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# An improved zero-free half-plane for Hecke $L$-functions over number fields and Quasi-Riemann Hypothesis

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**Abstract**

In this paper, we obtain a zero-free half-plane $\Re s > B_{\mathrm{tw}}$ for every finite-order Hecke $L$-function over every number field, where $B_{\mathrm{tw}} = 0.87405850379185900499\ldots$ is an explicitly defined algebraic number of degree six. The proof refines the cubic-theta comparison in three parts. In Part I, we require sufficient prime-product length for length restoration and fix the short side of the equal-product comparison rectangle at $1/4$. Direct and complementary Poisson estimates give an amplified Gauss norm with an additional prime product. Its separate applications to the numerator and the counting detector, combined with contour displacement and successive moment bounds, give the intermediate boundary $B' = 0.87422125049290208328\ldots$.

In Part II, complete prime removal is permitted even when a positive length excess remains. The no-prime moment estimate then controls the resulting ordinary sums, and the comparison rectangle's short side may vary. These changes remove the coupled condition on the remaining prime length and enlarge the allowed norm widths. The improved detector count and numerator estimate give successive proved bounds converging to $B_{\mathrm{ex}} = 0.87407284766682790777\ldots$. We prove optimality for this specified direct, detector and fourth-moment comparison system, retaining its geometric length inequality while allowing the sum of the length exponents to vary.

In Part III, selected unit primes of the additional prime product are not extended. On the unit locus they enter the Gauss sums as a moving cubic character, and the first length condition of the enlarged norm becomes $A_{\max} + (4-\rho)W < 1$ for $p = 6\kappa \in [112/25, 9/2]$ and $0 \le \rho < \rho_{\max}(p)$. The twist enlarges the returned sums. An extension of the enlarged norm to a moving radical, with an $S$-arithmetic additive large sieve in its terminal terms, gives these sums the prime-length bound $z < (1/p + \alpha)G$ for $0 < \alpha < \alpha_{\max}(p)$, which closes the return estimate. Successive bounds converge to $B_{\mathrm{tw}}$, and $\alpha_{\max}$, $\rho_{\max}$ and $B_{\mathrm{tw}}$ are extremal for the corresponding steps. All three parts use the established theta, moment and continuation results of the earlier QRH papers and their general-field formulation.

## Contents

**1 Introduction** **4**
1.1 The main theorem . . . . . 4
1.2 The exhaustion-range endpoint . . . . . 5
1.3 The intermediate enlarged-count endpoint . . . . . 5
1.4 Earlier bounds and the results used here . . . . . 6
1.5 The methods in Parts I–III . . . . . 7
1.6 Optimality and organization of the proof . . . . . 11

**2 Arithmetic notation and the earlier comparison theorems** **11**

**Part I The amplified norm and the enlarged class count** **17**

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**3 Local Gauss estimates and the prime products** **17**
3.1 A divisor bound for nonunit prime contributions . . . 17
3.2 A frequency change for fixed primes dividing $s$ . . . 22
3.3 Retaining the common-support length . . . 26
3.4 An amplified norm retaining the initial prime product . . . 29

**4 The enlarged amplified norm** **34**
4.1 An amplified norm with variable lengths . . . 34
4.2 A direct estimate for small transformed length exponents . . . 39
4.3 The complementary Poisson estimate . . . 42
4.4 The enlarged length range . . . 47

**5 The approximate functional equation and its contours** **49**
5.1 Smooth weights beyond the 1/10 contour shift . . . 49

**6 Amplitude-class counts and short summation ranges** **51**
6.1 Amplitude-dependent bounds for small-amplitude classes . . . 52
6.2 The lowest-amplitude classes . . . 53
6.3 Separate parameters for the class count and the numerator moment . . . 53

**7 Adding the moment and contour savings** **56**
7.1 Combining the moment bound and contour displacement . . . 56

**8 The algebraic boundary and the successive count bounds** **60**
8.1 An explicit rational bound . . . 60
8.2 The algebraic endpoint . . . 61

**9 Arbitrary number fields and the boundary $B_*$** **66**
9.1 The estimates over fields containing the twelfth roots of unity . . . 66
9.2 Descent to an arbitrary number field . . . 68

**10 The successive moment bounds with the earlier count** **69**
10.1 The variable numerator-moment parameter . . . 69
10.2 The amplified norm for a variable moment parameter . . . 70
10.3 The comparison system and its endpoint . . . 74
10.4 The starting boundary for the enlarged count . . . 78

**11 The enlarged class count and its algebraic boundary** **81**
11.1 The extended moment range . . . 81
11.2 The counting detector and the additional prime product . . . 83
11.3 The new comparison and its endpoint . . . 86
11.4 The strict comparison inequalities . . . 88
11.5 Iteration and proof of the enlarged-count theorem . . . 89

**Part II Prime exhaustion and variable comparison rectangles** **91**

**12 An enlarged product norm with a variable comparison rectangle** **91**
12.1 The moment estimates and the norm statement . . . 91
12.2 Restoration when the prime list is exhausted . . . 92
12.3 Proof of the enlarged norm . . . 93

**13 Selecting the original factors and amplifying the detector** **97**
13.1 Inverse moment and the long detector count . . . 99
13.2 The numerator's separate amplification . . . 99

**14 Exact scalar comparison and continuation** **100**
14.1 Parameters and notation . . . 100
14.2 Detector widths and a common cutoff . . . 100
14.3 Numerator gain on every ordinary length . . . 102
14.4 Exact inequalities over the complete parameter range . . . 103
14.5 Smooth weights, error terms and analytic continuation . . . 105

**15 Successive moment bounds in the exhaustion range** **106**
15.1 The algebraic parameters . . . 106
15.2 Endpoint signs and a strict family . . . 107
15.3 Iteration and descent . . . 109

**16 Optimality for the exhaustion-norm comparison** **110**

**Part III Unit twists and returned sums** **113**

**17 The unit-twisted norm and its statements** **113**

**18 The strict margin** **114**

**19 Selecting twisted positions** **115**

**20 The cubic unit twist** **116**

**21 The returned sums** **117**
21.1 The returned gap . . . 117
21.2 The active estimate . . . 117
21.3 The inactive estimate and the untwisted configurations . . . 119

**22 Uniform choices and the proof of Theorem 17.1** **119**

**23 The comparison with the twisted origin width** **120**
23.1 The modified comparison . . . 120
23.2 The algebraic endpoint . . . 121

**24 The strict comparison family** **122**

**25 Successive bounds and the proof of Theorem 1.1** **123**

**26 Optimality for the unit-twisted comparison** **124**

**27 Proportional margins and the limit of the method** **125**

**A Exact rational inequalities in the comparison** **125**
A.1 Polynomial inequalities on a rectangle . . . 125
A.2 The strict rational starting bound . . . 126
A.3 The algebraic parameter interval . . . 127
A.4 Computations for the successive moment bounds . . . 128

**B Exact inequalities for the enlarged count** **129**
B.1 Algebraic endpoint and the two branches . . . 129
B.2 Rational interval verification of the remaining signs . . . 130

**C The zero-detecting estimates and theta exponents** **131**

**D Exact inequalities for the exhaustion range** **133**
D.1 The rational starting comparison . . . 133
D.2 Algebraic enclosures and the endpoint quadratics . . . 133
D.3 The full parameter family . . . 134

**E Exact inequalities for the unit-twisted comparison** **136**
E.1 The return estimate . . . 136
E.2 Algebraic enclosures and the endpoint quadratics . . . 137
E.3 The strict family . . . 137
E.4 The optimality signs . . . 137

**F The enlarged norm with a moving radical** **138**

**G An additive large-sieve terminal estimate** **140**

**H Unbalanced and uncentered variants** **144**

# 1 Introduction

In this section, we state Theorem 1.1, recall the earlier bounds in Section 1.4, and describe the methods of Parts I–III in Sections 1.5–1.6. We write $\mu_n$ for the group of $n$-th roots of unity and $\zeta_n = e^{2\pi i/n}$ for its chosen primitive root.

## 1.1 The main theorem

In this subsection, we state the zero-free half-plane of Theorem 1.1. Section 25 proves it by successive moment estimates and abelian descent.

Let $B_{\mathrm{tw}}$ be the unique root in $(0.87405850, 0.87405851)$ of

$$\begin{aligned} 1473552B^6 + 30485106B^5 + 82242441B^4 - 400709406B^3 \\ + 471613016B^2 - 221603468B + 36759635 = 0. \end{aligned} \tag{1.1}$$

This polynomial is irreducible over $\mathbb{Q}$, and $B_{\mathrm{tw}} = 0.87405850379185900499688623776976 7\ldots$.

**Theorem 1.1.** *Let $K$ be a number field and let $\chi$ be a finite-order Hecke character of $K$. Then $L_K(s,\chi)$ has no zeros in $\Re s > B_{\mathrm{tw}}$. The pole of a principal character at $s = 1$ is permitted. In particular,*

$$\zeta_K(s) \neq 0 \qquad (\Re s > B_{\mathrm{tw}},\ s \neq 1).$$

*The numerical boundary is independent of $K$ and $\chi$; constants and thresholds may depend on the fixed arithmetic data. No assertion is made on the line $\Re s = B_{\mathrm{tw}}$.*

The proof is given in Section 25. It starts from the boundary $B_{\mathrm{ex}}$ of Theorem 1.2; the first further moment step gives $0.8740587907\ldots$, and the successive steps converge to $B_{\mathrm{tw}}$. Theorem 26.1 proves optimality for the unit-twisted comparison system with its retained geometric length inequality.

### 1.2 The exhaustion-range endpoint

In this subsection, we record the endpoint of Part II, proved in Section 15. Let $B_{\mathrm{ex}}$ be the unique root in $(0.87407284, 0.87407285)$ of

$$178848B^4 - 507168B^3 + 522846B^2 - 231405B + 37099 = 0.$$

This polynomial is irreducible over $\mathbb{Q}$, and $B_{\mathrm{ex}} = 0.874072847666827907778669374 0\ldots$.

**Theorem 1.2.** *Let $K$ be a number field and let $\chi$ be a finite-order Hecke character of $K$. Then $L_K(s,\chi)$ has no zeros in $\Re s > B_{\mathrm{ex}}$. The pole of a principal character at $s = 1$ is permitted. No assertion is made on the line $\Re s = B_{\mathrm{ex}}$.*

The proof is given in Section 15. Its starting comparison gives the rational boundary 0.87408. The first further moment step already gives a boundary below 0.874073; the successive steps converge to $B_{\mathrm{ex}}$. Theorem 16.1 proves optimality for that comparison system with its retained geometric length inequality.

### 1.3 The intermediate enlarged-count endpoint

In this subsection, we record the intermediate endpoint proved in Section 11. Let $B'$ be the unique root in $(0.87422125, 0.87422126)$ of

$$481464B^4 - 1308150B^3 + 1274868B^2 - 525330B + 77719 = 0. \tag{1.2}$$

This polynomial is irreducible over $\mathbb{Q}$, and $B' = 0.87422125049290208328\ldots$.

**Theorem 1.3.** *Let $K$ be a number field and let $\chi$ be a finite-order Hecke character of $K$. Then $L_K(s,\chi)$ has no zeros in $\Re s > B'$. The pole of a principal character at $s = 1$ is permitted. In particular,*

$$\zeta_K(s) \neq 0 \qquad (\Re s > B',\ s \neq 1).$$

*The boundary is independent of $K$ and $\chi$. No assertion is made on the line $\Re s = B'$.*

We use the earlier results cited in Section 2 as established theorems. The additional norm estimate and its applications to the numerator and class count are proved below. Constants and thresholds may depend on the fixed field $K$, its excluded places and the fixed character $\chi$, including on $[K:\mathbb{Q}]$ and $|D_K|$. The numerical boundary $B'$ is the same for every $K$ and $\chi$.

*The fixed-moment endpoint.* With the numerator-moment parameter fixed at $3/4$, the argument gives $B_* = 0.87446830769353045955\ldots$ (Theorem 9.2). This is the optimum of that comparison system (Corollary 9.3). Its exact definition is as follows.

Let $r_*$ be the unique root in $(9/20, 23/50)$ of

$$218582r^3 - 376469r^2 - 11376r + 61560 = 0. \tag{1.3}$$

Set

$$\ell_* = \frac{3410r_*^2 - 7127r_* + 2520}{2286r_*^2 - 5011r_* + 1800}, \qquad B_* = \frac{11}{12} - \frac{\ell_*}{4}. \tag{1.4}$$

The remaining cubic-endpoint parameters and a closed form for $r_*$ are given in Section 8.2.

### 1.4 Earlier bounds and the results used here

In this subsection, we recall the bounds preceding Theorem 1.1. The precise earlier results used in its proof are stated in Section 2; their numerical optimization is not needed as an assumption in the new moment argument.

The cubic-theta argument of OpenAI [Ope26a] gives the boundary 7/8 over $\mathbb{Q}(\sqrt{-3})$ and the corresponding Dirichlet family. Its shorter 11/12 argument and the cyclotomic construction are given in [Ope26c, Ope26b].

In the first version of [Liu26a], by varying the length parameters $b, \ell$, the author obtains

$$B_{\mathrm{r}} = \frac{34999}{40000} = 0.874975, \qquad B_{\mathrm{alg}} = \frac{1507 - 2\sqrt{921}}{1653} = 0.874957069799\ldots.$$

Contemporaneous parameter refinements were reported by Mermet [Mer26], Wei [Wei26], and Chakhkiev [Cha26]; see also [Liu26a, Section 21]. The two GitHub repositories report

$$B_{\mathrm{MW}} = \frac{4374785349}{5000000000} = 0.8749570698,$$

whereas Chakhkiev reports the conditional bound $10937/12500 = 0.87496$ under the analytic estimates specified there. These reports concern the Eisenstein-field and corresponding Dirichlet setting, rather than arbitrary number fields.

In version 2 of [Liu26a], also varying the moment parameter, the author obtains

$$B_{\mathrm{new}} = 0.874957019420098946\ldots.$$

Assuming the analytic estimates of [Ope26a], Xu and Sang [XS26] replace the pointwise treatment of the reflected numerator by an amplified average and obtain

$$B_{\mathrm{XS}} = \frac{8748867}{10^7} = 0.8748867$$

for the same Eisenstein-field and Dirichlet families. Newton [New26] studies the associated parameter optimization and gives the certified boundary

$$B_{\mathrm{N}} = \frac{43744332481}{50000000000} = 0.87488664962$$

under the analytic hypotheses and parameter restrictions specified there. This last result is a refinement of the comparison rather than a new mean-value estimate.

Cureton and Thorner [CT26] obtain the boundary 31/32 for Dedekind zeta functions and finite-order Hecke $L$-functions over arbitrary number fields. Combining the averaged-numerator method of [XS26] with the general-field metaplectic method of [CT26], the author obtains in [Liu26b]

$$B_\flat = 0.874886647039964\ldots, \tag{1.5}$$

where

$$133488B_\flat^4 - 376596B_\flat^3 + 384534B_\flat^2 - 167541B_\flat + 26231 = 0.$$

Cureton and Thorner [CT26] use their general-field argument to descend from $K(\zeta_{12})$ to $K$. In this paper, we use the corresponding formulation in [Liu26b, Part III]. These field scopes are distinct: a bound over the Eisenstein field alone does not imply the same bound over every number field.

### 1.5 The methods in Parts I–III

In this subsection, we compare the norm and contour arguments of Part I with the enlarged norm of Part II and its unit twist in Part III. Sections 7, 14 and 23 give the corresponding exponent estimates.

All three parts use the same cubic-theta comparison, original prime factors and previously justified moment parameters. Part I (Sections 3–11) uses the fixed comparison

$$(m, m) \longmapsto (1/4,\, 2m - 1/4)$$

and retains enough prime length for restoration after every fixing. The amplified norm and contour displacement first give $B_*$; successive moment bounds give $B_\natural$, and joint prime selection in the detector gives $B'$. Part II (Sections 12–16) removes the coupling between remaining prime length and length excess, and permits a variable comparison

$$(m, m) \longmapsto (t,\, 2m - t).$$

The no-prime moment controls the resulting ordinary sums even when a positive M2 length excess remains after all prime factors have been removed. In Part I, exhaustion occurs only after the coupled length inequality has made this excess nonpositive. The parameter $t$ is chosen within the reflection and exceptional-term margins. These changes enlarge the norm domain; applying it to the detector and numerator then gives $B_{\mathrm{ex}}$. We describe the common comparison and the two successive norm ranges below, and then the unit twist of Part III.

Let $Z > 1$ be the main scale parameter. In the smoothed cubic-theta average $I_\eta(X, Y, Z)$ of [Liu26b, Section 13.2], $\eta$ is the fixed target Hecke character, and $X, Y$ are the two averaging lengths; $Z$ sets the smoothed norm scale of the indices in the cubic-theta sum. We write

$$X = Z^{l_x}, \qquad Y = Z^{l_y}.$$

The subscripts $x, y$ refer to $X, Y$, respectively. The total prime-product length exponent $\ell$ and balancing parameter $b$ satisfy

$$l_x + l_y + \ell = 1, \qquad b = l_y - l_x.$$

Write $U = Z^d$, $d > 0$, for the scale of the character parameter and $D = U^m$ for the length of an ordinary dyadic sum. The balanced length exponents are

$$l_x = \frac{1 - b - \ell}{2}, \qquad l_y = \frac{1 + b - \ell}{2},$$
$$h = \frac{1 + b + 3\ell}{2}, \qquad B(\ell) = \frac{11}{12} - \frac{\ell}{4}.$$

The endpoint character-parameter scale is $U = Z^h$. With the other class data fixed, $\mathscr{B}$ groups character parameters with fixed detector and prime-sum amplitude ranges. If $a_{\mathrm{det}}$ is the real-part parameter in the zero-detecting decomposition, write $\delta = 2a_{\mathrm{det}} - 1 > 0$; this parameter is denoted by $a$ in the cited comparison. For prime-sum factors of lengths $U^{\lambda_i}$, let $g_i$ be their logarithmic amplitude exponents, as in Lemma 4.2. Put

$$\sum_i \lambda_i = \mathscr{L} = \ell/d, \qquad 0 \le g_i \le \delta/2,$$
$$g = \mathscr{L}^{-1} \sum_i \lambda_i g_i = \delta x, \qquad 0 \le x \le 1/2.$$

Thus $\delta$ describes the detector range and $x$ the normalized mean prime-sum amplitude. The contour shift $\theta$ specifies the numerator line $\Re w = 1/2 - \theta$. For the initial moment parameter

$\kappa_m = 3/4$, the reference shift is $\theta_0 = 2\delta x/9$. Let $\mathcal{E}_{\theta,\gamma}(d,m)$ be the full comparison exponent when the contour is displaced to $\theta$ and the selected moment saves $U^{-\gamma}$. The simultaneous estimate is

$$\mathcal{E}_{\theta,\gamma}(d,m) - \mathcal{E}_{\theta_0,0}(d,m) = (\theta-\theta_0)\{d(1-m) - l_y\} - d\gamma. \tag{1.6}$$

Both terms use the same dyadic sum and original prime coefficients. The prime-product lower bounds precede differentiation of the smooth weights.

For the fixed-moment comparison $\kappa_m = 3/4$, retain the coefficient and smooth-weight hypotheses of Theorem 4.9. On the selected classes of Lemma 7.3, assume

$$\frac{3}{10} \le \delta \le \frac{3}{5}, \qquad \frac{2}{5} \le x \le \frac{1}{2}, \qquad \mathscr{L} \ge \frac{21}{100}, \qquad \frac{57}{100} \le \frac{l_y}{h} \le \frac{11}{19}.$$

Let $Q_0$ be the initial prime product and $Q_W$ a disjoint additional prime product. For $3/8 < m < 1/2$, write $W > 0$ for an upper bound on the length exponent of $Q_W$ relative to $U$. The enlarged norm permits, with strict inequalities,

$$W < \frac{7}{54}(1-2m), \qquad W < \frac{8m-3}{9}. \tag{1.7}$$

The unused prime product has lower mean

$$a(\delta,x) = \delta\frac{189x-25}{139}, \qquad 0 < \gamma < \frac{a(\delta,x)W}{2}.$$

The limiting upper length $W_{\mathrm{cap}}(m)$ and the contour are chosen separately for each dyadic sum:

$$W_{\mathrm{cap}}(m) = \min\left\{\frac{7}{54}(1-2m), \frac{8m-3}{9}\right\}, \qquad m_\theta = 1 - \frac{l_y}{h},$$
$$\theta_*(m) = \begin{cases} 0, & m \le m_\theta, \\ \delta/2, & m > m_\theta. \end{cases}$$

Under these class and geometry hypotheses, Section 7 proves

$$\inf_{3/8\le m\le 1/2}\left\{\frac{aW_{\mathrm{cap}}(m)}{2} + (\theta_*(m)-\theta_0)(m-m_\theta)\right\} = \Gamma_N = \frac{7}{108}a(\delta,x)(2l_y/h-1).$$

Here $\Gamma_N$ is the limiting normalized gain at $m = m_\theta$. These are limiting scalar values; the analytic estimates use strict interior lengths. At the critical length,

$$d = h, \quad m = m_\theta \quad\Longrightarrow\quad h(1-m_\theta) - l_y = 0, \qquad \mathcal{E}_{\theta,\gamma} - \mathcal{E}_{\theta_0,0} = -h\gamma.$$

Thus the enlarged moment supplies the saving at $m_\theta$, and the contour displacement controls the remaining lengths.

*Origin of the limiting parameters.* The parameters in (1.4) and (8.1) are determined by the class for which the comparison is sharp. With $E_\kappa$ as in (2.2), define

$$\mathcal{Q}_{\kappa,b,\ell}(\delta,x) = E_\kappa(\delta,x) - \frac{7}{108}a(\delta,x)(2l_y - h).$$

The contour contribution vanishes at $m_\theta = 1 - l_y/h$. The extreme amplitude is $x = 1/2$. At the limiting parameters the comparison has a double zero in $\delta$, and its coefficient of the balancing parameter vanishes:

$$\begin{aligned} &\mathcal{Q}_{\kappa_*,b_*,\ell_*}(r_*,1/2) = 0, \qquad \partial_\delta\mathcal{Q}_{\kappa_*,b_*,\ell_*}(r_*,1/2) = 0, \\ &\partial_b\mathcal{Q}_{\kappa_*,b_*,\ell_*}(r_*,1/2) = 0, \qquad \kappa_* = \frac{5}{6} - \frac{\ell_*}{2}. \end{aligned} \tag{1.8}$$

Eliminating $b_*$ and $\kappa_*$ gives the cubic (1.3) and the rational formula for $\ell_*$. The stationarity equation in $\delta$ then determines $b_*$. Section 8.2 proves these identities and verifies that all the other classes satisfy the required inequality.

The count iteration is used only to reach this boundary from strict interior estimates. Let $B_{\mathrm{seed}} = 0.874558$ denote the rational starting bound of Theorem 8.1. Its first combined step gives

$$B_1 = B_* + \frac{2B_{\mathrm{seed}} - 1 - \kappa_*}{100} < B_* + 1.8 \cdot 10^{-6}.$$

The additional moment and contour saving supplies the principal improvement; the subsequent iteration removes this residual gap.

Through Section 9, the unamplified numerator moment remains at $\kappa_m = 3/4$, supplied by the earlier $7/8$ bound. The class count may use a separate parameter $\kappa_c$. Starting from a strict rational bound, the iteration in Section 8 satisfies

$$\kappa_c' = \kappa_* + \frac{\kappa_c - \kappa_*}{50}, \qquad \kappa_* = 2B_* - 1.$$

Every finite step has positive margins. Taking the decreasing sequence of proved boundaries yields (1.4).

*Successive moment bounds.* Fix a number field $F \supset \mu_{12}$, and let $\beta_*(F)$ be the zero supremum defined in Section 2. For $13/18 \le \kappa_m \le 1$, under the coefficient, exclusion, mesh and length hypotheses of [Liu26b, Theorem 12.16 and Lemma 12.7], the zero-free hypothesis of M2 is $\beta_*(F) \le (1+\kappa_m)/2$. Section 10 therefore puts $\kappa_m = \kappa_c = 2B_n - 1$, where $B_n$ is the $n$-th proved boundary and $B_0 = B_*$. With $c = 1/(6\kappa_m)$,

$$z_0(m) = c(1-2m), \qquad \theta_0 = c\,\delta x, \qquad W_{\mathrm{cap}}(m) = \min\Big\{\frac{(1-c)(1-2m)}{6}, \frac{8m-3}{12\kappa_m}\Big\}.$$

At the extreme amplitude $x = 1/2$, with $\kappa_c, b, \ell, \delta$ fixed, the dependence of the moment contribution on $\kappa_m$ is

$$\frac{\delta}{2}\Big(\frac{h}{2} - l_y\Big)\frac{1+5c}{6}, \qquad \frac{h}{2} - l_y < 0,$$

which decreases as $\kappa_m$ decreases. The resulting iteration satisfies $\kappa_{n+1} - \kappa_\natural = (\kappa_n - \kappa_\natural)/50$, and its limit is the quartic endpoint (10.2).

*The enlarged class count.* The preceding numerator argument leaves the earlier M2 count in place. Section 11 also inserts an additional prime product into the counting detector. At moment parameter $\kappa$, the joint selection of complete original factors gives

$$|Q_0 Q_W|^2 \ge U^{2g(z_S+w)-o(1)}, \qquad z_S = \frac{1-2m}{6\kappa}, \quad w < W_{\mathrm{cap}}(m;\kappa).$$

The enlarged norm therefore replaces

$$A_S(m) = 1 - 2\delta m - 2g z_S(m) \quad \text{by} \quad A_S'(m) = A_S(m) - 2g W_{\mathrm{cap}}(m;\kappa)_+.$$

Balancing with M1 and the long count gives $R_\kappa' \le R_\kappa$. At a fixed numerator moment estimate, Hölder supplies the additional saving

$$\Delta_U = \frac{3}{4}(R_\kappa - R_\kappa') \quad \text{in the } U\text{-exponent}, \qquad \Delta_Z(d) = d\Delta_U \quad (U = Z^d).$$

At $d = h$, $\Delta_Z(h) = 3h(R_\kappa - R_\kappa')/4$. The parameters of (11.12) yield

$$B_\natural = 0.87446749650084027305\ldots, \qquad B' = 0.87422125049290208328\ldots,$$

and the iteration starts at $B_\natural$ with $B_n - B' = 50^{-n}(B_\natural - B')$.

*Part II: prime exhaustion and variable comparison.* The Part I enlarged count above uses the width $W_1(m) = (1-1/p)(1-2m)/6$, with $p = 6\kappa$, and a comparison short length $1/4$. Section 12 also applies the no-prime branch of M2 when complete prime removal leaves a positive length excess. Together with a variable comparison rectangle, this permits

$$W_o(m) = \frac{1-1/p}{5}(1-2m), \qquad W_j(m) = \frac{1/2 + (1-3/p)(1-2m)}{p+12},$$
$$W_d(m) = \min\{W_o(m), W_j(m)\}.$$

The same joint selection is used in the detector. Its count and the numerator gain now give 0.87408 at fixed rational parameters. The moment iteration of Section 15 gives $B_{\mathrm{ex}}$. Thus Part II enlarges the allowed norm widths by changing the restoration and comparison rectangle; the theta identity, moment estimates and detector mechanism are retained.

*Part III: cubic unit twists.* Part III keeps the comparison of Part II and enlarges the origin width of Theorem 12.1. For a unit prime product $a$ with $(n,a) = 1$,

$$\mathcal{G}(n, a^4 h) = \overline{\chi_n(a)^4}\mathcal{G}(n,h) = \chi_n(a)^2 \mathcal{G}(n,h).$$

In Part II every unit prime of $Q_W$ is extended. Instead, we keep unit primes of total width $s$ as the cubic twist $\chi_n(a)^2$ of the common moving character. The completed parameters and exponents change by

$$K \mapsto K - 4s, \quad m_* \mapsto m_* + 4s, \quad q_* \mapsto q_* + s, \quad M_* \mapsto M_* + 5s,$$
$$\mathcal{R}_{\mathrm{loc}} \mapsto \mathcal{R}_{\mathrm{loc}} + 2s, \qquad \mathcal{O}_{\mathrm{total}} \mapsto \mathcal{O}_{\mathrm{total}} - 3s.$$

With $s = \rho w/3 + \vartheta_{\mathrm{sel}}$, where $0 \le \vartheta_{\mathrm{sel}} \le \nu/1000$ is the whole-factor overshoot of (19.4), the first length condition becomes

$$A_{\max} + 4W < 1 \;\longrightarrow\; A_{\max} + (4-\rho)W < 1.$$

The twist joins the moving radical of every returned sum and enlarges its width by $5s$. With $\lambda = 1 + p\alpha$, the coefficient of $w$ in the return estimate is

$$E_\alpha(p,\rho) = \frac{(7-5p\alpha)\rho}{18} - \frac{p\alpha(6-p-p\rho)}{6(p-1)}.$$

Lemma 12.2 allows the prime width $z < G/p$, that is $\alpha = 0$, and $E_0(p,\rho) = 7\rho/18 > 0$. Thus the twist needs a larger prime-length bound. Appendices F–H extend Theorem 12.1 to a moving radical, bound its terminal terms by an $S$-arithmetic additive large sieve, and split the returned sums by their relative gap $\gamma$ between the centered and uncentered norms. This gives

$$z < \Big(\frac{1}{p} + \alpha\Big)G \qquad \Big(\alpha < \alpha_{\max}(p) = \frac{4p-6}{p(p+24)}\Big),$$
$$E_{\alpha_{\max}}(p,\rho) \le 0 \iff \rho \le \rho_{\max}(p) = \frac{6(p-6)(2p-3)}{p^2 - 193p + 198}.$$

An absolute margin from the strict first condition closes the estimate where $E_\alpha$ vanishes.

In the comparison, only the origin width changes:

$$W_o(m) = \frac{1-1/p}{5}(1-2m) \;\longrightarrow\; \frac{1-1/p}{5-\rho}(1-2m),$$
$$C(p) = \frac{p+4}{20p} \;\longrightarrow\; \widetilde{C}(p) = \frac{p+4-\rho_{\max}(p)}{4p(5-\rho_{\max}(p))}.$$

The successive bounds converge to $B_{\mathrm{tw}}$. Lemmas H.4 and 21.1 show that $\alpha_{\max}$ and $\rho_{\max}$ are extremal, and Theorem 26.1 proves the optimality of $B_{\mathrm{tw}}$. As the width constant tends to $1/4$, the boundary of this comparison tends to $13/15$ (Section 27).

### 1.6 Optimality and organization of the proof

In this subsection, we specify the optimality statements for the successive comparison systems in Sections 8, 10, 16 and 26, and summarize the three parts leading to Theorem 1.1.

At the ordinary length $m_\theta = 1 - l_y/h$, the contour term vanishes:

$$h(1 - m_\theta) - l_y = 0.$$

At $x = 1/2$ one has $a(\delta, 1/2) = \delta/2$. The amplitude bound and the allowed norm width then give

$$h\gamma \le \frac{7\delta}{216}(2l_y - h) \qquad (x = 1/2).$$

At the critical class $(\delta, x) = (r_*, 1/2)$, this is exactly the limiting obstruction. It proves optimality for the comparison with the stated norm domain, count estimate and moment parameter $3/4$. It does not assert the best possible zero-free region for a field.

Section 10 leaves this system: there $\kappa_m < 3/4$ and the initial prime product is longer. If instead $\kappa_c, \kappa_m \ge 2B_n - 1$ and the norm of Lemma 10.1 is used, the same obstruction at $(\delta, x) = (r_\natural, 1/2)$ gives, for the comparison function (10.25) and every $b$,

$$\mathcal{Q}_{\kappa_\natural, \kappa_\natural, b, \ell}(r_\natural, \tfrac{1}{2}) = e_\natural(\ell - \ell_\natural), \qquad e_\natural > 1,$$

and $B_\natural$ is optimal in that system (Theorem 10.8). This statement fixes the earlier count of Proposition 6.4. The enlarged count of Proposition 11.3 changes that input. No optimality assertion is made here for this intermediate enlarged-count system. Section 16 proves the optimality of $B_{\mathrm{ex}}$ for the exhaustion-norm comparison of Part II, retaining the geometric length inequality (16.1). Section 26 proves the optimality of $B_{\mathrm{tw}}$ for the unit-twisted comparison of Part III, with the same inequality.

Section 2 states the common arithmetic conventions and earlier inputs. Part I consists of Sections 3–11. Sections 3–4 develop the local arithmetic and amplified norm. Sections 5–7 treat the contours, class counts and combined saving. Section 8 gives the endpoint and its stated optimality; Section 9 proves the boundary $B_*$ over arbitrary fields. Section 10 proves the starting boundary $B_\natural$ and its optimality with the earlier count. Section 11 proves the enlarged count, its endpoint inequalities and Theorem 1.3.

Part II consists of Sections 12–16. Sections 12–14 prove the exhaustion norm, detector count and rational starting bound. Section 15 proves Theorem 1.2, and Section 16 proves its stated optimality.

Part III consists of Sections 17–27. Sections 17–22 prove the unit-twisted norm, Theorem 17.1, with the bound for returned sums of Appendices F–H. Sections 23–25 give the modified comparison, its endpoint and Theorem 1.1; Section 26 proves Theorem 26.1, and Section 27 records the limit of the method. Appendix E contains the exact inequalities of Part III.

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## 2 Arithmetic notation and the earlier comparison theorems

In this section, we fix the notation used in Sections 3–7 and state precisely which earlier results are used. The new amplified norm is proved in Section 4; it is not included among the cited inputs.

Fix a number field $F \supset \mu_{12}$, hence totally complex, a sufficiently large finite set $S$ of places containing the archimedean places, and let $\mathcal{O}_{F,S}$ be the ring of $S$-integers, written $\mathcal{O}$, with $\mathcal{O}$ principal and self-dual for the chosen trace character. Put

$$q_{\mathfrak{a}} = N\mathfrak{a}, \qquad U_S = \mathcal{O}^{\times}, \qquad \Gamma = U_S^6, \qquad r_F = [F : \mathbb{Q}]/2.$$

Here $N\mathfrak{a}$ is the absolute ideal norm. For a nonzero allowed generator $a$, $q_a$ denotes the norm of its exterior ideal, with the factors at $S$ omitted. We use

$$t_+ = \max\{t, 0\}, \qquad \operatorname{rad}(\mathfrak{a}) = \prod_{p \mid \mathfrak{a}} p, \qquad \tau(\mathfrak{a}) = \#\{\mathfrak{d} : \mathfrak{d} \mid \mathfrak{a}\}.$$

The valuation $v_p(a)$ is the exponent of $p$ in the exterior ideal of $a$. Elements are taken modulo $\Gamma$ with the allowed-generator and finite-ray conventions of [Liu26b, Part III]. A subscript $p$ denotes a prime ideal outside $S$. All local characters retain their zero extensions at nonunits. Let $\Theta$ be the fixed finite group of finite-order Hecke characters, unramified outside $S$, used in the moment estimates. It contains all initial finite coefficient characters and the group $\Upsilon_S$ generated by the finite characters in the allowed-generator unit decomposition. Thus

$$\psi \in \Theta \iff \psi\upsilon \in \Theta \quad (\upsilon \in \Upsilon_S), \qquad \overline{\Theta} = \Theta.$$

Exceptional inducing characters are those belonging to $\Theta$. The symbol $\mathcal{U}$ denotes the sixth-power-free nonunit parameters whose associated primitive characters lie outside $\Theta$.

Sums over the arithmetic parameters retain their multiplicities. The theta parameters are $k, u$; the dual Gauss frequencies are $h, j$. Their norm ranges are distinct from primitive-character conductor ranges.

Length exponents are related by

$$U = Z^d, \qquad U^t = Z^{dt}.$$

We write $S_{m,i}$ for an ordinary centrally normalized sum of length $U^m$, $Q_0$ for the initial prime product, and $Q_W$ for a disjoint additional prime product. Their exact coefficients, finite twists and independent smooth weights are those in the moment families of [Liu26b]. The canonical comparison has the same prime factors and smooth weights:

$$P = S_{m,1}S_{m,2}Q_0, \qquad P_c = P - P_{\mathrm{cmp}}, \qquad (m, m) \longmapsto (1/4, 2m - 1/4).$$

For the actual ordinary scales, $X_1X_2 = X_{1,\mathrm{cmp}}X_{2,\mathrm{cmp}}$ exactly. For a nonnegative weight $\mathcal{W}$ on the summation parameters, $\mathfrak{S}_J(\mathcal{W})$ denotes the finite weighted Sobolev seminorm in the Poisson formula of [Liu26b, Section 8]. The complete Fourier measure is retained on both sides of each centered norm. The index $J$ in $\mathfrak{S}_J$ is a finite derivative order; $H \geq 0$ bounds the external height and norm-twist parameters. The local length expression also denoted by $J$ is defined in Section 3.3.

Put

$$\mathcal{Z}_F = \bigcup_{\chi \text{ primitive, finite order over } F} \{\rho : L_F(\rho, \chi) = 0,\ \rho \text{ nontrivial}\},$$

$$\beta_*(F) = \sup\big(\{1/2\} \cup \{\Re\rho : \rho \in \mathcal{Z}_F\}\big).$$

Principal poles are excluded. We use the following earlier results:

1. The cubic-theta coefficient formulas, reciprocal-$L$ identity, continuation criterion for the original integral and initial 7/8 bound of [Ope26a, Ope26b, Liu26b].

2. The inverse and ordinary moment estimates M1 and M2, together with their full Gauss-sum form, compact-quotient Poisson formula, reflection, common coprimality restrictions on the coefficient variables and exceptional means, as assembled in [Liu26b, Sections 6–13]. The quadratic and cubic sieve estimates are [GL13, BGL14]; the metaplectic construction is [KP84, KP85], with the global Fourier normalizations of [Wei64].

3. The averaged-numerator comparison of [Liu26b, Section 14], with

$$\beta_*(F) \le \frac{1+\kappa}{2} \implies \kappa_c = \kappa, \qquad \beta_*(F) \le \frac{7}{8} \implies \kappa_m = \frac{3}{4} \quad \text{for unamplified M2.}$$

4. The direct estimate with $B(\ell) = 11/12 - \ell/4$ for $1/6 \le \ell \le 1/5$, the principal correction, and the final continuation argument of [Liu26b]. A strict upper bound for every remaining amplitude class gives the corresponding bound for $\beta_*(F)$.

5. The general-field transfer of the moment and continuation arguments in [Liu26b, Theorem 18.2], and abelian descent in [Liu26b, Proposition 19.1]. All results numbered in the cyclotomic part of that paper are used with this transfer to $F \supset \mu_{12}$; constants may depend on the fixed field.

*The additional analytic estimates.* The following table separates the cited estimates from the extensions proved in this paper. The enlarged norm and contour range are conclusions of the indicated results, rather than additional hypotheses attributed to [Liu26b].

| Estimate | Earlier result used | Extension proved here |
|---|---|---|
| Amplified norm | Full Gauss and M2 formulas, [Liu26b, Sections 12.7–12.8 and Appendix C.4] | Propositions 4.6, 4.8 and Theorem 4.9: direct and complementary Poisson estimates; Theorem 12.1: variable comparison lengths and an empty remaining prime product. |
| AFE contour | The holomorphic region of [Liu26b, Lemma 13.10] | Lemmas 5.1–5.2: compact-strip smooth weights and $0 \le \theta \le \delta/2$. |
| Smallest amplitudes | The lowest-amplitude estimate in [Liu26b, Proposition 14.6] | Lemma 6.3: the margin $1/125$ in Part I; Section 14: the margin $1/250$ in Part II. |
| Class count | The count and numerator average in [Liu26b, Section 13.4 and Proposition 14.3] | Proposition 6.4 and Corollaries 6.6, 6.7: separate parameters and explicit compatibility bounds; Proposition 11.3: the enlarged norm in zero detection; Proposition 13.2: simultaneous selection in the exhaustion range. |

The exact full Gauss sums and the exceptional means in item (2) are used, with their original coefficient restrictions.

*Local estimates.* For each norm estimate with fixed $F, S, \Theta$ and fixed arithmetic conventions, choose the subsidiary parameters within its numerical strict margins. Fix the required smooth-weight and Fourier orders before the external height cutoffs and the eventual threshold. The constants and orders may depend on these fixed arithmetic data.

*Global continuation.* For each finite comparison step, use the order

$$
\begin{aligned}
&(F, \kappa_c, \kappa_m, b, \ell, \text{strict margins}) \\
&\quad \longrightarrow (\text{numerical losses, steps and meshes}) \\
&\quad \longrightarrow (N_{\mathrm{pr}}, \ell_i, \text{buffer } e) \\
&\quad \longrightarrow (\eta, \text{final } S, \Theta, \text{arithmetic data and tests}) \\
&\quad \longrightarrow \text{internal derivative and Fourier orders} \\
&\quad \longrightarrow \text{height powers, external cutoffs, tail orders and thresholds.}
\end{aligned}
$$

Here $\eta$ is the target character and $N_{\mathrm{pr}}$ is the number of original prime ranges. This is the order of [Liu26b, Remark 3.2 and Lemma 12.17]. In particular, the prime-length mesh is chosen independently of the subsequent target-dependent arithmetic data.

For the step $\sigma > 0$ of Section 3.1 and prescribed loss $\epsilon_{\mathrm{num}} > 0$, choose

$$
0 < \eta_{\mathrm{mesh}} < \min\left\{\eta_{\mathrm{M2}}, \frac{\sigma}{6}, \frac{\epsilon_{\mathrm{num}}}{C_{\mathrm{num}}}\right\},
$$

where $\eta_{\mathrm{M2}} > 0$ is the arithmetic-independent mesh of the cited lemma and $C_{\mathrm{num}} > 0$ bounds the numerical coefficients in the selection and length-restoration losses.

The choice depends only on these losses and bounded scalar parameters. Later sections impose further upper bounds on this same $\eta_{\mathrm{mesh}}$; it is taken below all of them. Take $d_{\min} = 1/100$. For $U = Z^d$, $d \geq d_{\min}$, require

$$\max_i \ell_i \leq d_{\min}\eta_{\mathrm{mesh}} \quad \Longrightarrow \quad \max_i \frac{\ell_i}{d} \leq \eta_{\mathrm{mesh}}.$$

For $d < d_{\min}$, use the small-parameter estimate of [Liu26b, Section 13.5]. After fixing the arithmetic data and original factors, retain their weight seminorms. For $H_N, \epsilon_N > 0$ with $|\theta_N| \leq H_N/\log U$,

$$Z \geq \exp\left(\frac{H_N}{d_{\min}\epsilon_N}\right) \implies |\theta_N| \leq \epsilon_N.$$

Thus the fixed mesh and prime lengths suffice; the threshold may depend on the target and finite comparison step.

For an exterior modulus $u$, the full normalized Gauss sum is

$$\mathcal{G}(u,h) = q_u^{-1/2} \sum_{a \bmod u} \chi_u(a) e(ha/u), \qquad \mathcal{G}(1,h) = 1.$$

Here $e$ is the fixed trace additive character and $\chi_u$ is the zero-extended sextic residue character. The imprimitive local factors and the correlation formulas are those of [Liu26b, Lemmas 12.9–12.10]; in particular they are not replaced by their primitive-unit values. Local lengths such as $c_C, d_D$ always mean logarithms of actual ideal norms to base $U$. In the Gauss calculations, $x_{\mathrm{aux}}$ denotes the auxiliary length removed from the initial prime product. In the amplified norm, this auxiliary length is zero; the initial prime product is retained.

*Local Gauss notation.* The local and global uses of these letters are distinguished by their sections:

| | Sections 3–4 | Sections 5–8 |
|---|---|---|
| $h$ | Gauss frequency | $(1+b+3\ell)/2$ |
| $a, b$ | local amplifier lengths | $a(\delta, x)$, balancing $b$ |
| $w$ | additional prime-product exponent | complex numerator variable |
| $T, S$ | local principal lengths | unamplified slope $S$. |

The fixed convolution ideals $C, D$ satisfy

$$\mathrm{rad}(C) = \mathrm{rad}(D) = \mathrm{rad}(CD),$$
$$c_C = \log_U q_C, \quad d_D = \log_U q_D, \quad \mu_{\mathrm{com}} = \min(c_C, d_D), \qquad 0 \leq R + E_0 \leq \mu_{\mathrm{com}}.$$

Their prime-power multiplicities may differ. Here $s$ is the fixed inner divisor and $m_{\mathrm{cut}}$ an ordinary-length cutoff; $R$ is the primitive common-character modulus exponent and $E_0$ the squarefree-divisor exponent in the complementary coprimality expansion. In the global comparisons, $R$ denotes a class-count exponent; for the earlier count with fixed $\kappa$, $R = R_\kappa(\delta, x)$. The symbol $D$ in $(C, D)$ denotes an ideal, whereas $D = U^m$ is an ordinary summation length. The local length excess in Section 3.1 is $D_{\mathrm{loc}}$; other excesses include $D_{\mathrm{free}}$ and the M2 excess $D$ of Lemma 12.2.

For later use, put

$$l_y = \frac{1+b-\ell}{2}, \quad h = \frac{1+b+3\ell}{2}, \quad l_x = \frac{1-b-\ell}{2}, \quad U = Z^d, \quad C_0 = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4}. \tag{2.1}$$

If $R_\kappa(\delta, x)$ is the class-count exponent defined in Section 6.3, the endpoint of the unamplified character-sum comparison at $d = h$ is

$$E_\kappa(\delta, x) = C_0 + \frac{\delta}{2} + \delta x \left\{ \ell + \frac{4}{9}\left(\frac{h}{4} - \frac{l_y}{2}\right)\right\} - \frac{3h}{4}(1 - R_\kappa(\delta, x)). \tag{2.2}$$

When $\kappa$ is fixed, the global comparison abbreviates $R = R_\kappa(\delta, x)$ and $E = E_\kappa(\delta, x)$. Away from $d = h$, the full comparison exponent is affine in $d$, with slope

$$S = \frac{3R_\kappa + 1}{4} + \frac{\theta_0}{2} - \frac{17}{50}.$$

For $0 \le m \le 1/2$, the reference initial length and actual prime-product exponent satisfy

$$z_0(m) = 2(1 - 2m)/9, \qquad 0 \le z \le z_0(m).$$

For $m > 1/2$, use the unamplified moment without selected prime factors, so $z = 0$. The available prime width is $\mathscr{L} = \ell/d$. A numerator saving $U^{-\gamma}$ subtracts $d\gamma$ from the full exponent.

# Part I
# The amplified norm and the enlarged class count

## 3 Local Gauss estimates and the prime products

In this section, we prove the local estimates used in the amplified norm of Section 4. We first count nonunit prime choices at the original frequency, then treat fixed unit primes and the common-support terms.

### 3.1 A divisor bound for nonunit prime contributions

In this subsection, we apply Cauchy before dividing the original Gauss frequency. The resulting divisor estimate is used in the local decomposition in Section 3.4.

The labels 'good' and 'bad' in the local formulas denote the unit and nonunit cases, respectively. Every prime with a nonunit Gauss contribution divides the original nonzero Gauss frequency. Counting its possible choices before dividing that frequency gives the following improvement. The frequency $h = 0$ is excluded throughout this argument. Its separate compact-quotient origin remains estimated by the earlier powerful-product count; no divisor bound is applied to zero.

We retain the full normalized Gauss sums, allowed generators modulo $\Gamma = U_S^6$, fixed finite ray-class data, Fourier measure for the complete centered coefficient sums and uniform orders of derivatives and height bounds of [Liu26b].

The Gauss norms with fixed arithmetic parameters are evaluated by the length-restoration argument of [Liu26b, Sections 12.7–12.8]: the auxiliary ordinary sums are placed in the allowed length range before M2 is applied. No Gauss mean outside the established length range is assumed.

The length notation for prime-factor removal is

| | |
|---|---|
| $u$ | total freely removed prime length |
| $v$ | actual nonunit-prime length |
| $c$ | fifth-power-correction length |
| $L$ | ordinary power-removal length |
| $T, S$ | principal upper/lower lengths |
| $D_{\mathrm{loc}}$ | length excess before restoration. |

In particular,

$$0 \le v \le u, \qquad L \ge v + 4T, \qquad c, T, S \ge 0.$$

For the original product and the auxiliary moment, respectively, write

$$z = \mathrm{length}(Q_0), \quad w = \mathrm{length}(Q_W), \quad A_t = 2m_{\mathrm{ord}} + z + w,$$
$$A = \text{coefficient-length exponent}, \quad m = \text{parameter-range exponent},$$
$$q = \text{displayed character-support exponent}, \qquad M = m + q.$$

Here $q$ includes canceled factors, and the auxiliary $m$ differs from the ordinary length $m_{\mathrm{ord}}$ in $S_{m_{\mathrm{ord}},i}$. The common-support contribution and inner-divisor exponents are

$$\widetilde{q} = q + R + E_0, \qquad B_C = (3c_C - 5d_D - R)_+/6,$$
$$B_D = (3d_D - 5c_C - R)_+/6,$$
$$s_0 = \log_U q_s, \qquad s_0' = \log_U q_{s'},$$

where $s'$ is the inner divisor after the local prime removals. Starred symbols $A_*, M_*, q_*, m_*$ denote the resulting parameters after the local identities; their updates are given in Section 3.2. The total freely removed unit-prime length is $g_{\mathrm{good}} = u - v - c$; the fixed valuation-zero primes have the separate length $f_0$. The actual frequency annulus has exponent $K$, while $K_0$ is its nominal cutoff exponent; $\delta_1 \geq 0$ is the support buffer and $\Delta = K_0 - K \geq -\delta_1$ records the cutoff discrepancy. The auxiliary lengths $J, g$ are specified in Section 3.3. The positive parameters $\sigma, \xi$ control the auxiliary step and support margins, and $\eta_{\mathrm{mesh}} > 0$ is the prime-length mesh. The total dyadic-support error is written $\theta_N = O(1/\log U)$.

The symbol $D_{\mathrm{loc}}$ never includes a second use of the favorable $-d_D$ term already included in the earlier argument's $J, g$ inequality.

### 3.1.1 Cauchy's inequality with a divisor bound

In this subsubsection, we prove the finite Cauchy inequality of Lemma 3.1 and apply it to nonunit-prime divisors. The resulting divisor bound is used for the local Gauss sums in Section 3.1.2.

**Lemma 3.1** (Cauchy's inequality for restricted finite sums)**.** *Let $\mathcal{H}, \mathcal{B}$ be finite sets, let $\mathcal{B}(h) \subset \mathcal{B}$ satisfy $\#\mathcal{B}(h) \leq N_{\mathcal{B}}$, and let $c_b, Z_b(h)$ be arbitrary complex numbers. Then*

$$\sum_{h\in\mathcal{H}} \left| \sum_{b\in\mathcal{B}(h)} c_b Z_b(h) \right|^2 \leq N_{\mathcal{B}} \sum_{b\in\mathcal{B}} |c_b|^2 \sum_{\substack{h\in\mathcal{H} \\ b\in\mathcal{B}(h)}} |Z_b(h)|^2. \tag{3.1}$$

*No common function $Z_b$ or common change of variable across $b$ is required.*

*Proof.* For each original $h$, Cauchy's inequality gives

$$\left| \sum_{b\in\mathcal{B}(h)} c_b Z_b(h) \right|^2 \leq \#\mathcal{B}(h) \sum_{b\in\mathcal{B}(h)} |c_b Z_b(h)|^2 \leq N_{\mathcal{B}} \sum_{b\in\mathcal{B}(h)} |c_b|^2 |Z_b(h)|^2.$$

Sum over $h$ and interchange the finite sums:

$$\sum_{h} \sum_{b\in\mathcal{B}(h)} |c_b|^2 |Z_b(h)|^2 = \sum_{b} |c_b|^2 \sum_{h:\, b\in\mathcal{B}(h)} |Z_b(h)|^2.$$

□

The order is

$$\text{Cauchy at fixed } h \quad \longrightarrow \quad \sum_{h,b} \quad \longrightarrow \quad \text{frequency division.}$$

The coefficient squares therefore carry no unrestricted count of $b$.

**Lemma 3.2** (Divisor bound for the nonunit prime choices)**.** *Fix the removed factor positions and their unit, nonunit and fifth-power-correction pattern. On the original bounded frequency range, the number of actual nonunit-prime tuples and their allowed ordinary valuation/allocation data is at most*

$$\tau(\mathfrak{h})^{C_N} (1 + \log q_h)^{C_N} \ll_\epsilon U^\epsilon (1 + H)^C.$$

*Here $\mathfrak{h}$ is the exterior ideal of the original nonzero frequency $h$, $N$ is the number of factors in the fixed original list, and all constants have the uniformity and order of choices specified in Section 2.*

*Proof.* For each actual nonunit prime, $p \mid \mathfrak{h}$. The full local table gives

| local character | ordinary multiplicity |
|---|---|
| nonprincipal or lower principal | $r = v_p(h) \geq 1$ |
| upper principal | $r+1 \leq v_p(h), \quad r \equiv 5 \pmod 6$. |

Disjoint prime ranges determine the factor positions. Allocating each $r$ between the two ordinary factors therefore gives, for fixed $N$,

$$\#\{\text{bad-prime tuples and allocations at } h\} \ll_N \tau(\mathfrak{h})^{C_N}(1+\log q_h)^{C_N}.$$

The fixed generator and unit classes change only the constant. Whole frequency dyads are retained before logarithmic Fourier separation, as in [Liu26b, Section 12.7.1]. Their cutoff satisfies

$$q_h \ll T_{\mathrm{sec}} U^{\xi/2} \ll U^{C_{\mathrm{cut}}}(1+H)^{A_{\mathrm{cut}}},$$

where $T_{\mathrm{sec}}$ is the supremum of the nominal frequency scale in the fixed outer sector, and $C_{\mathrm{cut}}, A_{\mathrm{cut}}$ are fixed independently of the later Fourier coordinate. The ideal divisor bound, with a sufficiently small exponent, therefore gives

$$\tau(\mathfrak{h})^{C_N}(1+\log q_h)^{C_N} \ll_\epsilon U^\epsilon (1+H)^C.$$

For the weighted Fourier estimates one may also use the larger frequency majorant, where $\boldsymbol{\nu}$ is the Fourier coordinate and $K_*, B$ are fixed polynomial-growth exponents,

$$q_h \ll U^{K_*}(1+\|\boldsymbol{\nu}\|)^B(1+H)^B \quad \Longrightarrow \quad \tau(\mathfrak{h})^{C_N}(1+\log q_h)^{C_N} \ll_\epsilon U^\epsilon(1+\|\boldsymbol{\nu}\|)^{C_\epsilon}(1+H)^{C_\epsilon}.$$

Choose the small divisor exponent after fixing $N$, and the required weighted Fourier order before integration and external heights. This preserves the original weighted measure; no uniform unweighted bound over all $\boldsymbol{\nu}$ is required. □

### 3.1.2 Local Gauss sums and the Fourier measure

In this subsubsection, we apply the divisor estimate of Section 3.1.1 before changing the nonunit-prime frequency. The resulting exponent is computed in Section 3.1.3.

Fix the original common $C, D, s$, finite ray components and Fourier coordinates for the complete coefficient sums before extracting primes. First process every removed $p \mid s$ with its divisor bound; the remaining base condition is $s' \mid n$. If that step divides the frequency by $B_s$, write $h = h_{\mathrm{initial}}/B_s$. The free nonunit primes avoid $B_s$, so

$$p \text{ free bad} \quad \Longrightarrow \quad p \mid h \iff p \mid h_{\mathrm{initial}}.$$

Classify the remaining free positions as unit, nonunit or fifth-power correction positions. Fix correction indices first; their $P^{-1}$ weights have absolute coefficient mass $\ll U^\epsilon$. No nonunit-prime-dependent frequency division precedes Cauchy.

For an actual nonunit prime put $r = v_p(n)$. The full local table is

| branch | condition at $h$ | division | relative squared factor |
|---|---|---|---|
| $6 \nmid r+1$ | $r \geq 1,\ v_p(h) = r$ | $p^r$ | $1$ |
| $6 \mid r+1$ | $v_p(h) = r$ | $p^r$ | $q_p^{-1}$ |
| $6 \mid r+1$ | $v_p(h) \geq r+1$ | $p^{r+1}$ | $q_p(1-q_p^{-1})^2 \leq q_p$. |

Thus $p \mid h$ in every case, and $r \geq 5$ in the principal cases. Their moving exponents are $r+2$ and, in the upper case, $r+1 \equiv 0 \pmod 6$. Retain the full natural-zero radical even when the active exponent is zero.

For the $i$-th prime range $\mathcal{P}_i$, let $P_i$ be its norm scale, $V_i$ its smooth weight and $c_i(p)$ its allowed finite-group coefficient. For a fixed nonunit-prime tuple $b$, let $B_b$ be its frequency divisor and $\zeta_b$ the bounded scalar from the fixed local factors. Put

$$c_b = \prod_{i \text{ bad}} P_i^{-1/2} c_i(p_i) V_i(q_{p_i}/P_i)\, \zeta_b, \qquad |\zeta_b| \le 1, \qquad q_{B_b} = U^{L+T+o(1)}.$$

Frequency-dependent phases belong to $Z_b(h)$ below. The actual finite-group coefficients satisfy $|c_i(p)| \le C_i$, with no unit-modulus assumption. Prime-ideal counting gives

$$\sum_{p \in \mathcal{P}_i} |P_i^{-1/2} c_i(p) V_i(q_p/P_i)|^2 \le P_i^{-1} C_i^2 \|V_i\|_\infty^2 \#\mathcal{P}_i \ll C_i^2 \|V_i\|_\infty^2,$$

and therefore

$$\sum_b |c_b|^2 \ll_\epsilon U^\epsilon \mathfrak{S}_J(\mathcal{W})(1+H)^C.$$

The ordinary and prime normalizations give

$$q_p^{-r/2} q_p^{r/2} = 1, \qquad P_i^{-1/2} \asymp q_p^{-1/2}$$

on the fixed list. The principal contribution is $U^{T-S+o(1)}$, separate from the coefficient energy.

For a free unit prime $p \nmid h$, extending the residual sum from $(n,p) = 1$ introduces only $v_p(n) = 5$: the full local Gauss formula gives

$$\big|\mathcal{G}(p^i, hp^4)\big| = \begin{cases} q_p^2, & i = 5, \\ 0, & i \ge 1,\ i \ne 5. \end{cases}$$

Thus the correction, including ordinary extraction and the removed prime normalization, has weight

$$q_p^2\, q_p^{-5/2}\, P_i^{-1/2} = q_p^{-1/2} P_i^{-1/2} \asymp P_i^{-1}.$$

Fix these correction primes first. Multiplicativity of the retained finite-component coefficients then leaves the same residual Gauss polynomial for all remaining unit-prime choices.

Let $a$ index the unit-prime choices. For fixed $b$, $F_b$ is the residual Gauss polynomial, common to all $a$; $e_{a,b}$ are the remaining unit-prime coefficients and $\omega_{a,b}(h)$ the phase factors, with their prescribed zeros. The factor $\alpha_b(h)$ contains the remaining principal Gauss-sum factors. On the sum with the exact divisibility conditions, retaining all Gauss phases,

$$Z_b(h) = \alpha_b(h) \sum_a e_{a,b} \omega_{a,b}(h) F_b\big((h/B_b)a^4\big), \qquad |\alpha_b(h)|^2 \le U^{T-S+o(1)}.$$

Apply Lemmas 3.1–3.2 at this $h$, then interchange the positive sums over $h, b$, and only then set $h = B_b h'$. For fixed $b$,

$$q_{h'} \ll U^{K-L-T+o(1)}, \qquad (a, B_b) = 1, \qquad j = h' a^4.$$

Disjoint prime ranges preserve the unit-prime unit conditions. Write $\mathcal{N}_{\text{pattern}}$ for this fixed prime-type contribution to the normalized Gauss norm. The orbit-sum argument, using the same polynomial $F_b$ for all unit-prime choices $a$, gives

$$\mathcal{N}_{\text{pattern}} \ll U^{T-S+\epsilon}(1+H)^C \sum_b |c_b|^2 \sum_{q_j \ll U^{K-L-T+4g_{\text{good}}+o(1)}} |F_b(j)|^2. \tag{3.2}$$

Its good coefficient energy satisfies, uniformly in $b$,

$$\sum_a |e_{a,b}|^2 \ll_\epsilon U^\epsilon \mathfrak{S}_J(\mathcal{W})(1+H)^C.$$

Unit-locus phases have modulus one; the natural zeros restrict the energy and the smooth weights remain fixed. Divisor Cauchy precedes the enlarged balls. For each $b$, use the same scale bounds, moving presentation and natural zeros, in the order

$$\text{single restoration} \longrightarrow \text{length and inhomogeneous-term bounds} \longrightarrow \text{M2 on ordinary sums.}$$

The displayed coefficient energy then gives (3.2).

For a real Mellin frequency $t$, the change of variable gives

$$q_h^{it} = q_{B_b}^{it} q_{h'}^{it}, \qquad |q_{B_b}^{it}| = 1.$$

Other unitary Mellin coordinates factor in the same way. Integrate polynomial derivative and height factors in the original common weighted Fourier measure at sufficiently high finite order. The original $C, D, s$ count is included once. All coefficient coprimality restrictions remain; positivity is used only after the original complete signed coefficient sum has been formed.

### 3.1.3 The resulting Gauss-norm exponent

In this subsubsection, we compute the one-sided exponent after the divisor estimate of Section 3.1.2. The resulting criterion is Proposition 3.3; fixed primes dividing $s$ are treated in Section 3.2.

The auxiliary arithmetic data $A_*, M_*, q_*$, the actual $K_0 - K$, the earlier $J, g$, and the additional length excess $D_{\mathrm{loc}} + 3u/2 - 5v - T$ are unchanged. Only the unrestricted count of nonunit prime choices $U^v$ has disappeared. Consequently the homogeneous relative exponent is now

$$C_{\mathrm{new}} = -u - L + v - 4c + T - S + \frac{(D_{\mathrm{loc}} + 3u/2 - 5v - T)_+}{6}. \tag{3.3}$$

Thus $C_{\mathrm{new}} = C_{\mathrm{old}} - v$, where $C_{\mathrm{old}}$ is the exponent of the earlier argument, which retains the count $U^v$. The remaining $+v$ counts the displayed moving radical and stays in the exponent.

**Proposition 3.3** (Improved criterion for removing additional prime factors)**.** *Retain the fixed coefficient formulas and analytic hypotheses above. For the nonunit-prime contribution in* (3.3)*,*

$$\begin{aligned} D_{\mathrm{loc}} \le 9u/2 \quad &\Longrightarrow \quad C_{\mathrm{new}} \le -5v/6 \le 0, \\ D_{\mathrm{loc}} \le 7u/2 \quad &\Longrightarrow \quad C_{\mathrm{new}} \le -(u+5v)/6. \end{aligned}$$

*The inhomogeneous terms in the one-sided Gauss norm retain their separate estimates.*

*Proof.* Put $e = L - v - 4T \ge 0$. If the positive part is active, the exponent in (3.3) equals

$$-\frac{3}{4}u + \frac{D_{\mathrm{loc}}}{6} - \frac{5}{6}v - \frac{19}{6}T - 4c - S - e.$$

If it is inactive, the exponent is $-u - 3T - 4c - S - e \le -u$.

Thus, on the active branch,

$$\begin{array}{l|c} D_{\mathrm{loc}} \le 9u/2 & C_{\mathrm{new}} \le -5v/6 \\ D_{\mathrm{loc}} \le 7u/2 & C_{\mathrm{new}} \le -(u+5v)/6. \end{array}$$

On the inactive branch, $0 \le v \le u$ gives $-u \le -(u+5v)/6 \le -5v/6$. □

The corresponding changes are

| | free-prime term | retained upper bound |
|---|---|---|
| second origin | $-3v \longmapsto -4v$ | $A_t - 1 + 4u$ |
| centered exceptional term | $-13v/6 \longmapsto -19v/6$ | $A_t - 1 + 3u.$ |

Length conditions, exact annular scales, empty or subunit ranges and prime restoration are checked as before. The improvement changes the prefactor, with the same Gauss mean estimate and $\kappa$.

## 3.2 A frequency change for fixed primes dividing $s$

In this subsection, we rewrite the Gauss frequency for fixed unit primes. The common coprimality restrictions are retained for the norm in Section 4.

We now fix a removed prime $p \mid s$ before varying the remaining frequency, and treat its valuation-zero contribution. This case requires a frequency identity rather than the divisor estimate of Section 3.1.

### 3.2.1 The exact fixed-prime norm comparison

In this subsubsection, we prove the frequency identity (3.4) for a fixed prime. Section 3.2.2 retains the coprimality conditions in this identity.

For $p \nmid h$ and $(p, n) = 1$, the full Gauss CRT formula gives

$$\mathcal{G}(pn, h) = \mathcal{G}(p, h)\mathcal{G}(n, hp^4). \tag{3.4}$$

Retain the finite ray and allowed-generator components. The exact valuation-zero allocation keeps $(p, n) = 1$, so no fifth-power correction occurs. Let $A(n)$ be the canonical centered convolution and, after fixing a character component and the Fourier parameters, let $\tau_{\mathrm{coef}}(n)$ be its common multiplicative character presentation, including its zero extension. Thus

$$\tau_{\mathrm{coef}}(ab) = \tau_{\mathrm{coef}}(a)\tau_{\mathrm{coef}}(b), \qquad |\tau_{\mathrm{coef}}(n)| \le 1.$$

This coefficient is distinct from the ideal divisor function $\tau(\mathfrak{a})$ of Section 2. Let $H_0 > 0$ denote the retained frequency-norm cutoff. For fixed $p$, put

$$c_p = P_i^{-1/2} c_i(p) V_i(q_p/P_i), \qquad |\tau_{\mathrm{coef}}(p)\mathcal{G}(p, h)| \le 1 \quad (p \nmid h).$$

Consequently

$$\begin{aligned} \sum_{\substack{q_h \le H_0 \\ p \nmid h}} \left| c_p \sum_{(n,p)=1} A(n)\tau_{\mathrm{coef}}(pn)\mathcal{G}(pn, h) \right|^2 \\ \le |c_p|^2 \sum_{q_j \le H_0 q_p^4} \left| \sum_{(n,p)=1} A(n)\tau_{\mathrm{coef}}(n)\mathcal{G}(n, j) \right|^2. \end{aligned} \tag{3.5}$$

The change $j = hp^4$ is injective on actual $\Gamma$-orbits:

$$q_h \le H_0,\ p \nmid h \quad \Longrightarrow \quad q_j \le H_0 q_p^4,\ v_p(j) = 4.$$

Drop the image condition only in the positive norm; retain $(n, p) = 1$. All scales and cutoffs are actual, with a single prescribed $O(1/\log U)$ annular loss.

Since $p \mid s$, divisor counting retains the squared normalization:

$$\sum_{\substack{p \in \mathcal{P}_i \\ p \mid s}} |c_p|^2 \ll_\epsilon U^\epsilon P_i^{-1} C_i^2 \|V_i\|_\infty^2.$$

For a varying unit-prime index $a$, the condition $(n, a) = 1$ would instead change the function in the orbit sum. Here $p$ is fixed before the frequency varies.

### 3.2.2 Preservation of the coprimality conditions

In this subsubsection, we carry the common coprimality restriction through both Poisson formulas of Section 3.2.1.

Let $\mathfrak{r}_*$ be the fixed modulus recording the retained coprimality restrictions on the coefficient variable. The residual polynomial and the enlarged coprimality modulus are

$$F_p(j) = \sum_n A(n)\tau_{\mathrm{coef}}(n)1_{(n,p)=1}\mathcal{G}(n,j), \qquad \mathfrak{r}_* \longmapsto \mathfrak{r}_* p.$$

Here $p$ is fixed before $j$ varies and $q_{\mathfrak{r}_* p}$ is polynomially bounded. The indicator comes directly from the valuation-zero allocation; no moving factor $\chi_p(n)^2$ has been introduced.

The common-coefficient identity of [Liu26b, Section 12.1 and Lemma 12.8] retains this indicator in both factors, both rectangles and every remaining prime factor. Its treatment is

| stage | $1_{(n,p)=1}$ |
|---|---|
| both Poisson formulas | retained in the full Gauss polynomial |
| ordinary nonexceptional sum | expanded by [Liu26b, Lemma 12.5] |
| fixed-character exceptional term | retained in both matching means. |

The ordinary expansion is used only after the finite Gauss coefficient has disappeared, before M2 or reflection.

The retained support implies $p \nmid n$ for every transformed coefficient variable. Hence $p$ cannot enter the genuine new supports $D_2, E_2$ or their fixed active factor. An artificial divisor allocation selecting $p$ vanishes before positivity. In the final ordinary expansion, $\psi_{h'}$ denotes the resulting ordinary character with its fixed twists. The factor $\psi_{h'}(p)$ is a bounded parameter factor with its original zero; its coefficient mass has arbitrarily small power growth.

Equation (3.4) uses the unfactored Gauss sum. The frequency is $j$ with its full natural zeros, and its enlarged range is counted below. A canceled displayed moving factor cannot be reclassified as an extra coprimality condition: choosing a presentation containing $\chi_p(n)^2$ would require retaining it.

### 3.2.3 Changes in the length exponents

In this subsubsection, we compute the changes in the fixed-prime lengths produced by the identity of Section 3.2.1; the resulting criterion is Corollary 3.4.

Write $\lambda_p = \log_U q_p$ up to the prescribed annular error. The fixed-prime frequency comparison has

$$A \mapsto A - \lambda_p, \quad K \mapsto K + 4\lambda_p, \quad M \mapsto M - 6\lambda_p, \quad q \mapsto q, \quad s_0 \mapsto s_0 - \lambda_p.$$

Consequently

$$(A - \lambda_p) + q + B_C - (s_0 - \lambda_p) = A + q + B_C - s_0, \qquad K_0 - K \text{ is unchanged.}$$

The fixed squared coefficient contributes $-\lambda_p$ and the global affine defect increases by $3\lambda_p/2$. The original $J, g$ calculation uses its favorable $-d_D$ once.

Let $w_f$ be the length of additional primes fixed in the first common support, and $z_{C,\mathrm{old}}$ the corresponding length from $Q_0$. The unused initial length is measured by $e_z = (9/2)(z_0(m) - z) \geq 0$. For primes dividing $s$ with zero ordinary valuation and total width $f_0 = x_{\mathrm{aux}} + w$, with no newly fixed common factors and no free removed positions, the homogeneous exponent is therefore bounded by

$$-f_0 + \frac{(\frac{9}{2}w + \frac{3}{2}f_0)_+}{6} = -\frac{3}{4}x_{\mathrm{aux}} \qquad (x_{\mathrm{aux}}, w \geq 0). \tag{3.6}$$

In particular this presentation makes the case of primes dividing $s$ with zero ordinary valuation neutral at $x_{\mathrm{aux}} = 0$, rather than leave the excess in the earlier local estimate $w/6$. This is a statement about this fixed-prime comparison, not a global theorem eliminating the initial prime product.

For reference, let $f_1, L_1, T_1, S_1$ denote the data for the remaining fixed primes dividing $s$; their ordinary valuations are positive. Here $f_1$ is their total prime length, $L_1$ the ordinary power-removal length, and $T_1, S_1$ their upper and lower principal lengths, respectively. Then $L_1 \geq f_1 + 4T_1$. The combined fixed-stage data given by these exact comparisons are

$$A_1 = A_t - f_0 - f_1 - L_1, \qquad M_1 = 1 - 6f_0 - f_1 - L_1 + T_1, \qquad q_1 = f_1,$$

with normalized coefficient exponent $-f_0 - L_1 + T_1 - S_1$. After this fixed stage, perform the free unit, nonunit and fifth-power-correction partition afresh at its current frequency on the enlarged Gauss family. The divisor Cauchy is applied there before any free nonunit-prime division; it is applied to the newly obtained frequency sum.

Adding the divisor-Cauchy free stage gives the following upper bound for the normalization exponent when the positive part is nonzero:

$$-\frac{3}{4}x_{\mathrm{aux}} + \frac{w_f}{6} - \frac{5}{6}(f_1 + v) - \frac{19}{6}(T_1 + T) - \frac{7}{12}z_{C,\mathrm{old}} - \frac{e_z}{6} - 4c - S_1 - S,$$

after dropping further nonpositive ordinary-removal terms. The case in which the positive part vanishes has a nonpositive exponent. More precisely, put $e_1 = L_1 - f_1 - 4T_1 \geq 0$ and $e = L - v - 4T \geq 0$. Its normalized exponent is

$$-f_0 - f_1 - u - 3(T_1 + T) - 4c - S_1 - S - e_1 - e \leq 0.$$

**Corollary 3.4** (A nonpositive contribution to the norm exponent)**.** *Retain the length conditions, centered-weight, common-measure and length-restoration hypotheses of Section 3.1, together with the fixed-unit comparison above. Then*

$$0 \leq w_f \leq w, \qquad x_{\mathrm{aux}} \geq 2w_f/9 \quad \Longrightarrow \quad \mathcal{C}_{\mathrm{act}}, \mathcal{C}_{\mathrm{inact}} \leq 0.$$

*In particular, $x_{\mathrm{aux}} \geq 2w/9$ suffices for every $w_f \leq w$. For a prescribed discretization margin $\rho_{\mathrm{step}} > 0$,*

$$x_{\mathrm{req}} \geq 2w/9 + \rho_{\mathrm{step}}, \qquad \eta_{\mathrm{mesh}} \leq \rho_{\mathrm{step}}/3, \qquad x_{\mathrm{aux}} \geq x_{\mathrm{req}} - \eta_{\mathrm{mesh}}$$
$$\Longrightarrow \qquad \mathcal{C}_{\mathrm{act}} \leq -\rho_{\mathrm{step}}/2,$$

*before the smaller subsidiary errors. Here $\mathcal{C}_{\mathrm{act}}$ and $\mathcal{C}_{\mathrm{inact}}$ denote the contributions with nonzero and zero positive part, respectively.*

*Proof.* The active and inactive contributions satisfy, respectively,

$$\mathcal{C}_{\mathrm{act}} \leq -3x_{\mathrm{aux}}/4 + w_f/6 \leq 0 \qquad (x_{\mathrm{aux}} \geq 2w_f/9),$$
$$\mathcal{C}_{\mathrm{inact}} = -f_0 - f_1 - u - 3(T_1 + T) - 4c - S_1 - S - e_1 - e \leq 0.$$

Since $w_f \leq w$, the stronger condition $x_{\mathrm{aux}} \geq 2w/9$ suffices. For a complete-factor choice $x_{\mathrm{aux}} \geq x_{\mathrm{req}} - \eta_{\mathrm{mesh}}$,

$$\mathcal{C}_{\mathrm{act}} \leq -\frac{3}{4}\left(\frac{2w}{9} + \rho_{\mathrm{step}} - \eta_{\mathrm{mesh}}\right) + \frac{w}{6} \leq -\frac{3}{4}\rho_{\mathrm{step}} + \frac{1}{4}\rho_{\mathrm{step}} = -\frac{1}{2}\rho_{\mathrm{step}}.$$

At $u = 0$, retain the full fixed coefficient exponent $-f_0 - L_1 + T_1 - S_1$ and the single restoration loss. No condition restricted to the free primes, such as $D_{\mathrm{loc}} \leq 9u/2$, is imposed on this fixed contribution. □

This criterion bounds the normalization exponent. Its other length hypotheses remain necessary when $x_{\mathrm{aux}}, w$ vary. The case in which every additional prime lies in the common support is not sufficient by itself: when $w_f = w$ and $x_{\mathrm{aux}} = e_z = z_{C,\mathrm{old}} = f_1 = v = 0$, necessarily $f_0 = u = 0$, and this upper bound for the exponent is $w/6$. This is not an arithmetic lower bound, but it prevents a uniform $x_{\mathrm{aux}} = 0$ conclusion from these bounds alone.

### 3.2.4 The remaining prime length

In this subsubsection, we compare the remaining prime length with the transformed length excess of Section 3.2.3.

Let $D_{\mathrm{free}}^{\mathrm{old}}$ denote the remaining length excess in the preceding fixed-$s$ table, evaluated at the same actual allocation. The fixed-unit change gives $D_{\mathrm{free}}^{\mathrm{new}} = D_{\mathrm{free}}^{\mathrm{old}} + 5f_0$. After substituting $u = x_{\mathrm{aux}} + w - w_f - f_0 - f_1$, its exact value is

$$D_{\mathrm{free}}^{\mathrm{new}} = \frac{3}{2}x_{\mathrm{aux}} + 6w - e_z - \frac{7}{2}z_{C,\mathrm{old}} - 5w_f - 5f_1 - T_1 - 5v - T.$$

Here $z_0 = z_0(m)$, and $z_{0,\min}$ below denotes its minimum over the chosen ordinary-length interval. Write $z = z_0 - 2e_z/9$ and $z_{\mathrm{init}} = z - z_{C,\mathrm{old}} - x_{\mathrm{aux}}$ for the remaining original prime width. Then

$$\begin{aligned}\frac{7}{2}z_{\mathrm{init}} - D_{\mathrm{free}}^{\mathrm{new}} &= \frac{7}{2}z_0 - 5x_{\mathrm{aux}} - 6w + \frac{2}{9}e_z + 5w_f + 5f_1 + T_1 + 5v + T \\ &\geq \frac{7}{2}z_{0,\min} - 5x_{\mathrm{aux}} - 6w. \end{aligned} \tag{3.7}$$

The $f_0$ term cancels. For example,

$$z_{0,\min} = 2/75, \quad x_{\mathrm{aux}} = 9/5000, \quad w = 3/500 \quad \Longrightarrow \quad \frac{7}{2}z_{0,\min} - 5x_{\mathrm{aux}} - 6w = \frac{29}{600}.$$

Decreasing $x_{\mathrm{aux}}$ increases the margin. Genuine and final-divisor fixing of widths $b_g, b_d$ reduce the defect by at least $7b_g/2 + 9b_d/2$. Thus restoration, including exhaustion, remains valid when (3.7) exceeds the mesh and amplifier errors, with the used prime widths subtracted at each stage.

### 3.2.5 Ordinary summation lengths and exceptional terms

In this subsubsection, we compute the actual summation lengths and the origin and exceptional exponents for the allocation of Section 3.2.4. Section 3.3 then retains the common-support length.

Put $t = f_0 + u$ and $a = f_1 + t \leq x_{\mathrm{aux}} + w$. After all fixed and free local comparisons the auxiliary global parameters are

$$\begin{aligned} A_* &= A_t - f_1 - t - L_1 - L - 5c, \\ M_* &= 1 - 6t - f_1 - L_1 - L - 5c + T_1 + 5v + T, \\ q_* &= f_1 + v + c, \qquad m_* = M_* - q_*. \end{aligned}$$

All nonunit and correction natural support restrictions remain in the displayed $q_*$; only the unfactored fixed unit primes are in the common extra radical. If $D_{\mathrm{ord}} = L_1 + L + 5c \leq 2m + o(1)$, these give

$$M_* \geq 1 - 2m - 6a - o(1), \qquad m_* \geq 1 - 2m - 7a - o(1), \qquad A_* - m_* \leq A_t - 1 + 6a + o(1).$$

The last estimate follows also from the exact identity $A_* - m_* = A_t - 1 + 5t + f_1 - T_1 - 4v + c - T$. For nonempty common supports,

$$c_C \leq A_* + o(1), \quad R + E_0 \leq d_D \quad \Longrightarrow \quad m_* - c_C + d_D - R - E_0 \geq m_* - A_* - o(1) > 0$$

whenever $m_* - A_*$ has a fixed positive margin. The ordinary mean estimate is applied only after this verification of the length conditions. The actual frequency shift is

$$K_0^* = K_0 + 4(f_0 + g_{\mathrm{good}}) - L_1 - T_1 - L - T, \qquad g_{\mathrm{good}} = u - v - c.$$

It agrees with the formal length identity, so $K_0 - K$ and the original earlier $J, g$ restoration calculation remain unchanged.

After including all normalized fixed coefficients, the second-origin excess is exactly

$$A_t - 1 + 4t - L_1 - S_1 - L - 4v - 4c - S \le A_t - 1 + 4t.$$

The corresponding centered exceptional excess is

$$A_t - \frac{5}{6} - \frac{f_1}{6} - \frac{7}{6}(L_1 + L) + \frac{T_1 + T}{6} - S_1 - S + 3t - \frac{19}{6}v - \frac{29}{6}c \\ - \frac{2}{3}\big(\tfrac{1}{4} - L_1 - L - 5c\big)_+ \le A_t - 1 + 3t.$$

Indeed, write $L_{\mathrm{tot}} = L_1 + L$ and let $\mathcal{X}$ denote the preceding exceptional excess. Since $L_1 \ge f_1 + 4T_1$ and $L \ge v + 4T$, one has $T_1 + T \le L_{\mathrm{tot}}/4$. Hence

$$\begin{aligned} \mathcal{X} - (A_t - 1 + 3t) &\le \frac{1}{6} - \frac{9L_{\mathrm{tot}}}{8} - \frac{29c}{6} - \frac{2}{3}(1/4 - L_{\mathrm{tot}} - 5c)_+ \\ &\le \begin{cases} -11L_{\mathrm{tot}}/24 - 3c/2, & L_{\mathrm{tot}} + 5c \le 1/4, \\ 1/6 - 2(L_{\mathrm{tot}} + 5c)/3, & L_{\mathrm{tot}} + 5c \ge 1/4 \end{cases} \le 0. \end{aligned}$$

Thus the same parameter regions, or smaller $x_{\mathrm{aux}}$ within them, retain the length, origin and exceptional margins. Exact annular scales, independent smooth weights, the common explicit ideal-norm Mellin twist and canonical centering remain those of the cited theorem; the stated zero-free boundaries are unchanged.

## 3.3 Retaining the common-support length

In this subsection, we retain the common ideal length in the restoration inequality. This gives the homogeneous estimate used in Section 3.4.

The first paired common-support term need not save a power at a prime occurring to the first power in both ordinary factors. The exact restoration expression, however, retains a favorable common length that its earlier coarse upper bound discarded. We use its full common-length term in the following estimate.

### 3.3.1 The common-support normalization

In this subsubsection, we compare the paired common-support term with the restoration term. The result, Lemma 3.5, is inserted in the length excess in Section 3.3.2.

In the notation of the paired common-support comparison, put $c_C = \log_U q_C$, $d_D = \log_U q_D$ and $p = \log_U q_{\mathrm{rad}(CD)}$. The paired normalization is

$$M + \tfrac{1}{2}\{B_C + B_D - (c_C + d_D - 2p - R)\}, \qquad B_C = (3c_C - 5d_D - R)_+/6.$$

A new $C$-fixed prime may divide the other ordinary factor with multiplicities $(1, 1)$. At that prime,

$$(3 - 5)_+ = 0, \qquad 1 + 1 - 2 = 0, \qquad q_p^{-1} q_p = 1,$$

where the last factors are the paired central normalization and the single radical-family count. Prime sparsity gives logarithms; this term alone gives no saving proportional to the new common width.

Write $a$ for the earlier amplifier extraction length $w$ and $b$ for its new-radical length $w_o$, to distinguish them from the new length of the original prime-ideal range. The choices in the cited formula are

$$\Delta = K_0 - K \ge -\delta_1, \quad J = d_D - c_C + \Delta - 2a + b, \quad g = J_+ + O(\sigma), \quad 0 \le b \le a.$$

They are the unchanged quantities of [Liu26b, Sections 12.7.2 and 12.8.1].

**Lemma 3.5** (Common-support saving in length restoration)**.** *Suppose*

$$c_C, d_D \geq 0, \qquad \delta_1 \geq 0, \qquad \Delta \geq -\delta_1, \qquad 0 \leq b \leq a,$$
$$J = d_D - c_C + \Delta - 2a + b, \qquad \mu_{\mathrm{com}} = \min(c_C, d_D).$$

*Then*

$$\begin{aligned} -d_D - \Delta - b + J_+ &= \max\{-d_D - \Delta - b, -c_C - 2a\} \\ &\leq -\mu_{\mathrm{com}} + \delta_1. \end{aligned} \tag{3.8}$$

*Proof.* Put $X = -d_D - \Delta - b$. Then

$$\begin{aligned} X + J_+ &= \max\{X, X + J\} = \max\{-d_D - \Delta - b, -c_C - 2a\} \\ &\leq \max\{-d_D + \delta_1, -c_C\} \leq -\min(c_C, d_D) + \delta_1, \end{aligned}$$

using $\Delta \geq -\delta_1$, $a, b \geq 0$. □

This replaces the earlier argument's coarse bound of this entire expression by $\delta_1$. It is not a second use of the favorable $-d_D$. For example, when $c_C = d_D = \lambda_{\mathrm{com}} > 0$ and $\Delta = a = b = 0$, the first common-support term can be zero while (3.8) is exactly $-\lambda_{\mathrm{com}}$. The two coarse equalities cannot be attained simultaneously on such a common support.

### 3.3.2 The transformed length excess

In this subsubsection, we insert the common-length inequality of Lemma 3.5 into the transformed length excess of [Liu26b, Section 12.8.1].

Before its coarse $J$ substitution, [Liu26b, Section 12.8.1] bounds the transformed length defect by

$$a - d_D - (K_0 - K) - b + g + 2\theta_N.$$

The earlier formal reconstruction of ordinary summation lengths adds its global length excess and the common-fixed-prime term $-(7/2)z_C$. All its local comparisons shift actual $K$ and formal $K_0$ identically; $K_0 - K$, $C, D$, the earlier $J, g$, and the single paired arithmetic count are unchanged. Lemma 3.5 therefore lowers the additional free defect by $\mu_{\mathrm{com}}$, using the same substitution exactly once. More explicitly the full transformed length defect is at most

$$D_{\mathrm{free}} - \mu_{\mathrm{com}} + a + (g - J_+) + \delta_1 + 2\theta_N.$$

The earlier-amplifier contribution satisfies

$$a + (g - J_+) \leq 6(a + \ell), \qquad \begin{cases} a = 0,\ g - J_+ = 2\sigma = 6\ell, & \text{main term}, \\ \ell = 0,\ g - J_+ = \sigma,\ a + \sigma \leq 6a, & \text{amplifier error}. \end{cases}$$

Using $(X + Y)_+ \leq X_+ + Y$ for $Y \geq 0$, the squared loss is at most

$$\frac{(D_{\mathrm{free}} - \mu_{\mathrm{com}})_+}{6} + a + \ell + \frac{\delta_1 + 2\theta_N}{6} + \eta_{\mathrm{mesh}}.$$

The earlier exponent allowance for the auxiliary sum $\Delta' \geq a + \ell$ absorbs its contribution from the original amplifier. The additional squared loss of restoring the lengths is therefore at most

$$\frac{(D_{\mathrm{free}} - \mu_{\mathrm{com}})_+}{6} \tag{3.9}$$

apart from the already prescribed dyadic, support and mesh errors.

The original amplifier estimate still accounts for its own contribution. The exponent allowance for the auxiliary sum remains $\Delta' = b_2 - p_2 + a + B_C + \ell$; none of these terms is reused.

Here $b_2 = (c_2 + d_2)/2$, where $c_2, d_2$ are the two full common-support lengths in the second Poisson formula, and $p_2$ is the length of their radical. Write $g_2$ for the length of their greatest common divisor. After dividing the frequency by this divisor, the primes with equal multiplicities not divisible by six split into unit and nonunit sets: $t_2$ is the radical length of the unit set, and $V$ the length of the product of the nonunit primes. All these lengths are relative to $U$; see Section 4.3.

Lowering the defect improves the previously verified coupled remaining-prime margin. Every later genuine or final-divisor fixing is counted in that same length-restoration argument for all prime factors. When no prime factor remains, the cited estimate for arbitrary ordinary lengths applies with the same order of parameter choices. In particular (3.9) is derived using ordinary sums allowed for M2, rather than assumed for a Gauss sum outside the established length range.

### 3.3.3 The nonpositive normalization exponent

In this subsubsection, we prove that the full homogeneous exponent is nonpositive (Corollary 3.6). This estimate is used in the amplified norm of Section 3.4.

Retain the fixed-unit rewriting and original-frequency divisor Cauchy of the preceding sections. Let $f_0$ denote fixed unit primes dividing $s$, $f_1, L_1, T_1, S_1$ the other data for fixed primes dividing $s$, $z_{C,\mathrm{old}}$ the initial first-common-support prime length, and $w_f$ the corresponding new width. All widths and nonnegative local losses have their earlier meanings. Thus

$$u = x_{\mathrm{aux}} + w - w_f - f_0 - f_1 \geq 0, \quad L_1 \geq f_1 + 4T_1, \quad L \geq v + 4T,$$

and

$$D_f = \frac{9}{2}w - e_z - \frac{7}{2}(z_{C,\mathrm{old}} + w_f) + \frac{3}{2}f_0 - \frac{7}{2}f_1 - T_1.$$

Write $c_{\mathrm{cor}}$ for the free correction width, distinct from $c_C$, and put $e_1 = L_1 - f_1 - 4T_1 \geq 0$, $e = L - v - 4T \geq 0$.

**Corollary 3.6** (A nonpositive exponent with the initial prime product retained)**.** *Let the local widths and losses defined above be nonnegative, with*

$$\begin{gathered} u = x_{\mathrm{aux}} + w - w_f - f_0 - f_1 \geq 0, \qquad L_1 \geq f_1 + 4T_1, \qquad L \geq v + 4T, \\ D_f = \frac{9}{2}w - e_z - \frac{7}{2}(z_{C,\mathrm{old}} + w_f) + \frac{3}{2}f_0 - \frac{7}{2}f_1 - T_1, \\ \mu_{\mathrm{com}} = \min(c_C, d_D) \geq w_f. \end{gathered}$$

*Then, on actual ideal scales,*

$$\begin{aligned} \mathcal{C} = & -f_0 - L_1 + T_1 - S_1 - u - L + v - 4c_{\mathrm{cor}} + T - S \\ & + \frac{(D_f + 3u/2 - 5v - T - \mu_{\mathrm{com}})_+}{6} \leq 0. \end{aligned} \tag{3.10}$$

*This algebraic inequality requires neither $x_{\mathrm{aux}} \geq 2w_f/9$ nor positivity of the formal summation length.*

*Proof.* Put $\mathcal{D} = D_f + 3u/2 - 5v - T - \mu_{\mathrm{com}}$ and

$$\mathcal{C}_{\mathrm{hom}} = -f_0 - f_1 - u - 3(T_1 + T) - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e.$$

Substitute $w = u - x_{\mathrm{aux}} + w_f + f_0 + f_1$ into $D_f$. Then

$$\mathcal{D} = 6u - \frac{9}{2}x_{\mathrm{aux}} + w_f + 6f_0 + f_1 - \frac{7}{2}z_{C,\mathrm{old}} - e_z - T_1 - 5v - T - \mu_{\mathrm{com}},$$

and $L_1 = f_1 + 4T_1 + e_1$, $L = v + 4T + e$ give

$$\mathcal{C} = \mathcal{C}_{\mathrm{hom}} + \mathcal{D}_+/6 = \max\{\mathcal{C}_{\mathrm{hom}}, \mathcal{C}_{\mathrm{hom}} + \mathcal{D}/6\}.$$

The two entries satisfy

$$\begin{aligned}
\mathcal{C}_{\mathrm{hom}} &= -f_0 - f_1 - u - 3(T_1 + T) - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e \le 0,\\
\mathcal{C}_{\mathrm{hom}} + \mathcal{D}/6 &= -\frac{3}{4}x_{\mathrm{aux}} + \frac{w_f - \mu_{\mathrm{com}}}{6} - \frac{5}{6}(f_1 + v) - \frac{19}{6}(T_1 + T)\\
&\quad - \frac{7}{12}z_{C,\mathrm{old}} - \frac{e_z}{6} - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e \le 0.
\end{aligned}$$

Here $x_{\mathrm{aux}} \ge 0$, $\mu_{\mathrm{com}} \ge w_f$, and all remaining widths and losses are nonnegative. The calculation also holds at $u = 0$: it uses the full coefficient exponent and the single restoration loss, with no additional free-prime condition. □

Its analytic length-restoration application retains the cited moment hypotheses, positive global and formal lengths, full local support and coprimality conditions, exact centered weights and equal scale products, common Fourier measure, and remaining-prime-length conditions, including the case with no remaining prime factor.

Distinct new first-common primes divide $\mathrm{rad}(CD)$, and the original prime ranges are disjoint. Hence

$$w_{f,\mathrm{actual}} \le p \le \min(c_C, d_D) = \mu_{\mathrm{com}}, \qquad w_f \le \mu_{\mathrm{com}} + \theta_N.$$

The nominal homogeneous exponent is at most $\theta_N/6$; together with the original $\delta_1/6$ and edge losses it is counted once, not once per prime factor. For the requested positive loss, let $C_* > 0$ be the fixed constant controlling the accumulated support errors, and choose

$$0 < \sigma \text{ inside the fixed margins}, \qquad C_*\xi < \sigma/2, \qquad \eta_{\mathrm{mesh}} < \sigma/6.$$

The unchanged $J, g$ give auxiliary-length decrease at least $\sigma/2$. Every M2 application remains in the established allowed family. Retain the fixed field, coefficient and unit classes, natural zeros, and weighted derivative/Fourier conventions of Section 2. Fix the finite positive-length prime list and tests, then the weighted orders, before external heights. Exact annular extractions and equal scale products are preserved; absorb the single vanishing aggregate support error after the fixed threshold. Zero-length bounded-mass factors contribute no removed positive width.

We now apply this homogeneous estimate together with the decomposition into all local prime cases.

*From the local estimates to the amplified norm.* The preceding result is a local restoration statement. The following subsection combines these estimates to prove the amplified norm on the stated parameter region. Throughout, $x_{\mathrm{aux}} = 0$ means no auxiliary removal of a subproduct of the initial prime product; the original $Q_0$ remains in the polynomial.

### 3.4 An amplified norm retaining the initial prime product

In this subsection, we prove the amplified norm with the initial prime product retained. We separate the additional prime factors according to their divisibility and valuation conditions, verify the resulting length bounds, and then apply the ordinary second-moment estimate M2 recalled in Section 2. Section 4.1 extends the argument to variable lengths.

Throughout, $x_{\mathrm{aux}} = 0$: the auxiliary comparison removes no part of $Q_0$. Retain the common-support decomposition and ordinary-length restoration. For every sign of the intermediate length excess, the order is

$$\text{all local identities for } Q_W \quad \longrightarrow \quad \text{M2}.$$

*The moment and its arithmetic data.* We use the following coefficient families and length ranges.

Retain the fixed field, excluded places, finite coefficient families and unit-class data, Fourier conventions for the complete coefficient sums and the earlier bound $\beta_*(F) \le 7/8$, which permits $\kappa = 3/4$. Assume precisely the same full Gauss-sum, compact-quotient Poisson, M2, reflection, common-coefficient, centered-mean and weighted smooth-weight and Fourier results as in Section 2. These are the established results cited in Section 2. First consider the parameter range

$$0.4 \le m \le 0.443, \quad 0.4 \le \delta \le 0.55, \quad 0.48 \le x_{\mathrm{amp}} \le 0.5, \quad 0.21 \le \mathscr{L} \le 0.22.$$

Let $P = S_{m,1}S_{m,2}Q_0$ be the original unrestricted product, $z \le z_0 = 2(1-2m)/9$ the nominal length exponent of the initial prime product, and $Q_W$ a disjoint original product of length $0 \le w \le 1/100$. All positive-length prime ranges are mutually disjoint. For each independent smooth-weight tuple, use the same two tests and set

$$(m,m) \longmapsto (1/4, 2m-1/4), \qquad P_{\mathrm{c}} = P - P_{\mathrm{cmp}}, \qquad X_1X_2 = X_{1,\mathrm{cmp}}X_{2,\mathrm{cmp}}.$$

Fix the comparison before differentiation; retain exact norms, powers, weight arguments and the scale-product equality above. Only the affine length formulas incur the single aggregate error $\theta_N = O(1/\log U)$ over the finite allocations.

**Theorem 3.7** (Amplified norm with the initial prime product retained)**.** *Under the hypotheses above, for every required finite independent derivative family and every $\epsilon > 0$,*

$$\sum_k \mathcal{W}(q_k/U)|P_{\mathrm{c}}(k)Q_W(k)|^2 \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C, \tag{3.11}$$

$$\sum_{k\in\mathcal{U}} \mathcal{W}(q_k/U)|P(k)Q_W(k)|^2 \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C. \tag{3.12}$$

*The centered norm is taken over the unrestricted positive sum in the comparison; $\mathcal{U}$ is the original sixth-power-free nonunit family inducing outside $\Theta$. No auxiliary subproduct is removed from the initial prime product: $x_{\mathrm{aux}} = 0$ throughout. The amplitude selection is treated separately in Section 4.1.*

*Proof. Step 1. Divisibility and valuation cases for the additional primes.* We first separate the factors of $Q_W$ according to common support, divisibility by $s$, and their valuations in the ordinary coefficient variables. We apply M2 only after all these factors have been treated and the resulting lengths have been checked in Steps 2 and 3.

First write $Q_W = Q_{W,0}Q_{W,+}$, $Q_0 = Q_{00}Q_{0,+}$, where the subscript 0 denotes zero nominal length. For each fixed derivative tuple,

$$P = Q_{00}P_+, \qquad P_{\mathrm{c}} = Q_{00}P_{\mathrm{c},+},$$

and the character factors have modulus at most one. Hence

$$|Q_{W,0}(k)Q_{00}(k)| \le \|c_{Q_{W,0}}\|_{\ell^1}\|c_{Q_{00}}\|_{\ell^1}. \tag{3.13}$$

Apply this bound to both positive norms before Poisson. Its bounded coefficient mass belongs to the retained seminorm and height factor. Relabel $Q_{W,+}, Q_{0,+}$ as $Q_W, Q_0$; their lengths are still $w, z$. Only positive-length factors enter the following valuations. The selected extra factors also have positive amplitude.

Fix the original $C, D, s$, finite components and common Fourier coordinates in the order of Section 2. Put

$$c_C = \log_U q_C, \quad d_D = \log_U q_D, \quad \mu_{\mathrm{com}} = \min(c_C, d_D), \quad z_C = z_{C,\mathrm{old}} + w_f.$$

Each new first-common prime divides both $C, D$. Thus

$$w_{f,\mathrm{actual}} \le \mu_{\mathrm{com}}, \qquad w_f \le \mu_{\mathrm{com}} + \theta_N, \quad \theta_N = O(1/\log U).$$

Split every remaining extra prime according to $p \mid s$ or $p \nmid s$, before Gauss evaluation or positivity. For $p \mid s$, fix $p$ as a divisor of $s$, retain its coefficient and use

$$1_{s|pn} = 1_{(s/p)|n}.$$

The complete valuation allocation has two cases:

| ordinary valuation | total width | identity or bound |
|---|---|---|
| $0$ | $f_0$ | $\mathcal{G}(pn,h) = \mathcal{G}(p,h)\mathcal{G}(n,hp^4)$ |
| $>0$ | $f_1$ | $L_1 \ge f_1 + 4T_1$. |

The first identity requires $(p,n) = 1$ and, on nonzero terms, $p \nmid h$. For the whole fixed valuation-zero product $a_0$, impose these conditions simultaneously, then change $h \mapsto ha_0^4$ injectively, before positive enlargement. Positive-valuation divisions use disjoint primes and preserve the unit conditions. The scalar has modulus at most one; its squared coefficient retains $q_{a_0}^{-1}$. Keep $(n, a_0) = 1$ in both centered rectangles as a fixed extra radical, until an allowed ordinary M2 application or the matching means. The full nonunit table retains every principal and zero branch. The fixed-divisor contribution and resulting parameters are

$$-f_0 - L_1 + T_1 - S_1, \qquad A_t = 2m + z + w,$$
$$A_1 = A_t - f_0 - f_1 - L_1, \quad M_1 = 1 - 6f_0 - f_1 - L_1 + T_1, \quad q_1 = f_1.$$

The remaining free index avoids the reduced $s$. Disjoint original prime ranges make its base condition $s' \mid n$ and fixed common supports independent of that index. Apply the full free identity, including the fifth-power correction, to every remaining position:

$$u = w - w_f - f_0 - f_1 = g_{\mathrm{good}} + v + c_{\mathrm{cor}} \ge 0,$$
$$L \ge v + 4T, \qquad \text{free coefficient exponent} = -u - L + v - 4c_{\mathrm{cor}} + T - S.$$

Retain the correction weights $q_p^{-1}$. After the fixed-unit enlargement, every actual nonunit prime still divides the current $h \ne 0$. Apply divisor Cauchy before dividing $h$, with squared coefficient energy. For fixed nonunit-prime data,

$$F_b \text{ is common to all unit-prime choices } a, \qquad a^4 \mid j.$$

The divisor bounds for fixed-divisor, nonunit-prime and correction choices contribute arbitrarily small power losses.

The classification is exhaustive:

$$w = w_f + f_0 + f_1 + g_{\mathrm{good}} + v + c_{\mathrm{cor}}.$$

It is completed for either sign of every intermediate length excess, including $u = 0$, before M2. The original $Q_0$ retains all annuli, coprimality conditions and its whole-product divisor condition.

*Step 2. The coefficient contribution and length restoration.* We combine the local coefficient bounds with Corollary 3.6.

Put $e_z = (9/2)(z_0 - z) \ge 0$. The allocation gives

$$D_f = \frac{9}{2}w - e_z - \frac{7}{2}(z_{C,\mathrm{old}} + w_f) + \frac{3}{2}f_0 - \frac{7}{2}f_1 - T_1,$$
$$D_{\mathrm{free}} = D_f + \frac{3}{2}u - 5v - T,$$
$$\mathcal{R}_{\mathrm{loc}} = -f_0 - L_1 + T_1 - S_1 - u - L + v - 4c_{\mathrm{cor}} + T - S.$$

Both actual and formal frequency changes equal

$$4(f_0 + g_{\mathrm{good}}) - L_1 - T_1 - L - T, \qquad K_0^* - K^* = K_0 - K.$$

Retain all displayed radicals, including canceled natural-zero factors. The original $C, D, J, g$, structural count and weighted Fourier measure are unchanged. Lemma 3.5 uses its $-d_D$ only in the original $J$ bracket and gives

$$\text{additional squared restoration loss} \le \frac{(D_{\mathrm{free}} - \mu_{\mathrm{com}})_+}{6}.$$

The original amplifier contribution is included once. Applying Corollary 3.6 at $x_{\mathrm{aux}} = 0$ gives

$$\mathcal{R}_{\mathrm{loc}} + \frac{(D_{\mathrm{free}} - \mu_{\mathrm{com}})_+}{6} \le \begin{cases} 0, & \text{actual scales,} \\ \theta_N/6, & \text{nominal annuli.} \end{cases}$$

Neither a positive intermediate defect nor $u > 0$ is required.

*Step 3. Verification of the length hypotheses for M2.* We check the available prime length and the resulting ordinary summation lengths before applying M2.

The partition of the prime factors and the exact coupled identity give

$$z_{\mathrm{init}} = z + w - z_C - f_0 - f_1 - u = z - z_{C,\mathrm{old}} \ge 0,$$

$$\begin{aligned} \frac{7}{2} z_{\mathrm{init}} - D_{\mathrm{free}} &= \frac{7}{2} z_0 - 6w + \frac{2}{9} e_z + 5w_f + 5f_1 + T_1 + 5v + T \\ &\ge \frac{7}{2}\frac{19}{750} - 6\frac{1}{100} = \frac{43}{1500} > 0. \end{aligned}$$

Choose $0 < \sigma < 10^{-6}$ and smaller edges so that

$$D_{\mathrm{temp}} \le D_{\mathrm{free}} - \mu_{\mathrm{com}} + \frac{3}{125000}, \qquad \frac{7}{2} z_{\mathrm{init}} - D_{\mathrm{temp}} > \frac{43}{1500} - \frac{3}{125000} > 0.$$

Genuine and final-divisor fixing of widths $b_g, b_d$ reduce the defect by at least $7b_g/2$ and $9b_d/2$, respectively; retain $-c_2 + g_2 - t_2 \le 0$. With $b = b_g + b_d \le z_{\mathrm{init}}$,

$$\frac{9}{2}(z_{\mathrm{init}} - b) - (D_{\mathrm{temp}} - 7b/2) \ge \frac{7}{2} z_{\mathrm{init}} - D_{\mathrm{temp}} > 0.$$

Thus a positive defect has sufficient remaining length of the initial prime product. For $z_{\mathrm{init}} > 0$, the auxiliary prime ranges for the amplifier avoid the original ranges; the amplifier error powers are assigned only to ordinary factors. At exhaustion, including $z = 0$, use

$$z_{\mathrm{init}} = 0 \Longrightarrow D_{\mathrm{free}} - \mu_{\mathrm{com}} \le -43/1500, \qquad g = J_+ + \sigma, \quad \ell = 0.$$

No amplifier prime factor is extracted and no average over auxiliary primes is taken. The strict margin absorbs $\sigma$ and the subsidiary edges. This also covers the cases in which all additional primes lie in the common support and $w = 0$.

For nonempty coefficients put $t = f_0 + u$, $a = f_1 + t \le w \le 1/100$. The completed parameters are

$$\begin{aligned} A_* &= A_t - f_1 - t - L_1 - L - 5c_{\mathrm{cor}}, \\ M_* &= 1 - 6t - f_1 - L_1 - L - 5c_{\mathrm{cor}} + T_1 + 5v + T, \\ q_* &= f_1 + v + c_{\mathrm{cor}}, \qquad m_* = M_* - q_*. \end{aligned}$$

All natural-zero supports remain in $q_*$. Fixed unit primes stay in the polynomially bounded extra coprimality radical. Ordinary removal is at most $2m + o(1)$, and $A_t \le 691/750$. Hence

$$\begin{aligned} M_* &\ge 1 - 2m - 6a - o(1) \ge 27/500 - o(1), \\ m_* &\ge 1 - 2m - 7a - o(1) \ge 11/250 - o(1), \\ A_* - m_* &= A_t - 1 + 5t + f_1 - T_1 - 4v + c_{\mathrm{cor}} - T \\ &\le A_t - 1 + 6a \le -7/375 + o(1). \end{aligned} \tag{3.14}$$

The same bounds hold at intermediate fixed stages, with M2 only after classification. Nonempty coefficient and frequency supports give

$$\begin{aligned} c_C &\le A_* + o(1), \qquad R + E_0 \le \min(c_C, d_D), \\ -o(1) \le K &\le K_0^* + o(1), \qquad K_0^* = 2A_* - c_C - d_D + R + E_0 - m_*. \end{aligned}$$

Consequently both the global and formal lengths are allowed:

$$\begin{aligned} d_D &\le 2A_* - m_* + o(1) \le A_* - 7/375 + o(1), \\ m_* - c_C + d_D - R - E_0 &\ge m_* - c_C \ge 7/375 - o(1) > 0. \end{aligned}$$

For the comparison product retain $L_{\mathrm{ref}} = (1/4 - L_1 - L - 5c_{\mathrm{cor}})_+$ and its volume branch. The origin and centered-exceptional ceilings satisfy

$$A_t + 4t \le 721/750 < 1, \qquad A_t + 3t \le 1427/1500 < 1.$$

These follow from the same full local formulas, with nonpositive terms discarded only in the upper-bound direction. Keep the original powerful-product count for the first zero-frequency term. Discard tails on the genuine coprime expression in the order of Section 2. Exact scale products, common support restrictions and both matching fixed-character means are retained through principal cancellation.

*Step 4. Integration and the uncentered norm.* We integrate the two estimates against the original Fourier measure and recover the norm on $\mathcal{U}$.

For each original $C, D, s$ and Fourier tuple, both the estimate with the stated ordering and the estimate with the ordinary factors interchanged fit

$$A_t - c_C + \widetilde{q} + B_C - s_0.$$

Integrate them in the same original weighted measure, including the paired arithmetic count once and retaining the nonpositive first paired term. The resulting integrated estimate gives (3.11).

On $\mathcal{U}$, reflection of the longer comparison factor gives

$$\text{total reflected length} \le \frac{5}{2} - 4(0.4) + \frac{9}{2}(0.01) = \frac{189}{200} < 1.$$

Apply the original M2 theorem to $P_{\mathrm{cmp}} Q_W$. In the positive norms with weight $\mathcal{W}(q_k/U)$,

$$\begin{aligned} \|PQ_W\|_{2,\mathcal{U}}^2 &\le 2\|P_{\mathrm{c}} Q_W\|_{2,\mathcal{U}}^2 + 2\|P_{\mathrm{cmp}} Q_W\|_{2,\mathcal{U}}^2 \\ &\le 2\|P_{\mathrm{c}} Q_W\|_{2,\mathrm{lift}}^2 + 2\|P_{\mathrm{cmp}} Q_W\|_{2,\mathcal{U}}^2 \\ &\ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C. \end{aligned}$$

This proves (3.12); the uncentered family still excludes parameters inducing principal characters.

For a requested $\epsilon > 0$, choose the positive step inside the strict margins, then subsidiary edges, and require

$$C_* \xi < \sigma/2, \qquad \eta_{\mathrm{mesh}} < \sigma/6, \qquad \eta_{\mathrm{mesh}} \le 10^{-7} \quad \text{for the selected estimate.}$$

Next fix the original finite prime-range and test-function list, request sufficiently small divisor, valuation and allocation losses, and fix the internal weighted derivative/Fourier orders, then the external height exponent and tail orders, and finally the threshold. The unchanged $J, g$ give auxiliary-length decrease at least $\sigma/2$; every ordinary sum is already in the established allowed M2 family. A nonpositive homogeneous exponent needs only the requested positive loss. Absorb the single total $\theta_N$ after the fixed large-$U$ threshold and take the finite maximum over allocation patterns. Constants remain independent of extracted ideals and integrated Fourier labels. For $U = Z^d$, retain the fixed positive compact $d$ range and check $\mathscr{L} = \ell/d \in [0.21, 0.22]$ separately. □

# 4 The enlarged amplified norm

In this section, we prove the additional amplified norm. The classification of Section 3 gives the homogeneous estimates; a direct bound for small formal summation lengths and the complementary Gauss calculation give the larger domain used in Section 7.

## 4.1 An amplified norm with variable lengths

In this subsection, we prove a parameter-dependent amplified norm from the classification of Section 3. We then estimate the unused prime amplitudes for the selection in Section 7.

This section retains the established cited results and the earlier bound $\beta_*(F) \le 7/8$, which permits $\kappa = 3/4$. The classification of Theorem 3.7 is independent of its numerical parameter range. We now give parameter-dependent sufficient inequalities; the comparison follows in Section 7. All powers below are relative to the actual scale of the character parameter $U$.

**Theorem 4.1** (Amplified norm with variable lengths)**.** *Retain the arithmetic and analytic conventions of Theorem 3.7 and Section 2. Fix $3/8 < m_- \le r < 1/2$ and $W \ge 0$ with*

$$\tau = 1 - 2r - 9W > 0, \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1. \tag{4.1}$$

*For disjoint products $Q_0, Q_W$ of complete prime-sum factors with length exponents $z, w$, put*

$$m \in [m_-, r], \qquad 0 \le z \le z_0(m) = 2(1-2m)/9, \qquad 0 \le w \le W,$$
$$P = S_{m,1} S_{m,2} Q_0, \qquad P_{\mathrm{c}} = P - P_{\mathrm{cmp}}.$$

*Use the canonical equal-product comparison and retain $Q_0$ with auxiliary removal length $x_{\mathrm{aux}} = 0$. Then, for every $\epsilon > 0$,*

$$\sum_k \mathcal{W}(q_k/U) |P_{\mathrm{c}}(k) Q_W(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C, \tag{4.2}$$

$$\sum_{k \in \mathcal{U}} \mathcal{W}(q_k/U) |P(k) Q_W(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C. \tag{4.3}$$

*The centered sum is unrestricted; $\mathcal{U}$ is the prescribed nonexceptional sixth-power-free nonunit family. The bounds hold for every required finite independent family of smooth weights and derivatives, with the original Fourier measure, zero and support conditions, and order of choices. Steps, meshes and internal losses are chosen inside the fixed strict margins.*

*Proof.* Repeat the exact classification of Theorem 3.7 with the original $C, D, s$ and weighted Fourier measure. Bound zero-length factors before Poisson, retain every coprimality and natural-zero factor, and classify all extra positions before M2. The local identities do not depend on the earlier parameter range.

All homogeneous identities are unchanged. In particular, with the notation of the proof using all local prime cases,

$$D_{\mathrm{free}} = 6w - e_z - \frac{7}{2}z_{C,\mathrm{old}} - 5w_f - 5f_1 - T_1 - 5v - T, \qquad \mu_{\mathrm{com}} = \min(c_C, d_D) \geq w_{f,\mathrm{actual}},$$

and the additional loss of restoring the lengths is $(D_{\mathrm{free}} - \mu_{\mathrm{com}})_+/6$. Corollary 3.6 makes the complete coefficient exponent plus this loss nonpositive on actual scales, with the same total $O(1/\log U)$ conversion discrepancy on nominal annuli. No second use of $-d_D$ or extra structural-family count is made.

The new numerical length conditions are as follows. Put

$$F_\tau = \frac{7}{9}\tau, \qquad z_{0,\min} = \frac{2}{9}(1-2r), \qquad A_{\max} = 2r + z_{0,\min} = \frac{14r+2}{9}.$$

The exact coupled remaining-prime margin is bounded below by

$$\begin{aligned}\frac{7}{2}z_{\mathrm{init}} - D_{\mathrm{free}} &= \frac{7}{2}z_0 - 6w + \frac{2}{9}e_z + 5w_f + 5f_1 + T_1 + 5v + T \\ &\geq \frac{7}{9}(1-2r) - 6W = F_\tau + W > 0.\end{aligned}$$

Put $D_{\mathrm{sharp}} = D_{\mathrm{free}} - \mu_{\mathrm{com}}$ and choose the amplifier and edge allowance so that $D_{\mathrm{temp}} - D_{\mathrm{sharp}} < F_\tau + W$. Genuine and final-divisor fixing reduce the defect by at least $7/2$ and $9/2$ times their respective widths. Hence, for $0 \leq b \leq z_{\mathrm{init}}$,

$$\frac{9}{2}(z_{\mathrm{init}} - b) - (D_{\mathrm{temp}} - 7b/2) \geq \frac{7}{2}z_{\mathrm{init}} - D_{\mathrm{temp}} > 0.$$

At exhaustion,

$$z_{\mathrm{init}} = 0 \Longrightarrow D_{\mathrm{sharp}} \leq -(F_\tau + W) < 0, \qquad g = J_+ + \sigma, \quad \ell = 0.$$

Use the case with no remaining prime factor, without averaging over auxiliary primes.

For $t = f_0 + u$ and $a = f_1 + t \leq w$, the unchanged auxiliary estimate formulas and ordinary-removal bound $L_1 + L + 5c_{\mathrm{cor}} \leq 2m + o(1)$ give

$$\begin{aligned}M_* &\geq 1 - 2r - 6W - o(1) = \tau + 3W - o(1), \\ m_* &\geq 1 - 2r - 7W - o(1) = \tau + 2W - o(1), \\ A_* - m_* &\leq A_{\max} + 7W - 1 + o(1) = -F_\tau + o(1).\end{aligned}$$

For nonempty supports,

$$c_C \leq A_* + o(1), \qquad R + E_0 \leq \min(c_C, d_D), \qquad -o(1) \leq K \leq 2A_* - d_D - m_* + o(1).$$

Together with $A_* - m_* \leq -F_\tau + o(1)$, this gives

$$d_D \leq A_* - F_\tau + o(1), \qquad m_* - c_C + d_D - R - E_0 \geq F_\tau - o(1) > 0.$$

The same conservative bounds hold at each exact comparison stage, and no M2 estimate precedes the completed classification.

The second-origin and centered-exceptional ceilings, including the volume reference branch, are respectively

$$A_{\max} + 5W = 1 - F_\tau - 2W < 1, \qquad A_{\max} + 4W = 1 - F_\tau - 3W < 1.$$

The original zero-frequency term in the first Poisson formula, powerful-product count, tails and exact equal-product mean cancellations are handled in the order of choices in Section 2. The

two current-side estimates fit the same Gauss-norm targets, are integrated in the same weighted Fourier measure and incur their paired structural count once. This proves (4.2).

For the restoration of the comparison product in the uncentered norm on $\mathcal{U}$, the reflected total length exponent is at most

$$\frac{5}{2} - 4m_- + \frac{9}{2}W = C_{\mathrm{add}} < 1.$$

The initial four ordinary scales have exponent at least $1/4$: $m_- > 3/8 > 1/4$ and $2m_- - 1/4 > 1/2$. Also $W < 1/45$, by the exact identity

$$1 - 45W = 4\tau + 2(1 - C_{\mathrm{add}}) + 8(r - m_-) > 0.$$

Thus the original M2 theorem and the triangle inequality give (4.3). No parameter inducing a principal character is added back. The fixed-unit coprimality restrictions are partial products of the original list and have norm at most $U^{W+o(1)}$; the retained polynomially uniform common-extra-radical result treats them in the same fixed data.

Choose a positive step, subsidiary edges, mesh for the prime-product lengths and requested losses inside $F_\tau$, $F_\tau + W$, the positive lower bounds for the summation lengths, the origin margins and the margin for restoring the comparison product. Then fix the original finite list of prime ranges and test functions, finite allocation maximum, weighted internal orders, then the external height exponent and tail orders, and finally the threshold. The strict decrease in the auxiliary summation length and a single total annular discrepancy are exactly those of the earlier proof. None depends on an extracted ideal or subsequent Fourier label. This establishes all uniformities asserted above. □

**Lemma 4.2** (Mean amplitude of the remaining prime factors)**.** *Let $Q_i$ be a finite family of original prime-sum factors on mutually disjoint prime-ideal ranges, with length exponents $\lambda_i \geq 0$ and amplitude exponents $g_i$ on $\mathscr{B}$. Retain the original lower bounds*

$$|Q_i(k)| \geq U^{\lambda_i g_i - o(1)} \qquad (k \in \mathscr{B},\ g_i > 0).$$

*Suppose*

$$\delta > 0, \qquad 0 \leq x \leq 1/2, \qquad 0 \leq g_i \leq \delta/2,$$
$$\sum_i \lambda_i = \mathscr{L} \geq L_0 > z_0(m), \qquad \sum_i \lambda_i g_i = \delta x \mathscr{L}.$$

*Remove an initial prime product of length at most $z_0(m)$. If*

$$a(\delta, x, m) = \delta \left\{ \frac{1}{2} - \frac{(1/2 - x)L_0}{L_0 - z_0(m)} \right\} = \delta \frac{xL_0 - z_0(m)/2}{L_0 - z_0(m)} > 0, \tag{4.4}$$

*the remaining positive-amplitude prime ranges have mean amplitude at least $a(\delta, x, m)$ and total length at least*

$$P(\delta, x, m) = 2xL_0 - z_0(m). \tag{4.5}$$

*For $0 < W < P(\delta, x, m)$ and $\eta_{\mathrm{mesh}}$ sufficiently small, with $\max_i \lambda_i \leq \eta_{\mathrm{mesh}}$, selection in decreasing amplitude gives a disjoint product $Q_W$ of length exponent $w_{\mathrm{sel}}$ with*

$$W - \eta_{\mathrm{mesh}} \leq w_{\mathrm{sel}} \leq W,$$
$$|Q_W(k)|^2 \geq U^{2a(\delta,x,m)(W - \eta_{\mathrm{mesh}}) - o(1)} \qquad (k \in \mathscr{B}).$$

*Proof.* Write $z \leq z_0(m)$ for the removed length and $D_{\mathrm{amp}} = \delta(1/2 - x)\mathscr{L}$ for the total deficit from maximal amplitude. The remaining mean $\overline{g}$ satisfies

$$\overline{g} \geq \frac{\delta}{2} - \frac{D_{\mathrm{amp}}}{\mathscr{L} - z} \geq \frac{\delta}{2} - \frac{\delta(1/2 - x)L_0}{L_0 - z_0(m)} = a(\delta, x, m).$$

Here

$$\frac{\partial}{\partial z}\left(\frac{\delta}{2}-\frac{D_{\mathrm{amp}}}{\mathscr{L}-z}\right)\le 0, \qquad \frac{d}{d\mathscr{L}}\frac{\mathscr{L}}{\mathscr{L}-z_0(m)}\le 0.$$

The original zero-amplitude length is at most $2D_{\mathrm{amp}}/\delta = (1-2x)\mathscr{L}$. Thus the unused positive length is at least

$$\mathscr{L}-z-(1-2x)\mathscr{L} = 2x\mathscr{L}-z \ge 2xL_0 - z_0(m).$$

Removing zero amplitudes increases a positive mean. Since $\lambda_i \ge 0$ and $\max_i \lambda_i \le \eta_{\mathrm{mesh}}$, selection of complete factors in decreasing amplitude gives length in $[W-\eta_{\mathrm{mesh}}, W]$ and

$$\log_U |Q_W(k)|^2 \ge 2a(\delta,x,m)(W-\eta_{\mathrm{mesh}}) - o(1).$$

Apply the original amplitude lower bounds before differentiating weights. □

The lower bound in (4.4) is sharp given only the upper amplitude bound, the total mean and the removed length of the initial prime product. At $\mathscr{L} = L_0$ take an initial prime product of width $z_0(m)$ and amplitude $\delta/2$, and give every remaining position the common amplitude $a(\delta,x,m)$. Its weighted total is exactly $\delta x L_0$, the decreasing-amplitude rule removes the factors of maximal amplitude first, and every unused subproduct has that same mean. This is a sharpness statement for the scalar amplitude data; it does not assert that such a formal amplitude class is nonempty arithmetically.

**Corollary 4.3** (A norm bound on amplitude classes and the resulting saving)**.** *Under Theorem 4.1, fix an amplitude class $\mathscr{B}$ with*

$$a = a(\delta,x,m) > 0, \qquad 0 < W < \min\{(1-2m)/9,\ (8m-3)/9,\ 2xL_0 - z_0(m)\}.$$

*Then*

$$s < 2aW \quad\Longrightarrow\quad \sum_{k\in\mathscr{B}} \mathcal{W}(q_k/U)|P(k)|^2 \ll U^{1-s+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C.$$

*After separately extracting the initial prime-product saving, the reflected fourth moment and Hölder inequality give any additional first-moment saving*

$$0 < \gamma(\delta,x,m) < \frac{a(\delta,x,m)W}{2}. \tag{4.6}$$

*Uniformity on parameter families requires positive lower bounds for the retained length and selection margins and for*

$$2aW - s, \qquad aW/2 - \gamma.$$

*Proof.* For any $0 < \gamma < aW/2$, choose $s$ and then the mesh and internal loss so that

$$4\gamma < s < 2a(W-\eta_{\mathrm{mesh}}) - o(1) < 2aW.$$

On the original selected class,

$$|Q_W(k)|^2 \ge U^{s-o(1)} \quad\Longrightarrow\quad \sum_{k\in\mathscr{B}} \mathcal{W}(q_k/U)|P(k)|^2 \ll U^{-s+\epsilon}\sum_{k\in\mathcal{U}} \mathcal{W}(q_k/U)|P(k)Q_W(k)|^2.$$

Equation (4.3) gives the claimed norm bound. Hölder of order four then gives

$$\gamma < s/4, \qquad s < 2aW \quad\Longrightarrow\quad 0 < \gamma < aW/2.$$

The amplitude bound precedes Sobolev and derivative expansion. Choose products from the original class data, then fix derivative weights and Fourier orders. Uniform positive gaps in the length, selection and gain inequalities make these choices uniform. □

These bounds use the original norm, independently of any older restricted-remainder error term.

For a rectangle $\delta \in [d_0, d_1]$, $x \in [x_0, 1/2]$, $m \in [m_-, r]$, a uniform lower mean is

$$a_{\mathrm{box}} = d_0 \left\{ \frac{1}{2} - \frac{(1/2 - x_0)L_0}{L_0 - z_0(m_-)} \right\} > 0,$$

and available length of positive-amplitude prime factors is at least $2x_0L_0 - z_0(m_-)$. The upper endpoint $d_1$ does not enter this sharper amplitude bound: the amplitude bound and its deficit have the same factor $\delta$. For $d_0 = 2/5$, $x_0 = 12/25$, $L_0 = 21/100$, $m_- = 2/5$, one obtains $a_{\mathrm{box}} = 3536/18625$. Bounding the deficit instead by the separate maxima $\delta \le 11/20$ and $\mathscr{L} \le 11/50$ of the box of Theorem 3.7 gives only

$$\frac{d_0}{2} - \frac{(11/20)(1/2 - x_0)(11/50)}{L_0 - z_0(m_-)} = \frac{1}{5} - \frac{1089}{74500} = \frac{13811}{74500} < a_{\mathrm{box}}.$$

The theorem also permits a requested prime-product length depending on the ordinary length $W(m)$, provided $1 - 2m - 9W(m)$ and the selected margin in the available positive-amplitude prime length have uniform positive lower bounds on the chosen compact band. At the level of these sufficient length conditions, the limiting pointwise gain is

$$\gamma_{\lim}(\delta, x, m) = \frac{a(\delta, x, m)}{2} \min \left\{ \frac{1 - 2m}{9}, \frac{8m - 3}{9}, \ 2xL_0 - z_0(m) \right\}.$$

This is a supremum for this particular norm-and-selection exponent calculation; its strict boundary is not asserted attainable and it is not an optimality theorem for other analytic estimates or zero-free methods.

**Lemma 4.4** (An allowed choice for each amplitude class)**.** *On a compact amplitude-class set $\mathcal{K}$, suppose*

$$\tau > 0, \qquad m_- > 3/8, \qquad m_- \le m_0 < (1 - \tau)/2,$$
$$a_0(\delta, x) = a(\delta, x, m_-) > 0, \qquad t \in C(\mathcal{K}; (0, \infty)).$$

*Set*

$$r(\delta, x) = \frac{a_0(1 - \tau)/18 + tm_0}{a_0/9 + t},$$
$$W(\delta, x) = \frac{1 - 2r(\delta, x) - \tau}{9}.$$

*Then*

$$m_- \le r < (1 - \tau)/2, \qquad W > 0,$$
$$1 - 2r - 9W = \tau, \qquad a_0 W/2 = t(r - m_0).$$

*If, uniformly on this set,*

$$W < \min\{(8m_- - 3)/9, \ 2xL_0 - z_0(m_-)\}$$

*with a positive gap, the norm on $m \in [m_-, r(\delta, x)]$ supplies every gain satisfying*

$$\gamma(\delta, x) > 0, \qquad \inf_{\mathcal{K}} \{a_0 W/2 - \gamma\} > 0,$$

*with uniform positive loss margins. The finite class selections may therefore use these class-dependent endpoints and lengths. For a fixed finite class set, all strict margins are made uniform by taking their finite minimum.*

*Proof.* Put $A = a_0/9 > 0$ and $M = (1-\tau)/2$. Then

$$r = \frac{AM + tm_0}{A+t}, \qquad r - m_0 = \frac{A(M-m_0)}{A+t} > 0, \qquad M - r = \frac{t(M-m_0)}{A+t} > 0.$$

Consequently

$$m_- \le m_0 < r < M, \qquad W = \frac{2(M-r)}{9} > 0, \qquad 1 - 2r - 9W = \tau, \qquad \frac{a_0 W}{2} = t(r - m_0).$$

For $m \ge m_-$, Lemma 4.2 gives $a(\delta, x, m) \ge a_0$. Theorem 4.1 and Corollary 4.3 apply with formal margin $7\tau/9$ and the stated remaining strict gaps. Continuity on $\mathcal{K}$ and the prescribed gain gap give

$$\inf_{\mathcal{K}} t > 0, \qquad \inf_{\mathcal{K}} W > 0, \qquad \inf_{\mathcal{K}} (a_0 W/2 - \gamma) > 0.$$

Together with the assumed upper-length gaps, these bounds permit common positive steps, meshes, orders and thresholds. All choices precede derivatives and Fourier integration; the separate contour conditions on $t$ still have to be checked. □

## 4.2 A direct estimate for small transformed length exponents

In this subsection, we bound the actual Gauss polynomial when its formal summation-length exponent is small. The complementary range is treated in Section 4.3.

*The length exponent and the frequency cutoff.* We distinguish the transformed summation-length exponent from the Fourier-frequency cutoff.

Write $A = A_*$, $m = m_*$, $q = q_*$ for the completed auxiliary parameters, $c = c_C$, $d = d_D$, and $u = R + E_0$. These identities are numerical exponent notation for the actual completed norm even when a formal ordinary moment family would not be allowed. Put

$$\alpha = A - c, \qquad t = d - u \ge 0, \qquad m_{\mathrm{f}} = m - c + t, \qquad \nu = -m_{\mathrm{f}}.$$

The exact cutoff identity is

$$K_0^* = 2A - c - d + u - m = 2\alpha + \nu.$$

Thus a negative formal exponent $m_{\mathrm{f}}$ does not imply an empty actual frequency annulus. For example, $\alpha = 0$ leaves a possible constant Gauss polynomial $\mathcal{G}(1,h) = 1$ and a nonempty actual frequency range of exponent $\nu > 0$. It is the true $h$ cutoff, not $m_{\mathrm{f}}$, that controls counting of the actual frequencies.

**Lemma 4.5** (A direct bound for the Gauss polynomial)**.** *Retain the common support, divisor parameters, coefficient families and Fourier coordinates of the local decomposition. The factor $\vartheta(n)$ below is the fixed coefficient factor in the cited Gauss polynomial, with its prescribed zeros; it is independent of the frequency $h$. Put*

$$\Phi(h) = U^{-\alpha/2} \sum_n B(n)\vartheta(n) 1_{s'|n} 1_{(n,\mathfrak{r}_*)=1} \mathcal{G}(n,h), \qquad s_0' = \log_U q_{s'}.$$

*Here $q_n \asymp U^\alpha$ on the two equal-product rectangles; $B(n)$ is the original finite convolution of smooth weights and fixed finite-group prime coefficients. Retain all support and zero-extension conditions. After extracting the fixed weight seminorms, let $0 \le \omega(h) \le 1$ be the normalized frequency cutoff. Then, for every retained frequency range of exponent $K$,*

$$\sum_{0<q_h \ll U^K} \omega(h)|\Phi(h)|^2 \ll U^{K_+ + 2\alpha - 2s_0' + \epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C, \qquad K_+ = \max(K, 0).$$

*The positive loss includes all dyadic-scale discrepancies; the Fourier orders are those of the original weighted estimate.*

*Proof.* For fixed factor count the coefficients satisfy an ideal-divisor bound, uniformly in the fixed arithmetic labels. With the original weight seminorms and height factors understood,

$$|\mathcal{G}(n,h)| \le q_n^{1/2}, \qquad \sum_{\substack{q_n \asymp U^{\alpha} \\ s' \mid n}} |B(n)| \ll U^{\alpha - s_0' + \epsilon}.$$

The two centered products have the same scale. Hence

$$\begin{aligned} |\Phi(h)| &\le U^{-\alpha/2} \sum_{\substack{q_n \asymp U^{\alpha} \\ s' \mid n}} |B(n)| q_n^{1/2} \\ &\ll U^{\alpha - s_0' + \epsilon} \mathfrak{S}_J(\mathcal{W})^{1/2} (1+H)^C. \end{aligned}$$

A nonempty product has $s_0' \le \alpha + O(1/\log U)$. The actual nonzero frequency orbits, including bounded unit frequencies, satisfy

$$\#\{h : 0 < q_h \ll U^K\} \ll U^{K_+ + \epsilon}.$$

Squaring the pointwise bound and summing proves the assertion. The zero-frequency term in the first Poisson formula is separate. Keep all natural zeros and support restrictions in the polynomial; only their absolute values are bounded by one. Integrate the polynomial Fourier-height factors in the original measure after fixing a sufficiently high order. No additional restricted moment family or family count is introduced. □

**Proposition 4.6** (Direct estimate for small transformed length exponents)**.** *Retain the actual coefficient-square bounds for all local prime cases and natural-zero conventions. Fix* $\rho_{\mathrm{term}} = 1/100$ *and suppose* $m_{\mathrm{f}} \le \rho_{\mathrm{term}}$*. Let the class parameters obey*

$$\frac{3}{8} < m_- \le r < \frac{1}{2}, \qquad \eta_{3/4} = \frac{7}{9}(1-2r) - 6W > 0, \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1,$$

*with actual extra length* $w \le W$*. Then, relative to the original one-sided Gauss target, this branch has exponent less than*

$$-\frac{7249}{18600} < -\frac{3}{8},$$

*before arbitrarily small requested losses.*

*Proof.* Put $B_C = (3c - 5d - R)_+/6$. The original one-sided exponent, before the local prime removals, is

$$\Lambda_{\mathrm{in}} = A_t - c + u + B_C - s_0, \qquad u = R + E_0, \tag{4.7}$$

where $A_t = 2m_{\mathrm{ord}} + z + w$ is the original product length, $m_{\mathrm{ord}}$ is its ordinary length before classification, and $s_0$ is the original inner divisor length. The $m$ in the completed Gauss parameters is instead the residual length $m_*$. This is the allowance of [Liu26b, equation (12.3)] at the original fixed $C, D$ data. For the completed polynomial,

$$s_0' = s_0 - f_0 - f_1, \qquad \Lambda_* = (A - c) + q + u + B_C - s_0'.$$

Consequently the exact change in the allowance is

$$\Lambda_* - \Lambda_{\mathrm{in}} = A - A_t + q + f_0 + f_1.$$

This notation does not assume a moment theorem at that target. The actual frequency cutoff gives

$$\begin{gathered} K \le 2\alpha + \nu + o(1), \qquad \nu \ge -\rho_{\mathrm{term}}, \quad \alpha \ge -o(1), \\ K_+ \le 2\alpha + \nu + \rho_{\mathrm{term}} + o(1). \end{gathered}$$

Subtracting $\Lambda_*$ from Lemma 4.5 gives at most

$$3\alpha + \nu - s_0' - q - u - B_C + \rho_{\mathrm{term}} + o(1).$$

The local coefficient contribution after classification, including the difference between this affine target and the given target, is

$$\mathcal{R}_{\mathrm{loc}} = -f_0 - L_1 + T_1 - S_1 - u_{\mathrm{free}} - L + v - 4c_{\mathrm{cor}} + T - S.$$

The length $u_{\mathrm{free}} = g_{\mathrm{good}} + v + c_{\mathrm{cor}}$ is the freely removed length denoted by $u$ in Sections 3 and 4.1. It is distinct from the common-support quantity $u = R + E_0$ used here. The actual squared coefficient exponent and the allowance change give

$$\begin{aligned}(-f_0 - f_1 + T_1 - S_1 + T - S) + (\Lambda_* - \Lambda_{\mathrm{in}}) \\ = A - A_t + q + T_1 - S_1 + T - S = \mathcal{R}_{\mathrm{loc}}.\end{aligned}$$

Here $s_0' = s_0 - f_0 - f_1$; the fixed-data coefficient energy is retained.

Since $R \le u$ and $d = u + t$,

$$u + B_C \ge u + \left(\frac{c}{2} - u - \frac{5t}{6}\right)_+ \ge \frac{c}{2} - \frac{5t}{6}.$$

Also $c \ge m + t - \rho_{\mathrm{term}}$. Dropping the favorable $-s_0'$ shows that the complete relative exponent is at most

$$\begin{aligned}&\mathcal{R}_{\mathrm{loc}} + 3(A - c) + (c - m - t) - q - u - B_C + \rho_{\mathrm{term}} \\ &\le \mathcal{R}_{\mathrm{loc}} + 3A - \frac{7}{2}m - q - \frac{8}{3}t + \frac{7}{2}\rho_{\mathrm{term}} \\ &\le \mathcal{R}_{\mathrm{loc}} + 3A - \frac{7}{2}m - q + \frac{7}{2}\rho_{\mathrm{term}}.\end{aligned}$$

Put $L_1 = f_1 + 4T_1 + e_1$ and $L = v + 4T + e$, with $e_1, e \ge 0$. The exact completed expressions give the identity

$$\begin{aligned}\mathcal{R}_{\mathrm{loc}} + 3A - \frac{7}{2}m - q &= 3A_t - \frac{7}{2} + 17f_0 + \frac{5}{2}f_1 + 17g_{\mathrm{good}} + \frac{5}{2}v + 18c_{\mathrm{cor}} \\ &\quad - \frac{9}{2}(T_1 + T) - S_1 - S - \frac{e_1 + e}{2} \\ &\le 3A_t - \frac{7}{2} + 18w.\end{aligned}$$

All variables here refer to the same completed allocation, before the direct bound is inserted. The fixed prime dividing $s$ choices satisfy a divisor bound; actual nonunit-prime indices were already treated by original-frequency divisor Cauchy, with their squared coefficient energy retained; correction indices retain their absolute coefficient mass of arbitrarily small power growth. The direct bound is uniform in those fixed data and introduces no new count.

The length constraint for the amplitude class gives

$$A_t \le \frac{14r + 2}{9} + W = 1 - 5W - \eta_{3/4}.$$

Moreover

$$1 - \frac{279}{7}W = \frac{36}{7}\eta_{3/4} + 2(1 - C_{\mathrm{add}}) + 8(r - m_-) > 0,$$

so $W < 7/279$. The total exponent is consequently less than

$$-\frac{1}{2} + 3W - 3\eta_{3/4} + \frac{7}{2}\rho_{\mathrm{term}} < -\frac{7249}{18600} < -\frac{3}{8},$$

before sufficiently small requested losses. It fits the original one-sided target with a fixed strict margin. This proof uses no earlier Gauss norm estimate on a family with a negative formal summation-length exponent. □

Here $\eta_{3/4}$ is $\eta_\kappa$ of Section 10 at $\kappa_m = 3/4$. No ordinary sum of positive formal length is reconstructed.

For $m_{\mathrm{f}} > \rho_{\mathrm{term}}$, we use the direct Gauss calculation of Proposition 4.8. Lemma 4.7 supplies a numerical comparison while retaining the actual outer ideal. These estimates are combined in Theorem 4.9.

### 4.3 The complementary Poisson estimate

In this subsection, we prove the complementary one-sided Gauss estimate. Together with Section 4.2, it gives the norm domain in Section 4.4.

This subsection concerns a Gauss norm already obtained by the actual first transform and the exact local Gauss-sum decompositions. It does not apply the original first-Poisson estimate to an invented ordinary family. In particular the original outer ideal $D$ is retained as a fixed label, not reconstructed as a new factor in the shorter completed coefficient sum.

**Lemma 4.7** (An auxiliary outer-length parameter)**.** *For the completed Gauss norm above, assume $c \le A$ and $R + E_0 \le \min(c, d)$. Set*

$$\widehat{d} = \min(d, A), \qquad \widehat{K}_0 = K_0 + d - \widehat{d}, \qquad \widehat{\Delta} = \widehat{K}_0 - K.$$

*Then*

$$\widehat{d} \le A, \qquad R + E_0 \le \min(c, \widehat{d}), \qquad K \le \widehat{K}_0 + \delta_1,$$
$$B_C(\widehat{d}) = B_C(d), \qquad \min(c, \widehat{d}) = \min(c, d),$$
$$\widehat{K}_0 = 2A - c - \widehat{d} + R + E_0 - m, \qquad \widehat{d} + \widehat{\Delta} = d + (K_0 - K).$$

*The local exceptional-character savings are denoted by $F_1^{\mathrm{loc}}, F_2^{\mathrm{loc}}$. The first is given in (4.11); the second is the saving of [Liu26b, Section 12.8.2], bounded below in the proof of Proposition 4.8. This numerical replacement preserves the $J, g, F_1^{\mathrm{loc}}$ formulas, origin and length-restoration exponents, prime lengths and target exponent. It changes neither the polynomial, the actual cutoff and fixed ideal $D$, nor its character factors, outer measure and previously discarded tails. With $\widehat{m}_{\mathrm{f}} = m - c + \widehat{d} - R - E_0$,*

$$\begin{aligned} &\widehat{m}_{\mathrm{f}} \le \rho_{\mathrm{term}}: && \textit{Proposition 4.6 applies with the same target,} \\ &\widehat{m}_{\mathrm{f}} > \rho_{\mathrm{term}}: && \widehat{m}_{\mathrm{f}} > 0, \quad \widehat{d} \le A. \end{aligned}$$

*No new coefficient family or ideal of norm $U^{\widehat{d}}$ is introduced.*

*Proof.* The case $d \le A$ is unchanged. If $d > A$, then

$$\widehat{d} = A, \qquad 3c - 5\widehat{d} - R \le -2A - R \le 0, \qquad R + E_0 \le c \le A = \widehat{d}.$$

Thus both positive parts defining $B_C$ vanish, and

$$B_C(\widehat{d}) = B_C(d) = 0, \qquad \min(c, \widehat{d}) = \min(c, d) = c, \qquad \widehat{d} + \widehat{K}_0 - K = d + K_0 - K.$$

These are precisely the outer-length invariants in the formulas below; $\widehat{K}_0 \ge K_0$ preserves the actual cutoff. The direct estimate uses the same cutoff bound, target and common-support inequalities. The polynomial, ideal $D$, kernels and discarded tails are unchanged. At nominal endpoints use the support-envelope construction below. □

*Dependence on the outer length in the cited Gauss calculation.* Set $D_0 = d + K_0 - K$. The formulas below are the corresponding expressions in [Liu26b, Sections 12.7.1–12.8.3 and Appendix

C.4], with its amplifier lengths $w, w_o$ written here as $a, a_o$. Its amplifier density $\ell$ is written $\ell_a$ to separate it from the global prime length.

$$J = D_0 - c - 2a + a_o, \qquad \Lambda_a = A - c + \widetilde{q} + a_o + B_C - s_0', \tag{4.8}$$

$$M' = M + J - g - g_2 + t_2, \qquad \Delta' = b_2 - p_2 + a + B_C + \ell_a, \tag{4.9}$$

$$\mathcal{D}_{\mathrm{ret}} \le a - D_0 - a_o + g + 2\theta_N, \qquad \mathcal{O}_{\mathrm{ret}} = A - M - D_0 - a_o - B_C + g - \ell_a, \tag{4.10}$$

$$F_1^{\mathrm{loc}} = \frac{c}{6} + \frac{5D_0}{6} + \frac{a}{3} + \frac{\widetilde{q}}{6} + a_o + B_C - \frac{5g}{6} + \ell_a. \tag{4.11}$$

Here $\mathcal{D}_{\mathrm{ret}}$ is the length excess before prime restoration and $\mathcal{O}_{\mathrm{ret}}$ is the excess for the zero-frequency term in the second Poisson formula before the local coefficient contribution. The first line is [Liu26b, equations (12.3)–(12.4)]; the transformed width and its normalization are [Liu26b, equation (12.6) and Appendix C.4]. The last two lines are the pre-restoration expression of [Liu26b, Section 12.8.1] and the origin and exceptional expressions of [Liu26b, Sections 12.7.3 and 12.8.2].

Under Lemma 4.7,

$$\widehat{d} + \widehat{K}_0 - K = D_0, \qquad B_C(\widehat{d}) = B_C(d), \qquad \min(c, \widehat{d}) = \mu_{\mathrm{com}}.$$

Thus (4.8)–(4.11) are invariant. The $F_2^{\mathrm{loc}}$ terms in [Liu26b, Section 12.8.2] use only the new second common support; the centered mean in [Liu26b, Section 12.8.3] uses the unchanged actual equal-product rectangles. The original outer $D$-sum and its weight stay fixed. Tails are estimated from actual kernels and cutoffs before this numerical replacement; no outer count is recomputed at $\widehat{d}$.

*Dyadic length exponents and support errors.* For nonempty nominal supports choose the single total $\theta_N = O(1/\log U)$ so that

$$c \le \overline{A} := A + \theta_N, \qquad \overline{A} \ge 0, \qquad \widehat{d}_N = \min(d, \overline{A}), \quad \widehat{K}_{0,N} = K_0 + d - \widehat{d}_N.$$

The common-support saving and the target exponent are preserved exactly:

| | $B_C(d) = B_C(\widehat{d}_N)$ | $\min(c,d) = \min(c,\widehat{d}_N)$ |
|---|---|---|
| $d > \overline{A}$ | $0$ | $c$ |
| $d \le \overline{A}$ | $B_C(d)$ | $\min(c,d)$. |

Also $\widehat{d}_N \le A + \theta_N$; relative to the idealized comparison, the numerical change is at most $\theta_N$ wherever the ideal lengths are nonnegative. Include it in the total support error. The envelope changes no ordinary test, normalization, smooth-weight argument or exact centered product.

If a nonempty endpoint has negative nominal $A$ or $\alpha = A - c$, its negative part is $O(\theta_N)$. In a nonempty regular branch,

$$m_{\mathrm{f}} > \rho_{\mathrm{term}}, \quad K \ge -o(1), \quad K \le 2\alpha - \rho_{\mathrm{term}} + O(\theta_N) \quad \Longrightarrow \quad \alpha \ge \rho_{\mathrm{term}}/2 - o(1).$$

Bounded coefficient-support endpoints therefore use the direct estimate, whose fixed saving absorbs the same support error. Actual centers and product identities are preserved.

**Proposition 4.8** (The complementary Gauss-norm bound)**.** *Retain the Gauss-sum, correlation-Poisson, weight, common-coefficient, zero-extension, exceptional-mean and M2 hypotheses of Section 2, with $\kappa_m = 3/4$. For a completed positive Gauss norm with fixed outer parameters $C, D, s$, put*

$$\alpha = A - c, \qquad \widetilde{q} = q + R + E_0, \qquad M = m + q, \qquad K_0 = 2A - c - d + R + E_0 - m.$$

*Here $c = \log_U q_C$ and $d = \log_U q_D$, or the auxiliary numerical value of Lemma 4.7. Assume*

$$R + E_0 \le \min(c, d), \qquad c \le A + o(1), \qquad K \le K_0 + \delta_1, \quad 0 \le \delta_1 < \xi,$$
$$m - c + d - R - E_0 > \rho_{\mathrm{term}} > 0.$$

*The actual global lengths are uniformly positive; all length exponents and fixed support norms lie in the original bounded polynomial ranges. Retain the exact coefficients and equal-product scales, the comparison factor after prime-power removal, and the full remaining-prime-length margin. For the original length $A_t$ and removed length $t_{\mathrm{loc}}$ of (4.13), also retain uniform strict margins*

$$A_t + 4t_{\mathrm{loc}} < 1, \qquad A_t + 3t_{\mathrm{loc}} < 1.$$

*Then the one-sided Poisson estimate has target*

$$\Lambda = \alpha + \widetilde{q} + B_C - s_0', \qquad B_C = (3c - 5d - R)_+/6.$$

*For every prescribed $\varepsilon_{\mathrm{loc}} > 0$, the additional squared length-restoration loss is at most*

$$\frac{(D_{\mathrm{free}} - \min(c,d))_+}{6} + \varepsilon_{\mathrm{loc}}.$$

*The steps, support errors and complete-factor excesses are chosen inside this loss and the fixed strict margins, in the order of Section 2. The origin and exceptional-term estimates and weighted orders are unchanged. M2 is applied only to auxiliary ordinary sums satisfying the moment length conditions; the inequality $d \le A$ is not required.*

*Proof.* Use the actual normalized Gauss polynomial of Lemma 4.5. The positive formal length selects this branch; an ordinary moment is used only on sums formed after the second Poisson transform. For each fixed tuple, the convolution coefficients and their multiplicative presentation are independent of $h$. Only the Gauss coefficient and an already bounded scalar depend on $h$; all natural zeros remain in the Gauss polynomial. Thus this is the same common polynomial for the fixed-divisor and unit-prime arguments.

*The frequency sum and the amplifier.* Keep the original bounded sum over $h$ through the sixth-power amplifier identity, averaging over allowed auxiliary primes and Jensen's inequality. With $R_{\mathrm{fix}}$ the product of the other fixed supports, the pool exclusions satisfy

$$\#\{\mathfrak{p} \text{ in the pool} : \mathfrak{p} \mid hs'CDR_{\mathrm{fix}}\} \ll \log U,$$

since all these supports have bounded polynomial norms. The pool lies beyond the remaining original prime ranges, after the step and mesh are fixed. Representation multiplicity uses the actual bounded frequency range and positive pool step.

Use $a$ for an extraction from the original amplifier and $a_o \le a$ for its new radical, reserving the letter $w$ for the original extra product. Set

$$a_0 = A - c - a, \quad J = d - c + (K_0 - K) - 2a + a_o.$$

When prime-sum factors remain, use [Liu26b, Section 12.7.2]:

| | $(a, a_o)$ | $\ell_a$ | $g$ |
|---|---|---|---|
| amplified main term | $(0,0)$ | $\sigma/3$ | $J_+ + 2\sigma$ |
| fixed amplifier error | original $(a, a_o)$ | 0 | $J_+ + \sigma$. |

If no prime-sum factor remains, use instead

$$a = a_o = \ell_a = 0, \qquad g = J_+ + \sigma. \tag{4.12}$$

There is then no average over auxiliary primes or density factor. The identity $g - J_+ = \sigma$ retains the strict decrease in the transformed summation length; the origin and exceptional allowances are no worse than for the prime-bearing choice. This is also the choice used when the available length of the initial prime product is exhausted, as in Theorem 4.9.

The corresponding normalizations, with no averaging in the no-prime case, give the allowance

$$\Lambda_a = a_0 + \widetilde{q} + a_o + a + B_C - s_0'.$$

These are identities for the original expression and amplifier errors; they require no second ordinary coefficient sum of length exponent $A$.

*Second Poisson and ordinary sums after the second Poisson transform.* Restore the complete sums before squaring. The full finite correlation identity gives the earlier second-Poisson formula, with prefactor $U^{K+g-\ell_a-a_0}$ and its inverse roots. Retain every fixed common coprimality restriction. Estimate high-frequency tails on this genuine expression before any artificial coprime extension. The common Fourier boxes are chosen before the new remaining labels, exactly as in the earlier proof with the original weighted seminorms. All smooth weights here are the actual transformed smooth weights, whose orders are uniform on bounded polynomial ranges.

For its genuine second common support use the earlier variables $b_2, p_2, g_2, t_2, V$, with their actual valuations and zero branches. The nonzero summation parameters after the second Poisson transform have

$$\begin{aligned} m' &= 2a_0 - K - g - g_2 - V, \\ q' &= \widetilde{q} + a_o + t_2 + V, \\ M' &= m' + q' = M + J - g - g_2 + t_2 \le M - \sigma. \end{aligned}$$

Substitute $K_0$ in the equality. For the decrease,

$$J - g \le -\sigma, \quad t_2 - g_2 \le 0 \quad \Longrightarrow \quad M' \le M - \sigma.$$

The nonzero frequency annulus is empty if its cutoff lies below the minimum exterior norm. At a bounded endpoint include the edge $0 \le \delta_2 < \xi$ before declaring its summation range:

$$\text{nonempty} \Longrightarrow m' + \delta_2 \ge 0, \qquad M'_{\text{act}} \le M' + \delta_2 < M - \sigma + \xi.$$

The exact second normalization is

$$\lambda_0 = K + g - \ell_a - a_0 + p_2 - s_0' + g_2 - t_2 - b_2, \qquad \Lambda_a - \lambda_0 = M' + \Delta',$$

where $\Delta' = b_2 - p_2 + a + B_C + \ell_a \ge a + \ell_a$. The complete last-divisor allocation keeps unrestricted quotients, both actual rectangle products and all surviving prime annuli. Its paired weight and count are the original weighted measures with arbitrarily small power growth. Original fixed $D$ is not summed again here; only the genuine new second common support is counted once. The restored ordinary sums have the radical in the original character formula, including canceled factors. Their exceptional split is made only after the final character has been formed. The established M2 theorem, rather than an induction hypothesis for the new amplified family, bounds each allowed auxiliary sum.

*Length restoration.* Before its coarse substitution, the earlier length defect is

$$a - d - (K_0 - K) - a_o + g + 2\theta_N.$$

Its exact common-length maximum is

$$-d - (K_0 - K) - a_o + J_+ = \max\{-d - (K_0 - K) - a_o, -c - 2a\} \le -\min(c, d) + \delta_1.$$

Only $c, d, a, a_o \ge 0$ and $K_0 - K \ge -\delta_1$ are used. The same extra global defect $D_{\text{free}}$ from the actual full local decomposition therefore gives the additional squared loss at most $(D_{\text{free}} - \min(c, d))_+/6 + \varepsilon_{\text{loc}}$.

For the prescribed $\varepsilon_{\text{loc}} > 0$, choose the positive steps and subordinate support and mesh losses so that the prescribed exponent losses, including the excess from one complete factor, sum

to less than $\varepsilon_{\mathrm{loc}}$. After fixing the finite prime-factor list, take $U$ sufficiently large to absorb $\theta_N = O(1/\log U)$ in this same loss.

The original amplifier estimate accounts for its own $a + \ell_a$ portion once. The unchanged coupled remaining-prime identity justifies every restoration of allowed lengths; prime exhaustion uses the cited estimate for arbitrary ordinary lengths with no remaining prime factor in the earlier order. No given Gauss estimate outside the established length range is assumed.

*The total length removed by the local identities.* In the origin and exceptional estimates below, put

$$\begin{aligned} u_{\mathrm{free}} &= g_{\mathrm{good}} + v + c_{\mathrm{cor}}, \\ t_{\mathrm{loc}} &= f_0 + u_{\mathrm{free}} = w - w_f - f_1, \qquad 0 \le t_{\mathrm{loc}} \le w \le W. \end{aligned} \tag{4.13}$$

This is the quantity denoted by $t$ in the decomposition into all local prime cases of Section 3.4. Here $u_{\mathrm{free}}$ is the removed prime width; it is distinct from the common-support length $u = R + E_0$ in Section 4.2. The displayed identity follows from the partition $w = w_f + f_0 + f_1 + u_{\mathrm{free}}$.

*The zero-frequency term in the second Poisson formula.* Its length excess in the cited Poisson formula before the local coefficient contribution is

$$A - M - d - (K_0 - K) - a_o - B_C + g - \ell_a + O(\theta_N).$$

The exact maximum and $g - J_+ \le 2\sigma$ give

$$\begin{aligned} \mathcal{O}_{\mathrm{ret}} &\le A - M - \min(c, d) + \delta_1 - B_C + 2\sigma - \ell_a + O(\theta_N) \\ &\le A - M + O(\sigma + \xi + \theta_N). \end{aligned}$$

Adding the original local coefficient contribution yields

$$\mathcal{O}_{\mathrm{total}} \le A_t - 1 + 4t_{\mathrm{loc}} + O(\sigma + \xi + \theta_N).$$

This uses the existing full origin formula.

*Sparse exceptional character parameters and their means.* The earlier local correlation table and natural zeros force the same prescribed exterior sixth-power-free part. All earlier supports of the fixed character factors and the additional restrictions still impose coprimality, so they cannot cancel a newly forced exponent two. The sparse count applies on the enlarged actual frequency range. Its first saving is

$$F_1^{\mathrm{loc}} = c/6 + 5(d + K_0 - K)/6 + a/3 + \widetilde{q}/6 + a_o + B_C - 5g/6 + \ell_a.$$

The original amplifier data satisfy

$$0 \le a_o \le a \le 7\sigma/3, \qquad \sigma \le g - J_+ \le 2\sigma, \qquad \ell_a \ge 0.$$

Put $D_0 = d + K_0 - K \ge d - \delta_1$. The common-support bound gives

$$B_C + \widetilde{q}/6 \ge (3c - 5D_0)_+/6 - 5\delta_1/6.$$

Hence

$$F_1^{\mathrm{loc}} \ge \begin{cases} c + 2a + \widetilde{q}/6 + a_o/6 + B_C + \ell_a - 5\sigma/3, & J \ge 0, \\ 2c/3 + a/3 - 5\sigma/3 - 5\delta_1/6, & J < 0. \end{cases}$$

In both cases,

$$F_1^{\mathrm{loc}} \ge \frac{2(c+a)}{3} - \frac{22\sigma}{9} - \frac{5\delta_1}{6} > \frac{2(c+a)}{3} - 3\sigma - \frac{5\xi}{6}.$$

The extraction bound is the one for the original amplifier errors; it is not an estimate for arbitrary independent $a$. The second local saving $F_2^{\mathrm{loc}} \ge 2b_2/3$ concerns only genuine new second common support, so it has no original $d$ condition.

Apply the retained fixed-character mean to the actual centered rectangles, with the comparison factor after prime-power removal $(1/4 - L_1 - L - 5c_{\mathrm{cor}})_+$ and its volume branch. The same character, fixed coprimality restriction and whole-product Mellin twist occur on both sides, so their leading products cancel exactly at the actual equal product. After the actual local coefficient contribution, the exceptional ceiling is $A_t - 1 + 3t_{\mathrm{loc}}$, as in the calculation including all local prime cases. This argument does not replace the comparison factor after prime-power removal by a formal $M/4$, and it does not require a larger-width moment theorem.

*Uniformity and integration.* Retain the original Fourier measures and paired normalizations for the diagonal, sparse exceptional, amplifier-error and auxiliary ordinary terms. The positive formal/global length, origin, exceptional and coupled-length margins permit the same ordered choices of step, edges, mesh and finite orders. Integrate fixed $C, D$ and the remaining parameters in the original outer measure, with its original $D$ count. This proves the one-sided Poisson bound throughout the stated range of $d$. □

### 4.4 The enlarged length range

In this subsection, we combine the two Gauss estimates to obtain the sufficient length inequalities used in Section 7.

The allowed width changes as follows:

| | | |
|---|---|---|
| Theorem 4.1 | $W < (1-2r)/9$ | $C_{\mathrm{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1.$ |
| Theorem 4.9 | $W < 7(1-2r)/54$ | |

The common condition $C_{\mathrm{add}} < 1$ restores the comparison product. The larger bound of Theorem 4.9 supplies the extra prime-product saving used in Lemma 7.1.

**Theorem 4.9** (Amplified norm under the coupled length constraints)**.** *Retain the arithmetic data, coefficient families, summation domains and exact equal-product comparison of Theorem 4.1, with $\kappa_m = 3/4$ and the initial prime product $Q_0$ retained. Suppose*

$$3/8 < m_- \le r < 1/2, \qquad W \ge 0,$$

*and*

$$\eta_{3/4} = \frac{7}{9}(1-2r) - 6W > 0, \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1. \tag{4.14}$$

*For complete, disjoint prime-sum factors with*

$$m \in [m_-, r], \qquad 0 \le z \le z_0(m) = 2(1-2m)/9, \qquad 0 \le w \le W,$$

*one has, for every $\epsilon > 0$,*

$$\sum_k \mathcal{W}(q_k/U)|P_{\mathrm{c}}(k)Q_W(k)|^2 \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C, \tag{4.15}$$

$$\sum_{k\in\mathcal{U}} \mathcal{W}(q_k/U)|P(k)Q_W(k)|^2 \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C. \tag{4.16}$$

*The two summation domains, finite independent smooth-weight and derivative families, and order of choices are those of Theorem 4.1 and Section 2.*

*Proof.* Use the exact classification of Theorem 3.7, with zero-length factors bounded before Poisson and every extra position classified before M2. Keep $C, D, s$, natural zeros and the original Fourier measure. The algebraic inequality of Corollary 3.6 gives the nonpositive normalization exponent and one $O(1/\log U)$ discrepancy. This use of the algebraic inequality requires no

positive formal length; the analytic M2 applications occur only after the split below. The length identities yield

$$\frac{7}{2}z_{\mathrm{init}} - D_{\mathrm{free}} \geq \frac{7}{9}(1-2r) - 6W = \eta_{3/4} > 0, \qquad z_{\mathrm{init}} = 0 \Longrightarrow D_{\mathrm{free}} \leq -\eta_{3/4}.$$

Thus the original restoration remains valid, including the no-prime choice (4.12). The global lengths satisfy

$$\begin{aligned} A_t &\leq (14r+2)/9 + W = 1 - 5W - \eta_{3/4}, \\ M_* &\geq (12W + 9\eta_{3/4})/7 - o(1) > 0, \\ m_* &\geq (5W + 9\eta_{3/4})/7 - o(1) > 0. \end{aligned}$$

Retain the actual outer ideal $D$. Lemma 4.7 replaces only its numerical exponent by $\widehat{d} = \min(d_D, A_*)$, using $A_* + \theta_N$ at nominal endpoints. For the actual Gauss polynomial set

$$m_{\mathrm{f}} = m_* - c_C + \widehat{d} - R - E_0.$$

The complete range is

| | |
|---|---|
| $m_{\mathrm{f}} \leq 1/100$ | Proposition 4.6: saving greater than 3/8 |
| $m_{\mathrm{f}} > 1/100$ | Proposition 4.8: ordinary restored sums. |

The first case includes negative exponents and bounded unit frequencies. In the second, $R + E_0 \leq \min(c_C, \widehat{d})$ and $\widehat{d} \leq A_* + O(\theta_N)$. Neither case changes the actual frequency cutoff, outer ideal, coefficients or discarded tails. Every M2 application is on a formed ordinary sum with its full radical. In Proposition 4.8, request

$$0 < \varepsilon_{\mathrm{loc}} < \min\{\epsilon/4, \eta_{3/4}/100, (1 - C_{\mathrm{add}})/100, 1/100\}.$$

This local loss is included in the final $\epsilon$ and remains inside all the strict length and origin margins.

By (4.13), $t_{\mathrm{loc}} \leq W$, so

$$A_t + 4t_{\mathrm{loc}} \leq 1 - W - \eta_{3/4} < 1, \qquad A_t + 3t_{\mathrm{loc}} \leq 1 - 2W - \eta_{3/4} < 1.$$

Retain the volume reference branch and both matching fixed-character means at the exact equal product. The zero-frequency term in the first Poisson formula and high-frequency tails use their original expressions. Both one-sided estimates therefore fit the same Gauss allowance; integrate them in the same measure and count the common support once. This proves (4.15).

Reflection of the comparison product on $\mathcal{U}$ has total length exponent $C_{\mathrm{add}} < 1$. The earlier M2 theorem applies, and, in the weighted $\ell^2$ norms with weight $\mathcal{W}(q_k/U)$,

$$\|PQ_W\|_{2,\mathcal{W},\mathcal{U}} \leq \|P_c Q_W\|_{2,\mathcal{W}} + \|P_{\mathrm{cmp}} Q_W\|_{2,\mathcal{W},\mathcal{U}}.$$

This gives (4.16) without principal character parameters. Moreover,

$$1 - \frac{279}{7}W = \frac{36}{7}\eta_{3/4} + 2(1 - C_{\mathrm{add}}) + 8(r - m_-) > 0.$$

Thus $W < 7/279$ and all support norms remain polynomially bounded. Choose steps, support edges, mesh and losses inside the positive length and origin margins, then fix the finite weight families, weighted orders, then external heights and tail orders, and finally the threshold. The original strict decrease in auxiliary length is retained, uniformly in the later extracted ideals and Fourier variables. □

**Corollary 4.10** (The amplitude-class saving in the enlarged length range)**.** *Under Theorem 4.9, let $a = a(\delta, x, m) > 0$ be given by Lemma 4.2. Suppose the remaining positive-amplitude prime length exceeds $W$ with a uniform margin. Then*

$$s < 2aW, \qquad 0 < \gamma < aW/2$$

*are allowed squared-norm and first-moment savings, respectively, with the original order of smooth weights and derivatives. For fixed $m_-$ and class-dependent $(r, W)$, the norm conditions are*

$$W < \frac{7}{54}(1-2r), \qquad W < \frac{8m_- - 3}{9}.$$

*For uniformity on parameter families, all retained length and selection margins and the gaps $2aW - s$, $aW/2 - \gamma$ have positive lower bounds.*

*Proof.* Apply Lemma 4.2 to the original factors, before derivative expansion. The conditions $\eta_{3/4} > 0$ and $C_{\mathrm{add}} < 1$ of (4.14) are equivalent to

$$W < \frac{7}{54}(1-2r), \qquad W < \frac{8m_- - 3}{9}.$$

Equation (4.16) and the proof of Corollary 4.3 give

$$0 < \gamma < aW/2 \quad \Longrightarrow \quad \exists s: \quad 4\gamma < s < 2aW.$$

Choose the mesh and internal losses within the strict gaps, and use

$$\gamma < s/4.$$

The stated uniform positive gaps permit common choices of the mesh and internal losses. The same finite independent weight families and order of choices apply. □

Compared with Corollary 4.3,

$$\frac{1-2r}{9} \longrightarrow \frac{7(1-2r)}{54};$$

the zero-free boundary requires the comparison of Section 7.

# 5 The approximate functional equation and its contours

In this section, we prove uniform estimates for the test functions on a compact positive strip. These estimates permit the contour displacements combined with the amplified norm in Section 7.

## 5.1 Smooth weights beyond the $1/10$ contour shift

In this subsection, we bound the AFE test functions on a compact positive strip. This supplies the full contour range used in Section 7.

The earlier restriction to shifts of size at most $1/10$ came from the strip chosen in the lemma on smooth weights for the approximate functional equation; the holomorphic region for the correction factors permits a larger shift of the original numerator contour.

**Lemma 5.1** (Smooth weights on compact substrips)**.** *Fix $0 < \sigma_0 < 1/2$. For a primitive nonprincipal finite-order Hecke character over the fixed totally complex field $F$, use the original approximate functional equation with*

$$V_w(v) = \frac{1}{2\pi i}\int_{(2)} e^{\xi^2}\left(\frac{\Gamma(w+\xi)}{\Gamma(w)}\right)^{r_F} v^{-\xi}\frac{d\xi}{\xi}, \qquad T_w = (1+|\Im w|)^{r_F}, \quad v > 0.$$

*Uniformly for $\sigma_0 \le \Re w \le 1-\sigma_0$ and fixed $j, N \ge 0$,*

$$|(v\partial_v)^j V_w(v)| \ll_{F,\sigma_0,j,N} (1+v/T_w)^{-N}.$$

*For $w = 1/2+\theta+it$, $0 \le \theta \le 1/2-\sigma_0$, put*

$$\widetilde{V}_w(v) = \left(\frac{\Gamma(1-w)}{\Gamma(w)}\right)^{r_F} v^{2\theta} V_{1-w}(v).$$

*After increasing the tail order by one, the same derivative bound holds for $\widetilde{V}_w$. Before contour displacement use*

$$H_w(v) = v^{2w-1}\left(\frac{\Gamma(1-w)}{\Gamma(w)}\right)^{r_F} V_{1-w}(v), \qquad H_w(v) = v^{2it}\widetilde{V}_w(v) \quad (w = 1/2+\theta+it),$$

*with the real logarithm of $v$. Here $\Re w = 1/2+\theta \ge 1/2$, so $\Re(1-w) = 1/2-\theta$. The functions $V_w, H_w$ are holomorphic on $0 < \Re w < 1$; the real-line factorization is used only after displacement. Their dyadic expansions converge normally on compact substrips for each fixed finite annulus of character parameters.*

*Proof.* Put $T = (1+|\Im w|)^{r_F}$. Euler differentiation inserts powers of $\xi$ in the defining integral. For $v \ge T$, shift to any fixed line $\Re\xi = B > 0$. Stirling's formula and the Gaussian $e^{\xi^2}$ give

$$|(v\partial_v)^j V_w(v)| \ll_{F,\sigma_0,j,B} (T/v)^B.$$

For $v \le T$, shift to $\Re\xi = -\sigma_0/2$. Then

$$\Re(w+\xi) \ge \sigma_0/2 > 0, \qquad (v\partial_v)^j V_w(v) = \mathbf{1}_{j=0} + O_{F,\sigma_0,j}((v/T)^{\sigma_0/2}).$$

Only $\xi = 0$ is crossed. These two estimates prove the required bound for every fixed $j, N$.

The completed nonprincipal function is entire, so its functional equation gives the same two exact AFE halves on the enlarged strip. For $w = 1/2+\theta+it$, Stirling gives

$$\left|\frac{\Gamma(1-w)}{\Gamma(w)}\right|^{r_F} \ll T^{-2\theta}, \qquad 0 \le 2\theta \le 1-2\sigma_0 < 1.$$

Hence the normalized dual factor is bounded using

$$(v/T)^{2\theta}(1+v/T)^{-N-1} \le (1+v/T)^{-N}.$$

Fixed Euler derivatives obey the same bounds. Before the contour move use the holomorphic factor $v^{2w-1}$; afterwards its unitary part is included in the Mellin twist. The exact conductor and gamma factors are retained, and the large-$v$ bounds give normal convergence of the dyadic tails on compact substrips. □

**Lemma 5.2** (Extended contours for the dyadic sums)**.** *Retain the nonprincipal family, correction/reflection identities and unamplified M2 with $\kappa_m = 3/4$. Fix $z$ with $\Re z = 17/50$, then assign the amplitude classes; keep $z$ fixed thereafter. For $3/10 \le \delta \le 3/5$, every dyadic sum in the approximate functional equation admits each fixed classwise shift*

$$0 \le \theta \le \delta/2, \qquad \Re w = 1/2-\theta, \qquad \theta_- = 0, \quad \theta_+ = \delta/2.$$

*Keep $s$ on its initial right-hand line and move the whole dyadic integrand before the local $s$-contour displacement, absolute values, conductor separation and conjugation of dual weights. Retain the prescribed positive separation margins. The M2 exponent changes by*

$$(\theta-\theta_0)\{d(1-m)-l_y\}, \qquad \theta_0 = 2\delta x/9,$$

*with the same central-line moment, initial prime product and prime-sum coefficients.*

*Proof.* Here $a = (1+\delta)/2 = a_{\mathrm{det}}$ is the detector parameter, and $s, w, z$ are the contour variables of [Liu26b, Lemma 13.10]; this $z$ is not the length exponent of $Q_0$, and this $s$ is not the inner divisor of Section 3. The entire swept range satisfies

$$0 \le \theta \le \delta/2, \qquad 1/5 \le \Re w = 1/2 - \theta \le 1/2, \qquad 1/2 \le \Re(1-w) \le 4/5.$$

Lemma 5.1 applies with $\sigma_0 = 1/6$. The connecting region of [Liu26b, Lemma 13.10] contains

$$\Re s \ge a + 16e, \quad \Re w \ge 1 - a - 6e, \quad \Re z \ge 17/50.$$

On the final local contours,

$$\Re w \ge 1 - a \quad \Longrightarrow \quad \Re(s+w) \ge 1 + 16e.$$

Thus the correction holomorphy and the original unit- and nonunit-prime bounds hold throughout. For each strict tuple,

$$q_p^{\theta} q_p^{-(j_p-1)\theta} = q_p^{-(j_p-2)\theta} \le 1 \qquad (j_p \ge 2,\ \theta \ge 0);$$

nonstrict errors retain their positive $e$ margins. If $S_{\mathrm{fin}}$ denotes the finite places of $S$, the fixed-place majorant is summable:

$$\sum_{\substack{\mathfrak{a} \text{ integral} \\ \mathrm{supp}(\mathfrak{a}) \subset S_{\mathrm{fin}}}} q_{\mathfrak{a}}^{-1/6} = \prod_{p \in S_{\mathrm{fin}}} (1 - q_p^{-1/6})^{-1} < \infty.$$

The independent AFE weights and derivatives therefore remain in the original M2 families with fixed polynomial height dependence.

Keep finite character and dyadic cutoffs. With $s$ first on its original right-hand line, perform the contour moves in this order:

| | |
|---|---|
| unsplit numerator | $\Re w:\ c_w \longrightarrow 1/2$ |
| exact two-part AFE | insert on $\Re w = 1/2$ |
| each holomorphic dyadic sum | $\Re w:\ 1/2 \longrightarrow 1/2 - \theta$ |
| local $s$-contour | $\Re s \ge a + 16e.$ |

Here $c_w$ is the initial right-hand line. The first move is valid because the numerator is entire, the reciprocal factor is independent of $w$, and the cited region contains the path. Mellin decay removes horizontal joins; locally uniform bounds permit removal of the cutoffs in the original weighted height order. Only then conjugate the dual half as a whole. Keep each actual conductor, root number and gamma ratio.

The amplitude partition depends only on the character parameter and the already fixed $z$; it is unchanged during the $w$- and local $s$-contour displacements. Thus its prime coefficients remain fixed, and the change of factor is

$$q_u^{\theta-\theta_0} D^{-(\theta-\theta_0)} Y^{-(\theta-\theta_0)}.$$

With $q_u \asymp U = Z^d$, $D = U^m$ and $Y = Z^{l_y}$, its exponent change is the one stated, with the bounded annular ratios in the original smooth weights. □

The wider contour range is combined with the amplified norm in Section 7. Section 8 verifies the resulting comparison inequalities and obtains the endpoint $B_*$.

## 6 Amplitude-class counts and short summation ranges

In this section, we sharpen the amplitude-dependent class count and separate its parameter from that of the numerator moment. We also record the lowest-amplitude estimate needed for the comparison in Section 8.

### 6.1 Amplitude-dependent bounds for small-amplitude classes

In this subsection, we sharpen the small-amplitude count by retaining its amplitude dependence. The improvement is used in the short-range comparisons of Section 8.

The earlier count is retained at the allowed value $\kappa = 3/4$. This section sharpens a scalar consequence of that count; it introduces no new arithmetic moment hypothesis.

**Lemma 6.1** (Sharp linear amplitude correction)**.** *For $0 \le \delta \le 5/6$ and $0 \le x \le 1/2$, put*

$$A = 5/6 - \delta, \quad D = 3 - 17x/9, \quad P = (2 - 8x/9)(1 - x), \quad J = AD + \delta P, \qquad R = 1 - \delta + \frac{A\delta P}{2J}.$$

*In the notation of Proposition 6.4 and Lemma 6.5,*

$$A = A_\delta, \qquad (D, P, J, R) = (D_{3/4}, P_{3/4}, J_{3/4}, R_{3/4}).$$

*Then $J > 0$ and*

$$R \le 1 - \frac{2\delta}{3} - \frac{22}{81}\delta x.$$

*The coefficient $22/81$ is the largest constant $\alpha$ for which $R \le 1 - 2\delta/3 - \alpha\delta x$ holds on this whole rectangle.*

*Proof.* Since $D, P > 0$,

$$J = \begin{cases} (5/6)D > 0, & \delta = 0, \\ (5/6 - \delta)D + \delta P > 0, & 0 < \delta < 5/6, \\ (5/6)P > 0, & \delta = 5/6. \end{cases}$$

Put $y = 1/2 - x$. The exact identity

$$2J - 3AP - \frac{44}{27}xJ = \frac{250x^2}{729} + \frac{\delta}{243}(352y^3 + 948y^2 + 984y + 199)$$

is nonnegative. Since

$$(1 - R) - \frac{2\delta}{3} = \frac{\delta}{6J}(2J - 3AP),$$

the claimed inequality follows, including $\delta = 0$ by continuity. For optimality take $x > 0$ and let $\delta \downarrow 0$:

$$\lim_{\delta \downarrow 0} \frac{(1 - R) - 2\delta/3}{\delta x} = \frac{2D - 3P}{6Dx} = \frac{22 - 12x}{81 - 51x}.$$

Letting $x \downarrow 0$ forces $\alpha \le 22/81$. This proves optimality only for the stated uniform linear count bound. □

**Proposition 6.2** (An improved comparison for small amplitudes)**.** *Use the earlier refined full exponent and the small-amplitude shift $\theta = g \min\{\ell/d, 1/(6\kappa)\}$, with $g = \delta x$ and $\kappa = 3/4$. Suppose the balanced geometry has $0 < \ell/h < 2/9$, $h < 1$ and the earlier monotonicity hypotheses hold. Set*

$$C_0 = -1/4 + b/6 + 5\ell/4, \qquad Q_{\mathrm{sm}} = \ell(1/2 + 2\ell/h) - 11h/54.$$

*For $0 \le \delta \le 1/8$, the exact endpoint obeys*

$$E_{\mathrm{small}}(\delta, x) \le C_0 + \delta\{(1 - h)/2 + Q_{\mathrm{sm}}x\} \le C_0 + \frac{1 - h + \max\{0, Q_{\mathrm{sm}}\}}{16}.$$

*In particular, if $Q_{\mathrm{sm}} \le 0$, a sufficient condition is*

$$C_0 + \frac{1 - h}{16} < -\frac{1}{250}.$$

*The same original $1/250$ margin is retained: if $C_0 + (1 - h + \max\{0, Q_{\mathrm{sm}}\})/16 < -1/250$ and $\zeta > 0$ is a buffer with $2\zeta < 1/1000$, then the extension to $d \le h + \zeta$ satisfies $E_{\mathrm{small}} + 2\zeta < -3/1000$.*

*Proof.* Lemma 6.1 gives

$$\begin{aligned} E_{\mathrm{small}} &= C_0 + \frac{\delta}{2} + \ell\delta x(1/2 + 2\ell/h) - \frac{3h}{4}(1-R) \\ &\le C_0 + \delta\left\{\frac{1-h}{2} + Q_{\mathrm{sm}}x\right\} \\ &\le C_0 + \frac{1-h+\max(0,Q_{\mathrm{sm}})}{16}. \end{aligned}$$

The last step uses $0 \le x \le 1/2$, $0 \le \delta \le 1/8$ and $h < 1$. The earlier monotonicity gives the extension to $d \le h + \zeta$. Under the stated $-1/250$ condition,

$$E_{\mathrm{small}} + 2\zeta < -\frac{1}{250} + \frac{1}{1000} = -\frac{3}{1000}.$$

The original count, shift and height argument are unchanged. □

## 6.2 The lowest-amplitude classes

In this subsection, we verify the sufficient lowest-amplitude margin required by the parameters of Section 8.2.

**Lemma 6.3** (Lowest-amplitude estimate with a 1/125 threshold)**.** *Retain the earlier direct, principal, detector, correction, moment, height and contour estimates, with their order of choices for nonprincipal characters. Suppose*

$$C_0 + (1/2 + 3\ell/2)/50 < -1/125, \qquad 2\zeta < 1/1000, \qquad 0 < \varepsilon_{\mathrm{cls}} \le 1/1000.$$

*Then the lowest-amplitude contribution has exponent*

$$\mathcal{E}_{\mathrm{low}}(d) < -7/1000 < -\varepsilon_{\mathrm{cls}} \qquad (d \le h + \zeta),$$

*before the prescribed subsidiary losses. It therefore satisfies the same character-sum remainder estimate with those losses chosen inside this strict margin.*

*Proof.* Write $\mathcal{E}_{\mathrm{low}}$ for the exponent in the cited lowest-amplitude estimate, and $\bar{g}$ for the weighted mean prime-sum amplitude in that class. Then

$$R = 1, \qquad \bar{g} \le 1/100, \qquad \mathcal{E}_{\mathrm{low}}(h) \le C_0 + (1/2 + 3\ell/2)/50 < -1/125.$$

Its extension to $d \le h + \zeta$ loses at most $2\zeta$. Therefore

$$\mathcal{E}_{\mathrm{low}}(d) < -\frac{1}{125} + 2\zeta < -\frac{1}{125} + \frac{1}{1000} = -\frac{7}{1000} < -\varepsilon_{\mathrm{cls}}.$$

This is the proof of [Liu26b, Proposition 14.6] with the weaker sufficient threshold and the same classification and contour identities. Choose the remaining nonprincipal losses inside the strict margin. The principal, small-norm and height/trace estimates retain their original hypotheses. □

## 6.3 Separate parameters for the class count and the numerator moment

In this subsection, we apply the class count with a separately established parameter. The numerator moment remains fixed, as required in Section 4.

Retain the enlarged original norm at $\kappa_{\mathrm{m}} = 3/4$, justified by 7/8. A separately proved global bound supplies the count parameter $\kappa_{\mathrm{c}}$ below; the numerator norm, selected initial prime product, length constraints, amplitude bounds and Gauss restoration remain fixed.

**Proposition 6.4** (The class-count parameter from an earlier zero-free bound)**.** *Suppose an already proved global bound is* $\beta_*(F) \le B_{\mathrm{seed}}$*. Fix*

$$\frac{13}{18} \le \kappa_{\mathrm{c}} \le \frac{3}{4}, \qquad 2B_{\mathrm{seed}} - 1 \le \kappa_{\mathrm{c}}.$$

*For the original detected amplitude classes with* $0 < \delta \le 5/6$*,* $0 \le x \le 1/2$*, and available length of the original prime product* $\mathscr{L} > 1/5$*, the earlier count is*

$$\#\mathscr{B} \ll U^{R_{\kappa_{\mathrm{c}}}(\delta,x)+\epsilon}(1+H)^{C},$$

*where*

$$\begin{aligned} c_{\mathrm{cnt}}(\kappa) &= \frac{1}{3\kappa}, \\ D_\kappa &= 3 - (1 + 2c_{\mathrm{cnt}}(\kappa))x, \\ P_\kappa &= (2 - 2c_{\mathrm{cnt}}(\kappa)x)(1-x), \\ J_\kappa &= (5/6 - \delta)D_\kappa + \delta P_\kappa, \\ R_\kappa &= 1 - \delta + \frac{(5/6-\delta)\delta P_\kappa}{2J_\kappa}. \end{aligned}$$

*The numerator moment parameter may simultaneously remain* $\kappa_{\mathrm{m}} = 3/4$*, justified by the earlier* $7/8$ *theorem.*

*Proof.* Use the three counts of [Liu26b, Section 13.4]. Their parameter is

$$\widetilde{c} = 2c_{\mathrm{cnt}}(\kappa_{\mathrm{c}}) = \frac{2}{3\kappa_{\mathrm{c}}}, \qquad \frac{13}{18} \le \kappa_{\mathrm{c}} \le 1 \quad \Longleftrightarrow \quad \frac{2}{3} \le \widetilde{c} \le \frac{12}{13}.$$

Here, as in [Liu26b, Section 13.4], $r$ is the length exponent of the inverse moment M1 and $m = t - r$. The crossing lengths satisfy

| estimate | requested prime length | crossing upper bound |
|---|---|---|
| M1 | $(1-r)/2$ | $5/26 < 1/5$ |
| M2 | $(1-2m)/(6\kappa_{\mathrm{c}})$ | $\widetilde{c}/12 + o(1) \le 1/13 + o(1) < 1/5.$ |

Thus $\mathscr{L} > 1/5$ permits the original decrements and mesh. Keep the original finite-group coefficients, detector weights and fourth-power amplitude bound in M2. The count-side hypothesis is

$$\beta_*(F) \le B_{\mathrm{seed}} \le (1+\kappa_{\mathrm{c}})/2,$$

while the numerator uses $\kappa_m = 3/4$. Balancing the three counts gives $R_{\kappa_{\mathrm{c}}}$ by [Liu26b, equation (13.16)]. The restriction $\kappa \ge 7497/10000$ of [Liu26b, Remark 14.1(1)] does not enter this count calculation. □

**Lemma 6.5** (Monotonicity of the count)**.** *For the preceding domain,* $J_\kappa > 0$ *and*

$$\frac{\partial R_\kappa}{\partial \kappa} = \frac{\delta x (5/6-\delta)^2 (1-x)^2}{3\kappa^2 J_\kappa^2} \ge 0. \tag{6.1}$$

*In particular* $R_{\kappa_{\mathrm{c}}} \le R_{3/4}$ *whenever* $\kappa_{\mathrm{c}} \le 3/4$*.*

*Proof.* Both $D_\kappa$ and $P_\kappa$ are positive on $13/18 \le \kappa \le 3/4$, $0 \le x \le 1/2$. At $\delta = 5/6$, $J_\kappa = \delta P_\kappa > 0$; elsewhere it is a positive sum. Differentiate the rational expression for $R$. Since

$$(\partial_\kappa P_\kappa)D_\kappa - P_\kappa(\partial_\kappa D_\kappa) = \frac{2x(1-x)^2}{3\kappa^2},$$

with $A_\delta = 5/6 - \delta$ one obtains

$$\partial_\kappa R = \frac{A_\delta \delta}{2J_\kappa^2}\{(\partial_\kappa P_\kappa)J_\kappa - P_\kappa \partial_\kappa J_\kappa\} = \frac{A_\delta^2 \delta x (1-x)^2}{3\kappa^2 J_\kappa^2} \ge 0.$$

This is (6.1). □

**Corollary 6.6** (Separate parameters in the numerator estimate)**.** *Use the count of Proposition 6.4 and retain*

$$\kappa_{\mathrm{m}} = 3/4, \qquad \theta_0 = \frac{\delta x}{6\kappa_{\mathrm{m}}}, \qquad z_0(m) = \frac{1-2m}{6\kappa_{\mathrm{m}}}, \qquad R = R_{\kappa_{\mathrm{c}}}.$$

*The numerator exponent and its amplified version are*

$$N = \frac{3R_{\kappa_{\mathrm{c}}}+1}{4} - \frac{\theta_0}{2}, \qquad N_{\mathrm{amp}} = N - \gamma.$$

*The character-sum comparison at its endpoint is*

$$E_{\mathrm{dec}}(\delta, x) = C_0 + \frac{\delta}{2} + \delta x \left\{ \ell + \frac{h/2 - l_y}{6\kappa_{\mathrm{m}}} \right\} - \frac{3h}{4}(1 - R_{\kappa_{\mathrm{c}}}),$$
$$C_0 = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4}, \qquad E_{\mathrm{dec}} \le E_{b,\ell,3/4}.$$

*The earlier small-amplitude comparisons using $R \le 1 - 2\delta/3$ remain valid with the smaller count $R_{\kappa_{\mathrm{c}}}$.*

*Proof.* Proposition 14.3 of [Liu26b] accepts the class-count exponent separately from the numerator moment parameter. Lemma 6.5 gives

$$R_{\kappa_{\mathrm{c}}} \le R_{3/4}, \qquad R_{\kappa_{\mathrm{c}}} \le (6\kappa_m - 2\delta x)\mathscr{L} \quad \text{whenever} \quad R_{3/4} \le (6\kappa_m - 2\delta x)\mathscr{L}.$$

Keep $\kappa_m = 3/4$, $\theta_0$ and $z_0$ fixed. In the earlier full exponent [Liu26b, equation (14.7)], substitution gives

$$N = \frac{3R_{\kappa_{\mathrm{c}}}+1}{4} - \frac{\theta_0}{2},$$
$$E_{\mathrm{dec}} - E_{b,\ell,3/4} = \frac{3h}{4}\left(R_{\kappa_{\mathrm{c}}} - R_{3/4}\right) \le 0.$$

Both AFE halves, all independent derivatives and the enlarged norm retain their original moment-side parameters and length constraints. The coarse count inequalities therefore remain valid. □

**Corollary 6.7** (Available prime-product length and the separated parameters)**.** *Under Proposition 6.4, retain $\kappa_{\mathrm{m}} = 3/4$. Suppose*

$$\frac{7}{20} \le \delta \le \frac{3}{5}, \qquad \frac{2}{5} \le x \le \frac{1}{2}, \qquad \mathscr{L} \ge \frac{21}{100}.$$

*Then the prime-length condition for the numerator estimate has the uniform margin*

$$(6\kappa_{\mathrm{m}} - 2\delta x)\mathscr{L} - R_{\kappa_{\mathrm{c}}}(\delta, x) \ge \frac{24683}{162000} > 0. \tag{6.2}$$

*The unamplified affine slope in the scale exponent $d$ of Section 2 satisfies*

$$S = \frac{3R_{\kappa_{\mathrm{c}}}+1}{4} + \frac{\theta_0}{2} - \frac{17}{50} \le \frac{52}{75} < 1. \tag{6.3}$$

*Proof.* By Lemmas 6.5 and 6.1,

$$R_{\kappa_{\mathrm{c}}} \le R_{3/4} \le 1 - \frac{2\delta}{3} - \frac{22}{81}\delta x \le 1.$$

Since $6\kappa_{\mathrm{m}} - 2\delta x > 0$, the left side of (6.2) is at least

$$\left(\frac{9}{2} - 2\delta x\right)\frac{21}{100} - 1 + \frac{2\delta}{3} + \frac{22}{81}\delta x = -\frac{11}{200} + \delta\left(\frac{2}{3} - \frac{601x}{4050}\right).$$

The last expression increases with $\delta$ and decreases with $x$ on the displayed rectangle. At $(\delta, x) = (7/20, 1/2)$ it equals $24683/162000$. Finally, $\theta_0 = 2\delta x/9 \le 1/15$, so

$$S \le 1 + \frac{1}{30} - \frac{17}{50} = \frac{52}{75}.$$

□

The sufficient restriction $\kappa \ge 7497/10000$ in the earlier numerator comparison is thus not imposed on $\kappa_c$. The numerator moment, initial prime product and contour parameter continue to use $\kappa_m = 3/4 > 7497/10000$. The class count uses its full proved interval, while (6.2)– (6.3) verify these two compatibility conditions directly on the class rectangle where the new estimate is applied. In Section 10, where $\kappa_m < 7497/10000$, the uses of this restriction are replaced as in Remark 10.4(b). The complementary amplitude-class and parameter-range comparisons are treated separately in Section 8.

# 7 Adding the moment and contour savings

In this section, we apply the norm of Section 4 on the contours of Section 5. A prime-product length chosen separately for each dyadic sum gives a common saving over all ordinary lengths; Section 8 uses that saving in the full comparison.

## 7.1 Combining the moment bound and contour displacement

In this subsection, we apply the selected moment on the displaced contour and minimize the resulting gain over ordinary lengths. Section 8.2 uses this common gain.

Theorem 4.9 is uniform over the independent AFE weights on every contour of Lemma 5.2. That lemma moves the whole integrand before absolute values. The selected-moment saving therefore remains available on the displaced contour with the same initial prime product and original coefficients.

**Lemma 7.1** (The combined exponent on a dyadic sum)**.** *Retain Theorem 4.9, the class-count exponent $R$, the initial prime product $Q_0$, and a disjoint product with selected first-moment saving $\gamma > 0$. Put*

$$g = \delta x, \quad \kappa_m = 3/4, \quad \theta_0 = \frac{g}{6\kappa_m} = \frac{2\delta x}{9}, \quad z_0(m) = \frac{1-2m}{6\kappa_m}, \quad \mathcal{K} = \frac{3R+1}{4}.$$

*For a dyadic sum of length $D = U^m$, $m < 1/2$, and every allowed shift $0 \le \theta \le \delta/2$, the full comparison exponent is*

$$\mathcal{E}_0(d) + (\theta - \theta_0)\{d(1-m) - l_y\} - d\gamma, \tag{7.1}$$

*where $\mathcal{E}_0(d)$ is the unamplified reference exponent. This holds for both halves of the approximate functional equation, with the actual conductors, fixed-place factors and simultaneous strict-error parameters retained. Neither the choice nor the length constraint of $Q_0$ changes.*

*Proof.* Lemmas 5.1–5.2 keep the two AFE weights and their independent derivatives in the norm family of Theorem 4.9. Fix the original amplitude class, initial prime product and extra product before differentiating these weights. Their lower bounds are independent of the contour displacement.

The selected fourth moment saves $2gz_0 + 4\gamma$ in its exponent. Hölder with the class count gives the first-moment exponent

$$\begin{aligned}\frac{3R}{4} + \frac{1 - 2gz_0 - 4\gamma}{4} - \theta m &= \mathcal{K} - \frac{gz_0}{2} - \gamma - \theta m \\ &= \mathcal{K} - \frac{\theta_0}{2} - (\theta - \theta_0)m - \gamma, \qquad \frac{gz_0}{2} = \frac{\theta_0(1-2m)}{2}.\end{aligned}$$

Arbitrarily small exponent losses and the fixed seminorm and height factors are understood. For $U = Z^d$, $D = U^m$ and $Y = Z^{l_y}$, the functional equation, AFE coefficients and outer Mellin factor therefore give the relative factor

$$U^{\theta-\theta_0}D^{-(\theta-\theta_0)}Y^{-(\theta-\theta_0)}U^{-\gamma} = Z^{(\theta-\theta_0)\{d(1-m)-l_y\}-d\gamma}.$$

This proves (7.1) for the first half.

Move the holomorphic dual expression before whole-dual conjugation. Lemma 5.1 gives the same $D^{-\theta}$ and the character of the original prime coefficients. Retain the actual conductor and one simultaneous strict-error tuple. Its finite prime set $\mathcal{P}_{\mathrm{str}}$ contributes

$$\prod_{\mathfrak{p}\in\mathcal{P}_{\mathrm{str}}}\left(q_{\mathfrak{p}}^{\theta}q_{\mathfrak{p}}^{-(j_{\mathfrak{p}}-1)\theta}\right) = \prod_{\mathfrak{p}\in\mathcal{P}_{\mathrm{str}}} q_{\mathfrak{p}}^{-(j_{\mathfrak{p}}-2)\theta} \le 1 \qquad (j_{\mathfrak{p}} \ge 2,\ \theta \ge 0).$$

Fixed-place factors have finite absolute mass on the enlarged strip. All contour displacements precede the local $s$-contour displacement, absolute values and conductor separation. The same positive norm controls the later derivatives; no amplitude lower bound is required of a differentiated weight. □

**Lemma 7.2** (An ideal-counting bound for short sums)**.** *Suppose $\kappa_c \le 3/4$, so that $R = R_{\kappa_c}$ is bounded by the count of Lemma 6.1. On $3/10 \le \delta \le 3/5$, $2/5 \le x \le 1/2$, put $N = (3R+1)/4 - \theta_0/2$ and $m_{\mathrm{cut}} = 3/40$. For $m \le m_{\mathrm{cut}}$, the ideal-count exponent of the reference estimate is at most*

$$R + (1/2-\theta_0)m_{\mathrm{cut}} \le N - \frac{23}{2700}.$$

*For $m \ge m_{\mathrm{cut}}$, the initial prime product required by the unamplified M2 estimate has length at most $z_0(m_{\mathrm{cut}}) = 17/90 < 21/100$.*

*Proof.* The ideal-count expression increases with $m$, since $\theta_0 < 1/2$. The sharp count bound gives

$$\begin{aligned} N - \{R + (1/2-\theta_0)m_{\mathrm{cut}}\} &= \frac{1-R}{4} - \frac{3}{80} - \frac{17\theta_0}{40} \\ &\ge \frac{\delta}{6} - \frac{3}{80} - \frac{43\delta x}{1620} \\ &\ge \frac{497\delta}{3240} - \frac{3}{80} \ge \frac{23}{2700}. \end{aligned}$$

Also

$$z_0(3/40) = \frac{2}{9}\left(1 - \frac{6}{40}\right) = \frac{17}{90}.$$

At exhaustion, the earlier selection rule and $z_0 \le \mathscr{L}$ give total original gain at least $g\mathscr{L} \ge gz_0$. □

**Lemma 7.3** (A combined bound with variable prime-product length)**.** *Retain the balanced lengths and assume*

$$\frac{57}{100} \le \lambda = \frac{l_y}{h} \le \frac{59}{100}, \qquad \frac{3}{10} \le \delta \le \frac{3}{5}, \qquad \frac{2}{5} \le x \le \frac{1}{2}.$$

*At every scale under consideration, assume*

$$\mathscr{L} = \ell/d \ge 21/100.$$

*Put $m_\theta = 1-\lambda$, $m^\sharp = 25/62$, and*

$$W_{\mathrm{cap}}(m) = \min\left\{\frac{7}{54}(1-2m), \frac{8m-3}{9}\right\}, \qquad 3/8 \le m \le 1/2.$$

*The remaining positive-amplitude factors have mean at least* $a = \delta(189x - 25)/139$. *Using* $\theta = 0$ *for* $m \le m_\theta$ *and* $\theta = \delta/2$ *for* $m \ge m_\theta$, *the limiting gain divided by* $d$ *at* $d = h$ *is*

$$\mathcal{F}(m) = \frac{aW_{\mathrm{cap}}(m)}{2} + \begin{cases} \theta_0(m_\theta - m), & m \le m_\theta, \\ (\delta/2 - \theta_0)(m - m_\theta), & m \ge m_\theta. \end{cases}$$

*Consequently,*

$$\begin{aligned} \Gamma(\delta, x) &= \min_{3/8 \le m \le 1/2} \mathcal{F}(m) = \min\{\Gamma_L, \Gamma_N\}, \\ \Gamma_L &= \theta_0(5/8 - \lambda), \qquad \Gamma_N = \frac{7a(2\lambda - 1)}{108}. \end{aligned} \tag{7.2}$$

*Here* $\Gamma_L = \mathcal{F}(3/8)$ *and* $\Gamma_N = \mathcal{F}(m_\theta)$ *are the two candidate minimum values. Moreover,*

$$\lambda \le 11/19 \quad \Longrightarrow \quad \Gamma_L \ge \Gamma_N, \qquad \Gamma = \Gamma_N.$$

*At* $\lambda = 11/19$,

$$\begin{gathered} b = 13\ell/2 - 1, \qquad h = 19\ell/4, \qquad l_y = 11\ell/4, \\ \Gamma_N = \frac{7\delta(189x - 25)}{95076}, \qquad \Gamma_N\big|_{x=1/2} = 7\delta/1368. \end{gathered}$$

*These are limiting lengths and gains; the norm theorem is applied only with the strict margins of Proposition 7.4.*

*Proof.* The largest initial prime product at $m = 3/8$ has width $1/18$. Lemma 4.2 with $L_0 = 21/100$ therefore gives

$$a \ge \delta \frac{(21/100)x - 1/36}{21/100 - 1/18} = \delta \frac{189x - 25}{139}.$$

The limiting width uses the branch for restoring the comparison product below $m^\sharp$ and the branch determined by the coupled length constraint above it. One has $3/8 < m^\sharp < m_\theta < 1/2$. The three successive slopes of $\mathcal{F}$ are

$$\frac{4a}{9} - \theta_0 > 0, \qquad -\frac{7a}{54} - \theta_0 < 0, \qquad \frac{\delta}{2} - \theta_0 - \frac{7a}{54} > 0.$$

The first sign follows from $239x - 50 > 0$; the last holds uniformly on the stated $x$ interval, and the middle is immediate. Thus the only possible minimum points are $m = 3/8$ and $m = m_\theta$. Their gains are respectively

$$\Gamma_L = \theta_0(m_\theta - 3/8), \qquad \Gamma_N = \frac{7a(1 - 2m_\theta)}{108}.$$

These are (7.2). When $\lambda \le 11/19$, put $y = 1/2 - x \in [0, 1/10]$ and $z = 11/19 - \lambda \ge 0$. The exact identity

$$\frac{\Gamma_L - \Gamma_N}{\delta} = \frac{175y}{47538} + z\left(\frac{19}{108} - \frac{997y}{2502}\right) \ge 0$$

gives the stronger assertion. The right endpoint $m = 1/2$ cannot be smaller, by the last positive slope. □

**Proposition 7.4** (Allowed lengths and contours with a uniform saving)**.** *Under Lemma 7.3, suppose*

$$E(\delta, x) - h\Gamma(\delta, x) \le -\Delta_{\mathrm{cmp}} < 0$$

*uniformly on the stated class rectangle. Retain strict margins for the complementary comparisons, geometric inequalities, short-sum estimate and earlier parameter ranges, and decrease* $\Delta_{\mathrm{cmp}}$ *if necessary to lie inside all these margins. Choose* $\zeta > 0$ *also so that*

$$\ell/(h + \zeta) \ge 21/100.$$

*Then allowed prime-product lengths and contours give*

$$\mathcal{E}(d) \le -\Delta_{\mathrm{cmp}} + C\varepsilon + 2\zeta + \epsilon_{\mathrm{rem}} < 0 \qquad (0 < d \le h + \zeta),$$

*after choosing $\varepsilon, \zeta$ and the remaining losses with $C\varepsilon + 2\zeta + \epsilon_{\mathrm{rem}} < \Delta_{\mathrm{cmp}}$. The coefficient families and order of analytic operations are unchanged.*

*Proof.* Write $E - h\Gamma \le -\Delta_{\mathrm{cmp}}$ for the fixed comparison margin. Choose $0 < \varepsilon < 1/1000$ and set

$$m_L = 3/8 + \varepsilon, \qquad W_\varepsilon(m) = (1-\varepsilon)W_{\mathrm{cap}}(m), \qquad m_L \le m \le 9/20.$$

Apply the norm with $m_- = r = m$. Since $W_{\mathrm{cap}}(m) \ge 8\varepsilon/9$,

$$\frac{7}{9}(1-2m) - 6W_\varepsilon(m) \ge 6\varepsilon W_{\mathrm{cap}}(m) \ge 16\varepsilon^2/3,$$
$$1 - \left(\frac{5}{2} - 4m + \frac{9}{2}W_\varepsilon(m)\right) \ge \frac{9}{2}\varepsilon W_{\mathrm{cap}}(m) \ge 4\varepsilon^2.$$

Every selected product fits, because

$$W_{\mathrm{cap}}(m) \le 7/279 < 2(2/5)(21/100) - 1/18.$$

Use

| | |
|---|---|
| $m < 3/40$ | ideal counting |
| $3/40 \le m < m_L$ | central unamplified M2 |
| $m_L \le m \le 9/20$ | selected norm on retained pieces |
| $m > 9/20$ | outer unamplified M2. |

The selection and lower-edge losses satisfy

$$\Delta_{\mathrm{sel}} \le a\varepsilon W_{\mathrm{cap}}/2, \qquad \Delta_{\mathrm{edge}} \le \theta_0\varepsilon.$$

At the upper endpoint the excess over $\Gamma_N$ is at least $7\delta/2160$: with $y = 1/2 - x \ge 0$, $z = \lambda - 57/100 \ge 0$,

$$\begin{aligned}(1/2 - 2x/9)(\lambda - 11/20) - \frac{7(189x-25)(2\lambda-1)}{108\cdot 139} - \frac{7}{2160} \\ = \frac{997}{2502}yz + \frac{4199}{250200}y + \frac{35}{108}z \ge 0.\end{aligned} \tag{7.3}$$

This holds on the full stated rectangle and $\lambda$ interval.

Lemma 7.1 applies on both sides of $m_\theta$. For each fixed selected dyadic sum, $S > 1/5$ and $\gamma < 1/250$ give

$$S - \theta_0(1-m) - \gamma > \frac{1}{5} - \frac{1}{15}\frac{5}{8} - \frac{1}{250} > \frac{3}{20},$$
$$S + (\delta/2 - \theta_0)(1-m) - \gamma > \frac{1}{5} - \frac{1}{250} > \frac{19}{100}.$$

Both slopes are below 2, so

$$d \le h + \zeta \quad \Longrightarrow \quad \mathcal{E}(d) \le \mathcal{E}(h) + 2\zeta.$$

At $m_\theta$ apply this separately to both contours; their slopes need not match. The unamplified ranges use their finite endpoint bounds, the dominating affine extension and normally convergent tails. For $d < d_{\min}$, use the small-parameter estimate recalled in Section 2.

All losses can be chosen to satisfy

$$C\varepsilon + 2\zeta + \epsilon_{\mathrm{rem}} < \Delta_{\mathrm{cmp}},$$

where $C$ depends on the fixed compact data and $\epsilon_{\mathrm{rem}}$ collects the remaining positive exponent losses, including the local loss of Proposition 4.8. Choose the step, mesh, finite allocation bound and weighted orders before later Fourier labels and external heights. Original prime lower bounds precede differentiation. Count the logarithmically many dyadic sums inside the prescribed loss and remove tails in the original order. This proves the full parameter-dependent estimate without continuity across a fixed lower band in $d$. □

# 8 The algebraic boundary and the successive count bounds

In this section, we first obtain a strict rational bound from the earlier 7/8 result. We then prove the algebraic endpoint by successive count improvements, and determine the optimum of the comparison system. Section 9 gives the resulting theorem over arbitrary fields.

## 8.1 An explicit rational bound

In this subsection, we obtain the rational bound that starts the successive count improvements in Section 8.2.

Here the ideal-count cutoff is $m_{\mathrm{seed}} = 1/10$. The final comparison of Section 7.1 uses the smaller cutoff $m_{\mathrm{cut}} = 3/40$ in order to obtain its uniform 23/2700 margin on the larger class rectangle. Both fit the available prime length:

$$z_0(1/10) = 8/45 < 21/100, \qquad z_0(3/40) = 17/90 < 21/100.$$

Appendix A verifies their separate complete-sum inequalities.

Before passing to the algebraic limit, we give an explicit interior example. It splits according to the direct-estimate range and uses the original theta detector. Keep $\kappa_c = \kappa_m = 3/4$ and set

$$B_{\mathrm{seed}} = \frac{437279}{500000} = 0.874558, \quad b = \frac{167}{2500}, \quad \ell = \frac{63163}{375000}, \quad h = \frac{196513}{250000}, \quad l_y = \frac{336887}{750000},$$
$$m_L = \frac{393}{1000}, \quad m_R = \frac{1097}{2500}, \quad W = \frac{317}{20000}, \quad \varepsilon_c = 10^{-4}, \quad \varepsilon_{\mathrm{amp}} = 10^{-6}, \quad \varepsilon_{\mathrm{cmp}} = 10^{-7}.$$

The actual norm length conditions are

$$\frac{7}{9}(1 - 2m_R) - 6W = \frac{1}{10000}, \qquad 1 - \left(\frac{5}{2} - 4m_L + \frac{9}{2}W\right) = \frac{27}{40000}.$$

Here

$$1 - 2m_R - 9W = -81/4000 < 0.$$

Theorem 4.9 therefore uses the direct and complementary Poisson estimates, with the actual outer $D$ retained, in place of that earlier sufficient length condition.

Use

$$\mathcal{R} = [7/20, 3/5] \times [9/20, 1/2], \qquad \theta_0 = 2\delta x/9, \qquad \theta_- = \varepsilon_c, \quad \theta_+ = \delta/2 - \varepsilon_c.$$

Put $t_- = \theta_0 - \varepsilon_c$ and $t_+ = \delta/2 - \varepsilon_c - \theta_0$. With $z_L = 2(1 - 2m_L)/9$, put

$$a = \delta\left\{\frac{1}{2} - \frac{(1/2 - x)(21/100)}{21/100 - z_L}\right\}, \qquad \gamma = aW/2 - \varepsilon_{\mathrm{amp}}.$$

Appendix A proves the strict rational inequalities for these parameters. They include available length of the additional prime product and restoration of the comparison product; the complete left, middle and right comparisons on $\mathcal{R}$; the earlier comparison on its three complementary rectangles; both endpoints of every near-$h$ affine comparison exponent; the short ideal-count comparison; and all remaining comparison geometry. The lowest-amplitude margin 1/100 and the small-norm margin 1/20 hold, and the small-amplitude calculation keeps the threshold 1/250.

**Theorem 8.1** (An explicit boundary from the enlarged norm)**.** *The finite-order Hecke family over the fixed field $F$ satisfies*

$$\beta_*(F) \le \frac{437279}{500000} = 0.874558.$$

*Proof.* Theorem 4.9 and Lemma 5.2 supply the analytic estimates. The completed Gauss sums are covered by

$$\begin{cases} m_{\mathrm{f}} \le 1/100: & \text{Proposition 4.6,} \\ m_{\mathrm{f}} > 1/100: & \text{Proposition 4.8.} \end{cases}$$

Take $\zeta = 10^{-8}$ as in Appendix A.2. For these parameters,

$$0 < \zeta \le 1/2000 \quad \Longrightarrow \quad \frac{\ell}{h+\zeta} \ge \frac{63163}{294957} > \frac{21}{100}.$$

Thus Lemma 4.2 applies with $L_0 = 21/100$. Use ideal counting for $m \le 1/10$, unamplified M2 on the two outer ranges, and the selected norm on $[m_L, m_R]$. Write $\mathcal{E}_{\mathrm{chosen}}$ for the full corrected exponent. The exact checks of Appendix A give

$$\mathcal{E}_{\mathrm{chosen}}(h) < -\varepsilon_{\mathrm{cmp}}, \qquad 0 < \partial_d \mathcal{E}_{\mathrm{chosen}} < 2, \qquad S > 1/5,$$
$$\frac{1}{5 \cdot 80} - \frac{1}{500} > \varepsilon_{\mathrm{cmp}}, \qquad 2\zeta < \varepsilon_{\mathrm{cmp}}/2.$$

Hence the two endpoints of the near-$h$ band and $d \le h - 1/80$ have strict negative exponents. Apply prime lower bounds before derivatives, retaining both actual-conductor AFE halves and strict tuples. Choose all subsidiary losses within these strict margins. The earlier direct estimate and nonvanishing principal correction then give

$$\beta_*(F) \le B(\ell) = B_{\mathrm{seed}}.$$

The 7/8 bound justifies both moment parameters used in this interior construction. □

## 8.2 The algebraic endpoint

In this subsection, we determine the algebraic limiting parameters and prove the resulting bound. The field descent is given in Section 9.

The amplified norm and contour displacement apply to the same ordinary dyadic sum. We insert their sum in the full comparison of Section 2.

**Definition 8.2** (The combined algebraic endpoint)**.** Let $r_*$ be the unique root in $(9/20, 23/50)$ of

$$218582r^3 - 376469r^2 - 11376r + 61560 = 0.$$

Put

$$\ell_* = \frac{3410r_*^2 - 7127r_* + 2520}{2286r_*^2 - 5011r_* + 1800}, \qquad \kappa_* = \frac{5}{6} - \frac{\ell_*}{2}, \qquad B_* = \frac{11}{12} - \frac{\ell_*}{4}.$$

To define $b_*$, use independent variables $r, k$ and

$$\begin{aligned} n_b &= -6858kr^2 + 15033kr - 5400k + 600r^2 - 1837r + 720, \\ n_e &= -702kr^2 + 34995kr - 40500k - 408r^2 - 3775r + 5400, \\ n_0 &= 4806kr^2 - 10887kr + 8100k - 696r^2 + 1403r - 1080. \end{aligned}$$

Set

$$b_* = -\frac{\ell_* \partial_r n_e(r_*, \kappa_*) + \partial_r n_0(r_*, \kappa_*)}{\partial_r n_b(r_*, \kappa_*)},$$

where the partial derivatives hold $k$ fixed. Numerically,

$$\begin{aligned} r_* &= 0.45057074672406073136\ldots, \\ \ell_* &= 0.16879343589254482845\ldots, \\ b_* &= 0.09179235229895088958\ldots, \\ \kappa_* &= 0.74893661538706091910\ldots, \\ B_* &= 0.87446830769353045955\ldots. \end{aligned}$$

In particular $\kappa_* = 2B_* - 1$ exactly.

The parameters also admit the explicit formulas

$$\begin{aligned} b_* &= \frac{\ell_*^2(11665-468r_*)+\ell_*(4528r_*-20554)+5113-4412r_*}{\ell_*(4572r_*-5011)+7127-6820r_*}, \\ \kappa_* &= \frac{5}{6}-\frac{\ell_*}{2}=2B_*-1. \end{aligned} \tag{8.1}$$

Here $\ell_*$ is the prime-product length parameter, $b_*$ balances $l_y$ and $h$ (see below), and $\kappa_*$ is the limiting class-count parameter. In Sections 2–9, $\kappa_m = 3/4$ is the parameter in the numerator moment estimate; $\kappa_c$ denotes the independent class-count parameter. The parameter $r_*$ is the critical value of the detector coordinate $\delta$ in the class $(\delta, x)$ defined in Section 1.5.

A closed form for the root is obtained by putting $\Delta = 149188674457$:

$$r_* = \frac{376469 + 2\sqrt{\Delta}\cos\Big(\frac{1}{3}\arccos\frac{17862636423556205}{\Delta^{3/2}} - \frac{2\pi}{3}\Big)}{655746}, \tag{8.2}$$

where arccos takes values in $[0, \pi]$. The triple-angle identity gives (8.2), and

$$\ell_*, b_*, \kappa_*, B_* \in \mathbb{Q}(r_*), \qquad [\mathbb{Q}(r_*) : \mathbb{Q}] = 3.$$

The derivative formula for $b_*$ gives (8.1).

**Lemma 8.3** (The comparison inequality and its strict margins)**.** *Put*

$$\mathcal{Q}_{\kappa,b,\ell}(\delta, x) = E_\kappa(\delta, x) - \frac{7}{108}a(\delta, x)(2l_y - h), \qquad \mathcal{R}_* = [7/20, 3/5] \times [2/5, 1/2].$$

*For the parameters of Definition 8.2,*

$$\begin{aligned} \mathcal{Q}_{\kappa_*,b_*,\ell_*} &\le 0 \quad \text{on } \mathcal{R}_*, \\ \mathcal{Q}_{\kappa_*,b_*,\ell_*}(\delta, x) = 0 \quad &\Longleftrightarrow \quad (\delta, x) = (r_*, 1/2). \end{aligned}$$

*Uniformly on $\mathcal{R}_*$ and*

$$\begin{gathered} 0.7489 \le \kappa \le 0.7495, \qquad 0.16876 \le \ell \le 0.16880, \\ 0.0917923522 \le b \le 0.0917923524, \end{gathered}$$

*one has*

$$\partial_\ell \mathcal{Q} > 3/4, \qquad 0 \le \partial_\kappa \mathcal{Q} < 1/100.$$

*The remaining geometric, complementary-class, ordinary-sum and short-sum comparisons have uniform strict margins on this family.*

*Proof.* Write $\mathcal{R}_* = [7/20, 3/5] \times [2/5, 1/2]$ and use the intervals $I_b, I_\ell, I_\kappa$ of Appendix A.3. At $x = 1/2$,

$$\mathcal{Q}_{k,b,\ell}(\delta, 1/2) = \frac{bn_b(\delta, k) + \ell n_e(\delta, k) + n_0(\delta, k)}{432(54k\delta - 75k - 6\delta + 10)}.$$

Polynomial division by (1.3) gives

$$n_b(r_*, \kappa_*) = 0, \qquad \ell_* n_e(r_*, \kappa_*) + n_0(r_*, \kappa_*) = 0.$$

Put $N_*(\delta) = b_* n_b(\delta, \kappa_*) + \ell_* n_e(\delta, \kappa_*) + n_0(\delta, \kappa_*)$. Definition 8.2 and the exact interval signs give

$$\begin{gathered} N_*(r_*) = N_*'(r_*) = 0, \qquad N_*(\delta) = A_{\mathcal{Q}}(\delta - r_*)^2, \\ A_{\mathcal{Q}} = (-6858\kappa_* + 600)b_* + (-702\kappa_* - 408)\ell_* \\ + 4806\kappa_* - 696 > 0, \\ 54k\delta - 75k - 6\delta + 10 < 0 \quad (\delta \in [7/20, 3/5],\ k \in I_\kappa). \end{gathered}$$

Hence $\mathcal{Q}_*(\delta, 1/2) \le 0$, with equality only at $\delta = r_*$. The exact root isolation is

$$0.450570746724060730 < r_* < 0.450570746724060733,$$
$$0.87446830769352 < B_* < 0.87446830769355.$$

Appendix A proves the following rational signs:

| quantity | parameter domain |
|---|---|
| $\partial_x \mathcal{Q} > 0$ | $\mathcal{R}_* \times [0.7489366153, 0.7489366155] \times I_b \times I_\ell$ |
| $\partial_\ell \mathcal{Q} > 3/4$ | $\mathcal{R}_* \times \{.7489\} \times I_b \times I_\ell$. |

Here each cleared numerator is affine in $(b, \ell)$; its four corners suffice. Exact Bernstein coefficients and rational subdivisions retain the full $\kappa$ coordinate. In particular,

$$x < 1/2 \quad \Longrightarrow \quad \mathcal{Q}_*(\delta, x) < \mathcal{Q}_*(\delta, 1/2) \le 0.$$

For the larger count interval,

$$\partial_\kappa \partial_\ell \mathcal{Q} = \frac{9}{8} \partial_\kappa R \ge 0 \quad \Longrightarrow \quad \partial_\ell \mathcal{Q} > 3/4,$$

and

$$0 \le \partial_\kappa \mathcal{Q} = \frac{h(5/6 - \delta)^2 \delta x (1 - x)^2}{4\kappa^2 J^2} < \frac{25600}{2686321} < \frac{1}{100},$$
$$h \le \frac{1 + 0.0917923524 + 3(0.16880)}{2} < \frac{4}{5}.$$

With $\mathcal{G} = [1/8, 5/6] \times [0, 1/2]$, the complementary rectangles are

$$\mathcal{C}_1 = [1/8, 5/6] \times [0, 2/5],$$
$$\mathcal{C}_2 = [1/8, 7/20] \times [2/5, 1/2],$$
$$\mathcal{C}_3 = [3/5, 5/6] \times [2/5, 1/2].$$

The same exact tests, at the adverse count endpoints, give

| | | |
|---|---|---|
| $\mathcal{G}$ | | $E < 1/500$ |
| $\bigcup_i \mathcal{C}_i$ | | $E < -10^{-5}$ |
| $\mathcal{R}_*$ | $S > 1/5,$ | $E + h\Delta_{\text{short}} < -10^{-5}$. |

The ratio $\ell/(h+\zeta) = 2\ell/(1+b+3\ell+2\zeta)$ increases with $\ell$ and decreases with $b, \zeta$. On $I_b \times I_\ell$,

$$0 < \zeta \le 1/2000 \quad \Longrightarrow \quad \frac{\ell}{h + \zeta} \ge \frac{843800000}{3997680881} > \frac{21}{100}.$$

This verifies the prime-length hypothesis of Lemma 7.3 and Proposition 7.4 throughout the parameter family. The ordinary-gain and prime-length conditions are

$$a > 0, \quad 4a/9 - \theta_0 > 0, \quad \delta/2 - \theta_0 - 7a/54 > 0, \quad \Gamma_L > \Gamma_N,$$
$$2x(21/100) - 1/18 > 7/279, \qquad 57/100 < l_y/h < 59/100, \quad l_y/h < 11/19.$$

For $m_{\text{cut}} = 3/40$,

$$N - \{R + (1/2 - \theta_0) m_{\text{cut}}\} \ge \delta(1/6 - 43x/1620) - 3/80 \ge 23/2700, \qquad z_0(m_{\text{cut}}) = 17/90 < 21/100.$$

The last lower bound holds already for $\delta \ge 3/10$. The remaining geometric, principal and small-amplitude conditions are the explicit inequalities of Appendix A, using the $1/125$ threshold of Lemma 6.3. Their analytic application is Lemma 7.1 and Proposition 7.4. □

**Theorem 8.4** (The algebraic zero-free boundary)**.** *Retain the established inputs of Section 2, the amplified norm of Theorem 4.9 and the contours of Lemma 5.2. The* 7/8 *bound and Theorem 8.1 give*

$$\beta_*(F) \le B_{\mathrm{seed}} = 0.874558 \quad \Longrightarrow \quad \boxed{\beta_*(F) \le B_* = 0.87446830769353045955\ldots.}$$

*Proof.* Put $\kappa_0 = 187279/250000 = 2B_{\mathrm{seed}} - 1$ and fix $b = b_*$. For $\kappa \in (\kappa_*, \kappa_0]$, set

$$\ell(\kappa) = \ell_* - \frac{\kappa - \kappa_*}{25}.$$

This segment lies in the family of Lemma 8.3, and

$$\frac{d}{d\kappa}\mathcal{Q}_{\kappa,b_*,\ell(\kappa)} = \partial_\kappa \mathcal{Q} - \frac{1}{25}\partial_\ell \mathcal{Q} < -\frac{1}{100}.$$

Together with $\mathcal{Q}_{\kappa_*,b_*,\ell_*} \le 0$, this gives, uniformly on the complete class rectangle,

$$\mathcal{Q}_{\kappa,b_*,\ell(\kappa)} < -\nu_\kappa, \qquad \nu_\kappa = \frac{\kappa - \kappa_*}{100} > 0.$$

Let $\epsilon_{\mathrm{rem}}$ collect the remaining positive exponent losses. Choose the losses, in the order of Proposition 7.4, so that

$$0 < \varepsilon < 1/1000, \qquad C\varepsilon + 2\zeta + \epsilon_{\mathrm{rem}} < \nu_\kappa.$$

Put $m_a = 3/8 + \varepsilon$, $m_b = 9/20$. Here $C$ depends only on the fixed compact data. The choice of lengths and contours in Proposition 7.4 uses

| ordinary length exponent | estimate |
|---|---|
| $m < 3/40$ | ideal counting |
| $3/40 \le m < m_a$ | unamplified M2 on the central contour |
| $m_a \le m \le m_b$ | selected moment and contour saving |
| $m > m_b$ | unamplified M2 on the outer contour. |

On the selected interval take $m_- = r = m$ and $W(m) = (1-\varepsilon)W_{\mathrm{cap}}(m)$. Then

$$\frac{7}{9}(1-2m) - 6W(m) \ge \frac{7}{90}\varepsilon > 0, \qquad 1 - \left(\frac{5}{2} - 4m + \frac{9}{2}W(m)\right) \ge 4\varepsilon^2 > 0.$$

The ideal gain changes by $O(\varepsilon)$; the outer range has the strict margin of Proposition 7.4. On each selected dyadic sum, the slopes of the full comparison exponent satisfy

$$S - \theta_0(1-m) - \gamma > \frac{1}{5} - \frac{1}{15}\frac{37}{40} - \frac{3}{10}\frac{7}{279}\frac{1}{2} > \frac{1}{8},$$
$$S + (\delta/2 - \theta_0)(1-m) - \gamma > \frac{1}{5} - \frac{3}{10}\frac{7}{279}\frac{1}{2} > \frac{1}{8}.$$

Both slopes are below 2. Hence the extension to $d \le h + \zeta$ loses at most $2\zeta$. Choose $\varepsilon$, then the mesh, smooth-weight, height and $\zeta$ losses inside $\nu_\kappa$. Apply the norm before differentiation and retain both actual-conductor AFE halves. Proposition 7.4 handles the complementary length ranges and tails; the earlier estimates apply for $d \le h - 1/80$. The lowest-amplitude, principal and direct comparisons retain their checked hypotheses.

The continuation argument now gives

$$\beta_*(F) \le B(\kappa) = B_* + \frac{\kappa - \kappa_*}{100}, \qquad 2B(\kappa) - 1 - \kappa_* = \frac{\kappa - \kappa_*}{50}.$$

Starting from $B_0 = B_{\mathrm{seed}}$ and setting $\kappa_n = 2B_n - 1$,

$$\beta_*(F) \le B_n \implies \beta_*(F) \le B_{n+1}, \qquad B_n - B_* = 50^{-n}(B_{\mathrm{seed}} - B_*) \longrightarrow 0.$$

Every finite step has strictly positive margins and its own threshold, while $\kappa_m = 3/4$ remains justified by 7/8. Taking the infimum over $n$ proves the theorem. □

**Theorem 8.5** (Optimality under the coupled norm length constraints)**.** *Fix the detector, class count, balanced-length hypotheses and direct theta estimate used above, and the norm of Theorem 4.9. On each amplitude class, allow dyadic-sum-dependent prime lengths and contours satisfying*

$$\kappa_m = 3/4, \qquad 0 \le W(m) \le \frac{7}{54}(1-2m), \qquad 0 \le \theta(m) \le \delta/2,$$
$$1/6 \le \ell \le 1/5, \qquad B(\ell) = 11/12 - \ell/4.$$

*Every allowed sequence of count improvements in this comparison system, starting at* $B_0 = B_{\mathrm{seed}}$*, with*

$$13/18 \le \kappa_n \le 3/4, \qquad \kappa_n \ge 2B_n - 1,$$

*satisfies*

$$B_n \ge B_* \quad (n \ge 0), \qquad \liminf_{n\to\infty} B_n \ge B_*.$$

*The sequence of Theorem 8.4 has* $B_n - B_* = 50^{-n}(B_{\mathrm{seed}} - B_*) \to 0$*. This optimality keeps the detector, moment estimates, moment parameter and allowed norm lengths fixed.*

*Proof.* Consider a first step below $B_*$. Its preceding bound $X \ge B_*$ gives

$$\kappa \ge 2X - 1 \ge \kappa_*, \qquad \ell > \ell_*.$$

The retained geometry gives

$$0 < b < \frac{3(1-3\ell)}{8} \le \frac{3}{16}, \qquad m_\theta = \frac{4\ell}{1+b+3\ell} > \frac{32}{81} > \frac{3}{8}, \qquad \frac{1}{2} - m_\theta = \frac{1+b-5\ell}{2(1+b+3\ell)} > 0.$$

At the critical class $(\delta, x) = (r_*, 1/2)$,

$$h(1-m_\theta) - l_y = 0, \qquad \gamma \le \frac{\delta W}{4}, \qquad h\gamma \le \frac{7\delta}{216}(2l_y - h).$$

Put $r = r_*$, $R = R_\kappa(r_*, 1/2)$ and $R_* = R_{\kappa_*}(r_*, 1/2)$. The buffered pointwise estimate and the optimistic amplified fourth-moment estimate have exponents

$$P_{\mathrm{pt}}(m; R) = R + \frac{rm}{2}, \qquad P_4(m; R, A) = \frac{3R+1}{4} - Ar(1-2m), \qquad A = A_0 := \frac{19}{216}. \tag{8.3}$$

Count monotonicity gives $R \ge R_*$. Therefore

$$P_{\mathrm{pt}}(m; R) - P_4(m; R, A_0) \ge \frac{35r_*}{108}(m - \mu_*), \qquad \mu_* := \frac{54(1-R_*) - 19r_*}{70r_*}.$$

The retained geometry also gives

$$m_\theta > g(\ell) := \frac{32\ell}{11+15\ell}, \qquad g'(\ell) = \frac{352}{(11+15\ell)^2} > 0. \tag{8.4}$$

The rational bounds (A.6) yield

$$\mu_* < \frac{399035}{10^6} < \frac{399159}{10^6} < g(\ell_*) < m_\theta.$$

Thus $P_{\mathrm{pt}}(m_\theta; R) > P_4(m_\theta; R, A_0)$. Absolute ideal counting gives $R + m_\theta/2 \ge P_{\mathrm{pt}}(m_\theta; R)$, and the unamplified moment omits the nonnegative selected saving. At $m_\theta$ the full contour term vanishes. Hence the buffered pointwise estimate, absolute ideal counting and the unamplified moment do not improve this optimistic amplified comparison at $m_\theta$, and $\mathcal{Q}_{\kappa,b,\ell}(r_*, 1/2) \le 0$ is

necessary. By the proof of Lemma 8.3, $n_b(r_*,\kappa_*)=0$ and $\ell_* n_e(r_*,\kappa_*)+n_0(r_*,\kappa_*)=0$. Hence the coefficient of $b$ vanishes at $\kappa=\kappa_*$, and

$$\mathcal{Q}_{\kappa_*,b,\ell}(r_*,1/2)=e_*(\ell-\ell_*),\qquad e_*=\frac{n_e(r_*,\kappa_*)}{432L_*},\quad L_*=54\kappa_* r_*-75\kappa_*-6r_*+10.$$

On $[4505,4506]\times[7489,7490]\cdot 10^{-4}\ni(r_*,\kappa_*)$, termwise bounds give $n_e<-15000$ and $-31<L_*<-30$. Thus

$$e_*>\frac{15000}{432\cdot 31}=\frac{15000}{13392}>1,\qquad e_*=1.13381952498716284524\ldots.$$

Since $\partial_\kappa\mathcal{Q}=(3h/4)\partial_\kappa R\geq 0$,

$$\kappa\geq\kappa_*,\quad \ell>\ell_*\quad\Longrightarrow\quad \mathcal{Q}_{\kappa,b,\ell}(r_*,1/2)>0.$$

This contradicts the necessary condition $\mathcal{Q}_{\kappa,b,\ell}(r_*,1/2)\leq 0$. Hence no step goes below $B_*$, and every finite iterate and its lower limit are at least $B_*$. Theorem 8.4 attains this infimum. □

# 9 Arbitrary number fields and the boundary $B_*$

In this section, we verify that the new estimates apply over every fixed field containing $\mu_{12}$. We then descend from $K(\zeta_{12})$ to $K$ and prove the boundary $B_*$ for every number field. The estimates are checked in Section 9.1, and the descent is given in Section 9.2.

## 9.1 The estimates over fields containing the twelfth roots of unity

In this subsection, we check the norm and contour estimates of Sections 3–7 over a fixed field containing $\mu_{12}$.

The general-field arithmetic is supplied by [Liu26b, Part III]. We verify that every new estimate above uses only that arithmetic and fixed-field estimates.

**Proposition 9.1** (The combined comparison over number fields)**.** *For every fixed $F\supset\mu_{12}$, the norm, smooth-weight, contour and combined-comparison estimates used in Theorem 8.4 hold with the same numerical exponents and length inequalities. Consequently,*

$$\beta_*(F)\leq B_*=0.87446830769353045955\ldots,$$

*with $B_*$ as in Definition 8.2. Constants, derivative orders and thresholds may depend on $F$ and its fixed arithmetic data.*

*Proof.* Fix $F\supset\mu_{12}$ and the arithmetic data of [Liu26b, Theorem 18.2]. Enlarge $S$ to contain the places above 6, the different and the fixed exceptional local data. All constants below may depend on these fixed choices.

*Local arithmetic and summation lengths.* For $\mathfrak{p}\notin S$,

$$\mu_{12}\hookrightarrow(\mathcal{O}_F/\mathfrak{p})^\times,\qquad 12\mid(q_\mathfrak{p}-1),\qquad \left(\frac{x}{\mathfrak{p}}\right)_n\equiv x^{(q_\mathfrak{p}-1)/n}\pmod{\mathfrak{p}}\quad(n=2,3,6),$$

for $x\notin\mathfrak{p}$, with value zero at $x\in\mathfrak{p}$. Thus the full local Gauss identities and natural zeros used in Sections 3–4 are unchanged. Put $F_0=\mathbb{Q}(\omega)$, where $\omega=\zeta_3=e^{2\pi i/3}$. Write $\Lambda_F$ for the unitary Eisenstein coefficient character denoted by $\Lambda$ in [Liu26b, Lemma 4.2]. It is given by

$$\Lambda_{F_0}(\mathfrak{a})=\frac{\jmath(\pi_+(\mathfrak{a}))}{|\jmath(\pi_+(\mathfrak{a}))|},\qquad \Lambda_F=\Lambda_{F_0}\circ N_{F/F_0}.$$

Here $\mathfrak{a}$ is an integral $F_0$-ideal prime to 3, $\pi_+(\mathfrak{a}) \equiv 1 \pmod 3$ is its primary generator, and $\jmath : F_0 \hookrightarrow \mathbb{C}$ is the fixed embedding used for the cubic residue character and Jacobi sums. The map $N_{F/F_0}$ is the ideal norm. The character $\Lambda_F$ need not have finite order.

The theta character is

$$\chi_{\theta,\eta} = \overline{\Lambda_F}\,\overline{G_0}\eta,$$

with $G_0$ the finite-order coefficient character of the cited lemma. These constructions require no normality of $F/F_0$. The fixed unit and ray-class sets contribute constants. For the ideal-divisor function $\tau_F$ and fixed $A$,

$$\tau_F(\mathfrak{a})^A \ll_{F,A,\epsilon} (N\mathfrak{a})^{\epsilon}.$$

Apply the proofs of Theorems 4.1 and 4.9 using these identities and the transferred Poisson and M2 estimates. Keep the actual frequency domain in Proposition 4.6, and the actual outer ideal and Fourier measure in Proposition 4.8. Their length conditions remain

$$\eta_{3/4} = \frac{7}{9}(1-2r) - 6W > 0, \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1.$$

Write $\mathcal{N}_{F,c}$ and $\mathcal{N}_{F,\mathcal{U}}$ for the left sides of (4.15)–(4.16). Then

$$\max\{\mathcal{N}_{F,c}, \mathcal{N}_{F,\mathcal{U}}\} \ll_{F,\epsilon} U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C,$$

with the same nonexceptional character parameters, centered means and exact equal-product comparison. No additional principal character parameters occur.

*Smooth weights and contours.* The field $F$ is totally complex, and its finite-order characters have trivial infinity type. With $r_F = [F:\mathbb{Q}]/2$, the gamma ratios in Lemma 5.1 are

$$\left(\frac{\Gamma(w+\xi)}{\Gamma(w)}\right)^{r_F}, \qquad \left(\frac{\Gamma(1-w)}{\Gamma(w)}\right)^{r_F}.$$

For fixed $j, N$ and $1/6 \le \Re w \le 5/6$, that lemma gives

$$|(v\partial_v)^j V_w(v)| \ll_{F,j,N} \left(1 + \frac{v}{(1+|\Im w|)^{r_F}}\right)^{-N}.$$

The original and reflected contours satisfy

$$\frac{1}{5} \le \Re w \le \frac{1}{2}, \qquad \frac{1}{2} \le \Re(1-w) \le \frac{4}{5}, \qquad q_{\mathfrak{p}}^{-(j_{\mathfrak{p}}-2)\theta} \le 1 \quad (j_{\mathfrak{p}} \ge 2,\ \theta \ge 0).$$

Retain the actual conductors and fixed-$S$ factors. The M2 estimate for arbitrary bounded length exponents of [Liu26b, Theorem 18.2] handles the degree-dependent long AFE sums; the short and selected ranges keep their original numerical cutoffs. Thus the dependence on $r_F$ changes constants and height powers, rather than powers of $U$.

*Selection and successive bounds.* Prime ideals in the fixed ray classes supply the original ranges. After fixing the field data, the same selection and mesh choices give

$$\theta_0 = \frac{2\delta x}{9}, \qquad \mathcal{E}_{\theta,\gamma} - \mathcal{E}_{\theta_0,0} = (\theta-\theta_0)\{d(1-m) - l_y\} - d\gamma.$$

The scalar estimates of Section 8 therefore remain unchanged. The transferred 7/8 theorem supplies $\kappa_m = 3/4$; Theorem 8.1, applied over $F$, gives $B_0 = B_{\mathrm{seed}}$. The finite-step implication is

$$\beta_*(F) \le B_n, \quad \kappa_n = 2B_n - 1 \quad \Longrightarrow \quad \beta_*(F) \le B_{n+1} = B_* + \frac{\kappa_n - \kappa_*}{100}.$$

Since $\kappa_* = 2B_* - 1$,

$$B_n - B_* = \frac{B_{\mathrm{seed}} - B_*}{50^n} \longrightarrow 0, \qquad \beta_*(F) \le \inf_n B_n = B_*.$$

Each finite step uses its own strictly positive margins and threshold; no threshold uniform in $n$ or $F$ is required. □

## 9.2 Descent to an arbitrary number field

In this subsection, we descend through $K(\zeta_{12})$ using abelian factorization. Section 10 applies the same descent to $B_\natural$.

**Theorem 9.2** (The zero-free half-plane over every number field)**.** *For every number field $K$ and every finite-order Hecke character $\chi$ of $K$, the full Hecke $L$-function has no zeros in*

$$\boxed{\Re s > B_* = 0.87446830769353045955\ldots.}$$

*The principal pole at $s = 1$ is permitted. In particular, for every number field $K$,*

$$\zeta_K(s) \neq 0 \qquad (\Re s > B_*,\ s \neq 1).$$

*The numerical boundary is independent of $K$ and $\chi$. No assertion is made on the line $\Re s = B_*$.*

*Proof.* For primitive $\chi$ of order $d$, let $M/K$ be its cyclic class-field extension and put $F = K(\zeta_{12})$, $E = MF$. Restriction gives

$$\mathrm{Gal}(E/F) \hookrightarrow \mathrm{Gal}(M/K), \qquad \mathrm{Gal}(E/M) \hookrightarrow (\mathbb{Z}/12\mathbb{Z})^\times.$$

For a finite abelian group $G$, write $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$. Both extensions are abelian. Proposition 9.1 and class-field factorization [Mil20], as in [Liu26b, Proposition 19.1], give

$$\begin{aligned}\zeta_E(s) &= \prod_{\psi \in \widehat{\mathrm{Gal}(E/F)}} L_F(s, \psi) \\ &= \zeta_M(s) \prod_{\substack{\omega \in \widehat{\mathrm{Gal}(E/M)} \\ \omega \neq 1}} L_M(s, \omega), \\ \zeta_M(s) &= \prod_{j=0}^{d-1} L_K(s, \chi^j).\end{aligned}$$

For $\Re s > B_*$, $s \neq 1$, the first product is nonzero and every complementary nonprincipal factor is entire. Hence

$$\zeta_E(s) \neq 0 \quad \Longrightarrow \quad \zeta_M(s) \neq 0 \quad \Longrightarrow \quad L_K(s, \chi) \neq 0.$$

At $s = 1$, for $d > 1$,

$$\prod_{j=1}^{d-1} L_K(1, \chi^j) = \frac{\mathrm{Res}_{s=1}\, \zeta_M(s)}{\mathrm{Res}_{s=1}\, \zeta_K(s)} > 0.$$

Thus each nonprincipal factor is nonzero; $d = 1$ permits the principal pole. An omitted Euler factor satisfies

$$1 - \alpha q_\mathfrak{p}^{-s} = 0, \quad |\alpha| = 1 \quad \Longrightarrow \quad \Re s = 0.$$

If $\alpha = 0$ the factor is one. This handles imprimitive functions. The full factorizations include ramified places and finite-order sign characters at real places of $K$. □

Section 10 lowers the boundary of Theorem 9.2 to $B_\natural$. Section 11 then proves Theorem 1.3 with the enlarged count.

**Corollary 9.3** (Optimality within the stated comparison system)**.** *Let $F \supset \mu_{12}$ be fixed. Every sequence of Theorem 8.5 over $F$ satisfies*

$$B_n \geq B_* \quad (n \geq 0), \qquad \liminf_{n\to\infty} B_n \geq B_* \qquad (\kappa_m = 3/4),$$

*and Proposition 9.1 gives $\beta_*(F) \leq B_*$. The descent of Theorem 9.2 preserves $\Re s > B_*$.*

*Proof.* For every fixed $F \supset \mu_{12}$, at the critical class $(\delta, x) = (r_*, 1/2)$,

$$h(1 - m_\theta) - l_y = 0, \qquad h\gamma \le \frac{7\delta}{216}(2l_y - h).$$

Proposition 9.1 preserves these quantities and the critical class; Theorem 8.5 gives the same scalar optimum. The descent in Theorem 9.2 preserves the strict half-plane exactly. □

# 10 The successive moment bounds with the earlier count

In this section, we let the numerator-moment parameter follow the successive zero-free bounds. We prove the amplified norm of Theorem 4.9 for every allowed $\kappa_m < 3/4$, solve the resulting comparison system, obtain the starting boundary $B_\natural$ for Section 11, and prove its optimality when the class count remains that of Proposition 6.4.

The auxiliary parameters for the boundary $B_\natural$ of (10.2) are defined as follows. Let $r_\natural$ be the unique root in $(9/20, 23/50)$ of

$$q_\natural(r) = 615108r^4 - 3239236r^3 + 4106021r^2 - 506420r - 334400. \tag{10.1}$$

The polynomial $q_\natural$ is irreducible over $\mathbb{Q}$. Set

$$\begin{gathered}\ell_\natural = -\frac{252833046}{2942707225}r_\natural^3 + \frac{2932032572}{2942707225}r_\natural^2 - \frac{159506318657}{76510387850}r_\natural + \frac{6990506117}{7651038785},\\ \kappa_\natural = \frac{5}{6} - \frac{\ell_\natural}{2} = 2B_\natural - 1, \qquad B_\natural = \frac{11}{12} - \frac{\ell_\natural}{4}.\end{gathered} \tag{10.2}$$

The numbers $\ell_\natural$ and $B_\natural$ are algebraic of degree four:

$$\begin{gathered}2229\ell_\natural^4 - 7370\ell_\natural^3 + 5771\ell_\natural^2 + 410\ell_\natural - 200 = 0,\\ 1925856B_\natural^4 - 5469552B_\natural^3 + 5643378B_\natural^2 - 2497497B_\natural + 399857 = 0.\end{gathered} \tag{10.3}$$

Thus

$$\begin{aligned}r_\natural &= 0.45053392487884345250\ldots, & \ell_\natural &= 0.16879668066330557444\ldots,\\ \kappa_\natural &= 0.74893499300168054611\ldots, & B_\natural &= 0.87446749650084027305\ldots,\end{aligned}$$
$$B_* - B_\natural = 8.1119269018\ldots \cdot 10^{-7}.$$

The balancing parameter $b_\natural = 0.09180139495752472774\ldots$ is defined in (10.28).

## 10.1 The variable numerator-moment parameter

In this subsection, we isolate the only source of $\kappa_m$, the estimate M2, and record the scalar data which depend on it. The corresponding norm is proved in Section 10.2.

The class count already uses $\kappa_c = 2B_n - 1$ (Proposition 6.4). The numerator moment uses $\kappa_m = 3/4$, justified by $\beta_*(F) \le 7/8$. The change is

$$\kappa_m = \frac{3}{4} \quad \longrightarrow \quad \kappa_m = \kappa_c = \kappa_n = 2B_n - 1. \tag{10.4}$$

Let $A$ be the total length exponent of a product, including its prime-sum lengths, $M$ its modulus exponent and $z_{\mathrm{pr}} = z$ its prime-sum length; with plain lengths $n_1, n_2$, $A + (6\kappa - 1)z \le M \iff n_1 + n_2 + 6\kappa z \le M$. On arithmetic parameters inducing outside $\Theta$, [Liu26b, Theorem 12.16 and Lemma 12.7] gives, for $13/18 \le \kappa \le 1$,

$$\beta_*(F) \le \frac{1+\kappa}{2} \ \ (\kappa < 1) \quad \Longrightarrow \quad \Big\{A + (6\kappa - 1)z \le M \ \Rightarrow\ \text{M2 holds}; \quad |Q_k|^2 \ll U^{\kappa z + \epsilon}\Big\}. \tag{10.5}$$

The hypothesis is used only iteratively:

$$\beta_*(F) \le B_n \quad\Longrightarrow\quad \beta_*(F) \le \frac{1+\kappa_n}{2} \quad\Longrightarrow\quad (10.5) \text{ at } \kappa = \kappa_n. \tag{10.6}$$

Every M2 application in the iteration of this section uses $\kappa > \kappa_\natural$, with $\kappa_\natural$ defined in (10.2). Write $\kappa_m = \kappa$ and

$$c_\kappa = \frac{1}{6\kappa}, \qquad k_6 = 6\kappa. \tag{10.7}$$

Thus $c_{\mathrm{cnt}}(\kappa) = 2c_\kappa$, where $c_{\mathrm{cnt}}$ is the count coefficient in Proposition 6.4. The letter $c_\kappa$ is distinct from the local lengths $c$, $c_{\mathrm{cor}}$ and $c_C$ of Sections 3–4. The $\kappa_m$-dependent scalar data become

$$\begin{aligned} &z_0(m) = c_\kappa(1-2m), \qquad \theta_0 = c_\kappa \delta x, \qquad z_L = z_0(3/8) = \frac{1}{24\kappa}, \\ &a(\delta, x) = \delta\,\frac{21x/100 - z_L/2}{21/100 - z_L}, \qquad W_{\mathrm{cap}}(m;\kappa) = \min\Big\{\frac{(1-c_\kappa)(1-2m)}{6}, \frac{8m-3}{12\kappa}\Big\}. \end{aligned} \tag{10.8}$$

At $\kappa = 3/4$,

$$\begin{aligned} c_\kappa = \frac{2}{9}, \quad z_0 = \frac{2(1-2m)}{9}, \quad \theta_0 = \frac{2\delta x}{9}, \quad z_L = \frac{1}{18}, \\ a = \delta\frac{189x-25}{139}, \quad W_{\mathrm{cap}} = \min\Big\{\frac{7}{54}(1-2m), \frac{8m-3}{9}\Big\}, \end{aligned}$$

as in (2.2), (1.7) and Lemma 7.3.

## 10.2 The amplified norm for a variable moment parameter

In this subsection, we prove Theorem 4.9 with $\kappa_m$ replaced by any $\kappa$ between the limiting value and 3/4. Section 10.3 applies this norm to the comparison.

**Lemma 10.1** (Amplified norm for a variable moment parameter)**.** *Let $\kappa_\natural \le \kappa \le 3/4$ and assume $\beta_*(F) \le (1+\kappa)/2$. Retain the arithmetic data, coefficient families, summation domains, smooth-weight families, order of choices and exact equal-product comparison of Theorem 4.1, with $\kappa_m = \kappa$ and the initial prime product $Q_0$ retained. Suppose $3/8 < m_- \le r < 1/2$, $W \ge 0$ and*

$$\eta_\kappa = (1-c_\kappa)(1-2r) - 6W > 0, \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + 6\kappa W < 1. \tag{10.9}$$

*For complete disjoint prime-sum factors with*

$$m \in [m_-, r], \qquad 0 \le z \le z_0(m) = c_\kappa(1-2m), \qquad 0 \le w \le W,$$

*the estimates (4.15) and (4.16) hold for every $\epsilon > 0$. In particular, for $m_- = r = m$,*

$$0 \le W < W_{\mathrm{cap}}(m;\kappa) \quad\Longrightarrow\quad (10.9). \tag{10.10}$$

*At $\kappa = 3/4$ this is Theorem 4.9.*

The implication (10.10) follows from

$$\eta_\kappa > 0 \iff W < \frac{(1-c_\kappa)(1-2m)}{6}, \qquad C_{\mathrm{add}} < 1 \iff 6\kappa W < 4m - \frac{3}{2} \iff W < \frac{8m-3}{12\kappa}.$$

*Proof.* We follow the proof of Theorem 4.9 and its inputs, Theorem 3.7, Proposition 3.3, Corollaries 3.4 and 3.6, Propositions 4.6 and 4.8, and Lemmas 4.7 and 5.2. We show that the substitutions of Table 3 are the only changes.

*Step 1: the single source of $\kappa$.* Put

$$\mathrm{Def} = A + (k_6 - 1)z_{\mathrm{pr}} - M. \tag{10.11}$$

By (10.5), M2 applies if $\mathrm{Def} \le 0$. If $\mathrm{Def} > 0$, remove prime length $\mathrm{Def}/k_6$, plus one mesh step. By the prime-length bound in (10.5),

$$\text{squared loss} = \kappa \cdot \frac{\mathrm{Def}}{k_6} = \frac{\mathrm{Def}}{6}. \tag{10.12}$$

Hence every restoration loss $(\cdot)_+/6$ in Sections 2–5 is independent of $\kappa$. Every application of M2 there (the restorations in Theorems 3.7, 4.1 and 4.9; the ordinary sums after the second Poisson formula in Proposition 4.8; the reflection in $C_{\mathrm{add}}$; the unamplified M2 in Lemma 5.2) uses the single parameter $\kappa_m$, and no other estimate fixed at $\kappa_m = 3/4$ occurs. The reflection of [Liu26b, Theorem 12.16] requires $6\kappa - 1 \ge 10/3$, and $6\kappa_\natural - 1 = 3.4936\ldots$.

*Step 2: changes of the length excess.* The length calculations of Sections 3.1–3.4 give the following changes of Def:

| operation | $\Delta\mathrm{Def}$ | at $\kappa = 3/4$ |
|---|---|---|
| initial family $A = 2m + z + w$, $M = 1$ (Sections 3.2, 3.4) | $\mathrm{Def} = k_6 w - e_z$, $e_z = k_6(z_0 - z)$ | $\frac{9}{2}w - e_z$ |
| fixed prime $f_0$: $(A, K, M, z_{\mathrm{pr}}) \mapsto$ $(A - \lambda_p, K + 4\lambda_p, M - 6\lambda_p, z_{\mathrm{pr}} - \lambda_p)$ (Section 3.2) | $(6 - k_6)\lambda_p$ | $\frac{3}{2}\lambda_p$ |
| fixed prime $f_1$ of positive valuation (Sections 3.2, 3.4) | $-(k_6 - 1)f_1 - T_1$ | $-\frac{7}{2}f_1 - T_1$ |
| free stage (Sections 3.1, 3.2) | $(6 - k_6)u - 5v - T$ | $\frac{3}{2}u - 5v - T$ |
| common fixing (Section 3.3) | $-(k_6 - 1)z_C$ | $-\frac{7}{2}z_C$ |
| genuine and final-divisor fixing (Section 3.2) | $-(k_6 - 1)b_g$, $-k_6 b_d$ | $-\frac{7}{2}b_g$, $-\frac{9}{2}b_d$ |

For the fixed prime $f_0$, $\Delta\mathrm{Def} = -\lambda_p + 6\lambda_p - (k_6 - 1)\lambda_p = (6 - k_6)\lambda_p$.

*Step 3: the length identities.* Summing the changes of Step 2 with $u = x_{\mathrm{aux}} + w - w_f - f_0 - f_1$ gives, in place of the display preceding (3.7),

$$D_{\mathrm{free}} = (6 - k_6)x_{\mathrm{aux}} + 6w - e_z - (k_6 - 1)z_{C,\mathrm{old}} - 5w_f - 5f_1 - T_1 - 5v - T. \tag{10.13}$$

With $z = z_0 - e_z/k_6$, $z_{\mathrm{init}} = z - z_{C,\mathrm{old}} - x_{\mathrm{aux}}$ and $(k_6 - 1)z_0 = (1 - c_\kappa)(1 - 2m)$,

$$\begin{aligned}(k_6 - 1)z_{\mathrm{init}} - D_{\mathrm{free}} &= (1 - c_\kappa)(1 - 2m) - 5x_{\mathrm{aux}} - 6w + \frac{e_z}{k_6} + 5w_f + 5f_1 + T_1 + 5v + T \\ &\ge (1 - c_\kappa)(1 - 2m) - 5x_{\mathrm{aux}} - 6w. \end{aligned} \tag{10.14}$$

The coefficients $6, 5, 5, 5$ do not depend on $\kappa$:

$$6 = k_6 + (6 - k_6), \qquad 5 = (k_6 - 1) + (6 - k_6),$$

and the terms in $f_0$ cancel. In the amplified norm $x_{\mathrm{aux}} = 0$, so

$$(k_6 - 1)z_{\mathrm{init}} - D_{\mathrm{free}} \ge (1 - c_\kappa)(1 - 2r) - 6W = \eta_\kappa, \qquad z_{\mathrm{init}} = 0 \Longrightarrow D_{\mathrm{free}} \le -\eta_\kappa.$$

This replaces the first display in the proof of Theorem 4.9. The remaining local identities become

$$-f_0 + \frac{(k_6 w + (6 - k_6)f_0)_+}{6} = -\kappa x_{\mathrm{aux}} \qquad (f_0 = x_{\mathrm{aux}} + w;\ \text{see (3.6)}), \tag{10.15}$$

$$\begin{aligned}\mathcal{C}_{\mathrm{hom}} + \frac{\mathcal{D}}{6} &= -\kappa x_{\mathrm{aux}} + \frac{w_f - \mu_{\mathrm{com}}}{6} - \frac{5}{6}(f_1 + v) - \frac{19}{6}(T_1 + T) - \frac{k_6 - 1}{6}z_{C,\mathrm{old}} - \frac{e_z}{6} \\ &\quad - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e \le 0 \qquad \text{(Corollary 3.6)}, \end{aligned} \tag{10.16}$$

$$C_{\mathrm{new}} = -\kappa u + \frac{D}{6} - \frac{5}{6}v - \frac{19}{6}T - 4c - S - e \qquad \text{(see (3.3), active branch)}. \tag{10.17}$$

By (10.17),

$$D \le k_6 u \implies C_{\mathrm{new}} \le -\frac{5v}{6}, \qquad D \le (k_6-1)u \implies C_{\mathrm{new}} \le -\frac{u+5v}{6}, \tag{10.18}$$

which is Proposition 3.3 with $9/2 \to k_6$ and $7/2 \to k_6 - 1$.

The identity in the proof of Proposition 4.6, with coefficients $17f_0$, $\frac{5}{2}f_1$, $18c_{\mathrm{cor}}$, $-\frac{9}{2}(T_1+T)$ and $-\frac{7}{2}m$, uses only $A_*, M_*, q_*$ and the Gauss exponents of $B_C$; it does not depend on $\kappa$. Its only dependence on $\kappa$ is through $A_t \le 2r + z_0(r) + W$. With

$$\begin{gathered} A_{\max} = 2r + z_0(r) = 1 - (1-c_\kappa)(1-2r), \\ A_{\max} + W = 1 - 5W - \eta_\kappa, \qquad M_* \ge \frac{\eta_\kappa + 6c_\kappa W}{1-c_\kappa} - o(1), \qquad m_* \ge \frac{\eta_\kappa + (7c_\kappa - 1)W}{1-c_\kappa} - o(1), \\ C_{\mathrm{add}} = \frac{5}{2} - 4m_- + k_6 W. \end{gathered} \tag{10.19}$$

The last identity follows from the reflected comparison

$$A + (k_6-1)(z+w) = \frac{3}{2} - 2m + k_6(z+w), \qquad k_6 z_0(m) = 1 - 2m.$$

At $\kappa = 3/4$, (10.19) gives the lower bounds $(12W + 9\eta_{3/4})/7$ and $(5W + 9\eta_{3/4})/7$ in the proof of Theorem 4.9.

*Step 4: restoration with fixings.* With $b_{\mathrm{fix}} = b_g + b_d$,

$$k_6(z_{\mathrm{init}} - b_{\mathrm{fix}}) - \big(D - (k_6-1)b_g - k_6 b_d\big) - \big((k_6-1)z_{\mathrm{init}} - D\big) = z_{\mathrm{init}} - b_g \ge 0. \tag{10.20}$$

Hence

$$(k_6-1)z_{\mathrm{init}} \ge D \quad \Longrightarrow \quad k_6(z_{\mathrm{init}} - b_{\mathrm{fix}}) \ge D - (k_6-1)b_g - k_6 b_d,$$

and the converse holds for $b_g = z_{\mathrm{init}}$. Thus the coefficient $1 - c_\kappa$ is necessary for this restoration argument when it imposes the inequality $(k_6-1)z_{\mathrm{init}} \ge D$ at every fixing. Lemma 12.2 applies the ordinary estimate M2 when all remaining prime factors have been removed; Theorem 12.1 therefore has the larger width domain. No necessity for that enlarged argument is asserted here.

*Step 5: the width and the margins.* The strict conditions (10.9) and $r \ge m_-$ give

$$1 - \frac{W}{W_{\max}(\kappa)} = \frac{4\eta_\kappa}{1-c_\kappa} + 2(1 - C_{\mathrm{add}}) + 8(r - m_-) > 0, \qquad W_{\max}(\kappa) = \frac{6\kappa - 1}{12\kappa(6\kappa+11)}. \tag{10.21}$$

At $\kappa = 3/4$ this is the identity $1 - \frac{279}{7}W = \frac{36}{7}\eta_{3/4} + 2(1 - C_{\mathrm{add}}) + 8(r - m_-)$ in the proofs of Proposition 4.6 and Theorem 4.9. For $\kappa \in [\kappa_\natural, 3/4]$, by exact rational bounds monotone in $\kappa$,

$$\begin{gathered} 1 - c_\kappa \ge 0.77746177, \qquad 7c_\kappa - 1 \ge \tfrac{5}{9}, \qquad W_{\max}(\kappa) \le 0.0250897556, \\ A_t + 4t_{\mathrm{loc}} \le 1 - W - \eta_\kappa < 1, \qquad A_t + 3t_{\mathrm{loc}} \le 1 - 2W - \eta_\kappa < 1, \\ -\frac{1}{2} + 3W - 3\eta_\kappa + \frac{7}{2}\rho_{\mathrm{term}} < -\frac{1}{2} + 3W_{\max}(\kappa) + \frac{7}{200} \le -0.3897307 < -\frac{3}{8}, \\ (k_6-1)z_0(0.443) - \tfrac{6}{100} \ge 0.0286306, \qquad \tfrac{5}{2} - 1.6 + \tfrac{k_6}{100} \le 0.945 < 1, \\ \tfrac{21}{100} > z_L = \tfrac{1}{24\kappa}, \qquad 2x\tfrac{21}{100} - z_L - W_{\max}(\kappa) \ge 0.08727 \quad (x \ge 2/5). \end{gathered} \tag{10.22}$$

Here $W_{\max}(\kappa) = W_{\mathrm{cap}}(m^\sharp; \kappa)$, where $m^\sharp = (k_6+8)/(2k_6+22)$ is the crossing point of the two branches. The contour region of Lemma 5.2, with $\Re z \ge 17/50$ and $\sigma_0 = 1/6$, does not depend on $\kappa$; there $\theta_0 = c_\kappa \delta x$.

*Order of choices.* Fix $\kappa = \kappa_n$ and the hypothesis (10.6); then the strict margins (10.9) and (10.22); then the steps of the proof of Theorem 4.9 in their original order: the classification of Theorem 3.7 before M2, the inequality (10.16), and the split of Lemma 4.7 between Propositions 4.6 and 4.8. Every displayed exponent has the $\kappa$-dependence specified in Table 3 and Remark 10.2, and every strict inequality keeps a positive margin by (10.22). □

Table 3: Dependence on $\kappa$ of the norm constants in Sections 2–5.

| value at $\kappa = 3/4$ | value for general $\kappa$ | occurrence |
|---|---|---|
| *Functions of* $6\kappa$ | | |
| $9/2$ | $6\kappa$ | $z_0$, $e_z$, $D$, final-divisor fixing, $C_{\mathrm{add}}$, $9u/2$ in Proposition 3.3 |
| $7/2$ | $6\kappa - 1$ | $z_C$, $f_1$, fixed primes dividing $s$, genuine fixing, allowance $\frac{7}{2}z_{\mathrm{init}}$, $7u/2$ in Proposition 3.3 |
| $3/2$ | $6 - 6\kappa$ | $3u/2$, $3f_0/2$ in Section 3.2 |
| $2/9$ | $c_\kappa$ | $z = z_0 - e_z/(6\kappa)$, $\theta_0 = c_\kappa \delta x$ |
| $7/9$ | $1 - c_\kappa$ | $\eta_{3/4} \to \eta_\kappa$, $A_{\max} = 1 - (1 - c_\kappa)(1 - 2r)$ |
| $7/12$, $3/4$ | $(6\kappa - 1)/6$, $\kappa$ | coefficients of $z_{C,\mathrm{old}}$ and $x_{\mathrm{aux}}$ in (10.16), (10.15) |
| $7/54$, $(8m - 3)/9$ | $(1 - c_\kappa)/6$, $(8m - 3)/(12\kappa)$ | $W_{\mathrm{cap}}$ |
| $7/279$ | $W_{\max}(\kappa)$ | Proposition 4.6, Theorem 4.9 |
| $4/9$ in (2.2) | $2c_\kappa$ | $E_\kappa$ |
| *Independent of* $\kappa$: $6w$, $5w_f$, $5f_1$, $5x_{\mathrm{aux}}$, $5v$, $19/6$, $5/6$, $4c$, $4c_{\mathrm{cor}}$, the restoration loss $1/6$, $M_*$, $A_*$, $q_*$, $6a$, $7a$, $4t$, $3t$, $B_C = (3c - 5d - R)_+/6$, the identity of Proposition 4.6 ($17f_0$, $\frac{5}{2}f_1$, $17g$, $18c_{\mathrm{cor}}$, $-\frac{9}{2}(T_1 + T)$, $\frac{7}{2}m$), $F_1^{\mathrm{loc}}, F_2^{\mathrm{loc}}, J, g, \Lambda_a, M', \Delta', \mathcal{O}_{\mathrm{ret}}$, $5/2 - 4m = (3/2 - 2m) + 6\kappa z_0$, and the contour data $17/50$, $1/6$. | | |
| *Chosen margins, unchanged*: $\rho_{\mathrm{term}} = 1/100$, $\sigma < 10^{-6}$, $3/125000$, $\rho_{\mathrm{step}}$. The $7/2$ in $-\frac{7}{2}m$ is a Gauss frequency exponent, not $6\kappa - 1$, and $-9/2$ in the identity of Proposition 4.6 is not $6\kappa$. | | |

*Remark* 10.2 (Constants in the variable-moment norm). The proof of the variable-moment norm uses the constants of Table 3 in the following forms.

(a) Proposition 4.6 and Theorem 4.9: the width satisfies $W < W_{\max}(\kappa)$, and the terminal exponent is at most $-\frac{1}{2} + 3W_{\max}(\kappa) + \frac{7}{200} \le -0.3897307 < -\frac{3}{8}$.

(b) Corollary 3.4, used only for $x_{\mathrm{aux}} > 0$: the sufficient length condition is

$$x_{\mathrm{aux}} \ge \frac{w_f}{6\kappa}, \qquad x_{\mathrm{req}} \ge \frac{w}{6\kappa} + \rho_{\mathrm{step}}, \qquad \eta_{\mathrm{mesh}} \le \frac{\rho_{\mathrm{step}}}{4}.$$

Indeed, for $w_f \le w$ and $x_{\mathrm{aux}} \ge x_{\mathrm{req}} - \eta_{\mathrm{mesh}}$,

$$\mathcal{C}_{\mathrm{act}} \le -\kappa x_{\mathrm{aux}} + \frac{w_f}{6} \le -\kappa(\rho_{\mathrm{step}} - \eta_{\mathrm{mesh}}) \le -\frac{3\kappa}{4}\rho_{\mathrm{step}} \le -\frac{\rho_{\mathrm{step}}}{2} \qquad (\kappa \ge \tfrac{2}{3}).$$

At $\kappa = 3/4$, $1/(6\kappa) = 2/9$ recovers the original length condition. The stronger mesh restriction is needed because

$$-\frac{2\kappa_\natural}{3}\rho_{\mathrm{step}} = -0.49929\ldots\rho_{\mathrm{step}} > -\frac{\rho_{\mathrm{step}}}{2}, \qquad -\frac{3\kappa}{4}\rho_{\mathrm{step}} \le -\frac{\rho_{\mathrm{step}}}{2} \quad (\kappa \ge \tfrac{2}{3}).$$

(c) The margin in Theorem 4.1 is $F_\kappa = (1 - c_\kappa)(1 - 2r) - 7W$, and $A_* - m_* \le W - \eta_\kappa + o(1) = -F_\kappa + o(1)$. The hypothesis $\tau = 1 - 2r - 9W > 0$ alone gives only

$$F_\kappa > (1 - 2r)\Big(\frac{2}{9} - c_\kappa\Big) \ge -3.2 \cdot 10^{-4}(1 - 2r) \qquad (\kappa \ge \kappa_\natural),$$

which does not give $A_* < m_*$. Theorem 4.9 uses Theorem 4.1 only for its arithmetic data and Corollary 4.3 only for the Hölder step.

(d) The numerical bounds in Theorem 3.7, with the worst case on $[\kappa_\natural, 3/4]$:

| $\frac{43}{1500}$ | $\frac{691}{750}$ | $-\frac{7}{375}$ | $\frac{721}{750}$ | $\frac{1427}{1500}$ | $\frac{189}{200}$ |
|---|---|---|---|---|---|
| $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $0.0286306$ | $0.9213694$ | $-0.0186306$ | $0.9613694$ | $0.9513694$ | $\frac{5}{2} - 1.6 + \frac{6\kappa}{100} \le 0.945$ |

(e) Corollary 4.3, Lemma 4.4, the identity $1 - 45W = 4\tau + \cdots$ and the example following (3.7) are specific to $\kappa = 3/4$. The hypothesis $\beta_*(F) \le 7/8$ in Sections 3.4, 4.1, 4.3, 4.4 and in Lemma 5.2 has the form $\beta_*(F) \le (1+\kappa)/2$.

### 10.3 The comparison system and its endpoint

In this subsection, we solve the comparison of Lemma 8.3 with independent count and moment parameters.

With $\kappa_m$ variable, (2.2), Lemma 7.3 and Lemma 8.3 become, with $c = c_{\kappa_m}$,

$$E_{\kappa_c,\kappa_m}(\delta, x) = C_0 + \frac{\delta}{2} + \delta x\Big\{\ell + c\Big(\frac{h}{2} - l_y\Big)\Big\} - \frac{3h}{4}\big(1 - R_{\kappa_c}(\delta, x)\big), \tag{10.23}$$

$$h\Gamma_N = \frac{1-c}{12}\, a(\delta, x)(2l_y - h), \tag{10.24}$$

$$\mathcal{Q}_{\kappa_c,\kappa_m,b,\ell}(\delta, x) = E_{\kappa_c,\kappa_m}(\delta, x) - \frac{1-c}{12}\, a(\delta, x)(2l_y - h), \tag{10.25}$$

with $R_\kappa$ of Proposition 6.4, $m_\theta = 1 - l_y/h$ and $a$ as in (10.8). For $a \ge 0$ and $3/8 \le m_\theta < 1/2$,

$$\frac{ah}{2} W_{\mathrm{cap}}(m_\theta; \kappa_m) \le h\Gamma_N, \qquad \text{equality if } m_\theta \ge m^\sharp = \frac{6\kappa_m + 8}{12\kappa_m + 22}.$$

Table 4 verifies this equality throughout the constructive parameter family. Outside that family, $h\Gamma_N$ remains an upper bound on the permitted selected saving, as used in Theorem 10.8. Thus $\mathcal{Q}_{\kappa,b,\ell} = \mathcal{Q}_{\kappa,3/4,b,\ell}$, since $c_{3/4}(h/2 - l_y) = \frac{4}{9}(h/4 - l_y/2)$ and $(1 - c_{3/4})/12 = 7/108$.

At $x = 1/2$, $a(\delta, 1/2) = \delta/2$ for every $z_L$, and

$$\mathcal{Q}_{\kappa_c,\kappa_m,b,\ell}(\delta, \tfrac{1}{2}) - \mathcal{Q}_{\kappa_c,b,\ell}(\delta, \tfrac{1}{2}) = \frac{\delta}{2}\Big(\frac{h}{2} - l_y\Big)\Big(\frac{1+5c}{6} - \frac{19}{54}\Big), \qquad \frac{h}{2} - l_y = \frac{5\ell - 1 - b}{4} < 0. \tag{10.26}$$

Hence

$$\kappa_m < \frac{3}{4} \iff c > \frac{2}{9} \iff \mathcal{Q}_{\kappa_c,\kappa_m,b,\ell}(\delta, \tfrac{1}{2}) < \mathcal{Q}_{\kappa_c,b,\ell}(\delta, \tfrac{1}{2}).$$

For $\kappa_m < 3/4$,

$$z_0(m) - \frac{2}{9}(1 - 2m) = \left(c - \frac{2}{9}\right)(1 - 2m), \qquad \theta_0 - \frac{2\delta x}{9} = \left(c - \frac{2}{9}\right)\delta x,$$
$$c > \frac{2}{9}, \qquad \frac{1-c}{6} < \frac{7}{54}.$$

Thus the gain uses the longer initial product of Lemma 10.1. Changing only the M2 parameter, with the norm of Theorem 4.9 retained, leaves $\mathcal{Q}(\cdot, 1/2)$ identically unchanged.

**Proposition 10.3** (The endpoint system and its quartic). *Put $\kappa_c = \kappa_m = 5/6 - \ell/2$, write $\mathcal{Q} = \mathcal{Q}_{\kappa_c,\kappa_m,b,\ell}$, and put*

$$D_\natural(\delta, \ell) = -48(3\ell - 5)(18\ell\delta - 25\ell - 26\delta + 35). \tag{10.27}$$

*(i)* $D_\natural\,\mathcal{Q}(\delta,1/2) = b\,p_b(\delta,\ell) + p_0(\delta,\ell)$, *where*

$$\begin{aligned} p_b &= (702\ell^2 - 2310\ell + 1880)\delta^2 + (-1587\ell^2 + 5029\ell - 3940)\delta \\ &\quad + 600\ell^2 - 1840\ell + 1400, \\ p_0 &= (378\ell^3 - 1428\ell^2 + 2162\ell - 1240)\delta^2 + (-4305\ell^3 + 14068\ell^2 - 13351\ell + 2860)\delta \\ &\quad + 4500\ell^3 - 14700\ell^2 + 13260\ell - 2100. \end{aligned}$$

*(ii) At* $(\delta,x) = (r,1/2)$, *with* $r$ *the critical value of* $\delta$, *the equations* $\mathcal{Q} = \partial_\delta\mathcal{Q} = \partial_b\mathcal{Q} = 0$ *are equivalent to*

$$p_b(r,\ell) = p_0(r,\ell) = 0, \qquad b\,\partial_\delta p_b(r,\ell) + \partial_\delta p_0(r,\ell) = 0,$$

*and*

$$\mathrm{Res}_\ell\big(p_b(r,\ell), p_0(r,\ell)\big) = 11520\,r^2(3r-5)(6r-5)^3\,q_\natural(r).$$

*The factors* $r^2(3r-5)(6r-5)^3$ *have no root in* $(9/20, 23/50)$, *and*

$$p_b(r,\ell_\natural(r)) \equiv p_0(r,\ell_\natural(r)) \equiv 0 \pmod{q_\natural(r)},$$

*where* $\ell_\natural(r)$ *is the cubic in* (10.2). *Put*

$$b_\natural = -\frac{\partial_\delta p_0}{\partial_\delta p_b}(r_\natural,\ell_\natural). \tag{10.28}$$

*(iii) At* $(b,\ell) = (b_\natural,\ell_\natural)$,

$$\mathcal{Q}(\delta,1/2) = \frac{A_\natural\,(\delta - r_\natural)^2}{D_\natural(\delta,\ell_\natural)}, \qquad A_\natural \in [-775.30309, -775.30308], \tag{10.29}$$

$$D_\natural(7/20,\ell_\natural) = 4905.62\ldots, \qquad D_\natural(3/5,\ell_\natural) = 3667.45\ldots,$$

*and* $D_\natural(\cdot,\ell_\natural)$ *is affine. Hence*

$$\mathcal{Q}(\delta,1/2) \le 0 \quad (7/20 \le \delta \le 3/5), \qquad \mathcal{Q}(\delta,1/2) = 0 \iff \delta = r_\natural.$$

*(iv)* $\partial_x\mathcal{Q} > 0$ *on* $\mathcal{R}_*$ *at the endpoint parameters. Hence* $\mathcal{Q}_{\kappa_\natural,\kappa_\natural,b_\natural,\ell_\natural} \le 0$ *on* $\mathcal{R}_*$, *with equality only at* $(r_\natural,1/2)$.

*(v) The conditions of Table 4 hold on the parameter boxes stated there.*

*At* $\kappa_m = 3/4$ *the same elimination gives* $432\,r(6r-5)^2(218582r^3 - 376469r^2 - 11376r + 61560)$, *that is, the cubic* (1.3) *and* $B_*$.

*Proof.* Parts (i)–(ii) are polynomial identities. Near the endpoint,

$$\begin{gathered} D_\natural \neq 0, \qquad \mathcal{Q} = \partial_b\mathcal{Q} = 0 \iff p_b = p_0 = 0, \\ p_b = p_0 = 0 \implies (\partial_\delta\mathcal{Q} = 0 \iff b\,\partial_\delta p_b + \partial_\delta p_0 = 0)\,. \end{gathered}$$

Section A.4 gives $\partial_\delta p_b(r_\natural,\ell_\natural) \neq 0$, defining (10.28). For (iii), the quadratic numerator has a double root at $r_\natural$; its leading coefficient is $A_\natural$, and the affine denominator is positive at both endpoints. For (iv), Table 4 gives

$$\mathcal{Q}(\delta,x) = \mathcal{Q}(\delta,1/2) - \int_x^{1/2} \partial_x\mathcal{Q}(\delta,t)\,dt \le \mathcal{Q}(\delta,1/2) - 2.67\cdot 10^{-4}(1/2-x).$$

Together with (iii), this proves (iv), including its equality claim. Part (v) is Table 4. The computations are described in Section A.4. □

For Table 4, define

$$\mathcal{P}_{\natural,0} = \{(\kappa_\natural, \kappa_\natural, \ell_\natural, b_\natural)\},$$
$$\mathcal{P}_\natural = \left\{ \begin{array}{c} \kappa_c, \kappa_m \in [\kappa_\natural, \kappa_*], \\ (\kappa_c, \kappa_m, \ell, b) : \ell \in [\ell_\natural - (\kappa_* - \kappa_\natural)/25, \ell_\natural], \\ |b - b_\natural| \le 10^{-30} \end{array} \right\}. \tag{10.30}$$

Let $\mathcal{P}_{\natural,3/4}$ denote the same family with the upper endpoint for $\kappa_m$ extended to $3/4$.

Table 4: Conditions for the endpoint parameters. Entries with inequality signs are rigorous bounds; entries without inequality signs are decimal approximations. The parameter sets are defined in (10.30); entries without an indicated set hold on $\mathcal{P}_\natural$. On $\mathcal{R}_*$, $\theta_0 \le 3/(60\kappa_m) < 0.06681$. Here $\gamma_{\max} = a\,W_{\max}(\kappa_m)/2$, $I_{\mathrm{short}} = R + (1/2 - \theta_0)m_{\mathrm{cut}}$ with $m_{\mathrm{cut}} = 3/40$ as in Lemma 7.2, and $Q_{\mathrm{sm}}$ is defined in Appendix A.

| condition | used in | bound or approximation |
|---|---|---|
| $\partial_x \mathcal{Q} > 0$ on $\mathcal{R}_*$ $(\mathcal{P}_{\natural,0})$ | Appendix A.3 | $\ge 2.67 \cdot 10^{-4}$ |
| $\partial_{\kappa_c}\mathcal{Q} + \partial_{\kappa_m}\mathcal{Q} - \frac{1}{25}\partial_\ell \mathcal{Q} < -\frac{1}{100}$ on $\mathcal{R}_*$ | Theorem 8.4 | $\le -0.01247$ |
| $\partial_\ell \mathcal{Q} > 3/4$ on $\mathcal{R}_*$ | Lemma 8.3 | $\ge 0.7932$ |
| $\partial_{\kappa_c}\mathcal{Q} \ge 0$, $\partial_{\kappa_m}\mathcal{Q} \ge 0$ on $\mathcal{R}_*$ | — | $\ge 4.3 \cdot 10^{-6}$, $\ge 9.8 \cdot 10^{-4}$ |
| $J > 0$ on $\mathcal{G}$ $(\mathcal{P}_{\natural,3/4})$ | Lemma 8.3 | $\ge 0.0971$ |
| $E < 1/500$ on $\mathcal{G}$ $(\mathcal{P}_{\natural,3/4})$ | Appendix A.3 | $\le 0.00199999$ |
| $E < -10^{-5}$ on $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ $(\mathcal{P}_{\natural,3/4})$ | Appendix A.3 | $\le -1.002$, $-1.064$, $-4.47$ (all $\times 10^{-5}$) |
| $S > 1/5$ on $\mathcal{R}_*$; $S < 1$ on $\mathcal{G}$ | Corollary 6.7 | $\ge 0.2397$; $\le 0.9412$ |
| $(6\kappa_m - 2\delta x)\frac{21}{100} - R_{\kappa_c} > 0$ on $\mathcal{R}_*$; on $\mathcal{G}$ | (6.2) | $\ge 0.0816$; $\ge 8.1 \cdot 10^{-5}$ |
| $\frac{21}{100}(6\kappa_m - \delta) - (1 - \frac{2\delta}{3}) > 0$, $\frac{1}{8} \le \delta \le \frac{5}{6}$ | [Liu26b, Proposition 14.6] | $\ge 7.41 \cdot 10^{-4}$ |
| $a > 0$; $\frac{a}{3\kappa_m} - \theta_0 > 0$; $\frac{\delta}{2} - \theta_0 - \frac{a(1-c)}{6} > 0$ | Lemma 7.3 | 0.127; $3.8 \cdot 10^{-3}$; 0.0694 |
| $\Gamma_L - \Gamma_N > 0$; $(\frac{\delta}{2} - \theta_0)(\lambda - \frac{11}{20}) - \Gamma_N > 0$ | Lemma 7.3, Proposition 7.4 | $\ge 1.9 \cdot 10^{-7}$; $\ge 9.7 \cdot 10^{-4}$ |
| $2x\frac{21}{100} - z_L - W_{\max} > 0$; $\gamma_{\max} < \frac{1}{250}$ | Lemma 4.2, Proposition 7.4 | $\ge 0.08727$; $< 0.0037635$ |
| $\frac{3}{8} < m^\sharp < m_\theta$, $\frac{25}{62} < m_\theta < \frac{9}{20}$, $\frac{57}{100} < \lambda < \frac{11}{19}$ | Lemma 7.3 | 0.403186, 0.422469, 0.577531 |
| $N - I_{\mathrm{short}} \ge \frac{23}{2700}$; $E + h\Delta_{\mathrm{short}} < -10^{-5}$ | Lemma 7.2 | $8.87 \cdot 10^{-3}$; $\le -5.9 \cdot 10^{-4}$ |
| $S + \Delta_{\mathrm{short}} > 0$ | Lemma 7.2 | $\ge 0.1452$ |
| $z_0(\frac{3}{40}), z_0(\frac{1}{10}), z_L < \frac{21}{100}$ | Lemma 7.2 | 0.189157, 0.178030, 0.05563 |
| $S - \frac{37}{40}\theta_0 - \gamma_{\max} > \frac{1}{8}$; $S - \gamma_{\max} > \frac{1}{8}$; slope $< 2$ | Theorem 8.4 | $\ge 0.1920$; $\ge 0.2360$; $\le 0.6997$ |
| $6\kappa_m \ell > 3/4$; $\frac{9}{2}\ell > \frac{3}{4}$ | [Liu26b, Remark 14.1] | 0.75851; 0.7596 |
| $l_x > \ell$; $l_y - \ell - \frac{11b}{6} > 0$; $l_y < \frac{1}{2}$; $h < 2l_y$ | Lemma 8.3 | 0.2009; 0.1244; 0.4615 |
| $\frac{\ell}{h+1/2000} > \frac{21}{100}$; $\frac{\ell}{h-1/80} < \frac{2}{9}$; $\frac{\ell}{h} < \frac{1}{6\kappa_m}$ | Proposition 6.4, Lemma 7.3 | 0.211102; 0.214591; via $\frac{\ell}{h-1/80} < \frac{2}{9} \le \frac{1}{6\kappa_m}$ |
| $Q_{\mathrm{sm}} < 0$; $C_0 + \frac{1-h}{16} < -\frac{1}{250}$ | Proposition 6.2 | $-7.07 \cdot 10^{-3}$; $-0.011147$ |
| $C_0 + \frac{1/2+3\ell/2}{50} < -\frac{1}{125}$ | Lemma 6.3 | $-0.0086400$ |
| $\frac{13h}{75} - \frac{l_y}{2} + \frac{1}{50} < -\frac{1}{20}$ | Appendix A | $-0.07224$ |

*Remark* 10.4 (Scalar bounds for the variable moment parameter). The scalar estimates used

below satisfy the following bounds on the parameter sets of Table 4.

(a) On $\mathcal{R}_*$, the contour parameter satisfies $\theta_0 = \delta x/(6\kappa_m) \le 3/(60\kappa_m) < 0.06681$ in Corollary 6.7, Proposition 7.4 and the slope display in the proof of Theorem 8.4.

(b) The three requirements of [Liu26b, Remark 14.1(1)] hold as follows: the crude supply of [Liu26b, Proposition 14.6] holds for $\kappa_m > 2263/3024 = 0.74834\ldots$; the slopes in [Liu26b, Lemma 14.5] are below 1 for $\kappa \ge 13/18$ (bounds 0.7081 and 0.7399 for $\ell \ge 1/6$; 0.7064 and 0.7324 on $\mathcal{P}_\natural$); and $6\kappa\ell > 3/4$.

(c) The constants in Lemmas 7.3, 7.2, 8.3, Proposition 7.4 and Appendix A.3 are

$$\frac{1-c}{6},\quad W_{\max}(\kappa_m),\quad \frac{a}{3\kappa_m},\quad \frac{a(1-c)}{6},\quad \frac{1}{24\kappa_m},\quad \frac{17c}{20},\quad \frac{4c}{5},\quad c.$$

(d) The norm conditions in Proposition 7.4 are $(1-c)(1-2m) - 6W > 0$ and $1 - (\frac{5}{2} - 4m + 6\kappa_m W) > 0$, as in (10.9).

(e) The critical-length bound in Theorem 8.5 is $h\gamma \le \frac{1-c}{12}a(2l_y - h)$ with $a = \delta/2$; this is (10.24).

The small-amplitude and lowest-amplitude estimates follow from Proposition 6.2 and Lemma 6.3, with the bounds of Table 4. For $\delta < 1/8$ one may keep $\kappa_m = 3/4$, since Proposition 6.2 uses $\ell/h < 2/9 = 1/(6\cdot 3/4)$ and $R_{\kappa_c} \le R_{3/4}$.

### 10.4 The starting boundary for the enlarged count

In this subsection, we iterate the comparison from $B_*$ over fields containing $\mu_{12}$ (Proposition 10.5), then descend to arbitrary number fields (Corollary 10.6). We also prove the optimality of $B_\natural$ in this comparison system (Theorem 10.8).

**Proposition 10.5** (The boundary over fields containing $\mu_{12}$)**.** *For every number field $F \supset \mu_{12}$,*

$$\beta_*(F) \le B_\natural = 0.87446749650084027305\ldots.$$

*Proof.* Fix $F \supset \mu_{12}$. The verification in the proof of Proposition 9.1 concerns the Gauss identities, natural zeros and summation lengths over $F$. It does not involve the value of $\kappa_m$, and Lemma 10.1 is proved by the same steps as Theorem 4.9. Hence Lemma 10.1 holds over $F$.

*Step 1: seed.* By Proposition 9.1,

$$\beta_*(F) \le B_0 = B_*, \qquad \kappa_0 = 2B_* - 1 = \kappa_* \in [\kappa_\natural, 3/4].$$

*Step 2: the family.* For $\kappa \in [\kappa_\natural, \kappa_*]$ put

$$\kappa_c = \kappa_m = \kappa, \qquad \ell(\kappa) = \ell_\natural - \frac{\kappa - \kappa_\natural}{25}, \qquad b = b_\natural, \qquad \mathcal{Q}_\kappa = \mathcal{Q}_{\kappa,\kappa,b_\natural,\ell(\kappa)}. \tag{10.31}$$

This family lies in $\mathcal{P}_\natural$. On $\mathcal{R}_*$, by Table 4,

$$\frac{d}{d\kappa}\mathcal{Q}_\kappa = \partial_{\kappa_c}\mathcal{Q} + \partial_{\kappa_m}\mathcal{Q} - \frac{1}{25}\partial_\ell \mathcal{Q} \le -0.01247 < -\frac{1}{100},$$

and $\mathcal{Q}_{\kappa_\natural} \le 0$ by Proposition 10.3(iv). Hence

$$\mathcal{Q}_\kappa \le -\nu_\kappa \quad \text{on } \mathcal{R}_*, \qquad \nu_\kappa = \frac{\kappa - \kappa_\natural}{100}. \tag{10.32}$$

*Step 3: one step.* Let $n \ge 0$ and assume $\beta_*(F) \le B_n$ with $\kappa_n = 2B_n - 1 \in (\kappa_\natural, \kappa_*]$. Then

$$\beta_*(F) \le \frac{1+\kappa_n}{2}, \qquad 2B_n - 1 \le \kappa_c = \kappa_n, \qquad \frac{13}{18} \le \kappa_n \le \frac{3}{4}.$$

Use Proposition 6.4 at $\kappa_c = \kappa_n$ and Lemma 10.1, unamplified M2 and Lemma 5.2 at $\kappa_m = \kappa_n$. In the continuation argument of Theorem 8.4, take, using Table 4,

$$c = c_{\kappa_n}, \qquad 0 < \varepsilon < 1/1000, \qquad C\varepsilon + 2\zeta + \epsilon_{\mathrm{rem}} < \nu_{\kappa_n},$$
$$m_a = 3/8 + \varepsilon, \quad m_b = 9/20, \quad m_- = r = m, \qquad W(m) = (1-\varepsilon)W_{\mathrm{cap}}(m;\kappa_n).$$

For $m \in [m_a, m_b]$, $1 - 2m \geq 1/10$ and $8m - 3 \geq 8\varepsilon$, hence

$$(1-c)(1-2m) - 6W(m) \geq \frac{(1-c)\varepsilon}{10} > 0, \qquad 1 - \Big(\frac{5}{2} - 4m + 6\kappa_n W(m)\Big) \geq \frac{\varepsilon(8m-3)}{2} \geq 4\varepsilon^2 > 0,$$

so (10.10) and Lemma 10.1 apply on every selected dyadic sum. By Table 4, the slopes of the full comparison exponent satisfy

$$S - \theta_0(1-m) - \gamma \geq 0.1920 > \frac{1}{8}, \qquad S + \Big(\frac{\delta}{2} - \theta_0\Big)(1-m) - \gamma \geq 0.2360 > \frac{1}{8},$$

and both are at most $0.6997 < 2$; the extension to $d \leq h + \zeta$ loses at most $2\zeta$. The complementary ranges, short sums, lowest amplitudes and the principal and direct comparisons use the entries of Table 4 on $\mathcal{P}_{\natural,3/4}$. By (10.32), the continuation argument in the proof of Theorem 8.4 and [Liu26b, Theorem 14.12],

$$\beta_*(F) \leq B_{n+1} = \frac{11}{12} - \frac{\ell(\kappa_n)}{4} = B_\natural + \frac{\kappa_n - \kappa_\natural}{100}. \tag{10.33}$$

*Step 4: iteration.* By (10.2) and (10.33),

$$\kappa_{n+1} - \kappa_\natural = 2B_{n+1} - 1 - \kappa_\natural = \frac{\kappa_n - \kappa_\natural}{50}, \qquad B_n - B_\natural = \frac{B_* - B_\natural}{50^n}.$$

Thus

$$\kappa_n \in (\kappa_\natural, \kappa_*], \qquad \nu_{\kappa_n} > 0 \quad (n \geq 0), \qquad \beta_*(F) \leq \inf_n B_n = B_\natural.$$

Step 3 has its own threshold at each finite $n$. □

**Corollary 10.6** (The earlier-count boundary over arbitrary fields)**.** *For every number field $K$ and finite-order Hecke character $\chi$, $L_K(s,\chi)$ has no zeros in $\Re s > B_\natural$. The principal pole at $s = 1$ is permitted.*

*Proof.* Apply the proof of Theorem 9.2 with $B_*$ replaced by $B_\natural$ and Proposition 9.1 replaced by Proposition 10.5. For $F = K(\zeta_{12})$, $E = MF$ and $\Re s > B_\natural$, $s \neq 1$,

$$\zeta_E(s) \neq 0 \quad \Longrightarrow \quad \zeta_M(s) \neq 0 \quad \Longrightarrow \quad L_K(s,\chi) \neq 0.$$

The nonvanishing at $s = 1$ of the nonprincipal factors and the treatment of imprimitive characters are unchanged. □

*Remark* 10.7 (Relation to Theorem 8.5). Theorem 8.5 assumes

$$\kappa_m = \frac{3}{4}, \qquad 0 \leq W(m) \leq \frac{7}{54}(1-2m),$$

and its proof uses $h\gamma \leq \frac{7\delta}{216}(2l_y - h)$ and $\theta_0 = r_*/9$. The proof of Proposition 10.5 uses $\kappa_m = \kappa_n < 3/4$, the initial length $(1-2m)/(6\kappa_n)$ and the norm of Lemma 10.1. It lies outside those hypotheses, so $B_\natural < B_*$ is consistent with Theorem 8.5. Conversely, by (10.26), changing $\kappa_m$ in M2 alone, with the norm of Theorem 4.9, leaves $\mathcal{Q}(\cdot, 1/2)$ unchanged. Pointwise $W_{\mathrm{cap}}(m;\kappa) \leq W_{\mathrm{cap}}(m;3/4)$ on the first branch, and the whole gain $B_* - B_\natural$ comes from $z_0(m) = (1-2m)/(6\kappa_m)$. Theorem 10.8 gives the corresponding optimality of $B_\natural$.

**Theorem 10.8** (Optimality with a variable moment parameter)**.** *Fix the detector, class count, balanced-length hypotheses and direct theta estimate used above. On each amplitude class, allow dyadic-sum-dependent prime lengths and contours, the count of Proposition 6.4 at $\kappa_c$, and the norm of Lemma 10.1 at $\kappa_m$ (Theorem 4.9 if $\kappa_m = 3/4$), with*

$$0 \le W(m) \le \frac{1-c_{\kappa_m}}{6}(1-2m), \qquad 0 \le \theta(m) \le \delta/2,$$
$$1/6 \le \ell \le 1/5, \qquad B(\ell) = 11/12 - \ell/4.$$

*Every allowed sequence of improvements in this comparison system, starting at $B_0 \ge B_\natural$, with*

$$2B_n - 1 \le \kappa_{c,n} \le \frac{3}{4}, \qquad 2B_n - 1 \le \kappa_{m,n} \le \frac{3}{4},$$

*satisfies*

$$B_n \ge B_\natural \quad (n \ge 0), \qquad \liminf_{n\to\infty} B_n \ge B_\natural.$$

*The sequence of Proposition 10.5 has $B_n - B_\natural = 50^{-n}(B_* - B_\natural) \to 0$. This optimality keeps the detector, moment estimates and allowed norm lengths fixed. The case $\kappa_m = 3/4$ is the system of Theorem 8.5.*

*Proof.* Suppose a first crossing satisfies $B_n \ge B_\natural$ and $B_{n+1} = B(\ell) < B_\natural$. Then

$$\kappa_c, \kappa_m \ge 2B_n - 1 \ge \kappa_\natural, \qquad \ell > \ell_\natural.$$

The retained geometry of the proof of Theorem 8.5 gives $m_\theta > 32/81 > 3/8$ and $1/2 - m_\theta > 0$. At the class $(\delta, x) = (r_\natural, 1/2)$, with $c = c_{\kappa_m}$ and $(1-2m_\theta)h = 2l_y - h$,

$$h(1-m_\theta) - l_y = 0, \qquad \gamma \le \frac{\delta W}{4}, \qquad h\gamma \le \frac{1-c}{12}\,\frac{\delta}{2}\,(2l_y - h) = h\Gamma_N,$$

by (10.24). Use the exponents in (8.3) with

$$r = r_\natural, \quad R = R_{\kappa_c}(r_\natural, 1/2), \quad R_\natural = R_{\kappa_\natural}(r_\natural, 1/2),$$
$$A = \frac{1+5c_{\kappa_m}}{24} \ge A_0 = \frac{19}{216}, \qquad R \ge R_\natural.$$

For $0 \le m \le 1/2$,

$$\begin{aligned} P_{\mathrm{pt}}(m;R) - P_4(m;R,A) &\ge \frac{R_\natural - 1}{4} + \frac{19r_\natural}{216}(1-2m) + \frac{r_\natural m}{2} \\ &= \frac{35r_\natural}{108}(m - \mu_\natural), \qquad \mu_\natural := \frac{54(1-R_\natural) - 19r_\natural}{70r_\natural}. \end{aligned}$$

Equations (8.4) and (A.6) give

$$\mu_\natural < \frac{399030}{10^6} < \frac{399165}{10^6} < g(\ell_\natural) < g(\ell) < m_\theta.$$

Hence the buffered pointwise estimate, absolute ideal counting and the unamplified moment do not improve this optimistic amplified comparison at $m_\theta$. The full contour contribution vanishes there, so

$$\mathcal{Q}_{\kappa_c,\kappa_m,b,\ell}(r_\natural, \tfrac{1}{2}) \le 0 \tag{10.34}$$

is necessary.

By (10.23), (10.24), $a(\delta, 1/2) = \delta/2$ and Lemma 6.5,

$$\partial_{\kappa_c}\mathcal{Q} = \frac{3h}{4}\,\partial_{\kappa_c}R_{\kappa_c} \ge 0, \qquad \partial_{\kappa_m}\mathcal{Q}(\delta, \tfrac{1}{2}) = \frac{5\delta(1+b-5\ell)}{288\kappa_m^2} > 0.$$

At $\kappa_c = \kappa_m = k$,

$$\mathcal{Q}_{k,k,b,\ell}(\delta, \tfrac{1}{2}) = \frac{b\, n_b^\natural(\delta,k) + \ell\, n_e^\natural(\delta,k) + n_0^\natural(\delta,k)}{288k\, L(\delta,k)}, \qquad L = 54k\delta - 75k - 6\delta + 10,$$

with

$$\begin{aligned} n_b^\natural &= -4212\delta^2k^2 + 90\delta^2k + 30\delta^2 + 9522\delta k^2 - 783\delta k - 50\delta - 3600k^2 + 480k, \\ n_e^\natural &= -2268\delta^2k^2 + 1278\delta^2k - 150\delta^2 + 25830\delta k^2 - 4725\delta k + 250\delta - 27000k^2 + 3600k, \\ n_0^\natural &= 3564\delta^2k^2 - 774\delta^2k + 30\delta^2 - 7758\delta k^2 + 1377\delta k - 50\delta + 5400k^2 - 720k. \end{aligned}$$

Substitute $\delta = r$, $k = 5/6 - \ell_\natural(r)/2$ with $\ell_\natural(r)$ the cubic of (10.2). Polynomial division by $q_\natural(r)$ gives

$$n_b^\natural(r_\natural, \kappa_\natural) = 0, \qquad \ell_\natural n_e^\natural(r_\natural, \kappa_\natural) + n_0^\natural(r_\natural, \kappa_\natural) = 0.$$

Hence, for every $b$,

$$\mathcal{Q}_{\kappa_\natural,\kappa_\natural,b,\ell}(r_\natural, \tfrac{1}{2}) = e_\natural(\ell - \ell_\natural), \qquad e_\natural = \frac{n_e^\natural(r_\natural, \kappa_\natural)}{288\kappa_\natural L(r_\natural, \kappa_\natural)}.$$

On $[4505, 4506] \times [7489, 7490] \cdot 10^{-4} \ni (r_\natural, \kappa_\natural)$, termwise bounds give $n_e^\natural < -7490$ and $-31 < L < -30$. Thus

$$e_\natural > \frac{7490}{288 \cdot \frac{3}{4} \cdot 31} = \frac{7490}{6696} > 1, \qquad e_\natural = 1.13390680121859172982\ldots.$$

Therefore

$$\kappa_c, \kappa_m \geq \kappa_\natural, \quad \ell > \ell_\natural \quad \Longrightarrow \quad \mathcal{Q}_{\kappa_c,\kappa_m,b,\ell}(r_\natural, \tfrac{1}{2}) \geq e_\natural(\ell - \ell_\natural) > 0,$$

contrary to (10.34). If $B_n \geq B_\natural$, then $\kappa_{c,n}, \kappa_{m,n} \geq 2B_n - 1 \geq \kappa_\natural$, so the next step has $\ell \leq \ell_\natural$ and $B_{n+1} = 11/12 - \ell/4 \geq B_\natural$. By induction $B_n \geq B_\natural$ for every $n$; neither a finite iteration nor a limit crosses $B_\natural$. The sequence of Proposition 10.5 reaches $B_\natural$. □

# 11 The enlarged class count and its algebraic boundary

In this section, we apply the enlarged amplified norm to the counting detector. The additional prime product improves the M2 count as well as the numerator estimate. Starting from Proposition 10.5, we obtain the boundary $B'$ of Theorem 1.3.

## 11.1 The extended moment range

In this subsection, we extend Lemma 10.1 to $\kappa_\flat \leq \kappa \leq 3/4$. Lemma 11.1 supplies the moment estimate used in Section 11.2.

Put $\kappa_\flat = 1871/2500 = 0.7484$ and retain $c_\kappa = 1/(6\kappa)$ and $z_S(m) = c_\kappa(1-2m) = z_0(m)$. Write

$$\begin{gathered} W_1(m) = \frac{(1-c_\kappa)(1-2m)}{6}, \qquad W_2(m) = \frac{8m-3}{12\kappa}, \qquad W_{\mathrm{cap}}(m;\kappa) = \min\{W_1(m), W_2(m)\}, \\ m^\sharp = \frac{6\kappa+8}{12\kappa+22}, \qquad W_{\max}(\kappa) = \frac{6\kappa-1}{12\kappa(6\kappa+11)}. \end{gathered} \tag{11.1}$$

Thus $W_1(m^\sharp) = W_2(m^\sharp) = W_{\max}$, and $W_2 > 0$ precisely when $m > 3/8$.

**Lemma 11.1** (The norm below the earlier endpoint)**.** *Let $\kappa_\flat \le \kappa \le 3/4$ and assume $\beta_*(F) \le (1+\kappa)/2$. Retain the arithmetic, coefficient, smooth-weight and summation conventions of Lemma 10.1. Suppose*

$$\begin{aligned} 3/8 < m_- \le r < 1/2, \qquad W \ge 0, \\ \eta_\kappa = (1-c_\kappa)(1-2r) - 6W > 0, \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + 6\kappa W < 1. \end{aligned} \tag{11.2}$$

*For $m \in [m_-, r]$, $0 \le z \le z_S(m)$, $0 \le w \le W$, and complete disjoint prime products $Q_0, Q_W$ of lengths $z, w$, the centered and original estimates (4.15)–(4.16) hold. In particular,*

$$\sum_{k \in \mathcal{U}} \mathcal{W}(q_k/U) |S_{m,1}(k) S_{m,2}(k) Q_0(k) Q_W(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C. \tag{11.3}$$

*The conclusion includes the finite independent derivative families and the order of choices of Theorem 4.9.*

*Proof.* Use the proof of Lemma 10.1 with $k_6 = 6\kappa$. The estimate M2 (10.5) is valid, since

$$\kappa \ge \kappa_\flat > 13/18, \qquad 6\kappa - 1 \ge 3.4904 > 10/3.$$

The identities (10.13)–(10.19) and (10.20) are unchanged. In particular,

$$\begin{gathered} (6\kappa-1) z_{\mathrm{init}} - D_{\mathrm{free}} \ge \eta_\kappa, \qquad z_{\mathrm{init}} = 0 \Longrightarrow D_{\mathrm{free}} \le -\eta_\kappa, \\ M_* \ge \frac{\eta_\kappa + 6c_\kappa W}{1-c_\kappa} - o(1), \qquad m_* \ge \frac{\eta_\kappa + (7c_\kappa - 1)W}{1-c_\kappa} - o(1), \\ A_t \le 1 - 5W - \eta_\kappa, \qquad 1 - \frac{W}{W_{\max}} = \frac{4\eta_\kappa}{1-c_\kappa} + 2(1 - C_{\mathrm{add}}) + 8(r - m_-) > 0. \end{gathered}$$

The restoration, homogeneous coefficient estimate and split between Propositions 4.6 and 4.8 therefore retain their strict margins. The uniform numerical bounds on this interval are those of Table 5 and the display above. Remark 10.2(b) uses only $\kappa \ge 2/3$, and Remark 10.2(c) is not used. For this table put

$$\rho_z(\kappa) = \frac{57}{500}(1-c_\kappa) - \frac{3}{50}.$$

Both $1 - c_\kappa$ and $\rho_z$ increase with $\kappa$, whereas $z_L = 1/(24\kappa)$ decreases. Moreover,

$$W'_{\max}(\kappa) = -\frac{36\kappa^2 - 12\kappa - 11}{12\kappa^2(6\kappa+11)^2} < 0, \qquad 36\kappa_\flat^2 - 12\kappa_\flat - 11 = 0.18289216 > 0.$$

Exact rational evaluation at $\kappa_\flat$ or $3/4$ gives every entry. The mesh condition in Remark 10.2(b) uses only $\kappa \ge 2/3$. Fix the strict margins, then the steps, meshes and local losses, then the finite test families and internal orders, before the external heights. This is the order of Lemma 10.1. □

Table 5: Uniform norm bounds on $[\kappa_\flat, 3/4]$. Terminating decimals are rational bounds.

| quantity | bound |
|---|---|
| $1-c_\kappa$; $7c_\kappa-1$ | $\ge 0.7773026$; $\ge 5/9$ |
| $W_{\max}(\kappa)$ | $\le 0.0250898199$ |
| $-1/2+3W_{\max}+7/200$ | $\le -0.3897305 < -3/8$ |
| $\rho_z(\kappa)$ | $\ge 0.0286125 > 0$ |
| $0.95-\rho_z$; $0.01-\rho_z$ | $\le 0.9213875$; $\le -0.0186125$ |
| $0.99-\rho_z$; $0.98-\rho_z$ | $\le 0.9613875$; $\le 0.9513875$ |
| $5/2-8/5+6\kappa/100$ | $\le 0.945 < 1$ |
| $z_L = 1/(24\kappa)$ | $\le 0.0556744 < 21/100$ |
| $2x(21/100)-z_L-W_{\max}$, $x\ge 2/5$ | $\ge 0.0872358 > 0$ |
| $(21/100)(6\kappa-1/8)-11/12$ (crude supply at $\delta=1/8$) | $\ge 0.0000673 > 0$ |
| $\theta_0 \le 3/(60\kappa)$ on $\mathcal{R}_*$ | $< 0.06681$ |

### 11.2 The counting detector and the additional prime product

In this subsection, we insert a disjoint additional prime product into the zero-detecting polynomial. Lemma 11.1 gives the improved count used in Section 11.3.

Let $\mathscr{B}$ be a detected amplitude class of [Liu26b, Section 13.4]. Its original prime factors satisfy

$$\sum_i \lambda_i = \mathscr{L} > \frac{1}{5}, \qquad \sum_i \lambda_i g_i = g\mathscr{L}, \qquad g=\delta x, \quad 0\le g_i\le \delta/2, \tag{11.4}$$

where $0<\delta\le 5/6$, $0\le x\le 1/2$ and $\lambda_i \le \eta_{\mathrm{mesh}}$. Positive-amplitude factors have their original lower bounds $|Q_i(k)| \ge U^{\lambda_i g_i - o(1)}$ on the class.

**Lemma 11.2** (Length conditions of the counting detector)**.** *Assume the hypotheses and lengths of Lemma 11.1. Fix a presentation, a detecting dyadic pair and disjoint products $Q_0, Q_W$ of complete original prime factors. Put*

$$P_k(\mathbf{t}) = S_m(k;\mathbf{t})^2 Q_0(k;\mathbf{t}) Q_W(k;\mathbf{t}), \qquad \mathbf{t}\in\mathcal{B}_T.$$

*Lemma 11.1 applies to every required mixed Sobolev derivative, and*

$$\sum_{k\in\mathcal{U}} \mathcal{W}(q_k/U) \sup_{\mathbf{t}\in\mathcal{B}_T} |P_k(\mathbf{t})|^2 \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C(1+T_{\mathcal{B}})^A,$$

*where $A, C, J$ depend only on the fixed derivative orders and data.*

*Proof.* By [Liu26b, Lemmas 13.7 and 13.9], after removing the deterministic scale phase, the centrally normalized weight satisfies

$$w_{\sigma,t}(y) = \rho_{\mathrm{ann}}(y) y^{-\sigma-it}, \qquad \partial_\sigma^a \partial_t^b w_{\sigma,t}(y) = (-1)^a(-i)^b(\log y)^{a+b} w_{\sigma,t}(y),$$

with fixed annular cutoff $\rho_{\mathrm{ann}}$. Retain the original conductor-dependent weight families. The fixed group $\Theta$ contains the detector's finite coefficient group and its conjugates. Conjugate the entire presentation, including both ordinary weights; retain its original prime coefficients and Mellin frequencies.

Fix the presentation and pair before $\mathcal{B}_T$. The numbers of presentations and pairs are $O(1)$ and $O((\log U)^2)$, respectively. Leibniz's rule gives

$$\partial^\alpha P_k = \sum_\iota c_{\alpha,\iota} S_{m,1}^{(\iota)}(k) S_{m,2}^{(\iota)}(k) Q_0^{(\iota)}(k) Q_W^{(\iota)}(k), \qquad \max_{\phi\in\mathbf{w}_{\alpha,\iota}} \|\phi\|_{C^J} \ll (1+T_{\mathcal{B}})^{A_J},$$

where $\mathbf{w}_{\alpha,\iota}$ consists of independently allowed smooth weights. Repeated Sobolev gives

$$\sum_k \sup_{\mathbf{t}\in\mathcal{B}_T} |P_k(\mathbf{t})|^2 \ll \sum_{\alpha\in\{0,1\}^{\dim \mathcal{B}_T}} \int_{\mathcal{B}_T^+} \sum_k |\partial^\alpha P_k(\mathbf{t})|^2 \, d\mathbf{t}.$$

Insert $\mathcal{W}(q_k/U)$ and apply (11.3) inside each integral. The volume and seminorm loss is $(1+T_{\mathcal{B}})^A$.

Separate the exceptional bounded unit classes. The remaining parameters lie in $\mathcal{U}$. Fix the finite height orders before choosing $T_{\mathcal{B}} = Z^{\tau_{\mathcal{B}}}$ to absorb its polynomial loss into the prescribed power of $U$. □

For $j = 0, 1, 2$ define

$$\begin{aligned}
\widetilde{c} = \frac{2}{3\kappa}, \qquad (\pi_0, \varpi_0) &= (\widetilde{c}x/2, 2 - \widetilde{c}x), \qquad (\pi_2, \varpi_2) = (-\widetilde{c}x/4, 2 + \widetilde{c}x), \\
(\pi_1, \varpi_1) &= \left( x\left(\frac{1}{3} + \frac{5\widetilde{c}}{12}\right), 2 - 2x\left(\frac{1}{3} + \frac{5\widetilde{c}}{12}\right)\right), \\
\mathcal{J}_j &= (5/6 - \delta)(1 - x + \varpi_j) + \delta(1 - x)\varpi_j, \\
t_j = 1 + \frac{\delta(1-x)(1-\pi_j)}{\mathcal{J}_j}, \qquad R_\kappa^{(j)} &= 1 - \delta + (5/6 - \delta)(t_j - 1), \\
m_*^{(j)} = \frac{(1-x)t_j + x - \pi_j}{1 - x + \varpi_j}, \qquad R'_\kappa &= \max\{R_\kappa^{(1)}, \min(R_\kappa^{(0)}, R_\kappa^{(2)})\}.
\end{aligned} \tag{11.5}$$

All denominators are positive on the stated class range.

**Proposition 11.3** (The enlarged class count)**.** *Suppose $\beta_*(F) \le B_{\mathrm{seed}}$ is already proved and*

$$\kappa_\flat \le \kappa_c \le 3/4, \qquad 2B_{\mathrm{seed}} - 1 \le \kappa_c.$$

*For the classes (11.4) and every $\epsilon > 0$,*

$$\#\mathscr{B} \ll U^{R'_{\kappa_c}(\delta,x)+\epsilon}(1+H)^C, \qquad 1 - \delta \le R'_{\kappa_c} \le R_{\kappa_c}. \tag{11.6}$$

*The M2 count exponent is replaced by*

$$A'_S(m) = 1 - 2\delta m - 2gz_S(m) - 2gW_{\mathrm{cap}}(m;\kappa_c)_+, \quad 0 \le m \le 1/2. \tag{11.7}$$

*For $m \le 3/8$ this is the old exponent $A_S(m)$.*

*Proof.* Write $\kappa = \kappa_c$. The premise gives $\beta_*(F) \le (1+\kappa)/2$. Retain the detector, M1 and long inverse count of [Liu26b, Section 13.4]; only M2 changes.

*Strict length margins.* Set aside $r \ge 1 - \epsilon_c$ and $m \ge 1/2 - \epsilon_c$; their estimates are given below after the balancing length is fixed. Choose $0 < \epsilon_1 < \epsilon_c/100$ and a finite grid $\mu_0 = 3/8 + \epsilon_1 < \cdots < \mu_N = 1/2 - \epsilon_c$ of step at most $\epsilon_1^2$. On $[\mu_{i-1}, \mu_i]$, put

$$W_i = (1 - \epsilon_1) \min\left\{ \frac{(1 - c_\kappa)(1 - 2\mu_i)}{6}, \frac{8\mu_{i-1} - 3}{12\kappa} \right\}.$$

For $m_- = \mu_{i-1}$ and $r = \mu_i$,

$$\eta_\kappa \ge \epsilon_1\epsilon_c, \qquad 1 - C_{\mathrm{add}} \ge 4\epsilon_1^2, \qquad 0 \le W_{\mathrm{cap}}(m;\kappa) - W_i \le W_{\max}\epsilon_1 + \epsilon_1^2\left(\frac{1}{3} + \frac{2}{3\kappa}\right).$$

For $m \le 3/8 + \epsilon_1$ retain the old M2. The enlarged range uses

$$z_S(m) + W_i \le \frac{1}{24\kappa} + W_{\max}(\kappa) < 0.081 < 1/5 < \mathscr{L}.$$

*Joint selection.* Fix the presentation and dyadic pair, and order the positive-amplitude original factors by decreasing $g_i$. Every such initial segment satisfies

$$\sum_{i\in I}\lambda_i g_i \ge g\sum_{i\in I}\lambda_i. \tag{11.8}$$

Put $z = z_S(m) - \epsilon_c/100$, $w = W_i$. If the positive length is at least $z+w$, take the shortest initial segment of complete factors of length at least $z+w-\eta_{\mathrm{mesh}}$. Split it into a maximal initial run of length at most $z$, forming $Q_0$, and the remaining factors, forming $Q_W$. The first length exceeds $z-\eta_{\mathrm{mesh}}$; the second is at most $w+\eta_{\mathrm{mesh}}$. Choose $\eta_{\mathrm{mesh}}$ after the grid margins, with $6\eta_{\mathrm{mesh}} < \epsilon_1\epsilon_c$ and $6\kappa\eta_{\mathrm{mesh}} < 4\epsilon_1^2$, so Lemma 11.1 applies with $W_i+\eta_{\mathrm{mesh}}$. If the positive length is smaller than $z+w$, retain all positive factors and make the same split. Their total amplitude is $g\mathscr{L} \ge g(z+w)$, and their lengths still fit. In either case,

$$|Q_0Q_W|^2 \ge U^{2g(z+w)-O(\eta_{\mathrm{mesh}})-o(1)}.$$

This lower bound concerns the joint product; it requires no separate lower mean for $Q_W$.

The detector gives $|S_m|^4 \ge U^{2\delta m-\epsilon_2}$. Let $\mathscr{B}_{D,N}\subseteq\mathscr{B}$ be the part detected with the fixed presentation and dyadic pair $(D,N)$. Choose the products before the common supremum and apply Lemma 11.2:

$$\#\mathscr{B}_{D,N}\,U^{2\delta m+2g(z+w)-O(\eta_{\mathrm{mesh}})-\epsilon_2} \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C(1+T_{\mathcal{B}})^A.$$

Sum over the logarithmic number of pairs and absorb the height, grid and mesh losses to obtain (11.7).

*Balancing the counts.* Put $f_j(m) = 1-\delta(\pi_j+\varpi_j m)$. Then

$$f_0 = A_S,\quad f_1 = A_S - 2gW_1,\quad f_2 = A_S - 2gW_2,\qquad A_S' = \max\{f_1, \min(f_0,f_2)\}.$$

The successive branches on $m\le 3/8$, $3/8\le m\le m^\sharp$ and $m\ge m^\sharp$ are $0,2,1$. Since $\widetilde{c}\le 12/13$, $\varpi_j>1$, and each branch decreases with $m$. For fixed $1\le t\le 3/2$, use M1 when $r\ge r_*(t)$ and M2 when $r\le r_*(t)$, where

$$A_I(r) = 1-\delta\{x+(1-x)r\},\qquad A_I(r_*) = A_S'(t-r_*).$$

The detector constraint $r+m\ge t-O(1/\log U)$ justifies M2 at $m=t-r_*$. For one branch, crossing also with the long count gives

$$r_* = \frac{\pi_j+\varpi_j t - x}{1-x+\varpi_j},\qquad A_I(r_*) = 1-\delta+(5/6-\delta)(t-1).$$

The solution is $t=t_j$, $m=m_*^{(j)}$ and count $R_\kappa^{(j)}$. For fixed $t$, let $C_j(t)$ be this short-count crossing value and put

$$C(t) = \max\{C_1(t), \min(C_0(t), C_2(t))\},\qquad L(t) = 1-\delta+(5/6-\delta)(t-1).$$

Each $C_j$ decreases strictly with $t$. Crossing the decreasing $A_I(r)$ with the increasing $f_j(t-r)$ preserves maxima and minima of the crossing values; hence $C(t)$ is the short-count bound. For $0<\delta<5/6$, $L$ increases strictly. Its crossing with $C$ is at

$$t_* = \max\{t_1, \min(t_0,t_2)\},\qquad L(t_*) = \max\{R_\kappa^{(1)}, \min(R_\kappa^{(0)}, R_\kappa^{(2)})\} = R_\kappa'.$$

Indeed, the sign of $C_j(t)-L(t)$ is the sign of $t_j-t$.

The lengths are allowed:

$$0\le t_j-1\le\frac{1-\pi_j}{\varpi_j}\le\frac{1}{2}.$$

At fixed $t$, $A'_S \le A_S$ moves the crossing to larger $r$, and the old crossing increases with $t$. At $t = 1$, with $\widetilde{c} \ge 2/3$ and $x \le 1/2$,

$$r_*^{(0)}(1) = \frac{2 - x - \widetilde{c}x/2}{3 - x - \widetilde{c}x} \ge \frac{2 - 4x/3}{3 - 5x/3} \ge \frac{8}{13},$$

since the left quotient increases with $\widetilde{c}$ and the middle one decreases with $x$. Thus $r_* \ge 8/13$, and the M1 prime length is at most $(1 - r_*)/2 \le 5/26$, as in the proof of Proposition 6.4. When branch 0 is active, the old crossing has $m_* \ge 1/3$ and $z_S(m_*) \le 1/13$; otherwise $m_* \ge 3/8$ and the 0.081 bound applies. Fix the detector with this $t_*$. The long inverse moment, with $e(r) = \max\{1, (1 + 5r)/6\}$, gives

$$\begin{aligned} r \ge 1: \quad e(r) - \delta r &= 1/6 + (5/6 - \delta)r \\ &\le 1/6 + (5/6 - \delta)t_* + o(1) = R'_\kappa + o(1), \end{aligned}$$

since $r \le t_* + O(1/\log U)$. For the two endpoint strips the unamplified estimates give

$$\begin{aligned} 1 - \epsilon_c \le r < 1: &\quad 1 - \delta r \le 1 - \delta + \delta\epsilon_c \le R'_\kappa + \delta\epsilon_c, \\ m \ge 1/2 - \epsilon_c: &\quad 1 - 2\delta m \le 1 - \delta + 2\delta\epsilon_c \le R'_\kappa + 2\delta\epsilon_c. \end{aligned}$$

Thus all three ranges satisfy the required count. Finally $R_\kappa^{(0)} = R_\kappa$, and $A'_S \le A_S$ gives $R'_\kappa \le R_\kappa$. At $x = 0$ or $\delta = 5/6$, the three $R^{(j)}$ coincide, so the old count gives the conclusion directly.

Fix $\kappa$ and the requested loss, then $\epsilon_c, \epsilon_1$, the grid and its strict margins, and then the prime mesh. Choose the detector buffer, finite internal orders and external height exponent in the order of Lemma 11.2. All losses are absorbed into $\epsilon$. □

### 11.3 The new comparison and its endpoint

In this subsection, we combine the improved count with the numerator estimate and determine $B'$. Appendix B verifies the identities and signs in Proposition 11.4.

Use $R'_{\kappa_c}$ in the comparison, independently of the moment parameter $\kappa_m$. With the notation of (10.8), put

$$\begin{aligned} E' &= C_0 + \frac{\delta}{2} + \delta x \left\{ \ell + c\left(\frac{h}{2} - l_y\right)\right\} - \frac{3h}{4}(1 - R'_{\kappa_c}), \\ \mathcal{Q}' &= E' - \frac{1 - c}{12} a(\delta, x)(2l_y - h), \qquad c = \frac{1}{6\kappa_m}. \end{aligned} \tag{11.9}$$

Write $E^{(j)}, \mathcal{Q}^{(j)}$ for these expressions with $R'$ replaced by $R^{(j)}$. Since $3h/4 > 0$,

$$\mathcal{Q}' = \max\{\mathcal{Q}^{(1)}, \min(\mathcal{Q}^{(0)}, \mathcal{Q}^{(2)})\} \le \max\{\mathcal{Q}^{(1)}, \mathcal{Q}^{(2)}\}. \tag{11.10}$$

The same identity holds for $E'$. The two uses of the enlarged norm combine by

$$\sum_{k \in \mathscr{B}} |N_k| \le (\#\mathscr{B})^{3/4} \left( \sum_{k \in \mathscr{B}} |N_k|^4 \right)^{1/4}.$$

The counting and numerator products are chosen separately, each from the original factors and within its own length inequalities.

**Proposition 11.4** (The enlarged-count endpoint). *Let $\delta'$ be the unique root in $(11/25, 23/50)$ of*

$$q_{\mathrm{ec}}(\delta) = 29562\delta^4 - 277427\delta^3 + 836708\delta^2 - 930160\delta + 272800. \tag{11.11}$$

*Put*

$$\begin{aligned}\ell' &= -\frac{142789387}{222821985}\delta'^3 + \frac{6250132787}{1336931910}\delta'^2 - \frac{1789275793}{222821985}\delta' + \frac{386361163}{133693191},\\ \kappa' &= \frac{5}{6} - \frac{\ell'}{2} = 2B' - 1, \qquad B' = \frac{11}{12} - \frac{\ell'}{4}.\end{aligned} \tag{11.12}$$

*For $\kappa_c = \kappa_m = 5/6 - \ell/2$ and $x = 1/2$, set*

$$\begin{aligned}D_2 &= -48(3\ell - 5)(54\delta\ell - 102\delta - 75\ell + 145),\\ D_1 &= -24(3\ell - 5)(144\delta\ell - 210\delta - 195\ell + 275),\\ p_b &= (2106\ell^2 - 7812\ell + 7230)\delta^2 + (-4761\ell^2 + 17538\ell - 16105)\delta\\ &\quad + 1800\ell^2 - 6480\ell + 5800,\\ p_0 &= (1134\ell^3 - 4626\ell^2 + 7302\ell - 5010)\delta^2\\ &\quad + (-12915\ell^3 + 48309\ell^2 - 51097\ell + 11495)\delta\\ &\quad + 13500\ell^3 - 51300\ell^2 + 53220\ell - 8700,\\ b' &= -\frac{\partial_\delta p_0}{\partial_\delta p_b}(\delta', \ell').\end{aligned}$$

*Then $D_2\mathcal{Q}^{(2)}(\delta, 1/2) = bp_b + p_0$,*

$$\mathrm{Res}_\ell(p_b, p_0) = -8398080\delta^2(2\delta - 5)(3\delta - 5)(6\delta - 5)^2 q_{\mathrm{ec}}(\delta),$$

*where the factors other than $q_{\mathrm{ec}}$ have no root in $(11/25, 23/50)$, and*

$$6687\ell'^4 - 25401\ell'^3 + 23297\ell'^2 + 2045\ell' - 900 = 0. \tag{11.13}$$

*The number $B'$ is the root specified in (1.2). Numerically,*

$$\begin{aligned}\delta' &= 0.44878702795803278950\ldots, & b' &= 0.08218969571904572506\ldots,\\ \ell' &= 0.16978166469505833354\ldots, & \kappa' &= 0.74844250098580416656\ldots,\\ B' &= 0.87422125049290208328\ldots, & B_\natural - B' &= 0.00024624600793818977\ldots.\end{aligned} \tag{11.14}$$

*At $(b', \ell', \kappa', \kappa')$,*

$$\mathcal{Q}'(\delta, 1/2) \le 0 \quad (7/20 \le \delta \le 3/5), \qquad \mathcal{Q}'(\delta, 1/2) = 0 \iff \delta = \delta'.$$

*Proof.* The numerator identity and resultant follow by rational polynomial arithmetic. Substitution of the cubic (11.12) gives $p_b(\delta', \ell') = p_0(\delta', \ell') = 0$ modulo $q_{\mathrm{ec}}$. The denominator defining $b'$ is nonzero. Hence the quadratic $b'p_b(\delta, \ell') + p_0(\delta, \ell')$ has a double zero at $\delta'$:

$$\mathcal{Q}^{(2)}(\delta, 1/2) = \frac{A'(\delta - \delta')^2}{D_2(\delta, \ell')}, \qquad -3407.843 < A' < -3407.842. \tag{11.15}$$

Both $D_i(\delta, \ell')$ are positive on $[7/20, 3/5]$. The quadratic $D_1\mathcal{Q}^{(1)}(\delta, 1/2) = A_1\delta^2 + B_1\delta + C_1$ satisfies

$$-2923.884 < A_1 < -2923.882, \qquad -906.675 < B_1^2 - 4A_1C_1 < -906.672 < 0.$$

Thus $\mathcal{Q}^{(1)} < 0$, $\mathcal{Q}^{(2)} \le 0$, and (11.10) applies. At $\delta = \delta'$, rational interval evaluation gives

$$R^{(1)} < R^{(2)} < R^{(0)}, \qquad 0.40249693 < m_*^{(2)} < 0.40249694 < m^\sharp.$$

Therefore $R' = R^{(2)}$ and equality holds there. Exact root isolation and the sign certificates are given in Appendix B. They also verify that the polynomials defining $\ell'$ and $B'$ are irreducible of degree four. □

## 11.4 The strict comparison inequalities

In this subsection, we verify strict inequalities for parameters approaching the endpoint of Proposition 11.4. Section 11.5 uses these inequalities at each finite iteration.

For the final iteration use

$$\begin{gathered}\kappa_c, \kappa_m \in [\kappa', \kappa_\natural], \qquad \ell \in [\ell' - (\kappa_\natural - \kappa')/25, \ell'], \qquad b = b', \\ \mathcal{R}_* = [7/20, 3/5] \times [2/5, 1/2], \qquad \mathcal{G} = [1/8, 5/6] \times [0, 1/2], \\ \mathcal{C}_1 = [1/8, 5/6] \times [0, 2/5], \quad \mathcal{C}_2 = [1/8, 7/20] \times [2/5, 1/2], \quad \mathcal{C}_3 = [3/5, 5/6] \times [2/5, 1/2].\end{gathered} \tag{11.16}$$

Call the parameter family in the first line $\mathcal{P}$, and write $\mathcal{P}_{3/4}$ when $\kappa_m$ is allowed up to $3/4$. Put

$$\begin{gathered}R = R'_{\kappa_c}, \quad \theta_0 = \frac{\delta x}{6\kappa_m}, \quad S = \frac{3R+1}{4} + \frac{\theta_0}{2} - \frac{17}{50}, \quad N = \frac{3R+1}{4} - \frac{\theta_0}{2}, \\ \lambda = \frac{l_y}{h}, \quad m_\theta = 1 - \lambda, \quad \Gamma_N = \frac{1-c}{12} a(2\lambda - 1), \quad \Gamma_L = \theta_0(5/8 - \lambda), \\ \gamma_{\max} = \frac{aW_{\max}(\kappa_m)}{2}, \qquad \Delta_{\text{short}} = R + (1/2 - \theta_0)\frac{3}{40} - N.\end{gathered} \tag{11.17}$$

**Lemma 11.5** (The comparison margins)**.** *The inequalities in Tables 6 and 7 hold on the indicated domains. At the endpoint parameters,*

$$\mathcal{Q}'_{\kappa',\kappa',b',\ell'}(\delta, x) \le 0 \quad \text{on } [7/20, 3/5] \times [0, 1/2], \tag{11.18}$$

*with equality only at* $(\delta', 1/2)$*. Along*

$$\kappa_c = \kappa_m = \kappa, \qquad \ell(\kappa) = \ell' - \frac{\kappa - \kappa'}{25}, \qquad b = b', \tag{11.19}$$

*one has, for* $\kappa' \le \kappa \le \kappa_\natural$*,*

$$\mathcal{Q}'_\kappa(\delta, x) \le -\frac{\kappa - \kappa'}{100} \quad \text{on } \mathcal{R}_*. \tag{11.20}$$

*Proof.* The rational interval calculation of Appendix B proves the table entries on enclosing rational boxes, treating the three rational branches separately. At the endpoint, $\partial_x \mathcal{Q}^{(j)} > 0.009$ for every $j$. Proposition 11.4 and (11.10) therefore give (11.18). On the parameter family,

$$\partial_{\kappa_c} \mathcal{Q}^{(j)} + \partial_{\kappa_m} \mathcal{Q}^{(j)} - \frac{1}{25} \partial_\ell \mathcal{Q}^{(j)} < -\frac{1}{100}.$$

For $\kappa_2 > \kappa_1$ in the family,

$$\mathcal{Q}^{(j)}_{\kappa_2} - \mathcal{Q}^{(j)}_{\kappa_1} \le -\frac{\kappa_2 - \kappa_1}{100} \quad \Longrightarrow \quad \mathcal{Q}'_{\kappa_2} - \mathcal{Q}'_{\kappa_1} \le -\frac{\kappa_2 - \kappa_1}{100}.$$

The implication follows from (11.10); setting $\kappa_1 = \kappa'$ proves (11.20), including branch changes. □

Table 6: Comparison inequalities with the enlarged count. Unless otherwise specified, the class domain is $\mathcal{R}_*$ and the parameter family is $\mathcal{P}$. Endpoint means $(\kappa_c, \kappa_m, b, \ell) = (\kappa', \kappa', b', \ell')$.

| condition | certified bound |
|---|---|
| $\partial_x \mathcal{Q}^{(j)}$, $j = 0, 1, 2$, endpoint, $[7/20, 3/5] \times [0, 1/2]$ | $> 0.009$ |
| $\partial_{\kappa_c} \mathcal{Q}^{(j)} + \partial_{\kappa_m} \mathcal{Q}^{(j)} - \partial_\ell \mathcal{Q}^{(j)}/25$ | $< -0.011 < -1/100$ |
| $\partial_{\kappa_c} \mathcal{Q}^{(j)}$; $\partial_{\kappa_m} \mathcal{Q}^{(j)}$; $\partial_\ell \mathcal{Q}^{(j)}$ | $> 0$; $> 0$; $> 3/4$ |
| $\mathcal{J}_j$ on $\mathcal{G} \times \mathcal{P}_{3/4}$ | $> 0.08$ |
| $E'$ on $\mathcal{G} \times \mathcal{P}_{3/4}$ | $< 1/500$ |
| $E'$ on each $\mathcal{C}_i \times \mathcal{P}_{3/4}$ | $< -10^{-5}$ |
| $S$ on $\mathcal{R}_*$; $S$ on $\mathcal{G} \times \mathcal{P}_{3/4}$ | $> 1/5$; $< 1$ |
| $(6\kappa_m - 2\delta x)(21/100) - R$ on $\mathcal{G} \times \mathcal{P}_{3/4}$ | $> 0.019$ |
| $(21/100)(6\kappa_m - \delta) - (1 - 2\delta/3)$, $1/8 \le \delta \le 5/6$ | $> 0.00012$ |
| $a$; $a/(3\kappa_m) - \theta_0$; $\delta/2 - \theta_0 - a(1-c)/6$ | $> 0.127$; $> 0.0127$; $> 0.105$ |
| $\Gamma_L - \Gamma_N$ | $> 0.000033$ |
| $(\delta/2 - \theta_0)(\lambda - 11/20) - \Gamma_N$ | $> 0.00116$ |
| $2x(21/100) - z_L - W_{\max}$; $\gamma_{\max}$ | $> 0.08723$; $< 1/250$ |
| $N - I_{\mathrm{short}} = -\Delta_{\mathrm{short}}$ | $> 0.0133 > 23/2700$ |
| $E' + h\Delta_{\mathrm{short}}$; $S + \Delta_{\mathrm{short}}$ | $< -0.0027$; $> 0$ |
| $S - 37\theta_0/40 - \gamma_{\max}$; $S - \gamma_{\max}$; $S + \delta/2 - \theta_0$ | $> 1/8$; $> 1/8$; $< 2$ |

Table 7: Geometric and complementary inequalities on $\mathcal{P}$. Here $Q_{\mathrm{sm}} = \ell(1/2 + 2\ell/h) - 11h/54$.

| condition | bound |
|---|---|
| $3/8 < m^\sharp < m_\theta$; $25/62 < m_\theta < 9/20$ | both hold |
| $57/100 < \lambda < 11/19 < 59/100$ | holds |
| $z_0(3/40), z_0(1/10), z_L < 21/100$ | largest $< 0.18929$ |
| $6\kappa_m \ell > 3/4$; $(9/2)\ell > 3/4$ | both hold |
| $l_x > \ell$; $l_y - \ell - 11b/6 > 0$; $l_y < 1/2$; $h < 2l_y$ | all hold |
| $\ell/(h + 1/2000) > 21/100$ | $> 0.21320$ |
| $\ell/(h - 1/80) < 2/9$ | $< 0.21677$ |
| $\ell/h < 1/(6\kappa_m)$ | gap $< -0.00918$ |
| $Q_{\mathrm{sm}} < 0$; $C_0 + (1-h)/16 < -1/250$ | both hold |
| $C_0 + (1/2 + 3\ell/2)/50 < -1/125$ | $< -0.00898$ |
| $13h/75 - l_y/2 + 1/50 < -1/20$ | $< -0.07016$ |
| $b > 0$; $1/6 < \ell < 1/5$; $B(\ell) > 437/500$ | all hold |

### 11.5 Iteration and proof of the enlarged-count theorem

In this subsection, we iterate the inequalities of Section 11.4 from $B_\natural$ to $B'$. The descent of Theorem 9.2 then gives Theorem 1.3.

**Proposition 11.6** (The improved boundary over fields containing $\mu_{12}$)**.** *For every number field $F \supset \mu_{12}$, one has $\beta_*(F) \le B'$.*

*Proof.* Fix $F$. Proposition 10.5 supplies $B_0 = B_\natural$, $\kappa_0 = 2B_0 - 1 = \kappa_\natural$. The field verification in Proposition 9.1 preserves the arithmetic families of Lemmas 11.1 and 11.2.

Suppose $\beta_*(F) \le B_n$ and $\kappa_n = 2B_n - 1 \in (\kappa', \kappa_\natural]$. Then Proposition 11.3 and Lemma 11.1 apply with $\kappa_c = \kappa_m = \kappa_n$. Put $b = b'$ and $\ell = \ell(\kappa_n)$ as in (11.19). Lemma 11.5 gives

$$\mathcal{Q}'_{\kappa_n} \le -\nu_n \quad \text{on } \mathcal{R}_*, \qquad \nu_n = \frac{\kappa_n - \kappa'}{100} > 0.$$

Apply Step 3 of Proposition 10.5 with the count of Proposition 11.3, the norm of Lemma 11.1 and Tables 6–7. The parameter sets are $\mathcal{P}$ and $\mathcal{P}_{3/4}$, and the strict comparison margin is (11.20). On $\mathcal{R}_* \times \mathcal{P}$, use $\theta_0 \le 3/(60\kappa_m) < 0.06681$. Its uses in Corollary 6.7, Proposition 7.4 and the slope display of Theorem 8.4 are the arithmetic parameters $S < 1$, $S - 37\theta_0/40 - \gamma_{\max} > 1/8$ and $S - \gamma_{\max} > 1/8$ of Table 6. On the selected numerator lengths set

$$m_a = 3/8 + \varepsilon, \quad m_b = 9/20, \qquad W(m) = (1-\varepsilon)W_{\mathrm{cap}}(m; \kappa_n).$$

The norm margins satisfy

$$(1-c)(1-2m) - 6W(m) \ge (1-c)\varepsilon/10 > 0, \qquad 1 - C_{\mathrm{add}} \ge 4\varepsilon^2 > 0.$$

Tables 6–7 justify the prime supply, both contour slopes, short sums, complementary classes, lowest amplitudes, principal remainder and direct comparison. For $\delta < 1/8$ retain the earlier comparison with $R'_{\kappa_c} \le R_{\kappa_c} \le R_{3/4}$. Choose the count loss and all remaining losses so that

$$C\varepsilon + 2\zeta + \epsilon_{\mathrm{rem}} < \nu_n, \qquad \ell/(h+\zeta) > 21/100.$$

The same continuation argument gives

$$\beta_*(F) \le B_{n+1} = \frac{11}{12} - \frac{\ell(\kappa_n)}{4} = B' + \frac{\kappa_n - \kappa'}{100}.$$

Consequently

$$\kappa_{n+1} - \kappa' = \frac{\kappa_n - \kappa'}{50}, \qquad B_n - B' = \frac{B_\natural - B'}{50^n}.$$

Hence $\kappa_n \in (\kappa', \kappa_\natural]$ for every $n$, and the step applies at every $n$. Every finite step uses a previously proved bound, with its own positive margins and threshold. Hence $\beta_*(F) \le \inf_n B_n = B'$. □

*Proof of Theorem 1.3.* Use the descent of Theorem 9.2 with Proposition 11.6. For a primitive finite-order character $\chi$ of $K$, let $M/K$ be its cyclic class-field extension, $F = K(\zeta_{12})$, and $E = MF$. The abelian factorizations give, for $\Re s > B'$ and $s \ne 1$,

$$\zeta_E(s) \ne 0 \quad \Longrightarrow \quad \zeta_M(s) \ne 0 \quad \Longrightarrow \quad L_K(s, \chi) \ne 0.$$

Nonprincipal factors are nonzero at $s = 1$; the principal pole is permitted. Deleted Euler factors of an imprimitive character have zeros only on $\Re s = 0$, as in the same descent proof. □

*Remark* 11.7 (Scope of the earlier optimality theorem). Theorem 10.8 fixes the count of Proposition 6.4. Proposition 11.3 replaces that input, so $B' < B_\natural$ is consistent with its conclusion. No optimality assertion for this intermediate enlarged comparison system is needed for Theorem 1.3. Section 16 treats the larger exhaustion-norm comparison separately.

# Part II
# Prime exhaustion and variable comparison rectangles

## 12 An enlarged product norm with a variable comparison rectangle

In this section, we prove Theorem 12.1 with comparison lengths $(t, 2m-t)$. Lemma 12.2 treats the empty prime product. Section 13 applies the resulting norm to zero detection and the numerator.

### 12.1 The moment estimates and the norm statement

In this subsection, we state the length conditions for Theorem 12.1, using the arithmetic estimates of Section 2. Lemma 12.2 gives the length-restoration estimate used in its proof.

Retain Section 2 and the moment, reflection, full Gauss, correlation and fixed-mean formulas of [Liu26b, Lemmas 12.3, 12.5–12.10, Theorem 12.16, Sections 12.7–12.8 and Appendix C.4]. At $z = 0$, (10.5) permits arbitrary bounded ordinary lengths, with principal inducing characters excluded. On parameters inducing outside $\Theta$, a prime product of length $z$ satisfies

$$|Q(k)|^2 \ll U^{\kappa z+\epsilon}.$$

Fixed-mean cancellation requires equal products of actual scales, the same character, coprimality conditions and explicit Mellin twist, and the two respective smooth weights.

Use Theorem 3.7, Corollary 3.6, the common-support inequality 3.5, and Propositions 4.6 and 4.8, under the length conditions below. Retain every moving radical, natural zero and polynomial norm-twist height factor.

Let $S_{v,i}(k)$ be the centrally normalized ordinary polynomial of length $U^v$ and weight $W_i$, in the common arithmetic parametrization. Let $Q_0, Q_W$ be products of distinct complete original prime factors on pairwise disjoint original prime ranges, with

$$\mathrm{length}_U(Q_0) = z, \qquad \mathrm{length}_U(Q_W) = w, \qquad \mathrm{supp}_{\mathrm{pr}}(Q_0) \cap \mathrm{supp}_{\mathrm{pr}}(Q_W) = \varnothing.$$

Here $\mathrm{supp}_{\mathrm{pr}}$ denotes the original prime ranges. Their finite-character coefficients belong to the original fixed coefficient family. Fix the factors and smooth weights before summation, allowing the required finite derivative families with one common explicit ideal-norm Mellin twist. Let $\mathcal{U}$ be the original nonunit sixth-power-free family inducing outside $\Theta$. Centered norms use the positive extension to the unrestricted arithmetic sum of [Liu26b, Lemma 12.1]; uncentered norms use $\mathcal{U}$.

**Theorem 12.1** (Enlarged norm with variable comparison)**.** *Assume $p = 6\kappa \in [112/25, 9/2]$ and $\beta_*(F) \le (1+\kappa)/2$. Fix a compact set of parameters*

$$0 < t \le \min(m_-, 1/4), \qquad m_- \le r < 1/2, \qquad 0 \le W \le 1/25,$$

*with uniform strict margins in*

$$A_{\max} + 4W < 1, \qquad A_{\max} = 2r + \frac{1-2r}{p} + W, \tag{12.1}$$

$$A_{\max} + 3W < \frac{5}{6} + \frac{2t}{3}, \tag{12.2}$$

$$2 + 2t - 4m_- + pW < 1. \tag{12.3}$$

*For each $\epsilon > 0$, require $\eta_{\mathrm{mesh}} > 0$ to be small in terms of the uniform strict margins, the lower bound for $t$ and the aggregate numerical length and modulus bounds; this requirement is fixed before the arithmetic data and the number of prime factors. Require*

$$\max_i \lambda_i \le \eta_{\mathrm{mesh}}, \qquad m \in [m_-, r], \qquad 0 \le z \le \frac{1-2m}{p}, \qquad 0 \le w \le W,$$

*where $\lambda_i$ are the individual original prime-factor widths. Put*

$$P = S_{m,1} S_{m,2} Q_0, \qquad P_{\mathrm{cmp}} = S_{t,1} S_{2m-t,2} Q_0, \qquad P_c = P - P_{\mathrm{cmp}}.$$

*Require exact equality of the products of the actual scales in each comparison. Then, for each required finite family of independently chosen smooth weights and their required derivatives,*

$$\sum_k \mathcal{W}(q_k/U) |P_c(k) Q_W(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C, \tag{12.4}$$

$$\sum_{k \in \mathcal{U}} \mathcal{W}(q_k/U) |P(k) Q_W(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C. \tag{12.5}$$

*Constants and thresholds have the same dependence on the fixed field, arithmetic data and finite family of smooth weights as the cited estimates. They are uniform on the indicated compact parameter set and in subsequent extracted ideals and Fourier labels.*

The hypotheses give

$$\kappa \ge \frac{56}{75} > \frac{13}{18}, \qquad p - 1 \ge \frac{87}{25} > 0, \qquad \inf t > 0,$$

and the direct terminal ceilings in the proof do not depend on $p$.

## 12.2 Restoration when the prime list is exhausted

In this subsection, we prove Lemma 12.2 by removing complete prime factors and using the ordinary case of (10.5) when none remain. The resulting estimate is used in the proof of Theorem 12.1.

**Lemma 12.2** (Removing all remaining prime factors)**.** *Let $P' = S_1 S_2 Q'$ be a product obtained from the families of Theorem 12.1 after the second correlation and complete genuine and final-divisor extraction. Here $S_1, S_2$ are ordinary annular sums and $Q'$ is the surviving prime product, with $\mathrm{length}_U(Q') = z'$. Retain the original coefficient measures, full moving radical and inducing character outside $\Theta$. Write $M_{\mathrm{new}} = m_{\mathrm{new}} + q_{\mathrm{new}}$, where $q_{h'} \ll U^{m_{\mathrm{new}}}$ is the buffered parameter range and $U^{q_{\mathrm{new}}}$ bounds the full displayed moving radical. Let $n_1, n_2$ be the declared lengths of $S_1, S_2$, and let*

$$D = n_1 + n_2 + pz' - M_{\mathrm{new}}, \qquad p = 6\kappa, \tag{12.6}$$

*be the M2 length excess. Suppose every original prime factor has length at most $\eta_{\mathrm{mesh}}$. Let $\mathcal{M}_2(P')$ denote the inner transformed squared sum over these parameters inducing outside $\Theta$, with the original positive coefficient measure and smooth tests. Then, for every $\epsilon > 0$,*

$$\mathcal{M}_2(P') \ll U^{M_{\mathrm{new}} + D_+/6 + \kappa\eta_{\mathrm{mesh}} + \epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C,$$

*where $\mathcal{W}$ denotes the retained smooth tests. No condition $D \le pz'$ is required.*

*Proof.* Treat the two one-sided positive Gauss means with their respective presentations, after the second correlation and complete genuine and last-divisor allocations. For fixed extracted data put

$$F_* = G_c V_{\mathrm{id}}, \qquad \psi_{h'}(n) = \tau_2(n) \chi_n(F_* h').$$

By [Liu26b, Lemmas 12.2 and 12.8, Section 12.7.3 and Appendix C.4], each comparison rectangle has two ordinary annular factors and the surviving complete prime annuli, with the common character $\psi_{h'}$ and one explicit ideal-norm Mellin twist. Let $c_{\mathrm{corr}}(h')$ be the normalized second-correlation scalar and $c_{\mathrm{ext}}(h')$ the product of extracted character values for the current allocation. On each part of the retained decomposition,

$$|c_{\mathrm{corr}}(h')| \le 1, \qquad |c_{\mathrm{ext}}(h')| \le 1.$$

Their natural zeros remain. Last-divisor extraction divides corresponding scales by the same exact norm, preserving unrestricted quotients, original annuli and finite-character prime coefficients. No Gauss coefficient remains on the ordinary factors.

Expand an additional fixed radical by [Liu26b, Lemma 12.5], keeping the full moving radical. Only after forming these ordinary products, split into

$$\begin{array}{l|l} \chi_{\mathrm{ind}} \in \Theta & \text{matching fixed-character means} \\ \chi_{\mathrm{ind}} \notin \Theta & \text{M2 and the factor removal below.} \end{array}$$

The lengths in (12.6) are declared after the chosen support buffers. For $D \le 0$, apply the appropriate clause of M2 without stripping. For $D > 0$, remove complete factors until their total length $v$ reaches $D/p$ or the list is exhausted. Then

$$0 \le v \le \min\{z', D/p + \eta_{\mathrm{mesh}}\}, \qquad pv \ge D \quad \text{or} \quad v = z', \qquad \kappa v \le D/6 + \kappa\eta_{\mathrm{mesh}}.$$

Put $z_{\mathrm{rem}} = z' - v$. The two clauses of [Liu26b, Theorem 12.16] give

| | length condition | moment estimate |
|---|---|---|
| $z_{\mathrm{rem}} > 0$ | $D - pv \le 0$ | prime-bearing M2 |
| $z_{\mathrm{rem}} = 0$ | actual bounded ordinary lengths | no-prime M2. |

Apply M2 separately to the two ordinary rectangles after genuine and final-divisor extraction. Their inducing character lies outside $\Theta \supset \{1\}$; all displayed radicals and natural zeros remain. The squared loss is at most $D_+/6 + \kappa\eta_{\mathrm{mesh}}$, and the subsidiary pointwise and M2 losses are chosen inside the final $\epsilon$ of Theorem 12.1. □

## 12.3 Proof of the enlarged norm

In this subsection, we prove Theorem 12.1 from the local identities of Section 3 and Lemma 12.2. Section 13 applies the resulting norm estimate.

*Proof of Theorem 12.1. First transform and exact local families.* Use the positive extension of [Liu26b, Lemma 12.1] and apply the first Poisson formula of [Liu26b, Section 12.7.1] to the complete product coefficients, with their common Mellin twist. The ordinary factors remain together. The comparison has

$$(v_1, v_2) \in \{(m, m), (t, 2m - t)\}, \qquad N_1 N_2 = N_1' N_2', \qquad A_t = 2m + z + w,$$

where $N_i, N_i'$ are the actual ordinary scales. At the first dual origin, the powerful-product condition gives $O(U^{A_t+\epsilon})$ coefficient pairs; paired normalization cancels this count. The origin is therefore $O(U^{1+\epsilon})$, independently of the particular comparison short length $t$.

For nonzero frequencies retain each original $C, D, s$, finite component and common Fourier coordinate. Apply the exact local classification before any ordinary moment: first-common extra width $w_f$; fixed unit and nonunit widths $f_0, f_1$; and free unit, nonunit and correction widths $g_{\mathrm{good}}, v, c_{\mathrm{cor}}$. Write

$$u_{\mathrm{free}} = g_{\mathrm{good}} + v + c_{\mathrm{cor}}, \qquad w = w_f + f_0 + f_1 + u_{\mathrm{free}}, \qquad t_{\mathrm{loc}} = f_0 + u_{\mathrm{free}}.$$

The width $u_{\mathrm{free}}$ is distinct from the common-support exponent $R + E_0$.

The ordinary power removals satisfy $L_1 \geq f_1 + 4T_1$ and $L \geq v + 4T$; $S_1, S, T_1, T \geq 0$. The exact identities and natural-zero branches are those of the cited local classification. In particular divisor Cauchy for nonunit choices is applied at the original frequency before any support-dependent division; the resulting squared coefficient energy is retained, without a new unrestricted count of those labels. Correction coefficients keep their absolute $q_p^{-1}$ mass. Unit choices use one common polynomial and divisor-bounded fibers. These operations preserve exact product scales and the same two smooth weights in their respective columns.

For the completed Gauss polynomial let $A = A_*, M = M_*, q = q_*$ and $m_* = M - q$. The identities are

$$\begin{aligned} A &= A_t - f_1 - t_{\mathrm{loc}} - L_1 - L - 5c_{\mathrm{cor}}, \\ M &= 1 - 6t_{\mathrm{loc}} - f_1 - L_1 - L - 5c_{\mathrm{cor}} + T_1 + 5v + T, \\ q &= f_1 + v + c_{\mathrm{cor}}, \\ \mathcal{R}_{\mathrm{loc}} &= -f_0 - L_1 + T_1 - S_1 - u_{\mathrm{free}} - L + v - 4c_{\mathrm{cor}} + T - S. \end{aligned}$$

Here $\mathcal{R}_{\mathrm{loc}}$ includes the change from the completed one-sided allowance to the original one-sided allowance. It is not an additional free label count. None of these identities depends on $p$ or on the chosen comparison short length $t$.

*Homogeneous restoration.* Set $e_z = 1 - 2m - pz \geq 0$ and let $z_{C,\mathrm{old}}$ be the initial first-common prime width. The completed defect is

$$D_f = 6w - e_z - (p-1)z_{C,\mathrm{old}} - 5w_f - 5f_1 - T_1 - 5v - T.$$

The exact common-support inequality lowers it by $\mu = \min(c_C, d_D) \geq w_f$ on actual scales. With $e_1 = L_1 - f_1 - 4T_1 \geq 0$ and $e = L - v - 4T \geq 0$, put

$$\mathcal{C}_{\mathrm{hom}} = -f_0 - f_1 - u_{\mathrm{free}} - 3(T_1 + T) - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e.$$

Direct substitution gives

$$\begin{aligned} \mathcal{C}_{\mathrm{hom}} + \frac{D_f - \mu}{6} = \frac{w_f - \mu}{6} - \frac{5}{6}(f_1 + v) - \frac{19}{6}(T_1 + T) \\ - \frac{p-1}{6} z_{C,\mathrm{old}} - \frac{e_z}{6} - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e \leq 0. \end{aligned}$$

Also $\mathcal{C}_{\mathrm{hom}} \leq 0$, so $\mathcal{C}_{\mathrm{hom}} + (D_f - \mu)_+/6 \leq 0$. After the second Poisson transform, genuine and final-divisor fixing only reduce the actual length defect. Lemma 12.2 pays for any remaining positive defect even at exhaustion. The original sixth-power amplifier contribution is still absorbed once by its allowance $\Delta' \geq a + \ell_a$. Its auxiliary step and mesh losses are included in $\epsilon$. No additional lower bound on the remaining prime-product length is needed for this argument.

*A direct branch for small global completed length.* Let $c = c_C$, $d = d_D$, $u = R + E_0$ and $B_C = (3c - 5d - R)_+/6$. The actual Gauss polynomial has coefficient scale $\alpha = A - c$, inner divisor length $s_0'$, and cutoff

$$K_0 = 2A - c - d + u - m_*.$$

Nonempty coefficient supports have $\alpha \geq -o(1)$. Retain the fixed frequency cutoff of Lemma 4.5 and a buffer $\xi > 0$. A nonempty nonzero frequency range satisfies $K_+ \leq K_0 + O(\xi) + o(1)$; otherwise its cutoff is eventually below the minimum nonzero norm. Bounded unit frequencies and nominal endpoints are retained with their single aggregate support discrepancy. Polynomial Fourier-height majorants remain in the original weighted Fourier measures. The raw bound of Lemma 4.5 has exponent $K_+ + 2\alpha - 2s_0'$. Subtract the completed allowance $\Lambda_* = (A - c) + q + u + B_C - s_0'$ and add $\mathcal{R}_{\mathrm{loc}}$. The result is at most

$$\mathcal{R}_{\mathrm{loc}} + 3A - 2c - d - M - B_C - s_0' + O(\xi) + o(1) \leq \mathcal{R}_{\mathrm{loc}} + 3A - M + O(\xi) + o(1).$$

The exact completed identities give

$$\mathcal{R}_{\mathrm{loc}} + 3A - M = 3A_t - 4 + 20t_{\mathrm{loc}} + 4f_1 + 4c_{\mathrm{cor}} - 3T_1 - 16v - 3T + 3m_* - S_1 - S.$$

For $m_* \le 1/100$, use $c_{\mathrm{cor}} \le t_{\mathrm{loc}}$, $t_{\mathrm{loc}} + f_1 \le w \le W$ and (12.1):

$$\mathcal{R}_{\mathrm{loc}} + 3A - M \le 3A_t - 4 + 24W + \frac{3}{100} < -1 + 12W + \frac{3}{100} \le -\frac{49}{100}.$$

Choose $\xi$ and the remaining losses inside this margin. The estimate applies directly to the actual Gauss polynomial, including negative formal $m_*$ or $M$.

*Formal-length split in the remaining branch.* Now $m_* > 1/100$ and $M > 1/100$. The auxiliary outer-length replacement of Lemma 4.7 uses $\widehat{d} = \min(d, A)$ with its support envelope and $\widehat{K}_0 = K_0 + d - \widehat{d}$. It changes no actual ideal, polynomial or frequency cutoff $K$, preserves $B_C$ and $\min(c, d)$, and satisfies $\widehat{d} + \widehat{K}_0 - K = d + K_0 - K$. Split by $m_f = m_* - c + \widehat{d} - u$. If $m_f \le 1/100$, the direct formal-terminal calculation in Proposition 4.6 gives

$$3A_t - \frac{7}{2} + 18W + \frac{7}{200} < -\frac{93}{200} + 6W \le -\frac{9}{40} < -\frac{1}{5}.$$

Its exact local identities do not depend on the initial comparison short length.

If $m_f > 1/100$, use the actual complementary Poisson calculation. Both global and formal lengths are then uniformly positive. The formulas for $J, g, M', \Delta'$ of Lemma 4.7 and Proposition 4.8, and the local sparse-character savings, are unchanged. Nonempty returned annuli have nonnegative actual length of the transformed arithmetic sum; the full moving radical remains in their combined width. A separate fixed polynomial-size extra coprimality radical is expanded completely using [Liu26b, Lemma 12.5] before a nonexceptional M2 call. The extracted divisor coefficients and their subpower total mass are retained; the actual plain lengths and remaining prime width decrease. This never deletes a displayed moving radical from the combined width. Exceptional centered terms instead retain the common coprimality condition in their fixed-character mean. Apply the established ordinary M2 only to these resulting ordinary sums, using exhaustion as above.

If no prime factor remains initially, use $a = a_o = \ell_a = 0$, $g = J_+ + \sigma$ with no auxiliary-prime average. The step still gives strict width decrease.

*Second origin and exceptional centered means.* For the second Poisson origin, (12.1) gives

$$A_t - 1 + 4t_{\mathrm{loc}} \le A_t - 1 + 4W < 0.$$

This positive diagonal is estimated separately from the equal-product mean cancellation.

For exceptional characters use the sparse-support count of [Liu26b, Section 12.8.2] and fixed-character means on the actual two rectangles. Every exact prime extraction divides both respective scales by the same ideal norm, so product equality survives. The same common character, coprimality conditions imposed by the character zeros and Mellin twist of the complete product occur on both rectangles. All four initial ordinary scales are at least $U^t$ because $t \le m$; after removals use $(t - L_1 - L - 5c_{\mathrm{cor}})_+$ and the original volume branch when it vanishes. Writing $L_{\mathrm{tot}} = L_1 + L$ and using $T_1 + T \le L_{\mathrm{tot}}/4$, the full local exceptional upper bound is

$$A_t - \frac{5}{6} + 3t_{\mathrm{loc}} - \frac{9L_{\mathrm{tot}}}{8} - \frac{29c_{\mathrm{cor}}}{6} - \frac{2}{3}(t - L_{\mathrm{tot}} - 5c_{\mathrm{cor}})_+.$$

For $L_{\mathrm{tot}}, c_{\mathrm{cor}} \ge 0$,

$$\frac{9L_{\mathrm{tot}}}{8} + \frac{29c_{\mathrm{cor}}}{6} + \frac{2}{3}(t - L_{\mathrm{tot}} - 5c_{\mathrm{cor}})_+ \ge \frac{2}{3}\{L_{\mathrm{tot}} + 5c_{\mathrm{cor}} + (t - L_{\mathrm{tot}} - 5c_{\mathrm{cor}})_+\} \ge \frac{2t}{3}.$$

Hence the exceptional exponent is at most $A_t - 5/6 - 2t/3 + 3t_{\mathrm{loc}}$. Condition (12.2) supplies a strict margin.

Differentiation changes only the smooth weights; the common character, coprimality conditions and explicit Mellin twist are retained. Thus the leading mean products cancel exactly.

*Integration and restoration of the comparison product.* Integrate both one-sided bounds in the original weighted Fourier measures, using their original Gauss allowances and the paired structural count once. The raw bounds are uniform in these labels; tails are discarded only from the genuine expression with fixed sufficient Fourier orders. Thus

$$\sum_k \mathcal{W}(q_k/U)|P_c(k)Q_W(k)|^2 \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C,$$

which is (12.4). Condition (12.3) implies $2m-t>1/2$. Reflect the long comparison factor with its actual conductor and natural zeros. Its ordinary length plus prime-bearing cost is at most

$$t+1-(2m-t)+p(z+w)\le 2+2t-4m_-+pW<1.$$

For the original detector and numerator presentations this reflection has only their natural zeros, covered by [Liu26b, Lemma 12.6]. There is no additional initial coprimality condition with polynomially bounded modulus in these applications.

If a fixed additional coprimality restriction is retained in a comparison product, its expansion must be treated with its coefficients rather than asserted to preserve the reflected margin. Put $n=2m-t$, $Z_{\mathrm{pr}}=z+w$ and $C_{\mathrm{refl}}=t+1-n+pZ_{\mathrm{pr}}<1$. A term removing plain widths $a_1,a_2$ and prime width $v$ has squared coefficient cost $U^{-a_1-a_2-v+o(1)}$. Reflect the second ordinary factor only when its surviving length $n-a_2$ exceeds $1/2$. Either choice gives a declared length at most $1-n+a_2+o(1)$, so the left-hand side of its moment length condition is at most

$$C_{\mathrm{refl}}+a_2-a_1-pv+\xi.$$

Its positive defect is at most $(a_2-a_1-pv+\xi)_+$. Lemma 12.2 restores it at squared cost at most that quantity divided by six, plus the mesh loss. The net exponent is at most

$$-a_1-a_2-v+\frac{(a_2-a_1-pv+\xi)_+}{6}\le -a_1-\frac{5}{6}a_2-v+\frac{\xi}{6}.$$

The mesh loss is included in $\epsilon$. Hilbert-space triangle and subpower divisor sums, with positive exponents at least $5/12$ in the unsquared coefficients, finish this comparison estimate with these coprimality restrictions. Nonempty subunit scales receive the single support allowance of [Liu26b, Section 12.8.1] in $\xi$. Thus the original M2 theorem, with this weighted restoration when needed, controls the comparison product on $\mathcal{U}$.

Hilbert-space triangle between $P_cQ_W$ and $P_{\mathrm{cmp}}Q_W$ gives (12.5).

*Uniform choices.* Let $\rho_{\mathrm{unif}}>0$ be a uniform lower bound for the three strict margins and the terminal saving $1/5$. Make the choices in the order

$$\begin{aligned}(\rho_{\mathrm{unif}},\epsilon) &\longrightarrow (\text{step, edges, moment losses, complete-factor mesh})\\ &\longrightarrow (\text{finite prime list, arithmetic data, smooth weights})\\ &\longrightarrow (\text{internal derivative and Fourier orders})\\ &\longrightarrow (\text{external heights, tail orders, threshold}).\end{aligned}$$

Use the arithmetic-independent M2 mesh. After fixing the list, the single aggregate support discrepancy $O(1/\log U)$ is absorbed. Compactness gives common choices. For varying $t(m),W(m)$, first make a uniform strict width decrement and then take a finite cover by small parameter neighborhoods. Each finite iteration has its own threshold. □

## 13 Selecting the original factors and amplifying the detector

In this section, we select disjoint products of original prime factors and apply Theorem 12.1 to the zero-detecting sums. We derive the count and numerator estimates used in Section 14.

We use $U$ for a dyadic scale of the arithmetic parameter. Fix a detected amplitude class, a presentation, and a dyadic detector pair before selecting factors. An original prime polynomial has width $\lambda_i$, meaning that its prime ideals lie in a dyadic norm range of scale $U^{\lambda_i}$, and amplitude $g_i$ on this class. For $g_i > 0$, the prescribed amplitude-subdivision losses give $|Q_i(u)| \geq U^{\lambda_i g_i - o(1)}$. We have

$$0 \leq g_i \leq \delta/2, \quad \sum_i \lambda_i = \mathscr{L}, \quad \sum_i \lambda_i g_i = g\mathscr{L}, \quad g = \delta x, \qquad 0 \leq x \leq \tfrac{1}{2}.$$

The notation $o(1)$ throughout this section denotes arbitrarily small exponent losses fixed before the final threshold, including amplitude-subdivision and mesh losses. It does not assert uniformity before those choices are made. For $d \leq h + \zeta$, Section 14 gives

$$\mathscr{L} = \ell/d \geq \ell/(h+\zeta) > 21/100.$$

**Lemma 13.1** (Simultaneous selection of prime factors)**.** *Suppose each original factor has width at most $\eta_{\mathrm{mesh}}$, and two upper length bounds $z, w \geq 0$ satisfy $z + w \leq \mathscr{L}$. There are disjoint products $Q_0, Q_W$ of complete positive-amplitude original factors, with respective widths at most $z, w$, for which*

$$|Q_0(u)Q_W(u)|^2 \geq U^{2g(z+w) - 2\delta\eta_{\mathrm{mesh}} - o(1)}$$

*throughout the fixed class. The number of factors does not multiply the displayed mesh loss.*

*Proof.* Order the positive-amplitude factors by decreasing $g_i$. Allowing a fraction of the last factor, take an initial segment of total length $z + w$. Its weighted mean is at least $g$. If the positive factors have smaller total length, retain all of them. In both cases, the unsquared amplitude exponent satisfies

$$A \geq g(z+w), \qquad A = \sum_{i \in I} \lambda_i g_i \quad \text{with a possible fractional endpoint.}$$

In the second case, this follows from $A = g\mathscr{L} \geq g(z+w)$.

Split the chosen segment after length $z$, or at its end if it is shorter. Discard any original factor cut by this division or by the endpoint of the chosen segment. There are at most two such factors. Assign the remaining complete factors to $Q_0$ and $Q_W$. Their respective lengths are at most $z, w$. Since $g_i \leq \delta/2$ and $\lambda_i \leq \eta_{\mathrm{mesh}}$, the discarded unsquared exponent is at most

$$2 \cdot \frac{\delta\eta_{\mathrm{mesh}}}{2} = \delta\eta_{\mathrm{mesh}}.$$

Consequently $|Q_0 Q_W|^2 \geq U^{2g(z+w) - 2\delta\eta_{\mathrm{mesh}} - o(1)}$. The previously fixed amplitude-subdivision loss gives the final $o(1)$. If the positive factors end before length $z$, take $Q_W = 1$. □

The selected lengths may be strictly smaller than the prescribed upper bounds. The enlarged norm holds for every

$$0 \leq z_{\mathrm{actual}} \leq z_0, \qquad 0 \leq w_{\mathrm{actual}} \leq W.$$

Thus the argument never requires additional factors when all positive-amplitude factors have been used. Decrease the prescribed bounds slightly before applying Lemma 13.1.

**Proposition 13.2** (Joint amplification of the ordinary detector)**.** *Fix a compact band of ordinary detector lengths $m \in (0, 1/2)$. Suppose the enlarged norm applies with initial upper length bound $z_0(m) = (1-2m)/p$ and additional upper length bound $W(m)$, with uniform strict margins in the length hypotheses, and suppose $z_0(m) + W(m) < \mathscr{L}$ uniformly. The arithmetic parameters for which the detector of [Liu26b, Lemma 13.7] gives $|S_m(u)|^2 \geq U^{\delta m - o(1)}$ have count*

$$\#\mathscr{B}_m \ll U^{1-\delta F_2(m)+\epsilon}(1+H)^C, \qquad F_2(m) = 2m + \frac{2x}{p}(1-2m) + 2xW(m). \tag{13.1}$$

*The exponent $C$ and the seminorm orders are finite and are fixed before the external height exponent is chosen.*

*Proof.* Apply Lemma 13.1 to the original factors with upper length bounds $z_0(m)$ and $W(m)$. For the original detector smooth weights,

$$|S_m(u)^2 Q_0(u) Q_W(u)|^2 \geq U^{2\delta m + 2\delta x(z_0(m)+W(m)) - o(1)}.$$

The enlarged norm therefore gives

$$\#\mathscr{B}_m\, U^{2\delta m + 2\delta x(z_0(m)+W(m)) - o(1)} \leq \sum_{u \in \mathscr{B}_m} |S_m(u)^2 Q_0(u) Q_W(u)|^2 \ll U^{1+\epsilon}(1+H)^C,$$

which is (13.1). The ordinary factor in this squared norm is $|S_m|^4$.

We justify its use for the actual detector, including its smooth weights depending on the summation parameter. The proof of [Liu26b, Lemma 13.9] first fixes the discrete presentation and dyadic pair, then applies a common parameter-box Sobolev inequality

$$\sum_u \sup_{\mathbf{t} \in \mathcal{B}_H} |P_u(\mathbf{t})|^2 \ll \sum_{\alpha \in \{0,1\}^{\dim \mathcal{B}_H}} \int_{\mathcal{B}_H^+} \sum_u |\partial^\alpha P_u(\mathbf{t})|^2 \, d\mathbf{t}.$$

Here replace its $P_u = S_m^2 Q_0$ by $S_m^2 Q_0 Q_W$. Each derivative is a finite sum of products of the two ordinary factors, with possibly different smooth weights, and the original prime factors with differentiated smooth weights. The enlarged norm is uniform for this finite family. It supplies the same inequality before the smooth-weight parameters giving the lower bound for each index are evaluated. Only the original, undifferentiated prime products require amplitude lower bounds; none are claimed for the differentiated factors.

Remove deterministic scale phases before differentiating. The derivatives then introduce logarithms of normalized annular ratios. Absorb the difference between the detector frequency and the original prime Mellin frequency into all prime smooth weights, including those of $Q_W$. The two ordinary factors retain the common explicit Mellin twist by the ideal norm used in the signed equal-product means. This does not require a new mean identity with independent Mellin twists of the ordinary factors. The parameter volume and the allowed incoming heights cost a fixed polynomial in $H$.

For arithmetic orientation, the whole detected presentation may be conjugated to $\psi = \psi_0 \bar{\chi}_u$. Its original prime coefficients are then $\bar{\psi}_0(\mathfrak{p}) 1_{\mathfrak{p} \in \mathcal{T}}$, expanded in the fixed finite coefficient group as in [Liu26b, Lemma 13.7]. The factors in $Q_W$ come from the same list with these same coefficients. Conjugate the entire product if the norm is stated in the opposite orientation. The fixed group is closed under conjugation. Exterior ramification of a nonunit sixth-power-free parameter cannot be canceled by an $S$-supported character, so the inducing character lies outside the exceptional group. Bounded unit parameters were separated in [Liu26b, Lemma 13.7]. All natural zeros and actual deleted Euler factors remain present.

The number of discrete presentations is fixed, and the number of dyadic pairs is logarithmic. Choose the loss in the moment estimate, factor decrements and mesh below the desired final exponent loss, then fix the finite derivative orders, choose the external height exponent to absorb $(1+H)^C$, and finally choose the threshold. This proves the stated uniform bound for the detector family of [Liu26b, Lemma 13.7]. □

### 13.1 Inverse moment and the long detector count

In this subsection, we combine Proposition 13.2 with the inverse moment estimate. Section 14 chooses a common truncation length before using the resulting count in the numerator estimate.

For a short inverse length $r \in (1/2, 1)$, the unchanged marked inverse moment uses the prime-product length bound $(1-r)/2$ and gives saving

$$F_1(r) = x + (1-x)r.$$

The long inverse exponent is

$$\frac{1}{6} + \left(\frac{5}{6} - \delta\right) T, \qquad 1 \le T \le \frac{3}{2}.$$

For each detected parameter, the two polynomial lower bounds hold and

$$r + m \ge T - o(1), \qquad m \le \frac{1}{2} + o(1), \qquad r_\times + m_\times = T \implies r \ge r_\times \text{ or } m \ge m_\times - o(1).$$

Use M1 in the first case and the ordinary norm in the second. Endpoint strips use the estimates without prime factors; the next section checks the common cutoff and the junction $m = 9/20$.

For each arithmetic parameter, use the inverse or ordinary moment according to its detecting lengths. The two estimates may select different subsets of the original prime factors. Their union gives the count before any estimate for the reflected numerator is used. Apply this count separately in Hölder's inequality for the numerator. Both estimates use the previously established moment parameter $\kappa$ of (14.1).

### 13.2 The numerator's separate amplification

In this subsection, we apply the enlarged norm to the numerator and pass from its fourth moment to its first moment by Hölder's inequality. Section 14 combines this gain with the contour displacement.

Joint selection would give

$$\sum_{u \in \mathscr{B}} |S_m(u)|^4 \ll U^{1-2g(z_0+W)+\epsilon}(1+H)^C,$$

and hence for a class of count $U^{R+\epsilon}$,

$$\sum_{u \in \mathscr{B}} |S_m(u)| \ll U^{(3R+1)/4 - gz_0/2 - gW/2 + \epsilon}(1+H)^C.$$

The numerator estimate below uses the weaker, already sufficient remaining-mean bound $a \le g$ for its additional gain $aW/2$. It does not need the sharper joint-selection gain there. In either version the approximate-functional-equation coefficient is treated separately; the existing baseline $gz_0/2$ is not counted twice.

The Sobolev argument for the actual conductor in [Liu26b, Proposition 14.3] also applies: for each dyad fix the original prime selection first, introduce the parameter of the conductor-dependent smooth weight on each of its $O(\log U)$ unit intervals, and differentiate the two ordinary smooth weights inside the norm. Dual blocks are wholly conjugated so their characters agree with the prime factors; root numbers disappear only after taking absolute values. Dyads above half length use the no-prime fourth moment, and sufficiently long tails are discarded only after fixing the external derivative orders. The short, supply-limited blocks and all resulting scalar inequalities required for their application are included in the next section's checks.

# 14 Exact scalar comparison and continuation

In this section, we use the enlarged norm of Theorem 12.1, the simultaneous factor selection of Lemma 13.1, the zero-detecting estimate of Proposition 13.2, and the cited theta/Mellin, M1, M2, contour and direct estimates. The numerical inequalities are verified by exact rational arithmetic in Appendix D.

## 14.1 Parameters and notation

In this subsection, we choose rational parameters whose moment hypothesis follows from Theorem 9.2. We define the exponents used in Proposition 14.2.

Fix

$$\begin{aligned} B &= \frac{5463}{6250} = 0.87408, \quad \ell = \frac{1597}{9375}, \quad b = \frac{1913}{20000}, \\ \kappa &= \frac{37446831}{50000000} = 0.74893662, \quad p = 6\kappa, \quad \varrho = 1 - \frac{1}{p}. \end{aligned} \tag{14.1}$$

Thus $B = 11/12 - \ell/4$. Theorem 9.2 gives $\beta_*(F) \le B_* < 0.87446831 = (1+\kappa)/2$, so the moment premise is already established. Put

$$\begin{aligned} l_x &= \frac{1-b-\ell}{2}, \quad l_y = \frac{1+b-\ell}{2} = \frac{277591}{600000}, \\ h &= \frac{1+b+3\ell}{2} = \frac{160669}{200000}, \quad C_0 = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4}, \quad T_y = 2l_y - h. \end{aligned}$$

The amplitude coordinates are $0 < \delta \le 5/6$, $0 \le x \le 1/2$, and $g = \delta x$. All prime lengths are relative to the scale of the arithmetic parameter $U = Z^d$. Write $L_0 = 21/100$ and $\theta_0 = \delta x/p$. For any allowed class-count exponent $R$, define

$$E(R) = C_0 + \frac{\delta}{2} + \delta x \left\{ \ell + \frac{h/2 - l_y}{p} \right\} - \frac{3h}{4}(1-R), \tag{14.2}$$

$$S(R) = \frac{3R+1}{4} + \frac{\theta_0}{2} - \frac{17}{50}, \tag{14.3}$$

$$N(R) = \frac{3R+1}{4} - \frac{\theta_0}{2}, \qquad D_{\mathrm{short}}(R) = R + (1/2 - \theta_0)\frac{3}{40} - N(R). \tag{14.4}$$

For fixed class data the unamplified reference exponent is $E(R) + (d-h)S(R)$. A selected gain $\gamma$ and a numerator contour $0 \le \theta \le \delta/2$ change this to

$$E(R) + (d-h)S(R) + (\theta - \theta_0)\{d(1-m) - l_y\} - d\gamma. \tag{14.5}$$

For $0 \le m \le 1/2$ this is the simultaneous moment/contour identity of Lemma 7.1, with its actual-conductor primal and dual smooth weights. Longer AFE blocks use the no-prime formula below.

## 14.2 Detector widths and a common cutoff

In this subsection, we choose the prime-product lengths in the ordinary zero-detecting sum and balance its count with the inverse estimates of Section 13. The resulting count is used in Proposition 14.2.

For $19/50 \le m \le 9/20$, let

$$W_o(m) = \frac{\varrho}{5}(1-2m), \qquad W_j(m) = \frac{1/2 + (1-3/p)(1-2m)}{p+12},$$
$$W_d(m) = \min\{W_o(m), W_j(m)\}.$$

The origin bound gives $W \le W_o$. The reflection and exceptional constraints on the comparison length $t$ are respectively

$$t < t_{\mathrm{high}} = 2m - \frac{1}{2} - \frac{p}{2}W, \qquad t > t_{\mathrm{low}} = 6W - \frac{3}{2}\varrho(1-2m) + \frac{1}{4}.$$

The interval is nonempty precisely when

$$t_{\mathrm{high}} - t_{\mathrm{low}} = \frac{p+12}{2}(W_j - W) > 0.$$

At $W = W_d$, take $t = \min\{1/4, (t_{\mathrm{low}} + t_{\mathrm{high}})/2\}$. The rational inequalities give

$$0 < t \le 1/4 < m, \qquad t > 5W_d, \qquad W_d < 1/25.$$

For the norm estimate use $W = (1-\epsilon_W)W_d$ with fixed sufficiently small $\epsilon_W > 0$; these auxiliary inequalities remain strict. The prime supply satisfies

$$z_0(m) + W_d(m) \le \frac{1 - 2(19/50)}{p} + \frac{1}{25} < L_0.$$

By Proposition 13.2, the ordinary detector saving is

$$F_2(m) = 2m + \frac{2x}{p}(1-2m) + 2xW_d(m).$$

The M1 saving for inverse length $r$ is $F_1(r) = x + (1-x)r$; the long inverse count at total cutoff $T_d$ is $1/6 + (5/6 - \delta)T_d$.

Write $W_i(m) = W_i(0) + W_i' m$ for $i = o, j$, with constant slopes $W_i'$, and define

$$\alpha_i = \frac{2x}{p} + 2xW_i(0), \quad \beta_i = 2 - \frac{4x}{p} + 2xW_i', \quad F_i(m) = \alpha_i + \beta_i m, \quad A_\delta = \frac{5}{6} - \delta, \quad H_\delta = \frac{A_\delta}{1-x} + \delta.$$

Thus $F_2 = \min(F_o, F_j)$. On the domain used below $A_\delta, H_\delta > 0$. Set

$$m_i = \frac{5/6 + A_\delta x/(1-x) - H_\delta \alpha_i}{A_\delta + H_\delta \beta_i}, \quad R_i = 1 - \delta F_i(m_i), \quad T_i = m_i + \frac{F_i(m_i) - x}{1-x}. \tag{14.6}$$

Every expression is rational with positive certified denominators.

**Lemma 14.1** (Shared detector cutoff)**.** *On $\mathcal{D} = [8/25, 3/5] \times [9/25, 1/2]$, the valid amplified class count is*

$$\#\mathscr{B} \ll U^{R_d + \epsilon}(1+H)^C, \qquad R_d = \max\{R_o, R_j\},$$

*with total detector cutoff $T_d = \max(T_o, T_j)$.*

*Proof.* The balance equation for branch $i$ is

$$-\frac{5}{6} + A_\delta m - \frac{A_\delta x}{1-x} + H_\delta F_i(m) = 0.$$

Its left side is increasing. Therefore the equation with $\min(F_o, F_j)$ is solved at $m_d = \max(m_o, m_j)$: the active branch is zero there and the other is nonnegative. At a branch root,

$$F_i(m_i) = \frac{5/6 + A_\delta x/(1-x) - A_\delta m_i}{H_\delta}.$$

Since $A_\delta, H_\delta, \delta > 0$,

$$m_i \le m_j \implies R_i \le R_j \implies T_i \le T_j, \qquad T_i = \frac{R_i - 1/6}{A_\delta}.$$

The same branch realizes all three maxima. With $r_d = T_d - m_d$,

$$1 - \delta F_1(r_d) = 1 - \delta F_2(m_d) = 1/6 + A_\delta T_d = R_d.$$

The exact calculations establish, for both branches,

$$19/50 < m_i < 9/20, \qquad 3/5 < T_i - m_i < 1, \qquad 1 < T_i < 3/2.$$

For inverse length at least $r_d$, use M1. If the detector presents an inverse length below $r_d$, its ordinary length is at least $m_d$ after the arbitrarily small detector losses of [Liu26b, Lemma 13.7]. Both $F_i$ increase, so the amplified M2 saving suffices through $9/20$. For larger lengths, the old M2 saving at $9/20$ already exceeds both $F_i(m_i)$, as certified explicitly; it is increasing thereafter. The long inverse count handles the remaining detector branch. The M1 selected-prime length is below $(1 - 3/5)/2 = 1/5 < L_0$, and its second length restriction has a strict margin.

Use a finite compact cover of the amplitude classes and dyadic lengths, then take the detector, prime and width meshes sufficiently small within these strict margins in the length hypotheses. Replacing limiting widths by strict interior widths changes the exponents by an arbitrarily small uniform amount. The original smooth weights and original prime lower bounds are retained before the common Sobolev step of Proposition 13.2. This gives the stated count with any prescribed positive exponent loss. □

### 14.3 Numerator gain on every ordinary length

In this subsection, we combine the numerator moment and contour displacement to obtain a uniform gain over the ordinary summation lengths. Proposition 14.2 then bounds the exponent in (14.5).

The numerator uses the fixed comparison length $t_n = 23/100$, with

$$m_L = (1 + 2t_n)/4 = 73/200, \quad z_L = (1 - 2m_L)/p, \quad a = \delta \frac{L_0 x - z_L/2}{L_0 - z_L}.$$

This conservative remaining-amplitude bound is sufficient; the stronger simultaneous factor selection for the numerator is not needed here. On $m_L \le m \le 9/20$ use the limiting width

$$W_n(m) = \min\left\{ \frac{\varrho}{5}(1 - 2m), \frac{4m - 1 - 2t_n}{p}, \frac{59}{2000} \right\}.$$

The exceptional constraint is redundant on this band: for $W \le \varrho(1 - 2m)/5$ and $m \le 9/20$,

$$t_{\mathrm{low}} \le \frac{1}{4} - \frac{3}{10}\varrho(1 - 2m) \le \frac{1}{4} - \frac{3\varrho}{100} < t_n,$$

where the last inequality is $\varrho/50 > 1/6 - 2t_n/3$. The constant cap is below 0.03. Put $m_\theta = 1 - l_y/h$ and

$$G_N = \frac{a\varrho}{10} T_y, \qquad G_L = \theta_0\{(1 - m_L)h - l_y\}.$$

Choose $\theta = 0$ for $m \le m_\theta$, and $\theta = \delta/2$ for $m \ge m_\theta$. At $d = h$ the gain in (14.5) is $h\mathcal{F}(m)$, where

$$\mathcal{F}(m) = \frac{aW_n(m)}{2} + \begin{cases} \theta_0(m_\theta - m), & m \le m_\theta, \\ (\delta/2 - \theta_0)(m - m_\theta), & m \ge m_\theta, \end{cases}$$

the function of Section 7.1 with $W_n$ in place of $W_{\mathrm{cap}}$. On each open branch with $m < m_\theta$,

$$\mathcal{F}'(m) = \begin{cases} 2a/p - \theta_0 > 0, & \text{reflection branch}, \\ -\theta_0 < 0, & \text{constant cap}, \\ -\varrho a/5 - \theta_0 < 0, & \text{origin branch}. \end{cases}$$

At $m_\theta$ the origin branch is active and below the constant cap. Where differentiable,

$$m > m_\theta \implies \mathcal{F}'(m) \geq \frac{\delta}{2} - \theta_0 - \frac{\varrho a}{5} > 0.$$

Thus

$$\min_{m_L \leq m \leq 9/20} h\mathcal{F}(m) = \min\{G_L, G_N\}.$$

Continuity handles branch crossings. The signs hold on both the main and bridge domains below. For $3/40 \leq m < m_L$, the central-contour unamplified estimate has gain at least $G_L$. For $m > 9/20$, the outer-contour unamplified estimate has gain at least its value at $9/20$, which is certified larger than $G_N$. For $m < 3/40$, use ideal counting and $D_{\mathrm{short}}(R)$ directly.

The outer assertion includes AFE lengths above $1/2$, which can be larger than one when the fixed field has large degree. For these blocks the no-prime first moment, including the AFE coefficient, is $(3R+1)/4-\theta m$. Relative to the reference exponent its complete contour correction is exactly

$$(\theta - \theta_0)(d/2 - l_y) - d\theta(m - 1/2), \qquad m \geq 1/2. \tag{14.7}$$

Let $\mathcal{E}_{\mathrm{out}}(m)$ be the unamplified exponent with $\theta = \delta/2$. The formulas agree at $m = 1/2$, and

$$\partial_m \mathcal{E}_{\mathrm{out}} = \begin{cases} -d(\delta/2 - \theta_0), & 9/20 < m < 1/2, \\ -d\delta/2, & m > 1/2, \end{cases} \qquad (d > 0).$$

Both derivatives are negative. Hence every outer AFE block is bounded by $\mathcal{E}_{\mathrm{out}}(9/20)$; the scale-slope estimate is applied to this single endpoint bound.

At $m = m_L$ the limiting selected width vanishes, so no uniform positive-width norm is applied at that endpoint. First choose a fixed small $\epsilon_L > 0$. Use the unamplified central-contour estimate through $m_L + \epsilon_L$. Its gain on the added strip is at least $G_L - h\theta_0\epsilon_L$, and $h\theta_0 < 0.054$ on the main and bridge domains. Since $E - G_L < -10^{-6}$ there, taking $0.054\epsilon_L < \nu/100$, with the margin $0 < \nu < 10^{-6}$ fixed in Section 14.5, preserves a strict saving. Apply the selected norm only on $[m_L + \epsilon_L, 9/20]$, where the limiting width has a uniform positive lower bound. Then choose the interior width loss and the prime and dyadic meshes inside its strict margins in the length hypotheses. The comparison-length and exceptional constraints also have uniform strict margins on this closed band after the interior decrement.

The contour endpoints 0 and $\delta/2$ are permitted by Lemma 5.2: the final $s$ buffer keeps $s + w$ strictly inside its holomorphy region. If strictly interior contours are used, take $\theta_- = \epsilon_\theta$ and $\theta_+ = \delta/2 - \epsilon_\theta$ instead. On the retained compact ranges the change in the exponent is at most $2\epsilon_\theta$, so choose this loss below $\nu/100$ as well. For outer blocks use their $m = 9/20$ bound first, so this constant does not depend on their possibly large lengths. Thus neither a zero-width endpoint nor a limiting contour is used without a positive-margin approximation.

### 14.4 Exact inequalities over the complete parameter range

In this subsection, we prove Proposition 14.2 and treat the remaining amplitude and summation ranges. Appendix D verifies the required rational inequalities.

Let $R_s = R_\kappa(\delta, x)$ be the count exponent of Proposition 6.4 at the current moment parameter $\kappa$; then $c_{\mathrm{cnt}}(\kappa) = 1/(3\kappa) = 2/p$.

Use $R_d$ on $\mathcal{D}$. On the bridge $\mathcal{H} = [3/10, 8/25] \times [9/25, 1/2]$, use $R_s$ with the amplified numerator. On the following complementary rectangles, use $R_s$ and the old numerator estimate:

$$\mathcal{C}_1 = [1/8, 5/6] \times [0, 9/25], \quad \mathcal{C}_2 = [1/8, 3/10] \times [9/25, 1/2], \quad \mathcal{C}_3 = [3/5, 5/6] \times [9/25, 1/2].$$

Boundary overlaps may use either valid estimate.

**Proposition 14.2** (Rational comparison inequalities)**.** *For the parameters* (14.1)*, one has*

$$\begin{aligned} E(R_i) - G_N &< -10^{-6}, & E(R_i) - G_L &< -10^{-6} & &(i = o, j;\ \mathcal{D}),\\ E(R_s) - G_N &< -10^{-6}, & E(R_s) - G_L &< -10^{-6} & &(\mathcal{H}),\\ E(R_s) &< -10^{-5} & & & &(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3). \end{aligned}$$

*The corresponding reference exponents on* $\mathcal{D}, \mathcal{H}$ *are below* $1/450$ *and their slopes satisfy* $1/5 < S < 1$*. On the complementary rectangles* $0 < S(R_s) < 1$*. On the main and bridge domains,*

$$E(R) + hD_{\mathrm{short}}(R) < -10^{-5}, \qquad 0 < S(R) + D_{\mathrm{short}}(R) < 2,$$

*and the moment supply* $(p - 2\delta x)L_0 > R$ *holds. The old moment supply is also certified on all of* $[1/8, 5/6] \times [0, 1/2]$*.*

*Proof.* Substitute (14.1). For each required rational inequality $f > 0$ on a closed rectangle $\mathcal{R}$, all inverted denominators exclude zero. With $c_X$ the midpoint and $[f](X)$ the natural interval evaluation, use

$$\mathcal{I}_f(X) = [f](X) \cap \left( f(c_X) + \sum_{j=1}^{2} [\partial_j f](X)(X_j - c_{X,j}) \right).$$

Outward rational arithmetic and a finite subdivision $\mathfrak{X}$ give

$$\mathcal{R} = \bigcup_{X \in \mathfrak{X}} X, \qquad \inf \mathcal{I}_f(X) > 0 \quad (X \in \mathfrak{X}) \quad \Longrightarrow \quad f > 0 \text{ on } \mathcal{R}.$$

The expressions in $R$ are affine, so the inequalities for $R_o, R_j$ hold for $R_d = \max(R_o, R_j)$. The same enclosures give the cutoff, supply, width, numerator-slope, short-sum and bridge conditions. □

The remaining ranges are treated as follows. For $d \le h - 1/80$ in the range of arithmetic parameters where the affine exponent formula [Liu26b, equation (14.7)] applies, the main and bridge estimates give

$$E(R) + (d - h)S(R) < 1/450 - 1/400 = -1/3600.$$

The complementary estimates improve as $d$ decreases, since $S(R_s) > 0$. On the main and bridge domains, $\delta \le 3/5$ gives $\theta_0 \le 3/(10p) < 0.067$, $(\delta/2 - \theta_0)(1 - m) \le 3/10$ and $\gamma = aW_n/2 \le (3/10)(59/2000)/2 < 0.0045$; also $1/5 < S < 1$. Since $\partial_d$ of (14.5) is $S + (\theta - \theta_0)(1 - m) - \gamma$, for each selected or unamplified comparison exponent $\mathcal{E}$ with $3/40 \le m \le 9/20$,

$$\frac{1}{8} < \frac{1}{5} - 0.067\frac{37}{40} - 0.0045 < \partial_d \mathcal{E} < 1 + 0.3 < 2.$$

The same bound applies to the outer endpoint $m = 9/20$; the short-sum slope is checked directly. Thus extending to $d \le h + \zeta$ costs at most $2\zeta$.

For $0 < \delta < 1/8$, keep the old numerator moment at $3/4$. The sharpened small-amplitude calculation gives

$$E_{\mathrm{small}} \le C_0 + (1 - h)/16 < -1/250;$$

its required coefficient $Q_{\mathrm{sm}} < 0$ is certified. The lowest-amplitude class endpoint satisfies

$$C_0 + (1/2 + 3\ell/2)/50 < -1/250.$$

The proof of Lemma 6.3, applied with this hypothesis, gives

$$\mathcal{E}_{\mathrm{low}}(d) \le C_0 + (1/2 + 3\ell/2)/50 + 2\zeta < -1/250 + 1/1000 = -3/1000.$$

The small-amplitude bound has the same extension. Choose the remaining losses for the lowest-amplitude classes inside this margin. The small-parameter estimate, including bounded nonprincipal unit parameters, has the certified bound

$$13h/75 - l_y/2 + 1/50 < -1/20.$$

The parameters in the far range use the right-hand contours of [Liu26b, Section 14.6] and sufficiently large $z_\infty$. The principal residues and their normalizer are unchanged. The savings of [Liu26b, Section 14.6] apply since $B - 437/500 = 1/12500 > 0$.

### 14.5 Smooth weights, error terms and analytic continuation

In this subsection, we choose the losses and analytic parameters in the required order and deduce $\beta_*(F) \le 5463/6250$ from Proposition 14.2. Section 15 starts the final iteration from this bound.

Fix $F$ and the rational parameters. Choose $0 < \nu < 10^{-6}$ below the common comparison, detector-crossing and supply margins. First choose the endpoint strips, strict interior widths, contour and comparison-length buffers, with total comparison loss $\epsilon_1 < \nu/4$. Their retained compact sets have a positive minimum norm margin. Choose the factor meshes and input losses below that minimum and the remaining comparison margin, and take a finite cover before the target-dependent arithmetic data.

The piecewise rational count, width and class expressions have bounded derivatives on these compact sets. Thus the width, amplitude, length, detector and fractional-factor losses can be chosen with total comparison cost $\epsilon_2$, before the target-dependent data, with

$$\epsilon_1 < \frac{\nu}{4}, \qquad \epsilon_2 < \frac{\nu}{4}, \qquad 0 < 2\zeta < \frac{\nu}{4}, \qquad \frac{\ell}{h+\zeta} > L_0.$$

For each original dyad use the order

$$\begin{aligned}(s,z) &\longrightarrow \text{amplitude classes} \longrightarrow \text{original prime selection}\\ &\longrightarrow \text{differentiation in the common Sobolev boxes.}\end{aligned}$$

These boxes include detector frequencies, original prime frequencies and actual conductors. Fix finite derivative orders before external heights. Move each complete holomorphic numerator dyad before taking absolute values; retain both AFE halves, root numbers, actual-conductor factors and every simultaneous strict error tuple.

Theorem 12.1 applies to the same independent ordinary and prime weights. Lemma 7.1 cancels the selected-prime conductor errors before the moment. The Euler numerator and principal coefficient remain unchanged as functions of the arithmetic summation parameter.

Include the finite count and numerator exponents in the nonprincipal error estimate of [Liu26b, Proposition 14.6]. The high-side inequalities, with thresholds $1/450$ and $1/250$, give a positive nonprincipal saving; the direct, small- and large-parameter estimates, principal correction and exact theta/Mellin identity retain their verified length hypotheses. Choose the remaining smooth-weight and contour loss $\epsilon_3 < \nu/4$, so that

$$\epsilon_1 + \epsilon_2 + 2\zeta + \epsilon_3 < \nu.$$

Then fix

$$\text{internal orders} \longrightarrow \text{target-dependent height exponent and tail order} \longrightarrow \text{threshold}.$$

The height factors consume less than a fixed fraction of the saving. This gives a common numerical $m_{\mathrm{save}} > 0$, with target-dependent constants and thresholds.

Suppose, for contradiction, that $\beta = \beta_*(F) > B$. The previous bound justifies the fixed moment parameter throughout. Choose $0 < \epsilon_{\mathrm{low}} < (\beta - B)/4$ and let $C_\theta(s) = s - (4+b)/6$. The identity of [Liu26b, Lemma 14.7], with the high-side estimates just proved, gives for every fixed target $\eta$ the same functions $J_\eta, f_\eta, H_\eta$ and the estimates

$$|J_\eta(Z)| \ll_\eta Z^{C_\theta(B)+\epsilon_{\mathrm{low}}}, \qquad |J_\eta(Z) - f_\eta(Z)| \ll_\eta Z^{C_\theta(\beta)-m_{\mathrm{save}}/2}.$$

Set

$$\sigma = \max\{B + \epsilon_{\mathrm{low}}, \beta - m_{\mathrm{save}}/2\} < \beta, \qquad f_\eta(Z) \ll_\eta Z^{\sigma-(4+b)/6}.$$

The Mellin integral over $[1,\infty)$ is holomorphic for $\Re s > \sigma$; shifting the original absolute-convergence line arbitrarily far right makes the integral over $(0,1]$ entire. Mellin inversion identifies this continuation with

$$e^{(s-5/6)^2}\frac{H_\eta(s)}{L_F^S(s,\eta)}.$$

Since $H_\eta$ is holomorphic and nonzero for $\Re s > 437/500$,

$$\sigma \geq B > \frac{437}{500} \quad \Longrightarrow \quad \mathcal{Z}_F \cap \{s : \Re s > \sigma\} = \varnothing.$$

The numerical $\sigma$ is common to every fixed target, although the constants and thresholds may depend on $\eta$. Hence

$$\beta_*(F) \leq \sigma < \beta_*(F),$$

a contradiction. Thus $\beta_*(F) \leq 0.87408$, with the moment premise supplied by Theorem 9.2.

*Remark* 14.3 (The one-step comparison criterion). Retain the formulas for $h, l_x, l_y, C_0$. If $p = 6\kappa \in [112/25, 9/2]$ has a previously established moment hypothesis and the same norm, detector, numerator and geometric inequalities have a positive common margin, the argument gives

$$\beta_*(F) \leq \frac{11}{12} - \frac{\ell}{4}.$$

Choose all fixed losses inside that margin. The rational values in (14.1) are needed only for the starting comparison.

# 15 Successive moment bounds in the exhaustion range

In this section, we iterate the rational bound of Section 14 with the same moment parameter in the zero-detecting estimates and numerator. Lemma 15.1 gives the strict inequalities at each step; abelian descent yields Theorem 1.2. Section 16 also permits the two moment parameters to differ when proving optimality.

## 15.1 The algebraic parameters

In this subsection, we define the algebraic parameters used in Lemma 15.1. Appendix D isolates their values and verifies their defining identities.

Let $\ell_{\mathrm{ex}}$ be the unique root in $(0.1703752759993550355, 0.1703752759993550356)$ of

$$f(\ell) = 621\ell^4 - 2064\ell^3 + 1657\ell^2 + 68\ell - 50. \tag{15.1}$$

Put

$$\begin{gathered} B_{\mathrm{ex}} = \frac{11}{12} - \frac{\ell_{\mathrm{ex}}}{4}, \qquad \kappa_{\mathrm{ex}} = 2B_{\mathrm{ex}} - 1, \qquad p_{\mathrm{ex}} = 6\kappa_{\mathrm{ex}} = 5 - 3\ell_{\mathrm{ex}}, \\ C(p) = \frac{p+4}{20p}, \qquad C_{\mathrm{ex}} = C(p_{\mathrm{ex}}), \\ r_{\mathrm{ex}} = -\frac{5(3\ell_{\mathrm{ex}} - 5)(9\ell_{\mathrm{ex}} - 5)}{6(21\ell_{\mathrm{ex}}^2 - 28\ell_{\mathrm{ex}} - 25)}. \end{gathered} \tag{15.2}$$

For an independent parameter $C$, define

$$R_C(\delta) = 1 - \delta + \frac{(5/6-\delta)\delta(1-4C)}{2\{(5/6-\delta)(5/2-8C)+\delta(1-4C)\}}, \tag{15.3}$$

$$\mathcal{Q}_{C,b,\ell}(\delta) = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4} + \frac{\delta}{2} + \frac{\delta\ell}{2} - C\delta T_y - \frac{3h}{4}(1 - R_C(\delta)), \tag{15.4}$$

where $h = (1+b+3\ell)/2$ and $T_y = (1+b-5\ell)/2$. The expression is affine in $b$. Define $b_{\mathrm{ex}}$ by

$$b_{\mathrm{ex}} = -\left.\frac{\partial_\delta \mathcal{Q}_{C,0,\ell}(\delta)}{\partial_\delta \partial_b \mathcal{Q}_{C,b,\ell}(\delta)}\right|_{C=C_{\mathrm{ex}},\,\ell=\ell_{\mathrm{ex}},\,\delta=r_{\mathrm{ex}}}. \tag{15.5}$$

The denominator is nonzero, as verified below. Numerically,

$$\begin{aligned} B_{\mathrm{ex}} &= 0.87407284766682790777786693740\ldots, \\ r_{\mathrm{ex}} &= 0.44469417311455635275466866926\ldots, \\ b_{\mathrm{ex}} &= 0.09567500764537642027548322 48\ldots, \\ C_{\mathrm{ex}} &= 0.09455460151844812985\ldots. \end{aligned} \tag{15.6}$$

The polynomial for $B_{\mathrm{ex}}$ is

$$178848B^4 - 507168B^3 + 522846B^2 - 231405B + 37099 = 0. \tag{15.7}$$

It is irreducible over $\mathbb{Q}$ and has a unique root in $(0.87407284, 0.87407285)$.

## 15.2 Endpoint signs and a strict family

In this subsection, we prove Lemma 15.1. The limiting exponent is nonpositive, and each parameter in (15.8) gives a strict saving. Proposition 15.2 uses these inequalities in the iteration.

For either detector branch $i = o, j$, let

$$\mathcal{Q}_i(\delta, x; \kappa, \ell, b) = E(R_i) - G_N$$

with the formulas of Section 14. At $x = 1/2$, $a = \delta/2$. The origin branch therefore satisfies

$$\mathcal{Q}_o(\delta, 1/2; \kappa, \ell, b) = \mathcal{Q}_{C(6\kappa),b,\ell}(\delta).$$

In particular, no remaining-amplitude loss is incurred in this class.

**Lemma 15.1** (The endpoint and the strict comparison family)**.** *At $\kappa = \kappa_{\mathrm{ex}}$, $\ell = \ell_{\mathrm{ex}}$ and $b = b_{\mathrm{ex}}$,*

$$\max\{\mathcal{Q}_o, \mathcal{Q}_j\} \le 0 \quad \text{on } \mathcal{D} = [8/25, 3/5] \times [9/25, 1/2].$$

*Equality holds only at $(\delta, x) = (r_{\mathrm{ex}}, 1/2)$. Put $\kappa_0 = 4676/6250 = 0.74816$ and, for $\kappa_{\mathrm{ex}} < \kappa \le \kappa_0$, set*

$$b = b_{\mathrm{ex}}, \qquad \ell(\kappa) = \ell_{\mathrm{ex}} - \frac{\kappa - \kappa_{\mathrm{ex}}}{25}. \tag{15.8}$$

*Then both branches satisfy*

$$\mathcal{Q}_i(\delta, x; \kappa, \ell(\kappa), b_{\mathrm{ex}}) < -\frac{\kappa - \kappa_{\mathrm{ex}}}{100} \qquad ((\delta, x) \in \mathcal{D}). \tag{15.9}$$

*The bridge, complementary-class, numerator-length and geometric inequalities used in the rational comparison continue to hold on this family, with the margins specified below.*

*Proof.* Rational polynomial arithmetic modulo (15.1) gives

$$\mathcal{Q}_{C_{\mathrm{ex}},b_{\mathrm{ex}},\ell_{\mathrm{ex}}}(r_{\mathrm{ex}}) = \partial_\delta \mathcal{Q}_{C_{\mathrm{ex}},b_{\mathrm{ex}},\ell_{\mathrm{ex}}}(r_{\mathrm{ex}}) = \partial_b \mathcal{Q}_{C_{\mathrm{ex}},b,\ell_{\mathrm{ex}}}(r_{\mathrm{ex}}) = 0.$$

In particular,

$$R_{C_{\mathrm{ex}}}(r_{\mathrm{ex}}) = \frac{5 + 12 C_{\mathrm{ex}} r_{\mathrm{ex}}}{9}. \tag{15.10}$$

On $8/25 \le \delta \le 3/5$, clearing positive denominators gives

$$\mathcal{Q}_o(\delta, 1/2) = \frac{A_o(\delta - r_{\mathrm{ex}})^2}{D_o(\delta)}, \qquad A_o < 0, \quad D_o(\delta) > 0,$$

$$\mathcal{Q}_j(\delta, 1/2) = \frac{A_j\delta^2 + B_j\delta + C_j}{D_j(\delta)}, \qquad A_j < 0, \quad B_j^2 - 4A_jC_j < 0, \quad D_j(\delta) > 0.$$

Hence $\mathcal{Q}_o \le 0$, with equality only at $r_{\mathrm{ex}}$, and $\mathcal{Q}_j < 0$.

Exact outward interval arithmetic gives

$$\partial_x \mathcal{Q}_i > 1/1000 \qquad \text{on } \mathcal{D} \text{ at the endpoint,} \tag{15.11}$$

$$\partial_\kappa \mathcal{Q}_i - \frac{1}{25}\partial_\ell \mathcal{Q}_i < -1/100 \qquad \text{on } \mathcal{D} \times \mathcal{P}, \tag{15.12}$$

where $\partial_\kappa$ differentiates the common parameter in both the count and numerator, and the enclosing parameter box is

$$\mathcal{P}: \quad \kappa \in [\kappa_{\mathrm{ex}}, \kappa_0], \quad \ell \in [\ell_{\mathrm{ex}} - (\kappa_0 - \kappa_{\mathrm{ex}})/25, \ell_{\mathrm{ex}}], \quad b = b_{\mathrm{ex}}.$$

The first inequality proves the endpoint assertion. Along (15.8), the common moment parameter satisfies

$$\frac{d}{d\kappa}\mathcal{Q}_i(\delta, x; \kappa, \ell(\kappa), b_{\mathrm{ex}}) = \partial_\kappa \mathcal{Q}_i - \frac{1}{25}\partial_\ell \mathcal{Q}_i < -\frac{1}{100}.$$

Integration from $\kappa_{\mathrm{ex}}$ gives (15.9). Here $\kappa = p/6$ is the M2 parameter, whereas $C(p)$ determines the prime-product length bound. The remaining assertions are the inequalities of Table 8, which the same rational interval calculation proves on $\mathcal{P}$ (Appendix D). □

All inequalities in Table 8 hold on $\mathcal{P}$. The definitions of $\mathcal{H}, \mathcal{C}_i$, $D_{\mathrm{short}}$ and $S$ are those of the rational comparison. Unless another domain is indicated, an amplitude inequality is on the main and bridge rectangles with the count prescribed there.

Table 8: Uniform inequalities for the exhaustion-norm iteration.

| quantity and domain | bound | used for |
|---|---|---|
| $E(R_i) - G_L$, $i = o, j$, on $\mathcal{D}$ | $< -10^{-7}$ | main saving |
| $E(R_s) - G_N$, $E(R_s) - G_L$ on $\mathcal{H}$ | $< -10^{-7}$ | main saving |
| $E(R_s)$ on each $\mathcal{C}_i$ | $< -10^{-5}$ | complementary classes |
| $E(R)$; $S(R)$ on the main and bridge rectangles | $< 1/450$; $1/5 < S < 1$ | the range $d \le h + \zeta$ |
| $E(R) + hD_{\mathrm{short}}(R)$; $S(R) + D_{\mathrm{short}}(R)$ | $< -10^{-5}$; $0 < S + D_{\mathrm{short}} < 2$ | short sums $m < 3/40$ |
| $(p - 2\delta x)L_0 - R$; $G_L - G_N$ | $> 0$; $> 0$ | moment supply; $\min\{G_L, G_N\} = G_N$ |
| $a$; $2a/p - \theta_0$; $\delta/2 - \theta_0 - \varrho a/5$ | all $> 0$ | signs of $\mathcal{F}'$ |
| $a(59/2000)h/2 - G_N$; $(\delta/2 - \theta_0)(l_y - 11h/20) - G_N$ | both $> 0$ | constant cap; outer blocks |
| $2xL_0 - z_L$ | $> 31/1000$ | numerator prime supply |
| $m_i$; $T_i - m_i$; $T_i$, $i = o, j$, on $\mathcal{D}$ | $(19/50, 9/20)$; $(3/5, 1)$; $(1, 3/2)$ | Lemma 14.1 |
| $9/10 + x/(5p) - F_i(m_i)$ | $> 0$ | old M2 beyond 9/20 |
| $S(R_s)$; $(p - 2\delta x)L_0 - R_s$ on the full class rectangle | $0 < S(R_s) < 1$; $> 0$ | complementary classes |
| $p\ell$; $\ell/(h + 10^{-9})$; $\ell/(h - 1/80)$ | $> 3/4$; $> 21/100$; $< 2/9$ | prime supply; cited length hypotheses |
| $l_x - \ell$; $l_y - \ell - 11b/6$; $1/2 - l_y$; $T_y$ | all $> 0$ | direct and geometric hypotheses |
| $m_\theta - (p + 4 + 10t_n)/(2p + 18)$; $59/2000 - \varrho(2l_y/h - 1)/5$ | both $> 0$ | origin branch active at $m_\theta$ |
| $\varrho/50 - (1/6 - 2t_n/3)$; $9/20 - m_\theta$ | both $> 0$ | exceptional constraint; $m_\theta < 9/20$ |
| $Q_{\mathrm{sm}}$; $C_0 + (1 - h)/16$; $C_0 + (1/2 + 3\ell/2)/50$ | $< 0$; $< -1/250$; $< -1/250$ | small and lowest amplitudes |
| $13h/75 - l_y/2 + 1/50$; $B(\ell) - 437/500$ | $< -1/20$; $> 0$ | small parameters; far range |

All relevant denominators are bounded away from zero. The detector width crossing is enclosed before the two width branches are checked; the comparison lengths satisfy the same strict auxiliary inequalities as in the rational case. Appendix D gives the exact arithmetic and its independent algebraic checks.

### 15.3 Iteration and descent

In this subsection, we prove Proposition 15.2 by iterating Lemma 15.1. The descent of Theorem 9.2 then proves Theorem 1.2. Every finite step uses a previously established moment hypothesis.

**Proposition 15.2.** *For every number field $F \supset \mu_{12}$, $\beta_*(F) \le B_{\mathrm{ex}}$.*

*Proof.* The comparison of Section 14 gives $B_0 = 5463/6250 = 0.87408$. Thus $\kappa_0 = 2B_0 - 1 = 4676/6250$ is a previously justified moment parameter. Suppose $\beta_*(F) \le B_n$ and $\kappa_n = 2B_n - 1 > \kappa_{\mathrm{ex}}$. Use $\ell(\kappa_n)$ and $b = b_{\mathrm{ex}}$, with count and numerator moment parameters $\kappa_c = \kappa_m = \kappa_n$. These

parameters lie in $\mathcal{P}$. Theorem 12.1 applies because $112/25 < 6\kappa_n < 9/2$, and Lemma 15.1 gives the strict main saving $\nu_n = (\kappa_n - \kappa_{\mathrm{ex}})/100$.

Choose all endpoint strips, interior width decrements and contour buffers within a fixed fraction of the lesser of $\nu_n$ and the strict side margins. Then take the positive compact norm margins, choose meshes and losses in the cited estimates inside them, and fix arithmetic data and finite families of smooth weights before the late heights and threshold. Section 14.5 applies with comparison margin $\nu_n > 0$. The small, far and principal terms satisfy Table 8. No original coefficient, principal numerator or normalizer changes. Consequently

$$\beta_*(F) \le B_{n+1} = \frac{11}{12} - \frac{\ell(\kappa_n)}{4} = B_{\mathrm{ex}} + \frac{\kappa_n - \kappa_{\mathrm{ex}}}{100}.$$

Hence

$$\kappa_{n+1} - \kappa_{\mathrm{ex}} = \frac{\kappa_n - \kappa_{\mathrm{ex}}}{50}, \qquad B_n - B_{\mathrm{ex}} = 50^{-n}(B_0 - B_{\mathrm{ex}}).$$

Each finite step has its previously proved premise, positive margins and threshold. Since

$$\beta_*(F) \le B_n \quad (n \ge 0), \qquad B_n \downarrow B_{\mathrm{ex}},$$

we obtain $\beta_*(F) \le \inf_n B_n = B_{\mathrm{ex}}$, independently of any uniformity of the thresholds in $n$. □

*Proof of Theorem 1.2.* Apply the descent in Theorem 9.2 to Proposition 15.2. For a finite-order character over $K$, use its cyclic class-field extension $M/K$ and $F = K(\zeta_{12})$. The abelian factorizations over $F$, over $M$, and over $K$ show that a zero of the given $L$-function in $\Re s > B_{\mathrm{ex}}$ would give a zero of $\zeta_{MF}$ there. Nonprincipal factors are entire, so they cannot cancel that zero by a pole. The case $s = 1$ follows from nonvanishing of nonprincipal finite-order Hecke $L$-functions; the principal pole is permitted. Deleted Euler factors have their zeros on $\Re s = 0$. □

The first step of the iteration in Proposition 15.2 already gives

$$B_1 = 0.8740729907134913496\ldots < 0.874073.$$

The distinction between this finite-step bound and the limiting quartic endpoint will be used in the optimality statement.

## 16 Optimality for the exhaustion-norm comparison

In this section, we prove Theorem 16.1: $B_{\mathrm{ex}}$ is the infimum for the specified norm, zero-detecting estimates and direct comparison. The ordinary comparison lengths and the two moment parameters may vary. An obstruction below means a necessary condition for this specified upper-bound comparison. It is not a lower bound for the real parts of zeros. We retain the direct theta majorant, the M1 and long-detector branches, the fourth-moment Hölder estimate, the original amplitude bounds, the geometric length inequality (16.1), and the second-origin width ceiling.

**Theorem 16.1** (Optimality in the specified comparison system)**.** *Let $\kappa_c$ and $\kappa_m$ be moment parameters for the zero-detecting count and the numerator, each justified by an already proved boundary, and put $p_d = 6\kappa_c$, $p_n = 6\kappa_m$. For $0 \le m \le 1/2$, suppose the ordinary zero-detecting sum (with $p = p_d$) and the numerator (with $p = p_n$) are estimated with initial prime-product length at most $z_0(m) = (1 - 2m)/p$ and additional prime-product length at most*

$$W_o(m) = \frac{1 - 1/p}{5}(1 - 2m).$$

*For $m > 1/2$, use the no-prime estimate with $z_0 = W = 0$. Retain the M1 and long-detector counts, fourth-moment Hölder inequality, buffered pointwise amplitude bounds and direct majorant used above. Permit arbitrary allowed comparison rectangles and arbitrary positive length*

*exponents $l_x, l_y, \ell$. Put $\Sigma = l_x + l_y$, $b = l_y - l_x \geq 0$ and $h = 1 - l_x + \ell > 0$, and retain the length inequality*

$$J_{\mathrm{geo}} := l_y - \ell - \frac{11b}{6} = \frac{\Sigma}{2} - \ell - \frac{4b}{3} > 0. \tag{16.1}$$

*Starting with a boundary at least $B_{\mathrm{ex}}$, no finite step in this comparison system can give a boundary below $B_{\mathrm{ex}}$. The successive bounds of Proposition 15.2 attain $B_{\mathrm{ex}}$ as an infimum. Thus $B_{\mathrm{ex}}$ is optimal for this system, including when the balanced relation $l_x + l_y + \ell = 1$ is not imposed.*

*Proof.* The condition (16.1) is among the strict inequalities of Table 8; hence the constructed sequence of bounds satisfies it. At $x = 1/2$, grant the full width $W_o$ at every ordinary length and discard the reflection, exceptional-character and prime-supply restrictions. This only strengthens the available comparison. The upper bound for the total prime-product length is

$$\frac{1-2m}{p} + W_o(m) = 4C(p)(1-2m), \qquad C(p) = \frac{p+4}{20p}.$$

Consequently the optimistic ordinary detector saving is $F_2(m) = 2m + 4C(p)(1-2m)$. Balancing it with $F_1(r) = 1/2 + r/2$ and the long-detector count gives (15.3). This is the best crossing of these three branches: the short count decreases and the long count increases with the total cutoff. Restoring any omitted length hypothesis can only increase the resulting count exponent. At the critical class $\delta = r_{\mathrm{ex}}$ this assertion also gives a necessary count for every cutoff $1 \leq \tau \leq 3/2$. The short crossing is

$$r(\tau) = \frac{4C_{\mathrm{ex}} + (2 - 8C_{\mathrm{ex}})\tau - 1/2}{5/2 - 8C_{\mathrm{ex}}}, \qquad m(\tau) = \tau - r(\tau),$$

and lies in $1/2 < r(\tau) \leq 1$, $0 < m(\tau) \leq 1/2$. The other allowed split is $(r, m) = (\tau, 0)$, by a limiting annulus. Here the ordinary count $1 - 4C_{\mathrm{ex}} r_{\mathrm{ex}}$ exceeds the long inverse count throughout the cutoff interval, since

$$-\frac{5}{12} + r_{\mathrm{ex}}(3/2 - 4C_{\mathrm{ex}}) > 0.$$

Thus the two adverse splits give precisely the decreasing short and increasing long bounds. The comparison must control this allowed amplitude class; no assertion that the arithmetic class is nonempty is used in the optimality argument.

For the numerator, at $m_\theta = 1 - l_y/h$ the contour coefficient vanishes. The amplitude bound $a \leq \delta/2$ and the optimistic width give the excess (15.4) in the balanced geometry. We now show that allowing the sum of the length exponents to vary does not remove the obstruction.

Let $\Sigma = l_x + l_y$, $b = l_y - l_x \geq 0$, $\ell > 0$, $h = 1 - l_x + \ell > 0$ and $l_y = (\Sigma + b)/2$. The free-length direct majorant of [Liu26b, Section 2.4], with its original normalization $C_\theta(s) = s - 5/6 + l_x/3 + \ell/6$, is

$$B_{\mathrm{low}} = \frac{5}{6} + \frac{l_x}{6} - \frac{\ell}{6} + G_{\mathrm{dir}} + \frac{1}{2}k(\Sigma, \ell),$$
$$G_{\mathrm{dir}} = \max\{0, b/12, b - l_y/2\}, \qquad k(\Sigma, \ell) = \max\{0, (1 + 3\ell - 2\Sigma)/4, \Sigma + \ell - 1, (3\ell - 1)/2\}.$$

The supplied comparison requires $B \geq B_{\mathrm{low}}$. Taking $G_{\mathrm{dir}} \geq b/12$ and the branches 0 and $\Sigma + \ell - 1$ of $k$ give

$$B \geq L_{\mathrm{dir}} = \frac{5}{6} + \frac{\Sigma}{12} - \frac{\ell}{6}, \quad B \geq L_K = \frac{1}{3} + \frac{7\Sigma}{12} + \frac{\ell}{3}.$$

No balanced relation $\Sigma + \ell = 1$ is assumed. The condition (16.1) is imposed separately from the direct majorant. Write $N_h = h - l_y = 1 - \Sigma + \ell$. Direct calculation gives

$$N_h - \frac{2}{5}h = \frac{341}{60} - \frac{35}{6}L_{\mathrm{dir}} - \frac{2}{3}L_K + \frac{3}{20}J_{\mathrm{geo}}. \tag{16.2}$$

An attempted improvement has $B < B_{\mathrm{ex}} < 8741/10000$. Therefore

$$N_h - \frac{2}{5}h > \frac{341}{60} - \frac{13}{2}B \geq \frac{101}{60000} > 0, \qquad m_\theta = \frac{N_h}{h} > \frac{2}{5}.$$

We must also compare with the pointwise estimate supplied by the buffered amplitude class. Put $r = r_{\mathrm{ex}}$, $C = C_{\mathrm{ex}}$ and $R = R_C(r)$. For $0 \leq m \leq 1/2$, the numerator and its allowed smooth weights satisfy, up to their fixed seminorms,

$$|S_m(u)|^2 \ll U^{rm+\epsilon}, \qquad \sum_{u\in\mathscr{B}} |S_m(u)| \ll U^{P_{\mathrm{pt}}(m;R)+\epsilon}, \qquad P_{\mathrm{pt}}(m;R) = R + \frac{rm}{2}.$$

The optimistic amplified fourth-moment estimate has first-moment exponent

$$P_4(m;R,C) = \frac{3R+1}{4} - Cr(1-2m).$$

The detector crossing is

$$m_d = \frac{1-R-4Cr}{2r(1-4C)} = \frac{2(1-12Cr)}{9r(1-4C)} \in (0.39816, 0.39818),$$

where the second equality uses (15.10). Thus

$$P_{\mathrm{pt}}(m;R) - P_4(m;R,C) = \frac{r(1-4C)}{2}(m-m_d) > 0 \qquad \left(\frac{2}{5} \leq m \leq \frac{1}{2}\right). \tag{16.3}$$

Since $0 < r < 1$, absolute ideal counting gives the still larger exponent $R + m/2 \geq P_{\mathrm{pt}}(m;R)$. Neither pointwise estimate improves on $P_4$ at the lengths used below.

First suppose $T_y = 2l_y - h > 0$, so $m_\theta < 1/2$. Evaluating the optimistic high-side comparison at $(\delta, x, d, m) = (r_{\mathrm{ex}}, 1/2, h, m_\theta)$ gives $B \geq H_{\mathrm{hi}}$, where

$$H_{\mathrm{hi}} = \frac{1+r_{\mathrm{ex}}}{2} - \frac{l_y}{2} + h\left(\frac{3R_{C_{\mathrm{ex}}}(r_{\mathrm{ex}})+1}{4} - \frac{1}{6}\right) + \frac{r_{\mathrm{ex}}-1}{2}\ell - C_{\mathrm{ex}}r_{\mathrm{ex}}T_y.$$

Using (15.10), the coefficient of $b$ vanishes and

$$H_{\mathrm{hi}} = 1 + (2C_{\mathrm{ex}} + 1/2)r_{\mathrm{ex}} - (1/2 + 2C_{\mathrm{ex}}r_{\mathrm{ex}})\Sigma + (2C_{\mathrm{ex}} + 1/2)r_{\mathrm{ex}}\ell.$$

Set $D_{\mathrm{hi}} = 16C_{\mathrm{ex}}r_{\mathrm{ex}} + 2r_{\mathrm{ex}} + 3$ and

$$\lambda_{\mathrm{dir}} = \frac{44C_{\mathrm{ex}}r_{\mathrm{ex}} + 7r_{\mathrm{ex}} + 4}{3D_{\mathrm{hi}}}, \qquad \lambda_K = \frac{4C_{\mathrm{ex}}r_{\mathrm{ex}} - r_{\mathrm{ex}} + 2}{3D_{\mathrm{hi}}}, \qquad \lambda_{\mathrm{hi}} = \frac{1}{D_{\mathrm{hi}}}.$$

These numbers are positive and sum to one. The exact dual identity

$$\lambda_{\mathrm{dir}}L_{\mathrm{dir}} + \lambda_K L_K + \lambda_{\mathrm{hi}}H_{\mathrm{hi}} = \frac{44C_{\mathrm{ex}}r_{\mathrm{ex}} + 7r_{\mathrm{ex}} + 7}{3D_{\mathrm{hi}}} = B_{\mathrm{ex}} \tag{16.4}$$

proves $B \geq B_{\mathrm{ex}}$ with $\Sigma$ free, the condition (16.1) retained, and the other auxiliary hypotheses of the norm estimate optimistically removed.

Now suppose $T_y \leq 0$. At $m = 1/2$, no prime width remains, and the optimal contour endpoint is $\theta = 0$. Equation (16.3) still excludes the buffered pointwise alternative at this endpoint. The necessary high-side majorant is now $H_{\mathrm{hi},0} = H_{\mathrm{hi}} + C_{\mathrm{ex}}r_{\mathrm{ex}}T_y$. Put

$$D_{\mathrm{hi},0} = 19 - (12C_{\mathrm{ex}} + 9)r_{\mathrm{ex}}, \quad \lambda_K^0 = \frac{3\{2 - (4C_{\mathrm{ex}} + 3)r_{\mathrm{ex}}\}}{D_{\mathrm{hi},0}},$$
$$\lambda_{\mathrm{hi}}^0 = \frac{13}{D_{\mathrm{hi},0}}, \quad \lambda_T^0 = \frac{18C_{\mathrm{ex}}r_{\mathrm{ex}} + 7r_{\mathrm{ex}} + 4}{2D_{\mathrm{hi},0}}.$$

All are positive and $\lambda_K^0 + \lambda_{\mathrm{hi}}^0 = 1$. An exact identity gives

$$\lambda_K^0 L_K + \lambda_{\mathrm{hi}}^0 H_{\mathrm{hi},0} + \lambda_T^0 T_y = \frac{52 + b(44C_{\mathrm{ex}}r_{\mathrm{ex}} + 7r_{\mathrm{ex}} + 4)}{4D_{\mathrm{hi},0}}.$$

Since $T_y \le 0$ and $b \ge 0$, this forces $B \ge 13/D_{\mathrm{hi},0} = 0.89697374236981439\ldots > B_{\mathrm{ex}}$. Thus neither edge of the free-length geometry avoids the obstruction.

*Separate moment parameters.* The count is monotone in its effective amplification parameter:

$$\frac{\partial R_C(r)}{\partial C} = -\frac{4r(6r-5)^2}{(48Cr - 80C - 18r + 25)^2} < 0, \qquad C'(p) = -\frac{1}{5p^2} < 0.$$

For fixed $m \le 1/2$, write

$$P_4(m; R, C(p_n)) = \frac{3R+1}{4} - C(p_n)r(1-2m), \qquad P_{\mathrm{pt}}(m; R) = R + \frac{rm}{2}.$$

The relevant derivatives are

$$\partial_R P_4 = \frac{3}{4}, \qquad \partial_R P_{\mathrm{pt}} = 1, \qquad \partial_{p_n} P_4(m; R, C(p_n)) = \frac{r(1-2m)}{5p_n^2} \ge 0, \qquad \partial_{p_d} R_{C(p_d)}(r) > 0.$$

Now take

$$m = \begin{cases} m_\theta, & T_y > 0, \\ 1/2, & T_y \le 0, \end{cases} \qquad R_{p_d} = R_{C(p_d)}(r).$$

For $p_d, p_n \ge p_{\mathrm{ex}}$,

$$\min\{P_4(m; R_{p_d}, C(p_n)), P_{\mathrm{pt}}(m; R_{p_d})\} \ge \min\{P_4(m; R, C_{\mathrm{ex}}), P_{\mathrm{pt}}(m; R)\} = P_4(m; R, C_{\mathrm{ex}}),$$

where (16.3) gives the equality at the chosen $m$. Thus the necessary comparisons remain $B \ge H_{\mathrm{hi}}$ and $B \ge H_{\mathrm{hi},0}$.

Starting from any known boundary at least $B_{\mathrm{ex}}$, the moment hypothesis $p_d, p_n \ge 12B_{\mathrm{known}} - 6$ prevents an iteration of this system from making its first crossing below $B_{\mathrm{ex}}$. Variable comparison lengths cannot change this conclusion because the proof already grants the full origin width everywhere.

The exact dual identities and their coefficient signs are verified in Appendix D. Proposition 15.2 proves attainability as an infimum with the actual hypotheses of the norm estimate. This completes the optimality statement. □

*Remark* 16.2. At the critical parameters the numerator length is $0.42413539306165\ldots$ and the detector crossing length is $0.39816812106788\ldots$. The origin ceiling is the binding width constraint; the optimized reflection and exceptional constraints have strict slack. At $x = 1/2$, the conservative numerator mean and mean of the selected factors both equal $\delta/2$. Further optimization of these choices therefore does not remove the contact obstruction. The theorem concerns the stated comparison system. A stronger moment, a better detector count, a sharper direct estimate, or a different theta average could leave this system and improve the boundary. Part III leaves it by enlarging the origin width $W_o$ through a cubic unit twist.

# Part III
# Unit twists and returned sums

## 17 The unit-twisted norm and its statements

In this section, we fix the prime-length constant and the relaxation parameter, recall the bound for returned sums from Appendix H, and state Theorem 17.1. Sections 18–22 prove it.

Sections 23–25 apply it to the comparison of Part II and prove Theorem 1.1; Section 26 proves the optimality of $B_{\mathrm{tw}}$.

We keep the notation of Section 12. Fix $p = 6\kappa \in [112/25, 9/2]$, $\varrho = 1 - 1/p$ and

$$\alpha_{\max}(p) = \frac{4p-6}{p(p+24)}, \qquad \lambda_{\max}(p) = 1 + p\alpha_{\max}(p) = \frac{5p+18}{p+24}, \tag{17.1}$$

$$\rho_{\max}(p) = \frac{6(p-6)(2p-3)}{p^2-193p+198}, \qquad \frac{d\rho_{\max}}{dp} = -\frac{42(53p^2-108p-72)}{(p^2-193p+198)^2} < 0. \tag{17.2}$$

Hence

$$\frac{24}{289} = \rho_{\max}(9/2) \le \rho_{\max}(p) \le \rho_{\max}(112/25) = \frac{5662}{67351} < \frac{1}{10}. \tag{17.3}$$

All numerical margins below use only $0 \le \rho < \rho_{\max}(p)$ and (17.3).

After ordinary reflection puts $0 \le n_1, n_2 \le M/2$, Corollary H.3 gives, for $G = M - n_1 - n_2 > 0$ and every $0 < \alpha < \alpha_{\max}(p)$,

$$z < \left(\frac{1}{p} + \alpha\right) G \implies \sum_{\psi_k \notin \Theta} |P(k)|^2 \ll U^{M+\epsilon}, \tag{17.4}$$

uniformly on compact parameter sets with positive margins; here the sum is over the parameters $k$ whose character $\psi_k$ induces outside $\Theta$. This replaces the bound $z < G/p$ of Lemma 12.2. For very small $G$ the remaining primes are removed and the no-prime M2 estimate is used.

**Theorem 17.1** (Unit-twisted amplified norm)**.** *Under the hypotheses and conventions of Theorem 12.1, with $0 \le \rho < \rho_{\max}(p)$, replace (12.1) by*

$$A_{\max} + (4-\rho)W < 1, \qquad A_{\max} = 2r + \frac{1-2r}{p} + W, \tag{17.5}$$

*and retain*

$$A_{\max} + 3W < \frac{5}{6} + \frac{2t}{3}, \qquad 2 + 2t - 4m_- + pW < 1, \qquad W \le \frac{1}{25}. \tag{17.6}$$

*Then (12.4) and (12.5) hold, uniformly on compact parameter sets with strict conditions.*

Lemma 21.1 shows that $\rho_{\max}(p)$ is the supremum of the relaxation parameters for which the return estimate closes, and Lemma H.4 shows that $\alpha_{\max}(p)$ is extremal for the decomposition of the returned sums. No uniformity is asserted as the margin in (17.5) or $\rho_{\max}(p) - \rho$ tends to zero.

With the restriction $W \le 1/25$ of Theorem 12.1 kept for the returned sums, one may take only $\alpha = 1/25$, and the same argument gives the weaker boundary 0.8740670193055 ....

## 18 The strict margin

In this section, we extract an absolute margin from the strict first condition (17.5). It is used for the second origin in Section 20 and for the returned gap in Section 21.

Put

$$\Delta = \varrho(1-2r) - (5-\rho)W > 0.$$

Set aside a fixed fraction of $\Delta$ for the edge and support errors, and choose $\nu > 0$ below the remaining margin. For $m \le r$, $w \le W$ and $z \le (1-2m)/p$, put

$$d_m = 1 - 2m, \qquad G_0 = d_m - 6w, \qquad A_t = 2m + z + w.$$

Then

$$\varrho d_m - (5-\rho)w \ge \Delta, \tag{18.1}$$

$$A_t - 1 + 4w \le \rho w - \nu, \tag{18.2}$$

$$G_0 \ge k_p w + \frac{\nu}{\varrho}, \qquad k_p = \frac{6 - p - p\rho}{p-1}. \tag{18.3}$$

Indeed,

$$A_t - 1 + 4w \le -\varrho d_m + 5w, \qquad G_0 = \frac{\varrho d_m - (5-\rho)w}{\varrho} + k_p w.$$

The second-origin term and the homogeneous return are separate summands, so (18.2) and (18.3) bound different terms.

## 19 Selecting twisted positions

In this section, we select the twisted unit positions before any coefficient extension. These positions are not extended; their primes enter only through the character $\chi_n(a)^2$. The effect of the twist is computed in Section 20.

Let $w_f, f_0, f_1, g_{\mathrm{good}}, v, c_{\mathrm{cor}}$ be the first-common, fixed-unit, fixed-nonunit, free-unit, free-nonunit and fifth-power-correction widths of the proof of Theorem 12.1:

$$w = w_f + f_0 + f_1 + g_{\mathrm{good}} + v + c_{\mathrm{cor}}, \qquad L_1 = f_1 + 4T_1 + e_1, \qquad L = v + 4T + e.$$

Put $\mu = \min(c_C, d_D) \ge w_f$ and $B_C = (3c_C - 5d_D - R)_+/6$. The origin loss and the homogeneous coefficient are

$$\mathcal{S} = 4w_f + 5f_1 + 5v + 4c_{\mathrm{cor}} + 4(T_1 + T) + S_1 + S + e_1 + e + \mu + B_C, \tag{19.1}$$

$$\mathcal{C}_{\mathrm{hom}} = -w + w_f - 3(T_1 + T) - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e, \tag{19.2}$$

and the second-origin excess is at most $A_t - 1 + 4w - \mathcal{S}$.

The complementary bound of Proposition 4.8 assumes $A_t + 4t_{\mathrm{loc}} < 1$ and $A_t + 3t_{\mathrm{loc}} < 1$. The first enters only through the zero-frequency term of the second Poisson formula,

$$\mathcal{O}_{\mathrm{total}} \le \mathcal{R}_{\mathrm{loc}} + A - M - \mu - B_C = A_t - 1 + 4w - \mathcal{S},$$

which is bounded directly below. The second follows from (18.2):

$$A_t + 3t_{\mathrm{loc}} \le A_t + 3w \le 1 - (1-\rho)w - \nu < 1.$$

Classify the first-common and nonunit positions before extending any free unit polynomial. No position is twisted if

$$\rho w \le \frac{\nu}{2} \qquad \text{or} \qquad w_f + f_1 + v \ge \frac{\rho w}{4},$$

since then the second origin has margin $\nu/2$, respectively $\mathcal{S} \ge \rho w$. Otherwise

$$w > \frac{\nu}{2\rho}, \qquad w - w_f - f_1 - v > \Big(1 - \frac{\rho}{4}\Big)w, \tag{19.3}$$

and the second expression is the width of the fixed and free unit positions before extension.

For each fixed local configuration, order the eligible original positions and select whole positions by their nominal widths until their total reaches $\rho w/3$. This is done after fixing the factor list and before the primes in those positions vary. Set

$$s = \frac{\rho w}{3} + \vartheta_{\mathrm{sel}}, \qquad 0 \le \vartheta_{\mathrm{sel}} \le \frac{\nu}{1000}, \tag{19.4}$$

after making the complete-factor mesh at most $\nu/1000$. This is possible, since by (17.3)

$$1-\frac{\rho}{4}>\frac{97}{100},\qquad \frac{s}{w}\le\frac{\rho}{3}+\frac{2\rho}{1000}=\frac{503}{1500}\rho<\frac{3}{100}. \tag{19.5}$$

The lower bound for $w$ in (19.3) lets the mesh be fixed before the factor list. The difference between nominal widths and prime norms is part of the $O_N(1/\log U)$ support error, $N$ the number of original positions. Only untwisted unit positions are extended, so the selection does not involve $c_{\mathrm{cor}}$.

## 20 The cubic unit twist

In this section, we compute the effect of the twist on the local terms of the proof of Theorem 12.1: the completed parameters, the second origin, the exceptional term and the terminal terms. The returned sums are treated in Section 21.

On the unit locus,

$$\mathcal{G}(n,a^4h)=\overline{\chi_n(a)^4}\mathcal{G}(n,h)=\chi_n(a)^2\mathcal{G}(n,h),$$

and the cubic character $\chi_n(a)^2$ joins the common character of every remaining ordinary and prime factor. The coprimality condition $(n,a)=1$ is kept.

For a free twisted tuple, Cauchy's inequality loses at most $U^{s+\epsilon}$; fixed unit tuples lose no more. With the tuple fixed, the untwisted orbit and divisor argument has the same coefficient energy, since the new phases have modulus at most one.

The nonunit divisor Cauchy is applied at the original frequency before any frequency enlargement, and fixed unit positions are chosen before their positive enlargement; these positions are disjoint from the nonunit supports. Every extended untwisted unit position keeps its fifth-power correction and its $q_p^{-1}$ coefficient mass. The twisted positions are not extended, and the auxiliary sixth-power prime pool excludes them within its logarithmic exclusion allowance.

The completed parameters change by

$$K\mapsto K-4s,\quad m_*\mapsto m_*+4s,\quad q_*\mapsto q_*+s,\quad M_*\mapsto M_*+5s, \tag{20.1}$$

and the nominal cutoff by $-4s$, so $K_0-K$, $D_0$, $J$ and the auxiliary step are unchanged. The one-sided allowance increases by $s$, and the count of free unit tuples contributes at most $s$:

$$\mathcal{R}_{\mathrm{loc}}\mapsto\mathcal{R}_{\mathrm{loc}}+2s,\qquad \mathcal{O}_{\mathrm{total}}\mapsto\mathcal{O}_{\mathrm{total}}-3s.$$

By (18.2) and (19.4),

$$\mathcal{O}_{\mathrm{total}}\le-\nu-3\vartheta_{\mathrm{sel}}-\mathcal{S}\le-\nu.$$

The exceptional mean exponent $F_1^{\mathrm{loc}}$ increases by $s/6$, so

$$\text{exceptional exponent}\mapsto\text{exceptional exponent}+2s-\frac{5}{6}\cdot5s-\frac{s}{6}=\text{exceptional exponent}-\frac{7s}{3},$$

and condition (12.2) remains sufficient.

The two raw terminal ceilings with the relaxed origin are, by (17.3),

$$-\frac{97}{100}+\frac{12+3\rho}{25}<-\frac{47}{100},\qquad -\frac{93}{200}+\frac{6+3\rho}{25}<-\frac{21}{100}. \tag{20.2}$$

The twist changes the two terminal expressions by

$$\mathcal{R}_{\mathrm{loc}}+3A-M_*:\ 2s-5s=-3s,\qquad \mathcal{R}_{\mathrm{loc}}+3A-\tfrac{7}{2}m_*-q_*:\ 2s-\tfrac{7}{2}\cdot4s-s=-13s.$$

A new residual or formal length below its cutoff makes the old one smaller by $4s$, so (20.2) remains valid, including small or negative completed lengths. The other terms satisfy the positive-length hypotheses of Proposition 4.8.

# 21 The returned sums

In this section, we bound the restoration exponent of every returned sum, using (17.4) and the margin (18.3) of Section 18, and we prove in Lemma 21.1 that $\rho_{\max}(p)$ is extremal.

## 21.1 The returned gap

In this subsection, we bound the gap of every returned sum below, using the shifts (20.1) of Section 20. The bound is used in Sections 21.2 and 21.3.

Fix $\rho < \rho_{\max}(p)$. Since $E_\alpha(p,\rho)$ below is continuous and decreasing in $\alpha$, and $E_{\alpha_{\max}}(p,\rho) < 0$ by Lemma 21.1, choose $\alpha < \alpha_{\max}(p)$ with

$$E_\alpha(p,\rho) \le 0, \qquad \frac{p\alpha}{6\varrho} - \frac{7-5p\alpha}{6000} > \frac{1}{12}, \tag{21.1}$$

which is possible because the second expression exceeds $1/12$ at $\alpha = \alpha_{\max}(p)$ (Appendix E.1).

In the formulas below, $M_*, q_*, m_*$ denote the completed values before the twist; the shifts in (20.1) are added once. Set

$$\lambda = 1 + p\alpha, \qquad e_z = d_m - pz \ge 0, \qquad T_* = 2m - L_1 - L - 5c_{\mathrm{cor}}.$$

Then

$$G_* = M_* - T_* = G_0 + 6w_f + 5f_1 + 5v + T_1 + T. \tag{21.2}$$

The first-common extraction has prime width $b_C = w_f + z_{C,\mathrm{old}}$ and ordinary removal $c_C - b_C$. An auxiliary extraction $a$ removes only ordinary length. On a second-common side, the full removal $c_2 \ge g_2$ contains prime width $z_2$ and ordinary removal $c_2 - z_2$.

The returned width and gap are

$$\begin{aligned} M' &= M_* + 5s + J - g_{\mathrm{aux}} - g_2 + t_2, \\ G' &= G_* + 5s + J - g_{\mathrm{aux}} - g_2 + t_2 + (c_C - b_C) + a + (c_2 - z_2). \end{aligned}$$

With $J = D_0 - c_C - 2a + a_o$ and

$$D_0 + a_o - J_+ = \min(D_0 + a_o, c_C + 2a) \ge \mu - O(\xi) - O(1/\log U), \qquad g_{\mathrm{aux}} - J_+ = O(\sigma),$$

dropping $t_2 \ge 0$ and $c_2 - g_2 \ge 0$ gives

$$G' \ge G_0 + 5w_f + 5f_1 + 5v + T_1 + T + \mu + 5s - z_{C,\mathrm{old}} - a - z_2 - O(\sigma + \xi + 1/\log U). \tag{21.3}$$

The remaining prime width is at most $z - z_{C,\mathrm{old}} - z_2$. A final ordinary removal increases the gap, a final prime removal lowers the prime width without lowering the gap, and ordinary reflection increases the gap. Thus (21.3) holds for every final returned sum.

## 21.2 The active estimate

In this subsection, we close the return estimate when the restoration exponent is positive, using (21.3) of Section 21.1 and the margin (18.3) of Section 18.

Choose an application parameter $\widehat{\alpha}$ with $\alpha < \widehat{\alpha} < \alpha_{\max}(p)$; the coefficient in the return estimate remains $\lambda = 1 + p\alpha$. Let $\eta_{\mathrm{mesh}}$ bound one complete prime factor, and put $D_\lambda = pz' - \lambda G'$. For $G' > \tau > 0$, strip complete factors of total width $v_{\mathrm{str}}$ until $pv_{\mathrm{str}} \ge (D_\lambda)_+$ or the list is empty. Then

$$v_{\mathrm{str}} \le \min\{z', (D_\lambda)_+/p + \eta_{\mathrm{mesh}}\}, \qquad \kappa v_{\mathrm{str}} \le (D_\lambda)_+/6 + \kappa\eta_{\mathrm{mesh}}.$$

If factors remain, their width satisfies

$$z' - v_{\mathrm{str}} \le \frac{\lambda}{p} G' < \Big(\frac{1}{p} + \widehat{\alpha}\Big) G',$$

with gap at least $(\widehat{\alpha}-\alpha)\tau$. Apply (17.4) with $\widehat{\alpha}$; if none remain, apply the no-prime M2. Thus the squared restoration exponent is bounded by

$$\mathcal{L}_{\mathrm{ret}} \le \frac{(pz'-\lambda G')_+}{6} + O(\epsilon), \tag{21.4}$$

with the mesh and subsidiary losses chosen inside $\epsilon$. The numerator before the positive part is at most

$$pz-\lambda(G_0+5w_f+5f_1+5v+T_1+T+\mu+5s)-(p-\lambda)(z_{C,\mathrm{old}}+z_2)+\lambda a+O(\sigma+\xi+1/\log U).$$

The normalization of Proposition 4.8 supplies $\Delta' \ge a+\ell_a \ge 0$. The exponent to be bounded is

$$\mathcal{B}_{\mathrm{ret}} = \mathcal{C}_{\mathrm{hom}}+2s+\mathcal{L}_{\mathrm{ret}}-\Delta', \qquad \frac{\lambda a}{6}-\Delta' \le -\Big(1-\frac{\lambda}{6}\Big)a-\ell_a \le 0,$$

since $\lambda<6$. Thus the allowance is subtracted once, including when $a=0$.

If the positive part in (21.4) is active, then, with $pz = G_0+6w-e_z$ and $\mu \ge w_f$,

$$\mathcal{C}_{\mathrm{hom}}+2s+\frac{pz'-\lambda G'}{6}-\Delta' \le -\frac{p\alpha}{6}G_0+\frac{7-5p\alpha}{6}s-\frac{e_z}{6}+O(\sigma+\xi+\epsilon+1/\log U), \tag{21.5}$$

besides $\lambda a/6-\Delta' \le 0$, the discarded terms being

$$-p\alpha w_f-\frac{\lambda}{6}(\mu-w_f)-\frac{5\lambda}{6}(f_1+v)-\Big(3+\frac{\lambda}{6}\Big)(T_1+T)-4c_{\mathrm{cor}}-S_1-S-e_1-e-\frac{p-\lambda}{6}(z_{C,\mathrm{old}}+z_2).$$

By (18.3) and (19.4),

$$\text{coefficient of } w = E_\alpha(p,\rho) = \frac{(7-5p\alpha)\rho}{18}-\frac{p\alpha(6-p-p\rho)}{6(p-1)}, \tag{21.6}$$

and, by (21.1) and $\vartheta_{\mathrm{sel}} \le \nu/1000$, the remaining terms give

$$E_\alpha(p,\rho)w-\frac{p\alpha\nu}{6\varrho}+\frac{7-5p\alpha}{6}\vartheta_{\mathrm{sel}} \le -\Big(\frac{p\alpha}{6\varrho}-\frac{7-5p\alpha}{6000}\Big)\nu < -\frac{\nu}{12}. \tag{21.7}$$

**Lemma 21.1** (The relaxation parameter is extremal)**.** *For $p \in [112/25, 9/2]$ and $0<\alpha\le\alpha_{\max}(p)$,*

$$\frac{\partial E_\alpha}{\partial\rho}=\frac{7-5p\alpha}{18}+\frac{p^2\alpha}{6(p-1)}>0, \qquad \frac{\partial E_\alpha}{\partial\alpha}=-\frac{p\{18-3p+(2p-5)\rho\}}{18(p-1)}<0 \quad (\rho\ge 0),$$
$$E_{\alpha_{\max}}(p,\rho)\le 0 \iff \rho\le\rho_{\max}(p).$$

*Let $\rho_{\max}(p)<\rho\le 1$ and $0<\alpha\le\alpha_{\max}(p)$. Then $E_\alpha(p,\rho)>0$, and there are scalar configurations satisfying the numerical length conditions (17.5) and (17.6) at this larger value of $\rho$, with origin margin $\Delta$ arbitrarily small compared with $w$, at which the right side of (21.5), with the smallest $s$ allowed by the second origin, is positive. Thus $\rho_{\max}(p)$ is the supremum of the relaxation parameters for which the return estimate (21.5) closes uniformly up to the origin boundary. This is not a lower bound for the underlying norms.*

*Proof.* $E_\alpha$ is affine in $\rho$ and in $\alpha$ with the displayed slopes; $18-3p>0$ and $2p-5>0$ on $[112/25, 9/2]$. The equation $E_{\alpha_{\max}}(p,\rho)=0$ is solved by (17.2). The coefficient of $s$ in (21.5) is $(7-5p\alpha)/6>0$. Consider the scalar configurations

$$w=W=\frac{1}{50}, \quad t=\frac{1}{4}, \quad d_m=\frac{(5-\rho)w+\Delta}{\varrho}, \quad m=r=\frac{1-d_m}{2}, \quad z=\frac{d_m}{p},$$

with free-unit width $g_{\mathrm{good}} = w$ and all other local widths and common supports zero. Thus the twisted unit primes are free, as in the Cauchy inequality of Section 20. For sufficiently small $\Delta > 0$, the remaining length conditions are strict; in particular

$$A_t + 3w = 1 - (1-\rho)w - \Delta < 1, \qquad 2 + 2t - 4m + pw = \frac{1}{2} + 2d_m + pw < 1.$$

Here $\mathcal{S} = e_z = 0$ and $G_0 = k_p w + \Delta/\varrho$ exactly. The actual origin requires $3s \geq \rho w - \Delta$. Substitution of its smallest allowed value into the right side of (21.5) gives

$$E_\alpha(p,\rho)w - \left\{\frac{p\alpha}{6\varrho} + \frac{7 - 5p\alpha}{18}\right\}\Delta.$$

Since $E_\alpha(p,\rho) > 0$, this is positive when $\Delta/w$ is sufficiently small. The argument uses the actual margin $\Delta$; it does not impose equality in the weakened bound with the margin $\nu < \Delta$ set aside in Section 18. Thus the return estimate does not close for a larger relaxation parameter uniformly up to the origin boundary. □

### 21.3 The inactive estimate and the untwisted configurations

In this subsection, we treat the inactive positive part and the two configurations of Section 19 without twist.

If the positive part is inactive, then $\Delta' \geq 0$ gives $\mathcal{B}_{\mathrm{ret}} \leq \mathcal{C}_{\mathrm{hom}} + 2s + O(\epsilon)$. By (19.3) and (19.5)

$$\mathcal{C}_{\mathrm{hom}} + 2s \leq -w + w_f + 2s \leq \Big(-1 + \frac{\rho}{4} + \frac{1006}{1500}\rho\Big)w = \Big(-1 + \frac{1381}{1500}\rho\Big)w < -\frac{9}{10}w. \tag{21.8}$$

Both twisted alternatives are therefore strictly negative before the losses set aside above, and (19.3) bounds $w$ below in terms of $\nu$.

Without twist, $s = 0$. With the completed defect

$$D_f = 6w - e_z - (p-1)z_{C,\mathrm{old}} - 5w_f - 5f_1 - T_1 - 5v - T,$$

(19.2) gives

$$\begin{aligned}\mathcal{C}_{\mathrm{hom}} + \frac{D_f - \mu}{6} = \frac{w_f - \mu}{6} - \frac{5}{6}(f_1 + v) - \frac{19}{6}(T_1 + T) - \frac{p-1}{6}z_{C,\mathrm{old}} - \frac{e_z}{6} \\ - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e \leq 0,\end{aligned}$$

and $\mathcal{C}_{\mathrm{hom}} \leq 0$, so $\mathcal{C}_{\mathrm{hom}} + (D_f - \mu)_+/6 \leq 0$. Lemma 12.2 applies to this defect, and its allowance contains the auxiliary extraction once. Section 19 gives the second-origin margin and (20.2) the terminal terms. These configurations close, including $w = 0$, with a nonpositive homogeneous exponent.

## 22 Uniform choices and the proof of Theorem 17.1

In this section, we fix the order of choices and complete the proof of Theorem 17.1 from Sections 18–21.

If a reflected returned sum has $0 \leq G' \leq \tau$, remove all remaining primes and use the no-prime M2, since

$$\kappa z' - \frac{(pz' - \lambda G')_+}{6} = \frac{\min\{pz', \lambda G'\}}{6} \leq \frac{\lambda\tau}{6}.$$

The same applies for $M' \leq \tau$, since $G' \leq M'$. On $G' > \tau$, the application parameter $\widehat{\alpha} > \alpha$ supplies the fixed strict margin proved in Section 21.2.

The reflection lemma includes a length buffer $\xi$. First replace bounded subunit ordinary scales by declared length zero; the change is at most the support error $\theta_N = O_N(1/\log U)$ per factor. The resulting nonnegative lengths satisfy $n_i \le M'/2+\xi+\theta_N$, and a nonempty parameter range has $m' \ge -\theta_N$. Set

$$\delta_{\mathrm{buf}} = 2\xi + 2\theta_N, \quad \widetilde{m} = m' + \delta_{\mathrm{buf}}, \quad \widetilde{M} = M' + \delta_{\mathrm{buf}}, \quad \widetilde{G} = G' + \delta_{\mathrm{buf}},$$

where $G' = M' - n_1 - n_2$ uses these nonnegative declared lengths. Keeping $q'$ unchanged gives

$$\widetilde{m} > 0, \qquad 0 \le n_i \le \widetilde{M}/2, \qquad 0 \le \widetilde{G} \le \widetilde{M}, \qquad q'/\widetilde{M} \le 1.$$

The norm exponent increases by $\delta_{\mathrm{buf}}$, and the gap increases by $\delta_{\mathrm{buf}}$. Including the clamping error, the net correction is $O(\xi + \theta_N)$ in (21.3)–(21.5). All applications of M2 and the returned norm above use the buffered widths. Choose $\xi$ and the threshold so that $\delta_{\mathrm{buf}} < \tau$ and this correction lies inside the set-aside loss and the strict active margin.

The choices are made in this order:

1. the geometry, a positive $\nu$ from the strict origin condition, and the strict exceptional and reflection margins;
2. the requested final loss, and $\alpha < \alpha_{\max}(p)$ satisfying (21.1), followed by $\widehat{\alpha} \in (\alpha, \alpha_{\max}(p))$;
3. the gap cutoff $\tau$, the auxiliary steps and the edge buffers, with total error in (21.5) below a fixed fraction of $\nu/12$, and inside the final loss without twist;
4. the complete-factor mesh, at most $\nu/1000$, and the other losses;
5. the factor list, the arithmetic data, the finite families of smooth weights and the derivative orders;
6. the height exponent and the threshold, with $1/\log U$ below the fixed margins.

The twisted tuple is fixed before the untwisted orbit argument. Once the mesh and the factor list are fixed, there are finitely many local configurations, and constants may depend on them and on $\nu$. All coprimality conditions and moving characters remain in the actual presentations. Integration against the weighted Fourier measures proves the centered norm, with the structural count once.

The comparison-product reflection and the expansion of the coprimality conditions use the unchanged third condition, and the triangle inequality with the comparison product gives the nonexceptional uncentered norm. This proves Theorem 17.1. No constant is uniform as $\nu \downarrow 0$; each finite iteration step has its own positive margins.

# 23 The comparison with the twisted origin width

In this section, we modify the scalar comparison of Section 14 by the origin width of Theorem 17.1, define the algebraic endpoint $B_{\mathrm{tw}}$ and prove its contact identities, Lemma 23.1. Section 24 gives the strict family, and Section 26 uses the same identities for optimality.

## 23.1 The modified comparison

In this subsection, we record the changes to Sections 13–15.

Lemma 13.1, Proposition 13.2 and the numerator estimate of Section 14 use Theorem 12.1 only through its conclusions (12.4) and (12.5) under its length conditions. They therefore hold with Theorem 17.1 in its place, under (17.5) and (17.6).

Put $\rho = \rho_{\max}(p)$ and $c = c(p) = 5 - \rho_{\max}(p)$, where $p$ is the moment parameter of the estimate in which the width occurs; Section 25 uses slightly smaller values of $\rho$. In Section 14 replace

$$W_o(m) = \frac{\varrho}{5}(1-2m) \;\longrightarrow\; \frac{\varrho}{c}(1-2m), \qquad G_N = \frac{a\varrho}{10}T_y \;\longrightarrow\; \frac{a\varrho}{2c}T_y, \tag{23.1}$$

and the origin branch $\varrho(1-2m)/5$ of $W_n$ by $\varrho(1-2m)/c$. The width $W_j$, the counts $R_i, R_s$, the exponents $E, S, D_{\text{short}}$, the crossings $m_i, T_i$ and the moment parameter $p$ are unchanged. Thus

$$z_0(m) + W_o(m) = 4C_\rho(p)(1-2m), \qquad C_\rho(p) = \frac{p+4-\rho}{4p(5-\rho)}, \tag{23.2}$$

and $C_\rho(p)$ at $\rho = 0$ is the constant $C(p)$ of (15.2).

The iteration starts from

$$B_0 = 0.87407285, \qquad \kappa_0 = 2B_0 - 1 = \frac{7481457}{10000000}.$$

Proposition 15.2 gives $\beta_*(F) \le B_{\text{ex}} < (1+\kappa_0)/2$, so $\kappa_0$ is a previously justified moment parameter.

## 23.2 The algebraic endpoint

In this subsection, we define $B_{\text{tw}}$ and the endpoint parameters, and prove Lemma 23.1.

Let $\ell_{\text{tw}}$ be the unique root in $(0.17043265, 0.17043266)$ of

$$\begin{aligned} f_{\text{tw}}(\ell) = {} & 184194\ell^6 - 19294821\ell^5 + 481077477\ell^4 - 1437647898\ell^3 \\ & + 1111054532\ell^2 + 46823455\ell - 33539275. \end{aligned} \tag{23.3}$$

Put

$$\begin{aligned} B_{\text{tw}} &= \frac{11}{12} - \frac{\ell_{\text{tw}}}{4}, \quad \kappa_{\text{tw}} = 2B_{\text{tw}} - 1, \quad p_{\text{tw}} = 6\kappa_{\text{tw}} = 5 - 3\ell_{\text{tw}}, \\ \widetilde{C}(p) &= C_{\rho_{\max}(p)}(p) = -\frac{p^2 - 200p - 684}{28p(p+126)}, \qquad C_{\text{tw}} = \widetilde{C}(p_{\text{tw}}). \end{aligned} \tag{23.4}$$

With $R_C$ and $\mathcal{Q}_{C,b,\ell}$ as in (15.3) and (15.4), define

$$r_{\text{tw}} = \frac{5 - 9\ell_{\text{tw}}}{6(1 + \ell_{\text{tw}} + 8C_{\text{tw}}\ell_{\text{tw}})}, \tag{23.5}$$

$$b_{\text{tw}} = -\left.\frac{\partial_\delta \mathcal{Q}_{C,0,\ell}(\delta)}{\partial_\delta \partial_b \mathcal{Q}_{C,b,\ell}(\delta)}\right|_{C=C_{\text{tw}},\, \ell=\ell_{\text{tw}},\, \delta=r_{\text{tw}}}. \tag{23.6}$$

The denominator in (23.6) is nonzero. Numerically,

$$\begin{aligned} \ell_{\text{tw}} &= 0.17043265149923064667912171558 7\ldots, \\ r_{\text{tw}} &= 0.44428449663005476928557852908 2\ldots, \\ b_{\text{tw}} &= 0.09579610595986973062810543980 8\ldots, \\ \kappa_{\text{tw}} &= 0.74811700758371800999377247553 9\ldots, \\ p_{\text{tw}} &= 4.48870204550230805\ldots, \qquad \rho_{\max}(p_{\text{tw}}) = 0.08362272823194174\ldots, \\ \alpha_{\max}(p_{\text{tw}}) &= 0.0934865580979076\ldots, \qquad C_{\text{tw}} = 0.0952172959982581\ldots. \end{aligned} \tag{23.7}$$

**Lemma 23.1** (Contact identities). *At $(C, b, \ell) = (C_{\text{tw}}, b_{\text{tw}}, \ell_{\text{tw}})$,*

$$\mathcal{Q}(r_{\text{tw}}) = \partial_\delta \mathcal{Q}(r_{\text{tw}}) = \partial_b \mathcal{Q}(r_{\text{tw}}) = 0, \qquad R_{C_{\text{tw}}}(r_{\text{tw}}) = \frac{5 + 12C_{\text{tw}}r_{\text{tw}}}{9}. \tag{23.8}$$

*The polynomials (23.3) and (1.1) are irreducible over $\mathbb{Q}$ and isolate the specified real roots.*

*Proof.* Clear the denominator of (15.4); its numerator $n$ is quadratic in $\delta$ and affine in $b$. Eliminating $\delta$ between $n$ and $\partial_b n$ gives, up to the nonzero factor $-24300(4C-1)^2(16C-5)(20C+7)$,

$$F(C,\ell) = 1600C^2\ell^2 - 320C^2\ell - 228C\ell^2 + 184C\ell - 20C - 103\ell^2 - 14\ell + 5. \tag{23.9}$$

Substituting $C = \widetilde{C}(5-3\ell)$, clearing denominators and taking the primitive part gives (23.3).

The eliminant alone does not prove contact. Substituting (23.5)–(23.6) in each expression of (23.8), clearing its denominator and reducing its numerator modulo $f_{\mathrm{tw}}$ gives zero; the denominators are nonzero on the isolating intervals. The substitution $\ell = 11/3 - 4B$ gives (1.1). Exact factorization and Sturm root counts prove the last assertion; the enclosures are in Appendix E. □

## 24 The strict comparison family

In this section, we verify the modified comparison of Section 23.1, with both detector widths, along a strict family through the endpoint of Section 23.2. Section 25 iterates it.

Let $\mathcal{Q}_i(\delta, x; \kappa, \ell, b, \rho) = E(R_i) - G_N$, $i = o, j$, with the modifications (23.1). At $x = 1/2$ one has $a = \delta/2$ and

$$\mathcal{Q}_o(\delta, 1/2; \kappa, \ell, b, \rho_{\max}(6\kappa)) = \mathcal{Q}_{\widetilde{C}(6\kappa),b,\ell}(\delta).$$

Define the strict family

$$b = b_{\mathrm{tw}}, \qquad \ell(\kappa) = \ell_{\mathrm{tw}} - \frac{\kappa - \kappa_{\mathrm{tw}}}{25}, \qquad \rho = \rho_{\max}(6\kappa), \qquad \kappa_{\mathrm{tw}} < \kappa \le \kappa_0, \tag{24.1}$$

and its closure

$$\mathcal{P}_{\mathrm{tw}} = \Big\{(\kappa, \ell(\kappa), b_{\mathrm{tw}}, \rho_{\max}(6\kappa)) : \ \kappa_{\mathrm{tw}} \le \kappa \le \kappa_0\Big\}.$$

**Lemma 24.1** (The endpoint and the strict family)**.** *At $(\kappa, \ell, b) = (\kappa_{\mathrm{tw}}, \ell_{\mathrm{tw}}, b_{\mathrm{tw}})$,*

$$\max\{\mathcal{Q}_o, \mathcal{Q}_j\} \le 0 \quad \text{on } \mathcal{D},$$

*with equality only at $(\delta, x) = (r_{\mathrm{tw}}, 1/2)$. On (24.1),*

$$\mathcal{Q}_i < -\frac{\kappa - \kappa_{\mathrm{tw}}}{100} \qquad (i = o, j, \ (\delta, x) \in \mathcal{D}), \tag{24.2}$$

*and*

$$\frac{\partial \mathcal{Q}_i}{\partial \rho} > -\frac{1}{100} \qquad \text{on } \mathcal{D} \times \mathcal{P}_{\mathrm{tw}}, \ \rho \in [0.083, 0.0841] \text{ free.} \tag{24.3}$$

*The inequalities of Table 8 hold on $\mathcal{P}_{\mathrm{tw}}$ with the modifications (23.1).*

*Proof.* At $x = 1/2$, clearing a positive denominator turns $\mathcal{Q}_o$ into a quadratic with a double zero at $r_{\mathrm{tw}}$ and negative leading coefficient, by Lemma 23.1. The branch $\mathcal{Q}_j$ has negative leading coefficient and negative discriminant, and its denominator is positive. Thus $\mathcal{Q}_o(\delta, 1/2) \le 0$ and $\mathcal{Q}_j(\delta, 1/2) < 0$.

Exact outward interval arithmetic gives

$$\partial_x \mathcal{Q}_i > \frac{1}{1000} \quad \text{on } \mathcal{D} \text{ at the endpoint}, \qquad \frac{d}{d\kappa}\mathcal{Q}_i < -\frac{1}{100} \quad \text{on } \mathcal{D} \times \mathcal{P}_{\mathrm{tw}},$$

where $d/d\kappa$ is the derivative along $\mathcal{P}_{\mathrm{tw}}$, and it gives (24.3). The first sign places the maximum at $x = 1/2$; integrating the second proves (24.2). Detector and numerator use the common moment parameter.

The inequalities of Table 8 that contain the origin coefficient 5 become

$$\frac{\delta}{2}-\theta_0-\frac{\varrho a}{c}>0,\qquad m_\theta-\frac{p-1+c(1+2t_n)}{2(p-1)+4c}>0,$$
$$\frac{59}{2000}-\frac{\varrho}{c}\Big(\frac{2l_y}{h}-1\Big)>0,\qquad \frac{\varrho}{10}\Big(1-\frac{4}{c}\Big)-\Big(\frac{1}{6}-\frac{2t_n}{3}\Big)>0;$$

at $c=5$ these are the corresponding inequalities of Table 8. In addition, $112/25<p<9/2$, $b>0$ and $1/6<\ell<1/5$ on $\mathcal{P}_{\mathrm{tw}}$. With $m_{oj}$ defined by $W_o(m_{oj})=W_j(m_{oj})$ and $W$ the active width on each side of $m_{oj}$, the comparison lengths of Section 14 satisfy

$$0<t_{\mathrm{low}}<\frac{1}{4},\qquad t_{\mathrm{high}}>0,\qquad t_{\mathrm{low}}>5W,\qquad 0<W<\frac{1}{25},\qquad L_0-\frac{1-2m}{p}-W>0.$$

The ordering $t_{\mathrm{low}}<t_{\mathrm{high}}$ follows from strictly interior widths. These checks cover the moving crossing $m_{oj}$, and Appendix E.3 proves all of them. □

## 25 Successive bounds and the proof of Theorem 1.1

In this section, we iterate Lemma 24.1 from the starting point $B_0$ of Section 23.1, using Theorem 17.1, and prove Theorem 1.1 by the descent of Theorem 9.2.

**Proposition 25.1.** *For every number field $F\supset\mu_{12}$, $\beta_*(F)\le B_{\mathrm{tw}}$.*

*Proof.* Start with $B_0=0.87407285$. Suppose $\beta_*(F)\le B_n$ with $B_n>B_{\mathrm{tw}}$, and put $\kappa_n=2B_n-1$ and $p_n=6\kappa_n$. Use the strict family (24.1) with count and numerator moment parameters $\kappa_c=\kappa_m=\kappa_n$, and the relaxation parameter

$$\rho_n=\rho_{\max}(p_n)-\eta_n,\qquad 0<\eta_n\le\frac{\kappa_n-\kappa_{\mathrm{tw}}}{2}.$$

Since $\kappa_0-\kappa_{\mathrm{tw}}<3\cdot10^{-5}$ and (17.3) holds,

$$0.083<\rho_{\max}(6\kappa_0)-\frac{\kappa_0-\kappa_{\mathrm{tw}}}{2}\le\rho_n<\rho_{\max}(p_n)<0.0841.$$

By (24.2) and (24.3),

$$\mathcal{Q}_i<-\frac{\kappa_n-\kappa_{\mathrm{tw}}}{100}+\frac{\eta_n}{100}\le-\nu_n,\qquad \nu_n=\frac{\kappa_n-\kappa_{\mathrm{tw}}}{200}.$$

The inequalities of Table 8 are strict on compact sets and continuous in $\rho$, so they persist for $\eta_n$ small enough. Since $\rho_n<\rho_{\max}(p_n)$, Theorem 17.1 applies.

Set aside a fixed fraction of the smaller of $\nu_n$ and these margins, and make the choices of the proof of Proposition 15.2 inside it. Section 14.5 applies with comparison margin $\nu_n>0$. Hence

$$\beta_*(F)\le B_{n+1}=\frac{11}{12}-\frac{\ell(\kappa_n)}{4}=B_{\mathrm{tw}}+\frac{\kappa_n-\kappa_{\mathrm{tw}}}{100}=B_{\mathrm{tw}}+\frac{B_n-B_{\mathrm{tw}}}{50},$$

and induction gives

$$B_n=B_{\mathrm{tw}}+50^{-n}(B_0-B_{\mathrm{tw}}).\tag{25.1}$$

Each finite step uses a previously proved moment bound and has its own positive margins and threshold. Taking the infimum over $n$ gives $\beta_*(F)\le B_{\mathrm{tw}}$. □

*Proof of Theorem 1.1.* Apply the descent in Theorem 9.2 to Proposition 25.1, as in the proof of Theorem 1.2. □

The first step of the iteration already gives

$$B_1=0.8740587907160218248\ldots<0.874059.$$

## 26 Optimality for the unit-twisted comparison

In this section, we prove Theorem 26.1: $B_{\mathrm{tw}}$ is the infimum of the comparison system of Theorem 16.1 when the origin width is enlarged by the twist. The proof is that of Theorem 16.1, and we record the changed quantities.

**Theorem 26.1** (Optimality for the unit-twisted comparison)**.** *In Theorem 16.1, replace the additional prime-product length $W_o(m)$ by*

$$W_o(m) = \frac{1-1/p}{5-\rho}(1-2m), \qquad 0 \le \rho \le \rho_{\max}(p),$$

*at each moment parameter $p \in \{p_d, p_n\}$, and retain the other hypotheses, including (16.1). Starting with a boundary at least $B_{\mathrm{tw}}$, no finite step in this comparison system gives a boundary below $B_{\mathrm{tw}}$. The successive bounds of Proposition 25.1 attain $B_{\mathrm{tw}}$ as an infimum.*

*Proof.* Follow the proof of Theorem 16.1, with $C(p)$ replaced by $C_\rho(p)$ of (23.2) and $(r_{\mathrm{ex}}, C_{\mathrm{ex}}, B_{\mathrm{ex}})$ by $(r_{\mathrm{tw}}, C_{\mathrm{tw}}, B_{\mathrm{tw}})$. The total prime-product length is $4C_\rho(p)(1-2m)$, and

$$\frac{\partial C_\rho(p)}{\partial \rho} = \frac{p-1}{4p(5-\rho)^2} > 0, \qquad \widetilde{C}'(p) = -\frac{163p^2+684p+43092}{14p^2(p+126)^2} < 0.$$

Hence, for $\rho \le \rho_{\max}(p)$, the width constant is at most $\widetilde{C}(p)$, which decreases in $p$. At a first crossing below $B_{\mathrm{tw}}$ the known boundary is at least $B_{\mathrm{tw}}$, so

$$p_d,\, p_n \ge 12B_{\mathrm{tw}} - 6 = p_{\mathrm{tw}}, \qquad \rho \le \rho_{\max}(p_{\mathrm{tw}}),$$

and the most favorable choice is $p_d = p_n = p_{\mathrm{tw}}$, $\rho = \rho_{\max}(p_{\mathrm{tw}})$, with constant $C_{\mathrm{tw}}$.

The relaxed count, its cutoff crossing $r(\tau)$ and the two adverse splits are unchanged in form, since

$$-\frac{5}{12} + r_{\mathrm{tw}}\Big(\frac{3}{2} - 4C_{\mathrm{tw}}\Big) > 0.$$

The separation identity (16.2) is unchanged, and $B < B_{\mathrm{tw}} < 8741/10000$ gives $m_\theta > 2/5$. By Lemma 23.1, the detector crossing is

$$m_d = \frac{2(1-12C_{\mathrm{tw}}r_{\mathrm{tw}})}{9r_{\mathrm{tw}}(1-4C_{\mathrm{tw}})} \in (0.39776, 0.39777),$$

so (16.3) holds with $(r_{\mathrm{tw}}, C_{\mathrm{tw}})$ for $2/5 \le m \le 1/2$.

For $T_y > 0$, the critical count identity of Lemma 23.1 gives the formula for $H_{\mathrm{hi}}$ with $(r_{\mathrm{tw}}, C_{\mathrm{tw}})$, and the weights $\lambda_{\mathrm{dir}}, \lambda_K, \lambda_{\mathrm{hi}}$, with $D_{\mathrm{hi}}$ at $(r_{\mathrm{tw}}, C_{\mathrm{tw}})$, are positive and sum to one. Reduction modulo $f_{\mathrm{tw}}$ gives

$$\lambda_{\mathrm{dir}}L_{\mathrm{dir}} + \lambda_K L_K + \lambda_{\mathrm{hi}}H_{\mathrm{hi}} = \frac{44C_{\mathrm{tw}}r_{\mathrm{tw}} + 7r_{\mathrm{tw}} + 7}{3D_{\mathrm{hi}}} = B_{\mathrm{tw}}. \tag{26.1}$$

For $T_y \le 0$, the weights $\lambda_K^0, \lambda_{\mathrm{hi}}^0, \lambda_T^0$ at $(r_{\mathrm{tw}}, C_{\mathrm{tw}})$ are positive and force

$$B \ge \frac{13}{D_{\mathrm{hi},0}} = 0.896935\ldots > B_{\mathrm{tw}}.$$

For separate moment parameters, the derivatives $\partial_R P_4 = 3/4$, $\partial_R P_{\mathrm{pt}} = 1$, $\partial_{p_n} P_4 = -\widetilde{C}'(p_n)r(1-2m) \ge 0$ and $\partial_{p_d} R_{\widetilde{C}(p_d)}(r) > 0$ give, for $p_d, p_n \ge p_{\mathrm{tw}}$ and the chosen $m$,

$$\min\{P_4(m; R_{p_d}, \widetilde{C}(p_n)), P_{\mathrm{pt}}(m; R_{p_d})\} \ge P_4(m; R, C_{\mathrm{tw}}).$$

Thus the necessary comparisons remain $B \ge H_{\mathrm{hi}}$ and $B \ge H_{\mathrm{hi},0}$, and no first crossing below $B_{\mathrm{tw}}$ occurs. The signs are verified in Appendix E.4, and Proposition 25.1 proves attainability. □

## 27 Proportional margins and the limit of the method

In this section, we compare the return estimate of Section 21 with an argument requiring proportional margins, and record the limit 13/15 of the direct-theta comparison.

For $w > 0$, write $u_s = s/w$. Without the absolute margin $\nu$, the two strict proportional requirements are

$$3u_s > \rho, \qquad \frac{7 - 5p\alpha}{6} u_s - \frac{p\alpha(6 - p - p\rho)}{6(p-1)} < 0.$$

The resulting interval for $u_s$ is nonempty if and only if $E_\alpha(p, \rho) < 0$; for $\alpha = \alpha_{\max}(p)$ this means $\rho < \rho_{\max}(p)$. With $p = 9/2$ and $\alpha = \alpha_{\max}(9/2) = 16/171$,

$$\frac{\rho}{3} < u_s < \frac{8(1 - 3\rho)}{217}, \qquad \text{of length } \frac{24 - 289\rho}{651},$$

which vanishes at $\rho = 24/289 = \rho_{\max}(9/2)$. Hence an argument requiring both proportional margins cannot reach $\rho_{\max}(p)$.

The proof of Section 21 keeps $\nu > 0$ in (18.2) and $\nu/\varrho$ in (18.3). The coefficient of $w$ may then vanish, and (21.7) remains negative after the excess width of the last complete factor. Equality in (17.5) is never used. By Lemmas 21.1 and H.4, no larger relaxation parameter or prime-length constant is available to this return estimate.

In the relaxed count of Theorem 26.1, the boundary depends only on the width constant $C$, through $F(C, \ell) = 0$ in (23.9) and $B = 11/12 - \ell/4$. Write $F(C, \ell) = a_C \ell^2 + b_C \ell + c_C$. For $C_{\mathrm{tw}} \le C \le 1/4$,

$$a_C = 1600C^2 - 228C - 103 < 0, \qquad c_C = 5(1 - 4C) \ge 0, \qquad F\Big(C, \frac{1}{5}\Big) = \frac{48}{25}(4C - 1) \le 0.$$

Hence the positive root

$$\ell_+(C) = \frac{b_C + \sqrt{b_C^2 - 4a_C c_C}}{2|a_C|} \in \Big(0, \frac{1}{5}\Big]$$

is continuous on $[C_{\mathrm{tw}}, 1/4]$. Since $b_{1/4} = 12 > 0$ and $F(1/4, \ell) = -12\ell(5\ell - 1)$, $\ell_+(1/4) = 1/5$. The boundary tends to $11/12 - 1/20 = 13/15$ as $C \uparrow 1/4$, that is, as $z_0(m) + W(m) \uparrow 1 - 2m$. Here $C_{\mathrm{tw}} = 0.09521\ldots$.

The branch $(1 + 3\ell - 2\Sigma)/4$ of $k(\Sigma, \ell)$ in the proof of Theorem 16.1 gives $B \ge L_J = 23/24 - \Sigma/6 + 5\ell/24$, and

$$\frac{17}{30} L_{\mathrm{dir}} + \frac{1}{30} L_K + \frac{2}{5} L_J = \frac{13}{15}, \qquad B \ge \max\{L_{\mathrm{dir}}, L_K, L_J\} \Longrightarrow B \ge \frac{13}{15}.$$

Hence a change of the norm alone, with these direct estimates fixed, cannot go below 13/15. This is a limitation of the stated estimates, not a lower bound for the true zero-free boundary.

# A Exact rational inequalities in the comparison

In this appendix, we give the rational computation used in Section 8. The method proves inequalities on complete closed rectangles; it does not replace any of the analytic estimates in Sections 3–7.

### A.1 Polynomial inequalities on a rectangle

In this subsection, we state the coefficient criterion used for the rational functions in Section 8.1. The same criterion, with one more variable, applies to the parameter intervals in Section 8.2.

For a polynomial $p$ on a rational rectangle, make the affine change of variables to $[0,1]^n$ and write

$$p(t) = \sum_{\alpha \le d} a_\alpha t^\alpha = \sum_{\beta \le d} b_\beta \prod_{j=1}^{n} \binom{d_j}{\beta_j} t_j^{\beta_j} (1-t_j)^{d_j - \beta_j}.$$

The exact conversion is

$$b_\beta = \sum_{\alpha \le \beta} a_\alpha \prod_{j=1}^{n} \frac{\binom{\beta_j}{\alpha_j}}{\binom{d_j}{\alpha_j}}. \tag{A.1}$$

For $t \in [0,1]^n$, the basis functions satisfy

$$B_\beta(t) := \prod_{j=1}^{n} \binom{d_j}{\beta_j} t_j^{\beta_j} (1-t_j)^{d_j - \beta_j} \ge 0, \qquad \sum_{\beta \le d} B_\beta(t) = 1.$$

Consequently

$$p(t) \ge \min_\beta b_\beta. \tag{A.2}$$

Rational bisection and (A.1) give finite covers with

$$X = \bigcup_\nu X_\nu, \qquad \min_\beta b_{\nu,\beta} > 0 \implies \inf_X p > 0.$$

For a rational function $p/q$, apply this criterion separately to numerator and denominator after fixing their common sign. A cleared numerator affine in each of $b, \ell$ attains its extrema at the four corners of their rectangle. Otherwise retain $b, \ell$ as polynomial variables unless monotonicity permits endpoint evaluation. Treat $\kappa$ in the same way.

## A.2 The strict rational starting bound

In this subsection, we specify the inequalities proving Theorem 8.1. The functions $E, S, R, \theta_0$ are those of Sections 2 and 6, evaluated at the rational parameters of Section 8.1.

Put $\varepsilon_{\mathrm{cmp}} = 10^{-7}$, $\zeta = 10^{-8}$, $m_{\mathrm{seed}} = 1/10$, $N = (3R+1)/4 - \theta_0/2$, and

$$\Delta_{\mathrm{short}} = R + (1/2 - \theta_0) m_{\mathrm{seed}} - N, \qquad I = [h - 1/80, h + \zeta].$$

Use $\mathcal{R} = [7/20, 3/5] \times [9/20, 1/2]$ and $\mathcal{G} = [1/8, 5/6] \times [0, 1/2]$. Its three complementary rectangles are

$$\begin{aligned} \mathcal{C}_1 &= [1/8, 5/6] \times [0, 9/20], \\ \mathcal{C}_2 &= [1/8, 7/20] \times [9/20, 1/2], \\ \mathcal{C}_3 &= [3/5, 5/6] \times [9/20, 1/2]. \end{aligned}$$

On $\mathcal{G}$, one has $J > 0$ and $E < 1/500$; on each $\mathcal{C}_i$, one has $E < -\varepsilon_{\mathrm{cmp}}$. On $\mathcal{R}$, one has

$$S > 1/5, \qquad a > 0, \quad \gamma > 0, \qquad 2x(21/100) - z_L - W > 0, \tag{A.3}$$

$$E - t_-\{h(1-m_L) - l_y\} < -\varepsilon_{\mathrm{cmp}}, \qquad E - h\gamma < -\varepsilon_{\mathrm{cmp}}, \tag{A.4}$$

$$E + t_+\{h(1-m_R) - l_y\} < -\varepsilon_{\mathrm{cmp}}, \qquad E + h\Delta_{\mathrm{short}} < -\varepsilon_{\mathrm{cmp}}. \tag{A.5}$$

For $d$ equal to either endpoint of $I$, each of

$$\begin{aligned} &E + (d-h)S + d\Delta_{\mathrm{short}}, \quad E + (d-h)S - t_-\{d(1-m_L) - l_y\}, \\ &E + (d-h)S - d\gamma, \quad E + (d-h)S + t_+\{d(1-m_R) - l_y\} \end{aligned}$$

is less than $-\varepsilon_{\mathrm{cmp}}/2$. Their slopes are positive and less than two on the ordinary left, middle and right ranges. Further,

$$\frac{1}{5}\frac{1}{80} - \frac{1}{500} > \varepsilon_{\mathrm{cmp}}, \qquad 2\zeta < \varepsilon_{\mathrm{cmp}}/2,$$

so parameters with smaller norm and the upper buffer in $d$ have the required strict bounds. The elementary remaining inequalities are

$$b > 0, \quad \frac{1}{6} < \ell < \frac{1}{5}, \quad l_x > \ell, \quad l_y - \ell > \frac{11b}{6}, \quad \frac{9}{2}\ell > \frac{3}{4},$$
$$\frac{21}{100} < \frac{\ell}{h+\zeta}, \qquad \frac{\ell}{h-1/80} < \frac{11}{50}, \qquad \frac{\ell}{h-1/80} < \frac{2}{9},$$
$$h < 2l_y, \qquad l_y < \frac{1}{2}, \qquad B > \frac{437}{500},$$
$$Q_{\mathrm{sm}} := \ell(1/2 + 2\ell/h) - 11h/54 < 0,$$
$$C_0 + (1-h)/16 < -1/250,$$
$$C_0 + (1/2 + 3\ell/2)/50 < -1/100,$$
$$13h/75 - l_y/2 + 1/50 < -1/20.$$

Substitution in (A.1)–(A.2) gives these signs and

$$(9/2 - 2\delta x)(21/100) > R, \qquad t_- > 0, \quad t_+ > 0.$$

The endpoint bounds above and affine dependence on $d$ prove the full comparison on $I$.

### A.3 The algebraic parameter interval

In this subsection, we describe the calculation used in Lemma 8.3. The algebraic identities at the critical class are proved by polynomial division by (1.3); the remaining signs are rational inequalities on the intervals below.

Use

$$I_b = [0.0917923522, 0.0917923524],$$
$$I_\ell = [0.16876, 0.16880], \qquad I_\kappa = [0.7489, 0.7495],$$
$$\mathcal{R}_* = [7/20, 3/5] \times [2/5, 1/2].$$

Every decimal endpoint here is an exact terminating decimal. The critical root is isolated in

$$0.450570746724060730 < r_* < 0.450570746724060733.$$

For $\mathcal{Q} = E_\kappa - 7a(2l_y - h)/108$, the coefficient test gives

$$\partial_x \mathcal{Q} > 0 \quad \text{on } \mathcal{R}_*$$

for $\kappa \in [0.7489366153, 0.7489366155]$ and $(b, \ell) \in I_b \times I_\ell$. It also gives $\partial_\ell \mathcal{Q} > 3/4$ at $\kappa = 0.7489$; the identity

$$\partial_\kappa \partial_\ell \mathcal{Q} = \frac{9}{8}\partial_\kappa R \geq 0$$

extends this inequality to $I_\kappa$. The formula in Lemma 8.3 gives $0 \leq \partial_\kappa \mathcal{Q} < 1/100$ throughout the larger interval.

Use the complementary rectangles of Lemma 8.3, with amplitude endpoint $2/5$. Lemma 6.5 gives

$$\partial_\kappa E = \frac{3h}{4}\partial_\kappa R \geq 0, \qquad \partial_\kappa S = \frac{3}{4}\partial_\kappa R \geq 0.$$

At the adverse count endpoints, the exact test gives

| domain | count parameter | bound |
|---|---|---|
| $\mathcal{G}$ | $\max I_\kappa$ | $E < 1/500$ |
| $\bigcup_i \mathcal{C}_i$ | $\max I_\kappa$ | $E < -10^{-5}$ |
| $\mathcal{R}_*$ | $\min I_\kappa$ | $S > 1/5$. |

The common ordinary gain satisfies

$$a > 0, \quad 4a/9 - \theta_0 > 0, \quad \delta/2 - \theta_0 - 7a/54 > 0,$$
$$\theta_0(5h/8 - l_y) > 7a(2l_y - h)/108,$$
$$(\delta/2 - \theta_0)(l_y - 11h/20) > 7a(2l_y - h)/108,$$
$$2x(21/100) - 1/18 > 7/279.$$

The short comparison uses $m_{\mathrm{cut}} = 3/40$ and $\Delta_{\mathrm{short}} = R + (1/2 - \theta_0)m_{\mathrm{cut}} - N$:

$$E + h\Delta_{\mathrm{short}} < -10^{-5}.$$

The geometric inequalities of the previous subsection persist, with the threshold for the lowest-amplitude estimate replaced by $1/125$. Finally

$$\frac{57}{100} < \frac{l_y}{h} < \frac{11}{19}, \qquad 25/62 < 1 - l_y/h < 9/20.$$

The two bounds on $l_y/h$ and the upper bound on $1 - l_y/h$ are checked by exact rational interval arithmetic on $I_b \times I_\ell$. The quantitative outer margin is the identity (7.3); an exact rational calculation gives it on the full parameter box.

These strict inequalities follow by outward-rounded rational interval arithmetic on finite rectangle covers.

## A.4 Computations for the successive moment bounds

In this subsection, we describe the calculations used in Section 10. They prove only the stated identities and inequalities; the analytic estimates are those of Lemma 10.1 and Section 8.

*Length identities.* Equations (10.13)–(10.19) and (10.20) are polynomial identities in $k_6$ and the lengths. At $k_6 = 9/2$ they give the formulas of Sections 3.1–4.4. The rational identity (10.21) specializes at $\kappa = 3/4$ to

$$1 - \frac{279}{7}W = \frac{36}{7}\eta_{3/4} + 2(1 - C_{\mathrm{add}}) + 8(r - m_-).$$

The bounds (10.22) are monotone in $\kappa$ on $[\kappa_\natural, 3/4]$ and were checked at the rational endpoints of an enclosure of $\kappa_\natural$.

*The endpoint.* The numerator in Proposition 10.3(i) and the resultant in (ii) were computed in exact rational arithmetic. The root $r_\natural$ was isolated in an interval of width $10^{-45}$; $\ell_\natural, b_\natural, B_\natural$ and $A_\natural$ were evaluated in interval arithmetic with outward rounding. In particular

$$B_\natural \in [0.87446749650084027305624 98,\ 0.87446749650084027305624 99],$$

$$B_* - B_\natural = 8.1119269018\ldots \cdot 10^{-7}.$$

Let $p_\ell$ and $p_B$ denote the two polynomials in (10.3). Exact polynomial division gives

$$p_\ell(\ell_\natural(r)) \equiv 0 \pmod{q_\natural(r)}, \qquad p_B(11/12 - \ell/4) = \frac{27}{8}p_\ell(\ell).$$

Both polynomials are irreducible over $\mathbb{Q}$. Thus their vanishing and the degree-four assertions are exact.

Exact rational root isolation gives the unique root of $q_\natural$ in $(9/20, 23/50)$; rational interval operations then evaluate $\ell_\natural$, $\kappa_\natural$ and $b_\natural = -\partial_\delta p_0/\partial_\delta p_b$. Every denominator is bounded away from zero. The resulting enclosures are contained in

$$\begin{aligned} r_\natural &\in [0.450533924878843452503328321900 32,\ 0.450533924878843452503328321900 33],\\ \ell_\natural &\in [0.168796680663305574441667419387 90,\ 0.168796680663305574441667419387 92],\\ b_\natural &\in [0.091801394957524727743432859327 25,\ 0.091801394957524727743432859327 27]. \end{aligned}$$

These enclosures, together with an exact enclosure of $\kappa_*$, give rational boxes containing $\mathcal{P}_{\natural,0}$, $\mathcal{P}_\natural$ and $\mathcal{P}_{\natural,3/4}$.

*The conditions of Table 4.* For each rational expression $F$ and finite rational covering $X = \bigcup_\nu X_\nu$, outward interval arithmetic gives

$$F(X_\nu) \subseteq [\underline{F}_\nu, \overline{F}_\nu] \quad\Longrightarrow\quad \min_\nu \underline{F}_\nu \le F(z) \le \max_\nu \overline{F}_\nu \qquad (z \in X).$$

The table bounds are rounded outwards from these extrema. In particular,

$$\theta_0 \le \frac{3}{60\kappa_m} < 0.06681 \qquad \text{(Remark 10.4(a))}.$$

*Iteration.* For the first steps of Proposition 10.5,

$$B_1 - B_\natural = 1.622\ldots\cdot 10^{-8}, \qquad B_2 - B_\natural = 3.245\ldots\cdot 10^{-10}, \qquad B_3 - B_\natural = 6.49\ldots\cdot 10^{-12},$$

in agreement with $B_n - B_\natural = 50^{-n}(B_* - B_\natural)$.

For $\mu_*$ and $\mu_\natural$ in the pointwise comparisons of Theorems 8.5 and 10.8, the root enclosures above and outward rational evaluation give

| $\bullet$ | $\mu_\bullet$ | $g(\ell_\bullet)$ |
|---|---|---|
| $*$ | $< 399035/10^6$ | $> 399159/10^6$ |
| $\natural$ | $< 399030/10^6$ | $> 399165/10^6$. |

(A.6)

# B Exact inequalities for the enlarged count

In this appendix, we give the algebraic enclosures and interval procedure used in Section 11. These calculations verify the stated identities and signs. The analytic estimates are Lemmas 11.1 and 11.2.

## B.1 Algebraic endpoint and the two branches

In this subsection, we isolate the parameters of Proposition 11.4 and verify its two rational expressions. The identity (11.15) treats the double zero exactly.

The identities of Proposition 11.4 follow by rational polynomial arithmetic. Sturm's rule and rational bisection give

$$\#\{\delta \in (11/25, 23/50) : q_{\mathrm{ec}}(\delta) = 0\} = 1, \qquad \delta' \in I_{\delta'}, \quad |I_{\delta'}| < 2\cdot 10^{-53}.$$

Substitution in (11.12) and the formula for $b'$ gives

$$\begin{aligned} \delta' &\in [0.448787027958032789505261396776 34,\ 0.448787027958032789505261396776 35],\\ \ell' &\in [0.169781664695058333542110877861 88,\ 0.169781664695058333542110877861 89],\\ b' &\in [0.082189695719045725067484724039 14,\ 0.082189695719045725067484724039 15],\\ \kappa' &\in [0.748442500985804166562277894402 39,\ 0.748442500985804166562277894402 40],\\ B' &\in [0.874221250492902083281138947201 19,\ 0.874221250492902083281138947201 20]. \end{aligned}$$

All endpoints are exact rationals. The algebraic calculation verifies containment of its narrower enclosures in each displayed interval. In particular,

$$-7911.143 < \partial_\delta p_b(\delta', \ell') < -7911.142, \qquad \kappa_\flat < \kappa' < \kappa_\natural.$$

The reductions of $p_b(\delta, \ell'(\delta))$ and $p_0(\delta, \ell'(\delta))$ modulo $q_{\mathrm{ec}}$ vanish identically. The same is true of the polynomial in (11.13) after substitution of $\ell'(\delta)$, and of the polynomial in (1.2) after $B = 11/12 - \ell'(\delta)/4$. The three quartics are irreducible over $\mathbb{Q}$.

At $(b', \ell')$, exact tangency yields (11.15). The signs of $A'$, $A_1$ and the branch-1 discriminant are the interval signs in Proposition 11.4. Since $D_i$ is affine in $\delta$, its endpoint bounds suffice:

| | $\delta = 7/20$ | $\delta = 3/5$ |
|---|---|---|
| $D_2$ | $> 21506$ | $> 16504$ |
| $D_1$ | $> 19070$ | $> 14071.$ |

For the smaller branch margin, complete the square:

$$A_1\delta^2 + B_1\delta + C_1 = A_1(\delta - v_1)^2 + M_1, \qquad v_1 = -\frac{B_1}{2A_1}, \quad M_1 = -\frac{B_1^2 - 4A_1C_1}{4A_1} < 0.$$

Interval evaluation on 4000 equal subintervals of $[7/20, 3/5]$ gives

$$\sup_{7/20 \le \delta \le 3/5} \mathcal{Q}^{(1)}(\delta, 1/2) < -4.6125 \cdot 10^{-6}.$$

Together with the branchwise $x$ derivatives in Table 6, this proves the full endpoint inequality without an interval test across the double zero of $\mathcal{Q}^{(2)}$.

## B.2 Rational interval verification of the remaining signs

In this subsection, we verify the inequalities of Section 11.4 on the full parameter domain by rational interval arithmetic. The enclosures of Proposition 11.4 are retained throughout.

Use $k_N^+ = 7489349931/10^{10} > \kappa_\natural$ and the enclosures above, widened by $10^{-50}$ at each end. If $I_\kappa, I_\ell, I_b$ are the widened intervals, the verification box contains

$$\kappa_c, \kappa_m \in [\inf I_\kappa, k_N^+], \qquad \ell \in [\inf I_\ell - (k_N^+ - \inf I_\kappa)/25, \sup I_\ell],$$
$$b \in I_b + [-10^{-30}, 10^{-30}].$$

For $\mathcal{P}_{3/4}$ replace the upper moment endpoint by $3/4$. All class endpoints are entered as exact rationals. Denominators are bounded away from zero on every retained box.

Every expression is rational in $(\delta, x, \kappa_c, \kappa_m, \ell, b)$. The integer-interval calculation verifies every displayed table threshold on the full enclosing rational box. It represents interval endpoints by integers divided by $N_{\mathrm{den}} = 2^{100}$ and rounds outwards after each arithmetic operation. For example, if $I = [u_1, u_2]/N_{\mathrm{den}}$ and $I' = [v_1, v_2]/N_{\mathrm{den}}$, put $\mu_- = \min(u_1v_1, u_1v_2, u_2v_1, u_2v_2)$ and $\mu_+ = \max(u_1v_1, u_1v_2, u_2v_1, u_2v_2)$. Then

$$II' \subseteq \left[\left\lfloor \frac{\mu_-}{N_{\mathrm{den}}} \right\rfloor, \left\lceil \frac{\mu_+}{N_{\mathrm{den}}} \right\rceil\right] / N_{\mathrm{den}}.$$

Addition is exact, and reciprocal bounds use integer division on an interval excluding zero. Rational constants are enclosed before use.

For a rational function $f$ and box $X$, intersect its natural interval extension with the mean-value enclosure

$$f(X) \subseteq f(x_0) + \sum_i (\partial_i f)(X)(X_i - x_{0,i}).$$

The $b$ interval may be retained throughout instead of centered. An undecided box is bisected at a rational midpoint; the two resulting subrectangles cover the original rectangle. Only a strict interval sign accepts a box. Thus the finite subdivision proves the sign on the complete domain.

For an expression $F$ increasing in the count exponent,

$$F(R') = \max\{F(R^{(1)}), \min(F(R^{(0)}), F(R^{(2)}))\}.$$

For decreasing $F$, interchange maximum and minimum. These identities are used for $E', S, N - I_{\mathrm{short}}$, the slopes and prime supply. Derivative inequalities are verified for each $\mathcal{Q}^{(j)}$ separately, and then used as finite-difference inequalities for their maximum and minimum. In particular, the calculation gives

$$\partial_x \mathcal{Q}^{(j)} > 0.009 \quad \text{at the endpoint},$$
$$\partial_{\kappa_c} \mathcal{Q}^{(j)} + \partial_{\kappa_m} \mathcal{Q}^{(j)} - \frac{1}{25}\partial_\ell \mathcal{Q}^{(j)} < -0.011,$$
$$E' + h\Delta_{\mathrm{short}} < -0.0027, \qquad (6\kappa_m - 2\delta x)(21/100) - R' > 0.019 \quad \text{on } \mathcal{G} \times \mathcal{P}_{3/4},$$

where the short-sum inequality is on $\mathcal{R}_* \times \mathcal{P}$. All remaining inequalities in Tables 6–7 follow by the same finite calculation. The crude supply is also immediate: its minimum occurs at $\delta = 1/8$ and $\kappa_m = \kappa'$, where it exceeds 0.00012.

Exact polynomial reduction verifies the resultant, both tangency identities and the active branch at the contact point. The remaining inequalities follow by outward-rounded rational interval arithmetic on finite rectangle covers.

## C The zero-detecting estimates and theta exponents

In this appendix, we derive the zero-detecting estimates and the exponent (14.5) from the moment lower bounds and theta identity. This verifies the normalization used in Section 14 and Appendix D. Write

$$g = \delta x, \qquad p = 6\kappa, \qquad \mathcal{K} = \frac{3R+1}{4}, \qquad \theta_0 = \frac{g}{p}, \qquad z_0(m) = \frac{1-2m}{p}, \qquad \sigma_{\mathrm{M}} = \frac{17}{50}.$$

Here $\sigma_{\mathrm{M}}$ is the fixed Mellin contour.

*The two detector counts.* For inverse length $r < 1$, the detector gives $|M_r|^2 \geq U^{\delta r - o(1)}$. A selected product of width $(1-r)/2$ has squared amplitude at least $U^{g(1-r)-o(1)}$. The marked inverse moment has upper bound $U^{1+o(1)}$, so division gives

$$1 - \delta r - g(1-r) = 1 - \delta\{x + (1-x)r\}.$$

For ordinary length $m$, the detector gives $|S_m|^2 \geq U^{\delta m - o(1)}$. The norm contains $|S_m^2|^2 = |S_m|^4$, whose lower exponent is $2\delta m$. The joint selected product adds $2g(z_0(m) + W(m))$. Therefore the ordinary count has exponent

$$1 - 2\delta m - 2g\{z_0(m) + W(m)\} = 1 - \delta F_2(m), \qquad F_2(m) = 2m + 2x\{z_0(m) + W(m)\}.$$

These are divisions of arithmetic parameter moments by lower bounds for each arithmetic parameter. Neither uses an identification of different arithmetic parameters with distinct primitive characters. A fourth root is not taken in a detector count.

For $r \geq 1$, the inverse moment exponent of [Liu26b, Lemma 10.10] $e(r) = (1+5r)/6$ instead gives count $1/6 + (5/6 - \delta)r$. Since $r \leq T + o(1)$ and $\delta \leq 5/6$, it is bounded by $1/6 + (5/6 - \delta)T$. Put $A_\delta = 5/6 - \delta$. Equal short counts give

$$r = \frac{F_2(m) - x}{1 - x}, \qquad T = r + m.$$

Equating their count with the long inverse bound yields

$$-\frac{5}{6}+A_\delta m-\frac{A_\delta x}{1-x}+\left(\frac{A_\delta}{1-x}+\delta\right)F_2(m)=0,$$

which is the balance equation used for each affine width branch.

*The reference exponent.* The comparison formula of [Liu26b, Section 14.1, equation (14.7)], before simplifying its constants and with an arbitrary valid class count $R$, is

$$\begin{aligned}\mathcal{E}_0(d)&=a_{\mathrm{bin}}-B-l_y(1/2+\theta_0)+h(\sigma_{\mathrm{M}}-1/6)-\ell/2+\ell g\\&\quad+d(\mathcal{K}+\theta_0/2-\sigma_{\mathrm{M}}),\qquad a_{\mathrm{bin}}=\frac{1+\delta}{2}.\end{aligned}\tag{C.1}$$

The arithmetic average, conductor factor and outside Mellin factor give

$$U^{\mathcal{K}-\theta_0/2}U^{\theta_0}Y^{-\theta_0}=U^{\mathcal{K}}Z^{\theta_0(d/2-l_y)},\qquad U=Z^d,\quad Y=Z^{l_y}.$$

At $d=h$, use $B=11/12-\ell/4$, $h=(1+b+3\ell)/2$, $l_y=(1+b-\ell)/2$ and $\mathcal{K}=1-3(1-R)/4$. The terms independent of $\delta,g,1-R$ reduce as follows:

$$\frac{1}{2}-B-\frac{l_y}{2}-\frac{\ell}{2}+\frac{5h}{6}=-\frac{1}{4}+\frac{b}{6}+\frac{5\ell}{4}=C_0.$$

Consequently

$$\mathcal{E}_0(h)=C_0+\frac{\delta}{2}+\delta x\left\{\ell+\frac{h/2-l_y}{p}\right\}-\frac{3h}{4}(1-R)=E(R),$$

and differentiating (C.1) gives

$$\mathcal{E}_0'(d)=\mathcal{K}+\frac{\theta_0}{2}-\frac{17}{50}=S(R).$$

This independently verifies $\mathcal{E}_0(d)=E(R)+(d-h)S(R)$.

*Hölder, the AFE and the displaced contour.* Let $\gamma$ be the additional first-moment gain beyond the baseline upper bound for the prime-product length $z_0$. The fourth-moment estimate has exponent $1-2gz_0-4\gamma$. Hölder with $U^{R+o(1)}$ arithmetic parameters gives the central first-moment exponent

$$\frac{3R}{4}+\frac{1-2gz_0-4\gamma}{4}=\mathcal{K}-\frac{gz_0}{2}-\gamma.$$

For a dyadic AFE block of length $U^m$ with $m\le 1/2$, the scalar $U^{-\theta m}$ changes this to

$$\mathcal{K}-\frac{gz_0}{2}-\gamma-\theta m=\mathcal{K}-\frac{\theta_0}{2}-(\theta-\theta_0)m-\gamma,$$

using $gz_0=\theta_0(1-2m)$. Relative to $\theta_0$, the conductor raising and outside Mellin factors are $U^{\theta-\theta_0}Y^{-(\theta-\theta_0)}$. Hence the full change is exactly

$$(\theta-\theta_0)\{d(1-m)-l_y\}-d\gamma.$$

This derivation applies to both AFE halves after the prescribed whole conjugation and actual-conductor normalization. It uses the baseline $gz_0/2$ once. In the conservative extra-prime selection $|Q_W|^2\ge U^{2aW-o(1)}$, with $a$ the remaining mean amplitude of the numerator section (not $a_{\mathrm{bin}}$), one has $\gamma=aW/2$; joint selection would replace $a$ by $g$.

For $m\ge 1/2$, use $z_0=0$ and the no-prime moment instead of continuing $(1-2m)/p$ to negative lengths. The actual correction is (14.7). Its excess below the formal continuation of the preceding display is $d\theta_0(m-1/2)\ge 0$. Every such outer block is bounded by the $m=9/20$ endpoint as proved in the scalar section.

Set $m_\theta = 1 - l_y/h$. At $d = h$, the contour saving is

$$\begin{cases} h\theta_0(m_\theta - m), & m \le m_\theta, \quad \theta = 0, \\ h(\delta/2 - \theta_0)(m - m_\theta), & m \ge m_\theta, \quad \theta = \delta/2. \end{cases}$$

The two endpoint gains are

$$\frac{ha}{2}\frac{\varrho(1-2m_\theta)}{5} = \frac{a\varrho T_y}{10} = G_N, \qquad \theta_0\{(1-m_L)h - l_y\} = G_L.$$

For $m < 3/40$, ideal counting and $\theta_0 < 1/2$ give

$$R + (1/2 - \theta_0)m - (\mathcal{K} - \theta_0/2) \le R + (1/2 - \theta_0)\frac{3}{40} - (\mathcal{K} - \theta_0/2) = D_{\text{short}}.$$

# D Exact inequalities for the exhaustion range

In this appendix, we verify Proposition 14.2 and Lemma 15.1 by rational identities and interval inequalities. The analytic inputs are those of Sections 12–14.

## D.1 The rational starting comparison

In this subsection, we verify Proposition 14.2 at (14.1). The second parameter choice below gives a direct one-step comparison: with the moment parameter $\kappa = 4676/6250$, justified by the bound $\beta_*(F) \le 5463/6250$ of Section 14.5, Remark 14.3 gives $\beta_*(F) \le 874073/10^6$ directly, without the family (15.8).

The parameters and comparison margins are

| $B$ | $\ell$ | $b$ | $\kappa$ | $\epsilon_{\text{cmp}}$ |
|---|---|---|---|---|
| $5463/6250$ | $1597/9375$ | $1913/20000$ | $37446831/50000000$ | $10^{-6}$ |
| $874073/10^6$ | $4(11/12 - B)$ | $3827/40000$ | $4676/6250$ | $10^{-7}$. |

The interval calculation below proves the strict comparisons; every inverted denominator excludes zero.

## D.2 Algebraic enclosures and the endpoint quadratics

In this subsection, we verify the contact identities and signs of Lemma 15.1, using the quartic (15.1).

Put

$$\ell_- = 0.17037527599935503555198917059375973764376115653239374775,$$
$$I_\ell = [\ell_-, \ell_- + 10^{-56}].$$

Sturm's theorem gives

$$\#\{\lambda \in I_\ell : f(\lambda) = 0\} = 1.$$

Rational evaluation of (15.2)–(15.5) on $I_\ell$ gives the enclosures

$$\kappa_- = 0.74814569533365581555733874803645346451145275506 7,$$
$$b_- = 0.09567500764537642027548322481699577004478313960 7,$$
$$I_\kappa = [\kappa_-, \kappa_- + 10^{-48}], \qquad I_b = [b_-, b_- + 10^{-48}],$$
$$\kappa_{\text{ex}} \in I_\kappa, \qquad b_{\text{ex}} \in I_b.$$

Put

$$r(\ell) = -\frac{5(3\ell - 5)(9\ell - 5)}{6(21\ell^2 - 28\ell - 25)}.$$

Before reducing the quotient for $b_{\mathrm{ex}}$, write

$$D_b(\ell) = \partial_\delta \partial_b \mathcal{Q}_o(\delta, 1/2)|_{\delta = r(\ell),\, \kappa = 5/6 - \ell/2} = \frac{N_b(\ell)}{T_b(\ell)}.$$

Exact polynomial reduction gives

$$\gcd(N_b, f) = \gcd(T_b, f) = 1 \quad \Longrightarrow \quad D_b(\ell_{\mathrm{ex}}) \in \mathbb{R}^\times.$$

All subsequent quotient reductions are in $\mathbb{Q}[\ell]/(f)$. At the endpoint substitutions, their denominators are units and

$$\mathcal{Q}_o(r_{\mathrm{ex}}, 1/2) = \partial_\delta \mathcal{Q}_o(r_{\mathrm{ex}}, 1/2) = \partial_b \mathcal{Q}_o(r_{\mathrm{ex}}, 1/2) = 0.$$

At $(\kappa_{\mathrm{ex}}, \ell_{\mathrm{ex}}, b_{\mathrm{ex}})$, for $8/25 \le \delta \le 3/5$, clearing denominators yields

$$\mathcal{Q}_i(\delta, 1/2) = \frac{A_i \delta^2 + B_i \delta + C_i}{D_i(\delta)}, \qquad D_i(\delta) > 0, \quad A_i < 0, \qquad i = o, j.$$

The remaining signs are

$$B_j^2 - 4A_jC_j < 0 \implies \mathcal{Q}_j(\delta, 1/2) < 0,$$
$$\mathcal{Q}_o(r_{\mathrm{ex}}, 1/2) = \partial_\delta \mathcal{Q}_o(r_{\mathrm{ex}}, 1/2) = 0 \implies \mathcal{Q}_o(\delta, 1/2) = \frac{A_o(\delta - r_{\mathrm{ex}})^2}{D_o(\delta)} \le 0.$$

For the quartic $P(B)$ in (15.7),

$$f(11/3 - 4B) = \frac{8}{9} P(B), \qquad P \text{ irreducible over } \mathbb{Q},$$
$$\#\{B \in (0.87407284, 0.87407285) : P(B) = 0\} = 1.$$

The last two assertions follow by rational factorization and Sturm's theorem.

### D.3 The full parameter family

In this subsection, we prove the inequalities along (15.8) used in Proposition 15.2 and verify the coefficient signs of Section 16.

Let $N_{\mathrm{den}} = 2^{220}$. For $I = [u_1, u_2]/N_{\mathrm{den}}$ and $I' = [v_1, v_2]/N_{\mathrm{den}}$, outward arithmetic gives

$$II' \subseteq \left[ \left\lfloor \frac{\min(u_1v_1, u_1v_2, u_2v_1, u_2v_2)}{N_{\mathrm{den}}} \right\rfloor, \left\lceil \frac{\max(u_1v_1, u_1v_2, u_2v_1, u_2v_2)}{N_{\mathrm{den}}} \right\rceil \right] / N_{\mathrm{den}},$$

and

$$0 \notin I \quad \Longrightarrow \quad I^{-1} \subseteq \left[ \left\lfloor \frac{N_{\mathrm{den}}^2}{u_2} \right\rfloor, \left\lceil \frac{N_{\mathrm{den}}^2}{u_1} \right\rceil \right] / N_{\mathrm{den}}.$$

Constants and algebraic enclosures are rounded outwards.

The rational parameter boxes are

$$Y_{\mathrm{ex}} = I_\kappa \times I_\ell \times I_b, \qquad \kappa_0 = 4676/6250,$$
$$\widehat{\mathcal{P}} = [\inf I_\kappa, \kappa_0] \times [\inf I_\ell - (\kappa_0 - \inf I_\kappa)/25, \sup I_\ell] \times I_b.$$

They contain the endpoint and $\mathcal{P}$, respectively. Here $\partial_\kappa$ acts on both detector and numerator.

Write $\mathbf{I}_F$ for outward interval evaluation of a reduced rational function $F$, using factored numerator and denominator and Horner's rule for the polynomial factors.

For $X = J_\delta \times J_x$, a fixed parameter box $Y$, and $c_X = \operatorname{mid} X$, put

$$F(X,Y) \subseteq \mathcal{J}_F(X,Y),$$
$$\mathcal{J}_F(X,Y) = \mathbf{I}_F(X,Y) \cap \left( \mathbf{I}_F(c_X,Y) + \sum_{j=1}^{2} \mathbf{I}_{\partial_j F}(X,Y)(X_j - c_{X,j}) \right).$$

A comparison inequality $G < -\epsilon$ is written as $F > 0$ with $F = -G - \epsilon$, and $G > \epsilon$ with $F = G - \epsilon$. Treat each branch and each indicated class rectangle separately. Use $\mathcal{B}_F = \mathcal{J}_F$ for the strict numerical inequalities of Proposition 14.2, Table 8 and the auxiliary detector-length inequalities below; use $\mathcal{B}_F = \mathbf{I}_F$ for

$$F_{i,x} = \partial_x \mathcal{Q}_i - 10^{-3}, \qquad F_{i,\kappa} = -10^{-2} - \partial_\kappa \mathcal{Q}_i + \frac{1}{25}\partial_\ell \mathcal{Q}_i, \qquad i = o, j.$$

Bisecting class rectangles at rational midpoints, with the parameter box $Y$ not subdivided, gives a finite cover $\mathfrak{X}_F$ with

$$\mathcal{D}_F \subseteq \bigcup_{X \in \mathfrak{X}_F} X, \qquad \min_{X \in \mathfrak{X}_F} \inf \mathcal{B}_F(X,Y) > 0 \implies F > 0 \text{ on } \mathcal{D}_F \times Y.$$

Since $F(X,Y) \subseteq \mathcal{B}_F(X,Y)$, these finite subdivisions give the following strict lower bounds for $\inf_{\mathcal{D}_F \times Y} F$:

| function and parameters | lower bound |
|---|---|
| rational comparison, first choice | $9 \cdot 10^{-7}$ |
| rational comparison, second choice | $5 \cdot 10^{-8}$ |
| auxiliary inequalities on $\widehat{\mathcal{P}}$ | $10^{-6}$ |
| $F_{o,x}$, $F_{j,x}$ on $Y_{\mathrm{ex}}$ | $8 \cdot 10^{-5}$, $2 \cdot 10^{-4}$ |
| $F_{o,\kappa}$, $F_{j,\kappa}$ on $\widehat{\mathcal{P}}$ | $4 \cdot 10^{-4}$, $10^{-3}$. |

In the first two entries the parameters are the respective choices of Section D.1, with their stated $\epsilon_{\mathrm{cmp}}$. Along (15.8), for $\kappa > \kappa_{\mathrm{ex}}$, the endpoint comparison of Lemma 15.1 and the bounds for $F_{i,x}$ and $F_{i,\kappa}$ give

$$E(R_i) - G_N < -(\kappa - \kappa_{\mathrm{ex}})/100, \qquad i = o, j.$$

At the endpoint the comparison is nonpositive. The auxiliary inequalities in the third entry hold on the full enclosing box $\widehat{\mathcal{P}}$.

For the detector length inequalities, let $m_{oj}(\kappa)$ be defined by $W_o(m_{oj}) = W_j(m_{oj})$. On its two sides use

$$m = 19/50 + (m_{oj}(\kappa) - 19/50)y \quad \text{or} \quad m = m_{oj}(\kappa) + (9/20 - m_{oj}(\kappa))y, \qquad 0 \le y \le 1.$$

The auxiliary inequalities are

$$19/50 < m_{oj} < 9/20,$$
$$0 < W_d < 1/25, \qquad 0 < t_{\mathrm{low}} < 1/4, \qquad t_{\mathrm{high}} > 0,$$
$$t_{\mathrm{low}} > 5W_d, \qquad z_0(m) + W_d(m) < L_0.$$

They involve only $m$ and $\kappa$ and hold on the respective intervals after subdivision in $y$ alone. Here $t_{\mathrm{low}}$ and $t_{\mathrm{high}}$ are evaluated at the limiting width $W_d$. Keep the comparison length $t$ chosen in Section 14. At the limiting width,

$$t_{\mathrm{low}}(W_d) \le t \le t_{\mathrm{high}}(W_d), \qquad W_d = W_j \implies t_{\mathrm{low}}(W_d) = t = t_{\mathrm{high}}(W_d).$$

For $W = (1 - \epsilon_W)W_d$, $0 < \epsilon_W < 1$, the identity $5W_o = (1 - 1/p)(1 - 2m)$ and $W_d \le W_o$ give

$$2m + (1 - 2m)/p + 5W \le 1 - 5\epsilon_W W_d < 1.$$

The comparison interval opens because

$$t_{\mathrm{low}}(W) = t_{\mathrm{low}}(W_d) - 6\epsilon_W W_d < t,$$
$$t_{\mathrm{high}}(W) = t_{\mathrm{high}}(W_d) + (p/2)\epsilon_W W_d > t.$$

Thus the origin, reflection and exceptional inequalities are strict at the interior width. Count balances and the identities of Section 16 follow after clearing denominators; the same rational evaluation gives their coefficient signs.

# E Exact inequalities for the unit-twisted comparison

In this appendix, we record the rational identities and interval inequalities used in Sections 21–26: the return estimate in Appendix E.1, the endpoint in Appendix E.2, the strict family in Appendix E.3, and the optimality signs in Appendix E.4.

## E.1 The return estimate

In this subsection, we verify the identities of Sections 19–21 and of Corollary H.3.

Exact symbolic computation gives

$$4 + 2p\alpha_{\max} = 12(\varrho - 4\alpha_{\max}), \qquad \frac{1}{2 + p\alpha_{\max}} = \frac{1}{6(\varrho - 4\alpha_{\max})}, \qquad E_{\alpha_{\max}}(p, \rho_{\max}(p)) = 0,$$

$$\frac{\partial E_{\alpha_{\max}}}{\partial \rho} = -\frac{p^2 - 193p + 198}{18(p-1)(p+24)} > 0, \qquad \frac{d\rho_{\max}}{dp} = -\frac{42(53p^2 - 108p - 72)}{(p^2 - 193p + 198)^2} < 0,$$

and solving $E_{\alpha_{\max}}(p, \rho) = 0$ for $\rho$ independently returns (17.2). On $[112/25, 9/2]$,

$$\alpha_{\max}(p) < \frac{\varrho}{5}, \qquad \frac{p\alpha_{\max}}{6\varrho} - \frac{7 - 5p\alpha_{\max}}{6000} > \frac{1}{12}, \qquad p > \lambda_{\max}(p), \qquad \frac{\lambda_{\max}(p)}{6} < 1.$$

With $\rho \le \rho_{\max}(112/25) = 5662/67351$,

$$\frac{503}{1500}\rho < \frac{3}{100}, \qquad 1 - \frac{\rho}{4} > \frac{97}{100}, \qquad -1 + \frac{1381}{1500}\rho < -\frac{9}{10},$$

and the two terminal ceilings (20.2) are below $-1/5$. The identities

$$\frac{5 - \rho}{\varrho} - 6 = k_p, \qquad t_{\mathrm{low}}(\gamma) = t_{\mathrm{high}}(\gamma) = 0 \quad \text{at } (\alpha, \gamma) = \left(\alpha_{\max}, \frac{1}{2 + p\alpha_{\max}}\right)$$

hold, and for $\alpha = \frac{99}{100}\alpha_{\max}(p)$ the interval of allowed $t$, the uncentered condition and the origin of Corollary H.3 have positive margins at the midpoint of $(\gamma_r, \gamma_c)$, at $p = 112/25$ and $p = 9/2$. The twisted and untwisted restoration identities of Section 21 are verified by direct substitution.

The signs on the whole interval $[112/25, 9/2]$ follow from Sturm root counts of the reduced numerators and denominators and one exact evaluation. For example,

$$\frac{p\alpha_{\max}}{6\varrho} - \frac{7 - 5p\alpha_{\max}}{6000} - \frac{1}{12} = \frac{3513p^2 - 17711p + 12198}{6000(p-1)(p+24)} > 0,$$

since the roots of the numerator lie below 4.22.

## E.2 Algebraic enclosures and the endpoint quadratics

In this subsection, we verify Lemma 23.1 and the endpoint signs used in Lemma 24.1. Put

$$\ell_- = \frac{17043265149923064667912171558759800240861152774490388350983 6}{10^{60}},$$
$$I_\ell = [\ell_-, \ell_- + 10^{-60}],$$

with $\ell_-$ the exact rational number displayed above. By Sturm's theorem, $f_{\mathrm{tw}}$ has exactly one zero in $I_\ell$. Directed rational evaluation of (23.5)–(23.6) on $I_\ell$ gives intervals of width below $10^{-59}$ containing

$$r_{\mathrm{tw}} = 0.44428449663005476928557852908290979870\ldots,$$
$$b_{\mathrm{tw}} = 0.09579610595986973062810543980817815339\ldots,$$
$$\kappa_{\mathrm{tw}} = 0.74811700758371800999377247553953433212\ldots,$$
$$B_{\mathrm{tw}} = 0.87405850379185900499688623776976716606\ldots.$$

The contact, tangency and $b$-stationarity numerators reduce to zero modulo $f_{\mathrm{tw}}$, the identity $\widetilde{C}(p) = -(p^2 - 200p - 684)/(28p(p + 126))$ holds, and both endpoint quadratics have positive denominators and negative leading coefficients on these intervals; the joint-width quadratic has negative discriminant. The polynomials (23.3) and (1.1) are irreducible, and (1.1) has one root in $(0.87405850, 0.87405851)$.

## E.3 The strict family

In this subsection, we describe the verification of Lemma 24.1, using the enclosures of Appendix E.2.

Along the family put $c_{\mathrm{fam}} = \ell_{\mathrm{tw}} + \kappa_{\mathrm{tw}}/25$, enclosed by the intervals of Appendix E.2, so that $\ell = c_{\mathrm{fam}} - \kappa/25$, and $\rho = \rho_{\max}(6\kappa)$. Each inequality of Table 8, with the modifications (23.1), is then a rational function of $(\delta, x, \kappa, c_{\mathrm{fam}}, b)$. It is evaluated in interval arithmetic with endpoints in $2^{-220}\mathbb{Z}$, rounded outward, on a subdivision of the stated $(\delta, x)$ domain and of $[\kappa_{\mathrm{tw}}, \kappa_0]$. On each box the lower endpoint is positive, and no denominator enclosure contains zero. A first-order Taylor enclosure in $(\delta, x, \kappa)$ is used for the derivative-free entries.

There are 104 table inequalities and four derivative inequalities, verified on 100672 boxes in all. The table inequalities have smallest lower margin $2.4 \times 10^{-7}$. The two iteration inequalities $d\mathcal{Q}_i/d\kappa < -10^{-2}$ have margins $1.1 \times 10^{-7}$ and $2.8 \times 10^{-5}$, and the two inequalities $\partial\mathcal{Q}_i/\partial\rho > -10^{-2}$, with $\rho \in [0.083, 0.0841]$ free, have margins $6.8 \times 10^{-6}$ and $5.2 \times 10^{-3}$. The two endpoint inequalities $\partial_x \mathcal{Q}_i > 10^{-3}$ have margins $1.9 \times 10^{-6}$ and $5.2 \times 10^{-7}$.

The two endpoint derivative checks use another 35370 boxes, separate from the 100672 family boxes.

A second evaluation with exact rational intervals and first-order derivative enclosures in $(\delta, x, \kappa, \ell)$ proves the main, bridge, complementary and full-class inequalities, the endpoint derivatives and the iteration derivatives: 32 inequalities on 9518 boxes.

## E.4 The optimality signs

In this subsection, we verify the signs used in Section 26.

The separation identity (16.2) and $341/60 - (13/2)(8741/10000) = 101/60000$ do not depend on the endpoint. The dual identity (26.1) and the edge identity at $(r_{\mathrm{tw}}, C_{\mathrm{tw}})$ are exact modulo $f_{\mathrm{tw}}$. At the endpoint enclosures,

$$m_d = 0.39776281583\ldots < \frac{2}{5}, \qquad r_{\mathrm{tw}}(1 - 4C_{\mathrm{tw}}) > 0, \qquad \frac{13}{D_{\mathrm{hi},0}} = 0.896935\ldots > B_{\mathrm{tw}},$$

and the weights $\lambda_{\mathrm{dir}}, \lambda_K, \lambda_{\mathrm{hi}}, \lambda_K^0, \lambda_{\mathrm{hi}}^0, \lambda_T^0$ are positive. The width constants satisfy

$$\widetilde{C}'(p) = -\frac{163p^2 + 684p + 43092}{14p^2(p+126)^2}, \qquad \frac{\partial C_\rho(p)}{\partial \rho} = \frac{p-1}{4p(5-\rho)^2},$$

with $684^2 - 4 \cdot 163 \cdot 43092 < 0$. The polynomial of $B_{\mathrm{ex}}$ has one root in $(0.87407284, 0.87407285)$, and $B_{\mathrm{tw}} < B_0 = 0.87407285$.

# F The enlarged norm with a moving radical

In this appendix, we extend the enlarged norm of Theorem 12.1, with its origin coefficient 4, to parameter families with a displayed moving radical. With the terminal estimate of Appendix G and the variants of Appendix H, it gives the bound (17.4) for the returned sums of Section 21. The proofs use only the results of Section 2 and Part II; no result below uses Theorem 17.1.

Fix $p = 6\kappa \in [112/25, 9/2]$ and the proved bound $\beta_*(F) \le (1+\kappa)/2$. Ordinary sums are centrally normalized, the original prime ranges are disjoint and complete, and the coefficients lie in the fixed group $\Theta$. Exact product scales, natural zeros, common norm twists, finite families of smooth weights and weighted Fourier measures are retained.

**Theorem F.1** (Small-moving-radical enlarged norm)**.** *Fix $0 \le q_0 \le 1/200$. On a parameter range $0 < q_k \ll U^{1-q_0}$, use the common presentation of [Liu26b, Section 12.4]*

$$\psi_k(n) = \nu_{\mathrm{fix}}(n) \prod_j \chi_{a_j}(n)^{e_j} \chi_n(k), \qquad \nu_{\mathrm{fix}} \in \Theta,$$

*whose full displayed moving radical has norm at most $U^{q_0}$. All zero extensions, including cancelled displayed factors, remain. Allow the same separate fixed common extra coprimality radical as [Liu26b]. Set $M_0 = (1-q_0) + q_0 = 1$.*

*Fix a compact parameter set with*

$$0 < t \le \min(m_-, 1/4), \quad m_- \le r < 1/2, \quad 0 \le W \le 1/25,$$

*bounded away from $t = 0$, and uniform strict margins in*

$$A_{\max} + 4W < 1, \qquad A_{\max} = 2r + (1-2r)/p + W, \tag{F.1}$$

$$A_{\max} + 3W < 5/6 + 2t/3, \tag{F.2}$$

$$2 + 2t - 4m_- + pW < 1. \tag{F.3}$$

*For $m \in [m_-, r]$, $0 \le z \le (1-2m)/p$ and $0 \le w \le W$, let*

$$P = S_{m,1} S_{m,2} Q_0, \quad P_{\mathrm{cmp}} = S_{t,1} S_{2m-t,2} Q_0, \quad P_c = P - P_{\mathrm{cmp}},$$

*with actual equal products of rectangle scales, and prime widths $z, w$ for $Q_0, Q_W$. Then, for a sufficiently fine complete-factor mesh,*

$$\sum_k \mathcal{W}(q_k/U^{1-q_0}) |P_c(k) Q_W(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C,$$

$$\sum_{\psi_k \text{ induces outside } \Theta} \mathcal{W}(q_k/U^{1-q_0}) |P(k) Q_W(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C.$$

*Both sums have the unrestricted parameter presentation of [Liu26b]; the second requires no sixth-power-free restriction on $k$. Uniformity and the order of choices are those of the moving presentations, with $q_0$ in the displayed compact interval.*

*Proof.* We follow the proof of Theorem 12.1 step by step and record, for each step, exactly how the incoming displayed radical of width $q_0$ enters. The conductor of $\psi_k$ is at most $q_k U^{q_0} \ll U^{M_0}$, $M_0 = (1-q_0)+q_0 = 1$.

*First transform and its origin.* The first bridge of [Liu26b] and one-sided allowance are stated for the common moving presentation, so the first Poisson formula of the proof of Theorem 12.1 applies with the same complete product coefficients and Mellin twist. At the first dual origin the powerful-product condition gives $O(U^{A_t+\epsilon})$ coefficient pairs, cancelled by the paired normalization, times the parameter length $U^{1-q_0}$. The origin is $O(U^{1-q_0+\epsilon}) \le O(U^{1+\epsilon})$. The displayed moving coefficient has modulus at most one and does not depend on $k$.

*Local classification.* A prime dividing the displayed radical is a natural zero of every column coefficient, so it can be neither a genuine common-support prime nor a classified prime. The widths $w_f, f_0, f_1, g_{\mathrm{good}}, v, c_{\mathrm{cor}}$, the power removals $L_1 \ge f_1 + 4T_1$, $L \ge v + 4T$, and the divisor Cauchy at the original frequency are those of the proof of Theorem 12.1. Put

$$w = w_f + f_0 + f_1 + u_{\mathrm{free}}, \qquad u_{\mathrm{free}} = g_{\mathrm{good}} + v + c_{\mathrm{cor}}, \qquad t_{\mathrm{loc}} = f_0 + u_{\mathrm{free}}.$$

*Completed parameters.* The incoming radical stays in the displayed radical, and the width normalization is $M_0 = 1$. Hence

$$\begin{aligned} A &= A_t - f_1 - t_{\mathrm{loc}} - L_1 - L - 5c_{\mathrm{cor}}, \\ M &= 1 - 6t_{\mathrm{loc}} - f_1 - L_1 - L - 5c_{\mathrm{cor}} + T_1 + 5v + T, \\ q &= q_0 + f_1 + v + c_{\mathrm{cor}}, \qquad m_* = M - q, \\ \mathcal{R}_{\mathrm{loc}} &= -f_0 - L_1 + T_1 - S_1 - u_{\mathrm{free}} - L + v - 4c_{\mathrm{cor}} + T - S. \end{aligned}$$

The one-sided allowance of [Liu26b, (12.3)] contains the displayed radical. Thus

$$\Lambda_{\mathrm{in}} = A_t - c + q_0 + u + B_C - s_0, \qquad \Lambda_* = (A-c) + q + u + B_C - s_0',$$

and $\Lambda_* - \Lambda_{\mathrm{in}} = A - A_t + f_1 + v + c_{\mathrm{cor}} + f_0 + f_1$ as in the proof of Theorem 12.1; $\mathcal{R}_{\mathrm{loc}}$ is unchanged. No displayed radical is reclassified as an extra coprimality condition.

*Homogeneous restoration.* Put $e_z = 1 - 2m - pz$ and $\mu = \min(c_C, d_D)$. With $e_1 = L_1 - f_1 - 4T_1 \ge 0$ and $e = L - v - 4T \ge 0$,

$$D_f = 6w - e_z - (p-1)z_{C,\mathrm{old}} - 5w_f - 5f_1 - T_1 - 5v - T,$$

$$\begin{aligned} \mathcal{R}_{\mathrm{loc}} + \frac{D_f - \mu}{6} &= \frac{w_f - \mu}{6} - \frac{5}{6}(f_1 + v) - \frac{19}{6}(T_1 + T) \\ &\quad - \frac{p-1}{6} z_{C,\mathrm{old}} - \frac{e_z}{6} - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e \le 0. \end{aligned}$$

So $\mathcal{R}_{\mathrm{loc}} + (D_f - \mu)_+/6 \le 0$. Lemma 12.2 is stated with the full displayed moving radical and applies on each returned sum, with the same single mesh excess.

*Direct terminal branches.* With $m_* = M - q$ the exact identity is

$$\begin{aligned} \mathcal{R}_{\mathrm{loc}} + 3A - M &= 3A_t - 4 + 20t_{\mathrm{loc}} + 4f_1 + 4c_{\mathrm{cor}} - 3T_1 - 16v - 3T \\ &\quad + 3m_* + 3q_0 - S_1 - S, \end{aligned}$$

and $\mathcal{R}_{\mathrm{loc}} + 3A - \frac{7}{2}m_* - q$ acquires $\frac{5}{2}q_0$. The direct bounds of that proof therefore give

$$3A_t - 4 + 24W + \tfrac{3}{100} + 3q_0 < -\tfrac{19}{40}, \qquad 3A_t - \tfrac{7}{2} + 18W + \tfrac{7}{200} + \tfrac{5}{2}q_0 < -\tfrac{17}{80} \qquad (q_0 \le 1/200).$$

Proposition G.4 removes the restriction on $q_0$.

*Complementary return.* Proposition 4.8 is stated for a completed Gauss norm with $M = m+q$ and $\widetilde{q} = q+R+E_0$, for an arbitrary displayed radical $q$. Its returned width $M' = M+J-g-g_2+t_2$

keeps the full moving radical in the combined width. Its formulas for $J, g, \Delta'$ and the sparse-character savings do not involve $q_0$ separately.

*Second origin.* By the completed parameters and the identity $\mathcal{R}_{\mathrm{loc}} + A - M - \mu - B_C = A_t - 1 + 4w - \mathcal{S}$, which does not contain $q_0$,

$$\mathcal{O}_{\mathrm{total}} \le A_t - 1 + 4t_{\mathrm{loc}} + O(\sigma + \xi + \theta_N) \le A_{\max} + 4W - 1 + o(1) < 0.$$

*Exceptional centered means.* The first sparse saving $F_1^{\mathrm{loc}}$ of that proof contains $\widetilde{q}/6 \ge q_0/6$, which only helps. The common character, mode and both actual rectangles are retained, so the bound $A_t - 5/6 - 2t/3 + 3t_{\mathrm{loc}}$ holds, and (F.2) gives a strict margin.

*Integration, reflection and coprimality conditions.* Integrate in the original weighted Fourier measures. For parameters $k$ with $\psi_k$ inducing outside $\Theta$, reflect the long comparison factor at its conductor $\ll U^{M_0} = U$; [Liu26b, Lemma 12.6] is uniform in the moving presentation. The reflected exponent is $t + 1 - (2m - t) + p(z + w) \le 2 + 2t - 4m_- + pW < 1$. A fixed extra coprimality condition is expanded with its coefficients as at the end of the proof of Theorem 12.1. Then $P = P_c + P_{\mathrm{cmp}}$ gives the second estimate, without a sixth-power-free restriction on $k$.

All margins are uniform on the compact parameter set, and the strict terminal savings absorb the finite support, mesh, derivative and Fourier losses in the order of choices of that proof. □

*Remark* F.2. Theorem F.1 is not an application of Theorem 12.1 to a family outside its stated hypotheses: the preceding proof checks the extra radical and the change of parameter family. It proves only the old coefficient 4. Its use for a returned sum, towards a stronger norm, requires the uniform return estimate of Section 21.

# G An additive large-sieve terminal estimate

In this appendix, we prove the additive large sieve for full Gauss sums, first over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-3})$ (Lemma G.1) and then in the $S$-arithmetic setting (Proposition G.2). We apply it to the terminal terms of Theorem F.1 in Proposition G.4 and Corollary G.5.

The next lemma is independent of the zero-free hypothesis and of the sextic-character moment. Write $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-3})$, $\mathcal{O}$ for its integer ring, and use a fixed additive character and fixed lattice normalization. In the imaginary quadratic case the finite unit multiplicity is fixed.

**Lemma G.1** (Full Gauss polynomials)**.** *For moduli $\mathfrak{n}$ of norm comparable to $N \ge 1$, let $\chi_{\mathfrak{n}}$ be any characters of the corresponding residue unit groups, extended by zero at nonunits; primitivity is not required. Put*

$$G_{\mathfrak{n}}(h) = (N\mathfrak{n})^{-1/2} \sum_{x \bmod \mathfrak{n}} \chi_{\mathfrak{n}}(x) e_F(hx/n), \qquad \Phi(h) = N^{-1/2} \sum_{\mathfrak{n}} b_{\mathfrak{n}} G_{\mathfrak{n}}(h),$$

*with a consistent generator convention. Let $d = 1$ or $2$ be the real dimension, and fix a nonnegative Schwartz function $\Omega$ majorizing the relevant fixed ball. For $H \ge 1$ put $\omega_H(h) = \Omega(h/H^{1/d})$. Then*

$$\sum_h \omega_H(h) |\Phi(h)|^2 \ll (H + N^2) \frac{1}{N} \sum_{\mathfrak{n}} |b_{\mathfrak{n}}|^2 \frac{\varphi(\mathfrak{n})}{N\mathfrak{n}}. \tag{G.1}$$

*The implied constant depends on fixed lattices, annular constants and finitely many weight seminorms. If $b_{\mathfrak{n}}$ is the divisor-bounded convolution supported on $\mathfrak{s} \mid \mathfrak{n}$, the energy on the right is $\ll (N\mathfrak{s})^{-1} N^{\epsilon}$ times its retained coefficient seminorms.*

*Proof.* Expand each full Gauss sum as written. A frequency $x/n$ with $(x, n) = 1$ has reduced denominator $\mathfrak{n}$, so frequencies from distinct moduli are distinct modulo the fixed additive dual lattice. Over $\mathbb{Q}$ their spacing is $\gg N^{-2}$. Over the Eisenstein lattice two distinct frequencies differ by a nonzero lattice element divided by $nm$, modulo the dual lattice; since $|n|, |m| \asymp N^{1/2}$,

their spacing is $\gg N^{-1}$. Fixed different and additive-character normalizations change only the constant.

For completeness, the needed separated-frequency inequality follows from a smooth Poisson majorant. In real dimension $d = 1$ or 2, let $R = H^{1/d}$, let $\delta_{\mathrm{sep}}$ be the above separation, and choose a fixed Schwartz function $\Omega$ from the statement. The Gram kernel from Poisson summation is bounded in absolute value by

$$C_J R^d \sum_{\varpi \in \Lambda^*} (1 + R|\phi - \phi' + \varpi|)^{-J}.$$

Packing the $\delta_{\mathrm{sep}}$-separated frequencies in each Euclidean shell gives a Schur sum $\ll R^d + \delta_{\mathrm{sep}}^{-d}$ for $J$ sufficiently large. The Schur bound, or $2|c_\phi c_{\phi'}| \le |c_\phi|^2 + |c_{\phi'}|^2$, therefore bounds the quadratic form by $(R^d + \delta_{\mathrm{sep}}^{-d}) \sum |c_\phi|^2$. In both dimensions $\delta_{\mathrm{sep}}^{-d} \ll N^2$.

The exact frequency coefficient is $c_{\mathfrak{n},x} = N^{-1/2} b_{\mathfrak{n}} (N\mathfrak{n})^{-1/2} \chi_{\mathfrak{n}}(x)$, hence

$$\sum |c_{\mathfrak{n},x}|^2 = \frac{1}{N} \sum_{\mathfrak{n}} |b_{\mathfrak{n}}|^2 \frac{\varphi(\mathfrak{n})}{N\mathfrak{n}}.$$

This proves (G.1). Divisor-bounded coefficients and the count of multiples of $\mathfrak{s}$ in a norm annulus give the last assertion. A finite number of ray components or fixed lattice translates is handled by a fixed Cauchy factor. □

**Proposition G.2** (The $S$-arithmetic additive large sieve)**.** *Let $F \supset \mu_{12}$, $S$, $R = \mathcal{O}_{F,S}$ and $\Gamma = U_S^6$ be the data of [Liu26b], so that $S$ contains the archimedean places, the places above 6 and the support of the different, and $R$ is self-dual for the trace character $\psi_S = \prod_{v \in S} \psi_v$ of $F_S = \prod_{v \in S} F_v$. Let $d = [F : \mathbb{Q}]$. For exterior generators $n$ modulo $\Gamma$ with $q_n \asymp N \ge 1$ and coefficients $b_n$, choose characters $\chi_n$ of $(R/nR)^\times$, extended by zero, with*

$$\chi_n(\gamma) = 1, \qquad \chi_{\gamma n} = \chi_n \quad (\gamma \in \Gamma),$$

*where the second equality uses $R/\gamma nR = R/nR$. The sextic characters used here satisfy these conditions. Put*

$$\mathcal{G}(n,h) = q_n^{-1/2} \sum_{a \bmod n} \chi_n(a) \psi_S(ha/n), \qquad \Phi(h) = N^{-1/2} \sum_n b_n \mathcal{G}(n,h).$$

*Then, summing over exterior generators $h$ modulo $\Gamma$,*

$$\sum_{0 < q_h \le H} |\Phi(h)|^2 \ll (H + N^2) \frac{1}{N} \sum_n |b_n|^2 \frac{\varphi(n)}{q_n}, \tag{G.2}$$

*for $H \ge 1$, with a constant depending on $F, S$ and the fixed annular constants. The same bound holds with $0 \le \omega \le 1$ supported in $q_h \le C_0 H$, with dependence also on $C_0$. More generally, if $|\omega(h)| \le C_A (1 + q_h/H)^{-A}$ for some $A > 1$, the bound holds for $\sum_{h \ne 0} |\omega(h)|\, |\Phi(h)|^2$, with dependence on $C_A, A$.*

*Proof. Invariance.* Changing the residue variable gives

$$\mathcal{G}(n, \epsilon h) = \overline{\chi_n(\epsilon)} \mathcal{G}(n,h) \qquad (\epsilon \in U_S).$$

Thus $\mathcal{G}(n, \gamma h) = \mathcal{G}(n,h)$ for $\gamma \in \Gamma$. Replacing $n$ by $\gamma n$ and $a$ by $\gamma a$ also leaves the sum unchanged, by the two compatibility conditions. Hence $\Phi$ is a function on the orbits $h\Gamma$, and $\mathcal{G}(n,h)$ depends only on $n\Gamma$. For the sextic characters this is the identity of [Liu26b, (3.7)].

*Balanced representatives.* By the product formula, $|y|_S = q_y$ for an exterior generator $y$. The $S$-unit theorem makes $\log|\Gamma|$ a full lattice in $\{x \in \mathbb{R}^S : \sum_v x_v = 0\}$. Hence each orbit $y\Gamma$ with $q_y = T$ contains $y$ with

$$|y_v|_v \asymp T^{d_v/d} \ (v \mid \infty), \qquad |y_v|_v \asymp 1 \ (v \in S_f),$$

where $d_v = [F_v : \mathbb{R}]$. Choose such representatives for the frequencies and moduli. Every representative of a frequency then lies in $\mathcal{B}_H = \{y : \|y_\infty\| \le c_0 H^{1/d},\ y_v \in \mathfrak{p}_v^{-k} \ (v \in S_f)\}$ for fixed $c_0, k$. Different orbits have different representatives.

*Frequencies.* For $(a,n) = 1$ put $\phi = a/n \in F_S/R$ and $c_{n,a} = N^{-1/2} b_n q_n^{-1/2} \chi_n(a)$. Then $\Phi(h) = \sum_{n,a} c_{n,a} \psi_S(h\phi)$ and

$$\sum_{n,a} |c_{n,a}|^2 = \frac{1}{N} \sum_n |b_n|^2 \frac{\varphi(n)}{q_n}.$$

If $a/n \equiv a'/n' \pmod R$, then $n \mid an'$ and $n' \mid a'n$, so $nR = n'R$; writing $n' = \epsilon n$ gives $a' \equiv \epsilon a \pmod n$. Each frequency class therefore arises from at most $[U_S : \Gamma]$ pairs. Grouping equal frequencies multiplies the energy by at most this fixed factor.

*Poisson summation.* Let $\Omega_S = \Omega_\infty \otimes \bigotimes_{v \in S_f} 1_{\mathfrak{p}_v^{-k}}$ with $\Omega_\infty(y) = e^{\pi c_0^2} \exp(-\pi \|y\|^2 / H^{2/d}) \ge 1$ on the archimedean part of $\mathcal{B}_H$. By positivity,

$$\sum_{0 < q_h \le H} |\Phi(h)|^2 \le \sum_{h \in R} \Omega_S(h) |\Phi(h)|^2 = \sum_{\phi, \phi'} c_\phi \overline{c_{\phi'}} K(\phi - \phi'),$$

$$K(\varsigma) = \sum_{h \in R} \Omega_S(h) \psi_S(h\varsigma) = \mathrm{vol}(F_S/R)^{-1} \sum_{\varpi \in R} \widehat{\Omega}_S(\varpi + \varsigma),$$

by self-duality of $R$. Here $\widehat{\Omega}_S = \widehat{\Omega}_\infty \otimes \bigotimes_v \mathrm{vol}(\mathfrak{p}_v^{-k}) 1_{L_v}$ with fixed compact open subgroups $L_v$, and $|\widehat{\Omega}_\infty(\xi)| \ll H \exp(-\pi H^{2/d} \|\xi\|^2)$.

*Separation.* Fix $\phi$ and let $P_\phi = \{(\phi - \phi' + \varpi)_\infty : (\phi - \phi' + \varpi)_v \in L_v \ (v \in S_f)\}$. If $\varsigma = \phi - \phi' + \varpi \ne 0$ with $\varsigma_v \in L_v$ for $v \in S_f$, then $nn'\varsigma \in R \setminus \{0\}$. Hence $|nn'\varsigma|_S \ge 1$, so $|\varsigma|_S \ge (q_n q_{n'})^{-1} \gg N^{-2}$. Since $|\varsigma_v|_v \ll 1$ on $S_f$,

$$\prod_{v \mid \infty} |\varsigma_v|_v \gg N^{-2}, \qquad \|\varsigma_\infty\| \gg N^{-2/d}.$$

The difference of two points of $P_\phi$ has the same form (with $\phi' = \phi''$ allowed, when it is a nonzero element of $R$ with $|\cdot|_S \ge 1$). So $P_\phi$ is $\delta_{\mathrm{sep}}$-separated in $F_\infty \cong \mathbb{R}^d$ with $\delta_{\mathrm{sep}} \gg N^{-2/d}$, and the map $(\phi', \varpi) \mapsto \varsigma_\infty$ is injective.

*Schur bound.* Counting $\delta_{\mathrm{sep}}$-separated points in shells,

$$\sum_{\phi'} |K(\phi - \phi')| \ll H \sum_{y \in P_\phi} \exp(-\pi H^{2/d} \|y\|^2) \ll H + \delta_{\mathrm{sep}}^{-d} \ll H + N^2.$$

The Schur test, $2|c_\phi c_{\phi'}| \le |c_\phi|^2 + |c_{\phi'}|^2$, gives (G.2). For a smooth weight $0 \le \omega \le 1$ the same majorant applies. For a rapidly decreasing weight, sum the bounds for $q_h \le 2^j H$ with weights $2^{-Aj}$, $A > 1$. □

*Remark* G.3. For $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-3})$ with $S$ the archimedean place, the same argument is Lemma G.1.

With $\alpha_{\mathrm{t}} = A - c$, the coefficient scale written $\alpha$ in the proof of Theorem 12.1, its terminal polynomial is $U^{-\alpha_{\mathrm{t}}/2} \sum_n B(n) \vartheta(n) 1_{s' \mid n} 1_{(n, \mathfrak{r}_*) = 1} \mathcal{G}(n, h)$ with the normalized $\mathcal{G}$, so it is $\Phi(h)$ above with $N = U^{\alpha_{\mathrm{t}}}$ and $b_n = B(n)\vartheta(n) 1_{s' \mid n} 1_{(n, \mathfrak{r}_*) = 1}$. Its energy is $\ll (N \mathfrak{s}')^{-1} U^{\epsilon} = U^{-s_0' + \epsilon}$ by the divisor bound.

**Proposition G.4** (Application to the terminal allowances)**.** *In the setting of Proposition G.2, both direct terminal branches in the proof of Theorem 12.1 close for every incoming displayed radical width* $0 \le q_0 \le 1$*, under the origin condition* $A_{\max} + 4W < 1$ *of Theorem F.1.*

*Proof.* Use the notation of that proof, writing $\alpha_{\mathrm{t}} = A - c$ for its $\alpha$; also $u = R + E_0$, $M = m_* + q$ and $m_{\mathrm{f}} = m_* - c + d - u$. The cutoff is $K_0 = 2\alpha_{\mathrm{t}} - m_{\mathrm{f}}$ and the allowance is

$$\Lambda_* = \alpha_{\mathrm{t}} + q + u + B_C - s_0'.$$

On a nonempty coefficient support, $\alpha_{\mathrm{t}} \ge -\theta_{\mathrm{supp}}$, where $\theta_{\mathrm{supp}} = O_{\mathcal{F}}(1/\log U)$ for the fixed original factor list $\mathcal{F}$. Use $N_{\mathrm{add}} = \max(1, U^{\alpha_{\mathrm{t}}})$ and $H_{\mathrm{add}} = \max(1, U^K)$ as the two scales in Proposition G.2, retaining the normalization $U^{-\alpha_{\mathrm{t}}/2}$. Its ratio to $N_{\mathrm{add}}^{-1/2}$ is bounded, and replacing $\alpha_{\mathrm{t}}$ by $(\alpha_{\mathrm{t}})_+$ changes the exponent by at most $\theta_{\mathrm{supp}}$. Empty supports contribute zero. As in Lemma 4.5, extract the fixed weight seminorms and height factor before applying the sieve: the retained frequency cutoff satisfies $0 \le \omega \le 1$, $\operatorname{supp}\omega \subset \{0 < q_h \ll U^K\}$. The discarded Fourier tails are estimated before this truncation; the zero frequency is a separate origin term. With $H$ still denoting the incoming norm-twist height, the coefficient energy $U^{-s_0'+\epsilon}$ gives

$$\sum_h \omega(h)|\Phi(h)|^2 \ll U^{\max(K_+, 2\alpha_{\mathrm{t}}) - s_0' + \epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C.$$

For a nonempty frequency range, $K_+ \le K_0 + O(\xi) + O(1/\log U)$. Subtracting $\Lambda_*$, the two relative exponents before $\mathcal{R}_{\mathrm{loc}}$ are

$$\begin{aligned} K_0 - \alpha_{\mathrm{t}} - q - u - B_C &= A - M - d - B_C, \\ \alpha_{\mathrm{t}} - q - u - B_C &= A - M + (m_* - c - u) - B_C. \end{aligned}$$

The second is at most $A - M + \tau$ if $m_* \le \tau$, and at most $A - M + \tau - d$ if $m_{\mathrm{f}} \le \tau$. In either terminal branch the relative exponent is at most

$$\mathcal{R}_{\mathrm{loc}} + A - M + \tau + O(\xi) + O(1/\log U) \le A_t - 1 + 4t_{\mathrm{loc}} + \tau + O(\xi) + O(1/\log U).$$

Choose $\tau$ and $\xi$ below one tenth of the margin in $1 - A_{\max} - 4W$, in place of the cutoff $1/100$ of that proof. Both terminal branches are then strictly negative after the threshold.

Otherwise both global and formal lengths exceed $\tau$, and Proposition 4.8 applies, since its proof uses the cutoff $1/100$ only through its positivity. The auxiliary outer-length replacement keeps its cutoff and allowance identities. □

**Corollary G.5** (Full moving-width extension)**.** *Theorem F.1 holds for* $0 \le q_0 \le 1$ *and without the condition* $W \le 1/25$*, in the fixed-field S-arithmetic realization, uniformly in its retained smooth weights and finite components.*

*Proof.* Proposition G.2 supplies the additive realization, and Proposition G.4 closes the two terminal branches using only $A_{\max} + 4W < 1$. In the proof of Theorem F.1, as in the proof of Theorem 12.1, the conditions $q_0 \le 1/200$ and $W \le 1/25$ enter only through the two direct terminal bounds

$$3A_t - 4 + 24W + \tfrac{3}{100} + 3q_0, \qquad 3A_t - \tfrac{7}{2} + 18W + \tfrac{7}{200} + \tfrac{5}{2}q_0.$$

The other steps use $W$ only through $A_{\max} + 4W < 1$, $A_{\max} + 3W < 5/6 + 2t/3$ and $2 + 2t - 4m_- + pW < 1$, and contain no upper bound on $q_0$ beyond the nonnegative incoming parameter length. □

*Remark* G.6 (Arithmetic setting). Lemma G.1 is an unconditional statement over $\mathbb{Q}$ and the Eisenstein field. The full arithmetic package is formulated over fields containing $\mu_{12}$; the Eisenstein field does not contain $\mu_{12}$. Consequently Corollary G.5 uses the field $F \supset \mu_{12}$ and the compatible realization checked in Proposition G.2. It is not a claim that Theorem 1.1 is proved over $\mathbb{Q}$ or the Eisenstein field without the general-field formulation.

# H Unbalanced and uncentered variants

In this appendix, we prove the unbalanced and uncentered variants of Theorem F.1 and Corollary G.5, and deduce the bound (17.4) for returned sums in Corollary H.3.

**Corollary H.1** (Unbalanced ordinary rectangle)**.** *In Theorem F.1 and Corollary G.5, replace* $S_{m,1}S_{m,2}$ *by* $S_{n_1,1}S_{n_2,2}$*, where* $n_1+n_2=2m$ *and* $\min(n_1,n_2)\ge t$*. Keep the same comparison* $S_{t,1}S_{2m-t,2}$ *at exact equal actual products. The same norm conclusions hold under the same conditions.*

*Proof.* Every local identity and homogeneous defect involves only $n_1+n_2$, and the first-origin and coefficient-square counts only the total product scale. The exceptional centered mean involves only

$$\min(n_1,n_2,t,2m-t)\ge t,$$

so its extraction and volume branches are unchanged. Reflection acts on the same long comparison factor. No other step uses $n_1=n_2$. □

**Proposition H.2** (Uncentered low-total enlarged norm)**.** *Fix* $p\in[112/25,9/2]$ *and* $\beta_*(F)\le(1+p/6)/2$*. Use the moving presentations and whole-product coefficient families of Theorem F.1, now with* $0\le q_0\le 1$*, parameter range* $q_k\ll U^{1-q_0}$ *and displayed radical at most* $U^{q_0}$*. Let* $n_1,n_2\ge 0$*,* $T_{\mathrm{ord}}=n_1+n_2\le T_{\max}<1$*,* $z\le(1-T_{\mathrm{ord}})/p$*,* $w\le W$*, and put* $A_{\max}=T_{\max}+(1-T_{\max})/p+W$*. Assume uniform strict margins in*

$$A_{\max}+4W<1,\qquad A_{\max}+3W<5/6.$$

*Then the uncentered norm of* $S_{n_1,1}S_{n_2,2}Q_0E$ *is* $\ll U^{1+\epsilon}$ *with the same seminorm and height dependence. No comparison rectangle, reflection condition, or positive lower bound on* $\min(n_1,n_2)$ *is needed. The estimate may include the original parameters inducing in* $\Theta$*.*

*Proof.* Apply the first transform and the local decomposition to the raw whole-product convolution. The origin, the coefficient energy, the homogeneous restoration and the nonexceptional return are as before, and Proposition G.4 applies without centering.

For a returned exceptional parameter use the raw-volume branch in the proof of Theorem 12.1 instead of a cancelled fixed-character mean. Here $T_1,T$ are the local principal lengths of Appendix F; $T_{\mathrm{ord}}$ denotes the ordinary total. The relative exponent before the local contribution is at most $A-5M/6-F_1-F_2$; the sparse-count savings of that proof satisfy $F_1,F_2\ge -O(\sigma+\xi)-O(1/\log U)$. Direct substitution gives

$$\begin{aligned}\mathcal{R}_{\mathrm{loc}}+A-5M/6&=A_t-5/6+3t_{\mathrm{loc}}-f_1/6-7(L_1+L)/6-29c_{\mathrm{cor}}/6\\&\quad+(T_1+T)/6-19v/6-S_1-S\\&\le A_t-5/6+3t_{\mathrm{loc}}.\end{aligned}$$

The discarded terms are nonpositive, since $T_1+T\le(L_1+L)/4$. Hence the second condition controls the exceptional terms without cancellation or a comparison rectangle. The argument starts from the full positive extension, so the original exceptional parameters are included. □

**Corollary H.3** (A prime-length bound for returned sums)**.** *Let* $M=m'+q'>0$ *be a returned total width with* $m',q'\ge 0$*, suppose ordinary reflection has placed two lengths* $0\le n_1,n_2\le M/2$*, and put* $T_{\mathrm{ord}}=n_1+n_2$*,* $g=M-T_{\mathrm{ord}}>0$ *and* $\gamma=g/M$*. For every* $0<\alpha<\alpha_{\max}(p)$*, the estimates of Corollary G.5 and Proposition H.2 admit any total prime width strictly below*

$$\Big(\frac{1}{p}+\alpha\Big)g,$$

*with a complete-factor mesh chosen inside the strict margins. The uncentered conclusion is for parameters whose final inducing character lies outside $\Theta$; inclusion of exceptional parameters is asserted only in the range of Proposition H.2. The statements are uniform on compact parameter sets with positive $M$, $\gamma$ and margins, and retain the moving radical and the coefficient families of Theorem 12.1.*

*Proof.* By Lemma 12.2 and the presentation of the returned sum in [Liu26b, Section 12.7.3], a returned sum has parameters $q_{h'} \ll U^{m'}$, a displayed moving radical of norm at most $U^{q'}$, $M = m' + q'$, centrally normalized annular ordinary factors with the original finite coefficients, and complete original prime factors on disjoint ranges. With $U^M$ in place of $U$ this is the family of Corollary G.5 and Proposition H.2, with $q_0 = q'/M \le 1$.

Rescale all exponents by $M$. Order the remaining complete prime factors and let $Q_0$ be a maximal initial segment of width at most $\gamma/p$; let $Q_W$ be the rest, of width at most $W = \alpha\gamma$ up to one complete factor. Then $Q_0$ and $Q_W$ have disjoint prime ranges, the mesh excess lies in the margin, and

$$A_{\max} \le 1 - (\varrho - \alpha)\gamma, \qquad A_{\max} + 4W \le 1 - (\varrho - 5\alpha)\gamma < 1,$$

since $\alpha < \alpha_{\max}(p) < \varrho/5$.

*Centered estimate.* Take $m_- = r = (1-\gamma)/2$. The exceptional and reflection conditions of Theorem F.1 read

$$t > t_{\mathrm{low}}(\gamma) = \frac{3}{2}\Big(\frac{1}{6} - (\varrho - 4\alpha)\gamma\Big), \qquad t < t_{\mathrm{high}}(\gamma) = \frac{1 - (2 + p\alpha)\gamma}{2}.$$

For $\gamma < 1/2$, the normalized ordinary lengths satisfy

$$\min(n_1, n_2) = 1 - \gamma - \max(n_1, n_2) \ge \frac{1}{2} - \gamma \ge t_{\mathrm{high}}(\gamma).$$

Also $t_{\mathrm{low}}(\gamma) < 1/4$, since $\varrho - 4\alpha > 0$. Thus Corollary H.1 applies with some $t \in (\max(t_{\mathrm{low}}, 0), \min(t_{\mathrm{high}}, 1/4))$ if and only if $t_{\mathrm{high}} > 0$ and $t_{\mathrm{low}} < t_{\mathrm{high}}$. With $D_\alpha = 4 + 2p\alpha - 6(\varrho - 4\alpha)$,

$$t_{\mathrm{low}} \ge t_{\mathrm{high}} \iff D_\alpha \gamma \ge 1,$$

so this holds whenever $\gamma < \gamma_c$, where

$$\gamma_c = \frac{1}{2 + p\alpha} \quad (D_\alpha \le 0), \qquad \gamma_c = \min\Big(\frac{1}{2 + p\alpha}, \frac{1}{D_\alpha}\Big) \quad (D_\alpha > 0).$$

*Uncentered estimate.* Proposition H.2 applies whenever

$$(\varrho - 4\alpha)\gamma > \frac{1}{6}, \qquad \text{that is,} \qquad \gamma > \gamma_r = \frac{1}{6(\varrho - 4\alpha)}.$$

*Coverage.* Since $D_\alpha < 6(\varrho - 4\alpha) \iff 2 + p\alpha < 6(\varrho - 4\alpha)$,

$$\gamma_r < \gamma_c \iff 2 + p\alpha < 6(\varrho - 4\alpha) \iff \alpha < \frac{4p - 6}{p(p + 24)} = \alpha_{\max}(p).$$

Choose $\gamma_s \in (\gamma_r, \gamma_c)$ and use the centered estimate for $\gamma \le \gamma_s$ and the uncentered one for $\gamma \ge \gamma_s$. On each compact range of $\gamma$ all conditions hold with positive margins, stable under the support and mesh losses. No uniformity is claimed as $\gamma \to 0$. □

**Lemma H.4** (The prime-length constant is extremal)**.** *Let $\alpha > \alpha_{\max}(p)$. Then there are a relative gap $\gamma \in (0, 1/2)$ and a total prime width strictly below $(1/p + \alpha)\gamma$ for which, whatever the division of the complete prime factors into $Q_0$ and $Q_W$, neither the conditions of Corollary H.1, for any $t > 0$, nor the exceptional condition of Proposition H.2 hold. The failure persists under small changes of the widths. Thus $\alpha_{\max}(p)$ is the supremum of the prime-length constants obtained from the centered and uncentered estimates.*

*Proof.* Choose $\alpha_0 \in (\alpha_{\max}(p), \alpha)$ and set

$$u = 2 + p\alpha_0, \qquad v = \varrho - 4\alpha_0, \qquad \gamma = 1/u \in (0, 1/2).$$

With $t_{\mathrm{low}}, t_{\mathrm{high}}$ of Corollary H.3 at $\alpha_0$ in place of $\alpha$,

$$u - 6v = (p+24)(\alpha_0 - \alpha_{\max}(p)) > 0, \qquad t_{\mathrm{high}} = 0, \qquad t_{\mathrm{low}} = \frac{u-6v}{4u} > 0, \qquad v\gamma = \frac{v}{u} < \frac{1}{6}.$$

In the normalized scale put $z_{\mathrm{tot}} = (1/p + \alpha_0)\gamma$ and $T_{\mathrm{ord}} = 1 - \gamma$. Every division of the complete prime factors with $z_{\mathrm{base}} \le \gamma/p$ has $W \ge z_{\mathrm{tot}} - \gamma/p = \alpha_0\gamma$. Thus even the actual coefficient length satisfies

$$T_{\mathrm{ord}} + z_{\mathrm{tot}} + 3W \ge 1 - v\gamma > 5/6,$$

so the raw exceptional condition fails. For a centered comparison, reflection of the long factor would require

$$t + 1 - (T_{\mathrm{ord}} - t) + pz_{\mathrm{tot}} = 2t + (2 + p\alpha_0)\gamma = 1 + 2t < 1,$$

which is impossible for $t > 0$. Moreover, the centered exceptional condition requires

$$t > \frac{3}{2}\big(T_{\mathrm{ord}} + z_{\mathrm{tot}} + 3W - 5/6\big) \ge \frac{u-6v}{4u} > 0.$$

Thus the two centered conditions have a positive separation, and both obstructions persist for sufficiently small changes of the widths. The width $(1/p + \alpha_0)g$ lies strictly below $(1/p + \alpha)g$, proving failure of uniform coverage of that open range, for every division of the complete prime factors. □

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