_{cd} + (K^2)_{da} K_{cb} - (K^2)_{db} K_{ca} + (K^2)_{ab} K_{cd} - (K^2)_{ad} K_{cb} = 0. \qquad \square$$

Part (2) is the defect form of the classical fact that the Newton tensors of a Codazzi tensor are divergence-free (Reilly 1973).

Theorem C (the count, and involutivity). *Let $g \in \mathcal{U}$, $p \in \Sigma$ and $K = \tilde{K}(g)(p)$.*

1. *(Identities.) $F(g)$ satisfies, identically in g:*
   - *the cyclic identity;*
   - *$B(F(g)) = 0$, with B formed from $\tilde{K}$, since $\nabla^a (\operatorname{adj} \tilde{K})_{ab} = \nabla^a \mathbb{M}_{ab} = -\nabla^a G_{ab} = 0$;*
   - *the first-order identity $\sum_{\mathrm{cyc}} \nabla_d F_{abc} = 0$, since $(g, \tilde{K})$ satisfies Gauss.*

2. *(Five conditions.) At every point, the four algebraic identities are independent, and they span the left null space of the order-3 symbol map $\tau_3 : S^3T_p^* \otimes \mathrm{Sym}^2 \to \Lambda^2 \otimes T^*$, which has rank 5. So $F = 0$ imposes exactly five conditions per point. That is two more than the three a metric has modulo diffeomorphisms.*

3. *(Cartan's test.) Let $g_q = \ker \tau_q$, and let $g_{q,k}$ be its elements independent of the first k coordinates of any orthonormal frame. Then $g_q = M_K^q$, the symbols of the linearised immersions defined below, and:*

   - $\dim g_3 = 55$, $\dim g_{3,1} = 19$, $\dim g_{3,2} = 4$;
   - *the prolonged symbol τ_4 has rank 12;*
   - $\dim g_4 = 78 = 55 + 19 + 4$.

   Cartan's inequality $\dim g_4 \le \dim g_3 + \dim g_{3,1} + \dim g_{3,2}$ therefore holds with equality, and g_3 is involutive. The left null space of τ_4 has dimension 15. It is spanned by the twelve prolongations of the algebraic identities and the three functionals $\xi \wedge C$, which are the symbol of the first-order identity in (1).

4. *(Involutivity.) By (1), every relation among the equations at the first prolongation is an exact identity, so the prolongation adds no integrability condition. With (3), the system $F = 0$ is involutive at p (Goldschmidt 1967; see Bryant et al. 1991 and Seiler 2010 for Cartan's test). Its last Cartan character is $\dim g_{3,2} = 4$. For real-analytic data, its general local solution therefore depends on four functions of three variables (Cartan–Kähler).*

Proof of (2). By Lemma 4.6 the four functionals annihilate the image of τ_3, so its rank is at most 5 once they are independent. In a principal frame of K their 4×4 minors include $\pm k_1k_2k_3$, so they are independent wherever $\det K \neq 0$. The image of τ_3 contains the images of $\sigma(e_1)$, $\sigma(e_2)$ and $\sigma(e_3)$, and a 5×5 minor of these is a nonzero rational multiple of a monomial in k_1, k_2, k_3 (`test_332` P1). So the rank is exactly 5 at every nondegenerate K. □

Proof of (3). Regard g_q as homogeneous polynomial tensors of degree q in x, annihilated by the constant-coefficient operator $\sigma(\partial) = \mathrm{Cod}_\partial \circ A_K^{-1} \circ E$, where $E = \sigma_2(D\mathbb{M})(\partial)$ is the linearised negative Einstein operator of flat space and $A_K = D\,\mathrm{adj}_K$.

The immersions lie in g_q. By Theorem B, $\sigma(\xi)$ kills $\xi \odot v$ and K itself for every ξ ($K - Q_\xi$ is of the form $\xi \odot w$). Hence g_q contains

$$M_K^q = \{\partial \odot \omega + \varphi K : \omega \text{ a homogeneous 1-form of degree } q+1,\ \varphi \text{ homogeneous of degree } q\}.$$

These are the symbols of the linearised immersions $\delta x = X + \varphi N$, which change the metric by $h = \mathcal{L}_X g - 2\varphi K$.

The map $(\omega, \varphi) \mapsto \partial\odot\omega + \varphi K$ is injective for $q \ge 2$. Suppose $\partial\odot\omega = -\varphi K$. The linearised curvature of the left side vanishes, so, at the level of constant-coefficient symbols, the Kulkarni–Nomizu product $\partial^2\varphi \mathbin{\bar\wedge} K$ vanishes. In dimension three, $A \mapsto A \mathbin{\bar\wedge} K$ has determinant $2(\det K)^2$ on Sym^2 (`test_326` P4), so $\partial^2\varphi = 0$, and $\varphi = 0$ by homogeneity. Then ω is a Killing form of degree at least 3, hence zero. So

$$\dim M_K^q = 3 \dim S^{q+1} + \dim S^q = (q+2)(2q+5),$$

which is $55, 78, 105, 136, \ldots$ for $q = 3, 4, 5, 6, \ldots$.

Nothing else lies in g_q. Two identities do the work. First, by Lemma 4.1(1)–(2), $\sigma_2(D\mathbb{M})(\xi)K = -\frac{1}{2}A_K(\xi \otimes \xi)$ for every ξ (`test_354` P1), so $E(fK) = -\frac{1}{2}A_K(\nabla^2 f)$ for every polynomial f. Second,

in dimension three $E(S) = 0$ iff the linearised curvature of S vanishes, iff $S = \partial \odot \omega$ (Saint-Venant; on polynomials there is no topology). For homogeneous S of degree $q \geq 1$ one can take $\omega = \frac{2}{q} Sx - \nabla(x^{\mathsf{T}} Sx)/q(q+1)$.

Let $h \in g_q$ with $q \geq 2$, and put $L = A_K^{-1} E(h)$, symmetric and homogeneous of degree $q-2$. Since $\sigma(\partial)h = \mathrm{Cod}_\partial L = 0$, $\partial_a L_{bc} = \partial_b L_{ac}$, so L is a Hessian: $L = \nabla^2 f$ with $f = x^{\mathsf{T}} Lx/q(q-1)$. Then $E(h + 2fK) = A_K L - A_K \nabla^2 f = 0$, so $h + 2fK = \partial \odot \omega$ and $h = \partial \odot \omega + \varphi K$ with $\varphi = -2f$. So $g_q = M_K^q$ at every nondegenerate K, and $\dim g_3 = 55$, $\dim g_4 = 78$.

Flags. Let $q \geq 3$ and let $h \in g_q$ be independent of x_i. Then so is L, so $\nabla^2(\partial_i f) = 0$, and $\partial_i f$, homogeneous of degree $q - 1 \geq 2$, vanishes. So $h + 2fK$ is independent of x_i, hence $\partial \odot (\partial_i \omega) = 0$, and $\partial_i \omega$ is a Killing form of degree $q \geq 3$, hence zero. The same holds for each excluded coordinate, in any orthonormal frame. So $g_{q,k}$ is M_K^q with ω and φ independent of the first k coordinates:

$$\dim g_{3,1} = 3 \dim S^4(\mathbb{R}^2) + \dim S^3(\mathbb{R}^2) = 19, \qquad \dim g_{3,2} = 3 \dim S^4(\mathbb{R}) + \dim S^3(\mathbb{R}) = 4.$$

Then $78 = 55 + 19 + 4$, and $\operatorname{rank} \tau_4 = 90 - 78 = 12$.

The fifteen functionals. The target of τ_4 has dimension 27, so its left null space has dimension 15. It contains the twelve first derivatives of the four algebraic identities and the three functionals $\xi \wedge C$ (Lemma 4.6). They are independent at every nondegenerate K. Write $C_{abc} = \epsilon_{abi} Z_{ic}$. The four algebraic identities are independent, by (2), and cut out a five-dimensional fibre, which in a principal frame is $\operatorname{tr} Z = 0$, KZ symmetric. So their twelve derivatives are independent. The symbol of the cyclic identity is, up to a nonzero factor, the divergence of Z. On derivatives $\eta \otimes Z_0$ with $Z_0 = \mathrm{diag}(1, 1, -2)$, which lies in the fibre, it gives $\eta^{\mathsf{T}} Z_0$, which ranges over all of $\mathbb{R}^3$. So the three are independent of the twelve, and the fifteen span the left null space.

Attribution and checks. This proof of (3) at every K is an independent machine-assisted review's (Appendix V). Before it, the dimensions were established by exact computation at sixteen shape operators, each in an exact rational frame: several generic ones, on both branches; the umbilic one, which is the round sphere's; two equal eigenvalues, on both branches; the TT-characteristic $\mathrm{diag}(2, 3, -3)$; the boundary case $k_1 + k_2 = 0$; and a near-degenerate one. Those computations are now checks: ranks modulo $2^{31} - 1$, squeezed against the lower bound $\dim g_q \geq \dim M_K^q$, give the dimensions over $\mathbb{Q}$ for $q \leq 5$ at all sixteen (`test_332` P2), and, in floating point, for $q \leq 9$ at two of them (`test_325` P3–P5, `test_326` P2–P3); the flag bounds are `test_344` P4. The two identities, the reconstruction of every element of g_3 as $\partial \odot \omega + \varphi K$, the dimensions at three further shape operators in non-principal frames, and the rank of the fifteen functionals are `test_354` P1–P4.

The hypotheses of (4). Goldschmidt's criterion needs locally constant ranks, a surjective first prolongation and an involutive symbol. At every point of $\mathcal{U}$:

- The ranks of τ_3, τ_4 and the flag restrictions are the same at every nondegenerate K, by (2) and (3). So g_3, g_4 and the $g_{3,k}$ are vector bundles. Since τ_3 has rank 5 on all of $\mathcal{U}$, by (2), $\mathcal{R}_3 = \{j^3 g : F(g) = 0\}$ is a submanifold of codimension 5 there.
- The fifteen functionals of (3) span the left null space of τ_4.
- Each is the symbol of an identity that $F(g)$ satisfies for **every** metric on $\mathcal{U}$, not only on solutions: the four algebraic identities of Lemma 4.6, their twelve first derivatives, and the cyclic identity $\sum_{\mathrm{cyc}(d,a,b)} \nabla_d F_{abc} = 0$. The last holds because $\tilde{K}$ satisfies the Gauss equation identically (Theorem A(2)), so the curvature term of Lemma 4.6(3) vanishes. The identities are linear in $(F, \nabla F)$, with coefficients in the jet of g. On $\mathcal{R}_3$, where $F = 0$, the functionals therefore annihilate the lower-order part of each prolonged equation. So $\mathcal{R}_4 \to \mathcal{R}_3$ is onto, and the curvature map of the criterion vanishes.

With the involutivity of g_3 from (3), Goldschmidt's theorem gives (4).

At every nondegenerate K, then, the formal solutions of the symbol are exactly the linearised immersions, and the count of Theorem C is the count of immersions: three functions for the tangential part and one for the normal displacement. The joint kernel of $\sigma(\xi)$ over all ξ is $\mathrm{span}\{K\}$, the normal direction $\delta g = -2\varphi K$ (`test_325` P5). The identities of Lemma 4.6 are checked exactly at a curved point (`test_325` P6).

Remark 4.7 (earlier counts). An earlier version reported $\operatorname{rank}\tau_4 = 15$ and Cartan characters $(2, 2, 1)$. The first came from a multiplicity factor in the prolongation that jet coordinates do not carry. The second stacked single covectors instead of restricting to a flag. Both numbers are wrong: the characters are $(36, 15, 4)$. The conclusion drawn from them, that $F = 0$ is involutive, is right, and is proved by (3)–(4) (Appendix C).

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5. Meagreness

Theorem D. *Let Σ be closed, $k \geq 3$, $\alpha \in (0, 1)$, and $\mathcal{Z} = \{g \in \mathcal{U}^{k,\alpha} : F(g) = 0\}$. Then $\mathcal{Z}$ is closed in $\mathcal{U}^{k,\alpha}$ and has empty interior. Hence it is nowhere dense, and meagre, in $\mathcal{U}^{k,\alpha}$ and in the space of all $C^{k,\alpha}$ metrics. This holds on both branches.*

Proof. F is continuous, so $\mathcal{Z}$ is closed in $\mathcal{U}^{k,\alpha}$.

$DF_g \neq 0$ at every $g \in \mathcal{U}^{k,\alpha}$. Fix p and ξ with $\sigma(\xi) \neq 0$ (Theorem B), and $h_0 \in \mathrm{Sym}^2 T_p^*$ with $\sigma(\xi)h_0 \neq 0$. Choose a cutoff χ near p and a function f with $df(p) = \xi$, and set $h_\lambda = \chi \cos(\lambda f)\, h_0$, with h_0 extended smoothly. The terms of $DF_g(h_\lambda)$ cubic in λ come only from the principal part, and equal $\lambda^3 \chi \sin(\lambda f)\, \sigma(df) h_0$. All other terms are $O(\lambda^2)$. So $DF_g(h_\lambda) \neq 0$ for large λ.

If $\mathcal{Z}$ contained an open set B, then $F \equiv 0$ on B would give $DF_g = 0$ for $g \in B$, a contradiction.

$\mathcal{Z}$ is closed in $\mathcal{U}^{k,\alpha}$ with empty interior there. $\mathcal{U}^{k,\alpha}$ is open in the space of all $C^{k,\alpha}$ metrics, since Σ is compact and $\det \mathbb{M}$ depends continuously on g in C^2. Let $\overline{\mathcal{Z}}$ be the closure of $\mathcal{Z}$ in the full space, and suppose that a nonempty open set V lies in $\overline{\mathcal{Z}}$. Then V meets $\mathcal{Z} \subset \mathcal{U}^{k,\alpha}$, so $V \cap \mathcal{U}^{k,\alpha}$ is a nonempty open subset of $\mathcal{U}^{k,\alpha}$ contained in $\overline{\mathcal{Z}} \cap \mathcal{U}^{k,\alpha} = \mathcal{Z}$, a contradiction. So $\overline{\mathcal{Z}}$ has empty interior. □

Attribution. This proof is the audit's, with the last step as a second independent read gave it. It replaces two earlier arguments. The first was local to the round sphere and used a lower bound for DF on TT tensors. The second appealed to a cokernel, which is not a proof: the zero map has a cokernel and vanishes everywhere. The audit also gave a one-direction proof at γ that needs only $DF_\gamma h \neq 0$ for one h. The step the proof uses, that the third-order part of DF_g is the symbol (4.1), is checked against the full nonlinear operator on both branches (`test_335` P5).

Remarks 5.1.

1. **Category, not measure.** The statement is about Baire category in a Banach space. There is no canonical measure on metrics, and none is used.
2. **Nothing outside $\mathcal{U}$.** Metrics with $\det \mathbb{M} = 0$ somewhere, realizable or not, are not covered. Section 8 treats that locus.
3. **No ellipticity used.** Theorem D holds wherever $\sigma \neq 0$, which is all of $\mathcal{U}$, on the TT-characteristic set included.

4. **Meagre is compatible with connected.** A line in the plane is nowhere dense and connected. The realizable metrics on the convex branch of S^3 form a path-connected, contractible set (the companion paper). Theorem C says how large the realizable set is locally: the immersions, four functions of three variables, against six for all metrics.

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6. The round sphere

Let γ be the unit round metric on S^3. Then $\mathbb{M}(\gamma) = \mathrm{Id}$, $\tilde{K}(\gamma) = \gamma$ (the unit sphere with outward normal), and $T_1 = 2\gamma$ is definite, so the TT slice serves at every point (Corollary 4.2). We use York's L^2-orthogonal decomposition (York 1973)

$$\mathrm{Sym}^2 T^*S^3 = TT \oplus \{\mathcal{L}_X\gamma - \tfrac{2}{3}(\operatorname{div} X)\gamma\} \oplus C^\infty(S^3)\,\gamma, \tag{6.1}$$

and the TT spectrum of §2.6: at level $n \geq 3$, $-\Delta = n^2 - 3$ with multiplicity $2(n^2-4)$. On S^3 the rough Laplacian preserves TT.

Theorem E (the linearisation at the round sphere).

1. *(Kernel.) $DF_\gamma(2u\gamma) = 0$ for every function u, and $DF_\gamma(\mathcal{L}_X\gamma) = 0$ for every vector field X. The kernel is exactly*
$$\ker DF_\gamma = \{2u\gamma\} + \{\mathcal{L}_X\gamma\}.$$
The sum is not direct. The intersection is $\{2u\gamma : u$ a first harmonic$\}$, and for a first harmonic f, $\mathcal{L}_{\nabla f}\gamma = -2f\gamma$.
2. *(Factorisation.) On TT tensors, $DF_\gamma = \mathrm{Cod} \circ \frac{1}{2}(-\Delta + 2)$.*
3. *(No TT Codazzi tensors.) If h is TT and $\mathrm{Cod}\, h = 0$, then $h = 0$.*
4. *(The L^2 identity.) For TT h, $\|\mathrm{Cod}\, h\|^2 = 2\|\nabla h\|^2 + 6\|h\|^2$. Hence*
$$\|DF_\gamma h\|^2 = \tfrac{1}{2}\langle h,\ (-\Delta + 3)(-\Delta + 2)^2 h\rangle.$$
At level n the factor is $\frac{1}{2}\big(n(n^2-1)\big)^2$. So $\|DF_\gamma h\|^2 \geq 288\,\|h\|^2$ on TT, with equality exactly on level 3.
5. *(Closed range.) For $k \geq 3$ and $\alpha \in (0,1)$, $DF_\gamma|_{TT} : C^{k,\alpha} \to C^{k-3,\alpha}$ is bounded below, with closed, complemented range.*

Proof. (1) The radial graphs $x = (1+u)\hat{n}$ have induced metric $(1+u)^2\gamma + du \otimes du$ exactly, because $\hat{n} \cdot d\hat{n} = 0$. So F vanishes along them, and differentiating at $u = 0$ gives $DF_\gamma(2u\gamma) = 0$. Naturality, $F(\phi^* g) = \phi^* F(g)$, gives the Lie statement.

For the intersection, suppose $2u\gamma = \mathcal{L}_X\gamma$. Then X is a conformal Killing field. On S^3 these are the Killing fields plus the gradients of first harmonics, so $u = 0$ or $u = -f$ with f a first harmonic. Obata's identity $\nabla^2 f = -f\gamma$ gives $\mathcal{L}_{\nabla f}\gamma = 2\nabla^2 f = -2f\gamma$.

That the kernel is no larger follows from (6.1), (2) and (3). DF_γ kills the last two summands of (6.1) by what was just shown, and is injective on TT.

(2) Three ingredients, for TT h.

- *The commutator.* On the unit sphere $R^e{}_{bda} = \delta^e{}_d\gamma_{ba} - \delta^e{}_a\gamma_{bd}$, so for trace-free h, $\gamma^{cd}[\nabla_d, \nabla_a]h_{bc} = 3h_{ab}$. With $\nabla^c h_{bc} = 0$ this gives $\nabla^c\nabla_a h_{bc} = 3h_{ab}$.

- *The curvature variation.* As $(0,2)$ tensors, $\delta\mathrm{Ric} = \frac{1}{2}\Delta_L h = 3h - \frac{1}{2}\Delta h$, since the Lichnerowicz Laplacian on TT tensors at the unit S^3 is $\Delta_L = -\Delta + 6$; and $\delta R = 0$. So $\delta G = \delta\mathrm{Ric} - \frac{1}{2}(\delta R)\gamma - \frac{1}{2}R(\gamma)h = 3h - \frac{1}{2}\Delta h - 3h = -\frac{1}{2}\Delta h$. For the endomorphism $\mathbb{M}^a{}_b = -g^{ac}G_{cb}$, with $G(\gamma) = -\gamma$ and $\delta g^{ac} = -h^{ac}$, this gives $\delta\mathbb{M} = -h + \frac{1}{2}\Delta h$, which is trace-free. At $\mathbb{M} = \mathrm{Id}$ this gives $\delta\tilde{K} = h - \frac{1}{2}\Delta h$ as an endomorphism. Lowering the index with $g = \gamma + h$ adds one more h, so the $(0,2)$ variation is $\delta K = \frac{1}{2}(-\Delta + 4)h$.
- *The connection.* At the umbilic background the variation of the Christoffel symbols inside Cod contributes $-\mathrm{Cod}\, h$: since $\mathrm{Cod}_g\, g = 0$ for every g, the variation of Cod_g applied to $\tilde{K}_0 = \gamma$ is $-\mathrm{Cod}\, h$.

So $DF_\gamma h = \mathrm{Cod}\, \frac{1}{2}(-\Delta + 4)h - \mathrm{Cod}\, h = \mathrm{Cod}\, \frac{1}{2}(-\Delta + 2)h$.

(3) Applying ∇^a to $\nabla_a h_{bc} = \nabla_b h_{ac}$ and using the commutator gives $\Delta h = 3h$. But $-\Delta \geq 0$ on TT, and S^3 is closed. The argument is global: locally there are nonzero trace-free Codazzi tensors, for instance $\nabla^2 f + f\gamma$ with $f = \cos 2r / \sin r$ on the punctured sphere (`test_354` P6).

(4) Expanding, $\|\mathrm{Cod}\, h\|^2 = 2\|\nabla h\|^2 - 2\int \nabla_a h_{bc}\nabla^b h^{ac}$. Integrate by parts and use $\nabla^a\nabla_b h_{ac} = 3h_{bc}$ to get $\int \nabla_a h_{bc}\nabla^b h^{ac} = -3\|h\|^2$. On TT, (2) and this give

$$\|DF_\gamma h\|^2 = 2\|\nabla k\|^2 + 6\|k\|^2 = 2\langle k, (-\Delta + 3)k\rangle, \qquad k = \tfrac{1}{2}(-\Delta + 2)h,$$

and $-\Delta$ commutes with itself.

(5) TT is not a subbundle, so the semi-Fredholm theorem for operators with injective symbol does not apply as it stands. Two standard repairs close it.

- *Augment.* $\mathcal{A} = (\mathrm{Cod}, \mathrm{div}, d \circ \mathrm{tr})$ is a first-order operator on the full symmetric bundle with injective symbol: the trace symbol removes $\mathrm{tr}\, H$, the divergence symbol removes $\xi^a H_{ab}$, and on what is left $\|\xi \wedge H\|^2 = 2|\xi|^2\|H\|^2$. Its kernel is $\mathbb{R}\gamma$, by (3). The elliptic estimate for $\mathcal{A}$ and compactness give a lower bound for $\mathrm{Cod} : C^{k-2,\alpha}_{TT} \to C^{k-3,\alpha}$. Since the rough Laplacian preserves TT and $-\Delta + 2 \geq 8$ there, $\frac{1}{2}(-\Delta + 2) : C^{k,\alpha}_{TT} \to C^{k-2,\alpha}_{TT}$ is an isomorphism, and (2) gives the lower bound for $DF_\gamma|_{TT}$.
- *Project.* Π_{TT} is a pseudodifferential operator of order zero, bounded on $C^{k,\alpha}$ for $\alpha \in (0,1)$, and $(DF_\gamma|_{TT})^* DF_\gamma|_{TT}$ is elliptic in that calculus.

Complementation is by $P = DF_\gamma\, (\Pi_{TT} DF_\gamma^* DF_\gamma \Pi_{TT})^{-1}\, \Pi_{TT} DF_\gamma^*$, the inverse taken on TT. By (4), $\Pi_{TT} DF_\gamma^* DF_\gamma \Pi_{TT} = \frac{1}{2}(-\Delta+3)(-\Delta+2)^2$ on TT, so the inverse is a function of the rough Laplacian, of order -6. The factor after DF_γ is then a pseudodifferential operator of order -3, bounded from $C^{k-3,\alpha}$ to $C^{k,\alpha}_{TT}$, and P is a bounded projection of $C^{k-3,\alpha}$ onto the range. □

Checks. (1)–(3) and (5) are the content of mathematics paper 1 (Thrussell 2026a), Theorem B. Part (2) is checked as an operator identity, not only on eigenmodes. On random tensors that are transverse-traceless to the order that matters at a point, the linearisation of the full nonlinear operator equals $\mathrm{Cod} \circ \frac{1}{2}(-\Delta + 2)$, exactly; a tensor whose trace has nonzero gradient fails it (`test_336` P1). Part (4) is the audit's (P1-X01). It is checked pointwise at level 3, on the constant trace-free tensors of a left-invariant frame of $S^3 = SU(2)$, where $|\nabla h|^2 = 6|h|^2$ and $|\mathrm{Cod}\, h|^2 = 18|h|^2$ (`test_336` P2). The augmented symbol of (5) is `test_336` P3. The kernel statements of (1) are checked with the full nonlinear operator, exactly, at a point (`test_327` P1–P2).

Remark 6.1 (the multipliers). The singular values of DF_γ on TT at level n are $n(n^2-1)/\sqrt{2}$. The numbers $n(n^2-1)$ are the eigenvalues, level by level, of Branson's third-order conformally covariant operator on functions on S^3, $P_3 = \sqrt{-\Delta + 1}\,(-\Delta)$ (Branson 1995; Beckner 1993). The

coincidence is an identity of operators. Let $C(g) = \mathrm{Cod}_g(\mathrm{Ric} - \frac{1}{4}Rg)$ be the Cotton tensor, the Codazzi tensor of the Schouten tensor in dimension three. At γ, $\tilde{K} = \gamma$ and the Schouten tensor is $\frac{1}{2}\gamma$, both Codazzi, and varying each along $\gamma + \varepsilon H$ gives the same tensor:

$$DF_\gamma(H) = DC_\gamma(H) = \mathrm{Cod}_\gamma(\dot{\mathrm{Ric}} - \tfrac{1}{4}\dot{R}\,\gamma - 2H)$$

pointwise, for every symmetric H. (For $\tilde{K}$: $\gamma\dot{\mathbb{M}} = -\dot{\mathrm{Ric}} + \frac{1}{2}\dot{R}\gamma + 2H$ and $\mathrm{tr}\,\dot{\mathbb{M}} = \frac{1}{2}\dot{R}$, so $\delta\tilde{K} = \dot{\mathrm{Ric}} - \frac{1}{4}\dot{R}\gamma - H$; for the Schouten tensor, $\dot{P} = \dot{\mathrm{Ric}} - \frac{1}{4}\dot{R}\gamma - \frac{3}{2}H$; and the variation of Cod acting on $c\gamma$ is $-c\,\mathrm{Cod}_\gamma H$.) So DF_γ is the linearised Cotton operator. The Cotton tensor is conformally invariant in dimension three, so its linearisation at the conformally flat γ commutes with the conformal group $SO(4,1)$, and it vanishes on the conformal and Lie directions on every open subset of S^3, since $C = 0$ on locally conformally flat metrics; the shell dipole of Proposition 6.2(3) is such a local direction. (`test_625`: the identity exactly at two rational points for a general polynomial H, with the conformal and Lie controls.)

Dualising the antisymmetric pair, $\star T_{cd} = \frac{1}{2}\epsilon_c{}^{ab}T_{abd}$, turns C into the Cotton–York tensor $Y = \star C$, which is symmetric, trace-free and divergence-free for every metric, with $Y(\gamma) = 0$. So $\star DF_\gamma = DY_\gamma$ maps all of Sym^2 into TT. On TT, $\star\mathrm{Cod}$ is the curl, $(\mathrm{curl}\,h)_{cd} = \epsilon_c{}^{ab}\nabla_a h_{bd}$; it satisfies $\mathrm{curl}^2 = \nabla^*\nabla + 3$ there, and at level n its eigenvalues are $+n$ and $-n$, one on each chiral summand, each of dimension $n^2 - 4$. Hence

$$2\star DF_\gamma|_{TT} = \mathrm{curl} \circ (-\Delta + 2) \quad \text{has eigenvalues} \quad \pm\, n(n^2-1),$$

Branson's multipliers exactly; the $1/\sqrt{2}$ of the singular values is $|C|^2 = 2|\star C|^2$. The covariance is explicit. For a conformal map Φ of S^3 with $\Phi^*\gamma = \Omega^2\gamma$, naturality and $C(\Omega^2 g) = C(g)$ give $DC_\gamma(\Omega^{-2}\Phi^*H) = \Phi^*DC_\gamma(H)$, and since $\epsilon_c{}^{ab}$ scales by Ω^{-1},

$$DY_\gamma(\Omega^{-2}\Phi^*H) = \Omega\,\Phi^*DY_\gamma(H),$$

with conformal weights 0 and 3: $2\star DF_\gamma$ is Branson's third-order intertwiner on trace-free tensors (Branson 1995). With Theorem E(1)–(3) it closes the exact sequence

$$0 \to \mathfrak{so}(4,1) \to \mathfrak{X}(S^3) \xrightarrow{\mathcal{K}} \mathrm{Sym}^2_0 \xrightarrow{2\star DF_\gamma} TT \to 0,$$

exact at Sym^2_0 by Theorem E and onto TT because $2\star DF_\gamma$ is invertible there. (`test_629`: the curl spectrum and $\mathrm{curl}^2 = \nabla^*\nabla + 3$ on TT for levels up to 7, by the Peter–Weyl decomposition of $SU(2)$; the covariance exactly for a Möbius dilation.)

The ζ-function $\sum_{n\geq 3} 2(n^2-4)\lambda_n^{-s}$ continues to $s = 0$ with the value

$$2\big[\zeta_H(-2,3) - 4\zeta_H(0,3)\big] = 10$$

for $\lambda_n = n(n^2-1)$, for $\lambda_n = \frac{1}{2}n^2(n^2-1)^2$, and indeed for any $\lambda_n = c\,n^p(1 + a_1 n^{-2} + a_2 n^{-4} + \cdots)$, a convergent expansion in even powers of $1/n$. The binomial terms of the continuation would need a pole at $2k - 2 = 1$ or $2k = 1$, and there is none. Odd powers do contribute: $\lambda_n = n^3(1 + a/n)$ gives $\zeta(0) = 10 + 8a/3 - 2a^3/9$ (`test_333`; the two multipliers above are `test_336` P4). Ten is the dimension of the conformal algebra $\mathfrak{so}(4,1)$ of S^3.

Why ten. The count is a theorem. On a closed manifold of odd dimension the heat expansion of a Laplace-type operator L has no t^0 term (Gilkey 1995), so $\zeta_L(0) = -\dim\ker L$; shifting L by a constant changes nothing, since ζ_L has poles only at half-integers. So the regularised count of all

the modes of such an operator is zero, and removing finitely many modes subtracts their number. On S^3 the trace-free tensors split L^2-orthogonally as $\mathrm{TT} \oplus \operatorname{im}\mathcal{K}$, where $\mathcal{K}\xi = \mathcal{L}_\xi\gamma - \frac{2}{3}(\operatorname{div}\xi)\gamma$ is the conformal Killing operator, injective on the complement of the ten conformal Killing fields. On an Einstein manifold the Lichnerowicz Laplacian intertwines the Lie derivative, $\Delta_L\mathcal{L}_\xi\gamma = \mathcal{L}_{\Delta_H\xi}\gamma$ with Δ_H the Hodge Laplacian on 1-forms, and commutes with the trace and with multiplication by γ (Lichnerowicz 1961; Besse 1987, Chapter 1). On trace-free tensors on the unit S^3, $\Delta_L = \nabla^*\nabla + 6$. Hence

$$(-\Delta + 2)\,\mathcal{K} = \mathcal{K}\,(\Delta_H - 4).$$

So $-\Delta+2$ preserves $\operatorname{im}\mathcal{K}$, and, being self-adjoint, preserves TT too. On $\operatorname{im}\mathcal{K}$ it is $\Delta_H - 4$ on 1-forms with the conformal Killing fields removed. Both are Laplace-type, so the trace-free tensors count 0, $\operatorname{im}\mathcal{K}$ counts $0-10$, and TT counts $0-(-10) = 10$ for the eigenvalues n^2-1 of $-\Delta+2$ on TT. By the continuation above, the same 10 holds for every λ_n in the even-power class, the multipliers $n(n^2-1)$ among them. The spectra confirm it sector by sector. The exact part of $\operatorname{im}\mathcal{K}$ is $\mathcal{K}(\nabla f) = 2\nabla_0^2 f$, with f at levels $n \geq 3$, multiplicity n^2, eigenvalue n^2-5: it counts $\zeta(-2) - 1 - 4 = -5$. The coexact part has levels $m \geq 3$, multiplicity $2(m^2-1)$, eigenvalue m^2-4: it counts -5, the six Killing fields removed (`test_627`).

Over all $n \geq 1$ the TT multiplicity counts 4, and the $n = 1$ term $2(1-4) = -6$ makes up the difference. The scalar count is the same theorem for Obata's operator, the trace-free Hessian $f \mapsto \nabla_0^2 f$, whose kernel is the constants and the first harmonics (Obata 1962): the scalar levels $n \geq 3$ count $0-1-4 = -5$. The determinant is $\ln\det_\zeta\,(n(n^2-1))|_{TT} = -14.0371968\ldots$, computed by differentiating the continuation and, independently, from a closed form in Hurwitz zeta values (`test_336` P4).

Proposition 6.2 (conformal directions modulo Lie, and the detector).

1. *Translating the unit sphere by εe gives, as a radial graph, $u = \varepsilon\langle e, \hat{n}\rangle + O(\varepsilon^2)$, a first harmonic. So the four ambient translations are conformal directions that lie in the Lie orbit, and they make the sum in Theorem E(1) non-direct.*
2. *On conformal directions, $\delta R(2u\gamma) = -4(\Delta+3)u$. Because $R(\gamma)$ is constant, δR vanishes on the Lie orbit. For $u \in C^\infty(S^3)$, $2u\gamma$ lies in the Lie orbit iff u lies in the span of the first harmonics iff $\delta R(2u\gamma) = 0$. So δR detects conformal directions modulo the Lie orbit.*
3. *(The shell dipole is local.) In hyperspherical coordinates $\gamma = dr^2 + \sin^2 r\,\hat{\gamma}$, on the punctured region $0 < r < \pi$, the tensor $h = 2\lambda\cos\theta\,dr^2$ equals $2w\gamma + \mathcal{L}_\eta\gamma$ with*

$$w = (\lambda + c\sin r)\cos\theta,$$

for any constant c. The $c\sin r$ member is a translation. On the λ member,

$$\delta R = 4\lambda(2\cot^2 r - 1)\cos\theta,$$

which grows like $8\lambda\cos\theta/r^2$ at the poles. So h is a local deformation on punctured shells, annihilated by DF_γ there and not in the Lie orbit. It does not extend to a smooth tensor on S^3.

Proof. (1) $|\hat{n} + \varepsilon e| = 1 + \varepsilon\langle e, \hat{n}\rangle + O(\varepsilon^2)$, and $\langle e, \hat{n}\rangle$ is a first harmonic.

(2) For $h = 2u\gamma$, $\delta R = -\Delta\operatorname{tr} h + \nabla^a\nabla^b h_{ab} - h^{ab}R_{ab} = -6\Delta u + 2\Delta u - 12u$. Invariance: $\delta_X R = \mathcal{L}_X R = 0$. The equivalence follows from the proof of Theorem E(1).

(3) Write the axisymmetric $\ell = 1$ generator as the 1-form $\eta = q(r)\cos\theta\,dr + k(r)\sin\theta\,d\theta$, with $w = s(r)\cos\theta$, and match components of $h = 2w\gamma + \mathcal{L}_\eta\gamma$, where $(\mathcal{L}_\eta\gamma)_{ab} = \nabla_a\eta_b + \nabla_b\eta_a$. The

rr component gives $q' = \lambda - s$. The $\theta\theta$ component, with which the $\varphi\varphi$ one agrees, gives $k = -s\sin^2 r - q\sin r\cos r$. Substituting both into the $r\theta$ component, $k' - q - 2k\cot r = 0$, the q terms cancel, leaving $-s'\sin r + (s-\lambda)\cos r = 0$. Finally $-\Delta\cos\theta = 2\cos\theta/\sin^2 r$. □

So the $\ell = 1$ kernel is not a direct sum of translations, moduli and Lie directions: the three overlap, and the shell dipoles are local. Theorem E's global kernel is unaffected. Parts (2) and (3) are checked with the full nonlinear operator, exactly, at a point: the detector, its vanishing on a first harmonic, and, on the dipole, $\delta R = 4\lambda(2\cot^2 r - 1)\cos\theta = 5\lambda/26$ at the test point $\cos r = 3/5$, $\cos\theta = 5/13$ (`test_327` P4). The decomposition in (3) is checked symbolically, for every c, with $\eta_r = (c\cos r + q_0)\cos\theta$ and $\eta_\theta = -(c\sin r + q_0\sin r\cos r + \lambda\sin^2 r)\sin\theta$ (`test_336` P5).

Proposition 6.3 (a kernel direction, two paths). *Let $f = x_0x_1|_{S^3}$, a harmonic at level $n = 3$. The paths*

$$g_{\mathrm{conf}}(t) = e^{2tf}\gamma, \qquad g_{\mathrm{rad}}(t) = e^{2tf}(\gamma + t^2\, df\otimes df)$$

have the same first variation $2f\gamma \in \ker DF_\gamma$. The second is the metric of the embedding $x_t(p) = e^{tf(p)}p$, so $F(g_{\mathrm{rad}}(t)) = 0$ for every t small enough that $g_{\mathrm{rad}}(t) \in \mathcal{U}$. For the first,

$$F(g_{\mathrm{conf}}(t)) = -t^2\, DF_\gamma(df\otimes df) + O(t^3), \qquad DF_\gamma(df\otimes df) \neq 0.$$

So the straight path along a kernel direction leaves the zero set at second order. The kernel is tangent to the realizable set, here along the radial lift, but the straight line is not.

Proof. $|d(e^{tf}p)|^2 = e^{2tf}(\gamma + t^2 df^2)$ because $p\cdot dp = 0$. Then $g_{\mathrm{conf}} = g_{\mathrm{rad}} - t^2 e^{2tf} df\otimes df$, and Taylor's theorem about $g_{\mathrm{rad}}(t)$ gives the expansion. That $DF_\gamma(df\otimes df) \neq 0$ is an exact computation at one point (`test_327` P3). There the whole expansion is also checked with the full nonlinear operator: the t^0 and t^1 terms vanish, the t^2 term is $-DF_\gamma(df\otimes df)$, and the radial lift gives zero through t^2. □

The expansion is the audit's (XP02). The companion paper gives a global form of the same fact: along g_{conf} the total absolute curvature has the deficit $J = 2\pi^2 - \pi^2 t^4/12 + O(t^6)$, which certifies non-realizability for small $t \neq 0$.

Remark 6.4 (local structure near γ). Theorem E(5) is the input an implicit-function argument needs: modulo the diffeomorphism action, the realizable set near γ should be a manifold modelled on the conformal directions. We do not carry this out. On the convex branch the companion paper describes the realizable set globally and at finite amplitude: the radial graphs, up to rigid motion. The equivariant construction is listed in §13.

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7. A worked non-round family: shells

This section treats metrics foliated by round 2-spheres. There F can be computed in closed form, and the realizable members can be written down. Everything is local on the region where the shells are nondegenerate, $\Phi > 0$.

7.1 Shell metrics and their realization

A **shell metric** is

$$g_0 = A(r)^2 dr^2 + \Phi(r)^2\hat{\gamma}, \qquad \hat{\gamma} \text{ the unit round metric on } S^2,$$

with $A, \Phi > 0$. Two sectional curvatures control it:

$$Q = \frac{A^2 - \Phi'^2}{A^2\Phi^2} \quad \text{(tangential)}, \qquad P = \frac{A'\Phi' - A\Phi''}{A^3\Phi} \quad \text{(radial)},$$

and in the orthonormal frame $\mathbb{M}_0 = \mathrm{diag}(Q, P, P)$. We work on the **positive branch** $Q, P > 0$. There g_0 is realized by the hypersurface of revolution $x = (\Phi(r)\hat{n}, Z(r)) \subset \mathbb{R}^3 \times \mathbb{R}$ with $Z' = \sqrt{A^2 - \Phi'^2}$. Its principal curvatures are

$$\kappa_{\mathrm{sph}} = \sqrt{Q} \text{ (twice)}, \qquad \kappa_{\mathrm{mer}} = P/\sqrt{Q},$$

so $\tilde{K}_0 = \mathrm{diag}(P/\sqrt{Q}, \sqrt{Q}, \sqrt{Q})$, and T_1 is positive definite: the TT slice serves along the family. On the round base, $A = 1$ and $\Phi = \sin r$, we have $Q = P = 1$.

7.2 First order: the radial-coefficient family

Deform the radial coefficient by a shell harmonic:

$$A \longmapsto A(1 + \varepsilon Y), \qquad -\hat{\Delta}Y = \Lambda Y, \quad \Lambda = \ell(\ell + 1),$$

that is, $h = 2A^2 Y\, dr^2$. For $\ell \geq 1$ the harmonic Y is singular where the shells degenerate, so the statement is local on $\{\Phi > 0\}$.

Theorem F (the shell classification). *On every base of the positive branch, the linearised residual $C = DF_{g_0}(h)$ has exactly the components*

$$C_{rar} = -\frac{\Lambda - 2}{2} \cdot \frac{A^3}{\Phi\sqrt{A^2 - \Phi'^2}}\, \partial_a Y, \qquad C_{abc} = -\frac{\Lambda - 2}{2} \cdot \frac{\sqrt{A^2 - \Phi'^2}}{A\Phi P}\big(\hat{\gamma}_{bc}\partial_a Y - \hat{\gamma}_{ac}\partial_b Y\big),$$

$$C_{rab} = \mathfrak{D}(r)\, T_{ab}[Y], \qquad T_{ab}[Y] = \big(\hat{\nabla}_a\hat{\nabla}_b - \tfrac{1}{2}\hat{\gamma}_{ab}\hat{\Delta}\big)Y,$$

with a, b, c angular, all other components zero, and

$$\mathfrak{D} = -\partial_r\Big(\frac{\sqrt{Q}}{P}\Big) + \frac{\Phi'}{\Phi} \cdot \frac{\sqrt{Q}}{P} - \frac{A\Phi'}{\sqrt{A^2 - \Phi'^2}},$$

a function of r alone that vanishes identically on the round base. Consequently:

1. *(Kernel.) $C = 0$ iff $\ell \in \{0, 1\}$. At $\ell = 0$ all three families vanish because $\partial_a Y = 0$ and $T[\mathrm{const}] = 0$. At $\ell = 1$ the first two vanish because $\Lambda = 2$, and the third by Obata's identity on S^2, $T[Y_1] = 0$.*
2. *(Exclusion, for this family.) For $\ell \geq 2$ the first family alone is nonzero wherever $\partial_a Y \neq 0$, since its prefactor is positive on the branch. So no deformation **of this family** with $\ell \geq 2$ is realizable at first order, on any base. Shapes outside the family are not barred: radial graphs are realizable for every ℓ (Theorem E(1)).*

Proof. Write the lapse of the r-foliation as $\mathcal{N} = A(1 + \varepsilon Y)$. The shell metric $\Phi^2\hat{\gamma}$ is unchanged, and the shells are umbilic with $\kappa_{ij} = \tilde{\theta} q_{ij}$, $\tilde{\theta} = \Phi'/(\mathcal{N}\Phi)$. Three identities of the foliation give the five components of $\mathbb{M}_1 = \delta\mathbb{M}$:

- the Gauss (Hamiltonian) identity gives $\mathbb{M}_1{}^r{}_r = 2Y\Phi'^2/(A^2\Phi^2)$;
- the momentum identity gives $\mathbb{M}_1{}^r{}_\theta = -\Phi'\partial_\theta Y/(A^2\Phi)$ and $\mathbb{M}_1{}^\theta{}_r = -\Phi'\partial_\theta Y/\Phi^3$;

- the radial Riccati identity gives $\mathbb{M}_1{}^{\theta}{}_{\theta} = -\cot\theta\,\partial_\theta Y/\Phi^2 - 2YP$ and $\mathbb{M}_1{}^{\varphi}{}_{\varphi} = -\partial_\theta^2 Y/\Phi^2 - 2YP$ (axisymmetric Y; the general case is the same with covariant derivatives). The $\theta\theta$ and $\varphi\varphi$ entries carry each other's Hessian components because $\mathbb{M}$ acts on the shell block as a 2×2 adjugate.

These are coordinate components. Lowered with g_0, $\mathbb{M}_1$ is symmetric, with $(\mathbb{M}_1)_{r\theta} = -\Phi'\partial_\theta Y/\Phi$ (`test_344` P2).

Differentiating $\tilde{K} = \sqrt{\det\mathbb{M}}\,\mathbb{M}^{-1}$ gives, for the endomorphism,

$$\delta\tilde{K} = \sqrt{\det\mathbb{M}_0}\,[S\,\mathbb{M}_0^{-1} - \mathbb{M}_0^{-1}\mathbb{M}_1\mathbb{M}_0^{-1}], \qquad S = \tfrac{1}{2}\mathrm{tr}(\mathbb{M}_0^{-1}\mathbb{M}_1) = \frac{Y\Phi'^2}{A^2-\Phi'^2} + \frac{\Lambda Y}{2\Phi^2 P} - 2Y.$$

Lower the index: $\tilde{h}_{ab} = \delta(\tilde{K}_{ab}) = g_{0,ac}\,\delta\tilde{K}^c{}_b + h_{ac}\tilde{K}_0{}^c{}_b$, whose metric term is $h_{rr}\tilde{K}_0{}^r{}_r = 2A^2YP/\sqrt{Q}$ in the rr slot. Then add the connection's variation. The Codazzi tensor has three $SO(3)$-covariant parts:

- The (r,a,r) part collapses through the factorisations $A^3\Phi'' - A^2A'\Phi' - A\Phi'^2\Phi'' + A'\Phi'^3 = -QPA^5\Phi^3$ and $(A'\Phi' - A\Phi'')\sqrt{A^2-\Phi'^2} = PA^4\Phi^2\sqrt{Q}$.
- In the (a,b,c) part the trace term S cancels. The factor $\Lambda - 2$ comes from the term $\Lambda Y/(2\Phi^2P)$ of S against $-2Y$.
- The (r,a,b) part sees only the trace-free part of $\tilde{h}$ on the shell, which is $-(\sqrt{Q}/P)\,T_{ab}[Y]$, and its (r,a) part $A\Phi'\partial_a Y/\sqrt{A^2-\Phi'^2}$. The contractions with $\Gamma^c{}_{ra} = (\Phi'/\Phi)\delta^c{}_a$ give the three terms of $\mathfrak{D}$. The trace parts, including the term in $\Gamma^r{}_{ab}\tilde{h}_{rr}$ with $\Gamma^r{}_{ab} = -(\Phi\Phi'/A^2)\hat{\gamma}_{ab}$, cancel.

Reflection $\varphi \mapsto -\varphi$ kills every component with an odd number of φ indices, so the list is complete. □

Three derivations and their checks. Mathematics paper 1 (Thrussell 2026a) derives Theorem F three ways, with disjoint inputs:

- the foliation identities above;
- direct linearisation of $g \mapsto \mathbb{M} \mapsto \tilde{K} \mapsto \mathrm{Cod}$;
- explicit embeddings.

All three families are reproduced from the full nonlinear chain, in exact jets, on six base–radius configurations: the round base, three non-round bases (one at two radii) and a constant-ratio base. This is done at $\ell = 0,1,2,3$ and at two angular points, together with the vanishing of the other five components and the kernel at $\ell \in \{0,1\}$ (`test_337` P1–P6). The independent audit had rebuilt the chain on a further base, $A = 1 + r^2$, $\Phi = r + r^3/10$, at $\ell = 1,2,3$ (P1-X02). The identity $\|T[Y_\ell]\|^2 = \frac{1}{2}\Lambda(\Lambda-2)\|Y_\ell\|^2$ (Bochner) shows that the third family vanishes exactly at $\ell \in \{0,1\}$, by two mechanisms: the gradient at $\ell = 0$, and Obata at $\ell = 1$.

Proposition 7.1 (the prefactor is a curvature invariant). *On the positive branch,*

$$\mathfrak{D} = \frac{1}{\kappa_{\mathrm{mer}}}\,\partial_r \ln\frac{\kappa_{\mathrm{mer}}}{\kappa_{\mathrm{sph}}}.$$

So $\mathfrak{D}(r_0) = 0$ iff the ratio of principal curvatures is stationary at r_0. And $\mathfrak{D} \equiv 0$ iff that ratio is a constant c, which is the two-parameter family $\Phi'^2 = A^2(1 - C_0^2\Phi^{2c})$, with the round base at $C_0 = c = 1$.

Proof. $\sqrt{Q}/P = 1/\kappa_{\mathrm{mer}}$. The Codazzi equation of the base, $\kappa'_{\mathrm{sph}} = (\Phi'/\Phi)(\kappa_{\mathrm{mer}} - \kappa_{\mathrm{sph}})$, collapses the three terms. □

This is checked symbolically on a general base and on the constant-ratio family, and against the full nonlinear chain on five bases, two of them (the round one and $A = 2$, $\Phi = 2r - r^2$) with $\mathfrak{D} \equiv 0$ (`test_337` P4, P8, P9).

Remark 7.2 (a quantitative form). Write $Z' = \sqrt{A^2 - \Phi'^2}$, and let $|C|^2 = C_{ijk}C^{ijk}$ be the full contraction. The first family alone gives, on an interior shell,

$$\int_{S^2_r} |C|^2 \, dA \;\geq\; \frac{\Lambda(\Lambda-2)^2}{2} \cdot \frac{A^2}{\Phi^2 Z'^2} \int_{S^2} Y_\ell^2 \, d\Omega$$

(the audit's P1-X05). The two index orders C_{rar} and $C_{arr} = -C_{rar}$ each contribute half. This is a bound on the linearised residual, not an exclusion at finite amplitude.

7.3 Hypersurfaces swept by round spheres

The $\ell = 1$ kernel of Theorem F is populated by actual hypersurfaces, and in two ways.

Let $x(r, \omega) = x_0(r) + \Phi(r)R(r)(\omega, 0)$, with $x_0(r) \in \mathbb{R}^4$, $R(r) \in SO(4)$ and $\omega \in S^2$: a one-parameter family of round 2-spheres in E^4. Write

$$R^{\mathsf{T}} x_0' = (a, \alpha) \in \mathbb{R}^3 \times \mathbb{R}, \qquad R^{\mathsf{T}} R' = \begin{pmatrix} \rho\times & -b \\ b^{\mathsf{T}} & 0 \end{pmatrix}.$$

So a is the in-plane velocity of the centre and α its velocity out of the shell's 3-plane, ρ is a rotation within the 3-plane, and b tilts the 3-plane toward its normal. Label points of each shell by a Möbius chart $z = p + iq$ with

$$E = \frac{|z - (u_1 + iu_2)|^2 + u_3^2}{2u_3}, \qquad \hat{n} = \Big(\frac{p-u_1}{E}, \frac{q-u_2}{E}, 1 - \frac{u_3}{E}\Big),$$

for functions $u_1, u_2, u_3 > 0$ of r, and write $\vec{W} = (u_1', u_2', u_3')/u_3$.

Lemma 7.3 (the chart flow). *$E'/E = -\vec{W} \cdot \hat{n}$ exactly. So the chart flow is a conformal Killing field of S^2 with no $\ell = 0$ part and no multipole above $\ell = 1$.*

Proof. Differentiate E, and use $(p - u_1)/E = n_1$, $(q - u_2)/E = n_2$, $u_3/E = 1 - n_3$. □

Proposition 7.4 (swept spheres).

1. *In coordinates (r, p, q) that follow the orthogonal trajectories of the shells, the induced metric is*
$$g = \big[H^2 + (\alpha + \Phi\, b \cdot \hat{n})^2\big] dr^2 + \frac{\Phi^2}{E^2}(dp^2 + dq^2), \qquad H = \Phi' + \Phi\, \vec{W} \cdot \hat{n},$$
with $a = \Phi\vec{W}$, the chart's rotation absorbed by ρ.
2. *Conversely, every metric of this form, for any functions $\Phi > 0$, α, b and chart data (u_1, u_2, u_3), with positive radial coefficient, is induced by such a family. The family is obtained by integrating $R' = R\Omega$ and $x_0' = R(\Phi\vec{W}, \alpha)$, where Ω is built from b and the chart's rotation $\rho = (u_2', -u_1', 0)/u_3$.*

Proof. $R^{\mathsf{T}}\partial_r x$ has three parts:

- along $\hat{n}$: $\Phi' + a \cdot \hat{n}$;
- along the 3-plane's normal: $\alpha + \Phi\, b \cdot \hat{n}$;

- tangential: $a - (a \cdot \hat{n})\hat{n} + \Phi\rho \times \hat{n}$.

The orthogonal-trajectory chart moves $\hat{n}$ by minus the tangential part, divided by Φ. That is a conformal Killing field (a gradient of a first harmonic plus a rotation), so the chart is Möbius at each r. The shells are umbilic in the hypersurface. With respect to the unit normal m of a shell within the hypersurface, its second fundamental form is $-(\hat{n} \cdot m)\, g_S/\Phi$, a multiple of the shell's induced metric g_S (here $\hat{n}$ stands for $R(\hat{n}, 0)$). So the conformal class of $dp^2 + dq^2$ is preserved along the trajectories, and $(\ln E^{-2})' = 2(a/\Phi) \cdot \hat{n}$.

- By Lemma 7.3, $a = \Phi\vec{W}$.
- The radial coefficient is the squared length of the normal part of $\partial_r x$.
- For (2), choose x_0 and R so that the three parts match the data, and run the computation backwards. The chart flow is $\partial_r \hat{n}|_{p,q} = -\vec{W} + (\vec{W} \cdot \hat{n})\hat{n} - \rho \times \hat{n}$ with $\rho = (u_2', -u_1', 0)/u_3$, so that is the rotation for which the r-lines are orthogonal trajectories (`test_344` P1). □

Checks. An explicit embedding, built by integrating these equations with α and b chosen independently of the drift, reproduces the metric to 10^{-10} (`test_338` P1). A member is an immersion wherever its radial coefficient is positive, but need not lie in $\mathcal{U}$: the flat members of Corollary 7.5(1) have $\mathbb{M} \equiv 0$. The polynomial forms $\mathcal{T}$ and $\mathcal{R}$ of §8 vanish on every member (Proposition 8.1(4)), and $F = 0$ wherever $\det \mathbb{M} > 0$. This is checked exactly, in polynomial form, on a general member with $\det \mathbb{M} > 0$ (`test_338` P3).

Corollary 7.5 (two subfamilies, and the first-order picture).

1. *(Drift.) With $b = 0$ and $\alpha = 0$ every shell stays in one 3-plane, so the hypersurface is a piece of that 3-plane and $H^2 dr^2 + \Phi^2 E^{-2}(dp^2 + dq^2)$ is Euclidean 3-space in non-concentric spherical coordinates. With $\alpha = s\Phi'$ and $b = s\vec{W}$, $s^2 = k/(1-k)$ and $0 \le k < 1$, it is*

$$\frac{H^2}{1-k}\, dr^2 + \frac{\Phi^2}{E^2}(dp^2 + dq^2),$$

   Szekeres's quasi-spherical metric (Szekeres 1975). Its embedding by shift, rotation and tilt is Buckley and Schlegel's (2020, §VIII); see also Hellaby and Buckley (2020).
2. *(Tilt.) With no drift, $\vec{W} = 0$, the metric $[\Phi'^2 + (\alpha + \Phi\, b \cdot \hat{n})^2]dr^2 + \Phi^2\hat{\gamma}$ is realizable for any α and b. If $\alpha \not\equiv 0$, its radial coefficient has the $\ell = 1$ part $2\alpha\Phi\, b \cdot \hat{n}$ at first order in b. On the positive branch $\alpha \neq 0$ automatically, since there $A_0^2 = \Phi'^2 + \alpha^2 > \Phi'^2$ (§7.1).*
3. *(First order.) About the shell metric with $A_0^2 = \Phi'^2 + \alpha^2$,*

$$\frac{\delta A}{A_0} = \frac{\Phi\,(\Phi'\vec{W} + \alpha\, b) \cdot \hat{n}}{A_0^2},$$

   which is regular where $\Phi' = 0$. The tilt changes nothing else. Its first-order variation is the $\ell = 1$ change of the radial coefficient with amplitude $\Phi\alpha b/A_0^2$: Theorem F's direction when b makes that amplitude constant, and in the first-order kernel for every b. The drift also shears the shell labels. In shell coordinates, with fixed labels on each shell, its variation is that direction plus an off-diagonal $\ell = 1$ term $\varepsilon\beta \sin\theta\,(dr\, d\theta + d\theta\, dr)$, which is itself in the first-order kernel. On the round base, for the radial modulation $2\varepsilon\cos\theta$, $\beta = -\tan r$.

Proof. (1) is Proposition 7.4 with those choices; flatness is the case $k = 0$. (2) and (3) are Proposition 7.4 and a first-order expansion. For the drift in shell coordinates, $x = x_0(r) + \Phi\hat{n}$ gives $g_{rr} = |x_0' + \Phi'\hat{n}|^2 + \ldots$ and $g_{r\theta} = \Phi\, x_0' \cdot \partial_\theta \hat{n}$. The drift's variation and the $\ell = 1$ direction both lie in the first-order kernel, so their difference, the off-diagonal term, does too. □

The shell-coordinate formulas, with the round-base value of β, are checked symbolically (`test_338` P8). The first-order kernel statements, for an r-dependent $\ell = 1$ amplitude and for the off-diagonal term alone, are checked with the full operator on three non-round bases (`test_337` P7).

Part (2) is checked with the full operator, exactly, on a non-round base: at two points, with $\mathbb{M}$ positive definite, and also for an axisymmetric tilt (`test_338` P2, P5, P6).

In plain terms. In three dimensions an $\ell = 1$ change of the radial coefficient of nested spheres can only come from translating their centres, and the translation also shears the labels on the shells. In E^4 it can come instead from tilting the spheres' 3-planes, and then nothing else changes. An earlier version asserted that the $\ell = 1$ amplitude must equal $\Phi\,\vec{W}\cdot\hat{n}$. Part (2) refutes this whenever $\alpha \not\equiv 0$ (Appendix C).

Proposition 7.6 (the Lorentz model of a shell; nesting). *Put*

$$V = \frac{1}{2u_3}(1,\ -u_1,\ -u_2,\ u_1^2+u_2^2+u_3^2) \in \mathbb{R}^4, \qquad \mathcal{Q}(V) = 4(V_0V_3 - V_1^2 - V_2^2),$$

a quadratic form of signature $(1,3)$*, with polarisation* $\mathcal{B}(X,Y) = \frac{1}{2}(\mathcal{Q}(X+Y) - \mathcal{Q}(X) - \mathcal{Q}(Y))$*, so that* $\mathcal{B}(X,X) = \mathcal{Q}(X)$*. Then:*

1. $\mathcal{Q}(V) = 1$*;*
2. $-\mathcal{Q}(dV) = (du_1^2 + du_2^2 + du_3^2)/u_3^2$*, so* (u_1,u_2,u_3) *is* V *on the hyperboloid model of* H^3 *and* $|\vec{W}| = |V'|$*;*
3. $E = \frac{1}{2}\mathcal{B}(V,N)$*, with* $N = (1,-p,-q,p^2+q^2)$ *null;*
4. *with* $\mathcal{Y} = \Phi V$*,* $\mathcal{Q}(\mathcal{Y}') = \Phi'^2 - \Phi^2|\vec{W}|^2$*.*

So $H > 0$ *in every direction iff* $\Phi' > \Phi|\vec{W}|$*, iff* $r \mapsto \Phi V(r)$ *is a future-directed timelike curve,* $\mathcal{B}(\mathcal{Y}', V) = \Phi' > 0$*. For the flat member of Corollary 7.5(1) this is the condition that the shells are nested round spheres. In Szekeres's setting it is the no-shell-crossing condition of Hellaby and Krasiński (2002, their eq. 92), restated as a causal condition. The Möbius freedom of the chart is* $SO^+(1,3)$ *acting on* V*.*

Proof. (1)–(4) are direct computation, and the last statement is Lemma 7.3 with $\min_{\hat{n}} \vec{W}\cdot\hat{n} = -|\vec{W}|$. □

Items (1)–(4) are checked symbolically in `test_330` P6, and Lemma 7.3 in `test_330` P5. Each shell is intrinsically round, with Gauss curvature $1/\Phi^2$ for every chart (`test_338` P7).

7.4 Second order at $\ell = 1$

The $\ell = 1$ direction of Theorem F is a first-order kernel, not a symmetry.

Theorem G (second order, round base). *On the round base, on interior shells* $0 < r < \pi$*, hold the deformation* $A \mapsto 1 + \varepsilon\cos\theta$ *pure to second order. Then the residual first appears at* $O(\varepsilon^2)$*:*

$$C^{(2)}_{r\theta r} = -\frac{2\sin\theta\cos\theta\cos^2 r}{\sin^4 r}, \quad C^{(2)}_{r\theta\theta} = -\frac{\sin^2\theta\cos r}{\sin^3 r}, \quad C^{(2)}_{r\varphi\varphi} = \frac{\sin^4\theta\cos r}{\sin^3 r}, \quad C^{(2)}_{\theta\varphi\varphi} = -\frac{2\sin^3\theta\cos\theta}{\tan^2 r}.$$

Its angular content is pure $\ell = 2$*. The mechanism is* $(\ell{=}1)\times(\ell{=}1) \to \ell{=}0 \oplus \ell{=}2$*, with the* $\ell = 0$ *part killed by the gradient prefactor. The obstruction is removed in two ways:*

1. *(Tilt.) A second-order correction* $\delta A/A = \varepsilon\cos\theta + \varepsilon^2\cot^2 r\,[\frac{1}{6} + \frac{1}{3}P_2(\cos\theta)]$*, with no off-diagonal term. This is the swept-sphere deformation of the round base with shells in tilting 3-planes,* $b = (\varepsilon/\sin^2 r)\hat{e}_3$ *(Corollary 7.5(2)), and the* r*-only* $\ell = 0$ *part never obstructs.*

2. *(Drift.) Write the metric as* $g_{rr} = 1 + 2\varepsilon\cos\theta + \varepsilon^2\mu(r)$ *and* $g_{r\theta} = \varepsilon\beta(r)\sin\theta$*, with the shells unchanged. The off-diagonal term is the one a drift of the centres carries in shell coordinates (Corollary 7.5(3)). The* $O(\varepsilon^2)$ *residual is quadratic in* β*. Its* $r\theta r$ *component is* $-(\beta + \tan r)^2/\tan^2 r$ *times* $2\sin\theta\cos\theta/\sin^4 r$*: a perfect square, with its double root at* $\beta = -\tan r$*, the value the drifting hypersurface carries. With* $\beta = -\tan r$ *and* $\mu = \sec^2 r$ *every component vanishes.*

Proof and scope.

- *Components.* They are computed with the full nonlinear chain in exact jets, at two radii and two angles, with the other five components zero (`test_339` P1). The audit had recomputed all four (P1-X03).
- *Route 1.* It removes all components with one function, and its identification with the tilt is the exact expansion of Corollary 7.5(2) (`test_338` P4). That it removes every component through $O(\varepsilon^2)$ is `test_339` P2.
- *Route 2.* It is computed exactly, for $s = 0, 1, 2, 3$ in $\beta = -s\tan r$ (`test_339` P3). The radial companion μ does not enter the residual.
- *Interior shells.* The statement is on $0 < r < \pi$, and route 2 also needs $\cos r \neq 0$: its $\beta = -\tan r$ and $\mu = \sec^2 r$ are singular on the equator. Both $\cos\theta$ and $\cot^2 r$ are singular at the poles, and nothing here is a smooth global deformation of S^3. □

The two routes are the two shellwise motions of Corollary 7.5. Second order selects neither: it rigidifies the extension within each class.

Theorem 7.7 (general bases). *Let* $A^2dr^2 + \Phi^2\hat{\gamma}$ *be a shell metric on an interval of the positive branch, so that* $\alpha = \sqrt{A^2 - \Phi'^2} > 0$*, and suppose* $\Phi' \neq 0$ *there. Give the drift of Corollary 7.5 the radial modulation* $2\varepsilon A^2\cos\theta$*: in shell coordinates* $g_{rr} = A^2(1 + 2\varepsilon\cos\theta) + \varepsilon^2\mu$ *and* $g_{r\theta} = \varepsilon\beta\sin\theta$*, with* $\beta_{\mathrm{pred}} = -\Phi A^2/\Phi'$ *and* $\mu_{\mathrm{pred}} = A^4/\Phi'^2$*. Let* $g(s)$ *have* $\beta = s\,\beta_{\mathrm{pred}}$*,* $\mu = \mu_{\mathrm{pred}}$ *and the shells unchanged. Then*

$$F(g(s)) = -\varepsilon^2(1-s)^2\,DF_{g_0}\Big(\frac{A^4}{\alpha^2}\cos^2\theta\,dr^2\Big) + O(\varepsilon^3).$$

So every $O(\varepsilon^2)$ *component is a constant multiple of* $(1-s)^2$*, with a double root at* $s = 1$*, where* $g(1)$ *is the metric of the drifting hypersurface; and a change of the radial coefficient by a function of* r *alone never obstructs.*

Proof. Combine the drift and the tilt in one swept family, with fixed shell labels and no rotation within the 3-planes: $a = \Phi\vec{W} = (\varepsilon s A^2/\Phi')\,\hat{e}_3$ and $b = (\varepsilon(1-s)A^2/(\alpha\Phi))\,\hat{e}_3$. As in the proof of Corollary 7.5, $R^{\mathsf{T}}\partial_r x = (a + \Phi'\hat{n},\ \alpha + \Phi\, b\cdot\hat{n})$, so its metric $\hat{g}(s)$ has the shells unchanged, $\hat{g}_{r\theta} = \Phi\, a\cdot\partial_\theta\hat{n} = \varepsilon s\beta_{\mathrm{pred}}\sin\theta$, and

$$\hat{g}_{rr} = A^2 + 2\varepsilon A^2\cos\theta + \varepsilon^2\Big[s^2\frac{A^4}{\Phi'^2} + (1-s)^2\frac{A^4}{\alpha^2}\cos^2\theta\Big].$$

For small ε, $\hat{g}(s)$ is realizable, so $F(\hat{g}(s)) = 0$, and

$$g(s) - \hat{g}(s) = -\varepsilon^2\Big[(s^2-1)\frac{A^4}{\Phi'^2} + (1-s)^2\frac{A^4}{\alpha^2}\cos^2\theta\Big]dr^2.$$

A variation $f(r)\,dr^2$ of the base keeps it a shell metric on the positive branch, which is realizable (§7.1), so $DF_{g_0}(f\,dr^2) = 0$. Expanding F about $\hat{g}(s)$, $F(g(s)) = DF_{g_0}(g(s) - \hat{g}(s)) + O(\varepsilon^3)$, which is the formula. □

The proof is an independent machine-assisted review's (Appendix V). The combined family is checked symbolically (`test_354` P5). The same structure is checked in exact arithmetic on three disjoint non-round bases, at eight base–radius pairs and three angular points, and on two bases with $\mathfrak{D} \equiv 0$: each $O(\varepsilon^2)$ component quadratic in s, with a double root at $s = 1$, and the residual unchanged when μ is rescaled or removed with β frozen (`test_340`).

If instead β and μ are frozen at their values at the point, the $(\Lambda - 2)$ families are unchanged. The $\mathfrak{D}$ family, however, acquires a simple root at $s = 1$, with a second root that varies from base to base. The freezing drops β', and only the $\mathfrak{D}$ family sees it. An earlier version reported that frozen picture, a transversal zero on a general base, as a property of the drift (Appendix C). On the round base $\beta_{\text{pred}} = -\tan r$, which is Theorem G(2).

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8. The degenerate locus

F is built from $\tilde{K} = \sqrt{\det \mathbb{M}}\, \mathbb{M}^{-1}$ and is undefined where $\det \mathbb{M} = 0$. This section asks what can be said there. Two polynomial forms of the equation remove the root and the inverse. They show exactly what survives at $\det \mathbb{M} = 0$, and on realizable metrics the answer is: nothing that $\mathbb{M}$ can see.

8.1 Two polynomial forms

Put $\mathbb{P} = \operatorname{adj} \mathbb{M}$, raised and lowered with g. By (2.3), on a realizable metric

$$\mathbb{P} = \operatorname{adj} \operatorname{adj} K = (\det K)\, K,$$

Thomas's identity (9.3). $\mathbb{P}$ is polynomial in the curvature and even under $K \mapsto -K$. On $\{\det \mathbb{M} > 0\}$, $\mathbb{P} = \sqrt{\det \mathbb{M}}\, \tilde{K}$, and $\mathbb{P}$ is positive definite iff $\tilde{K}$ is definite (`test_341` P1).

Proposition 8.1 (two polynomial forms). *For every metric g put*

$$\mathcal{T}(g) = 2(\det \mathbb{M}) \operatorname{Cod}(\mathbb{P}) - d(\det \mathbb{M}) \wedge \mathbb{P}, \qquad (d\varphi \wedge T)_{abc} = \partial_a \varphi\, T_{bc} - \partial_b \varphi\, T_{ac},$$

$$\mathcal{R}(g)_{ab}{}^{d} = \tfrac{1}{2}\big(\partial_a \det \mathbb{M}\ \delta_b{}^{d} - \partial_b \det \mathbb{M}\ \delta_a{}^{d}\big) - \big((\nabla_a \mathbb{M})\, \mathbb{P}\big)^{d}{}_{b} + \big((\nabla_b \mathbb{M})\, \mathbb{P}\big)^{d}{}_{a}.$$

1. *$\mathcal{T}$ is polynomial in the 3-jet of g and in g^{-1}: of degree four in the curvature and one in its derivative. On $\{\det \mathbb{M} > 0\}$, $\mathcal{T} = 2(\det \mathbb{M})^{3/2} F$.*
2. *$\mathcal{R}$ is polynomial of degree two in the curvature and one in its derivative. On $\{\det \mathbb{M} > 0\}$, $\mathcal{R}_{ab}{}^{d} = \sqrt{\det \mathbb{M}}\, F_{ab}{}^{c}\, \mathbb{M}_{c}{}^{d}$.*
3. *So on $\{\det \mathbb{M} > 0\}$, $\mathcal{T} = 0$ iff $\mathcal{R} = 0$ iff $F = 0$, on both branches. Both third-order symbols are $\sigma(F)$ composed with invertible algebraic maps, so Theorems B, C and D carry over to both on $\mathcal{U}$.*
4. *On a realizable metric both vanish everywhere, the degenerate locus included.*
5. *For every metric, the nine components of $\mathcal{R}$ satisfy the three linear identities $\mathcal{R}_{ab}{}^{b} = 0$, which are the contracted Bianchi identity, and the cyclic identity $\sum_{\text{cyc}(a,b,c)} \mathcal{R}_{ab}{}^{d}\, \mathbb{P}_{dc} = 0$.*

Proof. Write $\varphi = \sqrt{\det \mathbb{M}}$, so $\mathbb{P} = \varphi \tilde{K}$ and $2\varphi\, d\varphi = d \det \mathbb{M}$.

(1) $\operatorname{Cod}(\varphi T) = d\varphi \wedge T + \varphi \operatorname{Cod} T$ gives $2\varphi^2 \operatorname{Cod}(\varphi \tilde{K}) - 2\varphi\, d\varphi \wedge \varphi \tilde{K} = 2\varphi^3 \operatorname{Cod} \tilde{K}$.

(2) Differentiate $\mathbb{M}\tilde{K} = \varphi\, \mathrm{Id}$: $\mathbb{M}\, \nabla_a \tilde{K} = \partial_a \varphi\, \mathrm{Id} - (\nabla_a \mathbb{M})\tilde{K}$. Antisymmetrise in a, b and multiply by φ. Jacobi's formula $\partial_a \det \mathbb{M} = \operatorname{tr}\big(\mathbb{P}\, \nabla_a \mathbb{M}\big)$ shows that $\mathcal{R}$ is polynomial.

(3) $\mathbb{M}$ is invertible and $\varphi > 0$ there. Derivatives of the factors are of lower order.

(4) Where $\det \mathbb{M} > 0$ this is (3). At a point of a realizable metric where $\det \mathbb{M} = 0$, $\det K = 0$, so $\operatorname{rank} \mathbb{M} \le 1$ by Proposition 8.2(1), and both forms vanish there by Proposition 8.2(2). No derivative of K is used.

(5) Tracing over $b = d$ leaves $\left((\nabla_b \mathbb{M})\,\mathbb{P}\right)^b{}_a = (\operatorname{div} \mathbb{M})_c\, \mathbb{P}^c{}_a$, and $\operatorname{div} \mathbb{M} = -\operatorname{div} G = 0$. Where $\det \mathbb{M} > 0$, $F_{abc} = \mathcal{R}_{ab}{}^d\, \mathbb{P}_{dc}/\varphi^3$, so the cyclic identity of $\operatorname{Cod} \tilde{K}$ (Lemma 4.6(1)) gives the second. It is a polynomial identity in the jet that holds on an open set of jets, so it holds for every metric. □

$\mathcal{R}$ has the bidegree of Rivertz's six equalities (§1), and its six components have a tensor form. Dualise the antisymmetric pair, as one passes from the Cotton tensor to the Cotton–York tensor:

$$W^{cd} = \tfrac{1}{2}\, \epsilon^{cab}\, \mathcal{R}_{ab}{}^d, \qquad \mathcal{R}_{ab}{}^d = \epsilon_{cab} W^{cd}.$$

Then $\epsilon_{ecd} W^{cd} = \mathcal{R}_{de}{}^d$, so the three trace identities say exactly that W is symmetric: $W(g)$ is a symmetric tensor of bidegree $(2,1)$ in the curvature and its derivative, defined for every metric. The cyclic sum is totally antisymmetric, and $\epsilon^{abc} \sum_{\text{cyc}} \mathcal{R}_{ab}{}^d\, \mathbb{P}_{dc} = 6 \operatorname{tr}(W\mathbb{P})$, so the cyclic identity is the one scalar identity $\operatorname{tr}(W\mathbb{P}) = 0$, which leaves the five of Theorem C (`test_626`). The six components of W and Rivertz's six equalities span the same space (Theorem 8.6).

Thomas clears the root by the same factor $(\det \mathbb{M})^{3/2}$ as $\mathcal{T}$ (his §10). He then eliminates the unknown and arrives at a resultant system, which has no principal symbol to compute. $\mathcal{T}$ and $\mathcal{R}$ keep a tensor unknown. *Checks:* `test_341` P2 for $\mathcal{T}$, against F computed independently on both branches. For $\mathcal{R}$, `test_331`:

- on a generic metric with $\det \mathbb{M} > 0$, $\mathcal{R} = \sqrt{\det \mathbb{M}}\, F\mathbb{M}$ exactly, with F computed independently and nonzero, together with the identities (5);
- $\mathcal{R} = 0$ on two realizable metrics, one of them non-umbilic;
- $\mathcal{R}$ at the rank-one and rank-two points of §8.2.

8.2 At $\det \mathbb{M} = 0$

Proposition 8.2. *Let* $\det \mathbb{M}(p) = 0$.

1. *(Rank of the adjugate.)* $\operatorname{rank} \operatorname{adj} K$ *is 3, 1 or 0 as* $\operatorname{rank} K$ *is 3, 2 or at most 1. So a rank-2* $\mathbb{M}$ *is never* $\operatorname{adj} K$*, and on a realizable metric* $\det \mathbb{M}(p) = 0$ *forces* $\operatorname{rank} \mathbb{M}(p) \le 1$*:*

$$\operatorname{rank} K = 2 \iff \operatorname{rank} \mathbb{M} = 1, \qquad \operatorname{rank} K \le 1 \iff \mathbb{M} = 0.$$

2. *(Rank at most one.) If* $\operatorname{rank} \mathbb{M}(p) \le 1$*, then* $\mathbb{P}(p) = 0$*,* $d(\det \mathbb{M})(p) = 0$ *and* $\mathcal{T}(p) = \mathcal{R}(p) = 0$*. Both forms are silent at every such point, whatever the rank nearby.*
3. *(Non-realizable points, where the forms are not vacuous.) If* $\operatorname{rank} \mathbb{M}(p) = 2$*, with kernel spanned by a unit vector* v *and nonzero eigenvalues* m_1, m_2*, then* $\mathbb{P}(p) = m_1 m_2\, v \otimes v \neq 0$ *and*

$$\mathcal{T}(p) = -\, m_1 m_2\, d(\det \mathbb{M}) \wedge v^\flat \otimes v^\flat.$$

So $\mathcal{T}(p) = 0$ *iff* $d(\det \mathbb{M})(p)$ *is parallel to* $v^\flat$*. This is a genuine first-order condition, which a rank-2 point of a metric can fail.*

Proof. (1) In an eigenframe, $\operatorname{adj} \operatorname{diag}(k_1, k_2, k_3) = \operatorname{diag}(k_2 k_3, k_1 k_3, k_1 k_2)$. (2) $\mathbb{P}(p) = \operatorname{adj} \mathbb{M}(p) = 0$, and by Jacobi's formula $d(\det \mathbb{M})(p) = \operatorname{tr}\big(\mathbb{P}(p)\, \nabla \mathbb{M}(p)\big) = 0$. Every term of $\mathcal{T}$ and $\mathcal{R}$ carries an undifferentiated factor $\det \mathbb{M}$, $d \det \mathbb{M}$ or $\mathbb{P}$ (`test_344` P3, at points of rank one and zero with rank two or more nearby).

(3) At p the first term of $\mathcal{T}$ vanishes, and the second is $d(\det \mathbb{M}) \wedge \mathbb{P}$. □

Example. $g = dr^2 + (1 + r + r^3)^2 dx^2 + (1 + r + r^2)^2 dy^2$ at $r = 0$ has:

- $\mathbb{M} = \mathrm{diag}(-1, -2, 0)$, so $\det \mathbb{M} = 0$;
- $\partial_r \det \mathbb{M} = -12$;
- $\mathbb{P} = \mathrm{diag}(0, 0, 2)$;
- $\mathcal{T}_{ryy} = 24$, the only nonzero component up to antisymmetry in the first two indices;
- $\mathcal{R}_{rx}{}^{x} = -6$ and $\mathcal{R}_{ry}{}^{y} = 6$, likewise.

$v = \partial_y$ is not parallel to $d(\det \mathbb{M}) = -12\, dr$. Both forms are genuine conditions at a rank-two point, and they are different polynomial extensions of F there. The example is the audit's, rebuilt here (`test_329` P1–P2; `test_331` P3).

The symbol near the locus. On $\mathcal{U}$ the symbol of F uses $(D \operatorname{adj}_K)^{-1}$. By Lemma 4.1(2) its kernel condition can be written without the inverse, as $\sigma_2(D\mathbb{M})(\xi)h \in \operatorname{span}\{D \operatorname{adj}_K(\xi \otimes \xi)\} = \operatorname{span}\{\operatorname{adj}_2 Q_\xi\}$. That condition is polynomial in K and defined at $\det K = 0$. It describes F's kernel, not the symbols of $\mathcal{T}$ and $\mathcal{R}$, which vanish identically where $\operatorname{rank} \mathbb{M} \le 1$. Approaching the locus, the rank of $\sigma(\xi)$ stays 2 (checked down to $k_3 = 10^{-6}$). What degenerates is the constant. On the family $K = \mathrm{diag}(1, 2, k_3)$, on TT the largest singular value grows like $1/k_3$ at every covector; the smallest stays bounded at covectors orthogonal to the direction whose principal curvature vanishes, and grows like $1/k_3$ at the others; so their ratio grows like $1/k_3$ exactly at those orthogonal covectors (`test_341` P3). That describes this family, not every approach to the locus. On $K = \mathrm{diag}(-1, 1, -t)$ the covector e_3, along the vanishing direction, is TT-characteristic for every $t > 0$, since $k_1 + k_2 = 0$, so the TT slice has a kernel there however small t is (Theorem B(2); `test_354` P6).

8.3 The rank-one branch

On an open set where $\operatorname{rank} K = 2$, Proposition 8.2(1) gives $\mathbb{M} = \mu\, v \otimes v$ with $\mu = k_1 k_2 = R/2$ and v spanning $\ker K$, the direction of relative nullity.

Proposition 8.3 (the rank-one Einstein tensor). *Let (Σ^3, g) be any Riemannian three-manifold with $-G = \mu\, v \otimes v$ on an open set, $|v| = 1$, $\mu \neq 0$. Then the contracted Bianchi identity splits as*

$$\nabla_a(\mu v^a) = 0, \qquad \nabla_v v = 0.$$

So the integral curves of v are geodesics, and $\mu\, dV$ is conserved along them.

Proof. $0 = \nabla^a(\mu v_a v_b) = \nabla^a(\mu v_a)\, v_b + \mu\, \nabla_v v_b$. Contract with v^b and use $v^b \nabla v_b = 0$, which gives the first equation. The second follows because $\mu \neq 0$. □

For a hypersurface this recovers, from the intrinsic geometry alone, the classical fact that the leaves of relative nullity are totally geodesic. On the cone over a Clifford torus, $\mathbb{M} = \mathrm{diag}(-1/t^2, 0, 0)$: ∂_t is geodesic, $\operatorname{div}(\mu\, \partial_t) = 0$, and $\mathcal{T}$ and $\mathcal{R}$ vanish identically (`test_329` P3–P4; `test_331` P4). The split itself is checked at a point for general μ and v (`test_342` Q1–Q2).

Theorem 8.4 (the Codazzi block system). *On a rank-one region, let (e_1, e_2) be an orthonormal frame of $v^\perp$, parallel along v (possible because v is geodesic, so parallel transport along v preserves $v^\perp$), and $B_{ab} = \langle e_b, \nabla_{e_a} v \rangle$. B is intrinsic, since v is an eigenvector of the Einstein tensor. For a symmetric K with $Kv = 0$ and block $K_\perp$ on $v^\perp$, the Gauss and Codazzi equations are equivalent to*

$$\det K_\perp = \mu, \qquad B K_\perp = K_\perp B^{\mathsf{T}}, \qquad \nabla_v K_\perp = -B K_\perp,$$

together with the two Codazzi equations along $v^\perp$. Where the first two are independent they leave one free parameter of the two that $\mathbb{M}$ leaves, the ratio k_1/k_2 and the principal angle. Where B is scalar, as on the cone over a Clifford torus ($B = I/t$), the second is automatic and both remain, before the differential equations are imposed. The trace of the third is, by Jacobi's formula, $v(\ln|\mu|) = -\operatorname{tr} B$, which is Proposition 8.3's conservation law again.

Proof. Contract Codazzi with v and use $Kv = 0$: $(\nabla_a K)(e_b, v) = -(BK)_{ab}$. Symmetry in a, b gives the second equation, and the transport along v gives the third. □

Two consequences:

- Equal principal curvatures force B to be symmetric, so the leaves of $v^\perp$ have no twist.
- A nonzero pure rotation of the rulings (B skew, $B \neq 0$) forces $\operatorname{tr} K_\perp = 0$, because then $BK_\perp + K_\perp B = (\operatorname{tr} K_\perp)B$ must vanish. So $\mu = \det K_\perp < 0$ there.

On the cone over a Clifford torus all three equations hold exactly, with $B = I/t$ and $K_\perp = \operatorname{diag}(\tan\alpha, -\cot\alpha)/t$ (`test_342` P1–P5).

Corollary 8.5 (what F, $\mathcal{T}$ and $\mathcal{R}$ see on the degenerate locus).

- On a rank-one region of a realizable metric, F is $0/0$, and $\mathcal{T}$ and $\mathcal{R}$ vanish identically. Everything $\mathbb{M}$ determines there is (μ, v). Realizability passes to the block system of Theorem 8.4, which is the Sbrana–Cartan system.
- Generic hypersurfaces with an open rank-two region are rigid there. The isometrically deformable ones form the four classes of Sbrana (1909) and Cartan (1916): surface-like, ruled, continuous and discrete. For the modern treatment see Dajczer, Florit and Tojeiro (1998) and Dajczer and Tojeiro (2019). The classification describes the **deformable** subfamilies, not every rank-two hypersurface. These classes are invisible to F, $\mathcal{T}$ and $\mathcal{R}$, and visible to the block system.
- Across a *seam*, where one principal curvature passes through zero, $\tilde{K}$ flips sign while $K = \operatorname{sign}(\det K)\,\tilde{K}$ stays continuous. So K is fixed on the seam from either side, and the block system is needed only for an open rank-two region (`test_342` P6).

Proof. The first item is Proposition 8.2(2) with Theorem 8.4. The second is the cited classification. The third is Theorem A(3) on each side. □

8.4 Rivertz's equalities

Rivertz's six equalities $r_1, \ldots, r_6$ are polynomials of bidegree $(2, 1)$ in the components R_{ijkl} of the curvature and $S_{ijklm} = (\nabla_m R)_{ijkl}$ of its derivative (Rivertz 1999). Agaoka and Hashinaga (2022) print r_1 and r_2 in their introduction, and show that r_1 generates a six-dimensional $GL(3)$-irreducible space of such polynomials, of which $r_1, \ldots, r_6$ are a basis (their Remark 4.13(2)).

Theorem 8.6 (Rivertz's equalities span W). *At a point, in an orthonormal frame, the six components of W and Rivertz's six polynomials span the same six-dimensional space of polynomials in the 3-jet of the metric. Each r_a is a constant linear combination of the W^{cd}, and each W^{cd} of the r_a.*

Proof. Parametrise the 3-jet at a point by the six components of R and the eighteen of S, cut to fifteen by the second Bianchi identity. The six W^{cd} are linearly independent polynomials, and r_1 and r_2 each lie in their rational span. The images of r_1 and r_2 under the six relabellings of the frame span six dimensions. A relabelling is a permutation matrix, in $GL(3)$, so the images lie in

$\langle r_1, \dots, r_6 \rangle$. That space is six-dimensional, so the images span it. They also lie in $\langle W \rangle$, which is six-dimensional too. □

So $\mathcal{R}$ is Rivertz's system in tensor form, and $\operatorname{tr}(W\mathbb{P}) = 0$ is the one relation among its components. The reason is Agaoka and Hashinaga's Propositions 4.9 and 4.11. They write the Rivertz polynomials as $|A_0|$ times a tensor h^0, and the Codazzi defect of their Gauss solution α^0 as $-(\alpha^0 h^0)/|A_0|$. Here $\alpha^0 = \tilde{K}$ and $|A_0| = \det \tilde{K} = \sqrt{\det \mathbb{M}}$, and $\tilde{K}\mathbb{M} = \sqrt{\det \mathbb{M}}$, so Proposition 8.1(2) makes W a constant multiple of $\sqrt{\det \mathbb{M}}\, h^0$. Representation theory alone does not decide the theorem. $(\operatorname{tr} \mathbb{M})^2$ times the Cotton–York tensor is another symmetric tensor of the same bidegree, natural under $O(3)$, and it is not in $\langle W \rangle$.

Checks (`test_640`). On hypersurface jets, with $\beta = \nabla\alpha$ totally symmetric, W, r_1 and r_2 vanish; with β not totally symmetric they do not (P1). The span, symbolically (P2). The relabellings (P3). The Cotton–York control (P4).

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9. The chiral form

Let (e_1, e_2, e_3, N) be an adapted frame along a hypersurface, with coframe θ^i. The flat connection of E^4 restricts to an $\mathfrak{so}(4)$-valued 1-form Ω. Its tangential block is the Levi-Civita connection ω_{ij} of g, and its normal block is the shape operator, $\Omega_{Ni} = K_{ij}\theta^j$. In Euclidean signature $\mathfrak{so}(4) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$ splits into two commuting **real** halves. Write $A^\pm$ for the two parts of Ω. In components, $A^{\pm\, i} = \Gamma^i \pm K^i$, with $\Gamma^i = \frac{1}{2}\epsilon^{ijk}\omega_{jk}$ and $K^i = K_{ij}\theta^j$.

For any metric g and any symmetric K, define Ω by the same formulas. It is then an $\mathfrak{so}(4)$-connection on $T\Sigma \oplus \mathbb{R}$, and need not be flat. We write F_A for the curvature of a connection A; the subscript distinguishes it from the operator F.

Theorem 9.1 (Gauss and Codazzi as chiral flatness). $[A^+, A^-] = 0$, *and the curvatures of the two halves are*

$$F_{A^\pm} = \mathcal{G} \pm \mathcal{C},$$

where $\mathcal{G}$ is the $\mathfrak{su}(2)$-dual of the Gauss residual $R_{ij} - K^i \wedge K^j$, and $\mathcal{C} = d_\Gamma K$, the dual of the Codazzi residual. So (g, K) satisfies the Gauss and Codazzi equations iff both chiral connections are flat.

Proof. The curvature of Ω is $d\Omega + \Omega \wedge \Omega$. Its tangential block is the Gauss residual and its mixed block the Codazzi residual. Projecting onto the two $\mathfrak{su}(2)$ ideals gives the even and odd parts. □

Theorem 9.1 is the connection form of Lawn and Roth's spinorial characterization of hypersurfaces of $\mathbb{R}^4$ (Lawn and Roth 2010), as Proposition 9.4 makes explicit.

Corollary 9.2. *On $\mathcal{U}$, take $K = \tilde{K}$. Then $\mathcal{G} = 0$ identically (Theorem A(2)), so*

$$F_{A^\pm} = \pm F(g), \qquad F(g) = \tfrac{1}{2}(F_{A^+} - F_{A^-}).$$

The realizability operator is half the difference of the two chiral curvatures, which are equal and opposite.

Proposition 9.3 (reversing the normal swaps the halves). *Under $N \mapsto -N$, $K \mapsto -K$ and $\Omega \mapsto R\,\Omega R$ with $R = \operatorname{diag}(1, 1, 1, -1)$, which exchanges A^+ and A^-. On $\mathcal{U}$ this is the branch involution $\tilde{K} \mapsto -\tilde{K}$ of Theorem A(3).*

Proposition 9.4 (spinors). *Let Σ be spin, with Clifford multiplication normalised by $X \cdot X\cdot = -|X|^2$, and let K be symmetric. For one choice of orientation of $T\Sigma \oplus \mathbb{R}$, the connections that A^+ and A^- induce on the spinor bundle are*

$$\nabla^{\pm}_X = \nabla_X \mp \tfrac{1}{2} K(X)\cdot,$$

(the other orientation exchanges them), and the curvature of $\nabla^{\pm}$ is the image of $F_{A^{\pm}}$. Hence:

1. *A nonzero $\nabla^{\pm}$-parallel spinor exists near a point iff $F_{A^{\pm}} = 0$ there.*
2. *(Lawn and Roth 2010.) (g, K) satisfies the Gauss and Codazzi equations iff it carries nonzero parallel spinors for both ∇^+ and ∇^-. If K is Codazzi, one of them suffices: then $F_{A^+} = F_{A^-} = \mathcal{G}$.*
3. *For $g \in \mathcal{U}$ and $K = \tilde{K}(g)$, $F_{A^{\pm}} = \pm F(g)$. So, on a simply connected Σ, g is realizable iff it carries one nonzero spinor with $\nabla_X \psi = \frac{1}{2}\tilde{K}(X) \cdot \psi$, iff it carries one with the opposite sign. No Codazzi hypothesis is needed, because $\tilde{K}$ solves the Gauss equation by construction.*

Proof. Under $\mathfrak{so}(4) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$ each half-spin representation is trivial on one ideal. Restricted to $\mathfrak{so}(3)$ both are the spin representation, and the mixed generators act as $\pm\frac{1}{2}e_i\cdot$, with opposite signs on the two halves. So the connection that Ω induces on a half-spin bundle is $\nabla \mp \frac{1}{2}K\cdot$, and it depends only on $A^{\pm}$. Its curvature is the image of $F_{A^{\pm}}$, because the representation is a Lie algebra homomorphism.

(1) The image of a nonzero element of $\mathfrak{su}(2)$ is $v\cdot$ with $v \neq 0$, and $(v\cdot)^2 = -|v|^2$. So it is invertible, and annihilates no nonzero spinor. A parallel spinor forces the curvature to annihilate it. Conversely, a flat $\nabla^{\pm}$ has a two-dimensional space of local parallel spinors.

(2) is Theorem 9.1 with (1). When $\mathcal{C} = 0$, both curvatures equal $\mathcal{G}$.

(3) is Corollary 9.2 with (1). Simple connectivity makes local parallel spinors global. □

These spinors are the generalized Killing spinors of Bär, Gauduchon and Moroianu (2005). Their theorem embeds a metric that carries one, for a Codazzi tensor, in a Ricci-flat manifold with a parallel spinor. By (2), in dimension three that ambient manifold is flat near Σ, as Lawn and Roth also note. The Dirac operator sees less. With the convention above, $\sum_i e_i \cdot K(e_i)\cdot = -\operatorname{tr} K$, so $D\psi = \mp\frac{1}{2}(\operatorname{tr} K)\psi$ for a $\nabla^{\pm}$-parallel ψ. The sign depends on the Clifford convention; only $\operatorname{tr} K$ appears.

Remark 9.5 (why the split is real). In Euclidean signature $\mathrm{Spin}(4) = SU(2) \times SU(2)$ has two real factors, so $A^{\pm}$ are real connections. For spacelike hypersurfaces of $\mathbb{R}^{3,1}$ the structure algebra is $\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})$, which does not split over $\mathbb{R}$: its two halves are complex conjugates, and the analogous connection is complex. That complex connection is the self-dual connection of Ashtekar (1986).

Checks. All exact. Theorem 9.1 is checked on a generic metric with a generic K, with the Gauss and Codazzi blocks computed separately. On the paraboloid $z = |x|^2/2$ in $\mathbb{R}^4$ both halves are flat, and $1.1K$ breaks Gauss alone. Corollary 9.2 is checked on a non-realizable metric with $K = \tilde{K}$, and Proposition 9.3 directly (`test_343`). For Proposition 9.4 (`test_334`):

- the two half-spin representations;
- the spinor curvature as the image of $F_{A^{\pm}}$, with Gauss and Codazzi computed separately;
- parallel spinors of both signs on the paraboloid;
- Killing spinors on the round S^3, and none for $K = 2\,\mathrm{Id}$;
- the Dirac identity.

10. Where the problem sits (not load-bearing)

The classical programmes that settle local realizability in codimension one do not reach $n = 3$:

programme	requires	at $n = 3$
Thomas (1936, §6): Codazzi follows from Gauss	second fundamental form of rank ≥ 4	the rank is at most 3
Vilms (1977), via Chow (1949)	$n \geq 5$	not reached
conformal deformation of hypersurfaces (Cartan 1917), a related rigidity theory	$n \geq 5$	not reached

This is not an accident. The Gauss equation has $n^2(n^2-1)/12$ components and K has $n(n+1)/2$. The difference is

$$\frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2} = \frac{n(n-3)(n+1)(n+2)}{12},$$

which vanishes only at $n = 3$. Equivalently, $\dim \Lambda^2 = n$ only at $n = 3$, and Hodge duality identifies the curvature operator on Λ^2 with an endomorphism of $T\Sigma$. At $n = 3$ that endomorphism is $\mathbb{M}$: in an eigenframe of K the curvature operator is $e_i \wedge e_j \mapsto k_i k_j \, e_i \wedge e_j$, and its eigenvalues are the diagonal of $\operatorname{adj} K$. Vilms's determinant condition (his Theorem 3(ii)) is then the gate $\det \mathbb{M} > 0$. Fodor (2019) reaches the same sign structure through the logarithm of the curvature operator.

Thomas's rank condition is what makes Codazzi **redundant** in higher dimensions, not what makes the criterion **statable**. For $n \geq 4$ with rank at least 4, Codazzi follows from Gauss. At $n = 3$ it is independent, and that independence is the subject of this paper.

11. Regularity (not load-bearing)

F needs three derivatives of the metric: two for $\mathbb{M}$ and one for Cod. This is a different question from the thresholds of isometric embedding, which concern immersions of low regularity:

- In codimension one, $C^{1,\alpha}$ isometric immersions are flexible, in the sense of the h-principle (Gromov 1986), for $\alpha < 1/(1 + n^2 - n)$, which is $1/7$ at $n = 3$ (Cao, Hirsch and Inauen 2025). This improves Borisov's $1/(1 + n(n+1))$ (Borisov 1965, 2004; Conti, De Lellis and Székelyhidi 2012).
- For positively curved surfaces in $\mathbb{R}^3$ they are rigid for $\alpha > 2/3$ (Borisov 1959; Conti, De Lellis and Székelyhidi 2012). Smooth codimension-one immersions are not rigid in general: a cylinder over a plane curve bends, and §8 meets the deformable hypersurfaces of Sbrana and Cartan.
- For surfaces in $\mathbb{R}^3$ the critical exponent is conjectured to be $1/2$.

The results of this paper are statements at classical regularity.

Observation 11.1 (the chiral form's Gauss part). In Theorem 9.1, for the given smooth metric, $\mathcal{C} = d_\Gamma K$ is linear in K and passes to weak limits. The nonlinearity of $\mathcal{G}$ is $K \wedge K$, the 2×2 minors of $K = d\nu$: the Jacobian of the Gauss map ν. For an immersion of class $C^{1,\alpha}$, $\nu \in C^{0,\alpha}$. Its

2×2 minors $d\nu^a \wedge d\nu^b = d(\nu^a\, d\nu^b)$ are defined as distributions, and depend continuously on ν in $C^{0,\alpha}$, exactly when $\alpha > 1/2$ (Brezis and Nguyen 2011, sharp), the exponent conjectured for surfaces. Whether the full chiral system has a weak formulation at the same threshold is not settled here: reading K as a tangent tensor goes through the immersion's frame, which is only $C^{0,\alpha}$.

This is an observation about where the Gauss part can be read, not a rigidity or flexibility theorem. The estimate that would make it one is not attempted here.

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12. Higher dimensions (not load-bearing)

For a hypersurface $\Sigma^n \subset E^{n+1}$, write T_m for the Newton transformations of K. Then

$$\operatorname{adj} K = T_{n-1}, \qquad -G = T_2,$$

in every dimension, and the even-order T_{2k} are intrinsic, being polynomial in the curvature (Reilly 1973; Labbi 2007). For odd n, T_{n-1} is therefore intrinsic. On $\{\det T_{n-1} > 0\}$ the tensor

$$\tilde{K} = (\det T_{n-1})^{1/(n-1)}\, T_{n-1}^{-1}$$

satisfies $\operatorname{adj} \tilde{K} = T_{n-1}$.

For $n \geq 5$ this is a **contracted** identity, not the Gauss equation. The Riemann tensor has more components than T_{n-1} carries, and equality of the contracted quantities does not give $R = \tilde{K} \wedge \tilde{K}$. Full Gauss is a further algebraic condition on the metric. At $n = 3$ the two coincide, because the Riemann tensor is determined by the Einstein tensor and $T_2 = T_{n-1}$. For even n, $\operatorname{adj} K$ changes sign with K and is not intrinsic. So the reduction of §3 is special to $n = 3$, and the higher analogue of F is an operator on the metrics satisfying the extra algebraic conditions. It is not developed here.

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13. Open problems

1. **Local structure near γ.** The equivariant implicit-function construction that Theorem E(5) makes available: a manifold chart for the realizable set modulo diffeomorphisms (the Ebin–Marsden half).
2. **Round shells.** Is every realizable metric whose r-shells are intrinsically round spheres locally the metric of a hypersurface swept by round 2-spheres (Proposition 7.4)? An intrinsically round shell need not be extrinsically round.
3. **Quantitative realizability.** If $F(g)$ is small, is g close to a realizable metric? The relevant apparatus is the rigidity theory of non-Euclidean elasticity, for instance Kupferman, Maor and Shachar (2019). Theorem E(4) gives the linear answer at γ, on TT.
4. **Coercive estimates off umbilics.** An elliptic estimate for F in the gauge $\operatorname{div} h = 0$, $\langle h, \tilde{K} \rangle = 0$ of Corollary 4.2(2), on both branches. TT-characteristic covectors are no obstruction to it.
5. **Higher dimensions.** The operator built on T_{n-1} for odd $n \geq 5$, on the metrics where full Gauss holds (§12), and whether its linearisation at the round S^n factorises as in Theorem E.
6. **Attribution.** Rosenson (1940, 1941, 1943), cited by Vilms as a further solution of Gauss and Codazzi, has not been read here. If it contains the polynomial form of §8.1, the attribution there changes.

Appendix V. Verification index and disclosure

Every displayed result is reproduced by at least one of the scripts deposited with this paper. These are tests 325–327, 329–344, 354, 625–627, 629 and 640, with their library `ratjets.py` and a run log, at DOI 10.5281/zenodo.23765514. Each script states its expectations before computing and exits non-zero on any failure. Where a check had to be re-posed after its first run, the script records the first run, what failed and why. Arithmetic is exact (rational, symbolic, or exact algebraic) except where the index says floating point. A few zero tests on exact algebraic expressions fall back to evaluation at 60 digits, as the scripts state. The numbering continues that of the author's archive (concept DOI 10.5281/zenodo.21264118). Earlier archive scripts that covered the same results are superseded for this paper; the bundle's README lists them.

result	where	verification
$\det D\,\mathrm{adj}_K = -2(\det K)^2$	§2.4	`test_330` P1 (symbolic)
Theorem A	§3	`test_335` P1–P4 (symbolic); `test_330` P2
Example 3.3 (Flamm)	§3	`test_330` P3
Lemma 4.1	§4	`test_326` P1 (symbolic)
The symbol (4.1) against the full nonlinear operator	§4	`test_335` P5 (both branches)
Theorem B(1): rank, kernel	§4	`test_325` P1; `test_332` P3; with Lemma 4.1, `test_326` P1
Theorem B(2): the TT criterion and the counterexample	§4	`test_325` P2 (floating point, with the exact counterexample)
Corollary 4.2	§4	`test_332` P3
Lemma 4.6	§4	`test_325` P6
Theorem C(2): five conditions at every nondegenerate K	§4	`test_332` P1 (symbolic); `test_325` P3 (floating point)
Theorem C(3)–(4): Cartan's test, $\dim g_q$, $g_q = M_K^q$	§4	proof; its identities, the reconstruction of g_3, three further K and the fifteen functionals, `test_354` P1–P4; `test_332` P2 (exact, sixteen K); `test_325` P4–P5, `test_326` P2–P3 (floating point, $q \le 9$ at two K); flag bounds `test_344` P4
The Kulkarni–Nomizu determinant	§4	`test_326` P4 (symbolic)
Theorem D	§5	proof; `test_335` P5
Theorem E(1)	§6	`test_327` P1–P2
Theorem E(2)–(3)	§6	`test_336` P1 (operator identity on TT jets), P2; the local trace-free Codazzi tensor, `test_354` P6
Theorem E(4)	§6	`test_336` P2 (level 3)
Theorem E(5)	§6	proof; `test_336` P3 (the augmented symbol)
Remark 6.1: $DF_\gamma = DC_\gamma$	§6	proof; `test_625` P1–P2 (exact)
Remark 6.1: the curl spectrum, the intertwiner, the exact sequence	§6	proof; `test_629` P1 (floating point, the representation exact), P2 (exact)
Remark 6.1: $\zeta(0) = 10$, its scope, the determinant	§6	`test_336` P4; `test_333`
Remark 6.1: why ten, the sector counts	§6	proof; `test_627` P1–P5
Proposition 6.2	§6	`test_327` P4; `test_336` P5
Proposition 6.3	§6	`test_327` P3

result	where	verification
Theorem F	§7	`test_337` P1–P7; the components of $\mathbb{M}_1$, `test_344` P2
Proposition 7.1	§7	`test_337` P4, P8, P9
Lemma 7.3; Proposition 7.6	§7	`test_330` P5–P6; `test_338` P7
Proposition 7.4	§7	`test_338` P1, P3; the chart's rotation, `test_344` P1
Corollary 7.5	§7	`test_338` P2–P6, P8; `test_337` P7
Theorem G	§7	`test_339`; `test_338` P4; the tilted metric is not round, `test_354` P6
Theorem 7.7	§7	proof; the combined family, `test_354` P5; `test_340`
Proposition 8.1	§8	`test_341` P1–P2 ($\mathcal{T}$); `test_331` ($\mathcal{R}$)
§8, the dualisation W^{cd}	§8	`test_626` P1–P2 (symbolic)
Proposition 8.2 and its example	§8	`test_329` P1–P2; `test_331` P3; (2) pointwise, `test_344` P3
The symbol near the degenerate locus	§8	`test_341` P3 (rank exact; singular values in floating point); the characteristic covector, `test_354` P6
Proposition 8.3; Theorem 8.4; Corollary 8.5	§8	`test_342`; `test_329` P3–P4; `test_331` P4; scalar B, `test_354` P6
Theorem 9.1; Corollary 9.2; Proposition 9.3	§9	`test_343`
Proposition 9.4	§9	`test_334`
Theorem 8.6 and its controls	§8	`test_640` P1–P4 (exact, symbolic)
The count of §10	§10	`test_330` P4

Disclosure. Large-language-model systems (Anthropic's Claude) were used throughout, under the author's direction, for:

- derivation and derivation-checking;
- drafting;
- writing and running the verification scripts;
- an independent referee read of this draft, by a separate instance that had not seen it written, and further independent reads by Google Deep Review and by ChatGPT (October 2026). The last supplied the proofs of Theorem C(3) at every shape operator and of Theorem 7.7. A review by Gemini (November 2026) supplied the identity $DF_\gamma = DC_\gamma$ of Remark 6.1, the intertwiner and the count of ten as a theorem there, and the dualisation of $\mathcal{R}$ after Proposition 8.1. A second Gemini review (November 2026) proposed Theorem 8.6, by an argument from Schur's lemma that does not decide it alone (§8.4); the proof here is the finite check.

Every load-bearing claim is gated by the scripts above. An independent machine-assisted audit (October 2026) of the earlier mathematics papers preceded this draft. Results credited to it in the text are its own, and each is also checked by a script here. All errors remain the author's.

Appendix C. Changes from the deposited versions

This paper supersedes, for local questions, mathematics paper 1 (Thrussell 2026a, deposited as version 3), mathematics paper 2 (Thrussell 2026b, version 2) and mathematics paper 4 (Thrussell

2026d, draft 1). The statements below change. Labels such as "Theorem E" refer to the paper named, and the corresponding result here is given in brackets. The global results of papers 2, 3 and 5 (Thrussell 2026b, c, e) are treated in the companion paper.

Mathematics paper 1.

1. Theorem A(2)'s proof assumed every eigenvalue of $\mathbb{M}$ positive. It needs only nonzero eigenvalues with positive product (§3).
2. The example of the Riemannian Schwarzschild metric was called complete. The exterior sheet is not. The complete hypersurface is the two-ended Flamm surface (Example 3.3).
3. Global sufficiency needs a simply connected domain, or trivial monodromy. The round $\mathbb{RP}^3$ satisfies $F = 0$ and is not realizable (Remark 3.2).
4. Theorem D(2) [here Theorem F(2)] excludes $\ell \geq 2$ only within its own family. This was corrected in the repository's version 4, which is not deposited.
5. Proposition F's "invariant splitting" into translations, moduli and Lie directions overlaps, and the shell dipole is a local deformation on punctured shells, not a global smooth modulus (Proposition 6.2).
6. Theorem E [here Theorem G] is stated on interior shells. Its general-base statements are exact checks on three bases, not a theorem for every base. The transversal zero they report comes from freezing the drift's coefficients at a point; with the drift itself the zero is a double root on every base with $\Phi' \neq 0$ (Theorem 7.7). Its route 1 is now identified as the tilt (Corollary 7.5).
7. Appendix N's description of $\mathfrak{D}$'s zero set as open is withdrawn (Proposition 7.1). F's need for three derivatives is separated from the flexibility thresholds (§11).
8. §10 on higher dimensions: the reduction gives a contracted identity, not the Gauss equation (§12).

Mathematics paper 2. The meagreness proof local to the round sphere is replaced by one on the whole nondegenerate domain, on both branches (Theorem D). The TT levels are relabelled to the single convention of §2.6.

Mathematics paper 4.

1. "Nine Codazzi components against six": the count is nine components, four identities, five conditions, at every nondegenerate shape operator (Theorem C(2)).
2. The rank and kernel theorems were established at sample shape operators. They are now proved for every nondegenerate K (Theorem B).
3. TT-injectivity was claimed wherever $\det \mathbb{M} \neq 0$. It holds exactly where $\sigma_1 g - K$ is definite (Theorem B(2)). Overdetermined ellipticity was claimed on the same set. Modulo diffeomorphisms it holds nowhere, because the symbol also kills the normal direction K; modulo the linearised immersions it holds everywhere on $\mathcal{U}$ (Corollary 4.2).
4. "The fourth kernel direction is conformal iff $K \propto g$": at a fixed covector, iff $K|_{\xi^\perp}$ is scalar (Theorem B(3)).
5. Meagreness was argued from a cokernel. That argument is not a proof, and is replaced (Theorem D).
6. "P [here $\mathbb{P}$] vanishes at $\det \mathbb{M} = 0$ and the equation is vacuous there": true only where $\operatorname{rank} \mathbb{M} \leq 1$. "The same symbol": the same up to the factor $2(\det \mathbb{M})^{3/2}$ (Propositions 8.1–8.2).
7. Sbrana and Cartan classify the deformable rank-two hypersurfaces, not all of them (Corollary 8.5).
8. The Cartan characters $(2, 2, 1)$ and $\operatorname{rank} \tau_4 = 15$ are wrong. The correct values are $\operatorname{rank} \tau_4 = 12$

and characters $(36, 15, 4)$, from $\dim g_{3,k} = 55, 19, 4$. Cartan's test then holds with equality at every nondegenerate shape operator (Theorem C(3)), and the paper's conclusion, that the system is involutive, stands.

9. The claimed rise of the inverse-free symbol's rank to 3 at the degenerate locus is withdrawn. The degeneration of the injectivity constant on TT stands (§8.2).

Elsewhere in the author's archive. The claim that an $\ell = 1$ radial coefficient forces a chart drift of matching amplitude (the "amplitude half" of what the archive calls the canal tie) is false. When $\alpha \not\equiv 0$, which holds on the positive branch, an $\ell = 1$ radial coefficient is realized by a tilt with no chart drift (Corollary 7.5(2)).

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