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An improved zero-free half-plane for Hecke L-functions over number fields and Quasi-Riemann Hypothesis

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Abstract

In this paper, we obtain a zero-free half-plane $\Re s > B_{\mathrm{ex}}$ for every finite-order Hecke L-function over every number field, where $B_{\mathrm{ex}} = 0.87407284766682790777\ldots$ is an explicitly defined algebraic number of degree four. The proof refines the cubic-theta comparison in two parts. In Part I, we require sufficient prime-product length for length restoration and fix the short side of the equal-product comparison rectangle at $1/4$. Direct and complementary Poisson estimates give an amplified Gauss norm with an additional prime product. Its separate applications to the numerator and the counting detector, combined with contour displacement and successive moment bounds, give the intermediate boundary $B' = 0.87422125049290208328\ldots$.

In Part II, complete prime removal is permitted even when a positive length excess remains. The no-prime moment estimate then controls the resulting ordinary sums, and the comparison rectangle's short side may vary. These changes remove the coupled condition on the remaining prime length and enlarge the allowed norm widths. The improved detector count and numerator estimate give successive proved bounds converging to B_{ex}. We prove optimality for this specified direct, detector and fourth-moment comparison system, retaining its geometric length inequality while allowing the sum of the length exponents to vary. Both parts use the established theta, moment and continuation results of the earlier QRH papers and their general-field formulation.

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1 Introduction

In this section, we state Theorem 1.1, recall the earlier bounds in Section 1.3, and describe the methods of Parts I and II in Sections 1.4–1.5. We write μ_n for the group of n-th roots of unity and $\zeta_n = e^{2\pi i/n}$ for its chosen primitive root.

1.1 The main theorem

In this subsection, we state the zero-free half-plane of Theorem 1.1. Section 15 proves it by successive moment estimates and abelian descent.

Let B_{ex} be the unique root in $(0.87407284, 0.87407285)$ of

$$178848B^4 - 507168B^3 + 522846B^2 - 231405B + 37099 = 0.$$

This polynomial is irreducible over $\mathbb{Q}$, and $B_{\mathrm{ex}} = 0.874072847666827907778669374 0\ldots$.

Theorem 1.1. *Let K be a number field and let χ be a finite-order Hecke character of K. Then $L_K(s,\chi)$ has no zeros in $\Re s > B_{\mathrm{ex}}$. The pole of a principal character at $s = 1$ is permitted. In particular,*

$$\zeta_K(s) \neq 0 \qquad (\Re s > B_{\mathrm{ex}},\ s \neq 1).$$

The numerical boundary is independent of K and χ; constants and thresholds may depend on the fixed arithmetic data. No assertion is made on the line $\Re s = B_{\mathrm{ex}}$.

The proof is given in Section 15. Its starting comparison gives the rational boundary 0.87408. The first further moment step already gives a boundary below 0.874073; the successive steps converge to B_{ex}. Theorem 16.1 proves optimality for the stated comparison system with its retained geometric length inequality.

1.2 The intermediate enlarged-count endpoint

In this subsection, we record the intermediate endpoint proved in Section 11. Let B' be the unique root in $(0.87422125, 0.87422126)$ of

$$481464B^4 - 1308150B^3 + 1274868B^2 - 525330B + 77719 = 0. \tag{1.1}$$

This polynomial is irreducible over $\mathbb{Q}$, and $B' = 0.87422125049290208328\ldots$.

Theorem 1.2. *Let K be a number field and let χ be a finite-order Hecke character of K. Then $L_K(s,\chi)$ has no zeros in $\Re s > B'$. The pole of a principal character at $s = 1$ is permitted. In particular,*

$$\zeta_K(s) \neq 0 \qquad (\Re s > B',\ s \neq 1).$$

The boundary is independent of K and χ. No assertion is made on the line $\Re s = B'$.

We use the earlier results cited in Section 2 as established theorems. The additional norm estimate and its applications to the numerator and class count are proved below. Constants and thresholds may depend on the fixed field K, its excluded places and the fixed character χ, including on $[K:\mathbb{Q}]$ and $|D_K|$. The numerical boundary B' is the same for every K and χ.

The fixed-moment endpoint. With the numerator-moment parameter fixed at $3/4$, the argument gives $B_* = 0.874468307693530459 55\ldots$ (Theorem 9.2). This is the optimum of that comparison system (Corollary 9.3). Its exact definition is as follows.

Let r_* be the unique root in $(9/20, 23/50)$ of

$$218582r^3 - 376469r^2 - 11376r + 61560 = 0. \tag{1.2}$$

Set

$$\ell_* = \frac{3410r_*^2 - 7127r_* + 2520}{2286r_*^2 - 5011r_* + 1800}, \qquad B_* = \frac{11}{12} - \frac{\ell_*}{4}. \tag{1.3}$$

The remaining cubic-endpoint parameters and a closed form for r_* are given in Section 8.2.

1.3 Earlier bounds and the results used here

In this subsection, we recall the bounds preceding Theorem 1.1. The precise earlier results used in its proof are stated in Section 2; their numerical optimization is not needed as an assumption in the new moment argument.

The cubic-theta argument of OpenAI [Ope26a] gives the boundary $7/8$ over $\mathbb{Q}(\sqrt{-3})$ and the corresponding Dirichlet family. Its shorter $11/12$ argument and the cyclotomic construction are given in [Ope26c, Ope26b].

In the first version of [Liu26a], by varying the length parameters b, ℓ, the author obtains

$$B_{\mathrm{r}} = \frac{34999}{40000} = 0.874975, \qquad B_{\mathrm{alg}} = \frac{1507 - 2\sqrt{921}}{1653} = 0.874957069799\ldots.$$

Contemporaneous parameter refinements were reported by Mermet [Mer26], Wei [Wei26], and Chakhkiev [Cha26]; see also [Liu26a, Section 21]. The two GitHub repositories report

$$B_{\mathrm{MW}} = \frac{4374785349}{5000000000} = 0.8749570698,$$

whereas Chakhkiev reports the conditional bound $10937/12500 = 0.87496$ under the analytic estimates specified there. These reports concern the Eisenstein-field and corresponding Dirichlet setting, rather than arbitrary number fields.

In version 2 of [Liu26a], also varying the moment parameter, the author obtains

$$B_{\mathrm{new}} = 0.874957019420098946\ldots.$$

Assuming the analytic estimates of [Ope26a], Xu and Sang [XS26] replace the pointwise treatment of the reflected numerator by an amplified average and obtain

$$B_{\mathrm{XS}} = \frac{8748867}{10^7} = 0.8748867$$

for the same Eisenstein-field and Dirichlet families. Newton [New26] studies the associated parameter optimization and gives the certified boundary

$$B_{\mathrm{N}} = \frac{43744332481}{50000000000} = 0.87488664962$$

under the analytic hypotheses and parameter restrictions specified there. This last result is a refinement of the comparison rather than a new mean-value estimate.

Cureton and Thorner [CT26] obtain the boundary $31/32$ for Dedekind zeta functions and finite-order Hecke L-functions over arbitrary number fields. Combining the averaged-numerator method of [XS26] with the general-field metaplectic method of [CT26], the author obtains in [Liu26b]

$$B_\flat = 0.874886647039964\ldots, \tag{1.4}$$

where

$$133488B_\flat^4 - 376596B_\flat^3 + 384534B_\flat^2 - 167541B_\flat + 26231 = 0.$$

Cureton and Thorner [CT26] use their general-field argument to descend from $K(\zeta_{12})$ to K. In this paper, we use the corresponding formulation in [Liu26b, Part III]. These field scopes are distinct: a bound over the Eisenstein field alone does not imply the same bound over every number field.

1.4 The methods in Parts I and II

In this subsection, we compare the norm and contour arguments of Part I with the enlarged norm of Part II. Sections 7 and 14 give the corresponding exponent estimates.

Both parts use the same cubic-theta comparison, original prime factors and previously justified moment parameters. Part I (Sections 3–11) uses the fixed comparison

$$(m, m) \longmapsto (1/4,\ 2m - 1/4)$$

and retains enough prime length for restoration after every fixing. The amplified norm and contour displacement first give B_*; successive moment bounds give $B_\natural$, and joint prime selection in the detector gives B'. Part II (Sections 12–16) removes the coupling between remaining prime length and length excess, and permits a variable comparison

$$(m, m) \longmapsto (t,\ 2m - t).$$

The no-prime moment controls the resulting ordinary sums even when a positive M2 length excess remains after all prime factors have been removed. In Part I, exhaustion occurs only after the coupled length inequality has made this excess nonpositive. The parameter t is chosen within the reflection and exceptional-term margins. These changes enlarge the norm domain; applying it to the detector and numerator then gives B_{ex}. We describe the common comparison and the two successive norm ranges below.

Let $Z > 1$ be the main scale parameter. In the smoothed cubic-theta average $I_\eta(X, Y, Z)$ of [Liu26b, Section 13.2], η is the fixed target Hecke character, and X, Y are the two averaging lengths; Z sets the smoothed norm scale of the indices in the cubic-theta sum. We write

$$X = Z^{l_x}, \qquad Y = Z^{l_y}.$$

The subscripts x, y refer to X, Y, respectively. The total prime-product length exponent ℓ and balancing parameter b satisfy

$$l_x + l_y + \ell = 1, \qquad b = l_y - l_x.$$

Write $U = Z^d$, $d > 0$, for the scale of the character parameter and $D = U^m$ for the length of an ordinary dyadic sum. The balanced length exponents are

$$l_x = \frac{1 - b - \ell}{2}, \qquad l_y = \frac{1 + b - \ell}{2},$$
$$h = \frac{1 + b + 3\ell}{2}, \qquad B(\ell) = \frac{11}{12} - \frac{\ell}{4}.$$

The endpoint character-parameter scale is $U = Z^h$. With the other class data fixed, $\mathscr{B}$ groups character parameters with fixed detector and prime-sum amplitude ranges. If a_{det} is the real-part parameter in the zero-detecting decomposition, write $\delta = 2a_{\mathrm{det}} - 1 > 0$; this parameter is denoted by a in the cited comparison. For prime-sum factors of lengths U^{λ_i}, let g_i be their logarithmic amplitude exponents, as in Lemma 4.2. Put

$$\sum_i \lambda_i = \mathscr{L} = \ell/d, \qquad 0 \le g_i \le \delta/2,$$
$$g = \mathscr{L}^{-1} \sum_i \lambda_i g_i = \delta x, \qquad 0 \le x \le 1/2.$$

Thus δ describes the detector range and x the normalized mean prime-sum amplitude. The contour shift θ specifies the numerator line $\Re w = 1/2 - \theta$. For the initial moment parameter $\kappa_m = 3/4$, the reference shift is $\theta_0 = 2\delta x/9$. Let $\mathcal{E}_{\theta,\gamma}(d, m)$ be the full comparison exponent when the contour is displaced to θ and the selected moment saves $U^{-\gamma}$. The simultaneous estimate is

$$\mathcal{E}_{\theta,\gamma}(d, m) - \mathcal{E}_{\theta_0,0}(d, m) = (\theta - \theta_0)\{d(1 - m) - l_y\} - d\gamma. \tag{1.5}$$

Both terms use the same dyadic sum and original prime coefficients. The prime-product lower bounds precede differentiation of the smooth weights.

For the fixed-moment comparison $\kappa_m = 3/4$, retain the coefficient and smooth-weight hypotheses of Theorem 4.9. On the selected classes of Lemma 7.3, assume

$$\frac{3}{10} \le \delta \le \frac{3}{5}, \qquad \frac{2}{5} \le x \le \frac{1}{2}, \qquad \mathscr{L} \ge \frac{21}{100}, \qquad \frac{57}{100} \le \frac{l_y}{h} \le \frac{11}{19}.$$

Let Q_0 be the initial prime product and Q_W a disjoint additional prime product. For $3/8 < m < 1/2$, write $W > 0$ for an upper bound on the length exponent of Q_W relative to U. The enlarged norm permits, with strict inequalities,

$$W < \frac{7}{54}(1 - 2m), \qquad W < \frac{8m - 3}{9}. \tag{1.6}$$

The unused prime product has lower mean

$$a(\delta, x) = \delta\frac{189x - 25}{139}, \qquad 0 < \gamma < \frac{a(\delta, x)W}{2}.$$

The limiting upper length $W_{\mathrm{cap}}(m)$ and the contour are chosen separately for each dyadic sum:

$$W_{\mathrm{cap}}(m) = \min\left\{\frac{7}{54}(1-2m), \frac{8m-3}{9}\right\}, \qquad m_\theta = 1 - \frac{l_y}{h},$$
$$\theta_*(m) = \begin{cases} 0, & m \le m_\theta, \\ \delta/2, & m > m_\theta. \end{cases}$$

Under these class and geometry hypotheses, Section 7 proves

$$\inf_{3/8\le m\le 1/2}\left\{\frac{aW_{\mathrm{cap}}(m)}{2} + (\theta_*(m)-\theta_0)(m-m_\theta)\right\} = \Gamma_N = \frac{7}{108}a(\delta,x)(2l_y/h-1).$$

Here Γ_N is the limiting normalized gain at $m = m_\theta$. These are limiting scalar values; the analytic estimates use strict interior lengths. At the critical length,

$$d = h, \quad m = m_\theta \quad\Longrightarrow\quad h(1-m_\theta) - l_y = 0, \qquad \mathcal{E}_{\theta,\gamma} - \mathcal{E}_{\theta_0,0} = -h\gamma.$$

Thus the enlarged moment supplies the saving at m_θ, and the contour displacement controls the remaining lengths.

Origin of the limiting parameters. The parameters in (1.3) and (8.1) are determined by the class for which the comparison is sharp. With E_κ as in (2.2), define

$$\mathcal{Q}_{\kappa,b,\ell}(\delta,x) = E_\kappa(\delta,x) - \frac{7}{108}a(\delta,x)(2l_y - h).$$

The contour contribution vanishes at $m_\theta = 1 - l_y/h$. The extreme amplitude is $x = 1/2$. At the limiting parameters the comparison has a double zero in δ, and its coefficient of the balancing parameter vanishes:

$$\begin{aligned} &\mathcal{Q}_{\kappa_*,b_*,\ell_*}(r_*,1/2) = 0, \qquad \partial_\delta \mathcal{Q}_{\kappa_*,b_*,\ell_*}(r_*,1/2) = 0, \\ &\partial_b \mathcal{Q}_{\kappa_*,b_*,\ell_*}(r_*,1/2) = 0, \qquad \kappa_* = \frac{5}{6} - \frac{\ell_*}{2}. \end{aligned} \tag{1.7}$$

Eliminating b_* and κ_* gives the cubic (1.2) and the rational formula for ℓ_*. The stationarity equation in δ then determines b_*. Section 8.2 proves these identities and verifies that all the other classes satisfy the required inequality.

The count iteration is used only to reach this boundary from strict interior estimates. Let $B_{\mathrm{seed}} = 0.874558$ denote the rational starting bound of Theorem 8.1. Its first combined step gives

$$B_1 = B_* + \frac{2B_{\mathrm{seed}} - 1 - \kappa_*}{100} < B_* + 1.8\cdot 10^{-6}.$$

The additional moment and contour saving supplies the principal improvement; the subsequent iteration removes this residual gap.

Through Section 9, the unamplified numerator moment remains at $\kappa_m = 3/4$, supplied by the earlier 7/8 bound. The class count may use a separate parameter κ_c. Starting from a strict rational bound, the iteration in Section 8 satisfies

$$\kappa_c' = \kappa_* + \frac{\kappa_c - \kappa_*}{50}, \qquad \kappa_* = 2B_* - 1.$$

Every finite step has positive margins. Taking the decreasing sequence of proved boundaries yields (1.3).

Successive moment bounds. Fix a number field $F \supset \mu_{12}$, and let $\beta_*(F)$ be the zero supremum defined in Section 2. For $13/18 \le \kappa_m \le 1$, under the coefficient, exclusion, mesh and length hypotheses of [Liu26b, Theorem 12.16 and Lemma 12.7], the zero-free hypothesis of M2 is

$\beta_*(F) \le (1+\kappa_m)/2$. Section 10 therefore puts $\kappa_m = \kappa_c = 2B_n - 1$, where B_n is the n-th proved boundary and $B_0 = B_*$. With $c = 1/(6\kappa_m)$,

$$z_0(m) = c(1-2m), \qquad \theta_0 = c\,\delta x, \qquad W_{\mathrm{cap}}(m) = \min\Big\{\frac{(1-c)(1-2m)}{6}, \frac{8m-3}{12\kappa_m}\Big\}.$$

At the extreme amplitude $x = 1/2$, with $\kappa_c, b, \ell, \delta$ fixed, the dependence of the moment contribution on κ_m is

$$\frac{\delta}{2}\Big(\frac{h}{2} - l_y\Big)\frac{1+5c}{6}, \qquad \frac{h}{2} - l_y < 0,$$

which decreases as κ_m decreases. The resulting iteration satisfies $\kappa_{n+1} - \kappa_\natural = (\kappa_n - \kappa_\natural)/50$, and its limit is the quartic endpoint (10.2).

The enlarged class count. The preceding numerator argument leaves the earlier M2 count in place. Section 11 also inserts an additional prime product into the counting detector. At moment parameter κ, the joint selection of complete original factors gives

$$|Q_0 Q_W|^2 \ge U^{2g(z_S+w)-o(1)}, \qquad z_S = \frac{1-2m}{6\kappa}, \quad w < W_{\mathrm{cap}}(m;\kappa).$$

The enlarged norm therefore replaces

$$A_S(m) = 1 - 2\delta m - 2g z_S(m) \quad \text{by} \quad A'_S(m) = A_S(m) - 2gW_{\mathrm{cap}}(m;\kappa)_+.$$

Balancing with M1 and the long count gives $R'_\kappa \le R_\kappa$. At a fixed numerator moment estimate, Hölder supplies the additional saving

$$\Delta_U = \frac{3}{4}(R_\kappa - R'_\kappa) \quad \text{in the } U\text{-exponent}, \qquad \Delta_Z(d) = d\Delta_U \quad (U = Z^d).$$

At $d = h$, $\Delta_Z(h) = 3h(R_\kappa - R'_\kappa)/4$. The parameters of (11.12) yield

$$B_\natural = 0.87446749650084027305\ldots, \qquad B' = 0.87422125049290208328\ldots,$$

and the iteration starts at $B_\natural$ with $B_n - B' = 50^{-n}(B_\natural - B')$.

Part II: prime exhaustion and variable comparison. The Part I enlarged count above uses the width $W_1(m) = (1-1/p)(1-2m)/6$, with $p = 6\kappa$, and a comparison short length $1/4$. Section 12 also applies the no-prime branch of M2 when complete prime removal leaves a positive length excess. Together with a variable comparison rectangle, this permits

$$W_o(m) = \frac{1-1/p}{5}(1-2m), \qquad W_j(m) = \frac{1/2 + (1-3/p)(1-2m)}{p+12},$$
$$W_d(m) = \min\{W_o(m), W_j(m)\}.$$

The same joint selection is used in the detector. Its count and the numerator gain now give 0.87408 at fixed rational parameters. The moment iteration of Section 15 gives B_{ex}. Thus Part II enlarges the allowed norm widths by changing the restoration and comparison rectangle; the theta identity, moment estimates and detector mechanism are retained.

1.5 Optimality and organization of the proof

In this subsection, we specify the optimality statements for the successive comparison systems in Sections 8, 10 and 16, and summarize the two parts leading to Theorem 1.1.

At the ordinary length $m_\theta = 1 - l_y/h$, the contour term vanishes:

$$h(1 - m_\theta) - l_y = 0.$$

At $x = 1/2$ one has $a(\delta, 1/2) = \delta/2$. The amplitude bound and the allowed norm width then give

$$h\gamma \le \frac{7\delta}{216}(2l_y - h) \qquad (x = 1/2).$$

At the critical class $(\delta, x) = (r_*, 1/2)$, this is exactly the limiting obstruction. It proves optimality for the comparison with the stated norm domain, count estimate and moment parameter $3/4$. It does not assert the best possible zero-free region for a field.

Section 10 leaves this system: there $\kappa_m < 3/4$ and the initial prime product is longer. If instead $\kappa_c, \kappa_m \ge 2B_n - 1$ and the norm of Lemma 10.1 is used, the same obstruction at $(\delta, x) = (r_\natural, 1/2)$ gives, for the comparison function (10.25) and every b,

$$\mathcal{Q}_{\kappa_\natural, \kappa_\natural, b, \ell}(r_\natural, \tfrac{1}{2}) = e_\natural(\ell - \ell_\natural), \qquad e_\natural > 1,$$

and $B_\natural$ is optimal in that system (Theorem 10.8). This statement fixes the earlier count of Proposition 6.4. The enlarged count of Proposition 11.3 changes that input. No optimality assertion is made here for this intermediate enlarged-count system. Section 16 proves the optimality of B_{ex} for the exhaustion-norm comparison of Part II, retaining the geometric length inequality (16.1).

Section 2 states the common arithmetic conventions and earlier inputs. Part I consists of Sections 3–11. Sections 3–4 develop the local arithmetic and amplified norm. Sections 5–7 treat the contours, class counts and combined saving. Section 8 gives the endpoint and its stated optimality; Section 9 proves the boundary B_* over arbitrary fields. Section 10 proves the starting boundary $B_\natural$ and its optimality with the earlier count. Section 11 proves the enlarged count, its endpoint inequalities and Theorem 1.2.

Part II consists of Sections 12–16. Sections 12–14 prove the exhaustion norm, detector count and rational starting bound. Section 15 proves Theorem 1.1, and Section 16 proves its stated optimality.

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2 Arithmetic notation and the earlier comparison theorems

In this section, we fix the notation used in Sections 3–7 and state precisely which earlier results are used. The new amplified norm is proved in Section 4; it is not included among the cited inputs.

Fix a number field $F \supset \mu_{12}$, hence totally complex, a sufficiently large finite set S of places containing the archimedean places, and let $\mathcal{O}_{F,S}$ be the ring of S-integers, written $\mathcal{O}$, with $\mathcal{O}$ principal and self-dual for the chosen trace character. Put

$$q_{\mathfrak{a}} = N\mathfrak{a}, \qquad U_S = \mathcal{O}^\times, \qquad \Gamma = U_S^6, \qquad r_F = [F : \mathbb{Q}]/2.$$

Here $N\mathfrak{a}$ is the absolute ideal norm. For a nonzero allowed generator a, q_a denotes the norm of its exterior ideal, with the factors at S omitted. We use

$$t_+ = \max\{t, 0\}, \qquad \mathrm{rad}(\mathfrak{a}) = \prod_{p \mid \mathfrak{a}} p, \qquad \tau(\mathfrak{a}) = \#\{\mathfrak{d} : \mathfrak{d} \mid \mathfrak{a}\}.$$

The valuation $v_p(a)$ is the exponent of p in the exterior ideal of a. Elements are taken modulo Γ with the allowed-generator and finite-ray conventions of [Liu26b, Part III]. A subscript p denotes a prime ideal outside S. All local characters retain their zero extensions at nonunits.

Let Θ be the fixed finite group of finite-order Hecke characters, unramified outside S, used in the moment estimates. It contains all initial finite coefficient characters and the group Υ_S generated by the finite characters in the allowed-generator unit decomposition. Thus

$$\psi \in \Theta \iff \psi\upsilon \in \Theta \quad (\upsilon \in \Upsilon_S), \qquad \overline{\Theta} = \Theta.$$

Exceptional inducing characters are those belonging to Θ. The symbol $\mathcal{U}$ denotes the sixth-power-free nonunit parameters whose associated primitive characters lie outside Θ.

Sums over the arithmetic parameters retain their multiplicities. The theta parameters are k, u; the dual Gauss frequencies are h, j. Their norm ranges are distinct from primitive-character conductor ranges.

Length exponents are related by

$$U = Z^d, \qquad U^t = Z^{dt}.$$

We write $S_{m,i}$ for an ordinary centrally normalized sum of length U^m, Q_0 for the initial prime product, and Q_W for a disjoint additional prime product. Their exact coefficients, finite twists and independent smooth weights are those in the moment families of [Liu26b]. The canonical comparison has the same prime factors and smooth weights:

$$P = S_{m,1}S_{m,2}Q_0, \qquad P_c = P - P_{\mathrm{cmp}}, \qquad (m,m) \longmapsto (1/4, 2m - 1/4).$$

For the actual ordinary scales, $X_1X_2 = X_{1,\mathrm{cmp}}X_{2,\mathrm{cmp}}$ exactly. For a nonnegative weight $\mathcal{W}$ on the summation parameters, $\mathfrak{S}_J(\mathcal{W})$ denotes the finite weighted Sobolev seminorm in the Poisson formula of [Liu26b, Section 8]. The complete Fourier measure is retained on both sides of each centered norm. The index J in $\mathfrak{S}_J$ is a finite derivative order; $H \geq 0$ bounds the external height and norm-twist parameters. The local length expression also denoted by J is defined in Section 3.3.

Put

$$\mathcal{Z}_F = \bigcup_{\chi \text{ primitive, finite order over } F} \{\rho : L_F(\rho, \chi) = 0,\ \rho \text{ nontrivial}\},$$
$$\beta_*(F) = \sup\big(\{1/2\} \cup \{\Re\rho : \rho \in \mathcal{Z}_F\}\big).$$

Principal poles are excluded. We use the following earlier results:

1. The cubic-theta coefficient formulas, reciprocal-L identity, continuation criterion for the original integral and initial 7/8 bound of [Ope26a, Ope26b, Liu26b].

2. The inverse and ordinary moment estimates M1 and M2, together with their full Gauss-sum form, compact-quotient Poisson formula, reflection, common coprimality restrictions on the coefficient variables and exceptional means, as assembled in [Liu26b, Sections 6–13]. The quadratic and cubic sieve estimates are [GL13, BGL14]; the metaplectic construction is [KP84, KP85], with the global Fourier normalizations of [Wei64].

3. The averaged-numerator comparison of [Liu26b, Section 14], with

$$\beta_*(F) \leq \frac{1+\kappa}{2} \implies \kappa_c = \kappa, \qquad \beta_*(F) \leq \frac{7}{8} \implies \kappa_m = \frac{3}{4} \quad \text{for unamplified M2.}$$

4. The direct estimate with $B(\ell) = 11/12 - \ell/4$ for $1/6 \leq \ell \leq 1/5$, the principal correction, and the final continuation argument of [Liu26b]. A strict upper bound for every remaining amplitude class gives the corresponding bound for $\beta_*(F)$.

5. The general-field construction and abelian descent of [Liu26b, Part III].

The additional analytic estimates. The following table separates the cited estimates from the extensions proved in this paper. The enlarged norm and contour range are conclusions of the indicated results, rather than additional hypotheses attributed to [Liu26b].

Estimate	Earlier result used	Extension proved here
Amplified norm	Full Gauss and M2 formulas, [Liu26b, Sections 12.7–12.8 and Appendix C.4]	Propositions 4.6, 4.8 and Theorem 4.9: direct and complementary Poisson estimates; Theorem 12.1: variable comparison lengths and an empty remaining prime product.
AFE contour	The holomorphic region of [Liu26b, Lemma 13.10]	Lemmas 5.1–5.2: compact-strip smooth weights and $0 \le \theta \le \delta/2$.
Smallest amplitudes	The lowest-amplitude estimate in [Liu26b, Proposition 14.6]	Lemma 6.3: the margin $1/125$ in Part I; Section 14: the margin $1/250$ in Part II.
Class count	The count and numerator average in [Liu26b, Section 13.4 and Proposition 14.3]	Proposition 6.4 and Corollaries 6.6, 6.7: separate parameters and explicit compatibility bounds; Proposition 11.3: the enlarged norm in zero detection; Proposition 13.2: simultaneous selection in the exhaustion range.

The exact full Gauss sums and the exceptional means in item (2) are used, with their original coefficient restrictions.

Local estimates. For each norm estimate with fixed F, S, Θ and fixed arithmetic conventions, choose the subsidiary parameters within its numerical strict margins. Fix the required smooth-weight and Fourier orders before the external height cutoffs and the eventual threshold. The constants and orders may depend on these fixed arithmetic data.

Global continuation. For each finite comparison step, use the order

$$
\begin{aligned}
&(F, \kappa_c, \kappa_m, b, \ell, \text{strict margins}) \\
&\quad \longrightarrow (\text{numerical losses, steps and meshes}) \\
&\quad \longrightarrow (N_{\mathrm{pr}}, \ell_i, \text{buffer } e) \\
&\quad \longrightarrow (\eta, \text{final } S, \Theta, \text{arithmetic data and tests}) \\
&\quad \longrightarrow \text{internal derivative and Fourier orders} \\
&\quad \longrightarrow \text{height powers, external cutoffs, tail orders and thresholds.}
\end{aligned}
$$

Here η is the target character and N_{pr} is the number of original prime ranges. This is the order of [Liu26b, Remark 3.2 and Lemma 12.17]. In particular, the prime-length mesh is chosen independently of the subsequent target-dependent arithmetic data.

For the step $\sigma > 0$ of Section 3.1 and prescribed loss $\epsilon_{\mathrm{num}} > 0$, choose

$$
0 < \eta_{\mathrm{mesh}} < \min\left\{\eta_{\mathrm{M2}}, \frac{\sigma}{6}, \frac{\epsilon_{\mathrm{num}}}{C_{\mathrm{num}}}\right\},
$$

where $\eta_{\mathrm{M2}} > 0$ is the arithmetic-independent mesh of the cited lemma and $C_{\mathrm{num}} > 0$ bounds the numerical coefficients in the selection and length-restoration losses.

The choice depends only on these losses and bounded scalar parameters. Later sections impose further upper bounds on this same η_{mesh}; it is taken below all of them. Take $d_{\min} = 1/100$. For $U = Z^d$, $d \geq d_{\min}$, require

$$\max_i \ell_i \leq d_{\min}\eta_{\mathrm{mesh}} \quad \Longrightarrow \quad \max_i \frac{\ell_i}{d} \leq \eta_{\mathrm{mesh}}.$$

For $d < d_{\min}$, use the small-parameter estimate of [Liu26b, Section 13.5]. After fixing the arithmetic data and original factors, retain their weight seminorms. For $H_N, \epsilon_N > 0$ with $|\theta_N| \leq H_N/\log U$,

$$Z \geq \exp\left(\frac{H_N}{d_{\min}\epsilon_N}\right) \implies |\theta_N| \leq \epsilon_N.$$

Thus the fixed mesh and prime lengths suffice; the threshold may depend on the target and finite comparison step.

For an exterior modulus u, the full normalized Gauss sum is

$$\mathcal{G}(u,h) = q_u^{-1/2} \sum_{a \bmod u} \chi_u(a) e(ha/u), \qquad \mathcal{G}(1,h) = 1.$$

Here e is the fixed trace additive character and χ_u is the zero-extended sextic residue character. The imprimitive local factors and the correlation formulas are those of [Liu26b, Lemmas 12.9–12.10]; in particular they are not replaced by their primitive-unit values. Local lengths such as c_C, d_D always mean logarithms of actual ideal norms to base U. In the Gauss calculations, x_{aux} denotes the auxiliary length removed from the initial prime product. In the amplified norm, this auxiliary length is zero; the initial prime product is retained.

Local Gauss notation. The local and global uses of these letters are distinguished by their sections:

	Sections 3–4	Sections 5–8
h	Gauss frequency	$(1+b+3\ell)/2$
a, b	local amplifier lengths	$a(\delta, x)$, balancing b
w	additional prime-product exponent	complex numerator variable
T, S	local principal lengths	unamplified slope S.

The fixed convolution ideals C, D satisfy

$$\mathrm{rad}(C) = \mathrm{rad}(D) = \mathrm{rad}(CD),$$
$$c_C = \log_U q_C, \quad d_D = \log_U q_D, \quad \mu_{\mathrm{com}} = \min(c_C, d_D), \qquad 0 \leq R + E_0 \leq \mu_{\mathrm{com}}.$$

Their prime-power multiplicities may differ. Here s is the fixed inner divisor and m_{cut} an ordinary-length cutoff; R is the primitive common-character modulus exponent and E_0 the squarefree-divisor exponent in the complementary coprimality expansion. In the global comparisons, R denotes a class-count exponent; for the earlier count with fixed κ, $R = R_\kappa(\delta, x)$. The symbol D in (C, D) denotes an ideal, whereas $D = U^m$ is an ordinary summation length. The local length excess in Section 3.1 is D_{loc}; other excesses include D_{free} and the M2 excess D of Lemma 12.2.

For later use, put

$$l_y = \frac{1+b-\ell}{2}, \quad h = \frac{1+b+3\ell}{2}, \quad l_x = \frac{1-b-\ell}{2}, \quad U = Z^d, \quad C_0 = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4}. \tag{2.1}$$

If $R_\kappa(\delta, x)$ is the class-count exponent defined in Section 6.3, the endpoint of the unamplified character-sum comparison at $d = h$ is

$$E_\kappa(\delta, x) = C_0 + \frac{\delta}{2} + \delta x \left\{ \ell + \frac{4}{9}\left(\frac{h}{4} - \frac{l_y}{2}\right)\right\} - \frac{3h}{4}(1 - R_\kappa(\delta, x)). \tag{2.2}$$

When κ is fixed, the global comparison abbreviates $R = R_\kappa(\delta, x)$ and $E = E_\kappa(\delta, x)$. Away from $d = h$, the full comparison exponent is affine in d, with slope

$$S = \frac{3R_\kappa + 1}{4} + \frac{\theta_0}{2} - \frac{17}{50}.$$

For $0 \le m \le 1/2$, the reference initial length and actual prime-product exponent satisfy

$$z_0(m) = 2(1 - 2m)/9, \qquad 0 \le z \le z_0(m).$$

For $m > 1/2$, use the unamplified moment without selected prime factors, so $z = 0$. The available prime width is $\mathscr{L} = \ell/d$. A numerator saving $U^{-\gamma}$ subtracts $d\gamma$ from the full exponent.

Part I
The amplified norm and the enlarged class count

3 Local Gauss estimates and the prime products

In this section, we prove the local estimates used in the amplified norm of Section 4. We first count nonunit prime choices at the original frequency, then treat fixed unit primes and the common-support terms.

3.1 A divisor bound for nonunit prime contributions

In this subsection, we apply Cauchy before dividing the original Gauss frequency. The resulting divisor estimate is used in the local decomposition in Section 3.4.

The labels 'good' and 'bad' in the local formulas denote the unit and nonunit cases, respectively. Every prime with a nonunit Gauss contribution divides the original nonzero Gauss frequency. Counting its possible choices before dividing that frequency gives the following improvement. The frequency $h = 0$ is excluded throughout this argument. Its separate compact-quotient origin remains estimated by the earlier powerful-product count; no divisor bound is applied to zero.

We retain the full normalized Gauss sums, allowed generators modulo $\Gamma = U_S^6$, fixed finite ray-class data, Fourier measure for the complete centered coefficient sums and uniform orders of derivatives and height bounds of [Liu26b].

The Gauss norms with fixed arithmetic parameters are evaluated by the length-restoration argument of [Liu26b, Sections 12.7–12.8]: the auxiliary ordinary sums are placed in the allowed length range before M2 is applied. No Gauss mean outside the established length range is assumed.

The length notation for prime-factor removal is

u	total freely removed prime length
v	actual nonunit-prime length
c	fifth-power-correction length
L	ordinary power-removal length
T, S	principal upper/lower lengths
D_{loc}	length excess before restoration.

In particular,

$$0 \le v \le u, \qquad L \ge v + 4T, \qquad c, T, S \ge 0.$$

For the original product and the auxiliary moment, respectively, write

$$z = \mathrm{length}(Q_0), \quad w = \mathrm{length}(Q_W), \quad A_t = 2m_{\mathrm{ord}} + z + w,$$
$$A = \text{coefficient-length exponent}, \quad m = \text{parameter-range exponent},$$
$$q = \text{displayed character-support exponent}, \qquad M = m + q.$$

Here q includes canceled factors, and the auxiliary m differs from the ordinary length m_{ord} in $S_{m_{\mathrm{ord}},i}$. The common-support contribution and inner-divisor exponents are

$$\widetilde{q} = q + R + E_0, \qquad B_C = (3c_C - 5d_D - R)_+/6,$$
$$B_D = (3d_D - 5c_C - R)_+/6,$$
$$s_0 = \log_U q_s, \qquad s_0' = \log_U q_{s'},$$

where s' is the inner divisor after the local prime removals. Starred symbols A_*, M_*, q_*, m_* denote the resulting parameters after the local identities; their updates are given in Section 3.2. The total freely removed unit-prime length is $g_{\mathrm{good}} = u - v - c$; the fixed valuation-zero primes have the separate length f_0. The actual frequency annulus has exponent K, while K_0 is its nominal cutoff exponent; $\delta_1 \geq 0$ is the support buffer and $\Delta = K_0 - K \geq -\delta_1$ records the cutoff discrepancy. The auxiliary lengths J, g are specified in Section 3.3. The positive parameters σ, ξ control the auxiliary step and support margins, and $\eta_{\mathrm{mesh}} > 0$ is the prime-length mesh. The total dyadic-support error is written $\theta_N = O(1/\log U)$.

The symbol D_{loc} never includes a second use of the favorable $-d_D$ term already included in the earlier argument's J, g inequality.

3.1.1 Cauchy's inequality with a divisor bound

In this subsubsection, we prove the finite Cauchy inequality of Lemma 3.1 and apply it to nonunit-prime divisors. The resulting divisor bound is used for the local Gauss sums in Section 3.1.2.

Lemma 3.1 (Cauchy's inequality for restricted finite sums)**.** *Let $\mathcal{H}, \mathcal{B}$ be finite sets, let $\mathcal{B}(h) \subset \mathcal{B}$ satisfy $\#\mathcal{B}(h) \leq N_{\mathcal{B}}$, and let $c_b, Z_b(h)$ be arbitrary complex numbers. Then*

$$\sum_{h\in\mathcal{H}} \left| \sum_{b\in\mathcal{B}(h)} c_b Z_b(h) \right|^2 \leq N_{\mathcal{B}} \sum_{b\in\mathcal{B}} |c_b|^2 \sum_{\substack{h\in\mathcal{H} \\ b\in\mathcal{B}(h)}} |Z_b(h)|^2. \tag{3.1}$$

No common function Z_b or common change of variable across b is required.

Proof. For each original h, Cauchy's inequality gives

$$\left| \sum_{b\in\mathcal{B}(h)} c_b Z_b(h) \right|^2 \leq \#\mathcal{B}(h) \sum_{b\in\mathcal{B}(h)} |c_b Z_b(h)|^2 \leq N_{\mathcal{B}} \sum_{b\in\mathcal{B}(h)} |c_b|^2 |Z_b(h)|^2.$$

Sum over h and interchange the finite sums:

$$\sum_{h} \sum_{b\in\mathcal{B}(h)} |c_b|^2 |Z_b(h)|^2 = \sum_{b} |c_b|^2 \sum_{h:\, b\in\mathcal{B}(h)} |Z_b(h)|^2.$$

□

The order is

$$\text{Cauchy at fixed } h \quad \longrightarrow \quad \sum_{h,b} \quad \longrightarrow \quad \text{frequency division.}$$

The coefficient squares therefore carry no unrestricted count of b.

Lemma 3.2 (Divisor bound for the nonunit prime choices)**.** *Fix the removed factor positions and their unit, nonunit and fifth-power-correction pattern. On the original bounded frequency range, the number of actual nonunit-prime tuples and their allowed ordinary valuation/allocation data is at most*

$$\tau(\mathfrak{h})^{C_N} (1 + \log q_h)^{C_N} \ll_\epsilon U^\epsilon (1 + H)^C.$$

Here $\mathfrak{h}$ is the exterior ideal of the original nonzero frequency h, N is the number of factors in the fixed original list, and all constants have the uniformity and order of choices specified in Section 2.

Proof. For each actual nonunit prime, $p \mid \mathfrak{h}$. The full local table gives

local character	ordinary multiplicity
nonprincipal or lower principal	$r = v_p(h) \geq 1$
upper principal	$r+1 \leq v_p(h), \quad r \equiv 5 \pmod 6$.

Disjoint prime ranges determine the factor positions. Allocating each r between the two ordinary factors therefore gives, for fixed N,

$$\#\{\text{bad-prime tuples and allocations at } h\} \ll_N \tau(\mathfrak{h})^{C_N}(1+\log q_h)^{C_N}.$$

The fixed generator and unit classes change only the constant. Whole frequency dyads are retained before logarithmic Fourier separation, as in [Liu26b, Section 12.7.1]. Their cutoff satisfies

$$q_h \ll T_{\mathrm{sec}} U^{\xi/2} \ll U^{C_{\mathrm{cut}}}(1+H)^{A_{\mathrm{cut}}},$$

where T_{sec} is the supremum of the nominal frequency scale in the fixed outer sector, and $C_{\mathrm{cut}}, A_{\mathrm{cut}}$ are fixed independently of the later Fourier coordinate. The ideal divisor bound, with a sufficiently small exponent, therefore gives

$$\tau(\mathfrak{h})^{C_N}(1+\log q_h)^{C_N} \ll_\epsilon U^\epsilon (1+H)^C.$$

For the weighted Fourier estimates one may also use the larger frequency majorant, where $\boldsymbol{\nu}$ is the Fourier coordinate and K_*, B are fixed polynomial-growth exponents,

$$q_h \ll U^{K_*}(1+\|\boldsymbol{\nu}\|)^B(1+H)^B \quad \Longrightarrow \quad \tau(\mathfrak{h})^{C_N}(1+\log q_h)^{C_N} \ll_\epsilon U^\epsilon(1+\|\boldsymbol{\nu}\|)^{C_\epsilon}(1+H)^{C_\epsilon}.$$

Choose the small divisor exponent after fixing N, and the required weighted Fourier order before integration and external heights. This preserves the original weighted measure; no uniform unweighted bound over all $\boldsymbol{\nu}$ is required. □

3.1.2 Local Gauss sums and the Fourier measure

In this subsubsection, we apply the divisor estimate of Section 3.1.1 before changing the nonunit-prime frequency. The resulting exponent is computed in Section 3.1.3.

Fix the original common C, D, s, finite ray components and Fourier coordinates for the complete coefficient sums before extracting primes. First process every removed $p \mid s$ with its divisor bound; the remaining base condition is $s' \mid n$. If that step divides the frequency by B_s, write $h = h_{\mathrm{initial}}/B_s$. The free nonunit primes avoid B_s, so

$$p \text{ free bad} \quad \Longrightarrow \quad p \mid h \iff p \mid h_{\mathrm{initial}}.$$

Classify the remaining free positions as unit, nonunit or fifth-power correction positions. Fix correction indices first; their P^{-1} weights have absolute coefficient mass $\ll U^\epsilon$. No nonunit-prime-dependent frequency division precedes Cauchy.

For an actual nonunit prime put $r = v_p(n)$. The full local table is

branch	condition at h	division	relative squared factor
$6 \nmid r+1$	$r \geq 1,\ v_p(h) = r$	p^r	1
$6 \mid r+1$	$v_p(h) = r$	p^r	q_p^{-1}
$6 \mid r+1$	$v_p(h) \geq r+1$	p^{r+1}	$q_p(1-q_p^{-1})^2 \leq q_p$.

Thus $p \mid h$ in every case, and $r \geq 5$ in the principal cases. Their moving exponents are $r+2$ and, in the upper case, $r+1 \equiv 0 \pmod 6$. Retain the full natural-zero radical even when the active exponent is zero.

For the i-th prime range $\mathcal{P}_i$, let P_i be its norm scale, V_i its smooth weight and $c_i(p)$ its allowed finite-group coefficient. For a fixed nonunit-prime tuple b, let B_b be its frequency divisor and ζ_b the bounded scalar from the fixed local factors. Put

$$c_b = \prod_{i \text{ bad}} P_i^{-1/2} c_i(p_i) V_i(q_{p_i}/P_i)\, \zeta_b, \qquad |\zeta_b| \le 1, \qquad q_{B_b} = U^{L+T+o(1)}.$$

Frequency-dependent phases belong to $Z_b(h)$ below. The actual finite-group coefficients satisfy $|c_i(p)| \le C_i$, with no unit-modulus assumption. Prime-ideal counting gives

$$\sum_{p \in \mathcal{P}_i} |P_i^{-1/2} c_i(p) V_i(q_p/P_i)|^2 \le P_i^{-1} C_i^2 \|V_i\|_\infty^2 \#\mathcal{P}_i \ll C_i^2 \|V_i\|_\infty^2,$$

and therefore

$$\sum_b |c_b|^2 \ll_\epsilon U^\epsilon \mathfrak{S}_J(\mathcal{W})(1+H)^C.$$

The ordinary and prime normalizations give

$$q_p^{-r/2} q_p^{r/2} = 1, \qquad P_i^{-1/2} \asymp q_p^{-1/2}$$

on the fixed list. The principal contribution is $U^{T-S+o(1)}$, separate from the coefficient energy.

For a free unit prime $p \nmid h$, extending the residual sum from $(n,p) = 1$ introduces only $v_p(n) = 5$: the full local Gauss formula gives

$$\big|\mathcal{G}(p^i, hp^4)\big| = \begin{cases} q_p^2, & i = 5, \\ 0, & i \ge 1,\ i \ne 5. \end{cases}$$

Thus the correction, including ordinary extraction and the removed prime normalization, has weight

$$q_p^2\, q_p^{-5/2}\, P_i^{-1/2} = q_p^{-1/2} P_i^{-1/2} \asymp P_i^{-1}.$$

Fix these correction primes first. Multiplicativity of the retained finite-component coefficients then leaves the same residual Gauss polynomial for all remaining unit-prime choices.

Let a index the unit-prime choices. For fixed b, F_b is the residual Gauss polynomial, common to all a; $e_{a,b}$ are the remaining unit-prime coefficients and $\omega_{a,b}(h)$ the phase factors, with their prescribed zeros. The factor $\alpha_b(h)$ contains the remaining principal Gauss-sum factors. On the sum with the exact divisibility conditions, retaining all Gauss phases,

$$Z_b(h) = \alpha_b(h) \sum_a e_{a,b} \omega_{a,b}(h) F_b\big((h/B_b)a^4\big), \qquad |\alpha_b(h)|^2 \le U^{T-S+o(1)}.$$

Apply Lemmas 3.1–3.2 at this h, then interchange the positive sums over h, b, and only then set $h = B_b h'$. For fixed b,

$$q_{h'} \ll U^{K-L-T+o(1)}, \qquad (a, B_b) = 1, \qquad j = h' a^4.$$

Disjoint prime ranges preserve the unit-prime unit conditions. Write $\mathcal{N}_{\text{pattern}}$ for this fixed prime-type contribution to the normalized Gauss norm. The orbit-sum argument, using the same polynomial F_b for all unit-prime choices a, gives

$$\mathcal{N}_{\text{pattern}} \ll U^{T-S+\epsilon}(1+H)^C \sum_b |c_b|^2 \sum_{q_j \ll U^{K-L-T+4g_{\text{good}}+o(1)}} |F_b(j)|^2. \tag{3.2}$$

Its good coefficient energy satisfies, uniformly in b,

$$\sum_a |e_{a,b}|^2 \ll_\epsilon U^\epsilon \mathfrak{S}_J(\mathcal{W})(1+H)^C.$$

Unit-locus phases have modulus one; the natural zeros restrict the energy and the smooth weights remain fixed. Divisor Cauchy precedes the enlarged balls. For each b, use the same scale bounds, moving presentation and natural zeros, in the order

single restoration $\longrightarrow$ length and inhomogeneous-term bounds $\longrightarrow$ M2 on ordinary sums.

The displayed coefficient energy then gives (3.2).

For a real Mellin frequency t, the change of variable gives

$$q_h^{it} = q_{B_b}^{it} q_{h'}^{it}, \qquad |q_{B_b}^{it}| = 1.$$

Other unitary Mellin coordinates factor in the same way. Integrate polynomial derivative and height factors in the original common weighted Fourier measure at sufficiently high finite order. The original C, D, s count is included once. All coefficient coprimality restrictions remain; positivity is used only after the original complete signed coefficient sum has been formed.

3.1.3 The resulting Gauss-norm exponent

In this subsubsection, we compute the one-sided exponent after the divisor estimate of Section 3.1.2. The resulting criterion is Proposition 3.3; fixed primes dividing s are treated in Section 3.2.

The auxiliary arithmetic data A_*, M_*, q_*, the actual $K_0 - K$, the earlier J, g, and the additional length excess $D_{\mathrm{loc}} + 3u/2 - 5v - T$ are unchanged. Only the unrestricted count of nonunit prime choices U^v has disappeared. Consequently the homogeneous relative exponent is now

$$C_{\mathrm{new}} = -u - L + v - 4c + T - S + \frac{(D_{\mathrm{loc}} + 3u/2 - 5v - T)_+}{6}. \tag{3.3}$$

Thus $C_{\mathrm{new}} = C_{\mathrm{old}} - v$, where C_{old} is the exponent of the earlier argument, which retains the count U^v. The remaining $+v$ counts the displayed moving radical and stays in the exponent.

Proposition 3.3 (Improved criterion for removing additional prime factors)**.** *Retain the fixed coefficient formulas and analytic hypotheses above. For the nonunit-prime contribution in* (3.3)*,*

$$\begin{aligned} D_{\mathrm{loc}} \le 9u/2 \quad &\Longrightarrow \quad C_{\mathrm{new}} \le -5v/6 \le 0, \\ D_{\mathrm{loc}} \le 7u/2 \quad &\Longrightarrow \quad C_{\mathrm{new}} \le -(u+5v)/6. \end{aligned}$$

The inhomogeneous terms in the one-sided Gauss norm retain their separate estimates.

Proof. Put $e = L - v - 4T \ge 0$. If the positive part is active, the exponent in (3.3) equals

$$-\frac{3}{4}u + \frac{D_{\mathrm{loc}}}{6} - \frac{5}{6}v - \frac{19}{6}T - 4c - S - e.$$

If it is inactive, the exponent is $-u - 3T - 4c - S - e \le -u$.

Thus, on the active branch,

$$\begin{array}{l|c} D_{\mathrm{loc}} \le 9u/2 & C_{\mathrm{new}} \le -5v/6 \\ D_{\mathrm{loc}} \le 7u/2 & C_{\mathrm{new}} \le -(u+5v)/6. \end{array}$$

On the inactive branch, $0 \le v \le u$ gives $-u \le -(u+5v)/6 \le -5v/6$. □

The corresponding changes are

	free-prime term	retained upper bound
second origin	$-3v \longmapsto -4v$	$A_t - 1 + 4u$
centered exceptional term	$-13v/6 \longmapsto -19v/6$	$A_t - 1 + 3u.$

Length conditions, exact annular scales, empty or subunit ranges and prime restoration are checked as before. The improvement changes the prefactor, with the same Gauss mean estimate and κ.

3.2 A frequency change for fixed primes dividing s

In this subsection, we rewrite the Gauss frequency for fixed unit primes. The common coprimality restrictions are retained for the norm in Section 4.

We now fix a removed prime $p \mid s$ before varying the remaining frequency, and treat its valuation-zero contribution. This case requires a frequency identity rather than the divisor estimate of Section 3.1.

3.2.1 The exact fixed-prime norm comparison

In this subsubsection, we prove the frequency identity (3.4) for a fixed prime. Section 3.2.2 retains the coprimality conditions in this identity.

For $p \nmid h$ and $(p, n) = 1$, the full Gauss CRT formula gives

$$\mathcal{G}(pn, h) = \mathcal{G}(p, h)\mathcal{G}(n, hp^4). \tag{3.4}$$

Retain the finite ray and allowed-generator components. The exact valuation-zero allocation keeps $(p, n) = 1$, so no fifth-power correction occurs. Let $A(n)$ be the canonical centered convolution and, after fixing a character component and the Fourier parameters, let $\tau_{\mathrm{coef}}(n)$ be its common multiplicative character presentation, including its zero extension. Thus

$$\tau_{\mathrm{coef}}(ab) = \tau_{\mathrm{coef}}(a)\tau_{\mathrm{coef}}(b), \qquad |\tau_{\mathrm{coef}}(n)| \le 1.$$

This coefficient is distinct from the ideal divisor function $\tau(\mathfrak{a})$ of Section 2. Let $H_0 > 0$ denote the retained frequency-norm cutoff. For fixed p, put

$$c_p = P_i^{-1/2} c_i(p) V_i(q_p/P_i), \qquad |\tau_{\mathrm{coef}}(p)\mathcal{G}(p, h)| \le 1 \quad (p \nmid h).$$

Consequently

$$\begin{aligned} \sum_{\substack{q_h \le H_0 \\ p \nmid h}} \left| c_p \sum_{(n,p)=1} A(n)\tau_{\mathrm{coef}}(pn)\mathcal{G}(pn, h) \right|^2 \\ \le |c_p|^2 \sum_{q_j \le H_0 q_p^4} \left| \sum_{(n,p)=1} A(n)\tau_{\mathrm{coef}}(n)\mathcal{G}(n, j) \right|^2. \end{aligned} \tag{3.5}$$

The change $j = hp^4$ is injective on actual Γ-orbits:

$$q_h \le H_0,\ p \nmid h \quad \Longrightarrow \quad q_j \le H_0 q_p^4,\ v_p(j) = 4.$$

Drop the image condition only in the positive norm; retain $(n, p) = 1$. All scales and cutoffs are actual, with a single prescribed $O(1/\log U)$ annular loss.

Since $p \mid s$, divisor counting retains the squared normalization:

$$\sum_{\substack{p \in \mathcal{P}_i \\ p \mid s}} |c_p|^2 \ll_\epsilon U^\epsilon P_i^{-1} C_i^2 \|V_i\|_\infty^2.$$

For a varying unit-prime index a, the condition $(n, a) = 1$ would instead change the function in the orbit sum. Here p is fixed before the frequency varies.

3.2.2 Preservation of the coprimality conditions

In this subsubsection, we carry the common coprimality restriction through both Poisson formulas of Section 3.2.1.

Let $\mathfrak{r}_*$ be the fixed modulus recording the retained coprimality restrictions on the coefficient variable. The residual polynomial and the enlarged coprimality modulus are

$$F_p(j) = \sum_n A(n)\tau_{\mathrm{coef}}(n)1_{(n,p)=1}\mathcal{G}(n,j), \qquad \mathfrak{r}_* \longmapsto \mathfrak{r}_* p.$$

Here p is fixed before j varies and $q_{\mathfrak{r}_* p}$ is polynomially bounded. The indicator comes directly from the valuation-zero allocation; no moving factor $\chi_p(n)^2$ has been introduced.

The common-coefficient identity of [Liu26b, Section 12.1 and Lemma 12.8] retains this indicator in both factors, both rectangles and every remaining prime factor. Its treatment is

stage	$1_{(n,p)=1}$
both Poisson formulas	retained in the full Gauss polynomial
ordinary nonexceptional sum	expanded by [Liu26b, Lemma 12.5]
fixed-character exceptional term	retained in both matching means.

The ordinary expansion is used only after the finite Gauss coefficient has disappeared, before M2 or reflection.

The retained support implies $p \nmid n$ for every transformed coefficient variable. Hence p cannot enter the genuine new supports D_2, E_2 or their fixed active factor. An artificial divisor allocation selecting p vanishes before positivity. In the final ordinary expansion, $\psi_{h'}$ denotes the resulting ordinary character with its fixed twists. The factor $\psi_{h'}(p)$ is a bounded parameter factor with its original zero; its coefficient mass has arbitrarily small power growth.

Equation (3.4) uses the unfactored Gauss sum. The frequency is j with its full natural zeros, and its enlarged range is counted below. A canceled displayed moving factor cannot be reclassified as an extra coprimality condition: choosing a presentation containing $\chi_p(n)^2$ would require retaining it.

3.2.3 Changes in the length exponents

In this subsubsection, we compute the changes in the fixed-prime lengths produced by the identity of Section 3.2.1; the resulting criterion is Corollary 3.4.

Write $\lambda_p = \log_U q_p$ up to the prescribed annular error. The fixed-prime frequency comparison has

$$A \mapsto A - \lambda_p, \quad K \mapsto K + 4\lambda_p, \quad M \mapsto M - 6\lambda_p, \quad q \mapsto q, \quad s_0 \mapsto s_0 - \lambda_p.$$

Consequently

$$(A - \lambda_p) + q + B_C - (s_0 - \lambda_p) = A + q + B_C - s_0, \qquad K_0 - K \text{ is unchanged.}$$

The fixed squared coefficient contributes $-\lambda_p$ and the global affine defect increases by $3\lambda_p/2$. The original J, g calculation uses its favorable $-d_D$ once.

Let w_f be the length of additional primes fixed in the first common support, and $z_{C,\mathrm{old}}$ the corresponding length from Q_0. The unused initial length is measured by $e_z = (9/2)(z_0(m) - z) \geq 0$. For primes dividing s with zero ordinary valuation and total width $f_0 = x_{\mathrm{aux}} + w$, with no newly fixed common factors and no free removed positions, the homogeneous exponent is therefore bounded by

$$-f_0 + \frac{(\frac{9}{2}w + \frac{3}{2}f_0)_+}{6} = -\frac{3}{4}x_{\mathrm{aux}} \qquad (x_{\mathrm{aux}}, w \geq 0). \tag{3.6}$$

In particular this presentation makes the case of primes dividing s with zero ordinary valuation neutral at $x_{\mathrm{aux}} = 0$, rather than leave the excess in the earlier local estimate $w/6$. This is a statement about this fixed-prime comparison, not a global theorem eliminating the initial prime product.

For reference, let f_1, L_1, T_1, S_1 denote the data for the remaining fixed primes dividing s; their ordinary valuations are positive. Here f_1 is their total prime length, L_1 the ordinary power-removal length, and T_1, S_1 their upper and lower principal lengths, respectively. Then $L_1 \geq f_1 + 4T_1$. The combined fixed-stage data given by these exact comparisons are

$$A_1 = A_t - f_0 - f_1 - L_1, \qquad M_1 = 1 - 6f_0 - f_1 - L_1 + T_1, \qquad q_1 = f_1,$$

with normalized coefficient exponent $-f_0 - L_1 + T_1 - S_1$. After this fixed stage, perform the free unit, nonunit and fifth-power-correction partition afresh at its current frequency on the enlarged Gauss family. The divisor Cauchy is applied there before any free nonunit-prime division; it is applied to the newly obtained frequency sum.

Adding the divisor-Cauchy free stage gives the following upper bound for the normalization exponent when the positive part is nonzero:

$$-\frac{3}{4}x_{\mathrm{aux}} + \frac{w_f}{6} - \frac{5}{6}(f_1 + v) - \frac{19}{6}(T_1 + T) - \frac{7}{12}z_{C,\mathrm{old}} - \frac{e_z}{6} - 4c - S_1 - S,$$

after dropping further nonpositive ordinary-removal terms. The case in which the positive part vanishes has a nonpositive exponent. More precisely, put $e_1 = L_1 - f_1 - 4T_1 \geq 0$ and $e = L - v - 4T \geq 0$. Its normalized exponent is

$$-f_0 - f_1 - u - 3(T_1 + T) - 4c - S_1 - S - e_1 - e \leq 0.$$

Corollary 3.4 (A nonpositive contribution to the norm exponent)**.** *Retain the length conditions, centered-weight, common-measure and length-restoration hypotheses of Section 3.1, together with the fixed-unit comparison above. Then*

$$0 \leq w_f \leq w, \qquad x_{\mathrm{aux}} \geq 2w_f/9 \quad \Longrightarrow \quad \mathcal{C}_{\mathrm{act}}, \mathcal{C}_{\mathrm{inact}} \leq 0.$$

In particular, $x_{\mathrm{aux}} \geq 2w/9$ suffices for every $w_f \leq w$. For a prescribed discretization margin $\rho_{\mathrm{step}} > 0$,

$$x_{\mathrm{req}} \geq 2w/9 + \rho_{\mathrm{step}}, \qquad \eta_{\mathrm{mesh}} \leq \rho_{\mathrm{step}}/3, \qquad x_{\mathrm{aux}} \geq x_{\mathrm{req}} - \eta_{\mathrm{mesh}}$$
$$\Longrightarrow \qquad \mathcal{C}_{\mathrm{act}} \leq -\rho_{\mathrm{step}}/2,$$

before the smaller subsidiary errors. Here $\mathcal{C}_{\mathrm{act}}$ and $\mathcal{C}_{\mathrm{inact}}$ denote the contributions with nonzero and zero positive part, respectively.

Proof. The active and inactive contributions satisfy, respectively,

$$\mathcal{C}_{\mathrm{act}} \leq -3x_{\mathrm{aux}}/4 + w_f/6 \leq 0 \qquad (x_{\mathrm{aux}} \geq 2w_f/9),$$
$$\mathcal{C}_{\mathrm{inact}} = -f_0 - f_1 - u - 3(T_1 + T) - 4c - S_1 - S - e_1 - e \leq 0.$$

Since $w_f \leq w$, the stronger condition $x_{\mathrm{aux}} \geq 2w/9$ suffices. For a complete-factor choice $x_{\mathrm{aux}} \geq x_{\mathrm{req}} - \eta_{\mathrm{mesh}}$,

$$\mathcal{C}_{\mathrm{act}} \leq -\frac{3}{4}\left(\frac{2w}{9} + \rho_{\mathrm{step}} - \eta_{\mathrm{mesh}}\right) + \frac{w}{6} \leq -\frac{3}{4}\rho_{\mathrm{step}} + \frac{1}{4}\rho_{\mathrm{step}} = -\frac{1}{2}\rho_{\mathrm{step}}.$$

At $u = 0$, retain the full fixed coefficient exponent $-f_0 - L_1 + T_1 - S_1$ and the single restoration loss. No condition restricted to the free primes, such as $D_{\mathrm{loc}} \leq 9u/2$, is imposed on this fixed contribution. □

This criterion bounds the normalization exponent. Its other length hypotheses remain necessary when x_{aux}, w vary. The case in which every additional prime lies in the common support is not sufficient by itself: when $w_f = w$ and $x_{\mathrm{aux}} = e_z = z_{C,\mathrm{old}} = f_1 = v = 0$, necessarily $f_0 = u = 0$, and this upper bound for the exponent is $w/6$. This is not an arithmetic lower bound, but it prevents a uniform $x_{\mathrm{aux}} = 0$ conclusion from these bounds alone.

3.2.4 The remaining prime length

In this subsubsection, we compare the remaining prime length with the transformed length excess of Section 3.2.3.

Let $D_{\mathrm{free}}^{\mathrm{old}}$ denote the remaining length excess in the preceding fixed-s table, evaluated at the same actual allocation. The fixed-unit change gives $D_{\mathrm{free}}^{\mathrm{new}} = D_{\mathrm{free}}^{\mathrm{old}} + 5f_0$. After substituting $u = x_{\mathrm{aux}} + w - w_f - f_0 - f_1$, its exact value is

$$D_{\mathrm{free}}^{\mathrm{new}} = \frac{3}{2}x_{\mathrm{aux}} + 6w - e_z - \frac{7}{2}z_{C,\mathrm{old}} - 5w_f - 5f_1 - T_1 - 5v - T.$$

Here $z_0 = z_0(m)$, and $z_{0,\min}$ below denotes its minimum over the chosen ordinary-length interval. Write $z = z_0 - 2e_z/9$ and $z_{\mathrm{init}} = z - z_{C,\mathrm{old}} - x_{\mathrm{aux}}$ for the remaining original prime width. Then

$$\begin{aligned}\frac{7}{2}z_{\mathrm{init}} - D_{\mathrm{free}}^{\mathrm{new}} &= \frac{7}{2}z_0 - 5x_{\mathrm{aux}} - 6w + \frac{2}{9}e_z + 5w_f + 5f_1 + T_1 + 5v + T \\ &\geq \frac{7}{2}z_{0,\min} - 5x_{\mathrm{aux}} - 6w. \end{aligned} \tag{3.7}$$

The f_0 term cancels. For example,

$$z_{0,\min} = 2/75, \quad x_{\mathrm{aux}} = 9/5000, \quad w = 3/500 \quad \Longrightarrow \quad \frac{7}{2}z_{0,\min} - 5x_{\mathrm{aux}} - 6w = \frac{29}{600}.$$

Decreasing x_{aux} increases the margin. Genuine and final-divisor fixing of widths b_g, b_d reduce the defect by at least $7b_g/2 + 9b_d/2$. Thus restoration, including exhaustion, remains valid when (3.7) exceeds the mesh and amplifier errors, with the used prime widths subtracted at each stage.

3.2.5 Ordinary summation lengths and exceptional terms

In this subsubsection, we compute the actual summation lengths and the origin and exceptional exponents for the allocation of Section 3.2.4. Section 3.3 then retains the common-support length.

Put $t = f_0 + u$ and $a = f_1 + t \leq x_{\mathrm{aux}} + w$. After all fixed and free local comparisons the auxiliary global parameters are

$$\begin{aligned} A_* &= A_t - f_1 - t - L_1 - L - 5c, \\ M_* &= 1 - 6t - f_1 - L_1 - L - 5c + T_1 + 5v + T, \\ q_* &= f_1 + v + c, \qquad m_* = M_* - q_*. \end{aligned}$$

All nonunit and correction natural support restrictions remain in the displayed q_*; only the unfactored fixed unit primes are in the common extra radical. If $D_{\mathrm{ord}} = L_1 + L + 5c \leq 2m + o(1)$, these give

$$M_* \geq 1 - 2m - 6a - o(1), \qquad m_* \geq 1 - 2m - 7a - o(1), \qquad A_* - m_* \leq A_t - 1 + 6a + o(1).$$

The last estimate follows also from the exact identity $A_* - m_* = A_t - 1 + 5t + f_1 - T_1 - 4v + c - T$. For nonempty common supports,

$$c_C \leq A_* + o(1), \quad R + E_0 \leq d_D \quad \Longrightarrow \quad m_* - c_C + d_D - R - E_0 \geq m_* - A_* - o(1) > 0$$

whenever $m_* - A_*$ has a fixed positive margin. The ordinary mean estimate is applied only after this verification of the length conditions. The actual frequency shift is

$$K_0^* = K_0 + 4(f_0 + g_{\mathrm{good}}) - L_1 - T_1 - L - T, \qquad g_{\mathrm{good}} = u - v - c.$$

It agrees with the formal length identity, so $K_0 - K$ and the original earlier J, g restoration calculation remain unchanged.

After including all normalized fixed coefficients, the second-origin excess is exactly

$$A_t - 1 + 4t - L_1 - S_1 - L - 4v - 4c - S \le A_t - 1 + 4t.$$

The corresponding centered exceptional excess is

$$A_t - \frac{5}{6} - \frac{f_1}{6} - \frac{7}{6}(L_1 + L) + \frac{T_1 + T}{6} - S_1 - S + 3t - \frac{19}{6}v - \frac{29}{6}c \\ - \frac{2}{3}\big(\tfrac{1}{4} - L_1 - L - 5c\big)_+ \le A_t - 1 + 3t.$$

Indeed, write $L_{\mathrm{tot}} = L_1 + L$ and let $\mathcal{X}$ denote the preceding exceptional excess. Since $L_1 \ge f_1 + 4T_1$ and $L \ge v + 4T$, one has $T_1 + T \le L_{\mathrm{tot}}/4$. Hence

$$\begin{aligned} \mathcal{X} - (A_t - 1 + 3t) &\le \frac{1}{6} - \frac{9L_{\mathrm{tot}}}{8} - \frac{29c}{6} - \frac{2}{3}(1/4 - L_{\mathrm{tot}} - 5c)_+ \\ &\le \begin{cases} -11L_{\mathrm{tot}}/24 - 3c/2, & L_{\mathrm{tot}} + 5c \le 1/4, \\ 1/6 - 2(L_{\mathrm{tot}} + 5c)/3, & L_{\mathrm{tot}} + 5c \ge 1/4 \end{cases} \le 0. \end{aligned}$$

Thus the same parameter regions, or smaller x_{aux} within them, retain the length, origin and exceptional margins. Exact annular scales, independent Mellin twists and canonical centered weights remain those of the cited theorem; the stated zero-free boundaries are unchanged.

3.3 Retaining the common-support length

In this subsection, we retain the common ideal length in the restoration inequality. This gives the homogeneous estimate used in Section 3.4.

The first paired common-support term need not save a power at a prime occurring to the first power in both ordinary factors. The exact restoration expression, however, retains a favorable common length that its earlier coarse upper bound discarded. We use its full common-length term in the following estimate.

3.3.1 The common-support normalization

In this subsubsection, we compare the paired common-support term with the restoration term. The result, Lemma 3.5, is inserted in the length excess in Section 3.3.2.

In the notation of the paired common-support comparison, put $c_C = \log_U q_C$, $d_D = \log_U q_D$ and $p = \log_U q_{\mathrm{rad}(CD)}$. The paired normalization is

$$M + \tfrac{1}{2}\{B_C + B_D - (c_C + d_D - 2p - R)\}, \qquad B_C = (3c_C - 5d_D - R)_+/6.$$

A new C-fixed prime may divide the other ordinary factor with multiplicities $(1, 1)$. At that prime,

$$(3 - 5)_+ = 0, \qquad 1 + 1 - 2 = 0, \qquad q_p^{-1} q_p = 1,$$

where the last factors are the paired central normalization and the single radical-family count. Prime sparsity gives logarithms; this term alone gives no saving proportional to the new common width.

Write a for the earlier amplifier extraction length w and b for its new-radical length w_o, to distinguish them from the new length of the original prime-ideal range. The choices in the cited formula are

$$\Delta = K_0 - K \ge -\delta_1, \quad J = d_D - c_C + \Delta - 2a + b, \quad g = J_+ + O(\sigma), \quad 0 \le b \le a.$$

They are the unchanged quantities of [Liu26b, Sections 12.7.2 and 12.8.1].

Lemma 3.5 (Common-support saving in length restoration)**.** *Suppose*

$$c_C, d_D \geq 0, \qquad \delta_1 \geq 0, \qquad \Delta \geq -\delta_1, \qquad 0 \leq b \leq a,$$
$$J = d_D - c_C + \Delta - 2a + b, \qquad \mu_{\mathrm{com}} = \min(c_C, d_D).$$

Then

$$\begin{aligned} -d_D - \Delta - b + J_+ &= \max\{-d_D - \Delta - b, -c_C - 2a\} \\ &\leq -\mu_{\mathrm{com}} + \delta_1. \end{aligned} \tag{3.8}$$

Proof. Put $X = -d_D - \Delta - b$. Then

$$\begin{aligned} X + J_+ &= \max\{X, X + J\} = \max\{-d_D - \Delta - b, -c_C - 2a\} \\ &\leq \max\{-d_D + \delta_1, -c_C\} \leq -\min(c_C, d_D) + \delta_1, \end{aligned}$$

using $\Delta \geq -\delta_1$, $a, b \geq 0$. □

This replaces the earlier argument's coarse bound of this entire expression by δ_1. It is not a second use of the favorable $-d_D$. For example, when $c_C = d_D = \lambda_{\mathrm{com}} > 0$ and $\Delta = a = b = 0$, the first common-support term can be zero while (3.8) is exactly $-\lambda_{\mathrm{com}}$. The two coarse equalities cannot be attained simultaneously on such a common support.

3.3.2 The transformed length excess

In this subsubsection, we insert the common-length inequality of Lemma 3.5 into the transformed length excess of [Liu26b, Section 12.8.1].

Before its coarse J substitution, [Liu26b, Section 12.8.1] bounds the transformed length defect by

$$a - d_D - (K_0 - K) - b + g + 2\theta_N.$$

The earlier formal reconstruction of ordinary summation lengths adds its global length excess and the common-fixed-prime term $-(7/2)z_C$. All its local comparisons shift actual K and formal K_0 identically; $K_0 - K$, C, D, the earlier J, g, and the single paired arithmetic count are unchanged. Lemma 3.5 therefore lowers the additional free defect by μ_{com}, using the same substitution exactly once. More explicitly the full transformed length defect is at most

$$D_{\mathrm{free}} - \mu_{\mathrm{com}} + a + (g - J_+) + \delta_1 + 2\theta_N.$$

The earlier-amplifier contribution satisfies

$$a + (g - J_+) \leq 6(a + \ell), \qquad \begin{cases} a = 0,\ g - J_+ = 2\sigma = 6\ell, & \text{main term}, \\ \ell = 0,\ g - J_+ = \sigma,\ a + \sigma \leq 6a, & \text{amplifier error}. \end{cases}$$

Using $(X + Y)_+ \leq X_+ + Y$ for $Y \geq 0$, the squared loss is at most

$$\frac{(D_{\mathrm{free}} - \mu_{\mathrm{com}})_+}{6} + a + \ell + \frac{\delta_1 + 2\theta_N}{6} + \eta_{\mathrm{mesh}}.$$

The earlier exponent allowance for the auxiliary sum $\Delta' \geq a + \ell$ absorbs its contribution from the original amplifier. The additional squared loss of restoring the lengths is therefore at most

$$\frac{(D_{\mathrm{free}} - \mu_{\mathrm{com}})_+}{6} \tag{3.9}$$

apart from the already prescribed dyadic, support and mesh errors.

The original amplifier estimate still accounts for its own contribution. The exponent allowance for the auxiliary sum remains $\Delta' = b_2 - p_2 + a + B_C + \ell$; none of these terms is reused.

Here $b_2 = (c_2 + d_2)/2$, where c_2, d_2 are the two full common-support lengths in the second Poisson formula, and p_2 is the length of their radical. Write g_2 for the length of their greatest common divisor. After dividing the frequency by this divisor, the primes with equal multiplicities not divisible by six split into unit and nonunit sets: t_2 is the radical length of the unit set, and V the length of the product of the nonunit primes. All these lengths are relative to U; see Section 4.3.

Lowering the defect improves the previously verified coupled remaining-prime margin. Every later genuine or final-divisor fixing is counted in that same length-restoration argument for all prime factors. When no prime factor remains, the cited estimate for arbitrary ordinary lengths applies with the same order of parameter choices. In particular (3.9) is derived using ordinary sums allowed for M2, rather than assumed for a Gauss sum outside the established length range.

3.3.3 The nonpositive normalization exponent

In this subsubsection, we prove that the full homogeneous exponent is nonpositive (Corollary 3.6). This estimate is used in the amplified norm of Section 3.4.

Retain the fixed-unit rewriting and original-frequency divisor Cauchy of the preceding sections. Let f_0 denote fixed unit primes dividing s, f_1, L_1, T_1, S_1 the other data for fixed primes dividing s, $z_{C,\mathrm{old}}$ the initial first-common-support prime length, and w_f the corresponding new width. All widths and nonnegative local losses have their earlier meanings. Thus

$$u = x_{\mathrm{aux}} + w - w_f - f_0 - f_1 \geq 0, \quad L_1 \geq f_1 + 4T_1, \quad L \geq v + 4T,$$

and

$$D_f = \frac{9}{2}w - e_z - \frac{7}{2}(z_{C,\mathrm{old}} + w_f) + \frac{3}{2}f_0 - \frac{7}{2}f_1 - T_1.$$

Write c_{cor} for the free correction width, distinct from c_C, and put $e_1 = L_1 - f_1 - 4T_1 \geq 0$, $e = L - v - 4T \geq 0$.

Corollary 3.6 (A nonpositive exponent with the initial prime product retained)**.** *Let the local widths and losses defined above be nonnegative, with*

$$\begin{gathered} u = x_{\mathrm{aux}} + w - w_f - f_0 - f_1 \geq 0, \qquad L_1 \geq f_1 + 4T_1, \qquad L \geq v + 4T, \\ D_f = \frac{9}{2}w - e_z - \frac{7}{2}(z_{C,\mathrm{old}} + w_f) + \frac{3}{2}f_0 - \frac{7}{2}f_1 - T_1, \\ \mu_{\mathrm{com}} = \min(c_C, d_D) \geq w_f. \end{gathered}$$

Then, on actual ideal scales,

$$\begin{aligned} \mathcal{C} = & -f_0 - L_1 + T_1 - S_1 - u - L + v - 4c_{\mathrm{cor}} + T - S \\ & + \frac{(D_f + 3u/2 - 5v - T - \mu_{\mathrm{com}})_+}{6} \leq 0. \end{aligned} \tag{3.10}$$

This algebraic inequality requires neither $x_{\mathrm{aux}} \geq 2w_f/9$ nor positivity of the formal summation length.

Proof. Put $\mathcal{D} = D_f + 3u/2 - 5v - T - \mu_{\mathrm{com}}$ and

$$\mathcal{C}_{\mathrm{hom}} = -f_0 - f_1 - u - 3(T_1 + T) - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e.$$

Substitute $w = u - x_{\mathrm{aux}} + w_f + f_0 + f_1$ into D_f. Then

$$\mathcal{D} = 6u - \frac{9}{2}x_{\mathrm{aux}} + w_f + 6f_0 + f_1 - \frac{7}{2}z_{C,\mathrm{old}} - e_z - T_1 - 5v - T - \mu_{\mathrm{com}},$$

and $L_1 = f_1 + 4T_1 + e_1$, $L = v + 4T + e$ give

$$\mathcal{C} = \mathcal{C}_{\mathrm{hom}} + \mathcal{D}_+/6 = \max\{\mathcal{C}_{\mathrm{hom}}, \mathcal{C}_{\mathrm{hom}} + \mathcal{D}/6\}.$$

The two entries satisfy

$$\begin{aligned}
\mathcal{C}_{\mathrm{hom}} &= -f_0 - f_1 - u - 3(T_1 + T) - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e \le 0,\\
\mathcal{C}_{\mathrm{hom}} + \mathcal{D}/6 &= -\frac{3}{4}x_{\mathrm{aux}} + \frac{w_f - \mu_{\mathrm{com}}}{6} - \frac{5}{6}(f_1 + v) - \frac{19}{6}(T_1 + T)\\
&\quad - \frac{7}{12}z_{C,\mathrm{old}} - \frac{e_z}{6} - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e \le 0.
\end{aligned}$$

Here $x_{\mathrm{aux}} \ge 0$, $\mu_{\mathrm{com}} \ge w_f$, and all remaining widths and losses are nonnegative. The calculation also holds at $u = 0$: it uses the full coefficient exponent and the single restoration loss, with no additional free-prime condition. □

Its analytic length-restoration application retains the cited moment hypotheses, positive global and formal lengths, full local support and coprimality conditions, exact centered weights and equal scale products, common Fourier measure, and remaining-prime-length conditions, including the case with no remaining prime factor.

Distinct new first-common primes divide $\mathrm{rad}(CD)$, and the original prime ranges are disjoint. Hence

$$w_{f,\mathrm{actual}} \le p \le \min(c_C, d_D) = \mu_{\mathrm{com}}, \qquad w_f \le \mu_{\mathrm{com}} + \theta_N.$$

The nominal homogeneous exponent is at most $\theta_N/6$; together with the original $\delta_1/6$ and edge losses it is counted once, not once per prime factor. For the requested positive loss, let $C_* > 0$ be the fixed constant controlling the accumulated support errors, and choose

$$0 < \sigma \text{ inside the fixed margins}, \qquad C_*\xi < \sigma/2, \qquad \eta_{\mathrm{mesh}} < \sigma/6.$$

The unchanged J, g give auxiliary-length decrease at least $\sigma/2$. Every M2 application remains in the established allowed family. Retain the fixed field, coefficient and unit classes, natural zeros, and weighted derivative/Fourier conventions of Section 2. Fix the finite positive-length prime list and tests, then the weighted orders, before external heights. Exact annular extractions and equal scale products are preserved; absorb the single vanishing aggregate support error after the fixed threshold. Zero-length bounded-mass factors contribute no removed positive width.

We now apply this homogeneous estimate together with the decomposition into all local prime cases.

From the local estimates to the amplified norm. The preceding result is a local restoration statement. The following subsection combines these estimates to prove the amplified norm on the stated parameter region. Throughout, $x_{\mathrm{aux}} = 0$ means no auxiliary removal of a subproduct of the initial prime product; the original Q_0 remains in the polynomial.

3.4 An amplified norm retaining the initial prime product

In this subsection, we prove the amplified norm with the initial prime product retained. We separate the additional prime factors according to their divisibility and valuation conditions, verify the resulting length bounds, and then apply the ordinary second-moment estimate M2 recalled in Section 2. Section 4.1 extends the argument to variable lengths.

Throughout, $x_{\mathrm{aux}} = 0$: the auxiliary comparison removes no part of Q_0. Retain the common-support decomposition and ordinary-length restoration. For every sign of the intermediate length excess, the order is

$$\text{all local identities for } Q_W \quad \longrightarrow \quad \text{M2}.$$

The moment and its arithmetic data. We use the following coefficient families and length ranges.

Retain the fixed field, excluded places, finite coefficient families and unit-class data, Fourier conventions for the complete coefficient sums and the earlier bound $\beta_*(F) \le 7/8$, which permits $\kappa = 3/4$. Assume precisely the same full Gauss-sum, compact-quotient Poisson, M2, reflection, common-coefficient, centered-mean and weighted smooth-weight and Fourier results as in Section 2. These are the established results cited in Section 2. First consider the parameter range

$$0.4 \le m \le 0.443, \quad 0.4 \le \delta \le 0.55, \quad 0.48 \le x_{\mathrm{amp}} \le 0.5, \quad 0.21 \le \mathscr{L} \le 0.22.$$

Let $P = S_{m,1}S_{m,2}Q_0$ be the original unrestricted product, $z \le z_0 = 2(1-2m)/9$ the nominal length exponent of the initial prime product, and Q_W a disjoint original product of length $0 \le w \le 1/100$. All positive-length prime ranges are mutually disjoint. For each independent smooth-weight tuple, use the same two tests and set

$$(m,m) \longmapsto (1/4, 2m-1/4), \qquad P_{\mathrm{c}} = P - P_{\mathrm{cmp}}, \qquad X_1X_2 = X_{1,\mathrm{cmp}}X_{2,\mathrm{cmp}}.$$

Fix the comparison before differentiation; retain exact norms, powers, weight arguments and the scale-product equality above. Only the affine length formulas incur the single aggregate error $\theta_N = O(1/\log U)$ over the finite allocations.

Theorem 3.7 (Amplified norm with the initial prime product retained)**.** *Under the hypotheses above, for every required finite independent derivative family and every $\epsilon > 0$,*

$$\sum_k \mathcal{W}(q_k/U)|P_{\mathrm{c}}(k)Q_W(k)|^2 \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C, \tag{3.11}$$

$$\sum_{k\in\mathcal{U}} \mathcal{W}(q_k/U)|P(k)Q_W(k)|^2 \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C. \tag{3.12}$$

The centered norm is taken over the unrestricted positive sum in the comparison; $\mathcal{U}$ is the original sixth-power-free nonunit family inducing outside Θ. No auxiliary subproduct is removed from the initial prime product: $x_{\mathrm{aux}} = 0$ throughout. The amplitude selection is treated separately in Section 4.1.

Proof. Step 1. Divisibility and valuation cases for the additional primes. We first separate the factors of Q_W according to common support, divisibility by s, and their valuations in the ordinary coefficient variables. We apply M2 only after all these factors have been treated and the resulting lengths have been checked in Steps 2 and 3.

First write $Q_W = Q_{W,0}Q_{W,+}$, $Q_0 = Q_{00}Q_{0,+}$, where the subscript 0 denotes zero nominal length. For each fixed derivative tuple,

$$P = Q_{00}P_+, \qquad P_{\mathrm{c}} = Q_{00}P_{\mathrm{c},+},$$

and the character factors have modulus at most one. Hence

$$|Q_{W,0}(k)Q_{00}(k)| \le \|c_{Q_{W,0}}\|_{\ell^1}\|c_{Q_{00}}\|_{\ell^1}. \tag{3.13}$$

Apply this bound to both positive norms before Poisson. Its bounded coefficient mass belongs to the retained seminorm and height factor. Relabel $Q_{W,+}, Q_{0,+}$ as Q_W, Q_0; their lengths are still w, z. Only positive-length factors enter the following valuations. The selected extra factors also have positive amplitude.

Fix the original C, D, s, finite components and common Fourier coordinates in the order of Section 2. Put

$$c_C = \log_U q_C, \quad d_D = \log_U q_D, \quad \mu_{\mathrm{com}} = \min(c_C, d_D), \quad z_C = z_{C,\mathrm{old}} + w_f.$$

Each new first-common prime divides both C, D. Thus

$$w_{f,\mathrm{actual}} \le \mu_{\mathrm{com}}, \qquad w_f \le \mu_{\mathrm{com}} + \theta_N, \quad \theta_N = O(1/\log U).$$

Split every remaining extra prime according to $p \mid s$ or $p \nmid s$, before Gauss evaluation or positivity. For $p \mid s$, fix p as a divisor of s, retain its coefficient and use

$$1_{s|pn} = 1_{(s/p)|n}.$$

The complete valuation allocation has two cases:

ordinary valuation	total width	identity or bound
0	f_0	$\mathcal{G}(pn,h) = \mathcal{G}(p,h)\mathcal{G}(n,hp^4)$
>0	f_1	$L_1 \ge f_1 + 4T_1$.

The first identity requires $(p,n) = 1$ and, on nonzero terms, $p \nmid h$. For the whole fixed valuation-zero product a_0, impose these conditions simultaneously, then change $h \mapsto ha_0^4$ injectively, before positive enlargement. Positive-valuation divisions use disjoint primes and preserve the unit conditions. The scalar has modulus at most one; its squared coefficient retains $q_{a_0}^{-1}$. Keep $(n, a_0) = 1$ in both centered rectangles as a fixed extra radical, until an allowed ordinary M2 application or the matching means. The full nonunit table retains every principal and zero branch. The fixed-divisor contribution and resulting parameters are

$$-f_0 - L_1 + T_1 - S_1, \qquad A_t = 2m + z + w,$$
$$A_1 = A_t - f_0 - f_1 - L_1, \quad M_1 = 1 - 6f_0 - f_1 - L_1 + T_1, \quad q_1 = f_1.$$

The remaining free index avoids the reduced s. Disjoint original prime ranges make its base condition $s' \mid n$ and fixed common supports independent of that index. Apply the full free identity, including the fifth-power correction, to every remaining position:

$$u = w - w_f - f_0 - f_1 = g_{\mathrm{good}} + v + c_{\mathrm{cor}} \ge 0,$$
$$L \ge v + 4T, \qquad \text{free coefficient exponent} = -u - L + v - 4c_{\mathrm{cor}} + T - S.$$

Retain the correction weights q_p^{-1}. After the fixed-unit enlargement, every actual nonunit prime still divides the current $h \ne 0$. Apply divisor Cauchy before dividing h, with squared coefficient energy. For fixed nonunit-prime data,

$$F_b \text{ is common to all unit-prime choices } a, \qquad a^4 \mid j.$$

The divisor bounds for fixed-divisor, nonunit-prime and correction choices contribute arbitrarily small power losses.

The classification is exhaustive:

$$w = w_f + f_0 + f_1 + g_{\mathrm{good}} + v + c_{\mathrm{cor}}.$$

It is completed for either sign of every intermediate length excess, including $u = 0$, before M2. The original Q_0 retains all annuli, coprimality conditions and its whole-product divisor condition.

Step 2. The coefficient contribution and length restoration. We combine the local coefficient bounds with Corollary 3.6.

Put $e_z = (9/2)(z_0 - z) \ge 0$. The allocation gives

$$D_f = \frac{9}{2}w - e_z - \frac{7}{2}(z_{C,\mathrm{old}} + w_f) + \frac{3}{2}f_0 - \frac{7}{2}f_1 - T_1,$$
$$D_{\mathrm{free}} = D_f + \frac{3}{2}u - 5v - T,$$
$$\mathcal{R}_{\mathrm{loc}} = -f_0 - L_1 + T_1 - S_1 - u - L + v - 4c_{\mathrm{cor}} + T - S.$$

Both actual and formal frequency changes equal

$$4(f_0 + g_{\mathrm{good}}) - L_1 - T_1 - L - T, \qquad K_0^* - K^* = K_0 - K.$$

Retain all displayed radicals, including canceled natural-zero factors. The original C, D, J, g, structural count and weighted Fourier measure are unchanged. Lemma 3.5 uses its $-d_D$ only in the original J bracket and gives

$$\text{additional squared restoration loss} \le \frac{(D_{\mathrm{free}} - \mu_{\mathrm{com}})_+}{6}.$$

The original amplifier contribution is included once. Applying Corollary 3.6 at $x_{\mathrm{aux}} = 0$ gives

$$\mathcal{R}_{\mathrm{loc}} + \frac{(D_{\mathrm{free}} - \mu_{\mathrm{com}})_+}{6} \le \begin{cases} 0, & \text{actual scales,} \\ \theta_N/6, & \text{nominal annuli.} \end{cases}$$

Neither a positive intermediate defect nor $u > 0$ is required.

Step 3. Verification of the length hypotheses for M2. We check the available prime length and the resulting ordinary summation lengths before applying M2.

The partition of the prime factors and the exact coupled identity give

$$z_{\mathrm{init}} = z + w - z_C - f_0 - f_1 - u = z - z_{C,\mathrm{old}} \ge 0,$$

$$\begin{aligned} \frac{7}{2} z_{\mathrm{init}} - D_{\mathrm{free}} &= \frac{7}{2} z_0 - 6w + \frac{2}{9} e_z + 5w_f + 5f_1 + T_1 + 5v + T \\ &\ge \frac{7}{2}\frac{19}{750} - 6\frac{1}{100} = \frac{43}{1500} > 0. \end{aligned}$$

Choose $0 < \sigma < 10^{-6}$ and smaller edges so that

$$D_{\mathrm{temp}} \le D_{\mathrm{free}} - \mu_{\mathrm{com}} + \frac{3}{125000}, \qquad \frac{7}{2} z_{\mathrm{init}} - D_{\mathrm{temp}} > \frac{43}{1500} - \frac{3}{125000} > 0.$$

Genuine and final-divisor fixing of widths b_g, b_d reduce the defect by at least $7b_g/2$ and $9b_d/2$, respectively; retain $-c_2 + g_2 - t_2 \le 0$. With $b = b_g + b_d \le z_{\mathrm{init}}$,

$$\frac{9}{2}(z_{\mathrm{init}} - b) - (D_{\mathrm{temp}} - 7b/2) \ge \frac{7}{2} z_{\mathrm{init}} - D_{\mathrm{temp}} > 0.$$

Thus a positive defect has sufficient remaining length of the initial prime product. For $z_{\mathrm{init}} > 0$, the auxiliary prime ranges for the amplifier avoid the original ranges; the amplifier error powers are assigned only to ordinary factors. At exhaustion, including $z = 0$, use

$$z_{\mathrm{init}} = 0 \Longrightarrow D_{\mathrm{free}} - \mu_{\mathrm{com}} \le -43/1500, \qquad g = J_+ + \sigma, \quad \ell = 0.$$

No amplifier prime factor is extracted and no average over auxiliary primes is taken. The strict margin absorbs σ and the subsidiary edges. This also covers the cases in which all additional primes lie in the common support and $w = 0$.

For nonempty coefficients put $t = f_0 + u$, $a = f_1 + t \le w \le 1/100$. The completed parameters are

$$\begin{aligned} A_* &= A_t - f_1 - t - L_1 - L - 5c_{\mathrm{cor}}, \\ M_* &= 1 - 6t - f_1 - L_1 - L - 5c_{\mathrm{cor}} + T_1 + 5v + T, \\ q_* &= f_1 + v + c_{\mathrm{cor}}, \qquad m_* = M_* - q_*. \end{aligned}$$

All natural-zero supports remain in q_*. Fixed unit primes stay in the polynomially bounded extra coprimality radical. Ordinary removal is at most $2m + o(1)$, and $A_t \le 691/750$. Hence

$$\begin{aligned} M_* &\ge 1 - 2m - 6a - o(1) \ge 27/500 - o(1), \\ m_* &\ge 1 - 2m - 7a - o(1) \ge 11/250 - o(1), \\ A_* - m_* &= A_t - 1 + 5t + f_1 - T_1 - 4v + c_{\mathrm{cor}} - T \\ &\le A_t - 1 + 6a \le -7/375 + o(1). \end{aligned} \tag{3.14}$$

The same bounds hold at intermediate fixed stages, with M2 only after classification. Nonempty coefficient and frequency supports give

$$\begin{aligned} c_C &\le A_* + o(1), \qquad R + E_0 \le \min(c_C, d_D), \\ -o(1) \le K &\le K_0^* + o(1), \qquad K_0^* = 2A_* - c_C - d_D + R + E_0 - m_*. \end{aligned}$$

Consequently both the global and formal lengths are allowed:

$$\begin{aligned} d_D &\le 2A_* - m_* + o(1) \le A_* - 7/375 + o(1), \\ m_* - c_C + d_D - R - E_0 &\ge m_* - c_C \ge 7/375 - o(1) > 0. \end{aligned}$$

For the comparison product retain $L_{\mathrm{ref}} = (1/4 - L_1 - L - 5c_{\mathrm{cor}})_+$ and its volume branch. The origin and centered-exceptional ceilings satisfy

$$A_t + 4t \le 721/750 < 1, \qquad A_t + 3t \le 1427/1500 < 1.$$

These follow from the same full local formulas, with nonpositive terms discarded only in the upper-bound direction. Keep the original powerful-product count for the first zero-frequency term. Discard tails on the genuine coprime expression in the order of Section 2. Exact scale products, common support restrictions and both matching fixed-character means are retained through principal cancellation.

Step 4. Integration and the uncentered norm. We integrate the two estimates against the original Fourier measure and recover the norm on $\mathcal{U}$.

For each original C, D, s and Fourier tuple, both the estimate with the stated ordering and the estimate with the ordinary factors interchanged fit

$$A_t - c_C + \widetilde{q} + B_C - s_0.$$

Integrate them in the same original weighted measure, including the paired arithmetic count once and retaining the nonpositive first paired term. The resulting integrated estimate gives (3.11).

On $\mathcal{U}$, reflection of the longer comparison factor gives

$$\text{total reflected length} \le \frac{5}{2} - 4(0.4) + \frac{9}{2}(0.01) = \frac{189}{200} < 1.$$

Apply the original M2 theorem to $P_{\mathrm{cmp}} Q_W$. In the positive norms with weight $\mathcal{W}(q_k/U)$,

$$\begin{aligned} \|PQ_W\|_{2,\mathcal{U}}^2 &\le 2\|P_{\mathrm{c}} Q_W\|_{2,\mathcal{U}}^2 + 2\|P_{\mathrm{cmp}} Q_W\|_{2,\mathcal{U}}^2 \\ &\le 2\|P_{\mathrm{c}} Q_W\|_{2,\mathrm{lift}}^2 + 2\|P_{\mathrm{cmp}} Q_W\|_{2,\mathcal{U}}^2 \\ &\ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C. \end{aligned}$$

This proves (3.12); the uncentered family still excludes parameters inducing principal characters.

For a requested $\epsilon > 0$, choose the positive step inside the strict margins, then subsidiary edges, and require

$$C_* \xi < \sigma/2, \qquad \eta_{\mathrm{mesh}} < \sigma/6, \qquad \eta_{\mathrm{mesh}} \le 10^{-7} \quad \text{for the selected estimate.}$$

Next fix the original finite prime-range and test-function list, request sufficiently small divisor, valuation and allocation losses, and fix internal weighted derivative/Fourier orders and thresholds before the final external height exponent. The unchanged J, g give auxiliary-length decrease at least $\sigma/2$; every ordinary sum is already in the established allowed M2 family. A nonpositive homogeneous exponent needs only the requested positive loss. Absorb the single total θ_N after the fixed large-U threshold and take the finite maximum over allocation patterns. Constants remain independent of extracted ideals and integrated Fourier labels. For $U = Z^d$, retain the fixed positive compact d range and check $\mathscr{L} = \ell/d \in [0.21, 0.22]$ separately. □

4 The enlarged amplified norm

In this section, we prove the additional amplified norm. The classification of Section 3 gives the homogeneous estimates; a direct bound for small formal summation lengths and the complementary Gauss calculation give the larger domain used in Section 7.

4.1 An amplified norm with variable lengths

In this subsection, we prove a parameter-dependent amplified norm from the classification of Section 3. We then estimate the unused prime amplitudes for the selection in Section 7.

This section retains the established cited results and the earlier bound $\beta_*(F) \le 7/8$, which permits $\kappa = 3/4$. The classification of Theorem 3.7 is independent of its numerical parameter range. We now give parameter-dependent sufficient inequalities; the comparison follows in Section 7. All powers below are relative to the actual scale of the character parameter U.

Theorem 4.1 (Amplified norm with variable lengths)**.** *Retain the arithmetic and analytic conventions of Theorem 3.7 and Section 2. Fix $3/8 < m_- \le r < 1/2$ and $W \ge 0$ with*

$$\tau = 1 - 2r - 9W > 0, \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1. \tag{4.1}$$

For disjoint products Q_0, Q_W of complete prime-sum factors with length exponents z, w, put

$$m \in [m_-, r], \qquad 0 \le z \le z_0(m) = 2(1-2m)/9, \qquad 0 \le w \le W,$$
$$P = S_{m,1} S_{m,2} Q_0, \qquad P_{\mathrm{c}} = P - P_{\mathrm{cmp}}.$$

Use the canonical equal-product comparison and retain Q_0 with auxiliary removal length $x_{\mathrm{aux}} = 0$. Then, for every $\epsilon > 0$,

$$\sum_k \mathcal{W}(q_k/U) |P_{\mathrm{c}}(k) Q_W(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C, \tag{4.2}$$

$$\sum_{k \in \mathcal{U}} \mathcal{W}(q_k/U) |P(k) Q_W(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C. \tag{4.3}$$

The centered sum is unrestricted; $\mathcal{U}$ is the prescribed nonexceptional sixth-power-free nonunit family. The bounds hold for every required finite independent family of smooth weights and derivatives, with the original Fourier measure, zero and support conditions, and order of choices. Steps, meshes and internal losses are chosen inside the fixed strict margins.

Proof. Repeat the exact classification of Theorem 3.7 with the original C, D, s and weighted Fourier measure. Bound zero-length factors before Poisson, retain every coprimality and natural-zero factor, and classify all extra positions before M2. The local identities do not depend on the earlier parameter range.

All homogeneous identities are unchanged. In particular, with the notation of the proof using all local prime cases,

$$D_{\mathrm{free}} = 6w - e_z - \frac{7}{2} z_{C,\mathrm{old}} - 5w_f - 5f_1 - T_1 - 5v - T, \qquad \mu_{\mathrm{com}} = \min(c_C, d_D) \ge w_{f,\mathrm{actual}},$$

and the additional loss of restoring the lengths is $(D_{\mathrm{free}} - \mu_{\mathrm{com}})_+/6$. Corollary 3.6 makes the complete coefficient exponent plus this loss nonpositive on actual scales, with the same total $O(1/\log U)$ conversion discrepancy on nominal annuli. No second use of $-d_D$ or extra structural-family count is made.

The new numerical length conditions are as follows. Put

$$F_\tau = \frac{7}{9}\tau, \qquad z_{0,\min} = \frac{2}{9}(1-2r), \qquad A_{\max} = 2r + z_{0,\min} = \frac{14r+2}{9}.$$

The exact coupled remaining-prime margin is bounded below by

$$\begin{aligned}\frac{7}{2}z_{\mathrm{init}} - D_{\mathrm{free}} &= \frac{7}{2}z_0 - 6w + \frac{2}{9}e_z + 5w_f + 5f_1 + T_1 + 5v + T \\ &\geq \frac{7}{9}(1-2r) - 6W = F_\tau + W > 0.\end{aligned}$$

Put $D_{\mathrm{sharp}} = D_{\mathrm{free}} - \mu_{\mathrm{com}}$ and choose the amplifier and edge allowance so that $D_{\mathrm{temp}} - D_{\mathrm{sharp}} < F_\tau + W$. Genuine and final-divisor fixing reduce the defect by at least $7/2$ and $9/2$ times their respective widths. Hence, for $0 \leq b \leq z_{\mathrm{init}}$,

$$\frac{9}{2}(z_{\mathrm{init}} - b) - (D_{\mathrm{temp}} - 7b/2) \geq \frac{7}{2}z_{\mathrm{init}} - D_{\mathrm{temp}} > 0.$$

At exhaustion,

$$z_{\mathrm{init}} = 0 \Longrightarrow D_{\mathrm{sharp}} \leq -(F_\tau + W) < 0, \qquad g = J_+ + \sigma, \quad \ell = 0.$$

Use the case with no remaining prime factor, without averaging over auxiliary primes.

For $t = f_0 + u$ and $a = f_1 + t \leq w$, the unchanged auxiliary estimate formulas and ordinary-removal bound $L_1 + L + 5c_{\mathrm{cor}} \leq 2m + o(1)$ give

$$\begin{aligned}M_* &\geq 1 - 2r - 6W - o(1) = \tau + 3W - o(1), \\ m_* &\geq 1 - 2r - 7W - o(1) = \tau + 2W - o(1), \\ A_* - m_* &\leq A_{\max} + 7W - 1 + o(1) = -F_\tau + o(1).\end{aligned}$$

For nonempty supports,

$$c_C \leq A_* + o(1), \qquad R + E_0 \leq \min(c_C, d_D), \qquad -o(1) \leq K \leq 2A_* - d_D - m_* + o(1).$$

Together with $A_* - m_* \leq -F_\tau + o(1)$, this gives

$$d_D \leq A_* - F_\tau + o(1), \qquad m_* - c_C + d_D - R - E_0 \geq F_\tau - o(1) > 0.$$

The same conservative bounds hold at each exact comparison stage, and no M2 estimate precedes the completed classification.

The second-origin and centered-exceptional ceilings, including the volume reference branch, are respectively

$$A_{\max} + 5W = 1 - F_\tau - 2W < 1, \qquad A_{\max} + 4W = 1 - F_\tau - 3W < 1.$$

The original zero-frequency term in the first Poisson formula, powerful-product count, tails and exact equal-product mean cancellations are handled in the order of choices in Section 2. The two current-side estimates fit the same Gauss-norm targets, are integrated in the same weighted Fourier measure and incur their paired structural count once. This proves (4.2).

For the restoration of the comparison product in the uncentered norm on $\mathcal{U}$, the reflected total length exponent is at most

$$\frac{5}{2} - 4m_- + \frac{9}{2}W = C_{\mathrm{add}} < 1.$$

The initial four ordinary scales have exponent at least $1/4$: $m_- > 3/8 > 1/4$ and $2m_- - 1/4 > 1/2$. Also $W < 1/45$, by the exact identity

$$1 - 45W = 4\tau + 2(1 - C_{\mathrm{add}}) + 8(r - m_-) > 0.$$

Thus the original M2 theorem and the triangle inequality give (4.3). No parameter inducing a principal character is added back. The fixed-unit coprimality restrictions are partial products of the original list and have norm at most $U^{W+o(1)}$; the retained polynomially uniform common-extra-radical result treats them in the same fixed data.

Choose a positive step, subsidiary edges, mesh for the prime-product lengths and requested losses inside F_τ, $F_\tau + W$, the positive lower bounds for the summation lengths, the origin margins and the margin for restoring the comparison product. Then fix the original finite list of prime ranges and test functions, finite allocation maximum, weighted internal orders and thresholds, and finally the external height exponent. The strict decrease in the auxiliary summation length and a single total annular discrepancy are exactly those of the earlier proof. None depends on an extracted ideal or subsequent Fourier label. This establishes all uniformities asserted above. □

Lemma 4.2 (Mean amplitude of the remaining prime factors)**.** *Let Q_i be a finite family of original prime-sum factors on mutually disjoint prime-ideal ranges, with length exponents $\lambda_i \geq 0$ and amplitude exponents g_i on $\mathscr{B}$. Retain the original lower bounds*

$$|Q_i(k)| \geq U^{\lambda_i g_i - o(1)} \qquad (k \in \mathscr{B},\ g_i > 0).$$

Suppose

$$\delta > 0, \qquad 0 \leq x \leq 1/2, \qquad 0 \leq g_i \leq \delta/2,$$
$$\sum_i \lambda_i = \mathscr{L} \geq L_0 > z_0(m), \qquad \sum_i \lambda_i g_i = \delta x \mathscr{L}.$$

Remove an initial prime product of length at most $z_0(m)$. If

$$a(\delta, x, m) = \delta \left\{ \frac{1}{2} - \frac{(1/2 - x)L_0}{L_0 - z_0(m)} \right\} = \delta \frac{xL_0 - z_0(m)/2}{L_0 - z_0(m)} > 0, \tag{4.4}$$

the remaining positive-amplitude prime ranges have mean amplitude at least $a(\delta, x, m)$ and total length at least

$$P(\delta, x, m) = 2xL_0 - z_0(m). \tag{4.5}$$

For $0 < W < P(\delta, x, m)$ and η_{mesh} sufficiently small, with $\max_i \lambda_i \leq \eta_{\mathrm{mesh}}$, selection in decreasing amplitude gives a disjoint product Q_W of length exponent w_{sel} with

$$W - \eta_{\mathrm{mesh}} \leq w_{\mathrm{sel}} \leq W,$$
$$|Q_W(k)|^2 \geq U^{2a(\delta,x,m)(W - \eta_{\mathrm{mesh}}) - o(1)} \qquad (k \in \mathscr{B}).$$

Proof. Write $z \leq z_0(m)$ for the removed length and $D_{\mathrm{amp}} = \delta(1/2 - x)\mathscr{L}$ for the total deficit from maximal amplitude. The remaining mean $\overline{g}$ satisfies

$$\overline{g} \geq \frac{\delta}{2} - \frac{D_{\mathrm{amp}}}{\mathscr{L} - z} \geq \frac{\delta}{2} - \frac{\delta(1/2 - x)L_0}{L_0 - z_0(m)} = a(\delta, x, m).$$

Here

$$\frac{\partial}{\partial z}\left(\frac{\delta}{2} - \frac{D_{\mathrm{amp}}}{\mathscr{L} - z} \right) \leq 0, \qquad \frac{d}{d\mathscr{L}} \frac{\mathscr{L}}{\mathscr{L} - z_0(m)} \leq 0.$$

The original zero-amplitude length is at most $2D_{\mathrm{amp}}/\delta = (1 - 2x)\mathscr{L}$. Thus the unused positive length is at least

$$\mathscr{L} - z - (1 - 2x)\mathscr{L} = 2x\mathscr{L} - z \geq 2xL_0 - z_0(m).$$

Removing zero amplitudes increases a positive mean. Since $\lambda_i \geq 0$ and $\max_i \lambda_i \leq \eta_{\mathrm{mesh}}$, selection of complete factors in decreasing amplitude gives length in $[W - \eta_{\mathrm{mesh}}, W]$ and

$$\log_U |Q_W(k)|^2 \geq 2a(\delta, x, m)(W - \eta_{\mathrm{mesh}}) - o(1).$$

Apply the original amplitude lower bounds before differentiating weights. □

The lower bound in (4.4) is sharp given only the upper amplitude bound, the total mean and the removed length of the initial prime product. At $\mathscr{L} = L_0$ take an initial prime product of width $z_0(m)$ and amplitude $\delta/2$, and give every remaining position the common amplitude $a(\delta, x, m)$. Its weighted total is exactly $\delta x L_0$, the decreasing-amplitude rule removes the factors of maximal amplitude first, and every unused subproduct has that same mean. This is a sharpness statement for the scalar amplitude data; it does not assert that such a formal amplitude class is nonempty arithmetically.

Corollary 4.3 (A norm bound on amplitude classes and the resulting saving)**.** *Under Theorem 4.1, fix an amplitude class $\mathscr{B}$ with*

$$a = a(\delta, x, m) > 0, \qquad 0 < W < \min\{(1 - 2m)/9,\ (8m - 3)/9,\ 2xL_0 - z_0(m)\}.$$

Then

$$s < 2aW \quad \Longrightarrow \quad \sum_{k \in \mathscr{B}} \mathcal{W}(q_k/U)|P(k)|^2 \ll U^{1-s+\epsilon} \mathfrak{S}_J(\mathcal{W})(1 + H)^C.$$

After separately extracting the initial prime-product saving, the reflected fourth moment and Hölder inequality give any additional first-moment saving

$$0 < \gamma(\delta, x, m) < \frac{a(\delta, x, m)W}{2}. \tag{4.6}$$

Uniformity on parameter families requires positive lower bounds for the retained length and selection margins and for

$$2aW - s, \qquad aW/2 - \gamma.$$

Proof. For any $0 < \gamma < aW/2$, choose s and then the mesh and internal loss so that

$$4\gamma < s < 2a(W - \eta_{\mathrm{mesh}}) - o(1) < 2aW.$$

On the original selected class,

$$|Q_W(k)|^2 \geq U^{s-o(1)} \quad \Longrightarrow \quad \sum_{k \in \mathscr{B}} \mathcal{W}(q_k/U)|P(k)|^2 \ll U^{-s+\epsilon} \sum_{k \in \mathcal{U}} \mathcal{W}(q_k/U)|P(k)Q_W(k)|^2.$$

Equation (4.3) gives the claimed norm bound. Hölder of order four then gives

$$\gamma < s/4, \qquad s < 2aW \quad \Longrightarrow \quad 0 < \gamma < aW/2.$$

The amplitude bound precedes Sobolev and derivative expansion. Choose products from the original class data, then fix derivative weights and Fourier orders. Uniform positive gaps in the length, selection and gain inequalities make these choices uniform. □

These bounds use the original norm, independently of any older restricted-remainder error term.

For a rectangle $\delta \in [d_0, d_1]$, $x \in [x_0, 1/2]$, $m \in [m_-, r]$, a uniform lower mean is

$$a_{\mathrm{box}} = d_0 \left\{ \frac{1}{2} - \frac{(1/2 - x_0)L_0}{L_0 - z_0(m_-)} \right\} > 0,$$

and available length of positive-amplitude prime factors is at least $2x_0L_0 - z_0(m_-)$. The upper endpoint d_1 does not enter this sharper amplitude bound: the amplitude bound and its deficit have the same factor δ. For $d_0 = 2/5$, $x_0 = 12/25$, $L_0 = 21/100$, $m_- = 2/5$, one obtains $a_{\mathrm{box}} = 3536/18625$. Bounding the deficit instead by the separate maxima $\delta \le 11/20$ and $\mathscr{L} \le 11/50$ of the box of Theorem 3.7 gives only

$$\frac{d_0}{2} - \frac{(11/20)(1/2 - x_0)(11/50)}{L_0 - z_0(m_-)} = \frac{1}{5} - \frac{1089}{74500} = \frac{13811}{74500} < a_{\mathrm{box}}.$$

The theorem also permits a requested prime-product length depending on the ordinary length $W(m)$, provided $1 - 2m - 9W(m)$ and the selected margin in the available positive-amplitude prime length have uniform positive lower bounds on the chosen compact band. At the level of these sufficient length conditions, the limiting pointwise gain is

$$\gamma_{\lim}(\delta, x, m) = \frac{a(\delta, x, m)}{2} \min\left\{\frac{1-2m}{9}, \frac{8m-3}{9},\ 2xL_0 - z_0(m)\right\}.$$

This is a supremum for this particular norm-and-selection exponent calculation; its strict boundary is not asserted attainable and it is not an optimality theorem for other analytic estimates or zero-free methods.

Lemma 4.4 (An allowed choice for each amplitude class)**.** *On a compact amplitude-class set $\mathcal{K}$, suppose*

$$\tau > 0, \qquad m_- > 3/8, \qquad m_- \le m_0 < (1-\tau)/2,$$
$$a_0(\delta, x) = a(\delta, x, m_-) > 0, \qquad t \in C(\mathcal{K}; (0, \infty)).$$

Set

$$r(\delta, x) = \frac{a_0(1-\tau)/18 + tm_0}{a_0/9 + t},$$
$$W(\delta, x) = \frac{1 - 2r(\delta, x) - \tau}{9}.$$

Then

$$m_- \le r < (1-\tau)/2, \qquad W > 0,$$
$$1 - 2r - 9W = \tau, \qquad a_0 W/2 = t(r - m_0).$$

If, uniformly on this set,

$$W < \min\{(8m_- - 3)/9,\ 2xL_0 - z_0(m_-)\}$$

with a positive gap, the norm on $m \in [m_-, r(\delta, x)]$ supplies every gain satisfying

$$\gamma(\delta, x) > 0, \qquad \inf_{\mathcal{K}}\{a_0 W/2 - \gamma\} > 0,$$

with uniform positive loss margins. The finite class selections may therefore use these class-dependent endpoints and lengths. For a fixed finite class set, all strict margins are made uniform by taking their finite minimum.

Proof. Put $A = a_0/9 > 0$ and $M = (1-\tau)/2$. Then

$$r = \frac{AM + tm_0}{A + t}, \qquad r - m_0 = \frac{A(M - m_0)}{A + t} > 0, \qquad M - r = \frac{t(M - m_0)}{A + t} > 0.$$

Consequently

$$m_- \le m_0 < r < M, \qquad W = \frac{2(M-r)}{9} > 0, \qquad 1 - 2r - 9W = \tau, \qquad \frac{a_0 W}{2} = t(r - m_0).$$

For $m \geq m_-$, Lemma 4.2 gives $a(\delta, x, m) \geq a_0$. Theorem 4.1 and Corollary 4.3 apply with formal margin $7\tau/9$ and the stated remaining strict gaps. Continuity on $\mathcal{K}$ and the prescribed gain gap give

$$\inf_{\mathcal{K}} t > 0, \qquad \inf_{\mathcal{K}} W > 0, \qquad \inf_{\mathcal{K}} (a_0 W/2 - \gamma) > 0.$$

Together with the assumed upper-length gaps, these bounds permit common positive steps, meshes, orders and thresholds. All choices precede derivatives and Fourier integration; the separate contour conditions on t still have to be checked. □

4.2 A direct estimate for small transformed length exponents

In this subsection, we bound the actual Gauss polynomial when its formal summation-length exponent is small. The complementary range is treated in Section 4.3.

The length exponent and the frequency cutoff. We distinguish the transformed summation-length exponent from the Fourier-frequency cutoff.

Write $A = A_*$, $m = m_*$, $q = q_*$ for the completed auxiliary parameters, $c = c_C$, $d = d_D$, and $u = R + E_0$. These identities are numerical exponent notation for the actual completed norm even when a formal ordinary moment family would not be allowed. Put

$$\alpha = A - c, \qquad t = d - u \geq 0, \qquad m_{\mathrm{f}} = m - c + t, \qquad \nu = -m_{\mathrm{f}}.$$

The exact cutoff identity is

$$K_0^* = 2A - c - d + u - m = 2\alpha + \nu.$$

Thus a negative formal exponent m_{f} does not imply an empty actual frequency annulus. For example, $\alpha = 0$ leaves a possible constant Gauss polynomial $\mathcal{G}(1, h) = 1$ and a nonempty actual frequency range of exponent $\nu > 0$. It is the true h cutoff, not m_{f}, that controls counting of the actual frequencies.

Lemma 4.5 (A direct bound for the Gauss polynomial)**.** *Retain the common support, divisor parameters, coefficient families and Fourier coordinates of the local decomposition. The factor $\vartheta(n)$ below is the fixed coefficient factor in the cited Gauss polynomial, with its prescribed zeros; it is independent of the frequency h. Put*

$$\Phi(h) = U^{-\alpha/2} \sum_n B(n)\vartheta(n) 1_{s'|n} 1_{(n,\mathfrak{r}_*)=1} \mathcal{G}(n, h), \qquad s_0' = \log_U q_{s'}.$$

Here $q_n \asymp U^\alpha$ on the two equal-product rectangles; $B(n)$ is the original finite convolution of smooth weights and fixed finite-group prime coefficients. Retain all support and zero-extension conditions. After extracting the fixed weight seminorms, let $0 \leq \omega(h) \leq 1$ be the normalized frequency cutoff. Then, for every retained frequency range of exponent K,

$$\sum_{0 < q_h \ll U^K} \omega(h) |\Phi(h)|^2 \ll U^{K_+ + 2\alpha - 2s_0' + \epsilon} \mathfrak{S}_J(\mathcal{W})(1 + H)^C, \qquad K_+ = \max(K, 0).$$

The positive loss includes all dyadic-scale discrepancies; the Fourier orders are those of the original weighted estimate.

Proof. For fixed factor count the coefficients satisfy an ideal-divisor bound, uniformly in the fixed arithmetic labels. With the original weight seminorms and height factors understood,

$$|\mathcal{G}(n, h)| \leq q_n^{1/2}, \qquad \sum_{\substack{q_n \asymp U^\alpha \\ s'|n}} |B(n)| \ll U^{\alpha - s_0' + \epsilon}.$$

The two centered products have the same scale. Hence

$$|\Phi(h)| \le U^{-\alpha/2} \sum_{\substack{q_n \asymp U^{\alpha} \\ s' \mid n}} |B(n)| q_n^{1/2}$$
$$\ll U^{\alpha - s_0' + \epsilon} \mathfrak{S}_J(\mathcal{W})^{1/2} (1+H)^C.$$

A nonempty product has $s_0' \le \alpha + O(1/\log U)$. The actual nonzero frequency orbits, including bounded unit frequencies, satisfy

$$\#\{h : 0 < q_h \ll U^K\} \ll U^{K_+ + \epsilon}.$$

Squaring the pointwise bound and summing proves the assertion. The zero-frequency term in the first Poisson formula is separate. Keep all natural zeros and support restrictions in the polynomial; only their absolute values are bounded by one. Integrate the polynomial Fourier-height factors in the original measure after fixing a sufficiently high order. No additional restricted moment family or family count is introduced. □

Proposition 4.6 (Direct estimate for small transformed length exponents)**.** *Retain the actual coefficient-square bounds for all local prime cases and natural-zero conventions. Fix* $\rho_{\mathrm{term}} = 1/100$ *and suppose* $m_{\mathrm{f}} \le \rho_{\mathrm{term}}$*. Let the class parameters obey*

$$\frac{3}{8} < m_- \le r < \frac{1}{2}, \qquad \eta_{3/4} = \frac{7}{9}(1-2r) - 6W > 0, \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1,$$

with actual extra length $w \le W$*. Then, relative to the original one-sided Gauss target, this branch has exponent less than*

$$-\frac{7249}{18600} < -\frac{3}{8},$$

before arbitrarily small requested losses.

Proof. Put $B_C = (3c - 5d - R)_+/6$. The original one-sided exponent, before the local prime removals, is

$$\Lambda_{\mathrm{in}} = A_t - c + u + B_C - s_0, \qquad u = R + E_0, \tag{4.7}$$

where $A_t = 2m_{\mathrm{ord}} + z + w$ is the original product length, m_{ord} is its ordinary length before classification, and s_0 is the original inner divisor length. The m in the completed Gauss parameters is instead the residual length m_*. This is the allowance of [Liu26b, equation (12.3)] at the original fixed C, D data. For the completed polynomial,

$$s_0' = s_0 - f_0 - f_1, \qquad \Lambda_* = (A - c) + q + u + B_C - s_0'.$$

Consequently the exact change in the allowance is

$$\Lambda_* - \Lambda_{\mathrm{in}} = A - A_t + q + f_0 + f_1.$$

This notation does not assume a moment theorem at that target. The actual frequency cutoff gives

$$K \le 2\alpha + \nu + o(1), \qquad \nu \ge -\rho_{\mathrm{term}}, \quad \alpha \ge -o(1),$$
$$K_+ \le 2\alpha + \nu + \rho_{\mathrm{term}} + o(1).$$

Subtracting Λ_* from Lemma 4.5 gives at most

$$3\alpha + \nu - s_0' - q - u - B_C + \rho_{\mathrm{term}} + o(1).$$

The local coefficient contribution after classification, including the difference between this affine target and the given target, is

$$\mathcal{R}_{\mathrm{loc}} = -f_0 - L_1 + T_1 - S_1 - u_{\mathrm{free}} - L + v - 4c_{\mathrm{cor}} + T - S.$$

The length $u_{\mathrm{free}} = g_{\mathrm{good}} + v + c_{\mathrm{cor}}$ is the freely removed length denoted by u in Sections 3 and 4.1. It is distinct from the common-support quantity $u = R + E_0$ used here. The actual squared coefficient exponent and the allowance change give

$$\begin{aligned}(-f_0 - f_1 + T_1 - S_1 + T - S) + (\Lambda_* - \Lambda_{\mathrm{in}}) \\ = A - A_t + q + T_1 - S_1 + T - S = \mathcal{R}_{\mathrm{loc}}.\end{aligned}$$

Here $s_0' = s_0 - f_0 - f_1$; the fixed-data coefficient energy is retained.

Since $R \le u$ and $d = u + t$,

$$u + B_C \ge u + \left(\frac{c}{2} - u - \frac{5t}{6}\right)_+ \ge \frac{c}{2} - \frac{5t}{6}.$$

Also $c \ge m + t - \rho_{\mathrm{term}}$. Dropping the favorable $-s_0'$ shows that the complete relative exponent is at most

$$\begin{aligned}&\mathcal{R}_{\mathrm{loc}} + 3(A - c) + (c - m - t) - q - u - B_C + \rho_{\mathrm{term}} \\ &\le \mathcal{R}_{\mathrm{loc}} + 3A - \frac{7}{2}m - q - \frac{8}{3}t + \frac{7}{2}\rho_{\mathrm{term}} \\ &\le \mathcal{R}_{\mathrm{loc}} + 3A - \frac{7}{2}m - q + \frac{7}{2}\rho_{\mathrm{term}}.\end{aligned}$$

Put $L_1 = f_1 + 4T_1 + e_1$ and $L = v + 4T + e$, with $e_1, e \ge 0$. The exact completed expressions give the identity

$$\begin{aligned}\mathcal{R}_{\mathrm{loc}} + 3A - \frac{7}{2}m - q &= 3A_t - \frac{7}{2} + 17f_0 + \frac{5}{2}f_1 + 17g_{\mathrm{good}} + \frac{5}{2}v + 18c_{\mathrm{cor}} \\ &\quad - \frac{9}{2}(T_1 + T) - S_1 - S - \frac{e_1 + e}{2} \\ &\le 3A_t - \frac{7}{2} + 18w.\end{aligned}$$

All variables here refer to the same completed allocation, before the direct bound is inserted. The fixed prime dividing s choices satisfy a divisor bound; actual nonunit-prime indices were already treated by original-frequency divisor Cauchy, with their squared coefficient energy retained; correction indices retain their absolute coefficient mass of arbitrarily small power growth. The direct bound is uniform in those fixed data and introduces no new count.

The length constraint for the amplitude class gives

$$A_t \le \frac{14r + 2}{9} + W = 1 - 5W - \eta_{3/4}.$$

Moreover

$$1 - \frac{279}{7}W = \frac{36}{7}\eta_{3/4} + 2(1 - C_{\mathrm{add}}) + 8(r - m_-) > 0,$$

so $W < 7/279$. The total exponent is consequently less than

$$-\frac{1}{2} + 3W - 3\eta_{3/4} + \frac{7}{2}\rho_{\mathrm{term}} < -\frac{7249}{18600} < -\frac{3}{8},$$

before sufficiently small requested losses. It fits the original one-sided target with a fixed strict margin. This proof uses no earlier Gauss norm estimate on a family with a negative formal summation-length exponent. □

Here $\eta_{3/4}$ is η_κ of Section 10 at $\kappa_m = 3/4$. No ordinary sum of positive formal length is reconstructed.

For $m_{\mathrm{f}} > \rho_{\mathrm{term}}$, we use the direct Gauss calculation of Proposition 4.8. Lemma 4.7 supplies a numerical comparison while retaining the actual outer ideal. These estimates are combined in Theorem 4.9.

4.3 The complementary Poisson estimate

In this subsection, we prove the complementary one-sided Gauss estimate. Together with Section 4.2, it gives the norm domain in Section 4.4.

This subsection concerns a Gauss norm already obtained by the actual first transform and the exact local Gauss-sum decompositions. It does not apply the original first-Poisson estimate to an invented ordinary family. In particular the original outer ideal D is retained as a fixed label, not reconstructed as a new factor in the shorter completed coefficient sum.

Lemma 4.7 (An auxiliary outer-length parameter)**.** *For the completed Gauss norm above, assume $c \le A$ and $R + E_0 \le \min(c, d)$. Set*

$$\widehat{d} = \min(d, A), \qquad \widehat{K}_0 = K_0 + d - \widehat{d}, \qquad \widehat{\Delta} = \widehat{K}_0 - K.$$

Then

$$\widehat{d} \le A, \qquad R + E_0 \le \min(c, \widehat{d}), \qquad K \le \widehat{K}_0 + \delta_1,$$
$$B_C(\widehat{d}) = B_C(d), \qquad \min(c, \widehat{d}) = \min(c, d),$$
$$\widehat{K}_0 = 2A - c - \widehat{d} + R + E_0 - m, \qquad \widehat{d} + \widehat{\Delta} = d + (K_0 - K).$$

The local exceptional-character savings are denoted by $F_1^{\mathrm{loc}}, F_2^{\mathrm{loc}}$. The first is given in (4.11); the second is the saving of [Liu26b, Section 12.8.2], bounded below in the proof of Proposition 4.8. This numerical replacement preserves the J, g, F_1^{loc} formulas, origin and length-restoration exponents, prime lengths and target exponent. It changes neither the polynomial, the actual cutoff and fixed ideal D, nor its character factors, outer measure and previously discarded tails. With $\widehat{m}_{\mathrm{f}} = m - c + \widehat{d} - R - E_0$,

$$\widehat{m}_{\mathrm{f}} \le \rho_{\mathrm{term}}: \quad \textit{Proposition 4.6 applies with the same target,}$$
$$\widehat{m}_{\mathrm{f}} > \rho_{\mathrm{term}}: \quad \widehat{m}_{\mathrm{f}} > 0, \quad \widehat{d} \le A.$$

No new coefficient family or ideal of norm $U^{\widehat{d}}$ is introduced.

Proof. The case $d \le A$ is unchanged. If $d > A$, then

$$\widehat{d} = A, \qquad 3c - 5\widehat{d} - R \le -2A - R \le 0, \qquad R + E_0 \le c \le A = \widehat{d}.$$

Thus both positive parts defining B_C vanish, and

$$B_C(\widehat{d}) = B_C(d) = 0, \qquad \min(c, \widehat{d}) = \min(c, d) = c, \qquad \widehat{d} + \widehat{K}_0 - K = d + K_0 - K.$$

These are precisely the outer-length invariants in the formulas below; $\widehat{K}_0 \ge K_0$ preserves the actual cutoff. The direct estimate uses the same cutoff bound, target and common-support inequalities. The polynomial, ideal D, kernels and discarded tails are unchanged. At nominal endpoints use the support-envelope construction below. □

Dependence on the outer length in the cited Gauss calculation. Set $D_0 = d + K_0 - K$. The formulas below are the corresponding expressions in [Liu26b, Sections 12.7.1–12.8.3 and Appendix C.4], with its amplifier lengths w, w_o written here as a, a_o. Its amplifier density ℓ is written ℓ_a to separate it from the global prime length.

$$J = D_0 - c - 2a + a_o, \qquad \Lambda_a = A - c + \widetilde{q} + a_o + B_C - s_0', \tag{4.8}$$
$$M' = M + J - g - g_2 + t_2, \qquad \Delta' = b_2 - p_2 + a + B_C + \ell_a, \tag{4.9}$$
$$\mathcal{D}_{\mathrm{ret}} \le a - D_0 - a_o + g + 2\theta_N, \qquad \mathcal{O}_{\mathrm{ret}} = A - M - D_0 - a_o - B_C + g - \ell_a, \tag{4.10}$$

$$F_1^{\mathrm{loc}} = \frac{c}{6} + \frac{5D_0}{6} + \frac{a}{3} + \frac{\widetilde{q}}{6} + a_o + B_C - \frac{5g}{6} + \ell_a. \tag{4.11}$$

Here $\mathcal{D}_{\mathrm{ret}}$ is the length excess before prime restoration and $\mathcal{O}_{\mathrm{ret}}$ is the excess for the zero-frequency term in the second Poisson formula before the local coefficient contribution. The first line is [Liu26b, equations (12.3)–(12.4)]; the transformed width and its normalization are [Liu26b, equation (12.6) and Appendix C.4]. The last two lines are the pre-restoration expression of [Liu26b, Section 12.8.1] and the origin and exceptional expressions of [Liu26b, Sections 12.7.3 and 12.8.2].

Under Lemma 4.7,

$$\widehat{d} + \widehat{K}_0 - K = D_0, \qquad B_C(\widehat{d}) = B_C(d), \qquad \min(c, \widehat{d}) = \mu_{\mathrm{com}}.$$

Thus (4.8)–(4.11) are invariant. The F_2^{loc} terms in [Liu26b, Section 12.8.2] use only the new second common support; the centered mean in [Liu26b, Section 12.8.3] uses the unchanged actual equal-product rectangles. The original outer D-sum and its weight stay fixed. Tails are estimated from actual kernels and cutoffs before this numerical replacement; no outer count is recomputed at $\widehat{d}$.

Dyadic length exponents and support errors. For nonempty nominal supports choose the single total $\theta_N = O(1/\log U)$ so that

$$c \le \overline{A} := A + \theta_N, \qquad \overline{A} \ge 0, \qquad \widehat{d}_N = \min(d, \overline{A}), \quad \widehat{K}_{0,N} = K_0 + d - \widehat{d}_N.$$

The common-support saving and the target exponent are preserved exactly:

	$B_C(d) = B_C(\widehat{d}_N)$	$\min(c,d) = \min(c, \widehat{d}_N)$
$d > \overline{A}$	0	c
$d \le \overline{A}$	$B_C(d)$	$\min(c, d)$.

Also $\widehat{d}_N \le A + \theta_N$; relative to the idealized comparison, the numerical change is at most θ_N wherever the ideal lengths are nonnegative. Include it in the total support error. The envelope changes no ordinary test, normalization, smooth-weight argument or exact centered product.

If a nonempty endpoint has negative nominal A or $\alpha = A - c$, its negative part is $O(\theta_N)$. In a nonempty regular branch,

$$m_{\mathrm{f}} > \rho_{\mathrm{term}}, \quad K \ge -o(1), \quad K \le 2\alpha - \rho_{\mathrm{term}} + O(\theta_N) \quad \Longrightarrow \quad \alpha \ge \rho_{\mathrm{term}}/2 - o(1).$$

Bounded coefficient-support endpoints therefore use the direct estimate, whose fixed saving absorbs the same support error. Actual centers and product identities are preserved.

Proposition 4.8 (The complementary Gauss-norm bound)**.** *Retain the Gauss-sum, correlation-Poisson, weight, common-coefficient, zero-extension, exceptional-mean and M2 hypotheses of Section 2, with $\kappa_m = 3/4$. For a completed positive Gauss norm with fixed outer parameters C, D, s, put*

$$\alpha = A - c, \qquad \widetilde{q} = q + R + E_0, \qquad M = m + q, \qquad K_0 = 2A - c - d + R + E_0 - m.$$

Here $c = \log_U q_C$ and $d = \log_U q_D$, or the auxiliary numerical value of Lemma 4.7. Assume

$$R + E_0 \le \min(c, d), \qquad c \le A + o(1), \qquad K \le K_0 + \delta_1, \quad 0 \le \delta_1 < \xi,$$
$$m - c + d - R - E_0 > \rho_{\mathrm{term}} > 0.$$

The actual global lengths are uniformly positive; all length exponents and fixed support norms lie in the original bounded polynomial ranges. Retain the exact coefficients and equal-product scales, the comparison factor after prime-power removal, and the full remaining-prime-length margin. For the original length A_t and removed length t_{loc} of (4.13), also retain uniform strict margins

$$A_t + 4t_{\mathrm{loc}} < 1, \qquad A_t + 3t_{\mathrm{loc}} < 1.$$

Then the one-sided Poisson estimate has target

$$\Lambda = \alpha + \widetilde{q} + B_C - s_0', \qquad B_C = (3c - 5d - R)_+/6.$$

For every prescribed $\varepsilon_{\mathrm{loc}} > 0$, the additional squared length-restoration loss is at most

$$\frac{(D_{\mathrm{free}} - \min(c,d))_+}{6} + \varepsilon_{\mathrm{loc}}.$$

The steps, support errors and complete-factor excesses are chosen inside this loss and the fixed strict margins, in the order of Section 2. The origin and exceptional-term estimates and weighted orders are unchanged. M2 is applied only to auxiliary ordinary sums satisfying the moment length conditions; the inequality $d \le A$ is not required.

Proof. Use the actual normalized Gauss polynomial of Lemma 4.5. The positive formal length selects this branch; an ordinary moment is used only on sums formed after the second Poisson transform. For each fixed tuple, the convolution coefficients and their multiplicative presentation are independent of h. Only the Gauss coefficient and an already bounded scalar depend on h; all natural zeros remain in the Gauss polynomial. Thus this is the same common polynomial for the fixed-divisor and unit-prime arguments.

The frequency sum and the amplifier. Keep the original bounded sum over h through the sixth-power amplifier identity, averaging over allowed auxiliary primes and Jensen's inequality. With R_{fix} the product of the other fixed supports, the pool exclusions satisfy

$$\#\{\mathfrak{p} \text{ in the pool} : \mathfrak{p} \mid hs'CDR_{\mathrm{fix}}\} \ll \log U,$$

since all these supports have bounded polynomial norms. The pool lies beyond the remaining original prime ranges, after the step and mesh are fixed. Representation multiplicity uses the actual bounded frequency range and positive pool step.

Use a for an extraction from the original amplifier and $a_o \le a$ for its new radical, reserving the letter w for the original extra product. Set

$$a_0 = A - c - a, \quad J = d - c + (K_0 - K) - 2a + a_o.$$

When prime-sum factors remain, use [Liu26b, Section 12.7.2]:

	(a, a_o)	ℓ_a	g
amplified main term	$(0,0)$	$\sigma/3$	$J_+ + 2\sigma$
fixed amplifier error	original (a, a_o)	0	$J_+ + \sigma$.

If no prime-sum factor remains, use instead

$$a = a_o = \ell_a = 0, \qquad g = J_+ + \sigma. \tag{4.12}$$

There is then no average over auxiliary primes or density factor. The identity $g - J_+ = \sigma$ retains the strict decrease in the transformed summation length; the origin and exceptional allowances are no worse than for the prime-bearing choice. This is also the choice used when the available length of the initial prime product is exhausted, as in Theorem 4.9.

The corresponding normalizations, with no averaging in the no-prime case, give the allowance

$$\Lambda_a = a_0 + \widetilde{q} + a_o + a + B_C - s_0'.$$

These are identities for the original expression and amplifier errors; they require no second ordinary coefficient sum of length exponent A.

Second Poisson and ordinary sums after the second Poisson transform. Restore the complete sums before squaring. The full finite correlation identity gives the earlier second-Poisson formula, with prefactor $U^{K+g-\ell_a-a_0}$ and its inverse roots. Retain every fixed common coprimality

restriction. Estimate high-frequency tails on this genuine expression before any artificial coprime extension. The common Fourier boxes are chosen before the new remaining labels, exactly as in the earlier proof with the original weighted seminorms. All smooth weights here are the actual transformed smooth weights, whose orders are uniform on bounded polynomial ranges.

For its genuine second common support use the earlier variables b_2, p_2, g_2, t_2, V, with their actual valuations and zero branches. The nonzero summation parameters after the second Poisson transform have

$$\begin{aligned} m' &= 2a_0 - K - g - g_2 - V, \\ q' &= \widetilde{q} + a_o + t_2 + V, \\ M' &= m' + q' = M + J - g - g_2 + t_2 \le M - \sigma. \end{aligned}$$

Substitute K_0 in the equality. For the decrease,

$$J - g \le -\sigma, \quad t_2 - g_2 \le 0 \quad \Longrightarrow \quad M' \le M - \sigma.$$

The nonzero frequency annulus is empty if its cutoff lies below the minimum exterior norm. At a bounded endpoint include the edge $0 \le \delta_2 < \xi$ before declaring its summation range:

$$\text{nonempty} \Longrightarrow m' + \delta_2 \ge 0, \qquad M'_{\text{act}} \le M' + \delta_2 < M - \sigma + \xi.$$

The exact second normalization is

$$\lambda_0 = K + g - \ell_a - a_0 + p_2 - s'_0 + g_2 - t_2 - b_2, \qquad \Lambda_a - \lambda_0 = M' + \Delta',$$

where $\Delta' = b_2 - p_2 + a + B_C + \ell_a \ge a + \ell_a$. The complete last-divisor allocation keeps unrestricted quotients, both actual rectangle products and all surviving prime annuli. Its paired weight and count are the original weighted measures with arbitrarily small power growth. Original fixed D is not summed again here; only the genuine new second common support is counted once. The restored ordinary sums have the radical in the original character formula, including canceled factors. Their exceptional split is made only after the final character has been formed. The established M2 theorem, rather than an induction hypothesis for the new amplified family, bounds each allowed auxiliary sum.

Length restoration. Before its coarse substitution, the earlier length defect is

$$a - d - (K_0 - K) - a_o + g + 2\theta_N.$$

Its exact common-length maximum is

$$-d - (K_0 - K) - a_o + J_+ = \max\{-d - (K_0 - K) - a_o, -c - 2a\} \le -\min(c, d) + \delta_1.$$

Only $c, d, a, a_o \ge 0$ and $K_0 - K \ge -\delta_1$ are used. The same extra global defect D_{free} from the actual full local decomposition therefore gives the additional squared loss at most $(D_{\text{free}} - \min(c, d))_+/6 + \varepsilon_{\text{loc}}$.

For the prescribed $\varepsilon_{\text{loc}} > 0$, choose the positive steps and subordinate support and mesh losses so that the prescribed exponent losses, including the excess from one complete factor, sum to less than ε_{loc}. After fixing the finite prime-factor list, take U sufficiently large to absorb $\theta_N = O(1/\log U)$ in this same loss.

The original amplifier estimate accounts for its own $a + \ell_a$ portion once. The unchanged coupled remaining-prime identity justifies every restoration of allowed lengths; prime exhaustion uses the cited estimate for arbitrary ordinary lengths with no remaining prime factor in the earlier order. No given Gauss estimate outside the established length range is assumed.

The total length removed by the local identities. In the origin and exceptional estimates below, put

$$\begin{aligned} u_{\text{free}} &= g_{\text{good}} + v + c_{\text{cor}}, \\ t_{\text{loc}} &= f_0 + u_{\text{free}} = w - w_f - f_1, \qquad 0 \le t_{\text{loc}} \le w \le W. \end{aligned} \tag{4.13}$$

This is the quantity denoted by t in the decomposition into all local prime cases of Section 3.4. Here u_{free} is the removed prime width; it is distinct from the common-support length $u = R+E_0$ in Section 4.2. The displayed identity follows from the partition $w = w_f + f_0 + f_1 + u_{\mathrm{free}}$.

The zero-frequency term in the second Poisson formula. Its length excess in the cited Poisson formula before the local coefficient contribution is

$$A - M - d - (K_0 - K) - a_o - B_C + g - \ell_a + O(\theta_N).$$

The exact maximum and $g - J_+ \le 2\sigma$ give

$$\begin{aligned}\mathcal{O}_{\mathrm{ret}} &\le A - M - \min(c,d) + \delta_1 - B_C + 2\sigma - \ell_a + O(\theta_N)\\ &\le A - M + O(\sigma + \xi + \theta_N).\end{aligned}$$

Adding the original local coefficient contribution yields

$$\mathcal{O}_{\mathrm{total}} \le A_t - 1 + 4t_{\mathrm{loc}} + O(\sigma + \xi + \theta_N).$$

This uses the existing full origin formula.

Sparse exceptional character parameters and their means. The earlier local correlation table and natural zeros force the same prescribed exterior sixth-power-free part. All earlier supports of the fixed character factors and the additional restrictions still impose coprimality, so they cannot cancel a newly forced exponent two. The sparse count applies on the enlarged actual frequency range. Its first saving is

$$F_1^{\mathrm{loc}} = c/6 + 5(d + K_0 - K)/6 + a/3 + \widetilde{q}/6 + a_o + B_C - 5g/6 + \ell_a.$$

The original amplifier data satisfy

$$0 \le a_o \le a \le 7\sigma/3, \qquad \sigma \le g - J_+ \le 2\sigma, \qquad \ell_a \ge 0.$$

Put $D_0 = d + K_0 - K \ge d - \delta_1$. The common-support bound gives

$$B_C + \widetilde{q}/6 \ge (3c - 5D_0)_+/6 - 5\delta_1/6.$$

Hence

$$F_1^{\mathrm{loc}} \ge \begin{cases} c + 2a + \widetilde{q}/6 + a_o/6 + B_C + \ell_a - 5\sigma/3, & J \ge 0,\\ 2c/3 + a/3 - 5\sigma/3 - 5\delta_1/6, & J < 0.\end{cases}$$

In both cases,

$$F_1^{\mathrm{loc}} \ge \frac{2(c+a)}{3} - \frac{22\sigma}{9} - \frac{5\delta_1}{6} > \frac{2(c+a)}{3} - 3\sigma - \frac{5\xi}{6}.$$

The extraction bound is the one for the original amplifier errors; it is not an estimate for arbitrary independent a. The second local saving $F_2^{\mathrm{loc}} \ge 2b_2/3$ concerns only genuine new second common support, so it has no original d condition.

Apply the retained fixed-character mean to the actual centered rectangles, with the comparison factor after prime-power removal $(1/4 - L_1 - L - 5c_{\mathrm{cor}})_+$ and its volume branch. The same character, fixed coprimality restriction and whole-product Mellin twist occur on both sides, so their leading products cancel exactly at the actual equal product. After the actual local coefficient contribution, the exceptional ceiling is $A_t - 1 + 3t_{\mathrm{loc}}$, as in the calculation including all local prime cases. This argument does not replace the comparison factor after prime-power removal by a formal $M/4$, and it does not require a larger-width moment theorem.

Uniformity and integration. Retain the original Fourier measures and paired normalizations for the diagonal, sparse exceptional, amplifier-error and auxiliary ordinary terms. The positive formal/global length, origin, exceptional and coupled-length margins permit the same ordered choices of step, edges, mesh and finite orders. Integrate fixed C, D and the remaining parameters in the original outer measure, with its original D count. This proves the one-sided Poisson bound throughout the stated range of d. □

4.4 The enlarged length range

In this subsection, we combine the two Gauss estimates to obtain the sufficient length inequalities used in Section 7.

The allowed width changes as follows:

$$\begin{array}{l|l} \text{Theorem 4.1} & W < (1-2r)/9 \\ \text{Theorem 4.9} & W < 7(1-2r)/54 \end{array} \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1.$$

The common condition $C_{\mathrm{add}} < 1$ restores the comparison product. The larger bound of Theorem 4.9 supplies the extra prime-product saving used in Lemma 7.1.

Theorem 4.9 (Amplified norm under the coupled length constraints)**.** *Retain the arithmetic data, coefficient families, summation domains and exact equal-product comparison of Theorem 4.1, with $\kappa_m = 3/4$ and the initial prime product Q_0 retained. Suppose*

$$3/8 < m_- \le r < 1/2, \qquad W \ge 0,$$

and

$$\eta_{3/4} = \frac{7}{9}(1-2r) - 6W > 0, \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1. \tag{4.14}$$

For complete, disjoint prime-sum factors with

$$m \in [m_-, r], \qquad 0 \le z \le z_0(m) = 2(1-2m)/9, \qquad 0 \le w \le W,$$

one has, for every $\epsilon > 0$,

$$\sum_k \mathcal{W}(q_k/U)|P_{\mathrm{c}}(k)Q_W(k)|^2 \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C, \tag{4.15}$$

$$\sum_{k\in\mathcal{U}} \mathcal{W}(q_k/U)|P(k)Q_W(k)|^2 \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C. \tag{4.16}$$

The two summation domains, finite independent smooth-weight and derivative families, and order of choices are those of Theorem 4.1 and Section 2.

Proof. Use the exact classification of Theorem 3.7, with zero-length factors bounded before Poisson and every extra position classified before M2. Keep C, D, s, natural zeros and the original Fourier measure. The algebraic inequality of Corollary 3.6 gives the nonpositive normalization exponent and one $O(1/\log U)$ discrepancy. This use of the algebraic inequality requires no positive formal length; the analytic M2 applications occur only after the split below. The length identities yield

$$\frac{7}{2}z_{\mathrm{init}} - D_{\mathrm{free}} \ge \frac{7}{9}(1-2r) - 6W = \eta_{3/4} > 0, \qquad z_{\mathrm{init}} = 0 \Longrightarrow D_{\mathrm{free}} \le -\eta_{3/4}.$$

Thus the original restoration remains valid, including the no-prime choice (4.12). The global lengths satisfy

$$\begin{aligned} A_t &\le (14r+2)/9 + W = 1 - 5W - \eta_{3/4}, \\ M_* &\ge (12W + 9\eta_{3/4})/7 - o(1) > 0, \\ m_* &\ge (5W + 9\eta_{3/4})/7 - o(1) > 0. \end{aligned}$$

Retain the actual outer ideal D. Lemma 4.7 replaces only its numerical exponent by $\widehat{d} = \min(d_D, A_*)$, using $A_* + \theta_N$ at nominal endpoints. For the actual Gauss polynomial set

$$m_{\mathrm{f}} = m_* - c_C + \widehat{d} - R - E_0.$$

The complete range is

$m_{\mathrm{f}} \le 1/100$	Proposition 4.6: saving greater than 3/8
$m_{\mathrm{f}} > 1/100$	Proposition 4.8: ordinary restored sums.

The first case includes negative exponents and bounded unit frequencies. In the second, $R + E_0 \le \min(c_C, \widehat{d})$ and $\widehat{d} \le A_* + O(\theta_N)$. Neither case changes the actual frequency cutoff, outer ideal, coefficients or discarded tails. Every M2 application is on a formed ordinary sum with its full radical. In Proposition 4.8, request

$$0 < \varepsilon_{\mathrm{loc}} < \min\{\epsilon/4, \eta_{3/4}/100, (1 - C_{\mathrm{add}})/100, 1/100\}.$$

This local loss is included in the final ϵ and remains inside all the strict length and origin margins.

By (4.13), $t_{\mathrm{loc}} \le W$, so

$$A_t + 4t_{\mathrm{loc}} \le 1 - W - \eta_{3/4} < 1, \qquad A_t + 3t_{\mathrm{loc}} \le 1 - 2W - \eta_{3/4} < 1.$$

Retain the volume reference branch and both matching fixed-character means at the exact equal product. The zero-frequency term in the first Poisson formula and high-frequency tails use their original expressions. Both one-sided estimates therefore fit the same Gauss allowance; integrate them in the same measure and count the common support once. This proves (4.15).

Reflection of the comparison product on $\mathcal{U}$ has total length exponent $C_{\mathrm{add}} < 1$. The earlier M2 theorem applies, and, in the weighted ℓ^2 norms with weight $\mathcal{W}(q_k/U)$,

$$\|PQ_W\|_{2,\mathcal{W},\mathcal{U}} \le \|P_c Q_W\|_{2,\mathcal{W}} + \|P_{\mathrm{cmp}} Q_W\|_{2,\mathcal{W},\mathcal{U}}.$$

This gives (4.16) without principal character parameters. Moreover,

$$1 - \frac{279}{7}W = \frac{36}{7}\eta_{3/4} + 2(1 - C_{\mathrm{add}}) + 8(r - m_-) > 0.$$

Thus $W < 7/279$ and all support norms remain polynomially bounded. Choose steps, support edges, mesh and losses inside the positive length and origin margins, then fix the finite weight families, weighted orders and thresholds before external heights. The original strict decrease in auxiliary length is retained, uniformly in the later extracted ideals and Fourier variables. □

Corollary 4.10 (The amplitude-class saving in the enlarged length range)**.** *Under Theorem 4.9, let $a = a(\delta, x, m) > 0$ be given by Lemma 4.2. Suppose the remaining positive-amplitude prime length exceeds W with a uniform margin. Then*

$$s < 2aW, \qquad 0 < \gamma < aW/2$$

are allowed squared-norm and first-moment savings, respectively, with the original order of smooth weights and derivatives. For fixed m_- and class-dependent (r, W), the norm conditions are

$$W < \frac{7}{54}(1 - 2r), \qquad W < \frac{8m_- - 3}{9}.$$

For uniformity on parameter families, all retained length and selection margins and the gaps $2aW - s$, $aW/2 - \gamma$ have positive lower bounds.

Proof. Apply Lemma 4.2 to the original factors, before derivative expansion. The conditions $\eta_{3/4} > 0$ and $C_{\mathrm{add}} < 1$ of (4.14) are equivalent to

$$W < \frac{7}{54}(1 - 2r), \qquad W < \frac{8m_- - 3}{9}.$$

Equation (4.16) and the proof of Corollary 4.3 give

$$0 < \gamma < aW/2 \quad \Longrightarrow \quad \exists s: \quad 4\gamma < s < 2aW.$$

Choose the mesh and internal losses within the strict gaps, and use

$$\gamma < s/4.$$

The stated uniform positive gaps permit common choices of the mesh and internal losses. The same finite independent weight families and order of choices apply. □

Compared with Corollary 4.3,

$$\frac{1-2r}{9} \longrightarrow \frac{7(1-2r)}{54};$$

the zero-free boundary requires the comparison of Section 7.

5 The approximate functional equation and its contours

In this section, we prove uniform estimates for the test functions on a compact positive strip. These estimates permit the contour displacements combined with the amplified norm in Section 7.

5.1 Smooth weights beyond the $1/10$ contour shift

In this subsection, we bound the AFE test functions on a compact positive strip. This supplies the full contour range used in Section 7.

The earlier restriction to shifts of size at most $1/10$ came from the strip chosen in the lemma on smooth weights for the approximate functional equation; the holomorphic region for the correction factors permits a larger shift of the original numerator contour.

Lemma 5.1 (Smooth weights on compact substrips)**.** *Fix $0 < \sigma_0 < 1/2$. For a primitive nonprincipal finite-order Hecke character over the fixed totally complex field F, use the original approximate functional equation with*

$$V_w(v) = \frac{1}{2\pi i}\int_{(2)} e^{\xi^2}\left(\frac{\Gamma(w+\xi)}{\Gamma(w)}\right)^{r_F} v^{-\xi}\frac{d\xi}{\xi}, \qquad T_w = (1+|\Im w|)^{r_F}, \quad v > 0.$$

Uniformly for $\sigma_0 \le \Re w \le 1-\sigma_0$ and fixed $j, N \ge 0$,

$$|(v\partial_v)^j V_w(v)| \ll_{F,\sigma_0,j,N} (1+v/T_w)^{-N}.$$

For $w = 1/2+\theta+it$, $0 \le \theta \le 1/2-\sigma_0$, put

$$\widetilde{V}_w(v) = \left(\frac{\Gamma(1-w)}{\Gamma(w)}\right)^{r_F} v^{2\theta} V_{1-w}(v).$$

After increasing the tail order by one, the same derivative bound holds for $\widetilde{V}_w$. Before contour displacement use

$$H_w(v) = v^{2w-1}\left(\frac{\Gamma(1-w)}{\Gamma(w)}\right)^{r_F} V_{1-w}(v), \qquad H_w(v) = v^{2it}\widetilde{V}_w(v) \quad (w = 1/2+\theta+it),$$

with the real logarithm of v. Here $\Re w = 1/2+\theta \ge 1/2$, so $\Re(1-w) = 1/2-\theta$. The functions V_w, H_w are holomorphic on $0 < \Re w < 1$; the real-line factorization is used only after displacement. Their dyadic expansions converge normally on compact substrips for each fixed finite annulus of character parameters.

Proof. Put $T = (1 + |\Im w|)^{r_F}$. Euler differentiation inserts powers of ξ in the defining integral. For $v \geq T$, shift to any fixed line $\Re\xi = B > 0$. Stirling's formula and the Gaussian e^{ξ^2} give

$$|(v\partial_v)^j V_w(v)| \ll_{F,\sigma_0,j,B} (T/v)^B.$$

For $v \leq T$, shift to $\Re\xi = -\sigma_0/2$. Then

$$\Re(w + \xi) \geq \sigma_0/2 > 0, \qquad (v\partial_v)^j V_w(v) = \mathbf{1}_{j=0} + O_{F,\sigma_0,j}((v/T)^{\sigma_0/2}).$$

Only $\xi = 0$ is crossed. These two estimates prove the required bound for every fixed j, N.

The completed nonprincipal function is entire, so its functional equation gives the same two exact AFE halves on the enlarged strip. For $w = 1/2 + \theta + it$, Stirling gives

$$\left|\frac{\Gamma(1-w)}{\Gamma(w)}\right|^{r_F} \ll T^{-2\theta}, \qquad 0 \leq 2\theta \leq 1 - 2\sigma_0 < 1.$$

Hence the normalized dual factor is bounded using

$$(v/T)^{2\theta}(1 + v/T)^{-N-1} \leq (1 + v/T)^{-N}.$$

Fixed Euler derivatives obey the same bounds. Before the contour move use the holomorphic factor v^{2w-1}; afterwards its unitary part is included in the Mellin twist. The exact conductor and gamma factors are retained, and the large-v bounds give normal convergence of the dyadic tails on compact substrips. □

Lemma 5.2 (Extended contours for the dyadic sums)**.** *Retain the nonprincipal family, correction/reflection identities and unamplified M2 with $\kappa_m = 3/4$. Fix z with $\Re z = 17/50$, then assign the amplitude classes; keep z fixed thereafter. For $3/10 \leq \delta \leq 3/5$, every dyadic sum in the approximate functional equation admits each fixed classwise shift*

$$0 \leq \theta \leq \delta/2, \qquad \Re w = 1/2 - \theta, \qquad \theta_- = 0, \quad \theta_+ = \delta/2.$$

Keep s on its initial right-hand line and move the whole dyadic integrand before the local s-contour displacement, absolute values, conductor separation and conjugation of dual weights. Retain the prescribed positive separation margins. The M2 exponent changes by

$$(\theta - \theta_0)\{d(1-m) - l_y\}, \qquad \theta_0 = 2\delta x/9,$$

with the same central-line moment, initial prime product and prime-sum coefficients.

Proof. Here $a = (1+\delta)/2 = a_{\mathrm{det}}$ is the detector parameter, and s, w, z are the contour variables of [Liu26b, Lemma 13.10]; this z is not the length exponent of Q_0, and this s is not the inner divisor of Section 3. The entire swept range satisfies

$$0 \leq \theta \leq \delta/2, \qquad 1/5 \leq \Re w = 1/2 - \theta \leq 1/2, \qquad 1/2 \leq \Re(1-w) \leq 4/5.$$

Lemma 5.1 applies with $\sigma_0 = 1/6$. The connecting region of [Liu26b, Lemma 13.10] contains

$$\Re s \geq a + 16e, \quad \Re w \geq 1 - a - 6e, \quad \Re z \geq 17/50.$$

On the final local contours,

$$\Re w \geq 1 - a \quad \Longrightarrow \quad \Re(s + w) \geq 1 + 16e.$$

Thus the correction holomorphy and the original unit- and nonunit-prime bounds hold throughout. For each strict tuple,

$$q_p^{\theta} q_p^{-(j_p-1)\theta} = q_p^{-(j_p-2)\theta} \leq 1 \qquad (j_p \geq 2,\ \theta \geq 0);$$

nonstrict errors retain their positive e margins. If S_{fin} denotes the finite places of S, the fixed-place majorant is summable:

$$\sum_{\substack{\mathfrak{a}\ \text{integral}\\ \operatorname{supp}(\mathfrak{a})\subset S_{\mathrm{fin}}}} q_{\mathfrak{a}}^{-1/6} = \prod_{p\in S_{\mathrm{fin}}} (1-q_p^{-1/6})^{-1} < \infty.$$

The independent AFE weights and derivatives therefore remain in the original M2 families with fixed polynomial height dependence.

Keep finite character and dyadic cutoffs. With s first on its original right-hand line, perform the contour moves in this order:

unsplit numerator	$\Re w:\ c_w \longrightarrow 1/2$
exact two-part AFE	insert on $\Re w = 1/2$
each holomorphic dyadic sum	$\Re w:\ 1/2 \longrightarrow 1/2-\theta$
local s-contour	$\Re s \ge a + 16e.$

Here c_w is the initial right-hand line. The first move is valid because the numerator is entire, the reciprocal factor is independent of w, and the cited region contains the path. Mellin decay removes horizontal joins; locally uniform bounds permit removal of the cutoffs in the original weighted height order. Only then conjugate the dual half as a whole. Keep each actual conductor, root number and gamma ratio.

The amplitude partition depends only on the character parameter and the already fixed z; it is unchanged during the w- and local s-contour displacements. Thus its prime coefficients remain fixed, and the change of factor is

$$q_u^{\theta-\theta_0} D^{-(\theta-\theta_0)} Y^{-(\theta-\theta_0)}.$$

With $q_u \asymp U = Z^d$, $D = U^m$ and $Y = Z^{l_y}$, its exponent change is the one stated, with the bounded annular ratios in the original smooth weights. □

The wider contour range is combined with the amplified norm in Section 7. Section 8 verifies the resulting comparison inequalities and obtains the endpoint B_*.

6 Amplitude-class counts and short summation ranges

In this section, we sharpen the amplitude-dependent class count and separate its parameter from that of the numerator moment. We also record the lowest-amplitude estimate needed for the comparison in Section 8.

6.1 Amplitude-dependent bounds for small-amplitude classes

In this subsection, we sharpen the small-amplitude count by retaining its amplitude dependence. The improvement is used in the short-range comparisons of Section 8.

The earlier count is retained at the allowed value $\kappa = 3/4$. This section sharpens a scalar consequence of that count; it introduces no new arithmetic moment hypothesis.

Lemma 6.1 (Sharp linear amplitude correction)**.** *For $0 \le \delta \le 5/6$ and $0 \le x \le 1/2$, put*

$$A = 5/6-\delta, \quad D = 3-17x/9, \quad P = (2-8x/9)(1-x), \quad J = AD+\delta P, \qquad R = 1-\delta+\frac{A\delta P}{2J}.$$

In the notation of Proposition 6.4 and Lemma 6.5,

$$A = A_\delta, \qquad (D,P,J,R) = (D_{3/4}, P_{3/4}, J_{3/4}, R_{3/4}).$$

Then $J > 0$ and

$$R \le 1 - \frac{2\delta}{3} - \frac{22}{81}\delta x.$$

The coefficient $22/81$ is the largest constant α for which $R \le 1 - 2\delta/3 - \alpha\delta x$ holds on this whole rectangle.

Proof. Since $D, P > 0$,

$$J = \begin{cases} (5/6)D > 0, & \delta = 0, \\ (5/6 - \delta)D + \delta P > 0, & 0 < \delta < 5/6, \\ (5/6)P > 0, & \delta = 5/6. \end{cases}$$

Put $y = 1/2 - x$. The exact identity

$$2J - 3AP - \frac{44}{27}xJ = \frac{250x^2}{729} + \frac{\delta}{243}(352y^3 + 948y^2 + 984y + 199)$$

is nonnegative. Since

$$(1 - R) - \frac{2\delta}{3} = \frac{\delta}{6J}(2J - 3AP),$$

the claimed inequality follows, including $\delta = 0$ by continuity. For optimality take $x > 0$ and let $\delta \downarrow 0$:

$$\lim_{\delta \downarrow 0} \frac{(1 - R) - 2\delta/3}{\delta x} = \frac{2D - 3P}{6Dx} = \frac{22 - 12x}{81 - 51x}.$$

Letting $x \downarrow 0$ forces $\alpha \le 22/81$. This proves optimality only for the stated uniform linear count bound. □

Proposition 6.2 (An improved comparison for small amplitudes)**.** *Use the earlier refined full exponent and the small-amplitude shift $\theta = g\min\{\ell/d, 1/(6\kappa)\}$, with $g = \delta x$ and $\kappa = 3/4$. Suppose the balanced geometry has $0 < \ell/h < 2/9$, $h < 1$ and the earlier monotonicity hypotheses hold. Set*

$$C_0 = -1/4 + b/6 + 5\ell/4, \qquad Q_{\mathrm{sm}} = \ell(1/2 + 2\ell/h) - 11h/54.$$

For $0 \le \delta \le 1/8$, the exact endpoint obeys

$$E_{\mathrm{small}}(\delta, x) \le C_0 + \delta\{(1 - h)/2 + Q_{\mathrm{sm}}x\} \le C_0 + \frac{1 - h + \max\{0, Q_{\mathrm{sm}}\}}{16}.$$

In particular, if $Q_{\mathrm{sm}} \le 0$, a sufficient condition is

$$C_0 + \frac{1 - h}{16} < -\frac{1}{250}.$$

The same original $1/250$ margin is retained: if $C_0 + (1 - h + \max\{0, Q_{\mathrm{sm}}\})/16 < -1/250$ and $\zeta > 0$ is a buffer with $2\zeta < 1/1000$, then the extension to $d \le h + \zeta$ satisfies $E_{\mathrm{small}} + 2\zeta < -3/1000$.

Proof. Lemma 6.1 gives

$$\begin{aligned} E_{\mathrm{small}} &= C_0 + \frac{\delta}{2} + \ell\delta x(1/2 + 2\ell/h) - \frac{3h}{4}(1 - R) \\ &\le C_0 + \delta\left\{\frac{1 - h}{2} + Q_{\mathrm{sm}}x\right\} \\ &\le C_0 + \frac{1 - h + \max(0, Q_{\mathrm{sm}})}{16}. \end{aligned}$$

The last step uses $0 \le x \le 1/2$, $0 \le \delta \le 1/8$ and $h < 1$. The earlier monotonicity gives the extension to $d \le h + \zeta$. Under the stated $-1/250$ condition,

$$E_{\mathrm{small}} + 2\zeta < -\frac{1}{250} + \frac{1}{1000} = -\frac{3}{1000}.$$

The original count, shift and height argument are unchanged. □

6.2 The lowest-amplitude classes

In this subsection, we verify the sufficient lowest-amplitude margin required by the parameters of Section 8.2.

Lemma 6.3 (Lowest-amplitude estimate with a 1/125 threshold)**.** *Retain the earlier direct, principal, detector, correction, moment, height and contour estimates, with their order of choices for nonprincipal characters. Suppose*

$$C_0 + (1/2 + 3\ell/2)/50 < -1/125, \qquad 2\zeta < 1/1000, \qquad 0 < \varepsilon_{\mathrm{cls}} \le 1/1000.$$

Then the lowest-amplitude contribution has exponent

$$\mathcal{E}_{\mathrm{low}}(d) < -7/1000 < -\varepsilon_{\mathrm{cls}} \qquad (d \le h + \zeta),$$

before the prescribed subsidiary losses. It therefore satisfies the same character-sum remainder estimate with those losses chosen inside this strict margin.

Proof. Write $\mathcal{E}_{\mathrm{low}}$ for the exponent in the cited lowest-amplitude estimate, and $\bar{g}$ for the weighted mean prime-sum amplitude in that class. Then

$$R = 1, \qquad \bar{g} \le 1/100, \qquad \mathcal{E}_{\mathrm{low}}(h) \le C_0 + (1/2 + 3\ell/2)/50 < -1/125.$$

Its extension to $d \le h + \zeta$ loses at most 2ζ. Therefore

$$\mathcal{E}_{\mathrm{low}}(d) < -\frac{1}{125} + 2\zeta < -\frac{1}{125} + \frac{1}{1000} = -\frac{7}{1000} < -\varepsilon_{\mathrm{cls}}.$$

This is the proof of [Liu26b, Proposition 14.6] with the weaker sufficient threshold and the same classification and contour identities. Choose the remaining nonprincipal losses inside the strict margin. The principal, small-norm and height/trace estimates retain their original hypotheses. □

6.3 Separate parameters for the class count and the numerator moment

In this subsection, we apply the class count with a separately established parameter. The numerator moment remains fixed, as required in Section 4.

Retain the enlarged original norm at $\kappa_{\mathrm{m}} = 3/4$, justified by 7/8. A separately proved global bound supplies the count parameter κ_{c} below; the numerator norm, selected initial prime product, length constraints, amplitude bounds and Gauss restoration remain fixed.

Proposition 6.4 (The class-count parameter from an earlier zero-free bound)**.** *Suppose an already proved global bound is $\beta_*(F) \le B_{\mathrm{seed}}$. Fix*

$$\frac{13}{18} \le \kappa_{\mathrm{c}} \le \frac{3}{4}, \qquad 2B_{\mathrm{seed}} - 1 \le \kappa_{\mathrm{c}}.$$

For the original detected amplitude classes with $0 < \delta \le 5/6$, $0 \le x \le 1/2$, and available length of the original prime product $\mathscr{L} > 1/5$, the earlier count is

$$\#\mathscr{B} \ll U^{R_{\kappa_{\mathrm{c}}}(\delta,x)+\epsilon}(1+H)^{C},$$

where

$$\begin{aligned} c_{\mathrm{cnt}}(\kappa) &= \frac{1}{3\kappa}, \\ D_\kappa &= 3 - (1 + 2c_{\mathrm{cnt}}(\kappa))x, \\ P_\kappa &= (2 - 2c_{\mathrm{cnt}}(\kappa)x)(1 - x), \\ J_\kappa &= (5/6 - \delta)D_\kappa + \delta P_\kappa, \\ R_\kappa &= 1 - \delta + \frac{(5/6 - \delta)\delta P_\kappa}{2J_\kappa}. \end{aligned}$$

The numerator moment parameter may simultaneously remain $\kappa_{\mathrm{m}} = 3/4$, justified by the earlier 7/8 theorem.

Proof. Use the three counts of [Liu26b, Section 13.4]. Their parameter is

$$\widetilde{c} = 2c_{\mathrm{cnt}}(\kappa_{\mathrm{c}}) = \frac{2}{3\kappa_{\mathrm{c}}}, \qquad \frac{13}{18} \le \kappa_{\mathrm{c}} \le 1 \quad \Longleftrightarrow \quad \frac{2}{3} \le \widetilde{c} \le \frac{12}{13}.$$

Here, as in [Liu26b, Section 13.4], r is the length exponent of the inverse moment M1 and $m = t - r$. The crossing lengths satisfy

estimate	requested prime length	crossing upper bound
M1	$(1-r)/2$	$5/26 < 1/5$
M2	$(1-2m)/(6\kappa_{\mathrm{c}})$	$\widetilde{c}/12 + o(1) \le 1/13 + o(1) < 1/5.$

Thus $\mathscr{L} > 1/5$ permits the original decrements and mesh. Keep the original finite-group coefficients, detector weights and fourth-power amplitude bound in M2. The count-side hypothesis is

$$\beta_*(F) \le B_{\mathrm{seed}} \le (1 + \kappa_{\mathrm{c}})/2,$$

while the numerator uses $\kappa_m = 3/4$. Balancing the three counts gives $R_{\kappa_{\mathrm{c}}}$ by [Liu26b, equation (13.16)]. The restriction $\kappa \ge 7497/10000$ of [Liu26b, Remark 14.1(1)] does not enter this count calculation. □

Lemma 6.5 (Monotonicity of the count)**.** *For the preceding domain, $J_\kappa > 0$ and*

$$\frac{\partial R_\kappa}{\partial \kappa} = \frac{\delta x (5/6 - \delta)^2 (1-x)^2}{3\kappa^2 J_\kappa^2} \ge 0. \tag{6.1}$$

In particular $R_{\kappa_{\mathrm{c}}} \le R_{3/4}$ whenever $\kappa_{\mathrm{c}} \le 3/4$.

Proof. Both D_κ and P_κ are positive on $13/18 \le \kappa \le 3/4$, $0 \le x \le 1/2$. At $\delta = 5/6$, $J_\kappa = \delta P_\kappa > 0$; elsewhere it is a positive sum. Differentiate the rational expression for R. Since

$$(\partial_\kappa P_\kappa) D_\kappa - P_\kappa (\partial_\kappa D_\kappa) = \frac{2x(1-x)^2}{3\kappa^2},$$

with $A_\delta = 5/6 - \delta$ one obtains

$$\partial_\kappa R = \frac{A_\delta \delta}{2J_\kappa^2} \{ (\partial_\kappa P_\kappa) J_\kappa - P_\kappa \partial_\kappa J_\kappa \} = \frac{A_\delta^2 \delta x (1-x)^2}{3\kappa^2 J_\kappa^2} \ge 0.$$

This is (6.1). □

Corollary 6.6 (Separate parameters in the numerator estimate)**.** *Use the count of Proposition 6.4 and retain*

$$\kappa_{\mathrm{m}} = 3/4, \qquad \theta_0 = \frac{\delta x}{6\kappa_{\mathrm{m}}}, \qquad z_0(m) = \frac{1-2m}{6\kappa_{\mathrm{m}}}, \qquad R = R_{\kappa_{\mathrm{c}}}.$$

The numerator exponent and its amplified version are

$$N = \frac{3R_{\kappa_{\mathrm{c}}} + 1}{4} - \frac{\theta_0}{2}, \qquad N_{\mathrm{amp}} = N - \gamma.$$

The character-sum comparison at its endpoint is

$$E_{\mathrm{dec}}(\delta, x) = C_0 + \frac{\delta}{2} + \delta x \left\{ \ell + \frac{h/2 - l_y}{6\kappa_{\mathrm{m}}} \right\} - \frac{3h}{4}(1 - R_{\kappa_{\mathrm{c}}}),$$
$$C_0 = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4}, \qquad E_{\mathrm{dec}} \le E_{b,\ell,3/4}.$$

The earlier small-amplitude comparisons using $R \le 1 - 2\delta/3$ remain valid with the smaller count $R_{\kappa_{\mathrm{c}}}$.

Proof. Proposition 14.3 of [Liu26b] accepts the class-count exponent separately from the numerator moment parameter. Lemma 6.5 gives

$$R_{\kappa_c} \le R_{3/4}, \qquad R_{\kappa_c} \le (6\kappa_m - 2\delta x)\mathscr{L} \quad \text{whenever} \quad R_{3/4} \le (6\kappa_m - 2\delta x)\mathscr{L}.$$

Keep $\kappa_m = 3/4$, θ_0 and z_0 fixed. In the earlier full exponent [Liu26b, equation (14.7)], substitution gives

$$N = \frac{3R_{\kappa_c} + 1}{4} - \frac{\theta_0}{2},$$
$$E_{\mathrm{dec}} - E_{b,\ell,3/4} = \frac{3h}{4}\big(R_{\kappa_c} - R_{3/4}\big) \le 0.$$

Both AFE halves, all independent derivatives and the enlarged norm retain their original moment-side parameters and length constraints. The coarse count inequalities therefore remain valid. □

Corollary 6.7 (Available prime-product length and the separated parameters)**.** *Under Proposition 6.4, retain* $\kappa_{\mathrm{m}} = 3/4$. *Suppose*

$$\frac{7}{20} \le \delta \le \frac{3}{5}, \qquad \frac{2}{5} \le x \le \frac{1}{2}, \qquad \mathscr{L} \ge \frac{21}{100}.$$

Then the prime-length condition for the numerator estimate has the uniform margin

$$(6\kappa_{\mathrm{m}} - 2\delta x)\mathscr{L} - R_{\kappa_c}(\delta, x) \ge \frac{24683}{162000} > 0. \tag{6.2}$$

The unamplified affine slope in the scale exponent d of Section 2 satisfies

$$S = \frac{3R_{\kappa_c} + 1}{4} + \frac{\theta_0}{2} - \frac{17}{50} \le \frac{52}{75} < 1. \tag{6.3}$$

Proof. By Lemmas 6.5 and 6.1,

$$R_{\kappa_c} \le R_{3/4} \le 1 - \frac{2\delta}{3} - \frac{22}{81}\delta x \le 1.$$

Since $6\kappa_{\mathrm{m}} - 2\delta x > 0$, the left side of (6.2) is at least

$$\left(\frac{9}{2} - 2\delta x\right)\frac{21}{100} - 1 + \frac{2\delta}{3} + \frac{22}{81}\delta x = -\frac{11}{200} + \delta\left(\frac{2}{3} - \frac{601x}{4050}\right).$$

The last expression increases with δ and decreases with x on the displayed rectangle. At $(\delta, x) = (7/20, 1/2)$ it equals $24683/162000$. Finally, $\theta_0 = 2\delta x/9 \le 1/15$, so

$$S \le 1 + \frac{1}{30} - \frac{17}{50} = \frac{52}{75}.$$

□

The sufficient restriction $\kappa \ge 7497/10000$ in the earlier numerator comparison is thus not imposed on κ_c. The numerator moment, initial prime product and contour parameter continue to use $\kappa_{\mathrm{m}} = 3/4 > 7497/10000$. The class count uses its full proved interval, while (6.2)– (6.3) verify these two compatibility conditions directly on the class rectangle where the new estimate is applied. In Section 10, where $\kappa_m < 7497/10000$, the uses of this restriction are replaced as in Remark 10.4(b). The complementary amplitude-class and parameter-range comparisons are treated separately in Section 8.

7 Adding the moment and contour savings

In this section, we apply the norm of Section 4 on the contours of Section 5. A prime-product length chosen separately for each dyadic sum gives a common saving over all ordinary lengths; Section 8 uses that saving in the full comparison.

7.1 Combining the moment bound and contour displacement

In this subsection, we apply the selected moment on the displaced contour and minimize the resulting gain over ordinary lengths. Section 8.2 uses this common gain.

Theorem 4.9 is uniform over the independent AFE weights on every contour of Lemma 5.2. That lemma moves the whole integrand before absolute values. The selected-moment saving therefore remains available on the displaced contour with the same initial prime product and original coefficients.

Lemma 7.1 (The combined exponent on a dyadic sum)**.** *Retain Theorem 4.9, the class-count exponent R, the initial prime product Q_0, and a disjoint product with selected first-moment saving $\gamma > 0$. Put*

$$g = \delta x, \quad \kappa_m = 3/4, \quad \theta_0 = \frac{g}{6\kappa_m} = \frac{2\delta x}{9}, \quad z_0(m) = \frac{1-2m}{6\kappa_m}, \quad \mathcal{K} = \frac{3R+1}{4}.$$

For a dyadic sum of length $D = U^m$, $m < 1/2$, and every allowed shift $0 \le \theta \le \delta/2$, the full comparison exponent is

$$\mathcal{E}_0(d) + (\theta - \theta_0)\{d(1-m) - l_y\} - d\gamma, \tag{7.1}$$

where $\mathcal{E}_0(d)$ is the unamplified reference exponent. This holds for both halves of the approximate functional equation, with the actual conductors, fixed-place factors and simultaneous strict-error parameters retained. Neither the choice nor the length constraint of Q_0 changes.

Proof. Lemmas 5.1–5.2 keep the two AFE weights and their independent derivatives in the norm family of Theorem 4.9. Fix the original amplitude class, initial prime product and extra product before differentiating these weights. Their lower bounds are independent of the contour displacement.

The selected fourth moment saves $2gz_0 + 4\gamma$ in its exponent. Hölder with the class count gives the first-moment exponent

$$\begin{aligned}\frac{3R}{4} + \frac{1 - 2gz_0 - 4\gamma}{4} - \theta m &= \mathcal{K} - \frac{gz_0}{2} - \gamma - \theta m \\ &= \mathcal{K} - \frac{\theta_0}{2} - (\theta - \theta_0)m - \gamma, \qquad \frac{gz_0}{2} = \frac{\theta_0(1-2m)}{2}.\end{aligned}$$

Arbitrarily small exponent losses and the fixed seminorm and height factors are understood. For $U = Z^d$, $D = U^m$ and $Y = Z^{l_y}$, the functional equation, AFE coefficients and outer Mellin factor therefore give the relative factor

$$U^{\theta-\theta_0} D^{-(\theta-\theta_0)} Y^{-(\theta-\theta_0)} U^{-\gamma} = Z^{(\theta-\theta_0)\{d(1-m)-l_y\}-d\gamma}.$$

This proves (7.1) for the first half.

Move the holomorphic dual expression before whole-dual conjugation. Lemma 5.1 gives the same $D^{-\theta}$ and the character of the original prime coefficients. Retain the actual conductor and one simultaneous strict-error tuple. Its finite prime set $\mathcal{P}_{\mathrm{str}}$ contributes

$$\prod_{\mathfrak{p}\in\mathcal{P}_{\mathrm{str}}} \left(q_{\mathfrak{p}}^{\theta} q_{\mathfrak{p}}^{-(j_{\mathfrak{p}}-1)\theta}\right) = \prod_{\mathfrak{p}\in\mathcal{P}_{\mathrm{str}}} q_{\mathfrak{p}}^{-(j_{\mathfrak{p}}-2)\theta} \le 1 \qquad (j_{\mathfrak{p}} \ge 2,\ \theta \ge 0).$$

Fixed-place factors have finite absolute mass on the enlarged strip. All contour displacements precede the local s-contour displacement, absolute values and conductor separation. The same positive norm controls the later derivatives; no amplitude lower bound is required of a differentiated weight. □

Lemma 7.2 (An ideal-counting bound for short sums)**.** *Suppose $\kappa_c \le 3/4$, so that $R = R_{\kappa_c}$ is bounded by the count of Lemma 6.1. On $3/10 \le \delta \le 3/5$, $2/5 \le x \le 1/2$, put $N = (3R+1)/4 - \theta_0/2$ and $m_{\mathrm{cut}} = 3/40$. For $m \le m_{\mathrm{cut}}$, the ideal-count exponent of the reference estimate is at most*

$$R + (1/2 - \theta_0)m_{\mathrm{cut}} \le N - \frac{23}{2700}.$$

For $m \ge m_{\mathrm{cut}}$, the initial prime product required by the unamplified M2 estimate has length at most $z_0(m_{\mathrm{cut}}) = 17/90 < 21/100$.

Proof. The ideal-count expression increases with m, since $\theta_0 < 1/2$. The sharp count bound gives

$$\begin{aligned} N - \{R + (1/2 - \theta_0)m_{\mathrm{cut}}\} &= \frac{1-R}{4} - \frac{3}{80} - \frac{17\theta_0}{40} \\ &\ge \frac{\delta}{6} - \frac{3}{80} - \frac{43\delta x}{1620} \\ &\ge \frac{497\delta}{3240} - \frac{3}{80} \ge \frac{23}{2700}. \end{aligned}$$

Also

$$z_0(3/40) = \frac{2}{9}\left(1 - \frac{6}{40}\right) = \frac{17}{90}.$$

At exhaustion, the earlier selection rule and $z_0 \le \mathscr{L}$ give total original gain at least $g\mathscr{L} \ge gz_0$. □

Lemma 7.3 (A combined bound with variable prime-product length)**.** *Retain the balanced lengths and assume*

$$\frac{57}{100} \le \lambda = \frac{l_y}{h} \le \frac{59}{100}, \qquad \frac{3}{10} \le \delta \le \frac{3}{5}, \qquad \frac{2}{5} \le x \le \frac{1}{2}.$$

At every scale under consideration, assume

$$\mathscr{L} = \ell/d \ge 21/100.$$

Put $m_\theta = 1 - \lambda$, $m^\sharp = 25/62$, and

$$W_{\mathrm{cap}}(m) = \min\left\{\frac{7}{54}(1-2m), \frac{8m-3}{9}\right\}, \qquad 3/8 \le m \le 1/2.$$

The remaining positive-amplitude factors have mean at least $a = \delta(189x - 25)/139$. Using $\theta = 0$ for $m \le m_\theta$ and $\theta = \delta/2$ for $m \ge m_\theta$, the limiting gain divided by d at $d = h$ is

$$\mathcal{F}(m) = \frac{aW_{\mathrm{cap}}(m)}{2} + \begin{cases} \theta_0(m_\theta - m), & m \le m_\theta, \\ (\delta/2 - \theta_0)(m - m_\theta), & m \ge m_\theta. \end{cases}$$

Consequently,

$$\begin{aligned} \Gamma(\delta, x) &= \min_{3/8 \le m \le 1/2} \mathcal{F}(m) = \min\{\Gamma_L, \Gamma_N\}, \\ \Gamma_L &= \theta_0(5/8 - \lambda), \qquad \Gamma_N = \frac{7a(2\lambda - 1)}{108}. \end{aligned} \tag{7.2}$$

Here $\Gamma_L = \mathcal{F}(3/8)$ *and* $\Gamma_N = \mathcal{F}(m_\theta)$ *are the two candidate minimum values. Moreover,*

$$\lambda \le 11/19 \quad \Longrightarrow \quad \Gamma_L \ge \Gamma_N, \qquad \Gamma = \Gamma_N.$$

At $\lambda = 11/19$,

$$b = 13\ell/2 - 1, \qquad h = 19\ell/4, \qquad l_y = 11\ell/4,$$
$$\Gamma_N = \frac{7\delta(189x - 25)}{95076}, \qquad \Gamma_N\big|_{x=1/2} = 7\delta/1368.$$

These are limiting lengths and gains; the norm theorem is applied only with the strict margins of Proposition 7.4.

Proof. The largest initial prime product at $m = 3/8$ has width $1/18$. Lemma 4.2 with $L_0 = 21/100$ therefore gives

$$a \ge \delta \frac{(21/100)x - 1/36}{21/100 - 1/18} = \delta \frac{189x - 25}{139}.$$

The limiting width uses the branch for restoring the comparison product below $m^\sharp$ and the branch determined by the coupled length constraint above it. One has $3/8 < m^\sharp < m_\theta < 1/2$. The three successive slopes of $\mathcal{F}$ are

$$\frac{4a}{9} - \theta_0 > 0, \qquad -\frac{7a}{54} - \theta_0 < 0, \qquad \frac{\delta}{2} - \theta_0 - \frac{7a}{54} > 0.$$

The first sign follows from $239x - 50 > 0$; the last holds uniformly on the stated x interval, and the middle is immediate. Thus the only possible minimum points are $m = 3/8$ and $m = m_\theta$. Their gains are respectively

$$\Gamma_L = \theta_0(m_\theta - 3/8), \qquad \Gamma_N = \frac{7a(1 - 2m_\theta)}{108}.$$

These are (7.2). When $\lambda \le 11/19$, put $y = 1/2 - x \in [0, 1/10]$ and $z = 11/19 - \lambda \ge 0$. The exact identity

$$\frac{\Gamma_L - \Gamma_N}{\delta} = \frac{175y}{47538} + z\left(\frac{19}{108} - \frac{997y}{2502}\right) \ge 0$$

gives the stronger assertion. The right endpoint $m = 1/2$ cannot be smaller, by the last positive slope. □

Proposition 7.4 (Allowed lengths and contours with a uniform saving)**.** *Under Lemma 7.3, suppose*

$$E(\delta, x) - h\Gamma(\delta, x) \le -\Delta_{\mathrm{cmp}} < 0$$

uniformly on the stated class rectangle. Retain strict margins for the complementary comparisons, geometric inequalities, short-sum estimate and earlier parameter ranges, and decrease Δ_{cmp} *if necessary to lie inside all these margins. Choose* $\zeta > 0$ *also so that*

$$\ell/(h + \zeta) \ge 21/100.$$

Then allowed prime-product lengths and contours give

$$\mathcal{E}(d) \le -\Delta_{\mathrm{cmp}} + C\varepsilon + 2\zeta + \epsilon_{\mathrm{rem}} < 0 \qquad (0 < d \le h + \zeta),$$

after choosing ε, ζ *and the remaining losses with* $C\varepsilon + 2\zeta + \epsilon_{\mathrm{rem}} < \Delta_{\mathrm{cmp}}$*. The coefficient families and order of analytic operations are unchanged.*

Proof. Write $E - h\Gamma \le -\Delta_{\mathrm{cmp}}$ for the fixed comparison margin. Choose $0 < \varepsilon < 1/1000$ and set

$$m_L = 3/8 + \varepsilon, \qquad W_\varepsilon(m) = (1-\varepsilon)W_{\mathrm{cap}}(m), \qquad m_L \le m \le 9/20.$$

Apply the norm with $m_- = r = m$. Since $W_{\mathrm{cap}}(m) \ge 8\varepsilon/9$,

$$\frac{7}{9}(1-2m) - 6W_\varepsilon(m) \ge 6\varepsilon W_{\mathrm{cap}}(m) \ge 16\varepsilon^2/3,$$
$$1 - \left(\frac{5}{2} - 4m + \frac{9}{2}W_\varepsilon(m)\right) \ge \frac{9}{2}\varepsilon W_{\mathrm{cap}}(m) \ge 4\varepsilon^2.$$

Every selected product fits, because

$$W_{\mathrm{cap}}(m) \le 7/279 < 2(2/5)(21/100) - 1/18.$$

Use

$m < 3/40$	ideal counting
$3/40 \le m < m_L$	central unamplified M2
$m_L \le m \le 9/20$	selected norm on retained pieces
$m > 9/20$	outer unamplified M2.

The selection and lower-edge losses satisfy

$$\Delta_{\mathrm{sel}} \le a\varepsilon W_{\mathrm{cap}}/2, \qquad \Delta_{\mathrm{edge}} \le \theta_0\varepsilon.$$

At the upper endpoint the excess over Γ_N is at least $7\delta/2160$: with $y = 1/2 - x \ge 0$, $z = \lambda - 57/100 \ge 0$,

$$\begin{aligned}(1/2 - 2x/9)(\lambda - 11/20) - \frac{7(189x-25)(2\lambda-1)}{108\cdot 139} - \frac{7}{2160} \\ = \frac{997}{2502}yz + \frac{4199}{250200}y + \frac{35}{108}z \ge 0.\end{aligned} \tag{7.3}$$

This holds on the full stated rectangle and λ interval.

Lemma 7.1 applies on both sides of m_θ. For each fixed selected dyadic sum, $S > 1/5$ and $\gamma < 1/250$ give

$$S - \theta_0(1-m) - \gamma > \frac{1}{5} - \frac{1}{15}\frac{5}{8} - \frac{1}{250} > \frac{3}{20},$$
$$S + (\delta/2 - \theta_0)(1-m) - \gamma > \frac{1}{5} - \frac{1}{250} > \frac{19}{100}.$$

Both slopes are below 2, so

$$d \le h + \zeta \quad \Longrightarrow \quad \mathcal{E}(d) \le \mathcal{E}(h) + 2\zeta.$$

At m_θ apply this separately to both contours; their slopes need not match. The unamplified ranges use their finite endpoint bounds, the dominating affine extension and normally convergent tails. For $d < d_{\min}$, use the small-parameter estimate recalled in Section 2.

All losses can be chosen to satisfy

$$C\varepsilon + 2\zeta + \epsilon_{\mathrm{rem}} < \Delta_{\mathrm{cmp}},$$

where C depends on the fixed compact data and ϵ_{rem} collects the remaining positive exponent losses, including the local loss of Proposition 4.8. Choose the step, mesh, finite allocation bound and weighted orders before later Fourier labels and external heights. Original prime lower bounds precede differentiation. Count the logarithmically many dyadic sums inside the prescribed loss and remove tails in the original order. This proves the full parameter-dependent estimate without continuity across a fixed lower band in d. □

8 The algebraic boundary and the successive count bounds

In this section, we first obtain a strict rational bound from the earlier 7/8 result. We then prove the algebraic endpoint by successive count improvements, and determine the optimum of the comparison system. Section 9 gives the resulting theorem over arbitrary fields.

8.1 An explicit rational bound

In this subsection, we obtain the rational bound that starts the successive count improvements in Section 8.2.

Here the ideal-count cutoff is $m_{\mathrm{seed}} = 1/10$. The final comparison of Section 7.1 uses the smaller cutoff $m_{\mathrm{cut}} = 3/40$ in order to obtain its uniform 23/2700 margin on the larger class rectangle. Both fit the available prime length:

$$z_0(1/10) = 8/45 < 21/100, \qquad z_0(3/40) = 17/90 < 21/100.$$

Appendix A verifies their separate complete-sum inequalities.

Before passing to the algebraic limit, we give an explicit interior example. It splits according to the direct-estimate range and uses the original theta detector. Keep $\kappa_c = \kappa_m = 3/4$ and set

$$B_{\mathrm{seed}} = \frac{437279}{500000} = 0.874558, \quad b = \frac{167}{2500}, \quad \ell = \frac{63163}{375000}, \quad h = \frac{196513}{250000}, \quad l_y = \frac{336887}{750000},$$
$$m_L = \frac{393}{1000}, \quad m_R = \frac{1097}{2500}, \quad W = \frac{317}{20000}, \quad \varepsilon_c = 10^{-4}, \quad \varepsilon_{\mathrm{amp}} = 10^{-6}, \quad \varepsilon_{\mathrm{cmp}} = 10^{-7}.$$

The actual norm length conditions are

$$\frac{7}{9}(1 - 2m_R) - 6W = \frac{1}{10000}, \qquad 1 - \left(\frac{5}{2} - 4m_L + \frac{9}{2}W\right) = \frac{27}{40000}.$$

Here

$$1 - 2m_R - 9W = -81/4000 < 0.$$

Theorem 4.9 therefore uses the direct and complementary Poisson estimates, with the actual outer D retained, in place of that earlier sufficient length condition.

Use

$$\mathcal{R} = [7/20, 3/5] \times [9/20, 1/2], \qquad \theta_0 = 2\delta x/9, \qquad \theta_- = \varepsilon_c, \quad \theta_+ = \delta/2 - \varepsilon_c.$$

Put $t_- = \theta_0 - \varepsilon_c$ and $t_+ = \delta/2 - \varepsilon_c - \theta_0$. With $z_L = 2(1 - 2m_L)/9$, put

$$a = \delta\left\{\frac{1}{2} - \frac{(1/2 - x)(21/100)}{21/100 - z_L}\right\}, \qquad \gamma = aW/2 - \varepsilon_{\mathrm{amp}}.$$

Appendix A proves the strict rational inequalities for these parameters. They include available length of the additional prime product and restoration of the comparison product; the complete left, middle and right comparisons on $\mathcal{R}$; the earlier comparison on its three complementary rectangles; both endpoints of every near-h affine comparison exponent; the short ideal-count comparison; and all remaining comparison geometry. The lowest-amplitude margin 1/100 and the small-norm margin 1/20 hold, and the small-amplitude calculation keeps the threshold 1/250.

Theorem 8.1 (An explicit boundary from the enlarged norm)**.** *The finite-order Hecke family over the fixed field F satisfies*

$$\beta_*(F) \le \frac{437279}{500000} = 0.874558.$$

Proof. Theorem 4.9 and Lemma 5.2 supply the analytic estimates. The completed Gauss sums are covered by

$$\begin{cases} m_{\mathrm{f}} \le 1/100: & \text{Proposition 4.6,} \\ m_{\mathrm{f}} > 1/100: & \text{Proposition 4.8.} \end{cases}$$

Take $\zeta = 10^{-8}$ as in Appendix A.2. For these parameters,

$$0 < \zeta \le 1/2000 \quad \Longrightarrow \quad \frac{\ell}{h+\zeta} \ge \frac{63163}{294957} > \frac{21}{100}.$$

Thus Lemma 4.2 applies with $L_0 = 21/100$. Use ideal counting for $m \le 1/10$, unamplified M2 on the two outer ranges, and the selected norm on $[m_L, m_R]$. Write $\mathcal{E}_{\mathrm{chosen}}$ for the full corrected exponent. The exact checks of Appendix A give

$$\mathcal{E}_{\mathrm{chosen}}(h) < -\varepsilon_{\mathrm{cmp}}, \qquad 0 < \partial_d \mathcal{E}_{\mathrm{chosen}} < 2, \qquad S > 1/5,$$
$$\frac{1}{5 \cdot 80} - \frac{1}{500} > \varepsilon_{\mathrm{cmp}}, \qquad 2\zeta < \varepsilon_{\mathrm{cmp}}/2.$$

Hence the two endpoints of the near-h band and $d \le h - 1/80$ have strict negative exponents. Apply prime lower bounds before derivatives, retaining both actual-conductor AFE halves and strict tuples. Choose all subsidiary losses within these strict margins. The earlier direct estimate and nonvanishing principal correction then give

$$\beta_*(F) \le B(\ell) = B_{\mathrm{seed}}.$$

The 7/8 bound justifies both moment parameters used in this interior construction. □

8.2 The algebraic endpoint

In this subsection, we determine the algebraic limiting parameters and prove the resulting bound. The field descent is given in Section 9.

The amplified norm and contour displacement apply to the same ordinary dyadic sum. We insert their sum in the full comparison of Section 2.

Definition 8.2 (The combined algebraic endpoint)**.** Let r_* be the unique root in $(9/20, 23/50)$ of

$$218582r^3 - 376469r^2 - 11376r + 61560 = 0.$$

Put

$$\ell_* = \frac{3410r_*^2 - 7127r_* + 2520}{2286r_*^2 - 5011r_* + 1800}, \qquad \kappa_* = \frac{5}{6} - \frac{\ell_*}{2}, \qquad B_* = \frac{11}{12} - \frac{\ell_*}{4}.$$

To define b_*, use independent variables r, k and

$$\begin{aligned} n_b &= -6858kr^2 + 15033kr - 5400k + 600r^2 - 1837r + 720, \\ n_e &= -702kr^2 + 34995kr - 40500k - 408r^2 - 3775r + 5400, \\ n_0 &= 4806kr^2 - 10887kr + 8100k - 696r^2 + 1403r - 1080. \end{aligned}$$

Set

$$b_* = -\frac{\ell_* \partial_r n_e(r_*, \kappa_*) + \partial_r n_0(r_*, \kappa_*)}{\partial_r n_b(r_*, \kappa_*)},$$

where the partial derivatives hold k fixed. Numerically,

$$\begin{aligned} r_* &= 0.45057074672406073136\ldots, \\ \ell_* &= 0.16879343589254482845\ldots, \\ b_* &= 0.09179235229895088958\ldots, \\ \kappa_* &= 0.74893661538706091910\ldots, \\ B_* &= 0.87446830769353045955\ldots. \end{aligned}$$

In particular $\kappa_* = 2B_* - 1$ exactly.

The parameters also admit the explicit formulas

$$\begin{aligned} b_* &= \frac{\ell_*^2(11665-468r_*)+\ell_*(4528r_*-20554)+5113-4412r_*}{\ell_*(4572r_*-5011)+7127-6820r_*}, \\ \kappa_* &= \frac{5}{6}-\frac{\ell_*}{2}=2B_*-1. \end{aligned} \tag{8.1}$$

Here ℓ_* is the prime-product length parameter, b_* balances l_y and h (see below), and κ_* is the limiting class-count parameter. In Sections 2–9, $\kappa_m = 3/4$ is the parameter in the numerator moment estimate; κ_c denotes the independent class-count parameter. The parameter r_* is the critical value of the detector coordinate δ in the class (δ, x) defined in Section 1.4.

A closed form for the root is obtained by putting $\Delta = 149188674457$:

$$r_* = \frac{376469 + 2\sqrt{\Delta}\cos\left(\frac{1}{3}\arccos\frac{17862636423556205}{\Delta^{3/2}} - \frac{2\pi}{3}\right)}{655746}, \tag{8.2}$$

where arccos takes values in $[0, \pi]$. The triple-angle identity gives (8.2), and

$$\ell_*, b_*, \kappa_*, B_* \in \mathbb{Q}(r_*), \qquad [\mathbb{Q}(r_*) : \mathbb{Q}] = 3.$$

The derivative formula for b_* gives (8.1).

Lemma 8.3 (The comparison inequality and its strict margins)**.** *Put*

$$\mathcal{Q}_{\kappa,b,\ell}(\delta, x) = E_\kappa(\delta, x) - \frac{7}{108}a(\delta, x)(2l_y - h), \qquad \mathcal{R}_* = [7/20, 3/5] \times [2/5, 1/2].$$

For the parameters of Definition 8.2,

$$\begin{aligned} \mathcal{Q}_{\kappa_*,b_*,\ell_*} &\le 0 \quad \text{on } \mathcal{R}_*, \\ \mathcal{Q}_{\kappa_*,b_*,\ell_*}(\delta, x) = 0 \quad &\Longleftrightarrow \quad (\delta, x) = (r_*, 1/2). \end{aligned}$$

Uniformly on $\mathcal{R}_*$ and

$$\begin{gathered} 0.7489 \le \kappa \le 0.7495, \qquad 0.16876 \le \ell \le 0.16880, \\ 0.0917923522 \le b \le 0.0917923524, \end{gathered}$$

one has

$$\partial_\ell \mathcal{Q} > 3/4, \qquad 0 \le \partial_\kappa \mathcal{Q} < 1/100.$$

The remaining geometric, complementary-class, ordinary-sum and short-sum comparisons have uniform strict margins on this family.

Proof. Write $\mathcal{R}_* = [7/20, 3/5] \times [2/5, 1/2]$ and use the intervals I_b, I_ℓ, I_κ of Appendix A.3. At $x = 1/2$,

$$\mathcal{Q}_{k,b,\ell}(\delta, 1/2) = \frac{bn_b(\delta, k) + \ell n_e(\delta, k) + n_0(\delta, k)}{432(54k\delta - 75k - 6\delta + 10)}.$$

Polynomial division by (1.2) gives

$$n_b(r_*, \kappa_*) = 0, \qquad \ell_* n_e(r_*, \kappa_*) + n_0(r_*, \kappa_*) = 0.$$

Put $N_*(\delta) = b_* n_b(\delta, \kappa_*) + \ell_* n_e(\delta, \kappa_*) + n_0(\delta, \kappa_*)$. Definition 8.2 and the exact interval signs give

$$\begin{gathered} N_*(r_*) = N_*'(r_*) = 0, \qquad N_*(\delta) = A_{\mathcal{Q}}(\delta - r_*)^2, \\ A_{\mathcal{Q}} = (-6858\kappa_* + 600)b_* + (-702\kappa_* - 408)\ell_* \\ + 4806\kappa_* - 696 > 0, \\ 54k\delta - 75k - 6\delta + 10 < 0 \quad (\delta \in [7/20, 3/5],\ k \in I_\kappa). \end{gathered}$$

Hence $\mathcal{Q}_*(\delta, 1/2) \le 0$, with equality only at $\delta = r_*$. The exact root isolation is

$$0.450570746724060730 < r_* < 0.450570746724060733,$$
$$0.87446830769352 < B_* < 0.87446830769355.$$

Appendix A proves the following rational signs:

quantity	parameter domain
$\partial_x \mathcal{Q} > 0$	$\mathcal{R}_* \times [0.7489366153, 0.7489366155] \times I_b \times I_\ell$
$\partial_\ell \mathcal{Q} > 3/4$	$\mathcal{R}_* \times \{.7489\} \times I_b \times I_\ell$.

Here each cleared numerator is affine in (b, ℓ); its four corners suffice. Exact Bernstein coefficients and rational subdivisions retain the full κ coordinate. In particular,

$$x < 1/2 \quad \Longrightarrow \quad \mathcal{Q}_*(\delta, x) < \mathcal{Q}_*(\delta, 1/2) \le 0.$$

For the larger count interval,

$$\partial_\kappa \partial_\ell \mathcal{Q} = \frac{9}{8} \partial_\kappa R \ge 0 \quad \Longrightarrow \quad \partial_\ell \mathcal{Q} > 3/4,$$

and

$$0 \le \partial_\kappa \mathcal{Q} = \frac{h(5/6 - \delta)^2 \delta x (1 - x)^2}{4\kappa^2 J^2} < \frac{25600}{2686321} < \frac{1}{100},$$
$$h \le \frac{1 + 0.0917923524 + 3(0.16880)}{2} < \frac{4}{5}.$$

With $\mathcal{G} = [1/8, 5/6] \times [0, 1/2]$, the complementary rectangles are

$$\mathcal{C}_1 = [1/8, 5/6] \times [0, 2/5],$$
$$\mathcal{C}_2 = [1/8, 7/20] \times [2/5, 1/2],$$
$$\mathcal{C}_3 = [3/5, 5/6] \times [2/5, 1/2].$$

The same exact tests, at the adverse count endpoints, give

$\mathcal{G}$		$E < 1/500$
$\bigcup_i \mathcal{C}_i$		$E < -10^{-5}$
$\mathcal{R}_*$	$S > 1/5,$	$E + h\Delta_{\text{short}} < -10^{-5}$.

The ratio $\ell/(h+\zeta) = 2\ell/(1+b+3\ell+2\zeta)$ increases with ℓ and decreases with b, ζ. On $I_b \times I_\ell$,

$$0 < \zeta \le 1/2000 \quad \Longrightarrow \quad \frac{\ell}{h + \zeta} \ge \frac{843800000}{3997680881} > \frac{21}{100}.$$

This verifies the prime-length hypothesis of Lemma 7.3 and Proposition 7.4 throughout the parameter family. The ordinary-gain and prime-length conditions are

$$a > 0, \quad 4a/9 - \theta_0 > 0, \quad \delta/2 - \theta_0 - 7a/54 > 0, \quad \Gamma_L > \Gamma_N,$$
$$2x(21/100) - 1/18 > 7/279, \qquad 57/100 < l_y/h < 59/100, \quad l_y/h < 11/19.$$

For $m_{\text{cut}} = 3/40$,

$$N - \{R + (1/2 - \theta_0) m_{\text{cut}}\} \ge \delta(1/6 - 43x/1620) - 3/80 \ge 23/2700, \qquad z_0(m_{\text{cut}}) = 17/90 < 21/100.$$

The last lower bound holds already for $\delta \ge 3/10$. The remaining geometric, principal and small-amplitude conditions are the explicit inequalities of Appendix A, using the $1/125$ threshold of Lemma 6.3. Their analytic application is Lemma 7.1 and Proposition 7.4. □

Theorem 8.4 (The algebraic zero-free boundary)**.** *Retain the established inputs of Section 2, the amplified norm of Theorem 4.9 and the contours of Lemma 5.2. The* 7/8 *bound and Theorem 8.1 give*

$$\beta_*(F) \le B_{\mathrm{seed}} = 0.874558 \quad \Longrightarrow \quad \boxed{\beta_*(F) \le B_* = 0.87446830769353045955\ldots.}$$

Proof. Put $\kappa_0 = 187279/250000 = 2B_{\mathrm{seed}} - 1$ and fix $b = b_*$. For $\kappa \in (\kappa_*, \kappa_0]$, set

$$\ell(\kappa) = \ell_* - \frac{\kappa - \kappa_*}{25}.$$

This segment lies in the family of Lemma 8.3, and

$$\frac{d}{d\kappa}\mathcal{Q}_{\kappa,b_*,\ell(\kappa)} = \partial_\kappa \mathcal{Q} - \frac{1}{25}\partial_\ell \mathcal{Q} < -\frac{1}{100}.$$

Together with $\mathcal{Q}_{\kappa_*,b_*,\ell_*} \le 0$, this gives, uniformly on the complete class rectangle,

$$\mathcal{Q}_{\kappa,b_*,\ell(\kappa)} < -\nu_\kappa, \qquad \nu_\kappa = \frac{\kappa - \kappa_*}{100} > 0.$$

Let ϵ_{rem} collect the remaining positive exponent losses. Choose the losses, in the order of Proposition 7.4, so that

$$0 < \varepsilon < 1/1000, \qquad C\varepsilon + 2\zeta + \epsilon_{\mathrm{rem}} < \nu_\kappa.$$

Put $m_a = 3/8 + \varepsilon$, $m_b = 9/20$. Here C depends only on the fixed compact data. The choice of lengths and contours in Proposition 7.4 uses

ordinary length exponent	estimate
$m < 3/40$	ideal counting
$3/40 \le m < m_a$	unamplified M2 on the central contour
$m_a \le m \le m_b$	selected moment and contour saving
$m > m_b$	unamplified M2 on the outer contour.

On the selected interval take $m_- = r = m$ and $W(m) = (1-\varepsilon)W_{\mathrm{cap}}(m)$. Then

$$\frac{7}{9}(1-2m) - 6W(m) \ge \frac{7}{90}\varepsilon > 0, \qquad 1 - \left(\frac{5}{2} - 4m + \frac{9}{2}W(m)\right) \ge 4\varepsilon^2 > 0.$$

The ideal gain changes by $O(\varepsilon)$; the outer range has the strict margin of Proposition 7.4. On each selected dyadic sum, the slopes of the full comparison exponent satisfy

$$S - \theta_0(1-m) - \gamma > \frac{1}{5} - \frac{1}{15}\frac{37}{40} - \frac{3}{10}\frac{7}{279}\frac{1}{2} > \frac{1}{8},$$
$$S + (\delta/2 - \theta_0)(1-m) - \gamma > \frac{1}{5} - \frac{3}{10}\frac{7}{279}\frac{1}{2} > \frac{1}{8}.$$

Both slopes are below 2. Hence the extension to $d \le h + \zeta$ loses at most 2ζ. Choose ε, then the mesh, smooth-weight, height and ζ losses inside ν_κ. Apply the norm before differentiation and retain both actual-conductor AFE halves. Proposition 7.4 handles the complementary length ranges and tails; the earlier estimates apply for $d \le h - 1/80$. The lowest-amplitude, principal and direct comparisons retain their checked hypotheses.

The continuation argument now gives

$$\beta_*(F) \le B(\kappa) = B_* + \frac{\kappa - \kappa_*}{100}, \qquad 2B(\kappa) - 1 - \kappa_* = \frac{\kappa - \kappa_*}{50}.$$

Starting from $B_0 = B_{\mathrm{seed}}$ and setting $\kappa_n = 2B_n - 1$,

$$\beta_*(F) \le B_n \implies \beta_*(F) \le B_{n+1}, \qquad B_n - B_* = 50^{-n}(B_{\mathrm{seed}} - B_*) \longrightarrow 0.$$

Every finite step has strictly positive margins and its own threshold, while $\kappa_m = 3/4$ remains justified by 7/8. Taking the infimum over n proves the theorem. □

Theorem 8.5 (Optimality under the coupled norm length constraints)**.** *Fix the detector, class count, balanced-length hypotheses and direct theta estimate used above, and the norm of Theorem 4.9. On each amplitude class, allow dyadic-sum-dependent prime lengths and contours satisfying*

$$\kappa_m = 3/4, \qquad 0 \le W(m) \le \frac{7}{54}(1-2m), \qquad 0 \le \theta(m) \le \delta/2,$$
$$1/6 \le \ell \le 1/5, \qquad B(\ell) = 11/12 - \ell/4.$$

Every allowed sequence of count improvements in this comparison system, starting at $B_0 = B_{\mathrm{seed}}$*, with*

$$13/18 \le \kappa_n \le 3/4, \qquad \kappa_n \ge 2B_n - 1,$$

satisfies

$$B_n \ge B_* \quad (n \ge 0), \qquad \liminf_{n\to\infty} B_n \ge B_*.$$

The sequence of Theorem 8.4 has $B_n - B_* = 50^{-n}(B_{\mathrm{seed}} - B_*) \to 0$*. This optimality keeps the detector, moment estimates, moment parameter and allowed norm lengths fixed.*

Proof. Consider a first step below B_*. Its preceding bound $X \ge B_*$ gives

$$\kappa \ge 2X - 1 \ge \kappa_*, \qquad \ell > \ell_*.$$

The retained geometry gives

$$0 < b < \frac{3(1-3\ell)}{8} \le \frac{3}{16}, \qquad m_\theta = \frac{4\ell}{1+b+3\ell} > \frac{32}{81} > \frac{3}{8}, \qquad \frac{1}{2} - m_\theta = \frac{1+b-5\ell}{2(1+b+3\ell)} > 0.$$

At the critical class $(\delta, x) = (r_*, 1/2)$,

$$h(1-m_\theta) - l_y = 0, \qquad \gamma \le \frac{\delta W}{4}, \qquad h\gamma \le \frac{7\delta}{216}(2l_y - h).$$

Put $r = r_*$, $R = R_\kappa(r_*, 1/2)$ and $R_* = R_{\kappa_*}(r_*, 1/2)$. The buffered pointwise estimate and the optimistic amplified fourth-moment estimate have exponents

$$P_{\mathrm{pt}}(m; R) = R + \frac{rm}{2}, \qquad P_4(m; R, A) = \frac{3R+1}{4} - Ar(1-2m), \qquad A = A_0 := \frac{19}{216}. \tag{8.3}$$

Count monotonicity gives $R \ge R_*$. Therefore

$$P_{\mathrm{pt}}(m; R) - P_4(m; R, A_0) \ge \frac{35r_*}{108}(m - \mu_*), \qquad \mu_* := \frac{54(1-R_*) - 19r_*}{70r_*}.$$

The retained geometry also gives

$$m_\theta > g(\ell) := \frac{32\ell}{11+15\ell}, \qquad g'(\ell) = \frac{352}{(11+15\ell)^2} > 0. \tag{8.4}$$

The rational bounds (A.6) yield

$$\mu_* < \frac{399035}{10^6} < \frac{399159}{10^6} < g(\ell_*) < m_\theta.$$

Thus $P_{\mathrm{pt}}(m_\theta; R) > P_4(m_\theta; R, A_0)$. Absolute ideal counting gives $R + m_\theta/2 \ge P_{\mathrm{pt}}(m_\theta; R)$, and the unamplified moment omits the nonnegative selected saving. At m_θ the full contour term vanishes. Hence the buffered pointwise estimate, absolute ideal counting and the unamplified moment do not improve this optimistic amplified comparison at m_θ, and $\mathcal{Q}_{\kappa,b,\ell}(r_*, 1/2) \le 0$ is

necessary. By the proof of Lemma 8.3, $n_b(r_*,\kappa_*)=0$ and $\ell_* n_e(r_*,\kappa_*)+n_0(r_*,\kappa_*)=0$. Hence the coefficient of b vanishes at $\kappa=\kappa_*$, and

$$\mathcal{Q}_{\kappa_*,b,\ell}(r_*,1/2)=e_*(\ell-\ell_*),\qquad e_*=\frac{n_e(r_*,\kappa_*)}{432L_*},\quad L_*=54\kappa_* r_*-75\kappa_*-6r_*+10.$$

On $[4505,4506]\times[7489,7490]\cdot 10^{-4}\ni(r_*,\kappa_*)$, termwise bounds give $n_e<-15000$ and $-31<L_*<-30$. Thus

$$e_*>\frac{15000}{432\cdot 31}=\frac{15000}{13392}>1,\qquad e_*=1.13381952498716284524\ldots.$$

Since $\partial_\kappa\mathcal{Q}=(3h/4)\partial_\kappa R\geq 0$,

$$\kappa\geq\kappa_*,\quad \ell>\ell_*\quad\Longrightarrow\quad \mathcal{Q}_{\kappa,b,\ell}(r_*,1/2)>0.$$

This contradicts the necessary condition $\mathcal{Q}_{\kappa,b,\ell}(r_*,1/2)\leq 0$. Hence no step goes below B_*, and every finite iterate and its lower limit are at least B_*. Theorem 8.4 attains this infimum. □

9 Arbitrary number fields and the boundary B_*

In this section, we verify that the new estimates apply over every fixed field containing μ_{12}. We then descend from $K(\zeta_{12})$ to K and prove the boundary B_* for every number field. The estimates are checked in Section 9.1, and the descent is given in Section 9.2.

9.1 The estimates over fields containing the twelfth roots of unity

In this subsection, we check the norm and contour estimates of Sections 3–7 over a fixed field containing μ_{12}.

The general-field arithmetic is supplied by [Liu26b, Part III]. We verify that every new estimate above uses only that arithmetic and fixed-field estimates.

Proposition 9.1 (The combined comparison over number fields)**.** *For every fixed $F\supset\mu_{12}$, the norm, smooth-weight, contour and combined-comparison estimates used in Theorem 8.4 hold with the same numerical exponents and length inequalities. Consequently,*

$$\beta_*(F)\leq B_*=0.87446830769353045955\ldots,$$

with B_* as in Definition 8.2. Constants, derivative orders and thresholds may depend on F and its fixed arithmetic data.

Proof. Fix $F\supset\mu_{12}$ and the arithmetic data of [Liu26b, Theorem 18.2]. Enlarge S to contain the places above 6, the different and the fixed exceptional local data. All constants below may depend on these fixed choices.

Local arithmetic and summation lengths. For $\mathfrak{p}\notin S$,

$$\mu_{12}\hookrightarrow(\mathcal{O}_F/\mathfrak{p})^\times,\qquad 12\mid(q_\mathfrak{p}-1),\qquad \left(\frac{x}{\mathfrak{p}}\right)_n\equiv x^{(q_\mathfrak{p}-1)/n}\pmod{\mathfrak{p}}\quad(n=2,3,6),$$

for $x\notin\mathfrak{p}$, with value zero at $x\in\mathfrak{p}$. Thus the full local Gauss identities and natural zeros used in Sections 3–4 are unchanged. Put $F_0=\mathbb{Q}(\omega)$, where $\omega=\zeta_3=e^{2\pi i/3}$. Write Λ_F for the unitary Eisenstein coefficient character denoted by Λ in [Liu26b, Lemma 4.2]. It is given by

$$\Lambda_{F_0}(\mathfrak{a})=\frac{\jmath(\pi_+(\mathfrak{a}))}{|\jmath(\pi_+(\mathfrak{a}))|},\qquad \Lambda_F=\Lambda_{F_0}\circ N_{F/F_0}.$$

Here $\mathfrak{a}$ is an integral F_0-ideal prime to 3, $\pi_+(\mathfrak{a}) \equiv 1 \pmod 3$ is its primary generator, and $\jmath : F_0 \hookrightarrow \mathbb{C}$ is the fixed embedding used for the cubic residue character and Jacobi sums. The map N_{F/F_0} is the ideal norm. The character Λ_F need not have finite order.

The theta character is

$$\chi_{\theta,\eta} = \overline{\Lambda_F}\,\overline{G_0}\eta,$$

with G_0 the finite-order coefficient character of the cited lemma. These constructions require no normality of F/F_0. The fixed unit and ray-class sets contribute constants. For the ideal-divisor function τ_F and fixed A,

$$\tau_F(\mathfrak{a})^A \ll_{F,A,\epsilon} (N\mathfrak{a})^{\epsilon}.$$

Apply the proofs of Theorems 4.1 and 4.9 using these identities and the transferred Poisson and M2 estimates. Keep the actual frequency domain in Proposition 4.6, and the actual outer ideal and Fourier measure in Proposition 4.8. Their length conditions remain

$$\eta_{3/4} = \frac{7}{9}(1-2r) - 6W > 0, \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1.$$

Write $\mathcal{N}_{F,c}$ and $\mathcal{N}_{F,\mathcal{U}}$ for the left sides of (4.15)–(4.16). Then

$$\max\{\mathcal{N}_{F,c}, \mathcal{N}_{F,\mathcal{U}}\} \ll_{F,\epsilon} U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C,$$

with the same nonexceptional character parameters, centered means and exact equal-product comparison. No additional principal character parameters occur.

Smooth weights and contours. The field F is totally complex, and its finite-order characters have trivial infinity type. With $r_F = [F:\mathbb{Q}]/2$, the gamma ratios in Lemma 5.1 are

$$\left(\frac{\Gamma(w+\xi)}{\Gamma(w)}\right)^{r_F}, \qquad \left(\frac{\Gamma(1-w)}{\Gamma(w)}\right)^{r_F}.$$

For fixed j, N and $1/6 \le \Re w \le 5/6$, that lemma gives

$$|(v\partial_v)^j V_w(v)| \ll_{F,j,N} \left(1 + \frac{v}{(1+|\Im w|)^{r_F}}\right)^{-N}.$$

The original and reflected contours satisfy

$$\frac{1}{5} \le \Re w \le \frac{1}{2}, \qquad \frac{1}{2} \le \Re(1-w) \le \frac{4}{5}, \qquad q_{\mathfrak{p}}^{-(j_{\mathfrak{p}}-2)\theta} \le 1 \quad (j_{\mathfrak{p}} \ge 2,\ \theta \ge 0).$$

Retain the actual conductors and fixed-S factors. The M2 estimate for arbitrary bounded length exponents of [Liu26b, Theorem 18.2] handles the degree-dependent long AFE sums; the short and selected ranges keep their original numerical cutoffs. Thus the dependence on r_F changes constants and height powers, rather than powers of U.

Selection and successive bounds. Prime ideals in the fixed ray classes supply the original ranges. After fixing the field data, the same selection and mesh choices give

$$\theta_0 = \frac{2\delta x}{9}, \qquad \mathcal{E}_{\theta,\gamma} - \mathcal{E}_{\theta_0,0} = (\theta-\theta_0)\{d(1-m) - l_y\} - d\gamma.$$

The scalar estimates of Section 8 therefore remain unchanged. The transferred 7/8 theorem supplies $\kappa_m = 3/4$; Theorem 8.1, applied over F, gives $B_0 = B_{\mathrm{seed}}$. The finite-step implication is

$$\beta_*(F) \le B_n, \quad \kappa_n = 2B_n - 1 \quad \Longrightarrow \quad \beta_*(F) \le B_{n+1} = B_* + \frac{\kappa_n - \kappa_*}{100}.$$

Since $\kappa_* = 2B_* - 1$,

$$B_n - B_* = \frac{B_{\mathrm{seed}} - B_*}{50^n} \longrightarrow 0, \qquad \beta_*(F) \le \inf_n B_n = B_*.$$

Each finite step uses its own strictly positive margins and threshold; no threshold uniform in n or F is required. □

9.2 Descent to an arbitrary number field

In this subsection, we descend through $K(\zeta_{12})$ using abelian factorization. Section 10 applies the same descent to $B_\natural$.

Theorem 9.2 (The zero-free half-plane over every number field)**.** *For every number field K and every finite-order Hecke character χ of K, the full Hecke L-function has no zeros in*

$$\boxed{\Re s > B_* = 0.87446830769353045955\ldots.}$$

The principal pole at $s = 1$ is permitted. In particular, for every number field K,

$$\zeta_K(s) \neq 0 \qquad (\Re s > B_*,\ s \neq 1).$$

The numerical boundary is independent of K and χ. No assertion is made on the line $\Re s = B_*$.

Proof. For primitive χ of order d, let M/K be its cyclic class-field extension and put $F = K(\zeta_{12})$, $E = MF$. Restriction gives

$$\mathrm{Gal}(E/F) \hookrightarrow \mathrm{Gal}(M/K), \qquad \mathrm{Gal}(E/M) \hookrightarrow (\mathbb{Z}/12\mathbb{Z})^\times.$$

For a finite abelian group G, write $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$. Both extensions are abelian. Proposition 9.1 and class-field factorization [Mil20], as in [Liu26b, Proposition 19.1], give

$$\begin{aligned}\zeta_E(s) &= \prod_{\psi \in \widehat{\mathrm{Gal}(E/F)}} L_F(s, \psi) \\ &= \zeta_M(s) \prod_{\substack{\omega \in \widehat{\mathrm{Gal}(E/M)} \\ \omega \neq 1}} L_M(s, \omega), \\ \zeta_M(s) &= \prod_{j=0}^{d-1} L_K(s, \chi^j).\end{aligned}$$

For $\Re s > B_*$, $s \neq 1$, the first product is nonzero and every complementary nonprincipal factor is entire. Hence

$$\zeta_E(s) \neq 0 \quad \Longrightarrow \quad \zeta_M(s) \neq 0 \quad \Longrightarrow \quad L_K(s, \chi) \neq 0.$$

At $s = 1$, for $d > 1$,

$$\prod_{j=1}^{d-1} L_K(1, \chi^j) = \frac{\mathrm{Res}_{s=1}\, \zeta_M(s)}{\mathrm{Res}_{s=1}\, \zeta_K(s)} > 0.$$

Thus each nonprincipal factor is nonzero; $d = 1$ permits the principal pole. An omitted Euler factor satisfies

$$1 - \alpha q_\mathfrak{p}^{-s} = 0, \quad |\alpha| = 1 \quad \Longrightarrow \quad \Re s = 0.$$

If $\alpha = 0$ the factor is one. This handles imprimitive functions. The full factorizations include ramified places and finite-order sign characters at real places of K. □

Section 10 lowers the boundary of Theorem 9.2 to $B_\natural$. Section 11 then proves Theorem 1.2 with the enlarged count.

Corollary 9.3 (Optimality within the stated comparison system)**.** *Let $F \supset \mu_{12}$ be fixed. Every sequence of Theorem 8.5 over F satisfies*

$$B_n \geq B_* \quad (n \geq 0), \qquad \liminf_{n\to\infty} B_n \geq B_* \qquad (\kappa_m = 3/4),$$

and Proposition 9.1 gives $\beta_*(F) \leq B_*$. The descent of Theorem 9.2 preserves $\Re s > B_*$.

Proof. For every fixed $F \supset \mu_{12}$, at the critical class $(\delta, x) = (r_*, 1/2)$,

$$h(1 - m_\theta) - l_y = 0, \qquad h\gamma \le \frac{7\delta}{216}(2l_y - h).$$

Proposition 9.1 preserves these quantities and the critical class; Theorem 8.5 gives the same scalar optimum. The descent in Theorem 9.2 preserves the strict half-plane exactly. □

10 The successive moment bounds with the earlier count

In this section, we let the numerator-moment parameter follow the successive zero-free bounds. We prove the amplified norm of Theorem 4.9 for every allowed $\kappa_m < 3/4$, solve the resulting comparison system, obtain the starting boundary $B_\natural$ for Section 11, and prove its optimality when the class count remains that of Proposition 6.4.

The auxiliary parameters for the boundary $B_\natural$ of (10.2) are defined as follows. Let $r_\natural$ be the unique root in $(9/20, 23/50)$ of

$$q_\natural(r) = 615108r^4 - 3239236r^3 + 4106021r^2 - 506420r - 334400. \tag{10.1}$$

The polynomial $q_\natural$ is irreducible over $\mathbb{Q}$. Set

$$\begin{gathered}\ell_\natural = -\frac{252833046}{2942707225}r_\natural^3 + \frac{2932032572}{2942707225}r_\natural^2 - \frac{159506318657}{76510387850}r_\natural + \frac{6990506117}{7651038785},\\ \kappa_\natural = \frac{5}{6} - \frac{\ell_\natural}{2} = 2B_\natural - 1, \qquad B_\natural = \frac{11}{12} - \frac{\ell_\natural}{4}.\end{gathered} \tag{10.2}$$

The numbers $\ell_\natural$ and $B_\natural$ are algebraic of degree four:

$$\begin{gathered}2229\ell_\natural^4 - 7370\ell_\natural^3 + 5771\ell_\natural^2 + 410\ell_\natural - 200 = 0,\\ 1925856B_\natural^4 - 5469552B_\natural^3 + 5643378B_\natural^2 - 2497497B_\natural + 399857 = 0.\end{gathered} \tag{10.3}$$

Thus

$$\begin{aligned}r_\natural &= 0.45053392487884345250\ldots, & \ell_\natural &= 0.16879668066330557444\ldots,\\ \kappa_\natural &= 0.74893499300168054611\ldots, & B_\natural &= 0.87446749650084027305\ldots,\end{aligned}$$
$$B_* - B_\natural = 8.1119269018\ldots \cdot 10^{-7}.$$

The balancing parameter $b_\natural = 0.09180139495752472774\ldots$ is defined in (10.28).

10.1 The variable numerator-moment parameter

In this subsection, we isolate the only source of κ_m, the estimate M2, and record the scalar data which depend on it. The corresponding norm is proved in Section 10.2.

The class count already uses $\kappa_c = 2B_n - 1$ (Proposition 6.4). The numerator moment uses $\kappa_m = 3/4$, justified by $\beta_*(F) \le 7/8$. The change is

$$\kappa_m = \frac{3}{4} \quad \longrightarrow \quad \kappa_m = \kappa_c = \kappa_n = 2B_n - 1. \tag{10.4}$$

Let A be the total length exponent of a product, including its prime-sum lengths, M its modulus exponent and $z_{\mathrm{pr}} = z$ its prime-sum length; with plain lengths n_1, n_2, $A + (6\kappa - 1)z \le M \iff n_1 + n_2 + 6\kappa z \le M$. On arithmetic parameters inducing outside Θ, [Liu26b, Theorem 12.16 and Lemma 12.7] gives, for $13/18 \le \kappa \le 1$,

$$\beta_*(F) \le \frac{1+\kappa}{2} \ \ (\kappa < 1) \quad \Longrightarrow \quad \Big\{A + (6\kappa - 1)z \le M \ \Rightarrow\ \text{M2 holds}; \quad |Q_k|^2 \ll U^{\kappa z + \epsilon}\Big\}. \tag{10.5}$$

The hypothesis is used only iteratively:

$$\beta_*(F) \le B_n \quad\Longrightarrow\quad \beta_*(F) \le \frac{1+\kappa_n}{2} \quad\Longrightarrow\quad (10.5) \text{ at } \kappa = \kappa_n. \tag{10.6}$$

Every M2 application in the iteration of this section uses $\kappa > \kappa_\natural$, with $\kappa_\natural$ defined in (10.2). Write $\kappa_m = \kappa$ and

$$c_\kappa = \frac{1}{6\kappa}, \qquad k_6 = 6\kappa. \tag{10.7}$$

Thus $c_{\mathrm{cnt}}(\kappa) = 2c_\kappa$, where c_{cnt} is the count coefficient in Proposition 6.4. The letter c_κ is distinct from the local lengths c, c_{cor} and c_C of Sections 3–4. The κ_m-dependent scalar data become

$$\begin{aligned} &z_0(m) = c_\kappa(1-2m), \qquad \theta_0 = c_\kappa \delta x, \qquad z_L = z_0(3/8) = \frac{1}{24\kappa}, \\ &a(\delta, x) = \delta\,\frac{21x/100 - z_L/2}{21/100 - z_L}, \qquad W_{\mathrm{cap}}(m;\kappa) = \min\Big\{\frac{(1-c_\kappa)(1-2m)}{6}, \frac{8m-3}{12\kappa}\Big\}. \end{aligned} \tag{10.8}$$

At $\kappa = 3/4$,

$$\begin{aligned} c_\kappa = \frac{2}{9}, \quad z_0 = \frac{2(1-2m)}{9}, \quad \theta_0 = \frac{2\delta x}{9}, \quad z_L = \frac{1}{18}, \\ a = \delta\frac{189x-25}{139}, \quad W_{\mathrm{cap}} = \min\Big\{\frac{7}{54}(1-2m), \frac{8m-3}{9}\Big\}, \end{aligned}$$

as in (2.2), (1.6) and Lemma 7.3.

10.2 The amplified norm for a variable moment parameter

In this subsection, we prove Theorem 4.9 with κ_m replaced by any κ between the limiting value and 3/4. Section 10.3 applies this norm to the comparison.

Lemma 10.1 (Amplified norm for a variable moment parameter)**.** *Let $\kappa_\natural \le \kappa \le 3/4$ and assume $\beta_*(F) \le (1+\kappa)/2$. Retain the arithmetic data, coefficient families, summation domains, smooth-weight families, order of choices and exact equal-product comparison of Theorem 4.1, with $\kappa_m = \kappa$ and the initial prime product Q_0 retained. Suppose $3/8 < m_- \le r < 1/2$, $W \ge 0$ and*

$$\eta_\kappa = (1-c_\kappa)(1-2r) - 6W > 0, \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + 6\kappa W < 1. \tag{10.9}$$

For complete disjoint prime-sum factors with

$$m \in [m_-, r], \qquad 0 \le z \le z_0(m) = c_\kappa(1-2m), \qquad 0 \le w \le W,$$

the estimates (4.15) and (4.16) hold for every $\epsilon > 0$. In particular, for $m_- = r = m$,

$$0 \le W < W_{\mathrm{cap}}(m;\kappa) \quad\Longrightarrow\quad (10.9). \tag{10.10}$$

At $\kappa = 3/4$ this is Theorem 4.9.

The implication (10.10) follows from

$$\eta_\kappa > 0 \iff W < \frac{(1-c_\kappa)(1-2m)}{6}, \qquad C_{\mathrm{add}} < 1 \iff 6\kappa W < 4m - \frac{3}{2} \iff W < \frac{8m-3}{12\kappa}.$$

Proof. We follow the proof of Theorem 4.9 and its inputs, Theorem 3.7, Proposition 3.3, Corollaries 3.4 and 3.6, Propositions 4.6 and 4.8, and Lemmas 4.7 and 5.2. We show that the substitutions of Table 3 are the only changes.

Step 1: the single source of κ. Put

$$\mathrm{Def} = A + (k_6 - 1)z_{\mathrm{pr}} - M. \tag{10.11}$$

By (10.5), M2 applies if $\mathrm{Def} \le 0$. If $\mathrm{Def} > 0$, remove prime length Def/k_6, plus one mesh step. By the prime-length bound in (10.5),

$$\text{squared loss} = \kappa \cdot \frac{\mathrm{Def}}{k_6} = \frac{\mathrm{Def}}{6}. \tag{10.12}$$

Hence every restoration loss $(\cdot)_+/6$ in Sections 2–5 is independent of κ. Every application of M2 there (the restorations in Theorems 3.7, 4.1 and 4.9; the ordinary sums after the second Poisson formula in Proposition 4.8; the reflection in C_{add}; the unamplified M2 in Lemma 5.2) uses the single parameter κ_m, and no other estimate fixed at $\kappa_m = 3/4$ occurs. The reflection of [Liu26b, Theorem 12.16] requires $6\kappa - 1 \ge 10/3$, and $6\kappa_\natural - 1 = 3.4936\ldots$.

Step 2: changes of the length excess. The length calculations of Sections 3.1–3.4 give the following changes of Def:

operation	$\Delta\mathrm{Def}$	at $\kappa = 3/4$
initial family $A = 2m + z + w$, $M = 1$ (Sections 3.2, 3.4)	$\mathrm{Def} = k_6 w - e_z$, $e_z = k_6(z_0 - z)$	$\frac{9}{2}w - e_z$
fixed prime f_0: $(A, K, M, z_{\mathrm{pr}}) \mapsto$ $(A - \lambda_p, K + 4\lambda_p, M - 6\lambda_p, z_{\mathrm{pr}} - \lambda_p)$ (Section 3.2)	$(6 - k_6)\lambda_p$	$\frac{3}{2}\lambda_p$
fixed prime f_1 of positive valuation (Sections 3.2, 3.4)	$-(k_6 - 1)f_1 - T_1$	$-\frac{7}{2}f_1 - T_1$
free stage (Sections 3.1, 3.2)	$(6 - k_6)u - 5v - T$	$\frac{3}{2}u - 5v - T$
common fixing (Section 3.3)	$-(k_6 - 1)z_C$	$-\frac{7}{2}z_C$
genuine and final-divisor fixing (Section 3.2)	$-(k_6 - 1)b_g$, $-k_6 b_d$	$-\frac{7}{2}b_g$, $-\frac{9}{2}b_d$

For the fixed prime f_0, $\Delta\mathrm{Def} = -\lambda_p + 6\lambda_p - (k_6 - 1)\lambda_p = (6 - k_6)\lambda_p$.

Step 3: the length identities. Summing the changes of Step 2 with $u = x_{\mathrm{aux}} + w - w_f - f_0 - f_1$ gives, in place of the display preceding (3.7),

$$D_{\mathrm{free}} = (6 - k_6)x_{\mathrm{aux}} + 6w - e_z - (k_6 - 1)z_{C,\mathrm{old}} - 5w_f - 5f_1 - T_1 - 5v - T. \tag{10.13}$$

With $z = z_0 - e_z/k_6$, $z_{\mathrm{init}} = z - z_{C,\mathrm{old}} - x_{\mathrm{aux}}$ and $(k_6 - 1)z_0 = (1 - c_\kappa)(1 - 2m)$,

$$\begin{aligned}(k_6 - 1)z_{\mathrm{init}} - D_{\mathrm{free}} &= (1 - c_\kappa)(1 - 2m) - 5x_{\mathrm{aux}} - 6w + \frac{e_z}{k_6} + 5w_f + 5f_1 + T_1 + 5v + T \\ &\ge (1 - c_\kappa)(1 - 2m) - 5x_{\mathrm{aux}} - 6w. \end{aligned} \tag{10.14}$$

The coefficients $6, 5, 5, 5$ do not depend on κ:

$$6 = k_6 + (6 - k_6), \qquad 5 = (k_6 - 1) + (6 - k_6),$$

and the terms in f_0 cancel. In the amplified norm $x_{\mathrm{aux}} = 0$, so

$$(k_6 - 1)z_{\mathrm{init}} - D_{\mathrm{free}} \ge (1 - c_\kappa)(1 - 2r) - 6W = \eta_\kappa, \qquad z_{\mathrm{init}} = 0 \Longrightarrow D_{\mathrm{free}} \le -\eta_\kappa.$$

This replaces the first display in the proof of Theorem 4.9. The remaining local identities become

$$-f_0 + \frac{(k_6 w + (6 - k_6)f_0)_+}{6} = -\kappa x_{\mathrm{aux}} \qquad (f_0 = x_{\mathrm{aux}} + w;\ \text{see (3.6)}), \tag{10.15}$$

$$\begin{aligned}\mathcal{C}_{\mathrm{hom}} + \frac{\mathcal{D}}{6} &= -\kappa x_{\mathrm{aux}} + \frac{w_f - \mu_{\mathrm{com}}}{6} - \frac{5}{6}(f_1 + v) - \frac{19}{6}(T_1 + T) - \frac{k_6 - 1}{6}z_{C,\mathrm{old}} - \frac{e_z}{6} \\ &\quad - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e \le 0 \qquad \text{(Corollary 3.6)}, \end{aligned} \tag{10.16}$$

$$C_{\mathrm{new}} = -\kappa u + \frac{D}{6} - \frac{5}{6}v - \frac{19}{6}T - 4c - S - e \qquad (\text{see (3.3)},\ \text{active branch}). \tag{10.17}$$

By (10.17),

$$D \le k_6 u \implies C_{\mathrm{new}} \le -\frac{5v}{6}, \qquad D \le (k_6-1)u \implies C_{\mathrm{new}} \le -\frac{u+5v}{6}, \tag{10.18}$$

which is Proposition 3.3 with $9/2 \to k_6$ and $7/2 \to k_6 - 1$.

The identity in the proof of Proposition 4.6, with coefficients $17f_0$, $\frac{5}{2}f_1$, $18c_{\mathrm{cor}}$, $-\frac{9}{2}(T_1+T)$ and $-\frac{7}{2}m$, uses only A_*, M_*, q_* and the Gauss exponents of B_C; it does not depend on κ. Its only dependence on κ is through $A_t \le 2r + z_0(r) + W$. With

$$\begin{gathered} A_{\max} = 2r + z_0(r) = 1 - (1-c_\kappa)(1-2r), \\ A_{\max} + W = 1 - 5W - \eta_\kappa, \qquad M_* \ge \frac{\eta_\kappa + 6c_\kappa W}{1-c_\kappa} - o(1), \qquad m_* \ge \frac{\eta_\kappa + (7c_\kappa - 1)W}{1-c_\kappa} - o(1), \\ C_{\mathrm{add}} = \frac{5}{2} - 4m_- + k_6 W. \end{gathered} \tag{10.19}$$

The last identity follows from the reflected comparison

$$A + (k_6-1)(z+w) = \frac{3}{2} - 2m + k_6(z+w), \qquad k_6 z_0(m) = 1 - 2m.$$

At $\kappa = 3/4$, (10.19) gives the lower bounds $(12W + 9\eta_{3/4})/7$ and $(5W + 9\eta_{3/4})/7$ in the proof of Theorem 4.9.

Step 4: restoration with fixings. With $b_{\mathrm{fix}} = b_g + b_d$,

$$k_6(z_{\mathrm{init}} - b_{\mathrm{fix}}) - \big(D - (k_6-1)b_g - k_6 b_d\big) - \big((k_6-1)z_{\mathrm{init}} - D\big) = z_{\mathrm{init}} - b_g \ge 0. \tag{10.20}$$

Hence

$$(k_6-1)z_{\mathrm{init}} \ge D \quad \Longrightarrow \quad k_6(z_{\mathrm{init}} - b_{\mathrm{fix}}) \ge D - (k_6-1)b_g - k_6 b_d,$$

and the converse holds for $b_g = z_{\mathrm{init}}$. Thus the coefficient $1 - c_\kappa$ is necessary for this restoration argument when it imposes the inequality $(k_6-1)z_{\mathrm{init}} \ge D$ at every fixing. Lemma 12.2 applies the ordinary estimate M2 when all remaining prime factors have been removed; Theorem 12.1 therefore has the larger width domain. No necessity for that enlarged argument is asserted here.

Step 5: the width and the margins. The strict conditions (10.9) and $r \ge m_-$ give

$$1 - \frac{W}{W_{\max}(\kappa)} = \frac{4\eta_\kappa}{1-c_\kappa} + 2(1 - C_{\mathrm{add}}) + 8(r - m_-) > 0, \qquad W_{\max}(\kappa) = \frac{6\kappa - 1}{12\kappa(6\kappa+11)}. \tag{10.21}$$

At $\kappa = 3/4$ this is the identity $1 - \frac{279}{7}W = \frac{36}{7}\eta_{3/4} + 2(1 - C_{\mathrm{add}}) + 8(r - m_-)$ in the proofs of Proposition 4.6 and Theorem 4.9. For $\kappa \in [\kappa_\natural, 3/4]$, by exact rational bounds monotone in κ,

$$\begin{gathered} 1 - c_\kappa \ge 0.77746177, \qquad 7c_\kappa - 1 \ge \tfrac{5}{9}, \qquad W_{\max}(\kappa) \le 0.0250897556, \\ A_t + 4t_{\mathrm{loc}} \le 1 - W - \eta_\kappa < 1, \qquad A_t + 3t_{\mathrm{loc}} \le 1 - 2W - \eta_\kappa < 1, \\ -\frac{1}{2} + 3W - 3\eta_\kappa + \frac{7}{2}\rho_{\mathrm{term}} < -\frac{1}{2} + 3W_{\max}(\kappa) + \frac{7}{200} \le -0.3897307 < -\frac{3}{8}, \\ (k_6-1)z_0(0.443) - \tfrac{6}{100} \ge 0.0286306, \qquad \tfrac{5}{2} - 1.6 + \tfrac{k_6}{100} \le 0.945 < 1, \\ \tfrac{21}{100} > z_L = \tfrac{1}{24\kappa}, \qquad 2x\tfrac{21}{100} - z_L - W_{\max}(\kappa) \ge 0.08727 \quad (x \ge 2/5). \end{gathered} \tag{10.22}$$

Here $W_{\max}(\kappa) = W_{\mathrm{cap}}(m^\sharp; \kappa)$, where $m^\sharp = (k_6+8)/(2k_6+22)$ is the crossing point of the two branches. The contour region of Lemma 5.2, with $\Re z \ge 17/50$ and $\sigma_0 = 1/6$, does not depend on κ; there $\theta_0 = c_\kappa \delta x$.

Order of choices. Fix $\kappa = \kappa_n$ and the hypothesis (10.6); then the strict margins (10.9) and (10.22); then the steps of the proof of Theorem 4.9 in their original order: the classification of Theorem 3.7 before M2, the inequality (10.16), and the split of Lemma 4.7 between Propositions 4.6 and 4.8. Every displayed exponent has the κ-dependence specified in Table 3 and Remark 10.2, and every strict inequality keeps a positive margin by (10.22). □

Table 3: Dependence on κ of the norm constants in Sections 2–5.

value at $\kappa = 3/4$	value for general κ	occurrence
Functions of 6κ		
$9/2$	6κ	z_0, e_z, D, final-divisor fixing, C_{add}, $9u/2$ in Proposition 3.3
$7/2$	$6\kappa - 1$	z_C, f_1, fixed primes dividing s, genuine fixing, allowance $\frac{7}{2}z_{\mathrm{init}}$, $7u/2$ in Proposition 3.3
$3/2$	$6 - 6\kappa$	$3u/2$, $3f_0/2$ in Section 3.2
$2/9$	c_κ	$z = z_0 - e_z/(6\kappa)$, $\theta_0 = c_\kappa \delta x$
$7/9$	$1 - c_\kappa$	$\eta_{3/4} \to \eta_\kappa$, $A_{\max} = 1 - (1 - c_\kappa)(1 - 2r)$
$7/12$, $3/4$	$(6\kappa - 1)/6$, κ	coefficients of $z_{C,\mathrm{old}}$ and x_{aux} in (10.16), (10.15)
$7/54$, $(8m - 3)/9$	$(1 - c_\kappa)/6$, $(8m - 3)/(12\kappa)$	W_{cap}
$7/279$	$W_{\max}(\kappa)$	Proposition 4.6, Theorem 4.9
$4/9$ in (2.2)	$2c_\kappa$	E_κ
Independent of κ: $6w$, $5w_f$, $5f_1$, $5x_{\mathrm{aux}}$, $5v$, $19/6$, $5/6$, $4c$, $4c_{\mathrm{cor}}$, the restoration loss $1/6$, M_*, A_*, q_*, $6a$, $7a$, $4t$, $3t$, $B_C = (3c - 5d - R)_+/6$, the identity of Proposition 4.6 ($17f_0$, $\frac{5}{2}f_1$, $17g$, $18c_{\mathrm{cor}}$, $-\frac{9}{2}(T_1 + T)$, $\frac{7}{2}m$), $F_1^{\mathrm{loc}}, F_2^{\mathrm{loc}}, J, g, \Lambda_a, M', \Delta', \mathcal{O}_{\mathrm{ret}}$, $5/2 - 4m = (3/2 - 2m) + 6\kappa z_0$, and the contour data $17/50$, $1/6$.		
Chosen margins, unchanged: $\rho_{\mathrm{term}} = 1/100$, $\sigma < 10^{-6}$, $3/125000$, ρ_{step}. The $7/2$ in $-\frac{7}{2}m$ is a Gauss frequency exponent, not $6\kappa - 1$, and $-9/2$ in the identity of Proposition 4.6 is not 6κ.		

Remark 10.2 (Constants in the variable-moment norm). The proof of the variable-moment norm uses the constants of Table 3 in the following forms.

(a) Proposition 4.6 and Theorem 4.9: the width satisfies $W < W_{\max}(\kappa)$, and the terminal exponent is at most $-\frac{1}{2} + 3W_{\max}(\kappa) + \frac{7}{200} \le -0.3897307 < -\frac{3}{8}$.

(b) Corollary 3.4, used only for $x_{\mathrm{aux}} > 0$: the sufficient length condition is

$$x_{\mathrm{aux}} \ge \frac{w_f}{6\kappa}, \qquad x_{\mathrm{req}} \ge \frac{w}{6\kappa} + \rho_{\mathrm{step}}, \qquad \eta_{\mathrm{mesh}} \le \frac{\rho_{\mathrm{step}}}{4}.$$

Indeed, for $w_f \le w$ and $x_{\mathrm{aux}} \ge x_{\mathrm{req}} - \eta_{\mathrm{mesh}}$,

$$\mathcal{C}_{\mathrm{act}} \le -\kappa x_{\mathrm{aux}} + \frac{w_f}{6} \le -\kappa(\rho_{\mathrm{step}} - \eta_{\mathrm{mesh}}) \le -\frac{3\kappa}{4}\rho_{\mathrm{step}} \le -\frac{\rho_{\mathrm{step}}}{2} \qquad (\kappa \ge \tfrac{2}{3}).$$

At $\kappa = 3/4$, $1/(6\kappa) = 2/9$ recovers the original length condition. The stronger mesh restriction is needed because

$$-\frac{2\kappa_\natural}{3}\rho = -0.49929\ldots\rho > -\frac{\rho}{2}, \qquad -\frac{3\kappa}{4}\rho \le -\frac{\rho}{2} \quad (\kappa \ge \tfrac{2}{3}).$$

(c) The margin in Theorem 4.1 is $F_\kappa = (1 - c_\kappa)(1 - 2r) - 7W$, and $A_* - m_* \le W - \eta_\kappa + o(1) = -F_\kappa + o(1)$. The hypothesis $\tau = 1 - 2r - 9W > 0$ alone gives only

$$F_\kappa > (1 - 2r)\Big(\frac{2}{9} - c_\kappa\Big) \ge -3.2 \cdot 10^{-4}(1 - 2r) \qquad (\kappa \ge \kappa_\natural),$$

which does not give $A_* < m_*$. Theorem 4.9 uses Theorem 4.1 only for its arithmetic data and Corollary 4.3 only for the Hölder step.

(d) The numerical bounds in Theorem 3.7, with the worst case on $[\kappa_\natural, 3/4]$:

$\frac{43}{1500}$	$\frac{691}{750}$	$-\frac{7}{375}$	$\frac{721}{750}$	$\frac{1427}{1500}$	$\frac{189}{200}$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0.0286306	0.9213694	−0.0186306	0.9613694	0.9513694	$\frac{5}{2} - 1.6 + \frac{6\kappa}{100} \le 0.945$

(e) Corollary 4.3, Lemma 4.4, the identity $1 - 45W = 4\tau + \cdots$ and the example following (3.7) are specific to $\kappa = 3/4$. The hypothesis $\beta_*(F) \le 7/8$ in Sections 3.4, 4.1, 4.3, 4.4 and in Lemma 5.2 has the form $\beta_*(F) \le (1+\kappa)/2$.

10.3 The comparison system and its endpoint

In this subsection, we solve the comparison of Lemma 8.3 with independent count and moment parameters.

With κ_m variable, (2.2), Lemma 7.3 and Lemma 8.3 become, with $c = c_{\kappa_m}$,

$$E_{\kappa_c,\kappa_m}(\delta, x) = C_0 + \frac{\delta}{2} + \delta x\Big\{\ell + c\Big(\frac{h}{2} - l_y\Big)\Big\} - \frac{3h}{4}\big(1 - R_{\kappa_c}(\delta, x)\big), \tag{10.23}$$

$$h\Gamma_N = \frac{1-c}{12}\, a(\delta, x)(2l_y - h), \tag{10.24}$$

$$\mathcal{Q}_{\kappa_c,\kappa_m,b,\ell}(\delta, x) = E_{\kappa_c,\kappa_m}(\delta, x) - \frac{1-c}{12}\, a(\delta, x)(2l_y - h), \tag{10.25}$$

with R_κ of Proposition 6.4, $m_\theta = 1 - l_y/h$ and a as in (10.8). For $a \ge 0$ and $3/8 \le m_\theta < 1/2$,

$$\frac{ah}{2} W_{\mathrm{cap}}(m_\theta; \kappa_m) \le h\Gamma_N, \qquad \text{equality if } m_\theta \ge m^\sharp = \frac{6\kappa_m + 8}{12\kappa_m + 22}.$$

Table 4 verifies this equality throughout the constructive parameter family. Outside that family, $h\Gamma_N$ remains an upper bound on the permitted selected saving, as used in Theorem 10.8. Thus $\mathcal{Q}_{\kappa,b,\ell} = \mathcal{Q}_{\kappa,3/4,b,\ell}$, since $c_{3/4}(h/2 - l_y) = \frac{4}{9}(h/4 - l_y/2)$ and $(1 - c_{3/4})/12 = 7/108$.

At $x = 1/2$, $a(\delta, 1/2) = \delta/2$ for every z_L, and

$$\mathcal{Q}_{\kappa_c,\kappa_m,b,\ell}(\delta, \tfrac{1}{2}) - \mathcal{Q}_{\kappa_c,b,\ell}(\delta, \tfrac{1}{2}) = \frac{\delta}{2}\Big(\frac{h}{2} - l_y\Big)\Big(\frac{1+5c}{6} - \frac{19}{54}\Big), \qquad \frac{h}{2} - l_y = \frac{5\ell - 1 - b}{4} < 0. \tag{10.26}$$

Hence

$$\kappa_m < \frac{3}{4} \iff c > \frac{2}{9} \iff \mathcal{Q}_{\kappa_c,\kappa_m,b,\ell}(\delta, \tfrac{1}{2}) < \mathcal{Q}_{\kappa_c,b,\ell}(\delta, \tfrac{1}{2}).$$

For $\kappa_m < 3/4$,

$$z_0(m) - \frac{2}{9}(1 - 2m) = \left(c - \frac{2}{9}\right)(1 - 2m), \qquad \theta_0 - \frac{2\delta x}{9} = \left(c - \frac{2}{9}\right)\delta x,$$
$$c > \frac{2}{9}, \qquad \frac{1-c}{6} < \frac{7}{54}.$$

Thus the gain uses the longer initial product of Lemma 10.1. Changing only the M2 parameter, with the norm of Theorem 4.9 retained, leaves $\mathcal{Q}(\cdot, 1/2)$ identically unchanged.

Proposition 10.3 (The endpoint system and its quartic). *Put $\kappa_c = \kappa_m = 5/6 - \ell/2$, write $\mathcal{Q} = \mathcal{Q}_{\kappa_c,\kappa_m,b,\ell}$, and put*

$$D_\natural(\delta, \ell) = -48(3\ell - 5)(18\ell\delta - 25\ell - 26\delta + 35). \tag{10.27}$$

(i) $D_\natural\,\mathcal{Q}(\delta,1/2) = b\,p_b(\delta,\ell) + p_0(\delta,\ell)$, *where*

$$\begin{aligned} p_b &= (702\ell^2 - 2310\ell + 1880)\delta^2 + (-1587\ell^2 + 5029\ell - 3940)\delta \\ &\quad + 600\ell^2 - 1840\ell + 1400, \\ p_0 &= (378\ell^3 - 1428\ell^2 + 2162\ell - 1240)\delta^2 + (-4305\ell^3 + 14068\ell^2 - 13351\ell + 2860)\delta \\ &\quad + 4500\ell^3 - 14700\ell^2 + 13260\ell - 2100. \end{aligned}$$

(ii) At $(\delta,x) = (r,1/2)$, *with* r *the critical value of* δ, *the equations* $\mathcal{Q} = \partial_\delta\mathcal{Q} = \partial_b\mathcal{Q} = 0$ *are equivalent to*

$$p_b(r,\ell) = p_0(r,\ell) = 0, \qquad b\,\partial_\delta p_b(r,\ell) + \partial_\delta p_0(r,\ell) = 0,$$

and

$$\mathrm{Res}_\ell\big(p_b(r,\ell), p_0(r,\ell)\big) = 11520\,r^2(3r-5)(6r-5)^3\,q_\natural(r).$$

The factors $r^2(3r-5)(6r-5)^3$ *have no root in* $(9/20, 23/50)$, *and*

$$p_b(r,\ell_\natural(r)) \equiv p_0(r,\ell_\natural(r)) \equiv 0 \pmod{q_\natural(r)},$$

where $\ell_\natural(r)$ *is the cubic in* (10.2). *Put*

$$b_\natural = -\frac{\partial_\delta p_0}{\partial_\delta p_b}(r_\natural,\ell_\natural). \tag{10.28}$$

(iii) At $(b,\ell) = (b_\natural,\ell_\natural)$,

$$\mathcal{Q}(\delta,1/2) = \frac{A_\natural\,(\delta - r_\natural)^2}{D_\natural(\delta,\ell_\natural)}, \qquad A_\natural \in [-775.30309, -775.30308], \tag{10.29}$$

$$D_\natural(7/20,\ell_\natural) = 4905.62\ldots, \qquad D_\natural(3/5,\ell_\natural) = 3667.45\ldots,$$

and $D_\natural(\cdot,\ell_\natural)$ *is affine. Hence*

$$\mathcal{Q}(\delta,1/2) \le 0 \quad (7/20 \le \delta \le 3/5), \qquad \mathcal{Q}(\delta,1/2) = 0 \iff \delta = r_\natural.$$

(iv) $\partial_x\mathcal{Q} > 0$ *on* $\mathcal{R}_*$ *at the endpoint parameters. Hence* $\mathcal{Q}_{\kappa_\natural,\kappa_\natural,b_\natural,\ell_\natural} \le 0$ *on* $\mathcal{R}_*$, *with equality only at* $(r_\natural,1/2)$.

(v) The conditions of Table 4 hold on the parameter boxes stated there.

At $\kappa_m = 3/4$ *the same elimination gives* $432\,r(6r-5)^2(218582r^3 - 376469r^2 - 11376r + 61560)$, *that is, the cubic* (1.2) *and* B_*.

Proof. Parts (i)–(ii) are polynomial identities. Near the endpoint,

$$\begin{gathered} D_\natural \neq 0, \qquad \mathcal{Q} = \partial_b\mathcal{Q} = 0 \iff p_b = p_0 = 0, \\ p_b = p_0 = 0 \implies (\partial_\delta\mathcal{Q} = 0 \iff b\,\partial_\delta p_b + \partial_\delta p_0 = 0)\,. \end{gathered}$$

Section A.4 gives $\partial_\delta p_b(r_\natural,\ell_\natural) \neq 0$, defining (10.28). For (iii), the quadratic numerator has a double root at $r_\natural$; its leading coefficient is $A_\natural$, and the affine denominator is positive at both endpoints. For (iv), Table 4 gives

$$\mathcal{Q}(\delta,x) = \mathcal{Q}(\delta,1/2) - \int_x^{1/2} \partial_x\mathcal{Q}(\delta,t)\,dt \le \mathcal{Q}(\delta,1/2) - 2.67\cdot 10^{-4}(1/2-x).$$

Together with (iii), this proves (iv), including its equality claim. Part (v) is Table 4. The computations are described in Section A.4. □

For Table 4, define

$$\mathcal{P}_{\natural,0} = \{(\kappa_\natural, \kappa_\natural, \ell_\natural, b_\natural)\},$$
$$\mathcal{P}_\natural = \left\{ \begin{array}{c} \kappa_c, \kappa_m \in [\kappa_\natural, \kappa_*], \\ (\kappa_c, \kappa_m, \ell, b) : \ell \in [\ell_\natural - (\kappa_* - \kappa_\natural)/25, \ell_\natural], \\ |b - b_\natural| \le 10^{-30} \end{array} \right\}. \tag{10.30}$$

Let $\mathcal{P}_{\natural,3/4}$ denote the same family with the upper endpoint for κ_m extended to $3/4$.

Table 4: Conditions for the endpoint parameters. Entries with inequality signs are rigorous bounds; entries without inequality signs are decimal approximations. The parameter sets are defined in (10.30); entries without an indicated set hold on $\mathcal{P}_\natural$. On $\mathcal{R}_*$, $\theta_0 \le 3/(60\kappa_m) < 0.06681$. Here $\gamma_{\max} = a\,W_{\max}(\kappa_m)/2$, $I_{\mathrm{short}} = R + (1/2 - \theta_0)m_{\mathrm{cut}}$ with $m_{\mathrm{cut}} = 3/40$ as in Lemma 7.2, and Q_{sm} is defined in Appendix A.

condition	used in	bound or approximation
$\partial_x \mathcal{Q} > 0$ on $\mathcal{R}_*$ $(\mathcal{P}_{\natural,0})$	Appendix A.3	$\ge 2.67 \cdot 10^{-4}$
$\partial_{\kappa_c}\mathcal{Q} + \partial_{\kappa_m}\mathcal{Q} - \frac{1}{25}\partial_\ell \mathcal{Q} < -\frac{1}{100}$ on $\mathcal{R}_*$	Theorem 8.4	≤ -0.01247
$\partial_\ell \mathcal{Q} > 3/4$ on $\mathcal{R}_*$	Lemma 8.3	≥ 0.7932
$\partial_{\kappa_c}\mathcal{Q} \ge 0$, $\partial_{\kappa_m}\mathcal{Q} \ge 0$ on $\mathcal{R}_*$	—	$\ge 4.3 \cdot 10^{-6}$, $\ge 9.8 \cdot 10^{-4}$
$J > 0$ on $\mathcal{G}$ $(\mathcal{P}_{\natural,3/4})$	Lemma 8.3	≥ 0.0971
$E < 1/500$ on $\mathcal{G}$ $(\mathcal{P}_{\natural,3/4})$	Appendix A.3	≤ 0.00199999
$E < -10^{-5}$ on $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ $(\mathcal{P}_{\natural,3/4})$	Appendix A.3	≤ -1.002, -1.064, -4.47 (all $\times 10^{-5}$)
$S > 1/5$ on $\mathcal{R}_*$; $S < 1$ on $\mathcal{G}$	Corollary 6.7	≥ 0.2397; ≤ 0.9412
$(6\kappa_m - 2\delta x)\frac{21}{100} - R_{\kappa_c} > 0$ on $\mathcal{R}_*$; on $\mathcal{G}$	(6.2)	≥ 0.0816; $\ge 8.1 \cdot 10^{-5}$
$\frac{21}{100}(6\kappa_m - \delta) - (1 - \frac{2\delta}{3}) > 0$, $\frac{1}{8} \le \delta \le \frac{5}{6}$	[Liu26b, Proposition 14.6]	$\ge 7.41 \cdot 10^{-4}$
$a > 0$; $\frac{a}{3\kappa_m} - \theta_0 > 0$; $\frac{\delta}{2} - \theta_0 - \frac{a(1-c)}{6} > 0$	Lemma 7.3	0.127; $3.8 \cdot 10^{-3}$; 0.0694
$\Gamma_L - \Gamma_N > 0$; $(\frac{\delta}{2} - \theta_0)(\lambda - \frac{11}{20}) - \Gamma_N > 0$	Lemma 7.3, Proposition 7.4	$\ge 1.9 \cdot 10^{-7}$; $\ge 9.7 \cdot 10^{-4}$
$2x\frac{21}{100} - z_L - W_{\max} > 0$; $\gamma_{\max} < \frac{1}{250}$	Lemma 4.2, Proposition 7.4	≥ 0.08727; < 0.0037635
$\frac{3}{8} < m^\sharp < m_\theta$, $\frac{25}{62} < m_\theta < \frac{9}{20}$, $\frac{57}{100} < \lambda < \frac{11}{19}$	Lemma 7.3	0.403186, 0.422469, 0.577531
$N - I_{\mathrm{short}} \ge \frac{23}{2700}$; $E + h\Delta_{\mathrm{short}} < -10^{-5}$	Lemma 7.2	$8.87 \cdot 10^{-3}$; $\le -5.9 \cdot 10^{-4}$
$S + \Delta_{\mathrm{short}} > 0$	Lemma 7.2	≥ 0.1452
$z_0(\frac{3}{40}), z_0(\frac{1}{10}), z_L < \frac{21}{100}$	Lemma 7.2	0.189157, 0.178030, 0.05563
$S - \frac{37}{40}\theta_0 - \gamma_{\max} > \frac{1}{8}$; $S - \gamma_{\max} > \frac{1}{8}$; slope < 2	Theorem 8.4	≥ 0.1920; ≥ 0.2360; ≤ 0.6997
$6\kappa_m \ell > 3/4$; $\frac{9}{2}\ell > \frac{3}{4}$	[Liu26b, Remark 14.1]	0.75851; 0.7596
$l_x > \ell$; $l_y - \ell - \frac{11b}{6} > 0$; $l_y < \frac{1}{2}$; $h < 2l_y$	Lemma 8.3	0.2009; 0.1244; 0.4615
$\frac{\ell}{h+1/2000} > \frac{21}{100}$; $\frac{\ell}{h-1/80} < \frac{2}{9}$; $\frac{\ell}{h} < \frac{1}{6\kappa_m}$	Proposition 6.4, Lemma 7.3	0.211102; 0.214591; via $\frac{\ell}{h-1/80} < \frac{2}{9} \le \frac{1}{6\kappa_m}$
$Q_{\mathrm{sm}} < 0$; $C_0 + \frac{1-h}{16} < -\frac{1}{250}$	Proposition 6.2	$-7.07 \cdot 10^{-3}$; -0.011147
$C_0 + \frac{1/2+3\ell/2}{50} < -\frac{1}{125}$	Lemma 6.3	-0.0086400
$\frac{13h}{75} - \frac{l_y}{2} + \frac{1}{50} < -\frac{1}{20}$	Appendix A	-0.07224

Remark 10.4 (Scalar bounds for the variable moment parameter). The scalar estimates used

below satisfy the following bounds on the parameter sets of Table 4.

(a) On $\mathcal{R}_*$, the contour parameter satisfies $\theta_0 = \delta x/(6\kappa_m) \le 3/(60\kappa_m) < 0.06681$ in Corollary 6.7, Proposition 7.4 and the slope display in the proof of Theorem 8.4.

(b) The three requirements of [Liu26b, Remark 14.1(1)] hold as follows: the crude supply of [Liu26b, Proposition 14.6] holds for $\kappa_m > 2263/3024 = 0.74834\ldots$; the slopes in [Liu26b, Lemma 14.5] are below 1 for $\kappa \ge 13/18$ (bounds 0.7081 and 0.7399 for $\ell \ge 1/6$; 0.7064 and 0.7324 on $\mathcal{P}_\natural$); and $6\kappa\ell > 3/4$.

(c) The constants in Lemmas 7.3, 7.2, 8.3, Proposition 7.4 and Appendix A.3 are

$$\frac{1-c}{6},\quad W_{\max}(\kappa_m),\quad \frac{a}{3\kappa_m},\quad \frac{a(1-c)}{6},\quad \frac{1}{24\kappa_m},\quad \frac{17c}{20},\quad \frac{4c}{5},\quad c.$$

(d) The norm conditions in Proposition 7.4 are $(1-c)(1-2m) - 6W > 0$ and $1 - (\frac{5}{2} - 4m + 6\kappa_m W) > 0$, as in (10.9).

(e) The critical-length bound in Theorem 8.5 is $h\gamma \le \frac{1-c}{12}a(2l_y - h)$ with $a = \delta/2$; this is (10.24).

The small-amplitude and lowest-amplitude estimates follow from Proposition 6.2 and Lemma 6.3, with the bounds of Table 4. For $\delta < 1/8$ one may keep $\kappa_m = 3/4$, since Proposition 6.2 uses $\ell/h < 2/9 = 1/(6\cdot 3/4)$ and $R_{\kappa_c} \le R_{3/4}$.

10.4 The starting boundary for the enlarged count

In this subsection, we iterate the comparison from B_* over fields containing μ_{12} (Proposition 10.5), then descend to arbitrary number fields (Corollary 10.6). We also prove the optimality of $B_\natural$ in this comparison system (Theorem 10.8).

Proposition 10.5 (The boundary over fields containing μ_{12})**.** *For every number field $F \supset \mu_{12}$,*

$$\beta_*(F) \le B_\natural = 0.87446749650084027305\ldots.$$

Proof. Fix $F \supset \mu_{12}$. The verification in the proof of Proposition 9.1 concerns the Gauss identities, natural zeros and summation lengths over F. It does not involve the value of κ_m, and Lemma 10.1 is proved by the same steps as Theorem 4.9. Hence Lemma 10.1 holds over F.

Step 1: seed. By Proposition 9.1,

$$\beta_*(F) \le B_0 = B_*, \qquad \kappa_0 = 2B_* - 1 = \kappa_* \in [\kappa_\natural, 3/4].$$

Step 2: the family. For $\kappa \in [\kappa_\natural, \kappa_*]$ put

$$\kappa_c = \kappa_m = \kappa, \qquad \ell(\kappa) = \ell_\natural - \frac{\kappa - \kappa_\natural}{25}, \qquad b = b_\natural, \qquad \mathcal{Q}_\kappa = \mathcal{Q}_{\kappa,\kappa,b_\natural,\ell(\kappa)}. \tag{10.31}$$

This family lies in $\mathcal{P}_\natural$. On $\mathcal{R}_*$, by Table 4,

$$\frac{d}{d\kappa}\mathcal{Q}_\kappa = \partial_{\kappa_c}\mathcal{Q} + \partial_{\kappa_m}\mathcal{Q} - \frac{1}{25}\partial_\ell \mathcal{Q} \le -0.01247 < -\frac{1}{100},$$

and $\mathcal{Q}_{\kappa_\natural} \le 0$ by Proposition 10.3(iv). Hence

$$\mathcal{Q}_\kappa \le -\nu_\kappa \quad \text{on } \mathcal{R}_*, \qquad \nu_\kappa = \frac{\kappa - \kappa_\natural}{100}. \tag{10.32}$$

Step 3: one step. Let $n \ge 0$ and assume $\beta_*(F) \le B_n$ with $\kappa_n = 2B_n - 1 \in (\kappa_\natural, \kappa_*]$. Then

$$\beta_*(F) \le \frac{1+\kappa_n}{2}, \qquad 2B_n - 1 \le \kappa_c = \kappa_n, \qquad \frac{13}{18} \le \kappa_n \le \frac{3}{4}.$$

Use Proposition 6.4 at $\kappa_c = \kappa_n$ and Lemma 10.1, unamplified M2 and Lemma 5.2 at $\kappa_m = \kappa_n$. In the continuation argument of Theorem 8.4, take, using Table 4,

$$c = c_{\kappa_n}, \qquad 0 < \varepsilon < 1/1000, \qquad C\varepsilon + 2\zeta + \epsilon_{\mathrm{rem}} < \nu_{\kappa_n},$$
$$m_a = 3/8 + \varepsilon, \quad m_b = 9/20, \quad m_- = r = m, \qquad W(m) = (1-\varepsilon)W_{\mathrm{cap}}(m;\kappa_n).$$

For $m \in [m_a, m_b]$, $1 - 2m \geq 1/10$ and $8m - 3 \geq 8\varepsilon$, hence

$$(1-c)(1-2m) - 6W(m) \geq \frac{(1-c)\varepsilon}{10} > 0, \qquad 1 - \Big(\frac{5}{2} - 4m + 6\kappa_n W(m)\Big) \geq \frac{\varepsilon(8m-3)}{2} \geq 4\varepsilon^2 > 0,$$

so (10.10) and Lemma 10.1 apply on every selected dyadic sum. By Table 4, the slopes of the full comparison exponent satisfy

$$S - \theta_0(1-m) - \gamma \geq 0.1920 > \frac{1}{8}, \qquad S + \Big(\frac{\delta}{2} - \theta_0\Big)(1-m) - \gamma \geq 0.2360 > \frac{1}{8},$$

and both are at most $0.6997 < 2$; the extension to $d \leq h + \zeta$ loses at most 2ζ. The complementary ranges, short sums, lowest amplitudes and the principal and direct comparisons use the entries of Table 4 on $\mathcal{P}_{\natural,3/4}$. By (10.32), the continuation argument in the proof of Theorem 8.4 and [Liu26b, Theorem 14.12],

$$\beta_*(F) \leq B_{n+1} = \frac{11}{12} - \frac{\ell(\kappa_n)}{4} = B_\natural + \frac{\kappa_n - \kappa_\natural}{100}. \tag{10.33}$$

Step 4: iteration. By (10.2) and (10.33),

$$\kappa_{n+1} - \kappa_\natural = 2B_{n+1} - 1 - \kappa_\natural = \frac{\kappa_n - \kappa_\natural}{50}, \qquad B_n - B_\natural = \frac{B_* - B_\natural}{50^n}.$$

Thus

$$\kappa_n \in (\kappa_\natural, \kappa_*], \qquad \nu_{\kappa_n} > 0 \quad (n \geq 0), \qquad \beta_*(F) \leq \inf_n B_n = B_\natural.$$

Step 3 has its own threshold at each finite n. □

Corollary 10.6 (The earlier-count boundary over arbitrary fields)**.** *For every number field K and finite-order Hecke character χ, $L_K(s,\chi)$ has no zeros in $\Re s > B_\natural$. The principal pole at $s = 1$ is permitted.*

Proof. Apply the proof of Theorem 9.2 with B_* replaced by $B_\natural$ and Proposition 9.1 replaced by Proposition 10.5. For $F = K(\zeta_{12})$, $E = MF$ and $\Re s > B_\natural$, $s \neq 1$,

$$\zeta_E(s) \neq 0 \quad \Longrightarrow \quad \zeta_M(s) \neq 0 \quad \Longrightarrow \quad L_K(s,\chi) \neq 0.$$

The nonvanishing at $s = 1$ of the nonprincipal factors and the treatment of imprimitive characters are unchanged. □

Remark 10.7 (Relation to Theorem 8.5). Theorem 8.5 assumes

$$\kappa_m = \frac{3}{4}, \qquad 0 \leq W(m) \leq \frac{7}{54}(1-2m),$$

and its proof uses $h\gamma \leq \frac{7\delta}{216}(2l_y - h)$ and $\theta_0 = r_*/9$. The proof of Proposition 10.5 uses $\kappa_m = \kappa_n < 3/4$, the initial length $(1-2m)/(6\kappa_n)$ and the norm of Lemma 10.1. It lies outside those hypotheses, so $B_\natural < B_*$ is consistent with Theorem 8.5. Conversely, by (10.26), changing κ_m in M2 alone, with the norm of Theorem 4.9, leaves $\mathcal{Q}(\cdot, 1/2)$ unchanged. Pointwise $W_{\mathrm{cap}}(m;\kappa) \leq W_{\mathrm{cap}}(m;3/4)$ on the first branch, and the whole gain $B_* - B_\natural$ comes from $z_0(m) = (1-2m)/(6\kappa_m)$. Theorem 10.8 gives the corresponding optimality of $B_\natural$.

Theorem 10.8 (Optimality with a variable moment parameter)**.** *Fix the detector, class count, balanced-length hypotheses and direct theta estimate used above. On each amplitude class, allow dyadic-sum-dependent prime lengths and contours, the count of Proposition 6.4 at κ_c, and the norm of Lemma 10.1 at κ_m (Theorem 4.9 if $\kappa_m = 3/4$), with*

$$0 \le W(m) \le \frac{1 - c_{\kappa_m}}{6}(1 - 2m), \qquad 0 \le \theta(m) \le \delta/2,$$
$$1/6 \le \ell \le 1/5, \qquad B(\ell) = 11/12 - \ell/4.$$

Every allowed sequence of improvements in this comparison system, starting at $B_0 \ge B_\natural$, with

$$2B_n - 1 \le \kappa_{c,n} \le \frac{3}{4}, \qquad 2B_n - 1 \le \kappa_{m,n} \le \frac{3}{4},$$

satisfies

$$B_n \ge B_\natural \quad (n \ge 0), \qquad \liminf_{n\to\infty} B_n \ge B_\natural.$$

The sequence of Proposition 10.5 has $B_n - B_\natural = 50^{-n}(B_* - B_\natural) \to 0$. This optimality keeps the detector, moment estimates and allowed norm lengths fixed. The case $\kappa_m = 3/4$ is the system of Theorem 8.5.

Proof. Suppose a first crossing satisfies $B_n \ge B_\natural$ and $B_{n+1} = B(\ell) < B_\natural$. Then

$$\kappa_c, \kappa_m \ge 2B_n - 1 \ge \kappa_\natural, \qquad \ell > \ell_\natural.$$

The retained geometry of the proof of Theorem 8.5 gives $m_\theta > 32/81 > 3/8$ and $1/2 - m_\theta > 0$. At the class $(\delta, x) = (r_\natural, 1/2)$, with $c = c_{\kappa_m}$ and $(1 - 2m_\theta)h = 2l_y - h$,

$$h(1 - m_\theta) - l_y = 0, \qquad \gamma \le \frac{\delta W}{4}, \qquad h\gamma \le \frac{1 - c}{12}\,\frac{\delta}{2}\,(2l_y - h) = h\Gamma_N,$$

by (10.24). Use the exponents in (8.3) with

$$r = r_\natural, \quad R = R_{\kappa_c}(r_\natural, 1/2), \quad R_\natural = R_{\kappa_\natural}(r_\natural, 1/2),$$
$$A = \frac{1 + 5c_{\kappa_m}}{24} \ge A_0 = \frac{19}{216}, \qquad R \ge R_\natural.$$

For $0 \le m \le 1/2$,

$$\begin{aligned} P_{\mathrm{pt}}(m; R) - P_4(m; R, A) &\ge \frac{R_\natural - 1}{4} + \frac{19 r_\natural}{216}(1 - 2m) + \frac{r_\natural m}{2} \\ &= \frac{35 r_\natural}{108}(m - \mu_\natural), \qquad \mu_\natural := \frac{54(1 - R_\natural) - 19 r_\natural}{70 r_\natural}. \end{aligned}$$

Equations (8.4) and (A.6) give

$$\mu_\natural < \frac{399030}{10^6} < \frac{399165}{10^6} < g(\ell_\natural) < g(\ell) < m_\theta.$$

Hence the buffered pointwise estimate, absolute ideal counting and the unamplified moment do not improve this optimistic amplified comparison at m_θ. The full contour contribution vanishes there, so

$$\mathcal{Q}_{\kappa_c, \kappa_m, b, \ell}(r_\natural, \tfrac{1}{2}) \le 0 \tag{10.34}$$

is necessary.

By (10.23), (10.24), $a(\delta, 1/2) = \delta/2$ and Lemma 6.5,

$$\partial_{\kappa_c} \mathcal{Q} = \frac{3h}{4}\,\partial_{\kappa_c} R_{\kappa_c} \ge 0, \qquad \partial_{\kappa_m} \mathcal{Q}(\delta, \tfrac{1}{2}) = \frac{5\delta(1 + b - 5\ell)}{288\kappa_m^2} > 0.$$

At $\kappa_c = \kappa_m = k$,

$$\mathcal{Q}_{k,k,b,\ell}(\delta, \tfrac{1}{2}) = \frac{b\, n_b^\natural(\delta,k) + \ell\, n_e^\natural(\delta,k) + n_0^\natural(\delta,k)}{288k\, L(\delta,k)}, \qquad L = 54k\delta - 75k - 6\delta + 10,$$

with

$$\begin{aligned} n_b^\natural &= -4212\delta^2k^2 + 90\delta^2k + 30\delta^2 + 9522\delta k^2 - 783\delta k - 50\delta - 3600k^2 + 480k, \\ n_e^\natural &= -2268\delta^2k^2 + 1278\delta^2k - 150\delta^2 + 25830\delta k^2 - 4725\delta k + 250\delta - 27000k^2 + 3600k, \\ n_0^\natural &= 3564\delta^2k^2 - 774\delta^2k + 30\delta^2 - 7758\delta k^2 + 1377\delta k - 50\delta + 5400k^2 - 720k. \end{aligned}$$

Substitute $\delta = r$, $k = 5/6 - \ell_\natural(r)/2$ with $\ell_\natural(r)$ the cubic of (10.2). Polynomial division by $q_\natural(r)$ gives

$$n_b^\natural(r_\natural, \kappa_\natural) = 0, \qquad \ell_\natural n_e^\natural(r_\natural, \kappa_\natural) + n_0^\natural(r_\natural, \kappa_\natural) = 0.$$

Hence, for every b,

$$\mathcal{Q}_{\kappa_\natural,\kappa_\natural,b,\ell}(r_\natural, \tfrac{1}{2}) = e_\natural(\ell - \ell_\natural), \qquad e_\natural = \frac{n_e^\natural(r_\natural, \kappa_\natural)}{288\kappa_\natural L(r_\natural, \kappa_\natural)}.$$

On $[4505, 4506] \times [7489, 7490] \cdot 10^{-4} \ni (r_\natural, \kappa_\natural)$, termwise bounds give $n_e^\natural < -7490$ and $-31 < L < -30$. Thus

$$e_\natural > \frac{7490}{288 \cdot \frac{3}{4} \cdot 31} = \frac{7490}{6696} > 1, \qquad e_\natural = 1.133906801218591729 82\ldots.$$

Therefore

$$\kappa_c, \kappa_m \geq \kappa_\natural, \quad \ell > \ell_\natural \quad \Longrightarrow \quad \mathcal{Q}_{\kappa_c,\kappa_m,b,\ell}(r_\natural, \tfrac{1}{2}) \geq e_\natural(\ell - \ell_\natural) > 0,$$

contrary to (10.34). If $B_n \geq B_\natural$, then $\kappa_{c,n}, \kappa_{m,n} \geq 2B_n - 1 \geq \kappa_\natural$, so the next step has $\ell \leq \ell_\natural$ and $B_{n+1} = 11/12 - \ell/4 \geq B_\natural$. By induction $B_n \geq B_\natural$ for every n; neither a finite iteration nor a limit crosses $B_\natural$. The sequence of Proposition 10.5 reaches $B_\natural$. □

11 The enlarged class count and its algebraic boundary

In this section, we apply the enlarged amplified norm to the counting detector. The additional prime product improves the M2 count as well as the numerator estimate. Starting from Proposition 10.5, we obtain the boundary B' of Theorem 1.2.

11.1 The extended moment range

In this subsection, we extend Lemma 10.1 to $\kappa_\flat \leq \kappa \leq 3/4$. Lemma 11.1 supplies the moment estimate used in Section 11.2.

Put $\kappa_\flat = 1871/2500 = 0.7484$ and retain $c_\kappa = 1/(6\kappa)$ and $z_S(m) = c_\kappa(1-2m) = z_0(m)$. Write

$$\begin{gathered} W_1(m) = \frac{(1-c_\kappa)(1-2m)}{6}, \qquad W_2(m) = \frac{8m-3}{12\kappa}, \qquad W_{\mathrm{cap}}(m;\kappa) = \min\{W_1(m), W_2(m)\}, \\ m^\sharp = \frac{6\kappa+8}{12\kappa+22}, \qquad W_{\max}(\kappa) = \frac{6\kappa-1}{12\kappa(6\kappa+11)}. \end{gathered} \tag{11.1}$$

Thus $W_1(m^\sharp) = W_2(m^\sharp) = W_{\max}$, and $W_2 > 0$ precisely when $m > 3/8$.

Lemma 11.1 (The norm below the earlier endpoint)**.** *Let $\kappa_\flat \le \kappa \le 3/4$ and assume $\beta_*(F) \le (1+\kappa)/2$. Retain the arithmetic, coefficient, smooth-weight and summation conventions of Lemma 10.1. Suppose*

$$\begin{aligned} 3/8 < m_- \le r < 1/2, \qquad W \ge 0, \\ \eta_\kappa = (1-c_\kappa)(1-2r) - 6W > 0, \qquad C_{\mathrm{add}} = \frac{5}{2} - 4m_- + 6\kappa W < 1. \end{aligned} \tag{11.2}$$

For $m \in [m_-, r]$, $0 \le z \le z_S(m)$, $0 \le w \le W$, and complete disjoint prime products Q_0, Q_W of lengths z, w, the centered and original estimates (4.15)–(4.16) hold. In particular,

$$\sum_{k \in \mathcal{U}} \mathcal{W}(q_k/U) |S_{m,1}(k) S_{m,2}(k) Q_0(k) Q_W(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C. \tag{11.3}$$

The conclusion includes the finite independent derivative families and the order of choices of Theorem 4.9.

Proof. Use the proof of Lemma 10.1 with $k_6 = 6\kappa$. The estimate M2 (10.5) is valid, since

$$\kappa \ge \kappa_\flat > 13/18, \qquad 6\kappa - 1 \ge 3.4904 > 10/3.$$

The identities (10.13)–(10.19) and (10.20) are unchanged. In particular,

$$\begin{gathered} (6\kappa-1) z_{\mathrm{init}} - D_{\mathrm{free}} \ge \eta_\kappa, \qquad z_{\mathrm{init}} = 0 \Longrightarrow D_{\mathrm{free}} \le -\eta_\kappa, \\ M_* \ge \frac{\eta_\kappa + 6c_\kappa W}{1-c_\kappa} - o(1), \qquad m_* \ge \frac{\eta_\kappa + (7c_\kappa - 1)W}{1-c_\kappa} - o(1), \\ A_t \le 1 - 5W - \eta_\kappa, \qquad 1 - \frac{W}{W_{\max}} = \frac{4\eta_\kappa}{1-c_\kappa} + 2(1 - C_{\mathrm{add}}) + 8(r - m_-) > 0. \end{gathered}$$

The restoration, homogeneous coefficient estimate and split between Propositions 4.6 and 4.8 therefore retain their strict margins. The uniform numerical bounds on this interval are those of Table 5 and the display above. Remark 10.2(b) uses only $\kappa \ge 2/3$, and Remark 10.2(c) is not used. For this table put

$$\rho_z(\kappa) = \frac{57}{500}(1-c_\kappa) - \frac{3}{50}.$$

Both $1 - c_\kappa$ and ρ_z increase with κ, whereas $z_L = 1/(24\kappa)$ decreases. Moreover,

$$W'_{\max}(\kappa) = -\frac{36\kappa^2 - 12\kappa - 11}{12\kappa^2(6\kappa+11)^2} < 0, \qquad 36\kappa_\flat^2 - 12\kappa_\flat - 11 = 0.18289216 > 0.$$

Exact rational evaluation at $\kappa_\flat$ or $3/4$ gives every entry. The mesh condition in Remark 10.2(b) uses only $\kappa \ge 2/3$. Fix the strict margins, then the steps, meshes and local losses, then the finite test families and internal orders, before the external heights. This is the order of Lemma 10.1. □

Table 5: Uniform norm bounds on $[\kappa_\flat, 3/4]$. Terminating decimals are rational bounds.

quantity	bound
$1-c_\kappa$; $7c_\kappa-1$	≥ 0.7773026; $\geq 5/9$
$W_{\max}(\kappa)$	≤ 0.0250898199
$-1/2+3W_{\max}+7/200$	$\leq -0.3897305 < -3/8$
$\rho_z(\kappa)$	$\geq 0.0286125 > 0$
$0.95-\rho_z$; $0.01-\rho_z$	≤ 0.9213875; ≤ -0.0186125
$0.99-\rho_z$; $0.98-\rho_z$	≤ 0.9613875; ≤ 0.9513875
$5/2-8/5+6\kappa/100$	$\leq 0.945 < 1$
$z_L = 1/(24\kappa)$	$\leq 0.0556744 < 21/100$
$2x(21/100)-z_L-W_{\max}$, $x\geq 2/5$	$\geq 0.0872358 > 0$
$(21/100)(6\kappa-1/8)-11/12$ (crude supply at $\delta=1/8$)	$\geq 0.0000673 > 0$
$\theta_0 \leq 3/(60\kappa)$ on $\mathcal{R}_*$	< 0.06681

11.2 The counting detector and the additional prime product

In this subsection, we insert a disjoint additional prime product into the zero-detecting polynomial. Lemma 11.1 gives the improved count used in Section 11.3.

Let $\mathscr{B}$ be a detected amplitude class of [Liu26b, Section 13.4]. Its original prime factors satisfy

$$\sum_i \lambda_i = \mathscr{L} > \frac{1}{5}, \qquad \sum_i \lambda_i g_i = g\mathscr{L}, \qquad g=\delta x, \quad 0\leq g_i\leq \delta/2, \tag{11.4}$$

where $0<\delta\leq 5/6$, $0\leq x\leq 1/2$ and $\lambda_i \leq \eta_{\mathrm{mesh}}$. Positive-amplitude factors have their original lower bounds $|Q_i(k)| \geq U^{\lambda_i g_i - o(1)}$ on the class.

Lemma 11.2 (Length conditions of the counting detector)**.** *Assume the hypotheses and lengths of Lemma 11.1. Fix a presentation, a detecting dyadic pair and disjoint products Q_0, Q_W of complete original prime factors. Put*

$$P_k(\mathbf{t}) = S_m(k;\mathbf{t})^2 Q_0(k;\mathbf{t}) Q_W(k;\mathbf{t}), \qquad \mathbf{t}\in\mathcal{B}_T.$$

Lemma 11.1 applies to every required mixed Sobolev derivative, and

$$\sum_{k\in\mathcal{U}} \mathcal{W}(q_k/U) \sup_{\mathbf{t}\in\mathcal{B}_T} |P_k(\mathbf{t})|^2 \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C(1+T_{\mathcal{B}})^A,$$

where A, C, J depend only on the fixed derivative orders and data.

Proof. By [Liu26b, Lemmas 13.7 and 13.9], after removing the deterministic scale phase, the centrally normalized weight satisfies

$$w_{\sigma,t}(y) = \rho(y) y^{-\sigma-it}, \qquad \partial_\sigma^a \partial_t^b w_{\sigma,t}(y) = (-1)^a(-i)^b(\log y)^{a+b} w_{\sigma,t}(y),$$

with fixed annular cutoff ρ. Retain the original conductor-dependent weight families. The fixed group Θ contains the detector's finite coefficient group and its conjugates. Conjugate the entire presentation, including both ordinary weights; retain its original prime coefficients and Mellin frequencies.

Fix the presentation and pair before $\mathcal{B}_T$. The numbers of presentations and pairs are $O(1)$ and $O((\log U)^2)$, respectively. Leibniz's rule gives

$$\partial^\alpha P_k = \sum_\iota c_{\alpha,\iota} S_{m,1}^{(\iota)}(k) S_{m,2}^{(\iota)}(k) Q_0^{(\iota)}(k) Q_W^{(\iota)}(k), \qquad \max_{\phi\in\mathbf{w}_{\alpha,\iota}} \|\phi\|_{C^J} \ll (1+T_{\mathcal{B}})^{A_J},$$

where $\mathbf{w}_{\alpha,\iota}$ consists of independently allowed smooth weights. Repeated Sobolev gives

$$\sum_k \sup_{\mathbf{t}\in\mathcal{B}_T} |P_k(\mathbf{t})|^2 \ll \sum_{\alpha\in\{0,1\}^{\dim \mathcal{B}_T}} \int_{\mathcal{B}_T^+} \sum_k |\partial^\alpha P_k(\mathbf{t})|^2 \, d\mathbf{t}.$$

Insert $\mathcal{W}(q_k/U)$ and apply (11.3) inside each integral. The volume and seminorm loss is $(1+T_{\mathcal{B}})^A$.

Separate the exceptional bounded unit classes. The remaining parameters lie in $\mathcal{U}$. Fix the finite height orders before choosing $T_{\mathcal{B}} = Z^{\tau_{\mathcal{B}}}$ to absorb its polynomial loss into the prescribed power of U. □

For $j = 0, 1, 2$ define

$$\begin{aligned}
\widetilde{c} = \frac{2}{3\kappa}, \qquad (\pi_0, \varpi_0) &= (\widetilde{c}x/2, 2 - \widetilde{c}x), \qquad (\pi_2, \varpi_2) = (-\widetilde{c}x/4, 2 + \widetilde{c}x), \\
(\pi_1, \varpi_1) &= \left(x\left(\frac{1}{3} + \frac{5\widetilde{c}}{12}\right), 2 - 2x\left(\frac{1}{3} + \frac{5\widetilde{c}}{12}\right)\right), \\
\mathcal{J}_j &= (5/6 - \delta)(1 - x + \varpi_j) + \delta(1 - x)\varpi_j, \\
t_j = 1 + \frac{\delta(1-x)(1-\pi_j)}{\mathcal{J}_j}, \qquad R_\kappa^{(j)} &= 1 - \delta + (5/6 - \delta)(t_j - 1), \\
m_*^{(j)} = \frac{(1-x)t_j + x - \pi_j}{1 - x + \varpi_j}, \qquad R'_\kappa &= \max\{R_\kappa^{(1)}, \min(R_\kappa^{(0)}, R_\kappa^{(2)})\}.
\end{aligned} \tag{11.5}$$

All denominators are positive on the stated class range.

Proposition 11.3 (The enlarged class count)**.** *Suppose $\beta_*(F) \le B_{\mathrm{seed}}$ is already proved and*

$$\kappa_\flat \le \kappa_c \le 3/4, \qquad 2B_{\mathrm{seed}} - 1 \le \kappa_c.$$

For the classes (11.4) and every $\epsilon > 0$,

$$\#\mathscr{B} \ll U^{R'_{\kappa_c}(\delta,x)+\epsilon}(1+H)^C, \qquad 1 - \delta \le R'_{\kappa_c} \le R_{\kappa_c}. \tag{11.6}$$

The M2 count exponent is replaced by

$$A'_S(m) = 1 - 2\delta m - 2gz_S(m) - 2gW_{\mathrm{cap}}(m;\kappa_c)_+, \quad 0 \le m \le 1/2. \tag{11.7}$$

For $m \le 3/8$ this is the old exponent $A_S(m)$.

Proof. Write $\kappa = \kappa_c$. The premise gives $\beta_*(F) \le (1+\kappa)/2$. Retain the detector, M1 and long inverse count of [Liu26b, Section 13.4]; only M2 changes.

Strict length margins. Set aside $r \ge 1 - \epsilon_c$ and $m \ge 1/2 - \epsilon_c$; their estimates are given below after the balancing length is fixed. Choose $0 < \epsilon_1 < \epsilon_c/100$ and a finite grid $\mu_0 = 3/8 + \epsilon_1 < \cdots < \mu_N = 1/2 - \epsilon_c$ of step at most ϵ_1^2. On $[\mu_{i-1}, \mu_i]$, put

$$W_i = (1 - \epsilon_1) \min\left\{ \frac{(1 - c_\kappa)(1 - 2\mu_i)}{6}, \frac{8\mu_{i-1} - 3}{12\kappa} \right\}.$$

For $m_- = \mu_{i-1}$ and $r = \mu_i$,

$$\eta_\kappa \ge \epsilon_1\epsilon_c, \qquad 1 - C_{\mathrm{add}} \ge 4\epsilon_1^2, \qquad 0 \le W_{\mathrm{cap}}(m;\kappa) - W_i \le W_{\max}\epsilon_1 + \epsilon_1^2\left(\frac{1}{3} + \frac{2}{3\kappa}\right).$$

For $m \le 3/8 + \epsilon_1$ retain the old M2. The enlarged range uses

$$z_S(m) + W_i \le \frac{1}{24\kappa} + W_{\max}(\kappa) < 0.081 < 1/5 < \mathscr{L}.$$

Joint selection. Fix the presentation and dyadic pair, and order the positive-amplitude original factors by decreasing g_i. Every such initial segment satisfies

$$\sum_{i\in I}\lambda_i g_i \ge g\sum_{i\in I}\lambda_i. \tag{11.8}$$

Put $z = z_S(m) - \epsilon_c/100$, $w = W_i$. If the positive length is at least $z+w$, take the shortest initial segment of complete factors of length at least $z+w-\eta_{\mathrm{mesh}}$. Split it into a maximal initial run of length at most z, forming Q_0, and the remaining factors, forming Q_W. The first length exceeds $z-\eta_{\mathrm{mesh}}$; the second is at most $w+\eta_{\mathrm{mesh}}$. Choose η_{mesh} after the grid margins, with $6\eta_{\mathrm{mesh}} < \epsilon_1\epsilon_c$ and $6\kappa\eta_{\mathrm{mesh}} < 4\epsilon_1^2$, so Lemma 11.1 applies with $W_i+\eta_{\mathrm{mesh}}$. If the positive length is smaller than $z+w$, retain all positive factors and make the same split. Their total amplitude is $g\mathscr{L} \ge g(z+w)$, and their lengths still fit. In either case,

$$|Q_0Q_W|^2 \ge U^{2g(z+w)-O(\eta_{\mathrm{mesh}})-o(1)}.$$

This lower bound concerns the joint product; it requires no separate lower mean for Q_W.

The detector gives $|S_m|^4 \ge U^{2\delta m-\epsilon_2}$. Let $\mathscr{B}_{D,N}\subseteq\mathscr{B}$ be the part detected with the fixed presentation and dyadic pair (D,N). Choose the products before the common supremum and apply Lemma 11.2:

$$\#\mathscr{B}_{D,N}\,U^{2\delta m+2g(z+w)-O(\eta_{\mathrm{mesh}})-\epsilon_2} \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C(1+T_{\mathcal{B}})^A.$$

Sum over the logarithmic number of pairs and absorb the height, grid and mesh losses to obtain (11.7).

Balancing the counts. Put $f_j(m) = 1-\delta(\pi_j+\varpi_j m)$. Then

$$f_0 = A_S,\quad f_1 = A_S - 2gW_1,\quad f_2 = A_S - 2gW_2,\qquad A'_S = \max\{f_1, \min(f_0,f_2)\}.$$

The successive branches on $m\le 3/8$, $3/8\le m\le m^\sharp$ and $m\ge m^\sharp$ are $0,2,1$. Since $\widetilde{c}\le 12/13$, $\varpi_j>1$, and each branch decreases with m. For fixed $1\le t\le 3/2$, use M1 when $r\ge r_*(t)$ and M2 when $r\le r_*(t)$, where

$$A_I(r) = 1-\delta\{x+(1-x)r\},\qquad A_I(r_*) = A'_S(t-r_*).$$

The detector constraint $r+m\ge t-O(1/\log U)$ justifies M2 at $m=t-r_*$. For one branch, crossing also with the long count gives

$$r_* = \frac{\pi_j+\varpi_j t - x}{1-x+\varpi_j},\qquad A_I(r_*) = 1-\delta+(5/6-\delta)(t-1).$$

The solution is $t=t_j$, $m=m_*^{(j)}$ and count $R_\kappa^{(j)}$. For fixed t, let $C_j(t)$ be this short-count crossing value and put

$$C(t) = \max\{C_1(t), \min(C_0(t), C_2(t))\},\qquad L(t) = 1-\delta+(5/6-\delta)(t-1).$$

Each C_j decreases strictly with t. Crossing the decreasing $A_I(r)$ with the increasing $f_j(t-r)$ preserves maxima and minima of the crossing values; hence $C(t)$ is the short-count bound. For $0<\delta<5/6$, L increases strictly. Its crossing with C is at

$$t_* = \max\{t_1, \min(t_0,t_2)\},\qquad L(t_*) = \max\{R_\kappa^{(1)}, \min(R_\kappa^{(0)}, R_\kappa^{(2)})\} = R'_\kappa.$$

Indeed, the sign of $C_j(t)-L(t)$ is the sign of t_j-t.

The lengths are allowed:

$$0\le t_j-1\le\frac{1-\pi_j}{\varpi_j}\le\frac12.$$

At fixed t, $A'_S \le A_S$ moves the crossing to larger r, and the old crossing increases with t. At $t = 1$, with $\widetilde{c} \ge 2/3$ and $x \le 1/2$,

$$r_*^{(0)}(1) = \frac{2 - x - \widetilde{c}x/2}{3 - x - \widetilde{c}x} \ge \frac{2 - 4x/3}{3 - 5x/3} \ge \frac{8}{13},$$

since the left quotient increases with $\widetilde{c}$ and the middle one decreases with x. Thus $r_* \ge 8/13$, and the M1 prime length is at most $(1 - r_*)/2 \le 5/26$, as in the proof of Proposition 6.4. When branch 0 is active, the old crossing has $m_* \ge 1/3$ and $z_S(m_*) \le 1/13$; otherwise $m_* \ge 3/8$ and the 0.081 bound applies. Fix the detector with this t_*. The long inverse moment, with $e(r) = \max\{1, (1 + 5r)/6\}$, gives

$$\begin{aligned} r \ge 1: \quad e(r) - \delta r &= 1/6 + (5/6 - \delta)r \\ &\le 1/6 + (5/6 - \delta)t_* + o(1) = R'_\kappa + o(1), \end{aligned}$$

since $r \le t_* + O(1/\log U)$. For the two endpoint strips the unamplified estimates give

$$\begin{aligned} 1 - \epsilon_c \le r < 1: &\quad 1 - \delta r \le 1 - \delta + \delta\epsilon_c \le R'_\kappa + \delta\epsilon_c, \\ m \ge 1/2 - \epsilon_c: &\quad 1 - 2\delta m \le 1 - \delta + 2\delta\epsilon_c \le R'_\kappa + 2\delta\epsilon_c. \end{aligned}$$

Thus all three ranges satisfy the required count. Finally $R_\kappa^{(0)} = R_\kappa$, and $A'_S \le A_S$ gives $R'_\kappa \le R_\kappa$. At $x = 0$ or $\delta = 5/6$, the three $R^{(j)}$ coincide, so the old count gives the conclusion directly.

Fix κ and the requested loss, then ϵ_c, ϵ_1, the grid and its strict margins, and then the prime mesh. Choose the detector buffer, finite internal orders and external height exponent in the order of Lemma 11.2. All losses are absorbed into ϵ. □

11.3 The new comparison and its endpoint

In this subsection, we combine the improved count with the numerator estimate and determine B'. Appendix B verifies the identities and signs in Proposition 11.4.

Use R'_{κ_c} in the comparison, independently of the moment parameter κ_m. With the notation of (10.8), put

$$\begin{aligned} E' &= C_0 + \frac{\delta}{2} + \delta x \left\{ \ell + c\left(\frac{h}{2} - l_y\right)\right\} - \frac{3h}{4}(1 - R'_{\kappa_c}), \\ \mathcal{Q}' &= E' - \frac{1 - c}{12} a(\delta, x)(2l_y - h), \qquad c = \frac{1}{6\kappa_m}. \end{aligned} \tag{11.9}$$

Write $E^{(j)}, \mathcal{Q}^{(j)}$ for these expressions with R' replaced by $R^{(j)}$. Since $3h/4 > 0$,

$$\mathcal{Q}' = \max\{\mathcal{Q}^{(1)}, \min(\mathcal{Q}^{(0)}, \mathcal{Q}^{(2)})\} \le \max\{\mathcal{Q}^{(1)}, \mathcal{Q}^{(2)}\}. \tag{11.10}$$

The same identity holds for E'. The two uses of the enlarged norm combine by

$$\sum_{k \in \mathscr{B}} |N_k| \le (\#\mathscr{B})^{3/4} \left(\sum_{k \in \mathscr{B}} |N_k|^4 \right)^{1/4}.$$

The counting and numerator products are chosen separately, each from the original factors and within its own length inequalities.

Proposition 11.4 (The enlarged-count endpoint). *Let δ' be the unique root in $(11/25, 23/50)$ of*

$$q_{\mathrm{ec}}(\delta) = 29562\delta^4 - 277427\delta^3 + 836708\delta^2 - 930160\delta + 272800. \tag{11.11}$$

Put

$$\begin{aligned}\ell' &= -\frac{142789387}{222821985}\delta'^3 + \frac{6250132787}{1336931910}\delta'^2 - \frac{1789275793}{222821985}\delta' + \frac{386361163}{133693191},\\ \kappa' &= \frac{5}{6} - \frac{\ell'}{2} = 2B' - 1, \qquad B' = \frac{11}{12} - \frac{\ell'}{4}.\end{aligned} \tag{11.12}$$

For $\kappa_c = \kappa_m = 5/6 - \ell/2$ and $x = 1/2$, set

$$\begin{aligned}D_2 &= -48(3\ell - 5)(54\delta\ell - 102\delta - 75\ell + 145),\\ D_1 &= -24(3\ell - 5)(144\delta\ell - 210\delta - 195\ell + 275),\\ p_b &= (2106\ell^2 - 7812\ell + 7230)\delta^2 + (-4761\ell^2 + 17538\ell - 16105)\delta\\ &\quad + 1800\ell^2 - 6480\ell + 5800,\\ p_0 &= (1134\ell^3 - 4626\ell^2 + 7302\ell - 5010)\delta^2\\ &\quad + (-12915\ell^3 + 48309\ell^2 - 51097\ell + 11495)\delta\\ &\quad + 13500\ell^3 - 51300\ell^2 + 53220\ell - 8700,\\ b' &= -\frac{\partial_\delta p_0}{\partial_\delta p_b}(\delta', \ell').\end{aligned}$$

Then $D_2\mathcal{Q}^{(2)}(\delta, 1/2) = bp_b + p_0$,

$$\mathrm{Res}_\ell(p_b, p_0) = -8398080\delta^2(2\delta - 5)(3\delta - 5)(6\delta - 5)^2 q_{\mathrm{ec}}(\delta),$$

where the factors other than q_{ec} have no root in $(11/25, 23/50)$, and

$$6687\ell'^4 - 25401\ell'^3 + 23297\ell'^2 + 2045\ell' - 900 = 0. \tag{11.13}$$

The number B' is the root specified in (1.1). Numerically,

$$\begin{aligned}\delta' &= 0.44878702795803278950\ldots, \qquad b' = 0.08218969571904572506\ldots,\\ \ell' &= 0.16978166469505833354\ldots, \qquad \kappa' = 0.74844250098580416656\ldots,\\ B' &= 0.87422125049290208328\ldots, \qquad B_\natural - B' = 0.00024624600793818977\ldots.\end{aligned} \tag{11.14}$$

At $(b', \ell', \kappa', \kappa')$,

$$\mathcal{Q}'(\delta, 1/2) \le 0 \quad (7/20 \le \delta \le 3/5), \qquad \mathcal{Q}'(\delta, 1/2) = 0 \iff \delta = \delta'.$$

Proof. The numerator identity and resultant follow by rational polynomial arithmetic. Substitution of the cubic (11.12) gives $p_b(\delta', \ell') = p_0(\delta', \ell') = 0$ modulo q_{ec}. The denominator defining b' is nonzero. Hence the quadratic $b'p_b(\delta, \ell') + p_0(\delta, \ell')$ has a double zero at δ':

$$\mathcal{Q}^{(2)}(\delta, 1/2) = \frac{A'(\delta - \delta')^2}{D_2(\delta, \ell')}, \qquad -3407.843 < A' < -3407.842. \tag{11.15}$$

Both $D_i(\delta, \ell')$ are positive on $[7/20, 3/5]$. The quadratic $D_1\mathcal{Q}^{(1)}(\delta, 1/2) = A_1\delta^2 + B_1\delta + C_1$ satisfies

$$-2923.884 < A_1 < -2923.882, \qquad -906.675 < B_1^2 - 4A_1C_1 < -906.672 < 0.$$

Thus $\mathcal{Q}^{(1)} < 0$, $\mathcal{Q}^{(2)} \le 0$, and (11.10) applies. At $\delta = \delta'$, rational interval evaluation gives

$$R^{(1)} < R^{(2)} < R^{(0)}, \qquad 0.40249693 < m_*^{(2)} < 0.40249694 < m^\sharp.$$

Therefore $R' = R^{(2)}$ and equality holds there. Exact root isolation and the sign certificates are given in Appendix B. They also verify that the polynomials defining ℓ' and B' are irreducible of degree four. □

11.4 The strict comparison inequalities

In this subsection, we verify strict inequalities for parameters approaching the endpoint of Proposition 11.4. Section 11.5 uses these inequalities at each finite iteration.

For the final iteration use

$$\begin{gathered}\kappa_c, \kappa_m \in [\kappa', \kappa_\natural], \qquad \ell \in [\ell' - (\kappa_\natural - \kappa')/25, \ell'], \qquad b = b', \\ \mathcal{R}_* = [7/20, 3/5] \times [2/5, 1/2], \qquad \mathcal{G} = [1/8, 5/6] \times [0, 1/2], \\ \mathcal{C}_1 = [1/8, 5/6] \times [0, 2/5], \quad \mathcal{C}_2 = [1/8, 7/20] \times [2/5, 1/2], \quad \mathcal{C}_3 = [3/5, 5/6] \times [2/5, 1/2].\end{gathered} \tag{11.16}$$

Call the parameter family in the first line $\mathcal{P}$, and write $\mathcal{P}_{3/4}$ when κ_m is allowed up to $3/4$. Put

$$\begin{gathered}R = R'_{\kappa_c}, \quad \theta_0 = \frac{\delta x}{6\kappa_m}, \quad S = \frac{3R+1}{4} + \frac{\theta_0}{2} - \frac{17}{50}, \quad N = \frac{3R+1}{4} - \frac{\theta_0}{2}, \\ \lambda = \frac{l_y}{h}, \quad m_\theta = 1 - \lambda, \quad \Gamma_N = \frac{1-c}{12} a(2\lambda - 1), \quad \Gamma_L = \theta_0(5/8 - \lambda), \\ \gamma_{\max} = \frac{aW_{\max}(\kappa_m)}{2}, \qquad \Delta_{\text{short}} = R + (1/2 - \theta_0)\frac{3}{40} - N.\end{gathered} \tag{11.17}$$

Lemma 11.5 (The comparison margins)**.** *The inequalities in Tables 6 and 7 hold on the indicated domains. At the endpoint parameters,*

$$\mathcal{Q}'_{\kappa',\kappa',b',\ell'}(\delta, x) \le 0 \quad \text{on } [7/20, 3/5] \times [0, 1/2], \tag{11.18}$$

with equality only at $(\delta', 1/2)$*. Along*

$$\kappa_c = \kappa_m = \kappa, \qquad \ell(\kappa) = \ell' - \frac{\kappa - \kappa'}{25}, \qquad b = b', \tag{11.19}$$

one has, for $\kappa' \le \kappa \le \kappa_\natural$*,*

$$\mathcal{Q}'_\kappa(\delta, x) \le -\frac{\kappa - \kappa'}{100} \quad \text{on } \mathcal{R}_*. \tag{11.20}$$

Proof. The rational interval calculation of Appendix B proves the table entries on enclosing rational boxes, treating the three rational branches separately. At the endpoint, $\partial_x \mathcal{Q}^{(j)} > 0.009$ for every j. Proposition 11.4 and (11.10) therefore give (11.18). On the parameter family,

$$\partial_{\kappa_c} \mathcal{Q}^{(j)} + \partial_{\kappa_m} \mathcal{Q}^{(j)} - \frac{1}{25} \partial_\ell \mathcal{Q}^{(j)} < -\frac{1}{100}.$$

For $\kappa_2 > \kappa_1$ in the family,

$$\mathcal{Q}^{(j)}_{\kappa_2} - \mathcal{Q}^{(j)}_{\kappa_1} \le -\frac{\kappa_2 - \kappa_1}{100} \quad \Longrightarrow \quad \mathcal{Q}'_{\kappa_2} - \mathcal{Q}'_{\kappa_1} \le -\frac{\kappa_2 - \kappa_1}{100}.$$

The implication follows from (11.10); setting $\kappa_1 = \kappa'$ proves (11.20), including branch changes. □

Table 6: Comparison inequalities with the enlarged count. Unless otherwise specified, the class domain is $\mathcal{R}_*$ and the parameter family is $\mathcal{P}$. Endpoint means $(\kappa_c,\kappa_m,b,\ell)=(\kappa',\kappa',b',\ell')$.

condition	certified bound
$\partial_x\mathcal{Q}^{(j)}$, $j=0,1,2$, endpoint, $[7/20,3/5]\times[0,1/2]$	>0.009
$\partial_{\kappa_c}\mathcal{Q}^{(j)}+\partial_{\kappa_m}\mathcal{Q}^{(j)}-\partial_\ell\mathcal{Q}^{(j)}/25$	$<-0.011<-1/100$
$\partial_{\kappa_c}\mathcal{Q}^{(j)}$; $\partial_{\kappa_m}\mathcal{Q}^{(j)}$; $\partial_\ell\mathcal{Q}^{(j)}$	>0; >0; $>3/4$
$\mathcal{J}_j$ on $\mathcal{G}\times\mathcal{P}_{3/4}$	>0.08
E' on $\mathcal{G}\times\mathcal{P}_{3/4}$	$<1/500$
E' on each $\mathcal{C}_i\times\mathcal{P}_{3/4}$	$<-10^{-5}$
S on $\mathcal{R}_*$; S on $\mathcal{G}\times\mathcal{P}_{3/4}$	$>1/5$; <1
$(6\kappa_m-2\delta x)(21/100)-R$ on $\mathcal{G}\times\mathcal{P}_{3/4}$	>0.019
$(21/100)(6\kappa_m-\delta)-(1-2\delta/3)$, $1/8\le\delta\le5/6$	>0.00012
a; $a/(3\kappa_m)-\theta_0$; $\delta/2-\theta_0-a(1-c)/6$	>0.127; >0.0127; >0.105
$\Gamma_L-\Gamma_N$	>0.000033
$(\delta/2-\theta_0)(\lambda-11/20)-\Gamma_N$	>0.00116
$2x(21/100)-z_L-W_{\max}$; $\gamma_{\max}$	>0.08723; $<1/250$
$N-I_{\mathrm{short}}=-\Delta_{\mathrm{short}}$	$>0.0133>23/2700$
$E'+h\Delta_{\mathrm{short}}$; $S+\Delta_{\mathrm{short}}$	<-0.0027; >0
$S-37\theta_0/40-\gamma_{\max}$; $S-\gamma_{\max}$; $S+\delta/2-\theta_0$	$>1/8$; $>1/8$; <2

Table 7: Geometric and complementary inequalities on $\mathcal{P}$. Here $Q_{\mathrm{sm}}=\ell(1/2+2\ell/h)-11h/54$.

condition	bound
$3/8<m^\sharp<m_\theta$; $25/62<m_\theta<9/20$	both hold
$57/100<\lambda<11/19<59/100$	holds
$z_0(3/40),z_0(1/10),z_L<21/100$	largest <0.18929
$6\kappa_m\ell>3/4$; $(9/2)\ell>3/4$	both hold
$l_x>\ell$; $l_y-\ell-11b/6>0$; $l_y<1/2$; $h<2l_y$	all hold
$\ell/(h+1/2000)>21/100$	>0.21320
$\ell/(h-1/80)<2/9$	<0.21677
$\ell/h<1/(6\kappa_m)$	gap <-0.00918
$Q_{\mathrm{sm}}<0$; $C_0+(1-h)/16<-1/250$	both hold
$C_0+(1/2+3\ell/2)/50<-1/125$	<-0.00898
$13h/75-l_y/2+1/50<-1/20$	<-0.07016
$b>0$; $1/6<\ell<1/5$; $B(\ell)>437/500$	all hold

11.5 Iteration and proof of the enlarged-count theorem

In this subsection, we iterate the inequalities of Section 11.4 from $B_\natural$ to B'. The descent of Theorem 9.2 then gives Theorem 1.2.

Proposition 11.6 (The improved boundary over fields containing μ_{12})**.** *For every number field $F\supset\mu_{12}$, one has $\beta_*(F)\le B'$.*

Proof. Fix F. Proposition 10.5 supplies $B_0=B_\natural$, $\kappa_0=2B_0-1=\kappa_\natural$. The field verification in Proposition 9.1 preserves the arithmetic families of Lemmas 11.1 and 11.2.

Suppose $\beta_*(F) \le B_n$ and $\kappa_n = 2B_n - 1 \in (\kappa', \kappa_\natural]$. Then Proposition 11.3 and Lemma 11.1 apply with $\kappa_c = \kappa_m = \kappa_n$. Put $b = b'$ and $\ell = \ell(\kappa_n)$ as in (11.19). Lemma 11.5 gives

$$\mathcal{Q}'_{\kappa_n} \le -\nu_n \quad \text{on } \mathcal{R}_*, \qquad \nu_n = \frac{\kappa_n - \kappa'}{100} > 0.$$

Apply Step 3 of Proposition 10.5 with the count of Proposition 11.3, the norm of Lemma 11.1 and Tables 6–7. The parameter sets are $\mathcal{P}$ and $\mathcal{P}_{3/4}$, and the strict comparison margin is (11.20). On $\mathcal{R}_* \times \mathcal{P}$, use $\theta_0 \le 3/(60\kappa_m) < 0.06681$. Its uses in Corollary 6.7, Proposition 7.4 and the slope display of Theorem 8.4 are the arithmetic parameters $S < 1$, $S - 37\theta_0/40 - \gamma_{\max} > 1/8$ and $S - \gamma_{\max} > 1/8$ of Table 6. On the selected numerator lengths set

$$m_a = 3/8 + \varepsilon, \quad m_b = 9/20, \qquad W(m) = (1-\varepsilon)W_{\mathrm{cap}}(m; \kappa_n).$$

The norm margins satisfy

$$(1-c)(1-2m) - 6W(m) \ge (1-c)\varepsilon/10 > 0, \qquad 1 - C_{\mathrm{add}} \ge 4\varepsilon^2 > 0.$$

Tables 6–7 justify the prime supply, both contour slopes, short sums, complementary classes, lowest amplitudes, principal remainder and direct comparison. For $\delta < 1/8$ retain the earlier comparison with $R'_{\kappa_c} \le R_{\kappa_c} \le R_{3/4}$. Choose the count loss and all remaining losses so that

$$C\varepsilon + 2\zeta + \epsilon_{\mathrm{rem}} < \nu_n, \qquad \ell/(h+\zeta) > 21/100.$$

The same continuation argument gives

$$\beta_*(F) \le B_{n+1} = \frac{11}{12} - \frac{\ell(\kappa_n)}{4} = B' + \frac{\kappa_n - \kappa'}{100}.$$

Consequently

$$\kappa_{n+1} - \kappa' = \frac{\kappa_n - \kappa'}{50}, \qquad B_n - B' = \frac{B_\natural - B'}{50^n}.$$

Hence $\kappa_n \in (\kappa', \kappa_\natural]$ for every n, and the step applies at every n. Every finite step uses a previously proved bound, with its own positive margins and threshold. Hence $\beta_*(F) \le \inf_n B_n = B'$. □

Proof of Theorem 1.2. Use the descent of Theorem 9.2 with Proposition 11.6. For a primitive finite-order character χ of K, let M/K be its cyclic class-field extension, $F = K(\zeta_{12})$, and $E = MF$. The abelian factorizations give, for $\Re s > B'$ and $s \ne 1$,

$$\zeta_E(s) \ne 0 \quad \Longrightarrow \quad \zeta_M(s) \ne 0 \quad \Longrightarrow \quad L_K(s, \chi) \ne 0.$$

Nonprincipal factors are nonzero at $s = 1$; the principal pole is permitted. Deleted Euler factors of an imprimitive character have zeros only on $\Re s = 0$, as in the same descent proof. □

Remark 11.7 (Scope of the earlier optimality theorem). Theorem 10.8 fixes the count of Proposition 6.4. Proposition 11.3 replaces that input, so $B' < B_\natural$ is consistent with its conclusion. No optimality assertion for this intermediate enlarged comparison system is needed for Theorem 1.2. Section 16 treats the larger exhaustion-norm comparison separately.

Part II
Prime exhaustion and variable comparison rectangles

12 An enlarged product norm with a variable comparison rectangle

In this section, we prove Theorem 12.1 with comparison lengths $(t, 2m-t)$. Lemma 12.2 treats the empty prime product. Section 13 applies the resulting norm to zero detection and the numerator.

12.1 The moment estimates and the norm statement

In this subsection, we state the length conditions for Theorem 12.1, using the arithmetic estimates of Section 2. Lemma 12.2 gives the length-restoration estimate used in its proof.

Retain Section 2 and the moment, reflection, full Gauss, correlation and fixed-mean formulas of [Liu26b, Lemmas 12.3, 12.5–12.10, Theorem 12.16, Sections 12.7–12.8 and Appendix C.4]. At $z = 0$, (10.5) permits arbitrary bounded ordinary lengths, with principal inducing characters excluded. On parameters inducing outside Θ, a prime product of length z satisfies

$$|Q(k)|^2 \ll U^{\kappa z+\epsilon}.$$

Fixed-mean cancellation requires equal products of actual scales, the same character, coprimality conditions and explicit Mellin twist, and the two respective smooth weights.

Use Theorem 3.7, Corollary 3.6, the common-support inequality 3.5, and Propositions 4.6 and 4.8, under the length conditions below. Retain every moving radical, natural zero and polynomial norm-twist height factor.

Let $S_{v,i}(k)$ be the centrally normalized ordinary polynomial of length U^v and weight W_i, in the common arithmetic parametrization. Let Q_0, Q_W be products of distinct complete original prime factors on pairwise disjoint original prime ranges, with

$$\mathrm{length}_U(Q_0) = z, \qquad \mathrm{length}_U(Q_W) = w, \qquad \mathrm{supp}_{\mathrm{pr}}(Q_0) \cap \mathrm{supp}_{\mathrm{pr}}(Q_W) = \varnothing.$$

Here $\mathrm{supp}_{\mathrm{pr}}$ denotes the original prime ranges. Their finite-character coefficients belong to the original fixed coefficient family. Fix the factors and smooth weights before summation, allowing the required finite derivative families with one common explicit ideal-norm Mellin twist. Let $\mathcal{U}$ be the original nonunit sixth-power-free family inducing outside Θ. Centered norms use the positive extension to the unrestricted arithmetic sum of [Liu26b, Lemma 12.1]; uncentered norms use $\mathcal{U}$.

Theorem 12.1 (Enlarged norm with variable comparison)**.** *Assume $p = 6\kappa \in [112/25, 9/2]$ and $\beta_*(F) \le (1+\kappa)/2$. Fix a compact set of parameters*

$$0 < t \le \min(m_-, 1/4), \qquad m_- \le r < 1/2, \qquad 0 \le W \le 1/25,$$

with uniform strict margins in

$$A_{\max} + 4W < 1, \qquad A_{\max} = 2r + \frac{1-2r}{p} + W, \tag{12.1}$$

$$A_{\max} + 3W < \frac{5}{6} + \frac{2t}{3}, \tag{12.2}$$

$$2 + 2t - 4m_- + pW < 1. \tag{12.3}$$

For each $\epsilon > 0$, require $\eta_{\mathrm{mesh}} > 0$ to be small in terms of the uniform strict margins, the lower bound for t and the aggregate numerical length and modulus bounds; this requirement is fixed before the arithmetic data and the number of prime factors. Require

$$\max_i \lambda_i \le \eta_{\mathrm{mesh}}, \qquad m \in [m_-, r], \qquad 0 \le z \le \frac{1-2m}{p}, \qquad 0 \le w \le W,$$

where λ_i are the individual original prime-factor widths. Put

$$P = S_{m,1} S_{m,2} Q_0, \qquad P_{\mathrm{cmp}} = S_{t,1} S_{2m-t,2} Q_0, \qquad P_c = P - P_{\mathrm{cmp}}.$$

Require exact equality of the products of the actual scales in each comparison. Then, for each required finite family of independently chosen smooth weights and their required derivatives,

$$\sum_k \mathcal{W}(q_k/U) |P_c(k) Q_W(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C, \tag{12.4}$$

$$\sum_{k \in \mathcal{U}} \mathcal{W}(q_k/U) |P(k) Q_W(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C. \tag{12.5}$$

Constants and thresholds have the same dependence on the fixed field, arithmetic data and finite family of smooth weights as the cited estimates. They are uniform on the indicated compact parameter set and in subsequent extracted ideals and Fourier labels.

The hypotheses give

$$\kappa \ge \frac{56}{75} > \frac{13}{18}, \qquad p - 1 \ge \frac{87}{25} > 0, \qquad \inf t > 0,$$

and the direct terminal ceilings in the proof do not depend on p.

12.2 Restoration when the prime list is exhausted

In this subsection, we prove Lemma 12.2 by removing complete prime factors and using the ordinary case of (10.5) when none remain. The resulting estimate is used in the proof of Theorem 12.1.

Lemma 12.2 (Removing all remaining prime factors)**.** *Let $P' = S_1 S_2 Q'$ be a product obtained from the families of Theorem 12.1 after the second correlation and complete genuine and final-divisor extraction. Here S_1, S_2 are ordinary annular sums and Q' is the surviving prime product, with $\mathrm{length}_U(Q') = z'$. Retain the original coefficient measures, full moving radical and inducing character outside Θ. Write $M_{\mathrm{new}} = m_{\mathrm{new}} + q_{\mathrm{new}}$, where $q_{h'} \ll U^{m_{\mathrm{new}}}$ is the buffered parameter range and $U^{q_{\mathrm{new}}}$ bounds the full displayed moving radical. Let n_1, n_2 be the declared lengths of S_1, S_2, and let*

$$D = n_1 + n_2 + pz' - M_{\mathrm{new}}, \qquad p = 6\kappa, \tag{12.6}$$

be the M2 length excess. Suppose every original prime factor has length at most η_{mesh}. Let $\mathcal{M}_2(P')$ denote the inner transformed squared sum over these parameters inducing outside Θ, with the original positive coefficient measure and smooth tests. Then, for every $\epsilon > 0$,

$$\mathcal{M}_2(P') \ll U^{M_{\mathrm{new}} + D_+/6 + \kappa\eta_{\mathrm{mesh}} + \epsilon} \mathfrak{S}_J(\mathcal{W}) (1+H)^C,$$

where $\mathcal{W}$ denotes the retained smooth tests. No condition $D \le pz'$ is required.

Proof. Treat the two one-sided positive Gauss means with their respective presentations, after the second correlation and complete genuine and last-divisor allocations. For fixed extracted data put

$$F_* = G_c V_{\mathrm{id}}, \qquad \psi_{h'}(n) = \tau_2(n) \chi_n(F_* h').$$

By [Liu26b, Lemmas 12.2 and 12.8, Section 12.7.3 and Appendix C.4], each comparison rectangle has two ordinary annular factors and the surviving complete prime annuli, with the common character $\psi_{h'}$ and one explicit ideal-norm Mellin twist. Let $c_{\mathrm{corr}}(h')$ be the normalized second-correlation scalar and $c_{\mathrm{ext}}(h')$ the product of extracted character values for the current allocation. On each part of the retained decomposition,

$$|c_{\mathrm{corr}}(h')| \le 1, \qquad |c_{\mathrm{ext}}(h')| \le 1.$$

Their natural zeros remain. Last-divisor extraction divides corresponding scales by the same exact norm, preserving unrestricted quotients, original annuli and finite-character prime coefficients. No Gauss coefficient remains on the ordinary factors.

Expand an additional fixed radical by [Liu26b, Lemma 12.5], keeping the full moving radical. Only after forming these ordinary products, split into

$$\begin{array}{l|l} \chi_{\mathrm{ind}} \in \Theta & \text{matching fixed-character means} \\ \chi_{\mathrm{ind}} \notin \Theta & \text{M2 and the factor removal below.} \end{array}$$

The lengths in (12.6) are declared after the chosen support buffers. For $D \le 0$, apply the appropriate clause of M2 without stripping. For $D > 0$, remove complete factors until their total length v reaches D/p or the list is exhausted. Then

$$0 \le v \le \min\{z', D/p + \eta_{\mathrm{mesh}}\}, \qquad pv \ge D \quad \text{or} \quad v = z', \qquad \kappa v \le D/6 + \kappa\eta_{\mathrm{mesh}}.$$

Put $z_{\mathrm{rem}} = z' - v$. The two clauses of [Liu26b, Theorem 12.16] give

	length condition	moment estimate
$z_{\mathrm{rem}} > 0$	$D - pv \le 0$	prime-bearing M2
$z_{\mathrm{rem}} = 0$	actual bounded ordinary lengths	no-prime M2.

Apply M2 separately to the two ordinary rectangles after genuine and final-divisor extraction. Their inducing character lies outside $\Theta \supset \{1\}$; all displayed radicals and natural zeros remain. The squared loss is at most $D_+/6 + \kappa\eta_{\mathrm{mesh}}$, and the subsidiary pointwise and M2 losses are chosen inside the final ϵ of Theorem 12.1. □

12.3 Proof of the enlarged norm

In this subsection, we prove Theorem 12.1 from the local identities of Section 3 and Lemma 12.2. Section 13 applies the resulting norm estimate.

Proof of Theorem 12.1. First transform and exact local families. Use the positive extension of [Liu26b, Lemma 12.1] and apply the first Poisson formula of [Liu26b, Section 12.7.1] to the complete product coefficients, with their common Mellin twist. The ordinary factors remain together. The comparison has

$$(v_1, v_2) \in \{(m, m), (t, 2m - t)\}, \qquad N_1 N_2 = N_1' N_2', \qquad A_t = 2m + z + w,$$

where N_i, N_i' are the actual ordinary scales. At the first dual origin, the powerful-product condition gives $O(U^{A_t+\epsilon})$ coefficient pairs; paired normalization cancels this count. The origin is therefore $O(U^{1+\epsilon})$, independently of the particular comparison short length t.

For nonzero frequencies retain each original C, D, s, finite component and common Fourier coordinate. Apply the exact local classification before any ordinary moment: first-common extra width w_f; fixed unit and nonunit widths f_0, f_1; and free unit, nonunit and correction widths $g_{\mathrm{good}}, v, c_{\mathrm{cor}}$. Write

$$u_{\mathrm{free}} = g_{\mathrm{good}} + v + c_{\mathrm{cor}}, \qquad w = w_f + f_0 + f_1 + u_{\mathrm{free}}, \qquad t_{\mathrm{loc}} = f_0 + u_{\mathrm{free}}.$$

The width u_{free} is distinct from the common-support exponent $R + E_0$.

The ordinary power removals satisfy $L_1 \geq f_1 + 4T_1$ and $L \geq v + 4T$; $S_1, S, T_1, T \geq 0$. The exact identities and natural-zero branches are those of the cited local classification. In particular divisor Cauchy for nonunit choices is applied at the original frequency before any support-dependent division; the resulting squared coefficient energy is retained, without a new unrestricted count of those labels. Correction coefficients keep their absolute q_p^{-1} mass. Unit choices use one common polynomial and divisor-bounded fibers. These operations preserve exact product scales and the same two smooth weights in their respective columns.

For the completed Gauss polynomial let $A = A_*, M = M_*, q = q_*$ and $m_* = M - q$. The identities are

$$\begin{aligned} A &= A_t - f_1 - t_{\mathrm{loc}} - L_1 - L - 5c_{\mathrm{cor}}, \\ M &= 1 - 6t_{\mathrm{loc}} - f_1 - L_1 - L - 5c_{\mathrm{cor}} + T_1 + 5v + T, \\ q &= f_1 + v + c_{\mathrm{cor}}, \\ \mathcal{R}_{\mathrm{loc}} &= -f_0 - L_1 + T_1 - S_1 - u_{\mathrm{free}} - L + v - 4c_{\mathrm{cor}} + T - S. \end{aligned}$$

Here $\mathcal{R}_{\mathrm{loc}}$ includes the change from the completed one-sided allowance to the original one-sided allowance. It is not an additional free label count. None of these identities depends on p or on the chosen comparison short length t.

Homogeneous restoration. Set $e_z = 1 - 2m - pz \geq 0$ and let $z_{C,\mathrm{old}}$ be the initial first-common prime width. The completed defect is

$$D_f = 6w - e_z - (p-1)z_{C,\mathrm{old}} - 5w_f - 5f_1 - T_1 - 5v - T.$$

The exact common-support inequality lowers it by $\mu = \min(c_C, d_D) \geq w_f$ on actual scales. With $e_1 = L_1 - f_1 - 4T_1 \geq 0$ and $e = L - v - 4T \geq 0$, put

$$\mathcal{C}_{\mathrm{hom}} = -f_0 - f_1 - u_{\mathrm{free}} - 3(T_1 + T) - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e.$$

Direct substitution gives

$$\begin{aligned} \mathcal{C}_{\mathrm{hom}} + \frac{D_f - \mu}{6} = \frac{w_f - \mu}{6} - \frac{5}{6}(f_1 + v) - \frac{19}{6}(T_1 + T) \\ - \frac{p-1}{6} z_{C,\mathrm{old}} - \frac{e_z}{6} - 4c_{\mathrm{cor}} - S_1 - S - e_1 - e \leq 0. \end{aligned}$$

Also $\mathcal{C}_{\mathrm{hom}} \leq 0$, so $\mathcal{C}_{\mathrm{hom}} + (D_f - \mu)_+/6 \leq 0$. After the second Poisson transform, genuine and final-divisor fixing only reduce the actual length defect. Lemma 12.2 pays for any remaining positive defect even at exhaustion. The original sixth-power amplifier contribution is still absorbed once by its allowance $\Delta' \geq a + \ell_a$. Its auxiliary step and mesh losses are included in ϵ. No additional lower bound on the remaining prime-product length is needed for this argument.

A direct branch for small global completed length. Let $c = c_C$, $d = d_D$, $u = R + E_0$ and $B_C = (3c - 5d - R)_+/6$. The actual Gauss polynomial has coefficient scale $\alpha = A - c$, inner divisor length s_0', and cutoff

$$K_0 = 2A - c - d + u - m_*.$$

Nonempty coefficient supports have $\alpha \geq -o(1)$. Retain the fixed frequency cutoff of Lemma 4.5 and a buffer $\xi > 0$. A nonempty nonzero frequency range satisfies $K_+ \leq K_0 + O(\xi) + o(1)$; otherwise its cutoff is eventually below the minimum nonzero norm. Bounded unit frequencies and nominal endpoints are retained with their single aggregate support discrepancy. Polynomial Fourier-height majorants remain in the original weighted Fourier measures. The raw bound of Lemma 4.5 has exponent $K_+ + 2\alpha - 2s_0'$. Subtract the completed allowance $\Lambda_* = (A - c) + q + u + B_C - s_0'$ and add $\mathcal{R}_{\mathrm{loc}}$. The result is at most

$$\mathcal{R}_{\mathrm{loc}} + 3A - 2c - d - M - B_C - s_0' + O(\xi) + o(1) \leq \mathcal{R}_{\mathrm{loc}} + 3A - M + O(\xi) + o(1).$$

The exact completed identities give

$$\mathcal{R}_{\mathrm{loc}} + 3A - M = 3A_t - 4 + 20t_{\mathrm{loc}} + 4f_1 + 4c_{\mathrm{cor}} - 3T_1 - 16v - 3T + 3m_* - S_1 - S.$$

For $m_* \le 1/100$, use $c_{\mathrm{cor}} \le t_{\mathrm{loc}}$, $t_{\mathrm{loc}} + f_1 \le w \le W$ and (12.1):

$$\mathcal{R}_{\mathrm{loc}} + 3A - M \le 3A_t - 4 + 24W + \frac{3}{100} < -1 + 12W + \frac{3}{100} \le -\frac{49}{100}.$$

Choose ξ and the remaining losses inside this margin. The estimate applies directly to the actual Gauss polynomial, including negative formal m_* or M.

Formal-length split in the remaining branch. Now $m_* > 1/100$ and $M > 1/100$. The auxiliary outer-length replacement of Lemma 4.7 uses $\widehat{d} = \min(d, A)$ with its support envelope and $\widehat{K}_0 = K_0 + d - \widehat{d}$. It changes no actual ideal, polynomial or frequency cutoff K, preserves B_C and $\min(c, d)$, and satisfies $\widehat{d} + \widehat{K}_0 - K = d + K_0 - K$. Split by $m_f = m_* - c + \widehat{d} - u$. If $m_f \le 1/100$, the direct formal-terminal calculation in Proposition 4.6 gives

$$3A_t - \frac{7}{2} + 18W + \frac{7}{200} < -\frac{93}{200} + 6W \le -\frac{9}{40} < -\frac{1}{5}.$$

Its exact local identities do not depend on the initial comparison short length.

If $m_f > 1/100$, use the actual complementary Poisson calculation. Both global and formal lengths are then uniformly positive. The formulas for J, g, M', Δ' of Lemma 4.7 and Proposition 4.8, and the local sparse-character savings, are unchanged. Nonempty returned annuli have nonnegative actual length of the transformed arithmetic sum; the full moving radical remains in their combined width. A separate fixed polynomial-size extra coprimality radical is expanded completely using [Liu26b, Lemma 12.5] before a nonexceptional M2 call. The extracted divisor coefficients and their subpower total mass are retained; the actual plain lengths and remaining prime width decrease. This never deletes a displayed moving radical from the combined width. Exceptional centered terms instead retain the common coprimality condition in their fixed-character mean. Apply the established ordinary M2 only to these resulting ordinary sums, using exhaustion as above.

If no prime factor remains initially, use $a = a_o = \ell_a = 0$, $g = J_+ + \sigma$ with no auxiliary-prime average. The step still gives strict width decrease.

Second origin and exceptional centered means. For the second Poisson origin, (12.1) gives

$$A_t - 1 + 4t_{\mathrm{loc}} \le A_t - 1 + 4W < 0.$$

This positive diagonal is estimated separately from the equal-product mean cancellation.

For exceptional characters use the sparse-support count of [Liu26b, Section 12.8.2] and fixed-character means on the actual two rectangles. Every exact prime extraction divides both respective scales by the same ideal norm, so product equality survives. The same common character, coprimality conditions imposed by the character zeros and Mellin twist of the complete product occur on both rectangles. All four initial ordinary scales are at least U^t because $t \le m$; after removals use $(t - L_1 - L - 5c_{\mathrm{cor}})_+$ and the original volume branch when it vanishes. Writing $L_{\mathrm{tot}} = L_1 + L$ and using $T_1 + T \le L_{\mathrm{tot}}/4$, the full local exceptional upper bound is

$$A_t - \frac{5}{6} + 3t_{\mathrm{loc}} - \frac{9L_{\mathrm{tot}}}{8} - \frac{29c_{\mathrm{cor}}}{6} - \frac{2}{3}(t - L_{\mathrm{tot}} - 5c_{\mathrm{cor}})_+.$$

For $L_{\mathrm{tot}}, c_{\mathrm{cor}} \ge 0$,

$$\frac{9L_{\mathrm{tot}}}{8} + \frac{29c_{\mathrm{cor}}}{6} + \frac{2}{3}(t - L_{\mathrm{tot}} - 5c_{\mathrm{cor}})_+ \ge \frac{2}{3}\{L_{\mathrm{tot}} + 5c_{\mathrm{cor}} + (t - L_{\mathrm{tot}} - 5c_{\mathrm{cor}})_+\} \ge \frac{2t}{3}.$$

Hence the exceptional exponent is at most $A_t - 5/6 - 2t/3 + 3t_{\mathrm{loc}}$. Condition (12.2) supplies a strict margin.

Differentiation changes only the smooth weights; the common character, coprimality conditions and explicit Mellin twist are retained. Thus the leading mean products cancel exactly.

Integration and restoration of the comparison product. Integrate both one-sided bounds in the original weighted Fourier measures, using their original Gauss allowances and the paired structural count once. The raw bounds are uniform in these labels; tails are discarded only from the genuine expression with fixed sufficient Fourier orders. Thus

$$\sum_k \mathcal{W}(q_k/U)|P_c(k)Q_W(k)|^2 \ll U^{1+\epsilon}\mathfrak{S}_J(\mathcal{W})(1+H)^C,$$

which is (12.4). Condition (12.3) implies $2m-t>1/2$. Reflect the long comparison factor with its actual conductor and natural zeros. Its ordinary length plus prime-bearing cost is at most

$$t+1-(2m-t)+p(z+w)\le 2+2t-4m_-+pW<1.$$

For the original detector and numerator presentations this reflection has only their natural zeros, covered by [Liu26b, Lemma 12.6]. There is no additional initial coprimality condition with polynomially bounded modulus in these applications.

If a fixed additional coprimality restriction is retained in a comparison product, its expansion must be treated with its coefficients rather than asserted to preserve the reflected margin. Put $n=2m-t$, $Z_{\mathrm{pr}}=z+w$ and $C_{\mathrm{refl}}=t+1-n+pZ_{\mathrm{pr}}<1$. A term removing plain widths a_1,a_2 and prime width v has squared coefficient cost $U^{-a_1-a_2-v+o(1)}$. Reflect the second ordinary factor only when its surviving length $n-a_2$ exceeds $1/2$. Either choice gives a declared length at most $1-n+a_2+o(1)$, so the left-hand side of its moment length condition is at most

$$C_{\mathrm{refl}}+a_2-a_1-pv+\xi.$$

Its positive defect is at most $(a_2-a_1-pv+\xi)_+$. Lemma 12.2 restores it at squared cost at most that quantity divided by six, plus the mesh loss. The net exponent is at most

$$-a_1-a_2-v+\frac{(a_2-a_1-pv+\xi)_+}{6}\le -a_1-\frac{5}{6}a_2-v+\frac{\xi}{6}.$$

The mesh loss is included in ϵ. Hilbert-space triangle and subpower divisor sums, with positive exponents at least $5/12$ in the unsquared coefficients, finish this comparison estimate with these coprimality restrictions. Nonempty subunit scales receive the single support allowance of [Liu26b, Section 12.8.1] in ξ. Thus the original M2 theorem, with this weighted restoration when needed, controls the comparison product on $\mathcal{U}$.

Hilbert-space triangle between P_cQ_W and $P_{\mathrm{cmp}}Q_W$ gives (12.5).

Uniform choices. Let $\rho>0$ be a uniform lower bound for the three strict margins and the terminal saving $1/5$. Make the choices in the order

$$\begin{aligned}(\rho,\epsilon) &\longrightarrow (\text{step, edges, moment losses, complete-factor mesh})\\ &\longrightarrow (\text{finite prime list, arithmetic data, smooth weights})\\ &\longrightarrow (\text{internal derivative and Fourier orders})\\ &\longrightarrow (\text{external heights, tail orders, threshold}).\end{aligned}$$

Use the arithmetic-independent M2 mesh. After fixing the list, the single aggregate support discrepancy $O(1/\log U)$ is absorbed. Compactness gives common choices. For varying $t(m),W(m)$, first make a uniform strict width decrement and then take a finite cover by small parameter neighborhoods. Each finite iteration has its own threshold. □

13 Selecting the original factors and amplifying the detector

In this section, we select disjoint products of original prime factors and apply Theorem 12.1 to the zero-detecting sums. We derive the count and numerator estimates used in Section 14.

We use U for a dyadic scale of the arithmetic parameter. Fix a detected amplitude class, a presentation, and a dyadic detector pair before selecting factors. An original prime polynomial has width λ_i, meaning that its prime ideals lie in a dyadic norm range of scale U^{λ_i}, and amplitude g_i on this class. For $g_i > 0$, the prescribed amplitude-subdivision losses give $|Q_i(u)| \geq U^{\lambda_i g_i - o(1)}$. We have

$$0 \leq g_i \leq \delta/2, \quad \sum_i \lambda_i = \mathscr{L}, \quad \sum_i \lambda_i g_i = g\mathscr{L}, \quad g = \delta x, \qquad 0 \leq x \leq \tfrac{1}{2}.$$

The notation $o(1)$ throughout this section denotes arbitrarily small exponent losses fixed before the final threshold, including amplitude-subdivision and mesh losses. It does not assert uniformity before those choices are made. For $d \leq h + \zeta$, Section 14 gives

$$\mathscr{L} = \ell/d \geq \ell/(h+\zeta) > 21/100.$$

Lemma 13.1 (Simultaneous selection of prime factors)**.** *Suppose each original factor has width at most η_{mesh}, and two upper length bounds $z, w \geq 0$ satisfy $z + w \leq \mathscr{L}$. There are disjoint products Q_0, Q_W of complete positive-amplitude original factors, with respective widths at most z, w, for which*

$$|Q_0(u)Q_W(u)|^2 \geq U^{2g(z+w) - 2\delta\eta_{\mathrm{mesh}} - o(1)}$$

throughout the fixed class. The number of factors does not multiply the displayed mesh loss.

Proof. Order the positive-amplitude factors by decreasing g_i. Allowing a fraction of the last factor, take an initial segment of total length $z + w$. Its weighted mean is at least g. If the positive factors have smaller total length, retain all of them. In both cases, the unsquared amplitude exponent satisfies

$$A \geq g(z+w), \qquad A = \sum_{i \in I} \lambda_i g_i \quad \text{with a possible fractional endpoint.}$$

In the second case, this follows from $A = g\mathscr{L} \geq g(z+w)$.

Split the chosen segment after length z, or at its end if it is shorter. Discard any original factor cut by this division or by the endpoint of the chosen segment. There are at most two such factors. Assign the remaining complete factors to Q_0 and Q_W. Their respective lengths are at most z, w. Since $g_i \leq \delta/2$ and $\lambda_i \leq \eta_{\mathrm{mesh}}$, the discarded unsquared exponent is at most

$$2 \cdot \frac{\delta\eta_{\mathrm{mesh}}}{2} = \delta\eta_{\mathrm{mesh}}.$$

Consequently $|Q_0 Q_W|^2 \geq U^{2g(z+w) - 2\delta\eta_{\mathrm{mesh}} - o(1)}$. The previously fixed amplitude-subdivision loss gives the final $o(1)$. If the positive factors end before length z, take $Q_W = 1$. □

The selected lengths may be strictly smaller than the prescribed upper bounds. The enlarged norm holds for every

$$0 \leq z_{\mathrm{actual}} \leq z_0, \qquad 0 \leq w_{\mathrm{actual}} \leq W.$$

Thus the argument never requires additional factors when all positive-amplitude factors have been used. Decrease the prescribed bounds slightly before applying Lemma 13.1.

Proposition 13.2 (Joint amplification of the ordinary detector)**.** *Fix a compact band of ordinary detector lengths $m \in (0, 1/2)$. Suppose the enlarged norm applies with initial upper length bound $z_0(m) = (1-2m)/p$ and additional upper length bound $W(m)$, with uniform strict margins in the length hypotheses, and suppose $z_0(m) + W(m) < \mathscr{L}$ uniformly. The arithmetic parameters for which the detector of [Liu26b, Lemma 13.7] gives $|S_m(u)|^2 \geq U^{\delta m - o(1)}$ have count*

$$\#\mathscr{B}_m \ll U^{1-\delta F_2(m)+\epsilon}(1+H)^C, \qquad F_2(m) = 2m + \frac{2x}{p}(1-2m) + 2xW(m). \tag{13.1}$$

The exponent C and the seminorm orders are finite and are fixed before the external height exponent is chosen.

Proof. Apply Lemma 13.1 to the original factors with upper length bounds $z_0(m)$ and $W(m)$. For the original detector smooth weights,

$$|S_m(u)^2 Q_0(u) Q_W(u)|^2 \geq U^{2\delta m + 2\delta x(z_0(m)+W(m)) - o(1)}.$$

The enlarged norm therefore gives

$$\#\mathscr{B}_m\, U^{2\delta m + 2\delta x(z_0(m)+W(m)) - o(1)} \leq \sum_{u \in \mathscr{B}_m} |S_m(u)^2 Q_0(u) Q_W(u)|^2 \ll U^{1+\epsilon}(1+H)^C,$$

which is (13.1). The ordinary factor in this squared norm is $|S_m|^4$.

We justify its use for the actual detector, including its smooth weights depending on the summation parameter. The proof of [Liu26b, Lemma 13.9] first fixes the discrete presentation and dyadic pair, then applies a common parameter-box Sobolev inequality

$$\sum_u \sup_{\mathbf{t} \in \mathcal{B}_H} |P_u(\mathbf{t})|^2 \ll \sum_{\alpha \in \{0,1\}^{\dim \mathcal{B}_H}} \int_{\mathcal{B}_H^+} \sum_u |\partial^\alpha P_u(\mathbf{t})|^2 \, d\mathbf{t}.$$

Here replace its $P_u = S_m^2 Q_0$ by $S_m^2 Q_0 Q_W$. Each derivative is a finite sum of products of the two ordinary factors, with possibly different smooth weights, and the original prime factors with differentiated smooth weights. The enlarged norm is uniform for this finite family. It supplies the same inequality before the smooth-weight parameters giving the lower bound for each index are evaluated. Only the original, undifferentiated prime products require amplitude lower bounds; none are claimed for the differentiated factors.

Remove deterministic scale phases before differentiating. The derivatives then introduce logarithms of normalized annular ratios. Absorb the difference between the detector frequency and the original prime Mellin frequency into all prime smooth weights, including those of Q_W. The two ordinary factors retain the common explicit Mellin twist by the ideal norm used in the signed equal-product means. This does not require a new mean identity with independent Mellin twists of the ordinary factors. The parameter volume and the allowed incoming heights cost a fixed polynomial in H.

For arithmetic orientation, the whole detected presentation may be conjugated to $\psi = \psi_0 \bar{\chi}_u$. Its original prime coefficients are then $\bar{\psi}_0(\mathfrak{p}) 1_{\mathfrak{p} \in \mathcal{T}}$, expanded in the fixed finite coefficient group as in [Liu26b, Lemma 13.7]. The factors in Q_W come from the same list with these same coefficients. Conjugate the entire product if the norm is stated in the opposite orientation. The fixed group is closed under conjugation. Exterior ramification of a nonunit sixth-power-free parameter cannot be canceled by an S-supported character, so the inducing character lies outside the exceptional group. Bounded unit parameters were separated in [Liu26b, Lemma 13.7]. All natural zeros and actual deleted Euler factors remain present.

The number of discrete presentations is fixed, and the number of dyadic pairs is logarithmic. Choose the loss in the moment estimate, factor decrements and mesh below the desired final exponent loss, then fix the finite derivative orders, choose the external height exponent to absorb $(1+H)^C$, and finally choose the threshold. This proves the stated uniform bound for the detector family of [Liu26b, Lemma 13.7]. □

13.1 Inverse moment and the long detector count

In this subsection, we combine Proposition 13.2 with the inverse moment estimate. Section 14 chooses a common truncation length before using the resulting count in the numerator estimate.

For a short inverse length $r \in (1/2, 1)$, the unchanged marked inverse moment uses the prime-product length bound $(1-r)/2$ and gives saving

$$F_1(r) = x + (1-x)r.$$

The long inverse exponent is

$$\frac{1}{6} + \left(\frac{5}{6} - \delta\right) T, \qquad 1 \le T \le \frac{3}{2}.$$

For each detected parameter, the two polynomial lower bounds hold and

$$r + m \ge T - o(1), \qquad m \le \frac{1}{2} + o(1), \qquad r_\times + m_\times = T \implies r \ge r_\times \text{ or } m \ge m_\times - o(1).$$

Use M1 in the first case and the ordinary norm in the second. Endpoint strips use the estimates without prime factors; the next section checks the common cutoff and the junction $m = 9/20$.

For each arithmetic parameter, use the inverse or ordinary moment according to its detecting lengths. The two estimates may select different subsets of the original prime factors. Their union gives the count before any estimate for the reflected numerator is used. Apply this count separately in Hölder's inequality for the numerator. Both estimates use the previously established moment parameter κ of (14.1).

13.2 The numerator's separate amplification

In this subsection, we apply the enlarged norm to the numerator and pass from its fourth moment to its first moment by Hölder's inequality. Section 14 combines this gain with the contour displacement.

Joint selection would give

$$\sum_{u \in \mathscr{B}} |S_m(u)|^4 \ll U^{1-2g(z_0+W)+\epsilon}(1+H)^C,$$

and hence for a class of count $U^{R+\epsilon}$,

$$\sum_{u \in \mathscr{B}} |S_m(u)| \ll U^{(3R+1)/4 - gz_0/2 - gW/2 + \epsilon}(1+H)^C.$$

The numerator estimate below uses the weaker, already sufficient remaining-mean bound $a \le g$ for its additional gain $aW/2$. It does not need the sharper joint-selection gain there. In either version the approximate-functional-equation coefficient is treated separately; the existing baseline $gz_0/2$ is not counted twice.

The Sobolev argument for the actual conductor in [Liu26b, Proposition 14.3] also applies: for each dyad fix the original prime selection first, introduce the parameter of the conductor-dependent smooth weight on each of its $O(\log U)$ unit intervals, and differentiate the two ordinary smooth weights inside the norm. Dual blocks are wholly conjugated so their characters agree with the prime factors; root numbers disappear only after taking absolute values. Dyads above half length use the no-prime fourth moment, and sufficiently long tails are discarded only after fixing the external derivative orders. The short, supply-limited blocks and all resulting scalar inequalities required for their application are included in the next section's checks.

14 Exact scalar comparison and continuation

In this section, we use the enlarged norm of Theorem 12.1, the simultaneous factor selection of Lemma 13.1, the zero-detecting estimate of Proposition 13.2, and the cited theta/Mellin, M1, M2, contour and direct estimates. The numerical inequalities are verified by exact rational arithmetic in Appendix D.

14.1 Parameters and notation

In this subsection, we choose rational parameters whose moment hypothesis follows from Theorem 9.2. We define the exponents used in Proposition 14.2.

Fix

$$\begin{aligned} B &= \frac{5463}{6250} = 0.87408, \quad \ell = \frac{1597}{9375}, \quad b = \frac{1913}{20000}, \\ \kappa &= \frac{37446831}{50000000} = 0.74893662, \quad p = 6\kappa, \quad \varrho = 1 - \frac{1}{p}. \end{aligned} \tag{14.1}$$

Thus $B = 11/12 - \ell/4$. Theorem 9.2 gives $\beta_*(F) \le B_* < 0.87446831 = (1+\kappa)/2$, so the moment premise is already established. Put

$$\begin{aligned} l_x &= \frac{1-b-\ell}{2}, \quad l_y = \frac{1+b-\ell}{2} = \frac{277591}{600000}, \\ h &= \frac{1+b+3\ell}{2} = \frac{160669}{200000}, \quad C_0 = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4}, \quad T_y = 2l_y - h. \end{aligned}$$

The amplitude coordinates are $0 < \delta \le 5/6$, $0 \le x \le 1/2$, and $g = \delta x$. All prime lengths are relative to the scale of the arithmetic parameter $U = Z^d$. Write $L_0 = 21/100$ and $\theta_0 = \delta x/p$. For any allowed class-count exponent R, define

$$E(R) = C_0 + \frac{\delta}{2} + \delta x \left\{ \ell + \frac{h/2 - l_y}{p} \right\} - \frac{3h}{4}(1-R), \tag{14.2}$$

$$S(R) = \frac{3R+1}{4} + \frac{\theta_0}{2} - \frac{17}{50}, \tag{14.3}$$

$$N(R) = \frac{3R+1}{4} - \frac{\theta_0}{2}, \qquad D_{\mathrm{short}}(R) = R + (1/2 - \theta_0)\frac{3}{40} - N(R). \tag{14.4}$$

For fixed class data the unamplified reference exponent is $E(R) + (d-h)S(R)$. A selected gain γ and a numerator contour $0 \le \theta \le \delta/2$ change this to

$$E(R) + (d-h)S(R) + (\theta - \theta_0)\{d(1-m) - l_y\} - d\gamma. \tag{14.5}$$

For $0 \le m \le 1/2$ this is the simultaneous moment/contour identity of Lemma 7.1, with its actual-conductor primal and dual smooth weights. Longer AFE blocks use the no-prime formula below.

14.2 Detector widths and a common cutoff

In this subsection, we choose the prime-product lengths in the ordinary zero-detecting sum and balance its count with the inverse estimates of Section 13. The resulting count is used in Proposition 14.2.

For $19/50 \le m \le 9/20$, let

$$W_o(m) = \frac{\varrho}{5}(1-2m), \qquad W_j(m) = \frac{1/2 + (1-3/p)(1-2m)}{p+12},$$
$$W_d(m) = \min\{W_o(m), W_j(m)\}.$$

The origin bound gives $W \le W_o$. The reflection and exceptional constraints on the comparison length t are respectively

$$t < t_{\mathrm{high}} = 2m - \frac{1}{2} - \frac{p}{2}W, \qquad t > t_{\mathrm{low}} = 6W - \frac{3}{2}\varrho(1-2m) + \frac{1}{4}.$$

The interval is nonempty precisely when

$$t_{\mathrm{high}} - t_{\mathrm{low}} = \frac{p+12}{2}(W_j - W) > 0.$$

At $W = W_d$, take $t = \min\{1/4, (t_{\mathrm{low}} + t_{\mathrm{high}})/2\}$. The rational inequalities give

$$0 < t \le 1/4 < m, \qquad t > 5W_d, \qquad W_d < 1/25.$$

For the norm estimate use $W = (1-\epsilon_W)W_d$ with fixed sufficiently small $\epsilon_W > 0$; these auxiliary inequalities remain strict. The prime supply satisfies

$$z_0(m) + W_d(m) \le \frac{1 - 2(19/50)}{p} + \frac{1}{25} < L_0.$$

By Proposition 13.2, the ordinary detector saving is

$$F_2(m) = 2m + \frac{2x}{p}(1-2m) + 2xW_d(m).$$

The M1 saving for inverse length r is $F_1(r) = x + (1-x)r$; the long inverse count at total cutoff T_d is $1/6 + (5/6 - \delta)T_d$.

Write $W_i(m) = W_i(0) + W_i' m$ for $i = o, j$, with constant slopes W_i', and define

$$\alpha_i = \frac{2x}{p} + 2xW_i(0), \quad \beta_i = 2 - \frac{4x}{p} + 2xW_i', \quad F_i(m) = \alpha_i + \beta_i m, \quad A_\delta = \frac{5}{6} - \delta, \quad H_\delta = \frac{A_\delta}{1-x} + \delta.$$

Thus $F_2 = \min(F_o, F_j)$. On the domain used below $A_\delta, H_\delta > 0$. Set

$$m_i = \frac{5/6 + A_\delta x/(1-x) - H_\delta \alpha_i}{A_\delta + H_\delta \beta_i}, \quad R_i = 1 - \delta F_i(m_i), \quad T_i = m_i + \frac{F_i(m_i) - x}{1-x}. \tag{14.6}$$

Every expression is rational with positive certified denominators.

Lemma 14.1 (Shared detector cutoff). *On $\mathcal{D} = [8/25, 3/5] \times [9/25, 1/2]$, the valid amplified class count is*

$$\#\mathscr{B} \ll U^{R_d + \epsilon}(1+H)^C, \qquad R_d = \max\{R_o, R_j\},$$

with total detector cutoff $T_d = \max(T_o, T_j)$.

Proof. The balance equation for branch i is

$$-\frac{5}{6} + A_\delta m - \frac{A_\delta x}{1-x} + H_\delta F_i(m) = 0.$$

Its left side is increasing. Therefore the equation with $\min(F_o, F_j)$ is solved at $m_d = \max(m_o, m_j)$: the active branch is zero there and the other is nonnegative. At a branch root,

$$F_i(m_i) = \frac{5/6 + A_\delta x/(1-x) - A_\delta m_i}{H_\delta}.$$

Since $A_\delta, H_\delta, \delta > 0$,

$$m_i \le m_j \implies R_i \le R_j \implies T_i \le T_j, \qquad T_i = \frac{R_i - 1/6}{A_\delta}.$$

The same branch realizes all three maxima. With $r_d = T_d - m_d$,

$$1 - \delta F_1(r_d) = 1 - \delta F_2(m_d) = 1/6 + A_\delta T_d = R_d.$$

The exact calculations establish, for both branches,

$$19/50 < m_i < 9/20, \qquad 3/5 < T_i - m_i < 1, \qquad 1 < T_i < 3/2.$$

For inverse length at least r_d, use M1. If the detector presents an inverse length below r_d, its ordinary length is at least m_d after the arbitrarily small detector losses of [Liu26b, Lemma 13.7]. Both F_i increase, so the amplified M2 saving suffices through $9/20$. For larger lengths, the old M2 saving at $9/20$ already exceeds both $F_i(m_i)$, as certified explicitly; it is increasing thereafter. The long inverse count handles the remaining detector branch. The M1 selected-prime length is below $(1 - 3/5)/2 = 1/5 < L_0$, and its second length restriction has a strict margin.

Use a finite compact cover of the amplitude classes and dyadic lengths, then take the detector, prime and width meshes sufficiently small within these strict margins in the length hypotheses. Replacing limiting widths by strict interior widths changes the exponents by an arbitrarily small uniform amount. The original smooth weights and original prime lower bounds are retained before the common Sobolev step of Proposition 13.2. This gives the stated count with any prescribed positive exponent loss. □

14.3 Numerator gain on every ordinary length

In this subsection, we combine the numerator moment and contour displacement to obtain a uniform gain over the ordinary summation lengths. Proposition 14.2 then bounds the exponent in (14.5).

The numerator uses the fixed comparison length $t_n = 23/100$, with

$$m_L = (1 + 2t_n)/4 = 73/200, \quad z_L = (1 - 2m_L)/p, \quad a = \delta \frac{L_0 x - z_L/2}{L_0 - z_L}.$$

This conservative remaining-amplitude bound is sufficient; the stronger simultaneous factor selection for the numerator is not needed here. On $m_L \le m \le 9/20$ use the limiting width

$$W_n(m) = \min\left\{ \frac{\varrho}{5}(1 - 2m), \frac{4m - 1 - 2t_n}{p}, \frac{59}{2000} \right\}.$$

The exceptional constraint is redundant on this band: for $W \le \varrho(1 - 2m)/5$ and $m \le 9/20$,

$$t_{\mathrm{low}} \le \frac{1}{4} - \frac{3}{10}\varrho(1 - 2m) \le \frac{1}{4} - \frac{3\varrho}{100} < t_n,$$

where the last inequality is $\varrho/50 > 1/6 - 2t_n/3$. The constant cap is below 0.03. Put $m_\theta = 1 - l_y/h$ and

$$G_N = \frac{a\varrho}{10} T_y, \qquad G_L = \theta_0\{(1 - m_L)h - l_y\}.$$

Choose $\theta = 0$ for $m \le m_\theta$, and $\theta = \delta/2$ for $m \ge m_\theta$. At $d = h$ the gain in (14.5) is $h\mathcal{F}(m)$, where

$$\mathcal{F}(m) = \frac{aW_n(m)}{2} + \begin{cases} \theta_0(m_\theta - m), & m \le m_\theta, \\ (\delta/2 - \theta_0)(m - m_\theta), & m \ge m_\theta, \end{cases}$$

the function of Section 7.1 with W_n in place of W_{cap}. On each open branch with $m < m_\theta$,

$$\mathcal{F}'(m) = \begin{cases} 2a/p - \theta_0 > 0, & \text{reflection branch}, \\ -\theta_0 < 0, & \text{constant cap}, \\ -\varrho a/5 - \theta_0 < 0, & \text{origin branch}. \end{cases}$$

At m_θ the origin branch is active and below the constant cap. Where differentiable,

$$m > m_\theta \implies \mathcal{F}'(m) \geq \frac{\delta}{2} - \theta_0 - \frac{\varrho a}{5} > 0.$$

Thus

$$\min_{m_L \leq m \leq 9/20} h\mathcal{F}(m) = \min\{G_L, G_N\}.$$

Continuity handles branch crossings. The signs hold on both the main and bridge domains below. For $3/40 \leq m < m_L$, the central-contour unamplified estimate has gain at least G_L. For $m > 9/20$, the outer-contour unamplified estimate has gain at least its value at $9/20$, which is certified larger than G_N. For $m < 3/40$, use ideal counting and $D_{\mathrm{short}}(R)$ directly.

The outer assertion includes AFE lengths above $1/2$, which can be larger than one when the fixed field has large degree. For these blocks the no-prime first moment, including the AFE coefficient, is $(3R+1)/4-\theta m$. Relative to the reference exponent its complete contour correction is exactly

$$(\theta - \theta_0)(d/2 - l_y) - d\theta(m - 1/2), \qquad m \geq 1/2. \tag{14.7}$$

Let $\mathcal{E}_{\mathrm{out}}(m)$ be the unamplified exponent with $\theta = \delta/2$. The formulas agree at $m = 1/2$, and

$$\partial_m \mathcal{E}_{\mathrm{out}} = \begin{cases} -d(\delta/2 - \theta_0), & 9/20 < m < 1/2, \\ -d\delta/2, & m > 1/2, \end{cases} \qquad (d > 0).$$

Both derivatives are negative. Hence every outer AFE block is bounded by $\mathcal{E}_{\mathrm{out}}(9/20)$; the scale-slope estimate is applied to this single endpoint bound.

At $m = m_L$ the limiting selected width vanishes, so no uniform positive-width norm is applied at that endpoint. First choose a fixed small $\epsilon_L > 0$. Use the unamplified central-contour estimate through $m_L + \epsilon_L$. Its gain on the added strip is at least $G_L - h\theta_0\epsilon_L$, and $h\theta_0 < 0.054$ on the main and bridge domains. Since $E - G_L < -10^{-6}$ there, taking $0.054\epsilon_L < \nu/100$, with the margin $0 < \nu < 10^{-6}$ fixed in Section 14.5, preserves a strict saving. Apply the selected norm only on $[m_L + \epsilon_L, 9/20]$, where the limiting width has a uniform positive lower bound. Then choose the interior width loss and the prime and dyadic meshes inside its strict margins in the length hypotheses. The comparison-length and exceptional constraints also have uniform strict margins on this closed band after the interior decrement.

The contour endpoints 0 and $\delta/2$ are permitted by Lemma 5.2: the final s buffer keeps $s + w$ strictly inside its holomorphy region. If strictly interior contours are used, take $\theta_- = \epsilon_\theta$ and $\theta_+ = \delta/2 - \epsilon_\theta$ instead. On the retained compact ranges the change in the exponent is at most $2\epsilon_\theta$, so choose this loss below $\nu/100$ as well. For outer blocks use their $m = 9/20$ bound first, so this constant does not depend on their possibly large lengths. Thus neither a zero-width endpoint nor a limiting contour is used without a positive-margin approximation.

14.4 Exact inequalities over the complete parameter range

In this subsection, we prove Proposition 14.2 and treat the remaining amplitude and summation ranges. Appendix D verifies the required rational inequalities.

Let $R_s = R_\kappa(\delta, x)$ be the count exponent of Proposition 6.4 at the current moment parameter κ; then $c_{\mathrm{cnt}}(\kappa) = 1/(3\kappa) = 2/p$.

Use R_d on $\mathcal{D}$. On the bridge $\mathcal{H} = [3/10, 8/25] \times [9/25, 1/2]$, use R_s with the amplified numerator. On the following complementary rectangles, use R_s and the old numerator estimate:

$$\mathcal{C}_1 = [1/8, 5/6] \times [0, 9/25], \quad \mathcal{C}_2 = [1/8, 3/10] \times [9/25, 1/2], \quad \mathcal{C}_3 = [3/5, 5/6] \times [9/25, 1/2].$$

Boundary overlaps may use either valid estimate.

Proposition 14.2 (Rational comparison inequalities)**.** *For the parameters* (14.1)*, one has*

$$\begin{aligned} E(R_i) - G_N &< -10^{-6}, & E(R_i) - G_L &< -10^{-6} & &(i = o, j;\ \mathcal{D}),\\ E(R_s) - G_N &< -10^{-6}, & E(R_s) - G_L &< -10^{-6} & &(\mathcal{H}),\\ E(R_s) &< -10^{-5} & & & &(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3). \end{aligned}$$

The corresponding reference exponents on $\mathcal{D}, \mathcal{H}$ *are below* $1/450$ *and their slopes satisfy* $1/5 < S < 1$*. On the complementary rectangles* $0 < S(R_s) < 1$*. On the main and bridge domains,*

$$E(R) + hD_{\mathrm{short}}(R) < -10^{-5}, \qquad 0 < S(R) + D_{\mathrm{short}}(R) < 2,$$

and the moment supply $(p - 2\delta x)L_0 > R$ *holds. The old moment supply is also certified on all of* $[1/8, 5/6] \times [0, 1/2]$*.*

Proof. Substitute (14.1). For each required rational inequality $f > 0$ on a closed rectangle $\mathcal{R}$, all inverted denominators exclude zero. With c_X the midpoint and $[f](X)$ the natural interval evaluation, use

$$\mathcal{I}_f(X) = [f](X) \cap \left(f(c_X) + \sum_{j=1}^{2} [\partial_j f](X)(X_j - c_{X,j}) \right).$$

Outward rational arithmetic and a finite subdivision $\mathfrak{X}$ give

$$\mathcal{R} = \bigcup_{X \in \mathfrak{X}} X, \qquad \inf \mathcal{I}_f(X) > 0 \quad (X \in \mathfrak{X}) \quad \Longrightarrow \quad f > 0 \text{ on } \mathcal{R}.$$

The expressions in R are affine, so the inequalities for R_o, R_j hold for $R_d = \max(R_o, R_j)$. The same enclosures give the cutoff, supply, width, numerator-slope, short-sum and bridge conditions. □

The remaining ranges are treated as follows. For $d \le h - 1/80$ in the range of arithmetic parameters where the affine exponent formula [Liu26b, equation (14.7)] applies, the main and bridge estimates give

$$E(R) + (d - h)S(R) < 1/450 - 1/400 = -1/3600.$$

The complementary estimates improve as d decreases, since $S(R_s) > 0$. On the main and bridge domains, $\delta \le 3/5$ gives $\theta_0 \le 3/(10p) < 0.067$, $(\delta/2 - \theta_0)(1 - m) \le 3/10$ and $\gamma = aW_n/2 \le (3/10)(59/2000)/2 < 0.0045$; also $1/5 < S < 1$. Since ∂_d of (14.5) is $S + (\theta - \theta_0)(1 - m) - \gamma$, for each selected or unamplified comparison exponent $\mathcal{E}$ with $3/40 \le m \le 9/20$,

$$\frac{1}{8} < \frac{1}{5} - 0.067\frac{37}{40} - 0.0045 < \partial_d \mathcal{E} < 1 + 0.3 < 2.$$

The same bound applies to the outer endpoint $m = 9/20$; the short-sum slope is checked directly. Thus extending to $d \le h + \zeta$ costs at most 2ζ.

For $0 < \delta < 1/8$, keep the old numerator moment at $3/4$. The sharpened small-amplitude calculation gives

$$E_{\mathrm{small}} \le C_0 + (1 - h)/16 < -1/250;$$

its required coefficient $Q_{\mathrm{sm}} < 0$ is certified. The lowest-amplitude class endpoint satisfies

$$C_0 + (1/2 + 3\ell/2)/50 < -1/250.$$

The proof of Lemma 6.3, applied with this hypothesis, gives

$$\mathcal{E}_{\mathrm{low}}(d) \le C_0 + (1/2 + 3\ell/2)/50 + 2\zeta < -1/250 + 1/1000 = -3/1000.$$

The small-amplitude bound has the same extension. Choose the remaining losses for the lowest-amplitude classes inside this margin. The small-parameter estimate, including bounded nonprincipal unit parameters, has the certified bound

$$13h/75 - l_y/2 + 1/50 < -1/20.$$

The parameters in the far range use the right-hand contours of [Liu26b, Section 14.6] and sufficiently large z_∞. The principal residues and their normalizer are unchanged. The savings of [Liu26b, Section 14.6] apply since $B - 437/500 = 1/12500 > 0$.

14.5 Smooth weights, error terms and analytic continuation

In this subsection, we choose the losses and analytic parameters in the required order and deduce $\beta_*(F) \le 5463/6250$ from Proposition 14.2. Section 15 starts the final iteration from this bound.

Fix F and the rational parameters. Choose $0 < \nu < 10^{-6}$ below the common comparison, detector-crossing and supply margins. First choose the endpoint strips, strict interior widths, contour and comparison-length buffers, with total comparison loss $\epsilon_1 < \nu/4$. Their retained compact sets have a positive minimum norm margin. Choose the factor meshes and input losses below that minimum and the remaining comparison margin, and take a finite cover before the target-dependent arithmetic data.

The piecewise rational count, width and class expressions have bounded derivatives on these compact sets. Thus the width, amplitude, length, detector and fractional-factor losses can be chosen with total comparison cost ϵ_2, before the target-dependent data, with

$$\epsilon_1 < \frac{\nu}{4}, \qquad \epsilon_2 < \frac{\nu}{4}, \qquad 0 < 2\zeta < \frac{\nu}{4}, \qquad \frac{\ell}{h+\zeta} > L_0.$$

For each original dyad use the order

$$\begin{aligned}(s,z) &\longrightarrow \text{amplitude classes} \longrightarrow \text{original prime selection}\\ &\longrightarrow \text{differentiation in the common Sobolev boxes.}\end{aligned}$$

These boxes include detector frequencies, original prime frequencies and actual conductors. Fix finite derivative orders before external heights. Move each complete holomorphic numerator dyad before taking absolute values; retain both AFE halves, root numbers, actual-conductor factors and every simultaneous strict error tuple.

Theorem 12.1 applies to the same independent ordinary and prime weights. Lemma 7.1 cancels the selected-prime conductor errors before the moment. The Euler numerator and principal coefficient remain unchanged as functions of the arithmetic summation parameter.

Include the finite count and numerator exponents in the nonprincipal error estimate of [Liu26b, Proposition 14.6]. The high-side inequalities, with thresholds $1/450$ and $1/250$, give a positive nonprincipal saving; the direct, small- and large-parameter estimates, principal correction and exact theta/Mellin identity retain their verified length hypotheses. Choose the remaining smooth-weight and contour loss $\epsilon_3 < \nu/4$, so that

$$\epsilon_1 + \epsilon_2 + 2\zeta + \epsilon_3 < \nu.$$

Then fix

$$\text{internal orders} \longrightarrow \text{target-dependent height exponent and tail order} \longrightarrow \text{threshold.}$$

The height factors consume less than a fixed fraction of the saving. This gives a common numerical $m_{\mathrm{save}} > 0$, with target-dependent constants and thresholds.

Suppose, for contradiction, that $\beta = \beta_*(F) > B$. The previous bound justifies the fixed moment parameter throughout. Choose $0 < \epsilon_{\mathrm{low}} < (\beta - B)/4$ and let $C_\theta(s) = s - (4+b)/6$. The identity of [Liu26b, Lemma 14.7], with the high-side estimates just proved, gives for every fixed target η the same functions J_η, f_η, H_η and the estimates

$$|J_\eta(Z)| \ll_\eta Z^{C_\theta(B)+\epsilon_{\mathrm{low}}}, \qquad |J_\eta(Z) - f_\eta(Z)| \ll_\eta Z^{C_\theta(\beta)-m_{\mathrm{save}}/2}.$$

Set

$$\sigma = \max\{B + \epsilon_{\mathrm{low}}, \beta - m_{\mathrm{save}}/2\} < \beta, \qquad f_\eta(Z) \ll_\eta Z^{\sigma-(4+b)/6}.$$

The Mellin integral over $[1,\infty)$ is holomorphic for $\Re s > \sigma$; shifting the original absolute-convergence line arbitrarily far right makes the integral over $(0,1]$ entire. Mellin inversion identifies this continuation with

$$e^{(s-5/6)^2}\frac{H_\eta(s)}{L_F^S(s,\eta)}.$$

Since H_η is holomorphic and nonzero for $\Re s > 437/500$,

$$\sigma \geq B > \frac{437}{500} \quad \Longrightarrow \quad \mathcal{Z}_F \cap \{s : \Re s > \sigma\} = \varnothing.$$

The numerical σ is common to every fixed target, although the constants and thresholds may depend on η. Hence

$$\beta_*(F) \leq \sigma < \beta_*(F),$$

a contradiction. Thus $\beta_*(F) \leq 0.87408$, with the moment premise supplied by Theorem 9.2.

Remark 14.3 (The one-step comparison criterion). Retain the formulas for h, l_x, l_y, C_0. If $p = 6\kappa \in [112/25, 9/2]$ has a previously established moment hypothesis and the same norm, detector, numerator and geometric inequalities have a positive common margin, the argument gives

$$\beta_*(F) \leq \frac{11}{12} - \frac{\ell}{4}.$$

Choose all fixed losses inside that margin. The rational values in (14.1) are needed only for the starting comparison.

15 Successive moment bounds in the exhaustion range

In this section, we iterate the rational bound of Section 14 with the same moment parameter in the zero-detecting estimates and numerator. Lemma 15.1 gives the strict inequalities at each step; abelian descent yields Theorem 1.1. Section 16 also permits the two moment parameters to differ when proving optimality.

15.1 The algebraic parameters

In this subsection, we define the algebraic parameters used in Lemma 15.1. Appendix D isolates their values and verifies their defining identities.

Let ℓ_{ex} be the unique root in $(0.1703752759993550355, 0.1703752759993550356)$ of

$$f(\ell) = 621\ell^4 - 2064\ell^3 + 1657\ell^2 + 68\ell - 50. \tag{15.1}$$

Put

$$\begin{gathered} B_{\mathrm{ex}} = \frac{11}{12} - \frac{\ell_{\mathrm{ex}}}{4}, \qquad \kappa_{\mathrm{ex}} = 2B_{\mathrm{ex}} - 1, \qquad p_{\mathrm{ex}} = 6\kappa_{\mathrm{ex}} = 5 - 3\ell_{\mathrm{ex}}, \\ C(p) = \frac{p+4}{20p}, \qquad C_{\mathrm{ex}} = C(p_{\mathrm{ex}}), \\ r_{\mathrm{ex}} = -\frac{5(3\ell_{\mathrm{ex}} - 5)(9\ell_{\mathrm{ex}} - 5)}{6(21\ell_{\mathrm{ex}}^2 - 28\ell_{\mathrm{ex}} - 25)}. \end{gathered} \tag{15.2}$$

For an independent parameter C, define

$$R_C(\delta) = 1 - \delta + \frac{(5/6-\delta)\delta(1-4C)}{2\{(5/6-\delta)(5/2-8C)+\delta(1-4C)\}}, \tag{15.3}$$

$$\mathcal{Q}_{C,b,\ell}(\delta) = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4} + \frac{\delta}{2} + \frac{\delta\ell}{2} - C\delta T_y - \frac{3h}{4}(1 - R_C(\delta)), \tag{15.4}$$

where $h = (1+b+3\ell)/2$ and $T_y = (1+b-5\ell)/2$. The expression is affine in b. Define b_{ex} by

$$b_{\mathrm{ex}} = -\left.\frac{\partial_\delta \mathcal{Q}_{C,0,\ell}(\delta)}{\partial_\delta \partial_b \mathcal{Q}_{C,b,\ell}(\delta)}\right|_{C=C_{\mathrm{ex}},\,\ell=\ell_{\mathrm{ex}},\,\delta=r_{\mathrm{ex}}}. \tag{15.5}$$

The denominator is nonzero, as verified below. Numerically,

$$\begin{aligned} B_{\mathrm{ex}} &= 0.87407284766682790777786693740\ldots, \\ r_{\mathrm{ex}} &= 0.44469417311455635275466866926\ldots, \\ b_{\mathrm{ex}} &= 0.09567500764537642027548322 48\ldots, \\ C_{\mathrm{ex}} &= 0.09455460151844812985\ldots. \end{aligned} \tag{15.6}$$

The polynomial for B_{ex} is

$$178848B^4 - 507168B^3 + 522846B^2 - 231405B + 37099 = 0. \tag{15.7}$$

It is irreducible over $\mathbb{Q}$ and has a unique root in $(0.87407284, 0.87407285)$.

15.2 Endpoint signs and a strict family

In this subsection, we prove Lemma 15.1. The limiting exponent is nonpositive, and each parameter in (15.8) gives a strict saving. Proposition 15.2 uses these inequalities in the iteration.

For either detector branch $i = o, j$, let

$$\mathcal{Q}_i(\delta, x; \kappa, \ell, b) = E(R_i) - G_N$$

with the formulas of Section 14. At $x = 1/2$, $a = \delta/2$. The origin branch therefore satisfies

$$\mathcal{Q}_o(\delta, 1/2; \kappa, \ell, b) = \mathcal{Q}_{C(6\kappa),b,\ell}(\delta).$$

In particular, no remaining-amplitude loss is incurred in this class.

Lemma 15.1 (The endpoint and the strict comparison family)**.** *At $\kappa = \kappa_{\mathrm{ex}}$, $\ell = \ell_{\mathrm{ex}}$ and $b = b_{\mathrm{ex}}$,*

$$\max\{\mathcal{Q}_o, \mathcal{Q}_j\} \le 0 \quad \text{on } \mathcal{D} = [8/25, 3/5] \times [9/25, 1/2].$$

Equality holds only at $(\delta, x) = (r_{\mathrm{ex}}, 1/2)$. Put $\kappa_0 = 4676/6250 = 0.74816$ and, for $\kappa_{\mathrm{ex}} < \kappa \le \kappa_0$, set

$$b = b_{\mathrm{ex}}, \qquad \ell(\kappa) = \ell_{\mathrm{ex}} - \frac{\kappa - \kappa_{\mathrm{ex}}}{25}. \tag{15.8}$$

Then both branches satisfy

$$\mathcal{Q}_i(\delta, x; \kappa, \ell(\kappa), b_{\mathrm{ex}}) < -\frac{\kappa - \kappa_{\mathrm{ex}}}{100} \qquad ((\delta, x) \in \mathcal{D}). \tag{15.9}$$

The bridge, complementary-class, numerator-length and geometric inequalities used in the rational comparison continue to hold on this family, with the margins specified below.

Proof. Rational polynomial arithmetic modulo (15.1) gives

$$\mathcal{Q}_{C_{\mathrm{ex}},b_{\mathrm{ex}},\ell_{\mathrm{ex}}}(r_{\mathrm{ex}}) = \partial_\delta \mathcal{Q}_{C_{\mathrm{ex}},b_{\mathrm{ex}},\ell_{\mathrm{ex}}}(r_{\mathrm{ex}}) = \partial_b \mathcal{Q}_{C_{\mathrm{ex}},b,\ell_{\mathrm{ex}}}(r_{\mathrm{ex}}) = 0.$$

In particular,

$$R_{C_{\mathrm{ex}}}(r_{\mathrm{ex}}) = \frac{5 + 12C_{\mathrm{ex}}r_{\mathrm{ex}}}{9}. \tag{15.10}$$

On $8/25 \le \delta \le 3/5$, clearing positive denominators gives

$$\mathcal{Q}_o(\delta, 1/2) = \frac{A_o(\delta - r_{\mathrm{ex}})^2}{D_o(\delta)}, \qquad A_o < 0, \quad D_o(\delta) > 0,$$

$$\mathcal{Q}_j(\delta, 1/2) = \frac{A_j\delta^2 + B_j\delta + C_j}{D_j(\delta)}, \qquad A_j < 0, \quad B_j^2 - 4A_jC_j < 0, \quad D_j(\delta) > 0.$$

Hence $\mathcal{Q}_o \le 0$, with equality only at r_{ex}, and $\mathcal{Q}_j < 0$.

Exact outward interval arithmetic gives

$$\partial_x \mathcal{Q}_i > 1/1000 \qquad \text{on } \mathcal{D} \text{ at the endpoint,} \tag{15.11}$$

$$\partial_\kappa \mathcal{Q}_i - \frac{1}{25}\partial_\ell \mathcal{Q}_i < -1/100 \qquad \text{on } \mathcal{D} \times \mathcal{P}, \tag{15.12}$$

where ∂_κ differentiates the common parameter in both the count and numerator, and the enclosing parameter box is

$$\mathcal{P}: \quad \kappa \in [\kappa_{\mathrm{ex}}, \kappa_0], \quad \ell \in [\ell_{\mathrm{ex}} - (\kappa_0 - \kappa_{\mathrm{ex}})/25, \ell_{\mathrm{ex}}], \quad b = b_{\mathrm{ex}}.$$

The first inequality proves the endpoint assertion. Along (15.8), the common moment parameter satisfies

$$\frac{d}{d\kappa}\mathcal{Q}_i(\delta, x; \kappa, \ell(\kappa), b_{\mathrm{ex}}) = \partial_\kappa \mathcal{Q}_i - \frac{1}{25}\partial_\ell \mathcal{Q}_i < -\frac{1}{100}.$$

Integration from κ_{ex} gives (15.9). Here $\kappa = p/6$ is the M2 parameter, whereas $C(p)$ determines the prime-product length bound. The remaining assertions are the inequalities of Table 8, which the same rational interval calculation proves on $\mathcal{P}$ (Appendix D). □

All inequalities in Table 8 hold on $\mathcal{P}$. The definitions of $\mathcal{H}, \mathcal{C}_i$, D_{short} and S are those of the rational comparison. Unless another domain is indicated, an amplitude inequality is on the main and bridge rectangles with the count prescribed there.

Table 8: Uniform inequalities for the exhaustion-norm iteration.

quantity and domain	bound	used for
$E(R_i) - G_L$, $i = o, j$, on $\mathcal{D}$	$< -10^{-7}$	main saving
$E(R_s) - G_N$, $E(R_s) - G_L$ on $\mathcal{H}$	$< -10^{-7}$	main saving
$E(R_s)$ on each $\mathcal{C}_i$	$< -10^{-5}$	complementary classes
$E(R)$; $S(R)$ on the main and bridge rectangles	$< 1/450$; $1/5 < S < 1$	the range $d \le h + \zeta$
$E(R) + hD_{\mathrm{short}}(R)$; $S(R) + D_{\mathrm{short}}(R)$	$< -10^{-5}$; $0 < S + D_{\mathrm{short}} < 2$	short sums $m < 3/40$
$(p - 2\delta x)L_0 - R$; $G_L - G_N$	> 0; > 0	moment supply; $\min\{G_L, G_N\} = G_N$
a; $2a/p - \theta_0$; $\delta/2 - \theta_0 - \varrho a/5$	all > 0	signs of $\mathcal{F}'$
$a(59/2000)h/2 - G_N$; $(\delta/2 - \theta_0)(l_y - 11h/20) - G_N$	both > 0	constant cap; outer blocks
$2xL_0 - z_L$	$> 31/1000$	numerator prime supply
m_i; $T_i - m_i$; T_i, $i = o, j$, on $\mathcal{D}$	$(19/50, 9/20)$; $(3/5, 1)$; $(1, 3/2)$	Lemma 14.1
$9/10 + x/(5p) - F_i(m_i)$	> 0	old M2 beyond $9/20$
$S(R_s)$; $(p - 2\delta x)L_0 - R_s$ on the full class rectangle	$0 < S(R_s) < 1$; > 0	complementary classes
$p\ell$; $\ell/(h + 10^{-9})$; $\ell/(h - 1/80)$	$> 3/4$; $> 21/100$; $< 2/9$	prime supply; cited length hypotheses
$l_x - \ell$; $l_y - \ell - 11b/6$; $1/2 - l_y$; T_y	all > 0	direct and geometric hypotheses
$m_\theta - (p + 4 + 10t_n)/(2p + 18)$; $59/2000 - \varrho(2l_y/h - 1)/5$	both > 0	origin branch active at m_θ
$\varrho/50 - (1/6 - 2t_n/3)$; $9/20 - m_\theta$	both > 0	exceptional constraint; $m_\theta < 9/20$
Q_{sm}; $C_0 + (1 - h)/16$; $C_0 + (1/2 + 3\ell/2)/50$	< 0; $< -1/250$; $< -1/250$	small and lowest amplitudes
$13h/75 - l_y/2 + 1/50$; $B(\ell) - 437/500$	$< -1/20$; > 0	small parameters; far range

All relevant denominators are bounded away from zero. The detector width crossing is enclosed before the two width branches are checked; the comparison lengths satisfy the same strict auxiliary inequalities as in the rational case. Appendix D gives the exact arithmetic and its independent algebraic checks.

15.3 Iteration and descent

In this subsection, we prove Proposition 15.2 by iterating Lemma 15.1. The descent of Theorem 9.2 then proves Theorem 1.1. Every finite step uses a previously established moment hypothesis.

Proposition 15.2. *For every number field $F \supset \mu_{12}$, $\beta_*(F) \le B_{\mathrm{ex}}$.*

Proof. The comparison of Section 14 gives $B_0 = 5463/6250 = 0.87408$. Thus $\kappa_0 = 2B_0 - 1 = 4676/6250$ is a previously justified moment parameter. Suppose $\beta_*(F) \le B_n$ and $\kappa_n = 2B_n - 1 > \kappa_{\mathrm{ex}}$. Use $\ell(\kappa_n)$ and $b = b_{\mathrm{ex}}$, with count and numerator moment parameters $\kappa_c = \kappa_m = \kappa_n$. These

parameters lie in $\mathcal{P}$. Theorem 12.1 applies because $112/25 < 6\kappa_n < 9/2$, and Lemma 15.1 gives the strict main saving $\nu_n = (\kappa_n - \kappa_{\mathrm{ex}})/100$.

Choose all endpoint strips, interior width decrements and contour buffers within a fixed fraction of the lesser of ν_n and the strict side margins. Then take the positive compact norm margins, choose meshes and losses in the cited estimates inside them, and fix arithmetic data and finite families of smooth weights before the late heights and threshold. Section 14.5 applies with comparison margin $\nu_n > 0$. The small, far and principal terms satisfy Table 8. No original coefficient, principal numerator or normalizer changes. Consequently

$$\beta_*(F) \le B_{n+1} = \frac{11}{12} - \frac{\ell(\kappa_n)}{4} = B_{\mathrm{ex}} + \frac{\kappa_n - \kappa_{\mathrm{ex}}}{100}.$$

Hence

$$\kappa_{n+1} - \kappa_{\mathrm{ex}} = \frac{\kappa_n - \kappa_{\mathrm{ex}}}{50}, \qquad B_n - B_{\mathrm{ex}} = 50^{-n}(B_0 - B_{\mathrm{ex}}).$$

Each finite step has its previously proved premise, positive margins and threshold. Since

$$\beta_*(F) \le B_n \quad (n \ge 0), \qquad B_n \downarrow B_{\mathrm{ex}},$$

we obtain $\beta_*(F) \le \inf_n B_n = B_{\mathrm{ex}}$, independently of any uniformity of the thresholds in n. □

Proof of Theorem 1.1. Apply the descent in Theorem 9.2 to Proposition 15.2. For a finite-order character over K, use its cyclic class-field extension M/K and $F = K(\zeta_{12})$. The abelian factorizations over F, over M, and over K show that a zero of the given L-function in $\Re s > B_{\mathrm{ex}}$ would give a zero of ζ_{MF} there. Nonprincipal factors are entire, so they cannot cancel that zero by a pole. The case $s = 1$ follows from nonvanishing of nonprincipal finite-order Hecke L-functions; the principal pole is permitted. Deleted Euler factors have their zeros on $\Re s = 0$. □

The first step of the iteration in Proposition 15.2 already gives

$$B_1 = 0.8740729907134913496\ldots < 0.874073.$$

The distinction between this finite-step bound and the limiting quartic endpoint will be used in the optimality statement.

16 Optimality for the exhaustion-norm comparison

In this section, we prove Theorem 16.1: B_{ex} is the infimum for the specified norm, zero-detecting estimates and direct comparison. The ordinary comparison lengths and the two moment parameters may vary. An obstruction below means a necessary condition for this specified upper-bound comparison. It is not a lower bound for the real parts of zeros. We retain the direct theta majorant, the M1 and long-detector branches, the fourth-moment Hölder estimate, the original amplitude bounds, the geometric length inequality (16.1), and the second-origin width ceiling.

Theorem 16.1 (Optimality in the specified comparison system)**.** *Let κ_c and κ_m be moment parameters for the zero-detecting count and the numerator, each justified by an already proved boundary, and put $p_d = 6\kappa_c$, $p_n = 6\kappa_m$. For $0 \le m \le 1/2$, suppose the ordinary zero-detecting sum (with $p = p_d$) and the numerator (with $p = p_n$) are estimated with initial prime-product length at most $z_0(m) = (1 - 2m)/p$ and additional prime-product length at most*

$$W_o(m) = \frac{1 - 1/p}{5}(1 - 2m).$$

For $m > 1/2$, use the no-prime estimate with $z_0 = W = 0$. Retain the M1 and long-detector counts, fourth-moment Hölder inequality, buffered pointwise amplitude bounds and direct majorant used above. Permit arbitrary allowed comparison rectangles and arbitrary positive length

exponents l_x, l_y, ℓ. Put $\Sigma = l_x + l_y$, $b = l_y - l_x \geq 0$ and $h = 1 - l_x + \ell > 0$, and retain the length inequality

$$J_{\mathrm{geo}} := l_y - \ell - \frac{11b}{6} = \frac{\Sigma}{2} - \ell - \frac{4b}{3} > 0. \tag{16.1}$$

Starting with a boundary at least B_{ex}, no finite step in this comparison system can give a boundary below B_{ex}. The successive bounds of Proposition 15.2 attain B_{ex} as an infimum. Thus B_{ex} is optimal for this system, including when the balanced relation $l_x + l_y + \ell = 1$ is not imposed.

Proof. The condition (16.1) is among the strict inequalities of Table 8; hence the constructed sequence of bounds satisfies it. At $x = 1/2$, grant the full width W_o at every ordinary length and discard the reflection, exceptional-character and prime-supply restrictions. This only strengthens the available comparison. The upper bound for the total prime-product length is

$$\frac{1-2m}{p} + W_o(m) = 4C(p)(1-2m), \qquad C(p) = \frac{p+4}{20p}.$$

Consequently the optimistic ordinary detector saving is $F_2(m) = 2m + 4C(p)(1-2m)$. Balancing it with $F_1(r) = 1/2 + r/2$ and the long-detector count gives (15.3). This is the best crossing of these three branches: the short count decreases and the long count increases with the total cutoff. Restoring any omitted length hypothesis can only increase the resulting count exponent. At the critical class $\delta = r_{\mathrm{ex}}$ this assertion also gives a necessary count for every cutoff $1 \leq \tau \leq 3/2$. The short crossing is

$$r(\tau) = \frac{4C_{\mathrm{ex}} + (2 - 8C_{\mathrm{ex}})\tau - 1/2}{5/2 - 8C_{\mathrm{ex}}}, \qquad m(\tau) = \tau - r(\tau),$$

and lies in $1/2 < r(\tau) \leq 1$, $0 < m(\tau) \leq 1/2$. The other allowed split is $(r, m) = (\tau, 0)$, by a limiting annulus. Here the ordinary count $1 - 4C_{\mathrm{ex}} r_{\mathrm{ex}}$ exceeds the long inverse count throughout the cutoff interval, since

$$-\frac{5}{12} + r_{\mathrm{ex}}(3/2 - 4C_{\mathrm{ex}}) > 0.$$

Thus the two adverse splits give precisely the decreasing short and increasing long bounds. The comparison must control this allowed amplitude class; no assertion that the arithmetic class is nonempty is used in the optimality argument.

For the numerator, at $m_\theta = 1 - l_y/h$ the contour coefficient vanishes. The amplitude bound $a \leq \delta/2$ and the optimistic width give the excess (15.4) in the balanced geometry. We now show that allowing the sum of the length exponents to vary does not remove the obstruction.

Let $\Sigma = l_x + l_y$, $b = l_y - l_x \geq 0$, $\ell > 0$, $h = 1 - l_x + \ell > 0$ and $l_y = (\Sigma + b)/2$. The free-length direct majorant of [Liu26b, Section 2.4], with its original normalization $C_\theta(s) = s - 5/6 + l_x/3 + \ell/6$, is

$$B_{\mathrm{low}} = \frac{5}{6} + \frac{l_x}{6} - \frac{\ell}{6} + G_{\mathrm{dir}} + \frac{1}{2}k(\Sigma, \ell),$$
$$G_{\mathrm{dir}} = \max\{0, b/12, b - l_y/2\}, \qquad k(\Sigma, \ell) = \max\{0, (1 + 3\ell - 2\Sigma)/4, \Sigma + \ell - 1, (3\ell - 1)/2\}.$$

The supplied comparison requires $B \geq B_{\mathrm{low}}$. Taking $G_{\mathrm{dir}} \geq b/12$ and the branches 0 and $\Sigma + \ell - 1$ of k give

$$B \geq L_{\mathrm{dir}} = \frac{5}{6} + \frac{\Sigma}{12} - \frac{\ell}{6}, \quad B \geq L_K = \frac{1}{3} + \frac{7\Sigma}{12} + \frac{\ell}{3}.$$

No balanced relation $\Sigma + \ell = 1$ is assumed. The condition (16.1) is imposed separately from the direct majorant. Write $N_h = h - l_y = 1 - \Sigma + \ell$. Direct calculation gives

$$N_h - \frac{2}{5}h = \frac{341}{60} - \frac{35}{6}L_{\mathrm{dir}} - \frac{2}{3}L_K + \frac{3}{20}J_{\mathrm{geo}}. \tag{16.2}$$

An attempted improvement has $B < B_{\mathrm{ex}} < 8741/10000$. Therefore

$$N_h - \frac{2}{5}h > \frac{341}{60} - \frac{13}{2}B \geq \frac{101}{60000} > 0, \qquad m_\theta = \frac{N_h}{h} > \frac{2}{5}.$$

We must also compare with the pointwise estimate supplied by the buffered amplitude class. Put $r = r_{\mathrm{ex}}$, $C = C_{\mathrm{ex}}$ and $R = R_C(r)$. For $0 \leq m \leq 1/2$, the numerator and its allowed smooth weights satisfy, up to their fixed seminorms,

$$|S_m(u)|^2 \ll U^{rm+\epsilon}, \qquad \sum_{u\in\mathscr{B}} |S_m(u)| \ll U^{P_{\mathrm{pt}}(m;R)+\epsilon}, \qquad P_{\mathrm{pt}}(m;R) = R + \frac{rm}{2}.$$

The optimistic amplified fourth-moment estimate has first-moment exponent

$$P_4(m;R,C) = \frac{3R+1}{4} - Cr(1-2m).$$

The detector crossing is

$$m_d = \frac{1-R-4Cr}{2r(1-4C)} = \frac{2(1-12Cr)}{9r(1-4C)} \in (0.39816, 0.39818),$$

where the second equality uses (15.10). Thus

$$P_{\mathrm{pt}}(m;R) - P_4(m;R,C) = \frac{r(1-4C)}{2}(m-m_d) > 0 \qquad \left(\frac{2}{5} \leq m \leq \frac{1}{2}\right). \tag{16.3}$$

Since $0 < r < 1$, absolute ideal counting gives the still larger exponent $R + m/2 \geq P_{\mathrm{pt}}(m;R)$. Neither pointwise estimate improves on P_4 at the lengths used below.

First suppose $T_y = 2l_y - h > 0$, so $m_\theta < 1/2$. Evaluating the optimistic high-side comparison at $(\delta, x, d, m) = (r_{\mathrm{ex}}, 1/2, h, m_\theta)$ gives $B \geq H_{\mathrm{hi}}$, where

$$H_{\mathrm{hi}} = \frac{1+r_{\mathrm{ex}}}{2} - \frac{l_y}{2} + h\left(\frac{3R_{C_{\mathrm{ex}}}(r_{\mathrm{ex}})+1}{4} - \frac{1}{6}\right) + \frac{r_{\mathrm{ex}}-1}{2}\ell - C_{\mathrm{ex}}r_{\mathrm{ex}}T_y.$$

Using (15.10), the coefficient of b vanishes and

$$H_{\mathrm{hi}} = 1 + (2C_{\mathrm{ex}} + 1/2)r_{\mathrm{ex}} - (1/2 + 2C_{\mathrm{ex}}r_{\mathrm{ex}})\Sigma + (2C_{\mathrm{ex}} + 1/2)r_{\mathrm{ex}}\ell.$$

Set $D_{\mathrm{hi}} = 16C_{\mathrm{ex}}r_{\mathrm{ex}} + 2r_{\mathrm{ex}} + 3$ and

$$\lambda_{\mathrm{dir}} = \frac{44C_{\mathrm{ex}}r_{\mathrm{ex}} + 7r_{\mathrm{ex}} + 4}{3D_{\mathrm{hi}}}, \qquad \lambda_K = \frac{4C_{\mathrm{ex}}r_{\mathrm{ex}} - r_{\mathrm{ex}} + 2}{3D_{\mathrm{hi}}}, \qquad \lambda_{\mathrm{hi}} = \frac{1}{D_{\mathrm{hi}}}.$$

These numbers are positive and sum to one. The exact dual identity

$$\lambda_{\mathrm{dir}}L_{\mathrm{dir}} + \lambda_K L_K + \lambda_{\mathrm{hi}}H_{\mathrm{hi}} = \frac{44C_{\mathrm{ex}}r_{\mathrm{ex}} + 7r_{\mathrm{ex}} + 7}{3D_{\mathrm{hi}}} = B_{\mathrm{ex}} \tag{16.4}$$

proves $B \geq B_{\mathrm{ex}}$ with Σ free, the condition (16.1) retained, and the other auxiliary hypotheses of the norm estimate optimistically removed.

Now suppose $T_y \leq 0$. At $m = 1/2$, no prime width remains, and the optimal contour endpoint is $\theta = 0$. Equation (16.3) still excludes the buffered pointwise alternative at this endpoint. The necessary high-side majorant is now $H_{\mathrm{hi},0} = H_{\mathrm{hi}} + C_{\mathrm{ex}}r_{\mathrm{ex}}T_y$. Put

$$D_{\mathrm{hi},0} = 19 - (12C_{\mathrm{ex}} + 9)r_{\mathrm{ex}}, \quad \lambda_K^0 = \frac{3\{2 - (4C_{\mathrm{ex}} + 3)r_{\mathrm{ex}}\}}{D_{\mathrm{hi},0}},$$
$$\lambda_{\mathrm{hi}}^0 = \frac{13}{D_{\mathrm{hi},0}}, \quad \lambda_T^0 = \frac{18C_{\mathrm{ex}}r_{\mathrm{ex}} + 7r_{\mathrm{ex}} + 4}{2D_{\mathrm{hi},0}}.$$

All are positive and $\lambda_K^0 + \lambda_{\mathrm{hi}}^0 = 1$. An exact identity gives

$$\lambda_K^0 L_K + \lambda_{\mathrm{hi}}^0 H_{\mathrm{hi},0} + \lambda_T^0 T_y = \frac{52 + b(44C_{\mathrm{ex}}r_{\mathrm{ex}} + 7r_{\mathrm{ex}} + 4)}{4D_{\mathrm{hi},0}}.$$

Since $T_y \le 0$ and $b \ge 0$, this forces $B \ge 13/D_{\mathrm{hi},0} = 0.89697374236981439\ldots > B_{\mathrm{ex}}$. Thus neither edge of the free-length geometry avoids the obstruction.

Separate moment parameters. The count is monotone in its effective amplification parameter:

$$\frac{\partial R_C(r)}{\partial C} = -\frac{4r(6r-5)^2}{(48Cr - 80C - 18r + 25)^2} < 0, \qquad C'(p) = -\frac{1}{5p^2} < 0.$$

For fixed $m \le 1/2$, write

$$P_4(m; R, C(p_n)) = \frac{3R+1}{4} - C(p_n)r(1-2m), \qquad P_{\mathrm{pt}}(m; R) = R + \frac{rm}{2}.$$

The relevant derivatives are

$$\partial_R P_4 = \frac{3}{4}, \qquad \partial_R P_{\mathrm{pt}} = 1, \qquad \partial_{p_n} P_4(m; R, C(p_n)) = \frac{r(1-2m)}{5p_n^2} \ge 0, \qquad \partial_{p_d} R_{C(p_d)}(r) > 0.$$

Now take

$$m = \begin{cases} m_\theta, & T_y > 0, \\ 1/2, & T_y \le 0, \end{cases} \qquad R_{p_d} = R_{C(p_d)}(r).$$

For $p_d, p_n \ge p_{\mathrm{ex}}$,

$$\min\{P_4(m; R_{p_d}, C(p_n)), P_{\mathrm{pt}}(m; R_{p_d})\} \ge \min\{P_4(m; R, C_{\mathrm{ex}}), P_{\mathrm{pt}}(m; R)\} = P_4(m; R, C_{\mathrm{ex}}),$$

where (16.3) gives the equality at the chosen m. Thus the necessary comparisons remain $B \ge H_{\mathrm{hi}}$ and $B \ge H_{\mathrm{hi},0}$.

Starting from any known boundary at least B_{ex}, the moment hypothesis $p_d, p_n \ge 12B_{\mathrm{known}} - 6$ prevents an iteration of this system from making its first crossing below B_{ex}. Variable comparison lengths cannot change this conclusion because the proof already grants the full origin width everywhere.

The exact dual identities and their coefficient signs are verified in Appendix D. Proposition 15.2 proves attainability as an infimum with the actual hypotheses of the norm estimate. This completes the optimality statement. □

Remark 16.2. At the critical parameters the numerator length is $0.42413539306165\ldots$ and the detector crossing length is $0.39816812106788\ldots$. The origin ceiling is the binding width constraint; the optimized reflection and exceptional constraints have strict slack. At $x = 1/2$, the conservative numerator mean and mean of the selected factors both equal $\delta/2$. Further optimization of these choices therefore does not remove the contact obstruction. The theorem concerns the stated comparison system. A stronger moment, a better detector count, a sharper direct estimate, or a different theta average could leave this system and improve the boundary.

A Exact rational inequalities in the comparison

In this appendix, we give the rational computation used in Section 8. The method proves inequalities on complete closed rectangles; it does not replace any of the analytic estimates in Sections 3–7.

A.1 Polynomial inequalities on a rectangle

In this subsection, we state the coefficient criterion used for the rational functions in Section 8.1. The same criterion, with one more variable, applies to the parameter intervals in Section 8.2.

For a polynomial p on a rational rectangle, make the affine change of variables to $[0,1]^n$ and write

$$p(t) = \sum_{\alpha \le d} a_\alpha t^\alpha = \sum_{\beta \le d} b_\beta \prod_{j=1}^{n} \binom{d_j}{\beta_j} t_j^{\beta_j} (1-t_j)^{d_j-\beta_j}.$$

The exact conversion is

$$b_\beta = \sum_{\alpha \le \beta} a_\alpha \prod_{j=1}^{n} \frac{\binom{\beta_j}{\alpha_j}}{\binom{d_j}{\alpha_j}}. \tag{A.1}$$

For $t \in [0,1]^n$, the basis functions satisfy

$$B_\beta(t) := \prod_{j=1}^{n} \binom{d_j}{\beta_j} t_j^{\beta_j} (1-t_j)^{d_j-\beta_j} \ge 0, \qquad \sum_{\beta \le d} B_\beta(t) = 1.$$

Consequently

$$p(t) \ge \min_\beta b_\beta. \tag{A.2}$$

Rational bisection and (A.1) give finite covers with

$$X = \bigcup_\nu X_\nu, \qquad \min_\beta b_{\nu,\beta} > 0 \implies \inf_X p > 0.$$

For a rational function p/q, apply this criterion separately to numerator and denominator after fixing their common sign. A cleared numerator affine in each of b, ℓ attains its extrema at the four corners of their rectangle. Otherwise retain b, ℓ as polynomial variables unless monotonicity permits endpoint evaluation. Treat κ in the same way.

A.2 The strict rational starting bound

In this subsection, we specify the inequalities proving Theorem 8.1. The functions E, S, R, θ_0 are those of Sections 2 and 6, evaluated at the rational parameters of Section 8.1.

Put $\varepsilon_{\mathrm{cmp}} = 10^{-7}$, $\zeta = 10^{-8}$, $m_{\mathrm{seed}} = 1/10$, $N = (3R+1)/4 - \theta_0/2$, and

$$\Delta_{\mathrm{short}} = R + (1/2 - \theta_0) m_{\mathrm{seed}} - N, \qquad I = [h - 1/80, h + \zeta].$$

Use $\mathcal{R} = [7/20, 3/5] \times [9/20, 1/2]$ and $\mathcal{G} = [1/8, 5/6] \times [0, 1/2]$. Its three complementary rectangles are

$$\begin{aligned} \mathcal{C}_1 &= [1/8, 5/6] \times [0, 9/20], \\ \mathcal{C}_2 &= [1/8, 7/20] \times [9/20, 1/2], \\ \mathcal{C}_3 &= [3/5, 5/6] \times [9/20, 1/2]. \end{aligned}$$

On $\mathcal{G}$, one has $J > 0$ and $E < 1/500$; on each $\mathcal{C}_i$, one has $E < -\varepsilon_{\mathrm{cmp}}$. On $\mathcal{R}$, one has

$$S > 1/5, \qquad a > 0, \quad \gamma > 0, \qquad 2x(21/100) - z_L - W > 0, \tag{A.3}$$

$$E - t_-\{h(1-m_L) - l_y\} < -\varepsilon_{\mathrm{cmp}}, \qquad E - h\gamma < -\varepsilon_{\mathrm{cmp}}, \tag{A.4}$$

$$E + t_+\{h(1-m_R) - l_y\} < -\varepsilon_{\mathrm{cmp}}, \qquad E + h\Delta_{\mathrm{short}} < -\varepsilon_{\mathrm{cmp}}. \tag{A.5}$$

For d equal to either endpoint of I, each of

$$E+(d-h)S+d\Delta_{\mathrm{short}},\quad E+(d-h)S-t_-\{d(1-m_L)-l_y\},$$
$$E+(d-h)S-d\gamma,\quad E+(d-h)S+t_+\{d(1-m_R)-l_y\}$$

is less than $-\varepsilon_{\mathrm{cmp}}/2$. Their slopes are positive and less than two on the ordinary left, middle and right ranges. Further,

$$\frac{1}{5}\frac{1}{80}-\frac{1}{500}>\varepsilon_{\mathrm{cmp}},\qquad 2\zeta<\varepsilon_{\mathrm{cmp}}/2,$$

so parameters with smaller norm and the upper buffer in d have the required strict bounds. The elementary remaining inequalities are

$$b>0,\quad \frac{1}{6}<\ell<\frac{1}{5},\quad l_x>\ell,\quad l_y-\ell>\frac{11b}{6},\quad \frac{9}{2}\ell>\frac{3}{4},$$
$$\frac{21}{100}<\frac{\ell}{h+\zeta},\qquad \frac{\ell}{h-1/80}<\frac{11}{50},\qquad \frac{\ell}{h-1/80}<\frac{2}{9},$$
$$h<2l_y,\qquad l_y<\frac{1}{2},\qquad B>\frac{437}{500},$$
$$Q_{\mathrm{sm}}:=\ell(1/2+2\ell/h)-11h/54<0,$$
$$C_0+(1-h)/16<-1/250,$$
$$C_0+(1/2+3\ell/2)/50<-1/100,$$
$$13h/75-l_y/2+1/50<-1/20.$$

Substitution in (A.1)–(A.2) gives these signs and

$$(9/2-2\delta x)(21/100)>R,\qquad t_->0,\quad t_+>0.$$

The endpoint bounds above and affine dependence on d prove the full comparison on I.

A.3 The algebraic parameter interval

In this subsection, we describe the calculation used in Lemma 8.3. The algebraic identities at the critical class are proved by polynomial division by (1.2); the remaining signs are rational inequalities on the intervals below.

Use

$$I_b=[0.0917923522,0.0917923524],$$
$$I_\ell=[0.16876,0.16880],\qquad I_\kappa=[0.7489,0.7495],$$
$$\mathcal{R}_*=[7/20,3/5]\times[2/5,1/2].$$

Every decimal endpoint here is an exact terminating decimal. The critical root is isolated in

$$0.450570746724060730<r_*<0.450570746724060733.$$

For $\mathcal{Q}=E_\kappa-7a(2l_y-h)/108$, the coefficient test gives

$$\partial_x\mathcal{Q}>0\quad\text{on }\mathcal{R}_*$$

for $\kappa\in[0.7489366153,0.7489366155]$ and $(b,\ell)\in I_b\times I_\ell$. It also gives $\partial_\ell\mathcal{Q}>3/4$ at $\kappa=0.7489$; the identity

$$\partial_\kappa\partial_\ell\mathcal{Q}=\frac{9}{8}\partial_\kappa R\geq 0$$

extends this inequality to I_κ. The formula in Lemma 8.3 gives $0\leq\partial_\kappa\mathcal{Q}<1/100$ throughout the larger interval.

Use the complementary rectangles of Lemma 8.3, with amplitude endpoint 2/5. Lemma 6.5 gives

$$\partial_\kappa E = \frac{3h}{4}\partial_\kappa R \geq 0, \qquad \partial_\kappa S = \frac{3}{4}\partial_\kappa R \geq 0.$$

At the adverse count endpoints, the exact test gives

domain	count parameter	bound
$\mathcal{G}$	$\max I_\kappa$	$E < 1/500$
$\bigcup_i \mathcal{C}_i$	$\max I_\kappa$	$E < -10^{-5}$
$\mathcal{R}_*$	$\min I_\kappa$	$S > 1/5$.

The common ordinary gain satisfies

$$a > 0, \quad 4a/9 - \theta_0 > 0, \quad \delta/2 - \theta_0 - 7a/54 > 0,$$
$$\theta_0(5h/8 - l_y) > 7a(2l_y - h)/108,$$
$$(\delta/2 - \theta_0)(l_y - 11h/20) > 7a(2l_y - h)/108,$$
$$2x(21/100) - 1/18 > 7/279.$$

The short comparison uses $m_{\mathrm{cut}} = 3/40$ and $\Delta_{\mathrm{short}} = R + (1/2 - \theta_0)m_{\mathrm{cut}} - N$:

$$E + h\Delta_{\mathrm{short}} < -10^{-5}.$$

The geometric inequalities of the previous subsection persist, with the threshold for the lowest-amplitude estimate replaced by 1/125. Finally

$$\frac{57}{100} < \frac{l_y}{h} < \frac{11}{19}, \qquad 25/62 < 1 - l_y/h < 9/20.$$

The two bounds on l_y/h and the upper bound on $1 - l_y/h$ are checked by exact rational interval arithmetic on $I_b \times I_\ell$. The quantitative outer margin is the identity (7.3); an exact rational calculation gives it on the full parameter box.

These strict inequalities follow by outward-rounded rational interval arithmetic on finite rectangle covers.

A.4 Computations for the successive moment bounds

In this subsection, we describe the calculations used in Section 10. They prove only the stated identities and inequalities; the analytic estimates are those of Lemma 10.1 and Section 8.

Length identities. Equations (10.13)–(10.19) and (10.20) are polynomial identities in k_6 and the lengths. At $k_6 = 9/2$ they give the formulas of Sections 3.1–4.4. The rational identity (10.21) specializes at $\kappa = 3/4$ to

$$1 - \frac{279}{7}W = \frac{36}{7}\eta_{3/4} + 2(1 - C_{\mathrm{add}}) + 8(r - m_-).$$

The bounds (10.22) are monotone in κ on $[\kappa_\natural, 3/4]$ and were checked at the rational endpoints of an enclosure of $\kappa_\natural$.

The endpoint. The numerator in Proposition 10.3(i) and the resultant in (ii) were computed in exact rational arithmetic. The root $r_\natural$ was isolated in an interval of width 10^{-45}; $\ell_\natural, b_\natural, B_\natural$ and $A_\natural$ were evaluated in interval arithmetic with outward rounding. In particular

$$B_\natural \in [0.87446749650084027305624 98,\ 0.87446749650084027305624 99],$$

$$B_* - B_\natural = 8.1119269018\ldots \cdot 10^{-7}.$$

Let p_ℓ and p_B denote the two polynomials in (10.3). Exact polynomial division gives

$$p_\ell(\ell_\natural(r)) \equiv 0 \pmod{q_\natural(r)}, \qquad p_B(11/12 - \ell/4) = \frac{27}{8} p_\ell(\ell).$$

Both polynomials are irreducible over $\mathbb{Q}$. Thus their vanishing and the degree-four assertions are exact.

Exact rational root isolation gives the unique root of $q_\natural$ in $(9/20, 23/50)$; rational interval operations then evaluate $\ell_\natural$, $\kappa_\natural$ and $b_\natural = -\partial_\delta p_0/\partial_\delta p_b$. Every denominator is bounded away from zero. The resulting enclosures are contained in

$$\begin{aligned}
r_\natural &\in [0.450533924878843452503328321900 32, 0.450533924878843452503328321900 33],\\
\ell_\natural &\in [0.168796680663305574441667419387 90, 0.168796680663305574441667419387 92],\\
b_\natural &\in [0.091801394957524727743432859327 25, 0.091801394957524727743432859327 27].
\end{aligned}$$

These enclosures, together with an exact enclosure of κ_*, give rational boxes containing $\mathcal{P}_{\natural,0}$, $\mathcal{P}_\natural$ and $\mathcal{P}_{\natural,3/4}$.

The conditions of Table 4. For each rational expression F and finite rational covering $X = \bigcup_\nu X_\nu$, outward interval arithmetic gives

$$F(X_\nu) \subseteq [\underline{F}_\nu, \overline{F}_\nu] \quad \Longrightarrow \quad \min_\nu \underline{F}_\nu \le F(z) \le \max_\nu \overline{F}_\nu \qquad (z \in X).$$

The table bounds are rounded outwards from these extrema. In particular,

$$\theta_0 \le \frac{3}{60\kappa_m} < 0.06681 \qquad \text{(Remark 10.4(a))}.$$

Iteration. For the first steps of Proposition 10.5,

$$B_1 - B_\natural = 1.622\ldots \cdot 10^{-8}, \qquad B_2 - B_\natural = 3.245\ldots \cdot 10^{-10}, \qquad B_3 - B_\natural = 6.49\ldots \cdot 10^{-12},$$

in agreement with $B_n - B_\natural = 50^{-n}(B_* - B_\natural)$.

For μ_* and $\mu_\natural$ in the pointwise comparisons of Theorems 8.5 and 10.8, the root enclosures above and outward rational evaluation give

$\bullet$	$\mu_\bullet$	$g(\ell_\bullet)$
$*$	$< 399035/10^6$	$> 399159/10^6$
$\natural$	$< 399030/10^6$	$> 399165/10^6$.

(A.6)

B Exact inequalities for the enlarged count

In this appendix, we give the algebraic enclosures and interval procedure used in Section 11. These calculations verify the stated identities and signs. The analytic estimates are Lemmas 11.1 and 11.2.

B.1 Algebraic endpoint and the two branches

In this subsection, we isolate the parameters of Proposition 11.4 and verify its two rational expressions. The identity (11.15) treats the double zero exactly.

The identities of Proposition 11.4 follow by rational polynomial arithmetic. Sturm's rule and rational bisection give

$$\#\{\delta \in (11/25, 23/50) : q_{\mathrm{ec}}(\delta) = 0\} = 1, \qquad \delta' \in I_{\delta'}, \quad |I_{\delta'}| < 2 \cdot 10^{-53}.$$

Substitution in (11.12) and the formula for b' gives

$$\begin{aligned}
\delta' &\in [0.44878702795803278950526139677634,\ 0.44878702795803278950526139677635],\\
\ell' &\in [0.16978166469505833354211087786188,\ 0.16978166469505833354211087786189],\\
b' &\in [0.08218969571904572506748472403914,\ 0.08218969571904572506748472403915],\\
\kappa' &\in [0.74844250098580416656227789440239,\ 0.74844250098580416656227789440240],\\
B' &\in [0.87422125049290208328113894720119,\ 0.87422125049290208328113894720120].
\end{aligned}$$

All endpoints are exact rationals. The algebraic calculation verifies containment of its narrower enclosures in each displayed interval. In particular,

$$-7911.143 < \partial_\delta p_b(\delta', \ell') < -7911.142, \qquad \kappa_\flat < \kappa' < \kappa_\natural.$$

The reductions of $p_b(\delta, \ell'(\delta))$ and $p_0(\delta, \ell'(\delta))$ modulo q_{ec} vanish identically. The same is true of the polynomial in (11.13) after substitution of $\ell'(\delta)$, and of the polynomial in (1.1) after $B = 11/12 - \ell'(\delta)/4$. The three quartics are irreducible over $\mathbb{Q}$.

At (b', ℓ'), exact tangency yields (11.15). The signs of A', A_1 and the branch-1 discriminant are the interval signs in Proposition 11.4. Since D_i is affine in δ, its endpoint bounds suffice:

	$\delta = 7/20$	$\delta = 3/5$
D_2	> 21506	> 16504
D_1	> 19070	$> 14071.$

For the smaller branch margin, complete the square:

$$A_1\delta^2 + B_1\delta + C_1 = A_1(\delta - v_1)^2 + M_1, \qquad v_1 = -\frac{B_1}{2A_1}, \quad M_1 = -\frac{B_1^2 - 4A_1C_1}{4A_1} < 0.$$

Interval evaluation on 4000 equal subintervals of $[7/20, 3/5]$ gives

$$\sup_{7/20 \le \delta \le 3/5} \mathcal{Q}^{(1)}(\delta, 1/2) < -4.6125 \cdot 10^{-6}.$$

Together with the branchwise x derivatives in Table 6, this proves the full endpoint inequality without an interval test across the double zero of $\mathcal{Q}^{(2)}$.

B.2 Rational interval verification of the remaining signs

In this subsection, we verify the inequalities of Section 11.4 on the full parameter domain by rational interval arithmetic. The enclosures of Proposition 11.4 are retained throughout.

Use $k_N^+ = 7489349931/10^{10} > \kappa_\natural$ and the enclosures above, widened by 10^{-50} at each end. If I_κ, I_ℓ, I_b are the widened intervals, the verification box contains

$$\begin{gathered}
\kappa_c, \kappa_m \in [\inf I_\kappa, k_N^+], \qquad \ell \in [\inf I_\ell - (k_N^+ - \inf I_\kappa)/25, \sup I_\ell],\\
b \in I_b + [-10^{-30}, 10^{-30}].
\end{gathered}$$

For $\mathcal{P}_{3/4}$ replace the upper moment endpoint by $3/4$. All class endpoints are entered as exact rationals. Denominators are bounded away from zero on every retained box.

Every expression is rational in $(\delta, x, \kappa_c, \kappa_m, \ell, b)$. The integer-interval calculation verifies every displayed table threshold on the full enclosing rational box. It represents interval endpoints by integers divided by $N_{\mathrm{den}} = 2^{100}$ and rounds outwards after each arithmetic operation. For example, if $I = [u_1, u_2]/N_{\mathrm{den}}$ and $I' = [v_1, v_2]/N_{\mathrm{den}}$, put $\mu_- = \min(u_1v_1, u_1v_2, u_2v_1, u_2v_2)$ and $\mu_+ = \max(u_1v_1, u_1v_2, u_2v_1, u_2v_2)$. Then

$$II' \subseteq \left[\left\lfloor \frac{\mu_-}{N_{\mathrm{den}}} \right\rfloor, \left\lceil \frac{\mu_+}{N_{\mathrm{den}}} \right\rceil\right] / N_{\mathrm{den}}.$$

Addition is exact, and reciprocal bounds use integer division on an interval excluding zero. Rational constants are enclosed before use.

For a rational function f and box X, intersect its natural interval extension with the mean-value enclosure

$$f(X) \subseteq f(x_0) + \sum_i (\partial_i f)(X)(X_i - x_{0,i}).$$

The b interval may be retained throughout instead of centered. An undecided box is bisected at a rational midpoint; the two resulting subrectangles cover the original rectangle. Only a strict interval sign accepts a box. Thus the finite subdivision proves the sign on the complete domain.

For an expression F increasing in the count exponent,

$$F(R') = \max\{F(R^{(1)}), \min(F(R^{(0)}), F(R^{(2)}))\}.$$

For decreasing F, interchange maximum and minimum. These identities are used for $E', S, N - I_{\mathrm{short}}$, the slopes and prime supply. Derivative inequalities are verified for each $\mathcal{Q}^{(j)}$ separately, and then used as finite-difference inequalities for their maximum and minimum. In particular, the calculation gives

$$\partial_x \mathcal{Q}^{(j)} > 0.009 \quad \text{at the endpoint},$$
$$\partial_{\kappa_c} \mathcal{Q}^{(j)} + \partial_{\kappa_m} \mathcal{Q}^{(j)} - \frac{1}{25}\partial_\ell \mathcal{Q}^{(j)} < -0.011,$$
$$E' + h\Delta_{\mathrm{short}} < -0.0027, \qquad (6\kappa_m - 2\delta x)(21/100) - R' > 0.019 \quad \text{on } \mathcal{G} \times \mathcal{P}_{3/4},$$

where the short-sum inequality is on $\mathcal{R}_* \times \mathcal{P}$. All remaining inequalities in Tables 6–7 follow by the same finite calculation. The crude supply is also immediate: its minimum occurs at $\delta = 1/8$ and $\kappa_m = \kappa'$, where it exceeds 0.00012.

Exact polynomial reduction verifies the resultant, both tangency identities and the active branch at the contact point. The remaining inequalities follow by outward-rounded rational interval arithmetic on finite rectangle covers.

C The zero-detecting estimates and theta exponents

In this appendix, we derive the zero-detecting estimates and the exponent (14.5) from the moment lower bounds and theta identity. This verifies the normalization used in Section 14 and Appendix D. Write

$$g = \delta x, \qquad p = 6\kappa, \qquad \mathcal{K} = \frac{3R+1}{4}, \qquad \theta_0 = \frac{g}{p}, \qquad z_0(m) = \frac{1-2m}{p}, \qquad \sigma_{\mathrm{M}} = \frac{17}{50}.$$

Here σ_{M} is the fixed Mellin contour.

The two detector counts. For inverse length $r < 1$, the detector gives $|M_r|^2 \geq U^{\delta r - o(1)}$. A selected product of width $(1-r)/2$ has squared amplitude at least $U^{g(1-r)-o(1)}$. The marked inverse moment has upper bound $U^{1+o(1)}$, so division gives

$$1 - \delta r - g(1-r) = 1 - \delta\{x + (1-x)r\}.$$

For ordinary length m, the detector gives $|S_m|^2 \geq U^{\delta m - o(1)}$. The norm contains $|S_m^2|^2 = |S_m|^4$, whose lower exponent is $2\delta m$. The joint selected product adds $2g(z_0(m) + W(m))$. Therefore the ordinary count has exponent

$$1 - 2\delta m - 2g\{z_0(m) + W(m)\} = 1 - \delta F_2(m), \qquad F_2(m) = 2m + 2x\{z_0(m) + W(m)\}.$$

These are divisions of arithmetic parameter moments by lower bounds for each arithmetic parameter. Neither uses an identification of different arithmetic parameters with distinct primitive characters. A fourth root is not taken in a detector count.

For $r \geq 1$, the inverse moment exponent of [Liu26b, Lemma 10.10] $e(r) = (1+5r)/6$ instead gives count $1/6 + (5/6-\delta)r$. Since $r \leq T + o(1)$ and $\delta \leq 5/6$, it is bounded by $1/6 + (5/6-\delta)T$. Put $A_\delta = 5/6 - \delta$. Equal short counts give

$$r = \frac{F_2(m) - x}{1-x}, \qquad T = r + m.$$

Equating their count with the long inverse bound yields

$$-\frac{5}{6} + A_\delta m - \frac{A_\delta x}{1-x} + \left(\frac{A_\delta}{1-x} + \delta\right) F_2(m) = 0,$$

which is the balance equation used for each affine width branch.

The reference exponent. The comparison formula of [Liu26b, Section 14.1, equation (14.7)], before simplifying its constants and with an arbitrary valid class count R, is

$$\begin{aligned} \mathcal{E}_0(d) = a_{\mathrm{bin}} - B - l_y(1/2+\theta_0) + h(\sigma_{\mathrm{M}} - 1/6) - \ell/2 + \ell g \\ + d(\mathcal{K} + \theta_0/2 - \sigma_{\mathrm{M}}), \qquad a_{\mathrm{bin}} = \frac{1+\delta}{2}. \end{aligned} \tag{C.1}$$

The arithmetic average, conductor factor and outside Mellin factor give

$$U^{\mathcal{K}-\theta_0/2}\, U^{\theta_0} Y^{-\theta_0} = U^{\mathcal{K}} Z^{\theta_0(d/2-l_y)}, \qquad U = Z^d, \quad Y = Z^{l_y}.$$

At $d = h$, use $B = 11/12 - \ell/4$, $h = (1+b+3\ell)/2$, $l_y = (1+b-\ell)/2$ and $\mathcal{K} = 1 - 3(1-R)/4$. The terms independent of $\delta, g, 1-R$ reduce as follows:

$$\frac{1}{2} - B - \frac{l_y}{2} - \frac{\ell}{2} + \frac{5h}{6} = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4} = C_0.$$

Consequently

$$\mathcal{E}_0(h) = C_0 + \frac{\delta}{2} + \delta x \left\{ \ell + \frac{h/2 - l_y}{p} \right\} - \frac{3h}{4}(1-R) = E(R),$$

and differentiating (C.1) gives

$$\mathcal{E}_0'(d) = \mathcal{K} + \frac{\theta_0}{2} - \frac{17}{50} = S(R).$$

This independently verifies $\mathcal{E}_0(d) = E(R) + (d-h)S(R)$.

Hölder, the AFE and the displaced contour. Let γ be the additional first-moment gain beyond the baseline upper bound for the prime-product length z_0. The fourth-moment estimate has exponent $1 - 2gz_0 - 4\gamma$. Hölder with $U^{R+o(1)}$ arithmetic parameters gives the central first-moment exponent

$$\frac{3R}{4} + \frac{1 - 2gz_0 - 4\gamma}{4} = \mathcal{K} - \frac{gz_0}{2} - \gamma.$$

For a dyadic AFE block of length U^m with $m \leq 1/2$, the scalar $U^{-\theta m}$ changes this to

$$\mathcal{K} - \frac{gz_0}{2} - \gamma - \theta m = \mathcal{K} - \frac{\theta_0}{2} - (\theta - \theta_0)m - \gamma,$$

using $gz_0 = \theta_0(1-2m)$. Relative to θ_0, the conductor raising and outside Mellin factors are $U^{\theta-\theta_0} Y^{-(\theta-\theta_0)}$. Hence the full change is exactly

$$(\theta - \theta_0)\{d(1-m) - l_y\} - d\gamma.$$

This derivation applies to both AFE halves after the prescribed whole conjugation and actual-conductor normalization. It uses the baseline $gz_0/2$ once. In the conservative extra-prime selection $|Q_W|^2 \geq U^{2aW - o(1)}$, with a the remaining mean amplitude of the numerator section (not a_{bin}), one has $\gamma = aW/2$; joint selection would replace a by g.

For $m \ge 1/2$, use $z_0 = 0$ and the no-prime moment instead of continuing $(1-2m)/p$ to negative lengths. The actual correction is (14.7). Its excess below the formal continuation of the preceding display is $d\theta_0(m-1/2) \ge 0$. Every such outer block is bounded by the $m = 9/20$ endpoint as proved in the scalar section.

Set $m_\theta = 1 - l_y/h$. At $d = h$, the contour saving is

$$\begin{cases} h\theta_0(m_\theta - m), & m \le m_\theta, \quad \theta = 0, \\ h(\delta/2 - \theta_0)(m - m_\theta), & m \ge m_\theta, \quad \theta = \delta/2. \end{cases}$$

The two endpoint gains are

$$\frac{ha}{2}\frac{\varrho(1-2m_\theta)}{5} = \frac{a\varrho T_y}{10} = G_N, \qquad \theta_0\{(1-m_L)h - l_y\} = G_L.$$

For $m < 3/40$, ideal counting and $\theta_0 < 1/2$ give

$$R + (1/2-\theta_0)m - (\mathcal{K} - \theta_0/2) \le R + (1/2-\theta_0)\frac{3}{40} - (\mathcal{K} - \theta_0/2) = D_{\mathrm{short}}.$$

D Exact inequalities for the exhaustion range

In this appendix, we verify Proposition 14.2 and Lemma 15.1 by rational identities and interval inequalities. The analytic inputs are those of Sections 12–14.

D.1 The rational starting comparison

In this subsection, we verify Proposition 14.2 at (14.1). The second parameter choice below gives a direct one-step comparison: with the moment parameter $\kappa = 4676/6250$, justified by the bound $\beta_*(F) \le 5463/6250$ of Section 14.5, Remark 14.3 gives $\beta_*(F) \le 874073/10^6$ directly, without the family (15.8).

The parameters and comparison margins are

B	ℓ	b	κ	ϵ_{cmp}
$5463/6250$	$1597/9375$	$1913/20000$	$37446831/50000000$	10^{-6}
$874073/10^6$	$4(11/12 - B)$	$3827/40000$	$4676/6250$	10^{-7}.

The interval calculation below proves the strict comparisons; every inverted denominator excludes zero.

D.2 Algebraic enclosures and the endpoint quadratics

In this subsection, we verify the contact identities and signs of Lemma 15.1, using the quartic (15.1).

Put

$$\ell_- = 0.17037527599935503555198917059375973764376115653239374775,$$
$$I_\ell = [\ell_-, \ell_- + 10^{-56}].$$

Sturm's theorem gives

$$\#\{\lambda \in I_\ell : f(\lambda) = 0\} = 1.$$

Rational evaluation of (15.2)–(15.5) on I_ℓ gives the enclosures

$$\kappa_- = 0.74814569533365581555733874803645346451145275506 7,$$
$$b_- = 0.09567500764537642027548322481699577004478313960 7,$$
$$I_\kappa = [\kappa_-, \kappa_- + 10^{-48}], \qquad I_b = [b_-, b_- + 10^{-48}],$$
$$\kappa_{\mathrm{ex}} \in I_\kappa, \qquad b_{\mathrm{ex}} \in I_b.$$

Put

$$r(\ell) = -\frac{5(3\ell-5)(9\ell-5)}{6(21\ell^2-28\ell-25)}.$$

Before reducing the quotient for b_{ex}, write

$$D_b(\ell) = \partial_\delta \partial_b \mathcal{Q}_o(\delta, 1/2)|_{\delta=r(\ell),\,\kappa=5/6-\ell/2} = \frac{N_b(\ell)}{T_b(\ell)}.$$

Exact polynomial reduction gives

$$\gcd(N_b, f) = \gcd(T_b, f) = 1 \quad \Longrightarrow \quad D_b(\ell_{\mathrm{ex}}) \in \mathbb{R}^\times.$$

All subsequent quotient reductions are in $\mathbb{Q}[\ell]/(f)$. At the endpoint substitutions, their denominators are units and

$$\mathcal{Q}_o(r_{\mathrm{ex}}, 1/2) = \partial_\delta \mathcal{Q}_o(r_{\mathrm{ex}}, 1/2) = \partial_b \mathcal{Q}_o(r_{\mathrm{ex}}, 1/2) = 0.$$

At $(\kappa_{\mathrm{ex}}, \ell_{\mathrm{ex}}, b_{\mathrm{ex}})$, for $8/25 \le \delta \le 3/5$, clearing denominators yields

$$\mathcal{Q}_i(\delta, 1/2) = \frac{A_i\delta^2 + B_i\delta + C_i}{D_i(\delta)}, \qquad D_i(\delta) > 0, \quad A_i < 0, \qquad i = o, j.$$

The remaining signs are

$$B_j^2 - 4A_jC_j < 0 \implies \mathcal{Q}_j(\delta, 1/2) < 0,$$
$$\mathcal{Q}_o(r_{\mathrm{ex}}, 1/2) = \partial_\delta \mathcal{Q}_o(r_{\mathrm{ex}}, 1/2) = 0 \implies \mathcal{Q}_o(\delta, 1/2) = \frac{A_o(\delta - r_{\mathrm{ex}})^2}{D_o(\delta)} \le 0.$$

For the quartic $P(B)$ in (15.7),

$$f(11/3 - 4B) = \frac{8}{9}P(B), \qquad P \text{ irreducible over } \mathbb{Q},$$
$$\#\{B \in (0.87407284, 0.87407285) : P(B) = 0\} = 1.$$

The last two assertions follow by rational factorization and Sturm's theorem.

D.3 The full parameter family

In this subsection, we prove the inequalities along (15.8) used in Proposition 15.2 and verify the coefficient signs of Section 16.

Let $N_{\mathrm{den}} = 2^{220}$. For $I = [u_1, u_2]/N_{\mathrm{den}}$ and $I' = [v_1, v_2]/N_{\mathrm{den}}$, outward arithmetic gives

$$II' \subseteq \left[\left\lfloor \frac{\min(u_1v_1, u_1v_2, u_2v_1, u_2v_2)}{N_{\mathrm{den}}} \right\rfloor, \left\lceil \frac{\max(u_1v_1, u_1v_2, u_2v_1, u_2v_2)}{N_{\mathrm{den}}} \right\rceil\right]/N_{\mathrm{den}},$$

and

$$0 \notin I \quad \Longrightarrow \quad I^{-1} \subseteq \left[\left\lfloor \frac{N_{\mathrm{den}}^2}{u_2} \right\rfloor, \left\lceil \frac{N_{\mathrm{den}}^2}{u_1} \right\rceil\right]/N_{\mathrm{den}}.$$

Constants and algebraic enclosures are rounded outwards.

The rational parameter boxes are

$$Y_{\mathrm{ex}} = I_\kappa \times I_\ell \times I_b, \qquad \kappa_0 = 4676/6250,$$
$$\widehat{\mathcal{P}} = [\inf I_\kappa, \kappa_0] \times [\inf I_\ell - (\kappa_0 - \inf I_\kappa)/25, \sup I_\ell] \times I_b.$$

They contain the endpoint and $\mathcal{P}$, respectively. Here ∂_κ acts on both detector and numerator.

Write $\mathbf{I}_F$ for outward interval evaluation of a reduced rational function F, using factored numerator and denominator and Horner's rule for the polynomial factors.

For $X = J_\delta \times J_x$, a fixed parameter box Y, and $c_X = \operatorname{mid} X$, put

$$F(X,Y) \subseteq \mathcal{J}_F(X,Y),$$
$$\mathcal{J}_F(X,Y) = \mathbf{I}_F(X,Y) \cap \left(\mathbf{I}_F(c_X,Y) + \sum_{j=1}^{2} \mathbf{I}_{\partial_j F}(X,Y)(X_j - c_{X,j}) \right).$$

A comparison inequality $G < -\epsilon$ is written as $F > 0$ with $F = -G - \epsilon$, and $G > \epsilon$ with $F = G - \epsilon$. Treat each branch and each indicated class rectangle separately. Use $\mathcal{B}_F = \mathcal{J}_F$ for the strict numerical inequalities of Proposition 14.2, Table 8 and the auxiliary detector-length inequalities below; use $\mathcal{B}_F = \mathbf{I}_F$ for

$$F_{i,x} = \partial_x \mathcal{Q}_i - 10^{-3}, \qquad F_{i,\kappa} = -10^{-2} - \partial_\kappa \mathcal{Q}_i + \frac{1}{25}\partial_\ell \mathcal{Q}_i, \qquad i = o, j.$$

Bisecting class rectangles at rational midpoints, with the parameter box Y not subdivided, gives a finite cover $\mathfrak{X}_F$ with

$$\mathcal{D}_F \subseteq \bigcup_{X \in \mathfrak{X}_F} X, \qquad \min_{X \in \mathfrak{X}_F} \inf \mathcal{B}_F(X,Y) > 0 \implies F > 0 \text{ on } \mathcal{D}_F \times Y.$$

Since $F(X,Y) \subseteq \mathcal{B}_F(X,Y)$, these finite subdivisions give the following strict lower bounds for $\inf_{\mathcal{D}_F \times Y} F$:

function and parameters	lower bound
rational comparison, first choice	$9 \cdot 10^{-7}$
rational comparison, second choice	$5 \cdot 10^{-8}$
auxiliary inequalities on $\widehat{\mathcal{P}}$	10^{-6}
$F_{o,x},\ F_{j,x}$ on Y_{ex}	$8 \cdot 10^{-5},\ 2 \cdot 10^{-4}$
$F_{o,\kappa},\ F_{j,\kappa}$ on $\widehat{\mathcal{P}}$	$4 \cdot 10^{-4},\ 10^{-3}$.

In the first two entries the parameters are the respective choices of Section D.1, with their stated ϵ_{cmp}. Along (15.8), for $\kappa > \kappa_{\mathrm{ex}}$, the endpoint comparison of Lemma 15.1 and the bounds for $F_{i,x}$ and $F_{i,\kappa}$ give

$$E(R_i) - G_N < -(\kappa - \kappa_{\mathrm{ex}})/100, \qquad i = o, j.$$

At the endpoint the comparison is nonpositive. The auxiliary inequalities in the third entry hold on the full enclosing box $\widehat{\mathcal{P}}$.

For the detector length inequalities, let $m_{oj}(\kappa)$ be defined by $W_o(m_{oj}) = W_j(m_{oj})$. On its two sides use

$$m = 19/50 + (m_{oj}(\kappa) - 19/50)y \quad \text{or} \quad m = m_{oj}(\kappa) + (9/20 - m_{oj}(\kappa))y, \qquad 0 \le y \le 1.$$

The auxiliary inequalities are

$$19/50 < m_{oj} < 9/20,$$
$$0 < W_d < 1/25, \qquad 0 < t_{\mathrm{low}} < 1/4, \qquad t_{\mathrm{high}} > 0,$$
$$t_{\mathrm{low}} > 5W_d, \qquad z_0(m) + W_d(m) < L_0.$$

They involve only m and κ and hold on the respective intervals after subdivision in y alone. Here t_{low} and t_{high} are evaluated at the limiting width W_d. Keep the comparison length t chosen in Section 14. At the limiting width,

$$t_{\mathrm{low}}(W_d) \le t \le t_{\mathrm{high}}(W_d), \qquad W_d = W_j \implies t_{\mathrm{low}}(W_d) = t = t_{\mathrm{high}}(W_d).$$

For $W = (1-\epsilon_W)W_d$, $0 < \epsilon_W < 1$, the identity $5W_o = (1-1/p)(1-2m)$ and $W_d \le W_o$ give

$$2m + (1-2m)/p + 5W \le 1 - 5\epsilon_W W_d < 1.$$

The comparison interval opens because

$$t_{\mathrm{low}}(W) = t_{\mathrm{low}}(W_d) - 6\epsilon_W W_d < t,$$
$$t_{\mathrm{high}}(W) = t_{\mathrm{high}}(W_d) + (p/2)\epsilon_W W_d > t.$$

Thus the origin, reflection and exceptional inequalities are strict at the interior width. Count balances and the identities of Section 16 follow after clearing denominators; the same rational evaluation gives their coefficient signs.

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