

An improved zero-free half-plane for Hecke L -functions over number fields and Quasi-Riemann Hypothesis

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Abstract

In this paper, we obtain a zero-free half-plane $\Re s > B_*$ for every finite-order Hecke L -function over every number field, where $B_* = 0.87446830769353045955\dots$ is an explicitly defined algebraic number. This improves the Quasi-Riemann Hypothesis over arbitrary number fields. The proof refines the cubic-theta comparison. A sharper amplified Gauss-norm estimate permits a longer additional prime product, chosen separately for each dyadic summation length. Combining the resulting moment saving with contour displacement on the same dyadic sum gives the principal improvement. The enlarged moment controls the critical summation length, where the contour contribution vanishes. Successive improvements of the character-counting estimate then reach the algebraic endpoint, while the numerator-moment parameter remains fixed. We prove optimality within the resulting comparison system. The argument uses the established theta, moment and continuation results of the earlier QRH papers and the general-field formulation of those results.

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Date: 27 October 2026.

The research of the author is partially supported by the Simons Foundation Grant MPS-TSM-00006679.

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1 Introduction

In this section, we state an improved Quasi-Riemann Hypothesis for finite-order Hecke L -functions over arbitrary number fields and explain the additional moment estimate. The arithmetic estimates are proved in Sections 3–4; the comparison and its limiting parameters are treated in Sections 7–8. Section 9 gives the passage to arbitrary number fields. We write μ_n for the group of n -th roots of unity and $\zeta_n = e^{2\pi i/n}$ for its chosen primitive root.

1.1 The main theorem

In this subsection, we define the boundary and state the theorem proved in Section 9. Its exact expression follows from the comparison in Section 8.

Let r_* be the unique root in $(9/20, 23/50)$ of

$$218582r^3 - 376469r^2 - 11376r + 61560 = 0. \quad (1.1)$$

Set

$$\ell_* = \frac{3410r_*^2 - 7127r_* + 2520}{2286r_*^2 - 5011r_* + 1800}, \quad B_* = \frac{11}{12} - \frac{\ell_*}{4}. \quad (1.2)$$

The remaining limiting parameters are

$$b_* = \frac{\ell_*^2(11665 - 468r_*) + \ell_*(4528r_* - 20554) + 5113 - 4412r_*}{\ell_*(4572r_* - 5011) + 7127 - 6820r_*}, \quad (1.3)$$

$$\kappa_* = \frac{5}{6} - \frac{\ell_*}{2} = 2B_* - 1.$$

Here ℓ_* is the prime-product length parameter, b_* balances l_y and h (see below), and κ_* is the limiting class-count parameter. Here $\kappa_m = 3/4$ is the parameter in the numerator moment estimate; κ_c denotes the independent class-count parameter. The parameter r_* is the critical value of the detector coordinate δ in the class (δ, x) defined in Section 1.3.

A closed form for the root is obtained by putting $\Delta = 149188674457$:

$$r_* = \frac{376469 + 2\sqrt{\Delta} \cos\left(\frac{1}{3} \arccos \frac{17862636423556205}{\Delta^{3/2}} - \frac{2\pi}{3}\right)}{655746}, \quad (1.4)$$

where $\arccos$ takes values in $[0, \pi]$. Thus every parameter in (1.2)–(1.3) has an exact expression in this same cubic field. Formula (1.4) follows from the triple-angle identity; the expression for b_* is obtained by simplifying Definition 8.2.

Thus

$$\begin{aligned} r_* &= 0.45057074672406073136\dots, & \ell_* &= 0.16879343589254482845\dots, \\ b_* &= 0.09179235229895088958\dots, & \kappa_* &= 0.74893661538706091910\dots, \\ B_* &= 0.87446830769353045955\dots \end{aligned}$$

Theorem 1.1. *Let K be a number field and let χ be a finite-order Hecke character of K . Then $L_K(s, \chi)$ has no zeros in $\Re s > B_*$. The pole of a principal character at $s = 1$ is permitted. In particular,*

$$\zeta_K(s) \neq 0 \quad (\Re s > B_*, s \neq 1).$$

The boundary is independent of K and χ . No assertion is made on the line $\Re s = B_$.*

We use the earlier results cited in Section 2 as established theorems. The additional norm estimate and its use in the comparison are proved below. The constants and thresholds may depend on the fixed field K , its finite set of excluded places and the fixed character χ , including on the degree $[K : \mathbb{Q}]$ and the absolute field discriminant $|D_K|$. The numerical boundary B_* is the same for every K and χ .

1.2 Earlier bounds and the results used here

In this subsection, we recall the bounds preceding Theorem 1.1. The precise earlier results used in its proof are stated in Section 2; their numerical optimization is not needed as an assumption in the new moment argument.

The cubic-theta argument of OpenAI [Ope26a] gives the boundary $7/8$ over $\mathbb{Q}(\sqrt{-3})$ and the corresponding Dirichlet family. Its shorter $11/12$ argument and the cyclotomic construction are given in [Ope26c, Ope26b].

In the first version of [Liu26a], by varying the length parameters b, ℓ , the author obtains

$$B_r = \frac{34999}{40000} = 0.874975, \quad B_{\text{alg}} = \frac{1507 - 2\sqrt{921}}{1653} = 0.874957069799\dots$$

Contemporaneous parameter refinements were reported by Mermet [Mer26], Wei [Wei26], and Chakhkiev [Cha26]; see also [Liu26a, Section 21]. The two GitHub repositories report

$$B_{\text{MW}} = \frac{4374785349}{5000000000} = 0.8749570698,$$

whereas Chakhkiev reports the conditional bound $10937/12500 = 0.87496$ under the analytic inputs specified there. These reports concern the Eisenstein-field and corresponding Dirichlet setting, rather than arbitrary number fields.

In version 2 of [Liu26a], also varying the moment parameter, the author obtains

$$B_{\text{new}} = 0.874957019420098946\dots$$

Xu and Sang [XS26] replace the pointwise treatment of the reflected numerator by an amplified average and obtain

$$B_{\text{XS}} = \frac{8748867}{10^7} = 0.8748867$$

from the analytic inputs of [Ope26a]. Newton [New26] studies the associated parameter optimization and gives the certified boundary

$$B_{\text{N}} = \frac{43744332481}{50000000000} = 0.87488664962$$

under the analytic hypotheses and parameter restrictions specified there. This last result is a refinement of the comparison rather than a new mean-value estimate.

Cureton and Thorner [CT26] obtain the boundary 31/32 for Dedekind zeta functions and finite-order Hecke L -functions over arbitrary number fields. Following [XS26] and [CT26], in the general-field comparison in [Liu26b], the author obtains

$$B_{\mathfrak{p}} = 0.874886647039964 \dots, \quad (1.5)$$

where

$$133488B_{\mathfrak{p}}^4 - 376596B_{\mathfrak{p}}^3 + 384534B_{\mathfrak{p}}^2 - 167541B_{\mathfrak{p}} + 26231 = 0.$$

Their proof develops the general-field metaplectic argument and descends from $K(\zeta_{12})$ to K . In this paper, we use the corresponding formulation in [Liu26b, Part III]. These field scopes are distinct: a bound over the Eisenstein field alone does not imply the same bound over every number field.

1.3 The additional saving

In this subsection, we explain the two analytic ingredients of the improvement: the enlarged norm of Section 4 and its use with contour displacement on the same dyadic sum in Section 7. The count iteration of Section 8 then reaches the algebraic endpoint.

Let $Z > 1$ be the main scale parameter. In the smoothed cubic-theta average $I_{\eta}(X, Y, Z)$ of [Liu26b, Section 13.2], η is the fixed target Hecke character, and X, Y are the two averaging lengths; Z sets the smoothed norm scale of the indices in the cubic-theta sum. We write

$$X = Z^{l_x}, \quad Y = Z^{l_y}.$$

The subscripts x, y refer to X, Y , respectively. The total prime-product length exponent ℓ and balancing parameter b satisfy

$$l_x + l_y + \ell = 1, \quad b = l_y - l_x.$$

Write $U = Z^d$, $d > 0$, for the scale of the character parameter and $D = U^m$ for the length of an ordinary dyadic sum. The balanced length exponents are

$$\begin{aligned} l_x &= \frac{1 - b - \ell}{2}, & l_y &= \frac{1 + b - \ell}{2}, \\ h &= \frac{1 + b + 3\ell}{2}, & B(\ell) &= \frac{11}{12} - \frac{\ell}{4}. \end{aligned}$$

The endpoint character-parameter scale is $U = Z^h$. With the other class data fixed, $\mathcal{B}$ groups character parameters with fixed detector and prime-sum amplitude ranges. If a_{det} is the real-part parameter in the zero-detecting decomposition, write $\delta = 2a_{\text{det}} - 1 > 0$; this parameter is denoted by a in the cited comparison. For prime-sum factors of lengths U^{λ_i} , let g_i be their logarithmic amplitude exponents, as in Lemma 4.2. Put

$$\begin{aligned} \sum_i \lambda_i &= \mathcal{L} = \ell/d, & 0 \leq g_i &\leq \delta/2, \\ g &= \mathcal{L}^{-1} \sum_i \lambda_i g_i = \delta x, & 0 \leq x &\leq 1/2. \end{aligned}$$

Thus δ describes the detector range and x the normalized mean prime-sum amplitude. The contour shift θ specifies the numerator line $\Re w = 1/2 - \theta$, with reference shift $\theta_0 = 2\delta x/9$. Let $\mathcal{E}_{\theta, \gamma}(d, m)$ be the full comparison exponent when the contour is displaced to θ and the selected moment saves $U^{-\gamma}$. The simultaneous estimate is

$$\mathcal{E}_{\theta, \gamma}(d, m) - \mathcal{E}_{\theta_0, 0}(d, m) = (\theta - \theta_0)\{d(1 - m) - l_y\} - d\gamma. \quad (1.6)$$

Both terms use the same dyadic sum and original prime coefficients. The prime-product lower bounds precede differentiation of the smooth weights.

Let Q_0 be the initial prime product and E a disjoint additional prime product. Write $W \geq 0$ for an upper bound on the length exponent of E relative to U . The enlarged norm permits, with strict inequalities,

$$W < \frac{7}{54}(1 - 2m), \quad W < \frac{8m - 3}{9}. \quad (1.7)$$

The unused prime product has lower mean

$$a(\delta, x) = \delta \frac{189x - 25}{139}, \quad \gamma < \frac{a(\delta, x)W}{2}.$$

The limiting upper length $W_{\text{cap}}(m)$ and the contour are chosen separately for each dyadic sum:

$$W_{\text{cap}}(m) = \min \left\{ \frac{7}{54}(1 - 2m), \frac{8m - 3}{9} \right\}, \quad m_0 = 1 - \frac{l_y}{h},$$

$$\theta_*(m) = \begin{cases} 0, & m \leq m_0, \\ \delta/2, & m > m_0. \end{cases}$$

In the final parameter range, Section 7 proves

$$\inf_{3/8 \leq m \leq 1/2} \left\{ \frac{aW_{\text{cap}}(m)}{2} + (\theta_*(m) - \theta_0)(m - m_0) \right\} = \Gamma_N = \frac{7}{108}a(\delta, x)(2l_y/h - 1).$$

Here Γ_N is the limiting normalized gain at $m = m_0$. These are limiting scalar values; the analytic estimates use strict interior lengths. At the critical length,

$$d = h, \quad m = m_0 \implies h(1 - m_0) - l_y = 0, \quad \mathcal{E}_{\theta, \gamma} - \mathcal{E}_{\theta_0, 0} = -h\gamma.$$

Thus the enlarged moment supplies the saving at m_0 , and the contour displacement controls the remaining lengths.

Origin of the limiting parameters. The parameters in (1.2)–(1.3) are determined by the class for which the comparison is sharp. With E_κ as in (2.2), define

$$Q_{\kappa, b, \ell}(\delta, x) = E_\kappa(\delta, x) - \frac{7}{108}a(\delta, x)(2l_y - h).$$

The contour contribution vanishes at $m_0 = 1 - l_y/h$. The extreme amplitude is $x = 1/2$. At the limiting parameters the comparison has a double zero in δ , and its coefficient of the balancing parameter vanishes:

$$Q_{\kappa_*, b_*, \ell_*}(r_*, 1/2) = 0, \quad \partial_\delta Q_{\kappa_*, b_*, \ell_*}(r_*, 1/2) = 0,$$

$$\partial_b Q_{\kappa_*, b_*, \ell_*}(r_*, 1/2) = 0, \quad \kappa_* = \frac{5}{6} - \frac{\ell_*}{2}. \quad (1.8)$$

Eliminating b_* and κ_* gives the cubic (1.1) and the rational formula for ℓ_* . The stationarity equation in δ then determines b_* . Section 8.2 proves these identities and verifies that all the other classes satisfy the required inequality.

The count iteration is used only to reach this boundary from strict interior estimates. Let $B_{\text{seed}} = 0.874558$ denote the rational starting bound of Theorem 8.1. Its first combined step gives

$$B_1 = B_* + \frac{2B_{\text{seed}} - 1 - \kappa_*}{100} < B_* + 1.8 \cdot 10^{-6}.$$

The additional moment and contour saving supplies the principal improvement; the subsequent iteration removes this residual gap.

The unamplified numerator moment remains at $\kappa_m = 3/4$, supplied by the earlier $7/8$ bound. The class count may use a separate parameter κ_c . Starting from a strict rational bound, the iteration in Section 8 satisfies

$$\kappa'_c = \kappa_* + \frac{\kappa_c - \kappa_*}{50}, \quad \kappa_* = 2B_* - 1.$$

Every finite step has positive margins. Taking the decreasing sequence of proved boundaries yields (1.2).

1.4 Optimality and organization of the proof

In this subsection, we specify the optimality statement in Section 8 and summarize the steps leading to Theorem 1.1.

At the ordinary length $m_0 = 1 - l_y/h$, the contour term vanishes:

$$h(1 - m_0) - l_y = 0.$$

The amplitude bound and the admissible norm width then give

$$h\gamma \leq \frac{7\delta}{216}(2l_y - h).$$

At the critical class $(\delta, x) = (r_*, 1/2)$, this is exactly the limiting obstruction. It proves optimality for the comparison with the stated norm domain, count estimate and moment parameter $3/4$. It does not assert the best possible zero-free region for a field.

Sections 2–4 develop the arithmetic and amplified norm. Sections 5–7 treat the contours, class counts and combined saving. Section 8 gives the endpoint and its stated optimality; Section 9 proves Theorem 1.1 over arbitrary fields.

Acknowledgments

OpenAI Codex (GPT 6.1 Sol Medium-Ultra) and Anthropic Claude (Opus 5.5 Medium) assisted with drafting, checking references, the search for improved bounds, and auditing. The Lean formalization is in progress, based on the Lean files in [Liu26c] and in [CTLean].

2 Arithmetic notation and the earlier comparison theorems

In this section, we fix the notation used in Sections 3–7 and state precisely which earlier results are used. The new amplified norm is proved in Section 4; it is not included among the cited inputs.

Fix a number field $F \supset \mu_{12}$, hence totally complex, a sufficiently large finite set S of places containing the archimedean places, and let $R = \mathcal{O}_{F,S}$ be the ring of S -integers, with R principal and self-dual for the chosen trace character. Put

$$q_{\mathfrak{a}} = N\mathfrak{a}, \quad U_S = R^\times, \quad \Gamma = U_S^6, \quad r_F = [F : \mathbb{Q}]/2.$$

Here $N\mathfrak{a}$ is the absolute ideal norm. For a nonzero allowed generator a , q_a denotes the norm of its exterior ideal, with the factors at S omitted. We use

$$t_+ = \max\{t, 0\}, \quad \text{rad}(\mathfrak{a}) = \prod_{p|\mathfrak{a}} p, \quad \tau(\mathfrak{a}) = \#\{\mathfrak{d} : \mathfrak{d} \mid \mathfrak{a}\}.$$

The valuation $v_p(a)$ is the exponent of p in the exterior ideal of a . Elements are taken modulo Γ with the allowed-generator and finite-ray conventions of [Liu26b, Part III]. A subscript p denotes a prime ideal outside S . All local characters retain their zero extensions at nonunits. Let Θ be

the fixed finite group of finite-order Hecke characters, unramified outside S , used in the moment estimates. It contains all initial finite coefficient characters and the group Υ_S generated by the finite characters in the allowed-generator unit decomposition. Thus

$$\psi \in \Theta \iff \psi v \in \Theta \quad (v \in \Upsilon_S), \quad \overline{\Theta} = \Theta.$$

Exceptional inducing characters are those belonging to Θ . The symbol $\mathcal{U}$ denotes the sixth-power-free nonunit parameters whose associated primitive characters lie outside Θ .

The sums retain their arithmetic parameters; they are not replaced by sums over distinct primitive characters. In the theta expansion, k or u denotes a character parameter. In the local Gauss-sum calculations, h or j denotes a dual frequency. Their norm ranges are kept separate from the conductors of the associated primitive characters.

A length exponent t relative to U means a norm scale U^t ; the corresponding exponent relative to Z is dt , since $U = Z^d$. We write $S_{m,i}$ for an ordinary centrally normalized sum of length U^m , Q_0 for the initial prime product, and E for a disjoint additional prime product. Their exact coefficients, finite twists and independent smooth weights are those in the moment families of [Liu26b]. The canonical comparison has the same prime factors and smooth weights:

$$P = S_{m,1}S_{m,2}Q_0, \quad P_c = P - P_{\text{cmp}}, \quad (m, m) \mapsto (1/4, 2m - 1/4).$$

For the actual ordinary scales, $X_1X_2 = X_{1,\text{cmp}}X_{2,\text{cmp}}$ exactly. For a nonnegative weight $\mathcal{W}$ on the summation parameters, $\mathfrak{S}_J(\mathcal{W})$ denotes the finite weighted Sobolev seminorm in the Poisson formula of [loc. cit.]. The complete Fourier measure is retained on both sides of each centered norm. The index J in $\mathfrak{S}_J$ is a finite derivative order; $H \geq 0$ bounds the external height and norm-twist parameters. The local length expression also denoted by J is defined in Section 3.3. A scalar W bounds a prime-product length, whereas $W(\cdot)$ denotes a smooth test function.

Put

$$\begin{aligned} \mathcal{Z}_F &= \bigcup_{\chi \text{ primitive, finite order over } F} \{\rho : L_F(\rho, \chi) = 0, \rho \text{ nontrivial}\}, \\ \beta_*(F) &= \sup(\{1/2\} \cup \{\Re \rho : \rho \in \mathcal{Z}_F\}). \end{aligned}$$

Principal poles are excluded. We use the following earlier results:

1. The cubic-theta coefficient formulas, reciprocal- L identity, continuation criterion for the original integral and initial 7/8 bound of [Ope26a, Ope26b, Liu26b].
2. The inverse and ordinary moment estimates M1 and M2, together with their full Gauss-sum form, compact-quotient Poisson formula, reflection, common coprimality restrictions on the coefficient variables and exceptional means, as assembled in [Liu26b, Sections 6–13]. The quadratic and cubic sieve inputs are [GL13, BGL14]; the metaplectic construction is [KP84, KP85], with the global Fourier normalizations of [Wei64].
3. The averaged-numerator comparison of [Liu26b, Section 14]. In particular, a valid bound $\beta_*(F) \leq (1 + \kappa)/2$ permits the count parameter κ ; the unamplified moment at $\kappa_m = 3/4$ follows from 7/8.
4. The direct estimate with $B(\ell) = 11/12 - \ell/4$ for $1/6 \leq \ell \leq 1/5$, the principal correction, and the final continuation argument of [Liu26b]. A strict upper bound for every remaining amplitude class gives the corresponding bound for $\beta_*(F)$.
5. The general-field construction and abelian descent of [Liu26b, Part III].

The additional analytic estimates. The following table separates the cited estimates from the extensions proved in this paper. The enlarged norm and contour range are conclusions of the indicated results, rather than additional hypotheses attributed to [Liu26b].

Estimate	Earlier result used	Extension proved here
Amplified norm	Full Gauss and M2 formulas, [<i>Loc. cit.</i> , Sections 12.7–12.8 and Appendix C.4]	Propositions 4.6, 4.8 and Theorem 4.9: direct and complementary Poisson estimates for the enlarged lengths.
AFE contour	The holomorphic region of [<i>Loc. cit.</i> , Lemma 13.10]	Lemmas 5.1–5.2: compact-strip smooth weights and $0 \leq \theta \leq \delta/2$.
Smallest amplitudes	The lowest-amplitude estimate in [<i>Loc. cit.</i> , Proposition 14.6]	Lemma 6.3: the sufficient margin $1/125$.
Class count	The count and numerator average in [<i>Loc. cit.</i> , Section 13.4 and Proposition 14.3]	Proposition 6.4 and Corollaries 6.6, 6.7: separate parameters and explicit compatibility bounds.

The exact full Gauss sums and the exceptional means in item (2) are used, with their original coefficient restrictions.

Local estimates. For each norm estimate with fixed F, S, Θ and fixed arithmetic conventions, choose the subsidiary parameters within its numerical strict margins. Fix the required smooth-weight and Fourier orders before the external height cutoffs and the eventual threshold. The constants and orders may depend on these fixed arithmetic data.

Global continuation. For each finite comparison step, use the order

- $(F, \kappa_c, \kappa_m, b, \ell, \text{strict margins})$
- $\rightarrow$ (numerical losses, steps and meshes)
- $\rightarrow (K, \ell_i, \text{buffer } e)$
- $\rightarrow (\eta, \text{final } S, \Theta, \text{arithmetic data and tests})$
- $\rightarrow$ internal derivative and Fourier orders
- $\rightarrow$ height powers, external cutoffs, tail orders and thresholds.

Here η is the target character and K is the number of original prime ranges. This is the order of [Liu26b, Remark 3.3 and Lemma 12.17]. In particular, the prime-length mesh is chosen independently of the subsequent target-dependent arithmetic data.

Indeed, the new restrictions on that mesh come from the scalar length and saving gaps, $\eta_{\text{mesh}} < \sigma/6$, and the loss from one complete prime-sum factor. For the reserved positive loss ϵ_{num} , choose

$$0 < \eta_{\text{mesh}} < \min \left\{ \eta_{\text{M2}}, \frac{\sigma}{6}, \frac{\epsilon_{\text{num}}}{C_{\text{num}}} \right\},$$

where $\eta_{\text{M2}} > 0$ is the arithmetic-independent mesh of the cited lemma and $C_{\text{num}} > 0$ bounds the numerical coefficients in the selection and length-restoration losses. These choices depend only on the requested losses and bounded scalar parameters. On a retained scale range $U = Z^d$ with fixed $d \geq d_{\min} > 0$, choose the original lengths so that

$$\max_i \ell_i \leq d_{\min} \eta_{\text{mesh}} \implies \max_i \frac{\ell_i}{d} \leq \eta_{\text{mesh}}.$$

Smaller parameter norms use the earlier comparison. After fixing the arithmetic data and original factor list, their support constants enter through $\theta_N = H_N / \log U$ and the weight seminorms. Increasing the threshold absorbs θ_N without changing the chosen mesh or prime lengths. The thresholds may vary with the target and with the finite comparison step.

For an exterior modulus u , the full normalized Gauss sum is

$$\mathcal{G}(u, h) = q_u^{-1/2} \sum_{a \bmod u} \chi_u(a) e(ha/u), \quad \mathcal{G}(1, h) = 1.$$

Here e is the fixed trace additive character and χ_u is the zero-extended sextic residue character. The imprimitive local factors and the correlation formulas are those of [Liu26b, Lemmas 12.9–12.10]; in particular they are not replaced by their primitive-unit values. Local lengths such as c_C, d_D always mean logarithms of actual ideal norms to base U . In the Gauss calculations the auxiliary length removed from the initial prime product is also denoted by x ; it is distinct from the amplitude coordinate x in Sections 6–8. In the amplified norm, this auxiliary length is zero; the initial prime product is retained.

Local Gauss notation. The local and global uses of these letters are distinguished by their sections:

	Sections 3–4	Sections 5–8
h	Gauss frequency	$(1 + b + 3\ell)/2$
a, b	local amplifier lengths	$a(\delta, x)$, balancing b
w	additional prime-product exponent	complex numerator variable
T, S	local principal lengths	unamplified slope S .

In the local formulas, C, D are the two fixed common-support ideals of the coefficient convolutions, and s is the fixed inner divisor. Put $c_C = \log_U q_C$, $d_D = \log_U q_D$ and $\mu_{\text{com}} = \min(c_C, d_D)$; m_{cut} is an ordinary-length cutoff. The scalar R in these formulas is the norm exponent of the primitive common-character modulus. The scalar E_0 is the norm exponent of the squarefree divisor from the complementary coprimality expansion:

$$0 \leq R + E_0 \leq \min(c_C, d_D).$$

These scalar uses of R are distinguished from the ring $R = \mathcal{O}_{F,S}$ and the global class-count exponent R_κ . Likewise, $D = U^m$ is an ordinary summation length in the global comparison, while D in (C, D) is an ideal; length excesses have subscripts, such as D_{free} .

For later use, put

$$l_y = \frac{1 + b - \ell}{2}, \quad h = \frac{1 + b + 3\ell}{2}, \quad l_x = \frac{1 - b - \ell}{2}, \quad U = Z^d, \quad C_0 = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4}. \quad (2.1)$$

If $R_\kappa(\delta, x)$ is the class-count exponent defined in Section 6.3, the endpoint of the unamplified character-sum comparison at $d = h$ is

$$E_\kappa(\delta, x) = C_0 + \frac{\delta}{2} + \delta x \left\{ \ell + \frac{4}{9} \left(\frac{h}{4} - \frac{l_y}{2} \right) \right\} - \frac{3h}{4} (1 - R_\kappa(\delta, x)). \quad (2.2)$$

When κ is fixed, the global comparison abbreviates $R = R_\kappa(\delta, x)$ and $E = E_\kappa(\delta, x)$; this scalar E is distinguished from the additional prime product E . Away from $d = h$, the full comparison exponent is affine in d , with slope

$$S = \frac{3R_\kappa + 1}{4} + \frac{\theta_0}{2} - \frac{17}{50}.$$

For $0 \leq m \leq 1/2$, the reference initial length and actual prime-product exponent satisfy

$$z_0(m) = 2(1 - 2m)/9, \quad 0 \leq z \leq z_0(m).$$

For $m > 1/2$, use the unamplified moment without selected prime factors, so $z = 0$. The available prime width is $\mathcal{L} = \ell/d$. A numerator saving $U^{-\gamma}$ subtracts $d\gamma$ from the full exponent.

3 Local Gauss estimates and the prime products

In this section, we prove the local estimates used in the amplified norm of Section 4. We first count nonunit prime choices at the original frequency, then treat fixed unit primes and the common-support terms.

3.1 A divisor bound for nonunit prime contributions

In this subsection, we apply Cauchy before dividing the original Gauss frequency. The resulting divisor estimate is used in the local decomposition in Section 3.4.

The labels ‘good’ and ‘bad’ in the local formulas denote the unit and nonunit cases, respectively. Every prime with a nonunit Gauss contribution divides the original nonzero Gauss frequency. Counting its possible choices before dividing that frequency gives the following improvement. The frequency $h = 0$ is excluded throughout this argument. Its separate compact-quotient origin remains estimated by the earlier powerful-product count; no divisor bound is applied to zero.

We retain the full normalized Gauss sums, allowed generators modulo $\Gamma = U_S^6$, fixed finite ray-class data, Fourier measure for the complete centered coefficient sums and uniform orders of derivatives and height bounds of [Liu26b]. The Gauss norms with fixed arithmetic parameters are evaluated by the length-restoration argument of [*loc. cit.*]: the auxiliary ordinary sums are placed in the admissible length range before M2 is applied. No Gauss mean outside the established length range is assumed. The notation (u, L, v, c, T, S) and the length excess before restoration D are those of the earlier prime-factor removal argument: u is total freely removed length, v its actual nonunit-prime length, c its fifth-power correction length, L its ordinary power removal, and T, S its principal upper/lower lengths. In particular

$$0 \leq v \leq u, \quad L \geq v + 4T, \quad c, T, S \geq 0.$$

Write z, w for the length exponents of Q_0, E , respectively. The original total coefficient length is $A_t = 2m_{\text{ord}} + z + w$, where m_{ord} is the common length exponent of the two original ordinary factors. For the auxiliary moment, A is the total coefficient-length exponent, m the parameter-range exponent, and q the displayed character-support exponent, including canceled factors. Thus $M = m + q$ is the exponent in the moment bound; this auxiliary m is distinguished from the ordinary-length exponent in $S_{m,i}$. The common-support contribution and inner-divisor exponents are

$$\begin{aligned} \tilde{q} &= q + R + E_0, & B_C &= (3c_C - 5d_D - R)_+/6, \\ B_D &= (3d_D - 5c_C - R)_+/6, \\ s_0 &= \log_U q_s, & s'_0 &= \log_U q_{s'}, \end{aligned}$$

where s' is the inner divisor after the local prime removals. Starred symbols A_*, M_*, q_*, m_* denote the resulting parameters after the local identities; their updates are given in Section 3.2. The total freely removed unit-prime length is $g_{\text{good}} = u - v - c$; the fixed valuation-zero primes have the separate length f_0 . The actual frequency annulus has exponent K , while K_0 is its nominal cutoff exponent; $\delta_1 \geq 0$ is the support buffer and $\Delta = K_0 - K \geq -\delta_1$ records the cutoff discrepancy. The auxiliary lengths J, g are specified in Section 3.3. The positive parameters σ, ξ control the auxiliary step and support margins. A real $\eta > 0$ denotes the prime-length mesh, in contrast to the target character η in I_η . The total dyadic-support error is written $\theta_N = O(1/\log U)$.

The symbol D never includes a second use of the favorable $-d_D$ term already included in the earlier argument’s J, g inequality.

3.1.1 Cauchy’s inequality with a divisor bound

In this subsection, we prove the finite Cauchy inequality and apply it to nonunit-prime divisors.

Lemma 3.1 (Cauchy's inequality for restricted finite sums). *Let $\mathcal{H}, \mathcal{B}$ be finite sets, let $\mathcal{B}(h) \subset \mathcal{B}$ satisfy $\#\mathcal{B}(h) \leq M$, and let $c_b, Z_b(h)$ be arbitrary complex numbers. Then*

$$\sum_{h \in \mathcal{H}} \left| \sum_{b \in \mathcal{B}(h)} c_b Z_b(h) \right|^2 \leq M \sum_{b \in \mathcal{B}} |c_b|^2 \sum_{\substack{h \in \mathcal{H} \\ b \in \mathcal{B}(h)}} |Z_b(h)|^2. \quad (3.1)$$

No common function Z_b or common change of variable across b is required.

Proof. For each original h , Cauchy's inequality gives

$$\left| \sum_{b \in \mathcal{B}(h)} c_b Z_b(h) \right|^2 \leq \#\mathcal{B}(h) \sum_{b \in \mathcal{B}(h)} |c_b Z_b(h)|^2 \leq M \sum_{b \in \mathcal{B}(h)} |c_b|^2 |Z_b(h)|^2.$$

Sum over h and interchange the finite sums:

$$\sum_h \sum_{b \in \mathcal{B}(h)} |c_b|^2 |Z_b(h)|^2 = \sum_b |c_b|^2 \sum_{h: b \in \mathcal{B}(h)} |Z_b(h)|^2.$$

□

Cauchy's inequality is applied at each frequency before the finite sums are interchanged. One neither counts all b after summing the coefficient squares nor replaces h by a different quotient for each b before applying Cauchy's inequality.

Lemma 3.2 (Divisor bound for the nonunit prime choices). *Fix the removed factor positions and their unit, nonunit and fifth-power-correction pattern. On the original bounded frequency range, the number of actual nonunit-prime tuples and their allowed ordinary valuation/allocation data is at most*

$$\tau(\mathfrak{h})^{C_N} (1 + \log q_h)^{C_N} \ll_{\epsilon} U^{\epsilon} (1 + H)^C.$$

Here $\mathfrak{h}$ is the exterior ideal of the original nonzero frequency h , N is the number of factors in the fixed original list, and all constants have the uniformity and order of choices specified in Section 2.

Proof. For each actual nonunit prime, $p \mid \mathfrak{h}$. The full local table gives

local character	ordinary multiplicity
nonprincipal or lower principal	$r = v_p(h) \geq 1$
upper principal	$r + 1 \leq v_p(h), \quad r \equiv 5 \pmod{6}.$

Disjoint prime ranges determine the factor positions. Allocating each r between the two ordinary factors therefore gives, for fixed N ,

$$\#\{\text{bad-prime tuples and allocations at } h\} \ll_N \tau(\mathfrak{h})^{C_N} (1 + \log q_h)^{C_N}.$$

The fixed generator and unit classes change only the constant. Whole frequency dyads are retained before logarithmic Fourier separation, as in [Liu26b, Section 12.7.1]. Their cutoff satisfies

$$q_h \ll T_{\text{sec}} U^{\epsilon/2} \ll U^{C_{\text{cut}}} (1 + H)^{A_{\text{cut}}},$$

where T_{sec} is the supremum of the nominal frequency scale in the fixed outer sector, and $C_{\text{cut}}, A_{\text{cut}}$ are fixed independently of the later Fourier coordinate. The ideal divisor bound, with a sufficiently small exponent, therefore gives

$$\tau(\mathfrak{h})^{C_N} (1 + \log q_h)^{C_N} \ll_{\epsilon} U^{\epsilon} (1 + H)^C.$$

For the weighted Fourier estimates one may also use the larger frequency majorant, where ν is the Fourier coordinate and K_*, B are fixed polynomial-growth exponents,

$$q_h \ll U^{K_*}(1 + \|\nu\|)^B(1 + H)^B \implies \tau(\mathfrak{h})^{C_N}(1 + \log q_h)^{C_N} \ll_\epsilon U^\epsilon(1 + \|\nu\|)^{C_\epsilon}(1 + H)^{C_\epsilon}.$$

Choose the small divisor exponent after fixing N , and the required weighted Fourier order before integration and external heights. This preserves the original weighted measure; no uniform unweighted bound over all ν is required. $\square$

3.1.2 Local Gauss sums and the Fourier measure

In this subsection, we apply the divisor estimate before changing the nonunit-prime frequency, and then derive the exponent in the next subsection.

Fix the original common C, D, s , finite ray components and Fourier coordinates for the complete coefficient sums before extracting primes. First process every removed $p \mid s$ with its divisor bound; the remaining base condition is $s' \mid n$. If that step divides the frequency by B_s , write $h = h_{\text{initial}}/B_s$. The free nonunit primes avoid B_s , so

$$p \text{ free bad} \implies p \mid h \iff p \mid h_{\text{initial}}.$$

Classify the remaining free positions as unit, nonunit or fifth-power correction positions. Fix correction indices first; their P^{-1} weights have absolute coefficient mass $\ll U^\epsilon$. No nonunit-prime-dependent frequency division precedes Cauchy.

For an actual nonunit prime put $r = v_p(n)$. The full local table is

branch	condition at h	division	relative squared factor
$6 \nmid r + 1$	$r \geq 1, v_p(h) = r$	p^r	1
$6 \mid r + 1$	$v_p(h) = r$	p^r	q_p^{-1}
$6 \mid r + 1$	$v_p(h) \geq r + 1$	p^{r+1}	$q_p(1 - q_p^{-1})^2 \leq q_p$.

Thus $p \mid h$ in every case, and $r \geq 5$ in the principal cases. Their moving exponents are $r + 2$ and, in the upper case, $r + 1 \equiv 0 \pmod{6}$. Retain the full natural-zero radical even when the active exponent is zero.

For the i -th prime range $\mathcal{P}_i$, let P_i be its norm scale, V_i its smooth weight and $c_i(p)$ its allowed finite-group coefficient. For a fixed nonunit-prime tuple b , let B_b be its frequency divisor and ζ_b the bounded scalar from the fixed local factors. Put

$$c_b = \prod_{i \text{ bad}} P_i^{-1/2} c_i(p_i) V_i(q_{p_i}/P_i) \zeta_b, \quad |\zeta_b| \leq 1, \quad q_{B_b} = U^{L+T+o(1)}.$$

Frequency-dependent phases belong to $Z_b(h)$ below. The actual finite-group coefficients satisfy $|c_i(p)| \leq C_i$, with no unit-modulus assumption. Prime-ideal counting gives

$$\sum_{p \in \mathcal{P}_i} |P_i^{-1/2} c_i(p) V_i(q_p/P_i)|^2 \leq P_i^{-1} C_i^2 \|V_i\|_\infty^2 \#\mathcal{P}_i \ll C_i^2 \|V_i\|_\infty^2,$$

and therefore

$$\sum_b |c_b|^2 \ll_\epsilon U^\epsilon \mathfrak{S}_J(\mathcal{W})(1 + H)^C.$$

The ordinary normalization cancels the nonprincipal magnitude: $q_p^{-r/2} q_p^{r/2} = 1$. The remaining prime normalization is $P_i^{-1/2}$; actual-to-nominal ratios are bounded on the fixed list. Keep the principal contribution $U^{T-S+o(1)}$ separately from this coefficient energy.

For a free unit prime $p \nmid h$, extending the residual sum from $(n, p) = 1$ introduces only $v_p(n) = 5$: the full local Gauss formula gives

$$|\mathcal{G}(p^i, hp^4)| = \begin{cases} q_p^2, & i = 5, \\ 0, & i \geq 1, i \neq 5. \end{cases}$$

Thus the correction, including ordinary extraction and the removed prime normalization, has weight

$$q_p^2 q_p^{-5/2} P_i^{-1/2} = q_p^{-1/2} P_i^{-1/2} \asymp P_i^{-1}.$$

Fix these correction primes first. Multiplicativity of the retained finite-component coefficients then leaves the same residual Gauss polynomial for all remaining unit-prime choices.

Let a index the unit-prime choices. For fixed b , F_b is the residual Gauss polynomial, common to all a ; $e_{a,b}$ are the remaining unit-prime coefficients and $\omega_{a,b}(h)$ the phase factors, with their prescribed zeros. The factor $\alpha_b(h)$ contains the remaining principal Gauss-sum factors. On the sum with the exact divisibility conditions, retaining all Gauss phases,

$$Z_b(h) = \alpha_b(h) \sum_a e_{a,b} \omega_{a,b}(h) F_b((h/B_b)a^4), \quad |\alpha_b(h)|^2 \leq U^{T-S+o(1)}.$$

Apply Lemmas 3.1–3.2 at this h , then interchange the positive sums over h, b , and only then set $h = B_b h'$. For fixed b ,

$$q_{h'} \ll U^{K-L-T+o(1)}, \quad (a, B_b) = 1, \quad j = h'a^4.$$

Disjoint prime ranges preserve the unit-prime unit conditions. Write $\mathcal{N}_{\text{pattern}}$ for this fixed prime-type contribution to the normalized Gauss norm. The orbit-sum argument, using the same polynomial F_b for all unit-prime choices a , gives

$$\mathcal{N}_{\text{pattern}} \ll U^{T-S+\epsilon} (1+H)^C \sum_b |c_b|^2 \sum_{q_j \ll U^{K-L-T+4g_{\text{good}}+o(1)}} |F_b(j)|^2. \quad (3.2)$$

Its good coefficient energy satisfies, uniformly in b ,

$$\sum_a |e_{a,b}|^2 \ll_\epsilon U^\epsilon \mathfrak{S}_J(\mathcal{W}) (1+H)^C.$$

Cross phases have modulus one on their unit locus; natural zeros only restrict this energy. The good smooth weights are unchanged. The enlarged balls may lose the original divisibility conditions, but divisor Cauchy has already been completed. It is not applied again on those balls. The polynomial may vary with b , with the same uniform scales, moving presentation and natural zeros. Use the derived single restoration for each fixed b ; check admissibility and inhomogeneous terms before applying M2 to the restored ordinary sums. Equation (3.2) then uses the displayed coefficient energy, with no count of all nonunit-prime tuples.

For a real Mellin frequency t , the change of variable gives

$$q_h^{it} = q_{B_b}^{it} q_{h'}^{it}, \quad |q_{B_b}^{it}| = 1.$$

Other unitary Mellin coordinates factor in the same way. Integrate polynomial derivative and height factors in the original common weighted Fourier measure at sufficiently high finite order. The original C, D, s count is included once. All coefficient coprimality restrictions remain; positivity is used only after the original complete signed coefficient sum has been formed.

3.1.3 The resulting Gauss-norm exponent

In this subsection, we compute the one-sided exponent after the divisor estimate.

The auxiliary arithmetic data A_* , M_* , q_* , the actual $K_0 - K$, the earlier J, g , and the additional length excess $D + 3u/2 - 5v - T$ are unchanged. Only the unrestricted count of nonunit prime choices U^v has disappeared. Consequently the homogeneous relative exponent is now

$$C_{\text{new}} = -u - L + v - 4c + T - S + \frac{(D + 3u/2 - 5v - T)_+}{6}. \quad (3.3)$$

Thus $C_{\text{new}} = C_{\text{old}} - v$. The remaining $+v$ counts the displayed moving radical and stays in the exponent.

Proposition 3.3 (Improved criterion for removing additional prime factors). *Retain the fixed coefficient formulas and analytic hypotheses above. For the nonunit-prime contribution in (3.3),*

$$\begin{aligned} D \leq 9u/2 &\implies C_{\text{new}} \leq -5v/6 \leq 0, \\ D \leq 7u/2 &\implies C_{\text{new}} \leq -(u + 5v)/6. \end{aligned}$$

The inhomogeneous terms in the one-sided Gauss norm retain their separate estimates.

Proof. Put $e = L - v - 4T \geq 0$. If the positive part is active, the exponent in (3.3) equals

$$-\frac{3}{4}u + \frac{D}{6} - \frac{5}{6}v - \frac{19}{6}T - 4c - S - e.$$

If it is inactive, the exponent is $-u - 3T - 4c - S - e \leq -u$.

Thus, on the active branch,

$$\begin{aligned} D \leq 9u/2 &\Big| \quad C_{\text{new}} \leq -5v/6 \\ D \leq 7u/2 &\Big| \quad C_{\text{new}} \leq -(u + 5v)/6. \end{aligned}$$

On the inactive branch, $0 \leq v \leq u$ gives $-u \leq -(u + 5v)/6 \leq -5v/6$. □

The corresponding changes are

	free statistic	retained upper bound
second origin	$-3v \mapsto -4v$	$A_t - 1 + 4u$
centered exceptional term	$-13v/6 \mapsto -19v/6$	$A_t - 1 + 3u$.

Admissibility, exact annular scales, empty or subunit ranges and prime restoration are checked as before. The improvement changes the prefactor, with the same Gauss mean estimate and κ .

3.2 A frequency change for fixed primes dividing s

In this subsection, we rewrite the Gauss frequency for fixed unit primes. The common coprimality restrictions are retained for the norm in Section 4.

We now fix a removed prime $p \mid s$ before varying the remaining frequency, and treat its valuation-zero contribution. This case requires a frequency identity rather than the divisor estimate of Section 3.1.

3.2.1 The exact fixed-prime norm comparison

In this subsection, we prove the frequency identity for a fixed prime.

For $p \nmid h$ and $(p, n) = 1$, the full Gauss CRT formula gives

$$\mathcal{G}(pn, h) = \mathcal{G}(p, h)\mathcal{G}(n, hp^4). \quad (3.4)$$

Retain the finite ray and allowed-generator components. The exact valuation-zero allocation keeps $(p, n) = 1$, so no fifth-power correction occurs. Let $A(n)$ be the canonical centered convolution and, after fixing a character component and the Fourier parameters, let $\tau_{\text{coef}}(n)$ be its common multiplicative character presentation, including its zero extension. Thus

$$\tau_{\text{coef}}(ab) = \tau_{\text{coef}}(a)\tau_{\text{coef}}(b), \quad |\tau_{\text{coef}}(n)| \leq 1.$$

This coefficient is distinct from the ideal divisor function $\tau(\mathfrak{a})$ of Section 2. Let $H_0 > 0$ denote the retained frequency-norm cutoff. For fixed p , put

$$c_p = P_i^{-1/2} c_i(p) V_i(q_p/P_i), \quad |\tau_{\text{coef}}(p) \mathcal{G}(p, h)| \leq 1 \quad (p \nmid h).$$

Consequently

$$\begin{aligned} & \sum_{\substack{q_h \leq H_0 \\ p \nmid h}} \left| c_p \sum_{(n,p)=1} A(n) \tau_{\text{coef}}(pn) \mathcal{G}(pn, h) \right|^2 \\ & \leq |c_p|^2 \sum_{q_j \leq H_0 q_p^4} \left| \sum_{(n,p)=1} A(n) \tau_{\text{coef}}(n) \mathcal{G}(n, j) \right|^2. \end{aligned} \quad (3.5)$$

The change $j = hp^4$ is injective on actual Γ -orbits:

$$q_h \leq H_0, \quad p \nmid h \quad \implies \quad q_j \leq H_0 q_p^4, \quad v_p(j) = 4.$$

Drop the image condition only in the positive norm; retain $(n, p) = 1$. All scales and cutoffs are actual, with a single reserved $O(1/\log U)$ annular loss.

Since $p \mid s$, divisor counting retains the squared normalization:

$$\sum_{\substack{p \in \mathcal{P}_i \\ p \mid s}} |c_p|^2 \ll_{\epsilon} U^{\epsilon} P_i^{-1} C_i^2 \|V_i\|_{\infty}^2.$$

For a varying unit-prime index a , the condition $(n, a) = 1$ would instead change the function in the orbit sum. Here p is fixed before the frequency varies.

3.2.2 Preservation of the coprimality conditions

In this subsection, we carry the common coprimality restriction through both Poisson formulas.

Let R_* be the fixed modulus encoding the retained coprimality restrictions on the coefficient variable. The residual polynomial and the enlarged coprimality modulus are

$$F_p(j) = \sum_n A(n) \tau_{\text{coef}}(n) 1_{(n,p)=1} \mathcal{G}(n, j), \quad R_* \mapsto R_* p.$$

Here p is fixed before j varies and $q_{R_* p}$ is polynomially bounded. The indicator comes directly from the valuation-zero allocation; no moving factor $\chi_p(n)^2$ has been introduced.

The common-coefficient identity of [Liu26b, Section 12.1 and Lemma 12.8] retains this indicator in both factors, both rectangles and every remaining prime factor. Its treatment is

stage	$1_{(n,p)=1}$
both Poisson formulas	retained in the full Gauss polynomial
ordinary nonexceptional sum	expanded by [<i>loc. cit.</i> , Lemma 12.5]
fixed-character exceptional term	retained in both matching means.

The ordinary expansion is used only after the finite Gauss coefficient has disappeared, before M2 or reflection.

The retained support implies $p \nmid n$ for every transformed coefficient variable. Hence p cannot enter the genuine new supports D_2, E_2 or their fixed active factor. An artificial divisor allocation selecting p vanishes before positivity. In the final ordinary expansion, $\psi_{h'}$ denotes the resulting ordinary character with its fixed twists. The factor $\psi_{h'}(p)$ is a bounded parameter factor with its original zero; its coefficient mass has arbitrarily small power growth.

Equation (3.4) uses the unfactored Gauss sum. The frequency is j with its full natural zeros, and its enlarged range is charged below. A canceled displayed moving factor cannot be reclassified as an extra coprimality condition: choosing a presentation containing $\chi_p(n)^2$ would require retaining it.

3.2.3 Changes in the length exponents

In this subsection, we compute the changes in the fixed-prime lengths.

Write $\lambda_p = \log_U q_p$ up to the reserved annular error. The fixed-prime frequency comparison has

$$A \mapsto A - \lambda_p, \quad K \mapsto K + 4\lambda_p, \quad M \mapsto M - 6\lambda_p, \quad q \mapsto q, \quad s_0 \mapsto s_0 - \lambda_p.$$

Consequently

$$(A - \lambda_p) + q + B_C - (s_0 - \lambda_p) = A + q + B_C - s_0, \quad K_0 - K \text{ is unchanged.}$$

The fixed squared coefficient contributes $-\lambda_p$ and the global affine defect increases by $3\lambda_p/2$. The original J, g calculation uses its favorable $-d_D$ once.

Let w_f be the length of additional primes fixed in the first common support, and $z_{C,\text{old}}$ the corresponding length from Q_0 . The unused initial length is measured by $\eta_0 = (9/2)(z_0(m) - z) \geq 0$. For primes dividing s with zero ordinary valuation and total width $f_0 = x + w$, with no newly fixed common factors and no free removed positions, the homogeneous cost is therefore bounded by

$$-f_0 + \frac{(\frac{9}{2}w + \frac{3}{2}f_0)_+}{6} = -\frac{3}{4}x \quad (x, w \geq 0). \quad (3.6)$$

In particular this presentation makes the case of primes dividing s with zero ordinary valuation neutral at $x = 0$, rather than leave the excess in the earlier local estimate $w/6$. This is a statement about this fixed-prime comparison, not a global theorem eliminating the initial prime product.

For reference, let f_1, L_1, T_1, S_1 denote the data for the remaining fixed primes dividing s ; their ordinary valuations are positive. Here f_1 is their total prime length, L_1 the ordinary power-removal length, and T_1, S_1 their upper and lower principal lengths, respectively. Then $L_1 \geq f_1 + 4T_1$. The combined fixed-stage data given by these exact comparisons are

$$A_1 = A_t - f_0 - f_1 - L_1, \quad M_1 = 1 - 6f_0 - f_1 - L_1 + T_1, \quad q_1 = f_1,$$

with normalized coefficient exponent $-f_0 - L_1 + T_1 - S_1$. After this fixed stage, perform the free unit, nonunit and fifth-power-correction partition afresh at its current frequency on the enlarged Gauss family. The divisor Cauchy is applied there before any free nonunit-prime division; it does not assume that an old incidence graph survived positive enlargement. Adding the divisor-Cauchy free stage gives the following upper bound for the normalization exponent when the positive part is nonzero:

$$-\frac{3}{4}x + \frac{w_f}{6} - \frac{5}{6}(f_1 + v) - \frac{19}{6}(T_1 + T) - \frac{7}{12}z_{C,\text{old}} - \frac{\eta_0}{6} - 4c - S_1 - S,$$

after dropping further nonpositive ordinary-removal terms. The case in which the positive part vanishes has a nonpositive exponent. More precisely, put $e_1 = L_1 - f_1 - 4T_1 \geq 0$ and $e = L - v - 4T \geq 0$. Its normalized cost is

$$-f_0 - f_1 - u - 3(T_1 + T) - 4c - S_1 - S - e_1 - e \leq 0.$$

Corollary 3.4 (A nonpositive contribution to the norm exponent). *Retain the admissibility, centered-weight, common-measure and length-restoration hypotheses of Section 3.1, together with the fixed-unit comparison above. Then*

$$0 \leq w_f \leq w, \quad x \geq 2w_f/9 \implies \mathcal{C}_{\text{act}}, \mathcal{C}_{\text{inact}} \leq 0.$$

In particular, $x \geq 2w/9$ suffices for every $w_f \leq w$. For a prescribed discretization margin $\rho_{\text{step}} > 0$,

$$\begin{aligned} x_{\text{req}} \geq 2w/9 + \rho_{\text{step}}, \quad \eta_{\text{mesh}} \leq \rho_{\text{step}}/3, \quad x \geq x_{\text{req}} - \eta_{\text{mesh}} \\ \implies \mathcal{C}_{\text{act}} \leq -\rho_{\text{step}}/2, \end{aligned}$$

before the smaller subsidiary errors. Here $\mathcal{C}_{\text{act}}$ and $\mathcal{C}_{\text{inact}}$ denote the contributions with nonzero and zero positive part, respectively.

Proof. The active and inactive contributions satisfy, respectively,

$$\begin{aligned} \mathcal{C}_{\text{act}} &\leq -3x/4 + w_f/6 \leq 0 & (x \geq 2w_f/9), \\ \mathcal{C}_{\text{inact}} &= -f_0 - f_1 - u - 3(T_1 + T) - 4c - S_1 - S - e_1 - e \leq 0. \end{aligned}$$

Since $w_f \leq w$, the stronger condition $x \geq 2w/9$ suffices. For a complete-factor choice $x \geq x_{\text{req}} - \eta_{\text{mesh}}$,

$$\mathcal{C}_{\text{act}} \leq -\frac{3}{4} \left(\frac{2w}{9} + \rho_{\text{step}} - \eta_{\text{mesh}} \right) + \frac{w}{6} \leq -\frac{3}{4}\rho_{\text{step}} + \frac{1}{4}\rho_{\text{step}} = -\frac{1}{2}\rho_{\text{step}}.$$

At $u = 0$, retain the full fixed coefficient exponent $-f_0 - L_1 + T_1 - S_1$ and the single restoration cost. No condition restricted to the free primes, such as $D \leq 9u/2$, is imposed on this fixed contribution. $\square$

This criterion bounds the normalization exponent. Its other admissibility hypotheses remain necessary when x, w vary. The case in which every additional prime lies in the common support is not sufficient by itself: when $w_f = w$ and $x = \eta_0 = z_{C,\text{old}} = f_1 = v = 0$, necessarily $f_0 = u = 0$, and this upper bound for the exponent is $w/6$. This is not an arithmetic lower bound, but it prevents a uniform $x = 0$ conclusion from these bounds alone.

3.2.4 The remaining prime length

In this subsection, we compare the remaining prime length with the transformed length excess.

Let $D_{\text{free}}^{\text{old}}$ denote the remaining length excess in the preceding fixed- s table, evaluated at the same actual allocation. The fixed-unit change gives $D_{\text{free}}^{\text{new}} = D_{\text{free}}^{\text{old}} + 5f_0$. After substituting $u = x + w - w_f - f_0 - f_1$, its exact value is

$$D_{\text{free}}^{\text{new}} = \frac{3}{2}x + 6w - \eta_0 - \frac{7}{2}z_{C,\text{old}} - 5w_f - 5f_1 - T_1 - 5v - T.$$

Here $z_0 = z_0(m)$, and $z_{0,\text{min}}$ below denotes its minimum over the chosen ordinary-length interval. Write $z = z_0 - 2\eta_0/9$ and $z_{\text{init}} = z - z_{C,\text{old}} - x$ for the remaining original prime width. Then

$$\begin{aligned} \frac{7}{2}z_{\text{init}} - D_{\text{free}}^{\text{new}} &= \frac{7}{2}z_0 - 5x - 6w + \frac{2}{9}\eta_0 + 5w_f + 5f_1 + T_1 + 5v + T \\ &\geq \frac{7}{2}z_{0,\text{min}} - 5x - 6w. \end{aligned} \tag{3.7}$$

The f_0 term cancels. For example,

$$z_{0,\min} = 2/75, \quad x = 9/5000, \quad w = 3/500 \quad \implies \quad \frac{7}{2}z_{0,\min} - 5x - 6w = \frac{29}{600}.$$

Decreasing x increases the margin. Genuine and final-divisor fixing of widths b_g, b_d reduce the defect by at least $7b_g/2 + 9b_d/2$. Thus restoration, including exhaustion, remains valid when (3.7) exceeds the mesh and amplifier errors, with the used prime widths subtracted at each stage.

3.2.5 Ordinary summation lengths and exceptional terms

In this subsection, we compute the actual summation lengths and the origin and exceptional exponents.

Put $t = f_0 + u$ and $a = f_1 + t \leq x + w$. After all fixed and free local comparisons the auxiliary global parameters are

$$\begin{aligned} A_* &= A_t - f_1 - t - L_1 - L - 5c, \\ M_* &= 1 - 6t - f_1 - L_1 - L - 5c + T_1 + 5v + T, \\ q_* &= f_1 + v + c, \quad m_* = M_* - q_*. \end{aligned}$$

All nonunit and correction natural support restrictions remain in the displayed q_* ; only the unfactored fixed unit primes are in the common extra radical. If $D_{\text{ord}} = L_1 + L + 5c \leq 2m + o(1)$, these give

$$M_* \geq 1 - 2m - 6a - o(1), \quad m_* \geq 1 - 2m - 7a - o(1), \quad A_* - m_* \leq A_t - 1 + 6a + o(1).$$

The last estimate follows also from the exact identity $A_* - m_* = A_t - 1 + 5t + f_1 - T_1 - 4v + c - T$. For nonempty common supports,

$$c_C \leq A_* + o(1), \quad R + E_0 \leq d_D \quad \implies \quad m_* - c_C + d_D - R - E_0 \geq m_* - A_* - o(1) > 0$$

whenever $m_* - A_*$ has a fixed positive margin. The ordinary mean estimate is applied only after this admissibility check. The actual frequency shift is

$$K_0^* = K_0 + 4(f_0 + g_{\text{good}}) - L_1 - T_1 - L - T, \quad g_{\text{good}} = u - v - c.$$

It agrees with the formal length identity, so $K_0 - K$ and the original earlier J, g restoration calculation remain unchanged.

After including all normalized fixed coefficients, the second-origin excess is exactly

$$A_t - 1 + 4t - L_1 - S_1 - L - 4v - 4c - S \leq A_t - 1 + 4t.$$

The corresponding centered exceptional excess is

$$\begin{aligned} A_t - \frac{5}{6} - \frac{f_1}{6} - \frac{7}{6}(L_1 + L) + \frac{T_1 + T}{6} - S_1 - S + 3t - \frac{19}{6}v - \frac{29}{6}c \\ - \frac{2}{3}\left(\frac{1}{4} - L_1 - L - 5c\right)_+ \leq A_t - 1 + 3t. \end{aligned}$$

Indeed, write $B = L_1 + L$ and let $\mathcal{X}$ denote the preceding exceptional excess. Since $T_1 + T \leq B/5$,

$$\begin{aligned} \mathcal{X} - (A_t - 1 + 3t) &\leq \frac{1}{6} - \frac{17B}{15} - \frac{29c}{6} - \frac{2}{3}(1/4 - B - 5c)_+ \\ &\leq \begin{cases} -7B/15 - 3c/2, & B + 5c \leq 1/4, \\ 1/6 - 2(B + 5c)/3, & B + 5c \geq 1/4 \end{cases} \leq 0. \end{aligned}$$

Thus the same parameter regions, or smaller x within them, retain the length, origin and exceptional margins. Exact annular scales, independent Mellin twists and canonical centered weights remain those of the cited theorem; the stated zero-free boundaries are unchanged.

3.3 Retaining the common-support length

In this subsection, we retain the common ideal length in the restoration inequality. This gives the homogeneous estimate used in Section 3.4.

The first paired common-support term need not save a power at a prime occurring to the first power in both ordinary factors. The exact restoration expression, however, retains a favorable common length that its earlier coarse upper bound discarded. We use its full common-length term in the following estimate.

3.3.1 The common-support normalization

In this subsection, we compare the paired common-support term with the restoration term.

In the notation of the paired common-support comparison, put $c_C = \log_U q_C$, $d_D = \log_U q_D$ and $p = \log_U q_{\text{rad}(CD)}$. The paired normalization is

$$M + \frac{1}{2}\{B_C + B_D - (c_C + d_D - 2p - R)\}, \quad B_C = (3c_C - 5d_D - R)_+/6.$$

A new C -fixed prime may divide the other ordinary factor with multiplicities $(1, 1)$. At that prime,

$$(3 - 5)_+ = 0, \quad 1 + 1 - 2 = 0, \quad q_p^{-1}q_p = 1,$$

where the last factors are the paired central normalization and the single radical-family count. Prime sparsity gives logarithms; this term alone gives no saving proportional to the new common width.

Write a for the earlier amplifier extraction length w and b for its new-radical length w_o , to distinguish them from the new length of the original prime-ideal range. The choices in the cited formula are

$$\Delta = K_0 - K \geq -\delta_1, \quad J = d_D - c_C + \Delta - 2a + b, \quad g = J_+ + O(\sigma), \quad 0 \leq b \leq a.$$

They are the unchanged quantities of [Liu26b, Section 12.7.2] and [loc. cit., Section 12.8.1].

Lemma 3.5 (Common-support saving in length restoration). *Suppose*

$$\begin{aligned} c_C, d_D \geq 0, \quad \delta_1 \geq 0, \quad \Delta \geq -\delta_1, \quad 0 \leq b \leq a, \\ J = d_D - c_C + \Delta - 2a + b, \quad \mu_{\text{com}} = \min(c_C, d_D). \end{aligned}$$

Then

$$\begin{aligned} -d_D - \Delta - b + J_+ &= \max\{-d_D - \Delta - b, -c_C - 2a\} \\ &\leq -\mu_{\text{com}} + \delta_1. \end{aligned} \tag{3.8}$$

Proof. Put $X = -d_D - \Delta - b$. Then

$$\begin{aligned} X + J_+ &= \max\{X, X + J\} = \max\{-d_D - \Delta - b, -c_C - 2a\} \\ &\leq \max\{-d_D + \delta_1, -c_C\} \leq -\min(c_C, d_D) + \delta_1, \end{aligned}$$

using $\Delta \geq -\delta_1$, $a, b \geq 0$. □

This replaces the earlier argument's coarse bound of this entire expression by δ_1 . It is not a second use of the favorable $-d_D$. For example, when $c_C = d_D = \lambda_{\text{com}} > 0$ and $\Delta = a = b = 0$, the first common-support term can be zero while (3.8) is exactly $-\lambda_{\text{com}}$. The two coarse equalities cannot be attained simultaneously on such a common support.

3.3.2 The transformed length excess

In this subsection, we insert the common-length inequality into the transformed length excess.

Before its coarse J substitution, [Liu26b, Section 12.8.1] bounds the transformed length defect by

$$a - d_D - (K_0 - K) - b + g + 2\theta_N.$$

The earlier formal reconstruction of ordinary summation lengths adds its global length excess and the common-fixed-prime term $-(7/2)z_C$. All its local comparisons shift actual K and formal K_0 identically; $K_0 - K$, C , D , the earlier J , g , and the single paired arithmetic count are unchanged. Lemma 3.5 therefore lowers the additional free defect by μ_{com} , using the same substitution exactly once. More explicitly the full transformed length defect is at most

$$D_{\text{free}} - \mu_{\text{com}} + a + (g - J_+) + \delta_1 + 2\theta_N.$$

The earlier-amplifier contribution satisfies

$$a + (g - J_+) \leq 6(a + \ell), \quad \begin{cases} a = 0, & g - J_+ = 2\sigma = 6\ell, & \text{main term,} \\ \ell = 0, & g - J_+ = \sigma, & a + \sigma \leq 6a, & \text{amplifier error.} \end{cases}$$

Using $(X + Y)_+ \leq X_+ + Y$ for $Y \geq 0$, the squared cost is at most

$$\frac{(D_{\text{free}} - \mu_{\text{com}})_+}{6} + a + \ell + \frac{\delta_1 + 2\theta_N}{6} + \eta.$$

The earlier exponent allowance for the auxiliary sum $\Delta' \geq a + \ell$ absorbs its contribution from the original amplifier. The additional squared cost of restoring the lengths is therefore at most

$$\frac{(D_{\text{free}} - \mu_{\text{com}})_+}{6} \tag{3.9}$$

apart from the already reserved dyadic, support and mesh errors.

The original amplifier estimate still accounts for its own contribution. The exponent allowance for the auxiliary sum remains $\Delta' = b_2 - p_2 + a + B_C + \ell$; none of these terms is reused. Here $b_2 = (c_2 + d_2)/2$, where c_2, d_2 are the two full common-support lengths in the second Poisson formula, and p_2 is the length of their radical. Write g_2 for the length of their greatest common divisor. After dividing the frequency by this divisor, the primes with equal multiplicities not divisible by six split into unit and nonunit sets: t_2 is the radical length of the unit set, and V the length of the product of the nonunit primes. All these lengths are relative to U ; see Section 4.3. Lowering the defect improves the previously verified coupled remaining-prime margin. Every later genuine or final-divisor fixing is charged in that same length-restoration argument for all prime factors. When no prime factor remains, the cited estimate for arbitrary ordinary lengths applies with the same order of parameter choices. In particular (3.9) is derived using ordinary sums admissible for M2, rather than assumed for a Gauss sum outside the established length range.

3.3.3 The nonpositive normalization exponent

In this subsection, we prove that the full homogeneous exponent is nonpositive.

Retain the fixed-unit rewriting and original-frequency divisor Cauchy of the preceding sections. Let f_0 denote fixed unit primes dividing s , f_1, L_1, T_1, S_1 the other data for fixed primes dividing s , $z_{C,\text{old}}$ the initial first-common-support prime length, and w_f the corresponding new width. All widths and nonnegative local losses have their earlier meanings. Thus

$$u = x + w - w_f - f_0 - f_1 \geq 0, \quad L_1 \geq f_1 + 4T_1, \quad L \geq v + 4T,$$

and

$$D_f = \frac{9}{2}w - \eta_0 - \frac{7}{2}(z_{C,\text{old}} + w_f) + \frac{3}{2}f_0 - \frac{7}{2}f_1 - T_1.$$

Write c_{cor} for the free correction width, distinct from c_C , and put $e_1 = L_1 - f_1 - 4T_1 \geq 0$, $e = L - v - 4T \geq 0$.

Corollary 3.6 (A nonpositive exponent with the initial prime product retained). *Let the local widths and losses defined above be nonnegative, with*

$$\begin{aligned} u = x + w - w_f - f_0 - f_1 &\geq 0, & L_1 &\geq f_1 + 4T_1, & L &\geq v + 4T, \\ D_f &= \frac{9}{2}w - \eta_0 - \frac{7}{2}(z_{C,\text{old}} + w_f) + \frac{3}{2}f_0 - \frac{7}{2}f_1 - T_1, \\ \mu_{\text{com}} &= \min(c_C, d_D) \geq w_f. \end{aligned}$$

Then, on actual ideal scales,

$$\begin{aligned} \mathcal{C} = & -f_0 - L_1 + T_1 - S_1 - u - L + v - 4c_{\text{cor}} + T - S \\ & + \frac{(D_f + 3u/2 - 5v - T - \mu_{\text{com}})_+}{6} \leq 0. \end{aligned} \quad (3.10)$$

This algebraic inequality requires neither $x \geq 2w_f/9$ nor positivity of the formal summation length. Its analytic length-restoration application retains the cited moment hypotheses, positive global and formal lengths, full local support and coprimality conditions, exact centered weights and equal scale products, common Fourier measure, and remaining-prime-length conditions, including the case with no remaining prime factor.

Proof. Put $\mathcal{D} = D_f + 3u/2 - 5v - T - \mu_{\text{com}}$ and

$$\mathcal{C}_0 = -f_0 - f_1 - u - 3(T_1 + T) - 4c_{\text{cor}} - S_1 - S - e_1 - e.$$

Substitute $w = u - x + w_f + f_0 + f_1$ into D_f . Then

$$\mathcal{D} = 6u - \frac{9}{2}x + w_f + 6f_0 + f_1 - \frac{7}{2}z_{C,\text{old}} - \eta_0 - T_1 - 5v - T - \mu_{\text{com}},$$

and $L_1 = f_1 + 4T_1 + e_1$, $L = v + 4T + e$ give

$$\mathcal{C} = \mathcal{C}_0 + \mathcal{D}_+/6 = \max\{\mathcal{C}_0, \mathcal{C}_0 + \mathcal{D}/6\}.$$

The two entries satisfy

$$\begin{aligned} \mathcal{C}_0 &= -f_0 - f_1 - u - 3(T_1 + T) - 4c_{\text{cor}} - S_1 - S - e_1 - e \leq 0, \\ \mathcal{C}_0 + \mathcal{D}/6 &= -\frac{3}{4}x + \frac{w_f - \mu_{\text{com}}}{6} - \frac{5}{6}(f_1 + v) - \frac{19}{6}(T_1 + T) \\ &\quad - \frac{7}{12}z_{C,\text{old}} - \frac{\eta_0}{6} - 4c_{\text{cor}} - S_1 - S - e_1 - e \leq 0. \end{aligned}$$

Here $x \geq 0$, $\mu_{\text{com}} \geq w_f$, and all remaining widths and losses are nonnegative. The calculation also holds at $u = 0$: it uses the full coefficient exponent and the single restoration cost, with no additional free-prime condition. $\square$

Distinct new first-common primes divide $\text{rad}(CD)$, and the original prime ranges are disjoint. Hence

$$w_{f,\text{actual}} \leq p \leq \min(c_C, d_D) = \mu_{\text{com}}, \quad w_f \leq \mu_{\text{com}} + \theta_N.$$

The nominal homogeneous cost is at most $\theta_N/6$; together with the original $\delta_1/6$ and edge losses it is charged once, not once per prime factor. For the requested positive loss, let $C_* > 0$ be the fixed constant controlling the accumulated support errors, and choose

$$0 < \sigma \text{ inside the fixed margins,} \quad C_*\xi < \sigma/2, \quad \eta < \sigma/6.$$

The unchanged J, g give auxiliary-length decrease at least $\sigma/2$. Every M2 application remains in the established admissible family. Retain the fixed field, coefficient and unit classes, natural zeros, and weighted derivative/Fourier conventions of Section 2. Fix the finite positive-length prime list and tests, then the weighted orders, before external heights. Exact annular extractions and equal scale products are preserved; absorb the single vanishing aggregate support error after the fixed threshold. Zero-length bounded-mass factors contribute no removed positive width.

We now apply this homogeneous estimate together with the decomposition into all local prime cases.

From the local estimates to the amplified norm. The preceding result is a local restoration statement. The following subsection combines these estimates to prove the amplified norm on the stated parameter region. Throughout, $x = 0$ means no auxiliary removal of a subproduct of the initial prime product; the original Q_0 remains in the polynomial.

3.4 An amplified norm retaining the initial prime product

In this subsection, we prove the amplified norm with the initial prime product retained. We separate the additional prime factors according to their divisibility and valuation conditions, verify the resulting length bounds, and then apply the ordinary second-moment estimate M2 recalled in Section 2. Section 4.1 extends the argument to variable lengths.

Here $x = 0$ means that the auxiliary Gauss comparison absorbs no additional part of the initial prime product. The original polynomial still contains Q_0 . The common-support decomposition and subsequent restoration of ordinary summation lengths are retained. In particular, the condition does not delete Q_0 from the asserted original norm.

Every factor of E is treated by the appropriate local Gauss-sum identity before M2 is applied, even when an intermediate length excess is nonpositive.

The moment and its arithmetic data. We use the following coefficient families and length ranges.

Retain the fixed field, excluded places, finite coefficient families and unit-class data, Fourier conventions for the complete coefficient sums and the earlier $\beta_*(F) \leq 7/8$ premise allowing $\kappa = 3/4$. Assume precisely the same full Gauss-sum, compact-quotient Poisson, M2, reflection, common-coefficient, centered-mean and weighted smooth-weight and Fourier results as in Section 2. These are the established results cited in Section 2. First consider the parameter range

$$.4 \leq m \leq .443, \quad .4 \leq \delta \leq .55, \quad .48 \leq x_{\text{amp}} \leq .5, \quad .21 \leq \mathcal{L} \leq .22.$$

Let $P = S_{m,1}S_{m,2}Q_0$ be the original unrestricted product, $z \leq z_0 = (1 - 2m)/(9/2)$ the nominal length exponent of the initial prime product, and E a disjoint original product of length $0 \leq w \leq 1/100$. All positive-length prime ranges are mutually disjoint. For each independent smooth-weight tuple, use the same two tests and set

$$(m, m) \mapsto (1/4, 2m - 1/4), \quad P_c = P - P_{\text{cmp}}, \quad X_1X_2 = X_{1,\text{cmp}}X_{2,\text{cmp}}.$$

Choose the comparison before differentiation. Arithmetic identities retain exact norms, powers, weight arguments and scale products. Only affine length displays are nominal, with a single total $\theta_N = O(1/\log U)$ error over all finite allocations; centering always uses the exact product equality.

Theorem 3.7 (Amplified norm with the initial prime product retained). *Under the hypotheses above, for every required finite independent derivative family and every $\epsilon > 0$,*

$$\sum_k W(q_k/U) |P_c(k)E(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C, \quad (3.11)$$

$$\sum_{k \in \mathcal{U}} W(q_k/U) |P(k)E(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C. \quad (3.12)$$

The centered norm is taken over the unrestricted positive sum in the comparison; $\mathcal{U}$ is the original sixth-power-free nonunit family inducing outside Θ . No auxiliary subproduct is removed from the initial prime product: $x = 0$ throughout. The amplitude selection is treated separately in Section 4.1.

Proof. Step 1. Divisibility and valuation cases for the additional primes. We first separate the factors of E according to common support, divisibility by s , and their valuations in the ordinary coefficient variables. We apply M2 only after all these factors have been treated and the resulting lengths have been checked in Steps 2 and 3.

First write $E = E_0 E_+$, $Q_0 = Q_{00} Q_{0,+}$, where the subscript 0 denotes zero nominal length. Here $E_0(k)$ is a prime-product factor; the scalar E_0 in the local length formulas is the coprimality-divisor exponent defined in Section 2. For each fixed derivative tuple,

$$P = Q_{00} P_+, \quad P_c = Q_{00} P_{c,+},$$

and the character factors have modulus at most one. Hence

$$|E_0(k)Q_{00}(k)| \leq \|c_{E,0}\|_{\ell^1} \|c_{Q,0}\|_{\ell^1}. \quad (3.13)$$

Apply this bound to both positive norms before Poisson. Its bounded coefficient mass belongs to the retained seminorm and height factor. Relabel $E_+, Q_{0,+}$ as E, Q_0 ; their lengths are still w, z . Only positive-length factors enter the following valuations. The selected extra factors also have positive amplitude.

Fix the original C, D, s , finite components and common Fourier coordinates in the order of Section 2. Put

$$c_C = \log_U q_C, \quad d_D = \log_U q_D, \quad \mu_{\text{com}} = \min(c_C, d_D), \quad z_C = z_{C,\text{old}} + w_f.$$

Each new first-common prime divides both C, D . Thus

$$w_{f,\text{actual}} \leq \mu_{\text{com}}, \quad w_f \leq \mu_{\text{com}} + \theta_N, \quad \theta_N = O(1/\log U).$$

Split every remaining extra prime according to $p \mid s$ or $p \nmid s$, before Gauss evaluation or positivity. For $p \mid s$, fix p as a divisor of s , retain its coefficient and use

$$1_{s|pn} = 1_{(s/p)|n}.$$

The complete valuation allocation has two cases:

ordinary valuation	total width	identity or bound
0	f_0	$\mathcal{G}(pn, h) = \mathcal{G}(p, h) \mathcal{G}(n, hp^4)$
> 0	f_1	$L_1 \geq f_1 + 4T_1$.

The first identity requires $(p, n) = 1$ and, on nonzero terms, $p \nmid h$. For the whole fixed valuation-zero product a_0 , impose these conditions simultaneously, then change $h \mapsto ha_0^4$ injectively, before positive enlargement. Positive-valuation divisions use disjoint primes and preserve the unit conditions. The scalar has modulus at most one; its squared coefficient retains $q_{a_0}^{-1}$. Keep $(n, a_0) = 1$ in both centered rectangles as a fixed extra radical, until an admissible ordinary M2 application

or the matching means. The full nonunit table retains every principal and zero branch. The fixed-divisor contribution and resulting parameters are

$$\begin{aligned} -f_0 - L_1 + T_1 - S_1, \quad A_t &= 2m + z + w, \\ A_1 = A_t - f_0 - f_1 - L_1, \quad M_1 &= 1 - 6f_0 - f_1 - L_1 + T_1, \quad q_1 = f_1. \end{aligned}$$

The remaining free index avoids the reduced s . Disjoint original prime ranges make its base condition $s' \mid n$ and fixed common supports independent of that index. Apply the full free identity, including the fifth-power correction, to every remaining position:

$$\begin{aligned} u = w - w_f - f_0 - f_1 &= g_{\text{good}} + v + c_{\text{cor}} \geq 0, \\ L \geq v + 4T, \quad \text{free coefficient exponent} &= -u - L + v - 4c_{\text{cor}} + T - S. \end{aligned}$$

Correction indices retain their q_p^{-1} absolute coefficient weights. At the current nonzero frequency, every actual nonunit prime divides that frequency. Apply divisor Cauchy there, before nonunit-prime-dependent frequency division, retaining squared coefficient energy. This uses the incidence assertion after the fixed-unit enlargement. For fixed bad data, the unit-prime indices use one common polynomial F_b and divisor-bounded fibers $a^4 \mid j$. Fixed-divisor, bad and correction labels contribute only arbitrarily small power losses, not an unrestricted prime-family count.

The classification is exhaustive:

$$w = w_f + f_0 + f_1 + g_{\text{good}} + v + c_{\text{cor}}.$$

It is completed for either sign of every intermediate length excess, including $u = 0$, before M2. The original Q_0 retains all annuli, coprimality conditions and its whole-product divisor condition.

Step 2. The coefficient contribution and length restoration. We combine the local coefficient bounds with Corollary 3.6.

Put $\eta_0 = (9/2)(z_0 - z) \geq 0$. The allocation gives

$$\begin{aligned} D_f &= \frac{9}{2}w - \eta_0 - \frac{7}{2}(z_{C,\text{old}} + w_f) + \frac{3}{2}f_0 - \frac{7}{2}f_1 - T_1, \\ D_{\text{free}} &= D_f + \frac{3}{2}u - 5v - T, \\ R_{\text{loc}} &= -f_0 - L_1 + T_1 - S_1 - u - L + v - 4c_{\text{cor}} + T - S. \end{aligned}$$

Both actual and formal frequency changes equal

$$4(f_0 + g_{\text{good}}) - L_1 - T_1 - L - T, \quad K_0^* - K^* = K_0 - K.$$

Retain all displayed radicals, including canceled natural-zero factors. The original C, D, J, g , structural count and weighted Fourier measure are unchanged. Lemma 3.5 uses its $-d_D$ only in the original J bracket and gives

$$\text{additional squared restoration cost} \leq \frac{(D_{\text{free}} - \mu_{\text{com}})_+}{6}.$$

The original amplifier contribution is included once. Applying Corollary 3.6 at $x = 0$ gives

$$R_{\text{loc}} + \frac{(D_{\text{free}} - \mu_{\text{com}})_+}{6} \leq \begin{cases} 0, & \text{actual scales,} \\ \theta_N/6, & \text{nominal annuli.} \end{cases}$$

Neither a positive intermediate defect nor $u > 0$ is required.

Step 3. Verification of the length hypotheses for M2. We check the available prime length and the resulting ordinary summation lengths before applying M2.

The partition of the prime factors and the exact coupled identity give

$$z_{\text{init}} = z + w - z_C - f_0 - f_1 - u = z - z_{C,\text{old}} \geq 0,$$

$$\begin{aligned} \frac{7}{2}z_{\text{init}} - D_{\text{free}} &= \frac{7}{2}z_0 - 6w + \frac{2}{9}\eta_0 + 5w_f + 5f_1 + T_1 + 5v + T \\ &\geq \frac{7}{2} \frac{19}{750} - 6 \frac{1}{100} = \frac{43}{1500} > 0. \end{aligned}$$

Choose $0 < \sigma < 10^{-6}$ and smaller edges so that

$$D_{\text{temp}} \leq D_{\text{free}} - \mu_{\text{com}} + \frac{3}{125000}, \quad \frac{7}{2}z_{\text{init}} - D_{\text{temp}} > \frac{43}{1500} - \frac{3}{125000} > 0.$$

Genuine and final-divisor fixing of widths b_g, b_d reduce the defect by at least $7b_g/2$ and $9b_d/2$, respectively; retain $-c_2 + g_2 - t_2 \leq 0$. With $b = b_g + b_d \leq z_{\text{init}}$,

$$\frac{9}{2}(z_{\text{init}} - b) - (D_{\text{temp}} - 7b/2) \geq \frac{7}{2}z_{\text{init}} - D_{\text{temp}} > 0.$$

Thus a positive defect has sufficient remaining length of the initial prime product. For $z_{\text{init}} > 0$, the auxiliary prime ranges for the amplifier avoid the original ranges; the amplifier error powers are assigned only to ordinary factors. At exhaustion, including $z = 0$, use

$$z_{\text{init}} = 0 \implies D_{\text{free}} - \mu_{\text{com}} \leq -43/1500, \quad g = J_+ + \sigma, \quad \ell = 0.$$

No amplifier prime factor is extracted and no average over auxiliary primes is taken. The strict margin absorbs σ and the subsidiary edges. This also covers the cases in which all additional primes lie in the common support and $w = 0$.

For nonempty coefficients put $t = f_0 + u$, $a = f_1 + t \leq w \leq 1/100$. The completed parameters are

$$\begin{aligned} A_* &= A_t - f_1 - t - L_1 - L - 5c_{\text{cor}}, \\ M_* &= 1 - 6t - f_1 - L_1 - L - 5c_{\text{cor}} + T_1 + 5v + T, \\ q_* &= f_1 + v + c_{\text{cor}}, \quad m_* = M_* - q_*. \end{aligned}$$

All natural-zero supports remain in q_* . Fixed unit primes stay in the polynomially bounded extra coprimality radical. Ordinary removal is at most $2m + o(1)$, and $A_t \leq 691/750$. Hence

$$\begin{aligned} M_* &\geq 1 - 2m - 6a - o(1) \geq 27/500 - o(1), \\ m_* &\geq 1 - 2m - 7a - o(1) \geq 11/250 - o(1), \\ A_* - m_* &= A_t - 1 + 5t + f_1 - T_1 - 4v + c_{\text{cor}} - T \\ &\leq A_t - 1 + 6a \leq -7/375 + o(1). \end{aligned} \tag{3.14}$$

The same bounds hold at intermediate fixed stages, with M2 only after classification. Nonempty coefficient and frequency supports give

$$\begin{aligned} c_C &\leq A_* + o(1), \quad R + E_0 \leq \min(c_C, d_D), \\ -o(1) &\leq K \leq K_0^* + o(1), \quad K_0^* = 2A_* - c_C - d_D + R + E_0 - m_*. \end{aligned}$$

Consequently both the global and formal lengths are admissible:

$$\begin{aligned} d_D &\leq 2A_* - m_* + o(1) \leq A_* - 7/375 + o(1), \\ m_* - c_C + d_D - R - E_0 &\geq m_* - c_C \geq 7/375 - o(1) > 0. \end{aligned}$$

For the comparison product retain $L_{\text{ref}} = (1/4 - L_1 - L - 5c_{\text{cor}})_+$ and its volume branch. The origin and centered-exceptional ceilings satisfy

$$A_t + 4t \leq 721/750 < 1, \quad A_t + 3t \leq 1427/1500 < 1.$$

These follow from the same full local formulas, with nonpositive terms discarded only in the upper-bound direction. Keep the original powerful-product count for the first zero-frequency term. Discard tails on the genuine coprime expression in the order of Section 2. Exact scale products, common support restrictions and both matching fixed-character means are retained through principal cancellation.

Step 4. Integration and the uncentered norm. We integrate the two estimates against the original Fourier measure and recover the norm on $\mathcal{U}$.

For each original C, D, s and Fourier tuple, both the estimate with the stated ordering and the estimate with the ordinary factors interchanged fit

$$A_t - c_C + \tilde{q} + B_C - s_0.$$

Integrate them in the same original weighted measure, including the paired arithmetic count once and retaining the nonpositive first paired term. The resulting integrated estimate gives (3.11).

On $\mathcal{U}$, reflection of the longer comparison factor gives

$$\text{total reflected length} \leq \frac{5}{2} - 4(.4) + \frac{9}{2}(.01) = \frac{189}{200} < 1.$$

Apply the original M2 theorem to $P_{\text{cmp}}E$. In the positive norms with weight $W(q_k/U)$,

$$\begin{aligned} \|PE\|_{2,\mathcal{U}}^2 &\leq 2\|P_cE\|_{2,\mathcal{U}}^2 + 2\|P_{\text{cmp}}E\|_{2,\mathcal{U}}^2 \\ &\leq 2\|P_cE\|_{2,\text{lift}}^2 + 2\|P_{\text{cmp}}E\|_{2,\mathcal{U}}^2 \\ &\ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C. \end{aligned}$$

This proves (3.12); the uncentered family still excludes parameters inducing principal characters.

For a requested $\epsilon > 0$, choose the positive step inside the strict margins, then subsidiary edges and mesh satisfying

$$C_*\xi < \sigma/2, \quad \eta < \sigma/6, \quad \eta \leq 10^{-7} \quad \text{for the selected estimate.}$$

Next fix the original finite prime-range and test-function list, request sufficiently small divisor, valuation and allocation losses, and fix internal weighted derivative/Fourier orders and thresholds before the final external height exponent. The unchanged J, g give auxiliary-length decrease at least $\sigma/2$; every ordinary sum is already in the established admissible M2 family. Nonpositive homogeneous cost needs only the requested positive loss. Absorb the single total θ_N after the fixed large- U threshold and take the finite maximum over allocation patterns. Constants remain independent of extracted ideals and integrated Fourier labels. For $U = Z^d$, retain the fixed positive compact d range and check $\mathcal{L} = \ell/d \in [.21, .22]$ separately. $\square$

4 The enlarged amplified norm

In this section, we prove the additional amplified norm. The classification of Section 3 gives the homogeneous estimates; a direct bound for small formal summation lengths and the complementary Gauss calculation give the larger domain used in Section 7.

4.1 An amplified norm with variable lengths

In this subsection, we prove a parameter-dependent amplified norm from the classification of Section 3. We then estimate the unused prime amplitudes for the selection in Section 7.

This section retains the established cited results and the earlier 7/8 premise at $\kappa = 3/4$. The classification of Theorem 3.7 is independent of its numerical parameter range. We now give parameter-dependent sufficient inequalities; the comparison follows in Section 7. All powers below are relative to the actual scale of the character parameter U .

Theorem 4.1 (Amplified norm with variable lengths). *Retain the arithmetic and analytic conventions of Theorem 3.7 and Section 2. Fix $3/8 < m_- \leq r < 1/2$ and $W \geq 0$ with*

$$\tau = 1 - 2r - 9W > 0, \quad C_{\text{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1. \quad (4.1)$$

For disjoint products Q_0, E of complete prime-sum factors with length exponents z, w , put

$$m \in [m_-, r], \quad 0 \leq z \leq z_0(m) = 2(1 - 2m)/9, \quad 0 \leq w \leq W, \\ P = S_{m,1}S_{m,2}Q_0, \quad P_c = P - P_{\text{cmp}}.$$

Use the canonical equal-product comparison and retain Q_0 with auxiliary removal length $x = 0$. Then, for every $\epsilon > 0$,

$$\sum_k \mathcal{W}(q_k/U) |P_c(k)E(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C, \quad (4.2)$$

$$\sum_{k \in \mathcal{U}} \mathcal{W}(q_k/U) |P(k)E(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C. \quad (4.3)$$

The centered sum is unrestricted; $\mathcal{U}$ is the prescribed nonexceptional sixth-power-free nonunit family. The bounds hold for every required finite independent family of smooth weights and derivatives, with the original Fourier measure, zero and support conditions, and order of choices. Steps, meshes and internal losses are chosen inside the fixed strict margins.

Proof. Repeat the exact classification of Theorem 3.7 with the original C, D, s and weighted Fourier measure. Bound zero-length factors before Poisson, retain every coprimality and natural-zero factor, and classify all extra positions before M2. The local identities do not depend on the earlier parameter range.

All homogeneous identities are unchanged. In particular, with the notation of the proof using all local prime cases,

$$D_{\text{free}} = 6w - \eta_0 - \frac{7}{2}z_{C,\text{old}} - 5w_f - 5f_1 - T_1 - 5v - T, \quad \mu_{\text{com}} = \min(c_C, d_D) \geq w_{f,\text{actual}},$$

and the additional cost of restoring the lengths is $(D_{\text{free}} - \mu_{\text{com}})_+/6$. Corollary 3.6 makes the complete coefficient exponent plus this cost nonpositive on actual scales, with just the same total $O(1/\log U)$ conversion discrepancy on nominal annuli. No second use of $-d_D$ or extra structural-family count is made.

Here are all new numerical admissibility conditions. Put

$$F = \frac{7}{9}\tau, \quad z_{0,\min} = \frac{2}{9}(1 - 2r), \quad A_{\max} = 2r + z_{0,\min} = \frac{14r + 2}{9}.$$

The exact coupled remaining-prime margin is bounded below by

$$\begin{aligned} \frac{7}{2}z_{\text{init}} - D_{\text{free}} &= \frac{7}{2}z_0 - 6w + \frac{2}{9}\eta_0 + 5w_f + 5f_1 + T_1 + 5v + T \\ &\geq \frac{7}{9}(1 - 2r) - 6W = F + W > 0. \end{aligned}$$

Put $D_{\text{sharp}} = D_{\text{free}} - \mu_{\text{com}}$ and choose the amplifier and edge allowance so that $D_{\text{temp}} - D_{\text{sharp}} < F + W$. Genuine and final-divisor fixing reduce the defect by at least $7/2$ and $9/2$ times their respective widths. Hence, for $0 \leq b \leq z_{\text{init}}$,

$$\frac{9}{2}(z_{\text{init}} - b) - (D_{\text{temp}} - 7b/2) \geq \frac{7}{2}z_{\text{init}} - D_{\text{temp}} > 0.$$

At exhaustion,

$$z_{\text{init}} = 0 \implies D_{\text{sharp}} \leq -(F + W) < 0, \quad g = J_+ + \sigma, \quad \ell = 0.$$

Use the case with no remaining prime factor, without averaging over auxiliary primes.

For $t = f_0 + u$ and $a = f_1 + t \leq w$, the unchanged auxiliary input formulas and ordinary-removal bound $L_1 + L + 5c_{\text{cor}} \leq 2m + o(1)$ give

$$\begin{aligned} M_* &\geq 1 - 2r - 6W - o(1) = \tau + 3W - o(1), \\ m_* &\geq 1 - 2r - 7W - o(1) = \tau + 2W - o(1), \\ A_* - m_* &\leq A_{\max} + 7W - 1 + o(1) = -F + o(1). \end{aligned}$$

For nonempty supports,

$$c_C \leq A_* + o(1), \quad R + E_0 \leq \min(c_C, d_D), \quad -o(1) \leq K \leq 2A_* - d_D - m_* + o(1).$$

Together with $A_* - m_* \leq -F + o(1)$, this gives

$$d_D \leq A_* - F + o(1), \quad m_* - c_C + d_D - R - E_0 \geq F - o(1) > 0.$$

The same conservative bounds hold at each exact comparison stage, and no M2 estimate precedes the completed classification.

The second-origin and centered-exceptional ceilings, including the volume reference branch, are respectively

$$A_{\max} + 5W = 1 - F - 2W < 1, \quad A_{\max} + 4W = 1 - F - 3W < 1.$$

The original zero-frequency term in the first Poisson formula, powerful-product count, tails and exact equal-product mean cancellations are handled in the order of choices in Section 2. The two current-side estimates fit the same Gauss-norm targets, are integrated in the same weighted Fourier measure and pay their paired structural count once. This proves (4.2).

For the restoration of the comparison product in the uncentered norm on $\mathcal{U}$, the reflected total length exponent is at most

$$\frac{5}{2} - 4m_- + \frac{9}{2}W = C_{\text{add}} < 1.$$

The initial four ordinary scales have exponent at least $1/4$: $m_- > 3/8 > 1/4$ and $2m_- - 1/4 > 1/2$. Also $W < 1/45$, by the exact identity

$$1 - 45W = 4\tau + 2(1 - C_{\text{add}}) + 8(r - m_-) > 0.$$

Thus the original M2 theorem and the triangle inequality give (4.3). No parameter inducing a principal character is added back. The fixed-unit coprimality restrictions are partial products of the original list and have norm at most $U^{W+o(1)}$; the retained polynomially uniform common-extra-radical result treats them in the same fixed data.

Choose a positive step, subsidiary edges, mesh for the prime-product lengths and requested losses inside F , $F + W$, the positive lower bounds for the summation lengths, the origin margins and the margin for restoring the comparison product. Then fix the original finite list of prime ranges and test functions, finite allocation maximum, weighted internal orders and thresholds, and finally the external height exponent. The strict decrease in the auxiliary summation length and a single total annular discrepancy are exactly those of the earlier proof. None depends on an extracted ideal or subsequent Fourier label. This establishes all uniformities asserted above. $\square$

Lemma 4.2 (Mean amplitude of the remaining prime factors). *Let Q_i be a finite family of original prime-sum factors on mutually disjoint prime-ideal ranges, with length exponents $\lambda_i \geq 0$ and amplitude exponents g_i on $\mathcal{B}$. Retain the original lower bounds*

$$|Q_i(k)| \geq U^{\lambda_i g_i - o(1)} \quad (k \in \mathcal{B}, g_i > 0).$$

Suppose

$$\delta > 0, \quad 0 \leq x \leq 1/2, \quad 0 \leq g_i \leq \delta/2, \\ \sum_i \lambda_i = \mathcal{L} \geq L_0 > z_0(m), \quad \sum_i \lambda_i g_i = \delta x \mathcal{L}.$$

Remove an initial prime product of length at most $z_0(m)$. If

$$a(\delta, x, m) = \delta \left\{ \frac{1}{2} - \frac{(1/2 - x)L_0}{L_0 - z_0(m)} \right\} = \delta \frac{xL_0 - z_0(m)/2}{L_0 - z_0(m)} > 0, \quad (4.4)$$

the remaining positive-amplitude prime ranges have mean amplitude at least $a(\delta, x, m)$ and total length at least

$$P(\delta, x, m) = 2xL_0 - z_0(m). \quad (4.5)$$

For $0 < W < P(\delta, x, m)$ and a sufficiently fine mesh η with $\max_i \lambda_i \leq \eta$, selection in decreasing amplitude gives a disjoint product E of length exponent w_E with

$$W - \eta \leq w_E \leq W, \\ |E(k)|^2 \geq U^{2a(\delta, x, m)(W - \eta) - o(1)} \quad (k \in \mathcal{B}).$$

Proof. Write $z \leq z_0(m)$ for the removed length and $D_{\text{amp}} = \delta(1/2 - x)\mathcal{L}$ for the total deficit from maximal amplitude. The remaining mean $\bar{g}$ satisfies

$$\bar{g} \geq \frac{\delta}{2} - \frac{D_{\text{amp}}}{\mathcal{L} - z} \geq \frac{\delta}{2} - \frac{\delta(1/2 - x)L_0}{L_0 - z_0(m)} = a(\delta, x, m).$$

Here

$$\frac{\partial}{\partial z} \left(\frac{\delta}{2} - \frac{D_{\text{amp}}}{\mathcal{L} - z} \right) \leq 0, \quad \frac{d}{d\mathcal{L}} \frac{\mathcal{L}}{\mathcal{L} - z_0(m)} \leq 0.$$

The original zero-amplitude length is at most $2D_{\text{amp}}/\delta = (1 - 2x)\mathcal{L}$. Thus the unused positive length is at least

$$\mathcal{L} - z - (1 - 2x)\mathcal{L} = 2x\mathcal{L} - z \geq 2xL_0 - z_0(m).$$

Removing zero amplitudes increases a positive mean. Since $\lambda_i \geq 0$ and $\max_i \lambda_i \leq \eta$, selection of complete factors in decreasing amplitude gives length in $[W - \eta, W]$ and

$$\log_U |E(k)|^2 \geq 2a(\delta, x, m)(W - \eta) - o(1).$$

Apply the original amplitude lower bounds before differentiating weights. $\square$

The lower bound in (4.4) is sharp given only the upper amplitude bound, the total mean and the removed length of the initial prime product. At $\mathcal{L} = L_0$ take an initial prime product of width $z_0(m)$ and amplitude $\delta/2$, and give every remaining position the common amplitude $a(\delta, x, m)$. Its weighted total is exactly $\delta x L_0$, the decreasing-amplitude rule removes the factors of maximal amplitude first, and every unused subproduct has that same mean. This is a sharpness statement for the scalar amplitude data; it does not assert that such a formal amplitude class is nonempty arithmetically.

Corollary 4.3 (A norm bound on amplitude classes and the resulting saving). *Under Theorem 4.1, fix an amplitude class $\mathcal{B}$ with*

$$a = a(\delta, x, m) > 0, \quad 0 < W < \min\{(1 - 2m)/9, (8m - 3)/9, 2xL_0 - z_0(m)\}.$$

Then

$$s < 2aW \implies \sum_{k \in \mathcal{B}} \mathcal{W}(q_k/U) |P(k)|^2 \ll U^{1-s+\epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C.$$

After separately extracting the initial prime-product saving, the reflected fourth moment and Hölder inequality give any additional first-moment saving

$$0 < \gamma(\delta, x, m) < \frac{a(\delta, x, m)W}{2}. \quad (4.6)$$

These bounds use the original norm, independently of any older restricted-remainder error term. Uniformity on parameter families requires positive lower bounds for the retained length and selection margins and for

$$2aW - s, \quad aW/2 - \gamma.$$

Proof. For any $0 < \gamma < aW/2$, choose s and then the mesh and internal loss so that

$$4\gamma < s < 2a(W - \eta) - o(1) < 2aW.$$

On the original selected class,

$$|E(k)|^2 \geq U^{s-o(1)} \implies \sum_{k \in \mathcal{B}} \mathcal{W}(q_k/U) |P(k)|^2 \ll U^{-s+\epsilon} \sum_{k \in \mathcal{U}} \mathcal{W}(q_k/U) |P(k)E(k)|^2.$$

Equation (4.3) gives the claimed norm bound. Hölder of order four then gives

$$\gamma < s/4, \quad s < 2aW \implies 0 < \gamma < aW/2.$$

The amplitude bound precedes Sobolev and derivative expansion. Choose products from the original class data, then fix derivative weights and Fourier orders. Uniform positive gaps in the length, selection and gain inequalities make these choices uniform. $\square$

For a rectangle $\delta \in [d_0, d_1]$, $x \in [x_0, 1/2]$, $m \in [m_-, r]$, a uniform lower mean is

$$a_{\text{box}} = d_0 \left\{ \frac{1}{2} - \frac{(1/2 - x_0)L_0}{L_0 - z_0(m_-)} \right\} > 0,$$

and available length of positive-amplitude prime factors is at least $2x_0L_0 - z_0(m_-)$. The upper endpoint d_1 does not enter this sharper amplitude bound: the amplitude bound and its deficit have the same factor δ . For $d_0 = 2/5$, $x_0 = 12/25$, $L_0 = 21/100$, $m_- = 2/5$, one obtains $a_{\text{box}} = 3536/18625$. Using separate maxima for the two parameters instead would give only 13811/74500.

The theorem also permits a requested prime-product length depending on the ordinary length $W(m)$, provided $1 - 2m - 9W(m)$ and the selected margin in the available positive-amplitude prime length have uniform positive lower bounds on the chosen compact band. At the level of these sufficient admissibility conditions, the limiting pointwise gain is

$$\gamma_{\text{lim}}(\delta, x, m) = \frac{a(\delta, x, m)}{2} \min \left\{ \frac{1 - 2m}{9}, \frac{8m - 3}{9}, 2xL_0 - z_0(m) \right\}.$$

This is a supremum for this particular norm-and-selection exponent calculation; its strict boundary is not asserted attainable and it is not an optimality theorem for other analytic estimates or zero-free methods.

Lemma 4.4 (An admissible choice for each amplitude class). *On a compact amplitude-class set $\mathcal{K}$, suppose*

$$\begin{aligned} \tau > 0, \quad m_- > 3/8, \quad m_- \leq m_0 < (1 - \tau)/2, \\ a_0(\delta, x) = a(\delta, x, m_-) > 0, \quad t \in C(\mathcal{K}; (0, \infty)). \end{aligned}$$

Set

$$r(\delta, x) = \frac{a_0(1 - \tau)/18 + tm_0}{a_0/9 + t},$$

$$W(\delta, x) = \frac{1 - 2r(\delta, x) - \tau}{9}.$$

Then

$$m_- \leq r < (1 - \tau)/2, \quad W > 0,$$

$$1 - 2r - 9W = \tau, \quad a_0W/2 = t(r - m_0).$$

If, uniformly on this set,

$$W < \min\{(8m_- - 3)/9, 2xL_0 - z_0(m_-)\}$$

with a positive gap, the norm on $m \in [m_-, r(\delta, x)]$ supplies every gain satisfying

$$\gamma(\delta, x) > 0, \quad \inf_{\mathcal{K}} \{a_0W/2 - \gamma\} > 0,$$

with uniform positive loss margins. The finite class selections may therefore use these class-dependent endpoints and lengths. For a fixed finite class set, all strict margins are made uniform by taking their finite minimum.

Proof. Put $A = a_0/9 > 0$ and $M = (1 - \tau)/2$. Then

$$r = \frac{AM + tm_0}{A + t}, \quad r - m_0 = \frac{A(M - m_0)}{A + t} > 0, \quad M - r = \frac{t(M - m_0)}{A + t} > 0.$$

Consequently

$$m_- \leq m_0 < r < M, \quad W = \frac{2(M - r)}{9} > 0, \quad 1 - 2r - 9W = \tau, \quad \frac{a_0W}{2} = t(r - m_0).$$

For $m \geq m_-$, Lemma 4.2 gives $a(\delta, x, m) \geq a_0$. Theorem 4.1 and Corollary 4.3 apply with formal margin $7\tau/9$ and the stated remaining strict gaps. Continuity on $\mathcal{K}$ and the prescribed gain gap give

$$\inf_{\mathcal{K}} t > 0, \quad \inf_{\mathcal{K}} W > 0, \quad \inf_{\mathcal{K}} (a_0W/2 - \gamma) > 0.$$

Together with the assumed upper-length gaps, these bounds permit common positive steps, meshes, orders and thresholds. All choices precede derivatives and Fourier integration; the separate contour conditions on t still have to be checked. $\square$

4.2 A direct estimate for small transformed length exponents

In this subsection, we bound the actual Gauss polynomial when its formal summation-length exponent is small. The complementary range is treated in Section 4.3.

The length exponent and the frequency cutoff. We distinguish the transformed summation-length exponent from the Fourier-frequency cutoff.

Write $A = A_*$, $m = m_*$, $q = q_*$ for the completed auxiliary parameters, $c = c_C$, $d = d_D$, and $u = R + E_0$. These identities are numerical exponent notation for the actual completed norm even when a formal ordinary moment family would not be admissible. Put

$$\alpha = A - c, \quad t = d - u \geq 0, \quad m_f = m - c + t, \quad \nu = -m_f.$$

The exact cutoff identity is

$$K_0^* = 2A - c - d + u - m = 2\alpha + \nu.$$

Thus a negative formal exponent m_f does not imply an empty actual frequency annulus. For example, $\alpha = 0$ leaves a possible constant Gauss polynomial $\mathcal{G}(1, h) = 1$ and a nonempty actual frequency range of exponent $\nu > 0$. It is the true h cutoff, not m_f , that controls counting of the actual frequencies.

Lemma 4.5 (A direct bound for the Gauss polynomial). *Retain the common support, divisor parameters, coefficient families and Fourier coordinates of the local decomposition. The factor $\vartheta(n)$ below is the fixed coefficient factor in the cited Gauss polynomial, with its prescribed zeros; it is independent of the frequency h . Put*

$$F(h) = U^{-\alpha/2} \sum_n B(n) \vartheta(n) 1_{s'|n} 1_{(n, R_*)=1} \mathcal{G}(n, h), \quad s'_0 = \log_U q_{s'}.$$

Here $q_n \asymp U^\alpha$ on the two equal-product rectangles; $B(n)$ is the original finite convolution of smooth weights and fixed finite-group prime coefficients. Retain all support and zero-extension conditions. After extracting the fixed weight seminorms, let $0 \leq \omega(h) \leq 1$ be the normalized frequency cutoff. Then, for every retained frequency range of exponent K ,

$$\sum_{0 < q_h \ll U^K} \omega(h) |F(h)|^2 \ll U^{K_+ + 2\alpha - 2s'_0 + \epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C, \quad K_+ = \max(K, 0).$$

The positive loss includes all dyadic-scale discrepancies; the Fourier orders are those of the original weighted estimate.

Proof. For fixed factor count the coefficients satisfy an ideal-divisor bound, uniformly in the fixed arithmetic labels. With the original weight seminorms and height factors understood,

$$|\mathcal{G}(n, h)| \leq q_n^{1/2}, \quad \sum_{\substack{q_n \asymp U^\alpha \\ s'|n}} |B(n)| \ll U^{\alpha - s'_0 + \epsilon}.$$

The two centered products have the same scale. Hence

$$\begin{aligned} |F(h)| &\leq U^{-\alpha/2} \sum_{\substack{q_n \asymp U^\alpha \\ s'|n}} |B(n)| q_n^{1/2} \\ &\ll U^{\alpha - s'_0 + \epsilon} \mathfrak{S}_J(\mathcal{W})^{1/2} (1+H)^C. \end{aligned}$$

A nonempty product has $s'_0 \leq \alpha + O(1/\log U)$. The actual nonzero frequency orbits, including bounded unit frequencies, satisfy

$$\#\{h : 0 < q_h \ll U^K\} \ll U^{K_+ + \epsilon}.$$

Squaring the pointwise bound and summing proves the assertion. The zero-frequency term in the first Poisson formula is separate. Keep all natural zeros and support restrictions in the polynomial; only their absolute values are bounded by one. Integrate the polynomial Fourier-height factors in the original measure after fixing a sufficiently high order. No additional restricted moment family or family count is introduced. $\square$

Proposition 4.6 (Direct estimate for small transformed length exponents). *Retain the actual coefficient-square bounds for all local prime cases and natural-zero conventions. Fix $\rho_{\text{term}} = 1/100$ and suppose $m_f \leq \rho_{\text{term}}$. Let the class parameters obey*

$$\frac{3}{8} < m_- \leq r < \frac{1}{2}, \quad \eta = \frac{7}{9}(1 - 2r) - 6W > 0, \quad C_{\text{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1,$$

with actual extra length $w \leq W$. Then this branch fits the original one-sided Gauss target with a fixed strict power saving. In particular it does not require a reconstruction of an ordinary sum with positive formal length.

Proof. Put $B_C = (3c - 5d - R)_+/6$. The original one-sided exponent, before the local prime removals, is

$$\Lambda_{\text{in}} = A_t - c + u + B_C - s_0, \quad u = R + E_0, \quad (4.7)$$

where $A_t = 2m_{\text{ord}} + z + w$ is the original product length, m_{ord} is its ordinary length before classification, and s_0 is the original inner divisor length. The m in the completed Gauss parameters is instead the residual length m_* . This is the allowance of [Lin26b, equation (12.3)] at the original fixed C, D data. For the completed polynomial,

$$s'_0 = s_0 - f_0 - f_1, \quad \Lambda_* = (A - c) + q + u + B_C - s'_0.$$

Consequently the exact change in the allowance is

$$\Lambda_* - \Lambda_{\text{in}} = A - A_t + q + f_0 + f_1.$$

This notation does not assume a moment theorem at that target. The actual frequency cutoff gives

$$K \leq 2\alpha + \nu + o(1), \quad \nu \geq -\rho_{\text{term}}, \quad \alpha \geq -o(1), \\ K_+ \leq 2\alpha + \nu + \rho_{\text{term}} + o(1).$$

Subtracting Λ_* from Lemma 4.5 gives at most

$$3\alpha + \nu - s'_0 - q - u - B_C + \rho_{\text{term}} + o(1).$$

The local coefficient contribution after classification, including the difference between this affine target and the given target, is

$$\mathcal{R}_{\text{loc}} = -f_0 - L_1 + T_1 - S_1 - u_{\text{free}} - L + v - 4c_{\text{cor}} + T - S.$$

For clarity, $u_{\text{free}} = g_{\text{good}} + v + c_{\text{cor}}$ is unrelated to the common-support quantity $u = R + E_0$. The physical squared coefficient exponent and the allowance change give

$$(-f_0 - f_1 + T_1 - S_1 + T - S) + (\Lambda_* - \Lambda_{\text{in}}) \\ = A - A_t + q + T_1 - S_1 + T - S = \mathcal{R}_{\text{loc}}.$$

Here $s'_0 = s_0 - f_0 - f_1$; the fixed-data coefficient energy is retained.

Since $R \leq u$ and $d = u + t$,

$$u + B_C \geq u + \left(\frac{c}{2} - u - \frac{5t}{6} \right)_+ \geq \frac{c}{2} - \frac{5t}{6}.$$

Also $c \geq m + t - \rho_{\text{term}}$. Dropping the favorable $-s'_0$ shows that the complete relative exponent is at most

$$\mathcal{R}_{\text{loc}} + 3(A - c) + (c - m - t) - q - u - B_C + \rho_{\text{term}} \\ \leq \mathcal{R}_{\text{loc}} + 3A - \frac{7}{2}m - q - \frac{8}{3}t + \frac{7}{2}\rho_{\text{term}} \\ \leq \mathcal{R}_{\text{loc}} + 3A - \frac{7}{2}m - q + \frac{7}{2}\rho_{\text{term}}.$$

Put $L_1 = f_1 + 4T_1 + e_1$ and $L = v + 4T + e$, with $e_1, e \geq 0$. The exact completed inputs give the identity

$$\mathcal{R}_{\text{loc}} + 3A - \frac{7}{2}m - q = 3A_t - \frac{7}{2} + 17f_0 + \frac{5}{2}f_1 + 17g_{\text{good}} + \frac{5}{2}v + 18c_{\text{cor}} \\ - \frac{9}{2}(T_1 + T) - S_1 - S - \frac{e_1 + e}{2} \\ \leq 3A_t - \frac{7}{2} + 18w.$$

All variables here refer to the same completed allocation, before the direct bound is inserted. The fixed prime dividing s choices satisfy a divisor bound; actual nonunit-prime indices were already treated by original-frequency divisor Cauchy, with their squared coefficient energy retained; correction indices retain their absolute coefficient mass of arbitrarily small power growth. The direct bound is uniform in those fixed data and introduces no new count.

The length constraint for the amplitude class gives

$$A_t \leq \frac{14r+2}{9} + W = 1 - 5W - \eta.$$

Moreover

$$1 - \frac{279}{7}W = \frac{36}{7}\eta + 2(1 - C_{\text{add}}) + 8(r - m_-) > 0,$$

so $W < 7/279$. The total exponent is consequently less than

$$-\frac{1}{2} + 3W - 3\eta + \frac{7}{2}\rho_{\text{term}} < -\frac{7249}{18600} < -\frac{3}{8},$$

before sufficiently small requested losses. It fits the original one-sided target with a fixed strict margin. This proof uses no earlier Gauss norm estimate on a family with a negative formal summation-length exponent. $\square$

For $m_f > \rho_{\text{term}}$, we use the direct Gauss calculation of Proposition 4.8. Lemma 4.7 supplies a numerical comparison while retaining the actual outer ideal. These estimates are combined in Theorem 4.9.

4.3 The complementary Poisson estimate

In this subsection, we prove the complementary one-sided Gauss estimate. Together with Section 4.2, it gives the norm domain in Section 4.4.

This subsection concerns a Gauss norm already obtained by the actual first transform and the exact local Gauss-sum decompositions. It does not apply the original first-Poisson estimate to an invented ordinary family. In particular the original outer ideal D is retained as a fixed label, not reconstructed as a new factor in the shorter completed coefficient sum.

Lemma 4.7 (An auxiliary outer-length parameter). *For the completed Gauss norm above, assume $c \leq A$ and $R + E_0 \leq \min(c, d)$. Set*

$$\hat{d} = \min(d, A), \quad \hat{K}_0 = K_0 + d - \hat{d}, \quad \hat{\Delta} = \hat{K}_0 - K.$$

Then

$$\begin{aligned} \hat{d} &\leq A, & R + E_0 &\leq \min(c, \hat{d}), & K &\leq \hat{K}_0 + \delta_1, \\ B_C(\hat{d}) &= B_C(d), & \min(c, \hat{d}) &= \min(c, d), \\ \hat{K}_0 &= 2A - c - \hat{d} + R + E_0 - m, & \hat{d} + \hat{\Delta} &= d + (K_0 - K). \end{aligned}$$

The symbols F_1, F_2 denote the two local savings in the exceptional-character estimate, computed in Section 4.3. This numerical replacement preserves the J, g, F_1 formulas, origin and length-restoration exponents, prime lengths and target exponent. It changes neither the polynomial, the actual cutoff and fixed ideal D , nor its character factors, outer measure and previously discarded tails. With $\hat{m}_f = m - c + \hat{d} - R - E_0$,

$$\begin{aligned} \hat{m}_f &\leq \rho_{\text{term}} : \text{ Proposition 4.6 applies with the same target,} \\ \hat{m}_f &> \rho_{\text{term}} : \hat{m}_f > 0, \quad \hat{d} \leq A. \end{aligned}$$

No new coefficient family or ideal of norm $U^{\hat{d}}$ is introduced.

Proof. The case $d \leq A$ is unchanged. If $d > A$, then

$$\hat{d} = A, \quad 3c - 5\hat{d} - R \leq -2A - R \leq 0, \quad R + E_0 \leq c \leq A = \hat{d}.$$

Thus both positive parts defining B_C vanish, and

$$B_C(\hat{d}) = B_C(d) = 0, \quad \min(c, \hat{d}) = \min(c, d) = c, \quad \hat{d} + \hat{K}_0 - K = d + K_0 - K.$$

These are precisely the outer-length invariants in the formulas below; $\hat{K}_0 \geq K_0$ preserves the actual cutoff. The direct estimate uses the same cutoff bound, target and common-support inequalities. The polynomial, ideal D , kernels and discarded tails are unchanged. At nominal endpoints use the support-envelope construction below. $\square$

Dependence on the outer length in the cited Gauss calculation. For completeness, set $D_0 = d + K_0 - K$. The formulas below are the corresponding expressions in [Liu26b, Sections 12.7.1–12.8.3 and Appendix C.4], with its amplifier lengths w, w_o written here as a, a_o . Its amplifier density ℓ is written ℓ_a to separate it from the global prime length.

$$J = D_0 - c - 2a + a_o, \quad \Lambda_a = A - c + \tilde{q} + a_o + B_C - s'_0, \quad (4.8)$$

$$M' = M + J - g - g_2 + t_2, \quad \Delta' = b_2 - p_2 + a + B_C + \ell_a, \quad (4.9)$$

$$\mathcal{D}_{\text{ret}} \leq a - D_0 - a_o + g + 2\theta_N, \quad \mathcal{O}_{\text{ret}} = A - M - D_0 - a_o - B_C + g - \ell_a, \quad (4.10)$$

$$F_1 = \frac{c}{6} + \frac{5D_0}{6} + \frac{a}{3} + \frac{\tilde{q}}{6} + a_o + B_C - \frac{5g}{6} + \ell_a. \quad (4.11)$$

Here $\mathcal{D}_{\text{ret}}$ is the length excess before prime restoration and $\mathcal{O}_{\text{ret}}$ is the excess for the zero-frequency term in the second Poisson formula before the local coefficient contribution. The first line is [*loc. cit.*, equations (12.3)–(12.4)]; the transformed width and its normalization are [*loc. cit.*, equation (12.6) and Appendix C.4]. The last two lines are the pre-restoration expression of [*loc. cit.*, Section 12.8.1] and the origin and exceptional expressions of [*loc. cit.*, Sections 12.7.3 and 12.8.2].

Under Lemma 4.7,

$$\hat{d} + \hat{K}_0 - K = D_0, \quad B_C(\hat{d}) = B_C(d), \quad \min(c, \hat{d}) = \mu_{\text{com}}.$$

Thus (4.8)–(4.11) are invariant. The F_2 terms in [Liu26b, Section 12.8.2] use only the new second common support; the centered mean in [*loc. cit.*, Section 12.8.3] uses the unchanged actual equal-product rectangles. The original outer D -sum and its weight stay fixed. Tails are estimated from actual kernels and cutoffs before this numerical replacement; no outer count is recomputed at $\hat{d}$.

Dyadic length exponents and support errors. For nonempty nominal supports choose the single total $\theta_N = O(1/\log U)$ so that

$$c \leq \bar{A} := A + \theta_N, \quad \bar{A} \geq 0, \quad \hat{d}_N = \min(d, \bar{A}), \quad \hat{K}_{0,N} = K_0 + d - \hat{d}_N.$$

The common-support saving and the target exponent are preserved exactly:

$d > \bar{A}$	$B_C(d) = B_C(\hat{d}_N)$	$\min(c, d) = \min(c, \hat{d}_N)$
$d \leq \bar{A}$	0	c
	$B_C(d)$	$\min(c, d).$

Also $\hat{d}_N \leq A + \theta_N$; relative to the idealized comparison, the numerical change is at most θ_N wherever the ideal lengths are nonnegative. Include it in the total support error. The envelope changes no ordinary test, normalization, smooth-weight argument or exact centered product.

If a nonempty endpoint has negative nominal A or $\alpha = A - c$, its negative part is $O(\theta_N)$. In a nonempty regular branch,

$$m_f > \rho_{\text{term}}, \quad K \geq -o(1), \quad K \leq 2\alpha - \rho_{\text{term}} + O(\theta_N) \implies \alpha \geq \rho_{\text{term}}/2 - o(1).$$

Bounded coefficient-support endpoints therefore use the direct estimate, whose fixed saving absorbs the same support error. Actual centers and product identities are preserved.

Proposition 4.8 (The complementary Gauss-norm bound). *Retain the Gauss-sum, correlation-Poisson, weight, common-coefficient, zero-extension, exceptional-mean and M2 hypotheses of Section 2, with $\kappa_m = 3/4$. For a completed positive Gauss norm with fixed outer parameters C, D, s , put*

$$\alpha = A - c, \quad \tilde{q} = q + R + E_0, \quad M = m + q, \quad K_0 = 2A - c - d + R + E_0 - m.$$

Here $c = \log_U q_C$ and $d = \log_U q_D$, or the auxiliary numerical value of Lemma 4.7. Assume

$$R + E_0 \leq \min(c, d), \quad c \leq A + o(1), \quad K \leq K_0 + \delta_1, \quad 0 \leq \delta_1 < \xi, \\ m - c + d - R - E_0 > \rho_{\text{term}} > 0.$$

The actual global lengths are uniformly positive; all length exponents and fixed support norms lie in the original bounded polynomial ranges. Retain the exact coefficients and equal-product scales, the comparison factor after prime-power removal, and the full remaining-prime-length margin. Then the one-sided Poisson estimate has target

$$\Lambda = \alpha + \tilde{q} + B_C - s'_0, \quad B_C = (3c - 5d - R)_+/6.$$

For every prescribed $\varepsilon_{\text{loc}} > 0$, the additional squared length-restoration cost is at most

$$\frac{(D_{\text{free}} - \min(c, d))_+}{6} + \varepsilon_{\text{loc}}.$$

The steps, support errors and complete-factor excesses are chosen inside this loss and the fixed strict margins, in the order of Section 2. The origin and exceptional-term estimates and weighted orders are unchanged. M2 is applied only to admissible auxiliary ordinary sums; the inequality $d \leq A$ is not required.

Proof. Use the actual normalized Gauss polynomial of Lemma 4.5. The positive formal length selects this branch; an ordinary moment is used only on sums formed after the second Poisson transform. For each fixed tuple, the convolution coefficients and their multiplicative presentation are independent of h . Only the Gauss coefficient and an already bounded scalar depend on h ; all natural zeros remain in the Gauss polynomial. Thus this is the same common polynomial for the fixed-divisor and unit-prime arguments.

The frequency sum and the amplifier. Keep the original bounded sum over h through the sixth-power amplifier identity, averaging over admissible auxiliary primes and Jensen's inequality. With R_{fix} the product of the other fixed supports, the pool exclusions satisfy

$$\#\{\mathfrak{p} \text{ in the pool} : \mathfrak{p} \mid hs'CDR_{\text{fix}}\} \ll \log U,$$

since all these supports have bounded polynomial norms. The pool lies beyond the remaining original prime ranges, after the step and mesh are fixed. Representation multiplicity uses the actual bounded frequency range and positive pool step; $d \leq A$ is unnecessary.

Use a for an extraction from the original amplifier and $a_o \leq a$ for its new radical, reserving the letter w for the original extra product. Set

$$a_0 = A - c - a, \quad J = d - c + (K_0 - K) - 2a + a_o.$$

When prime-sum factors remain, use [Liu26b, Section 12.7.2]:

	(a, a_o)	ℓ_a	g
amplified main term	$(0, 0)$	$\sigma/3$	$J_+ + 2\sigma$
fixed amplifier error	original (a, a_o)	0	$J_+ + \sigma$.

If no prime-sum factor remains, use instead

$$a = a_o = \ell_a = 0, \quad g = J_+ + \sigma. \quad (4.12)$$

There is then no average over auxiliary primes or density factor. The identity $g - J_+ = \sigma$ retains the strict decrease in the transformed summation length; the origin and exceptional allowances are no worse than for the prime-bearing choice. This is also the choice used when the available length of the initial prime product is exhausted, as in Theorem 4.9.

The corresponding normalizations, with no averaging in the no-prime case, give the allowance

$$\Lambda_a = a_o + \tilde{q} + a_o + a + B_C - s'_0.$$

These are identities for the actual input and amplifier errors; they require no second ordinary coefficient sum of length exponent A .

Second Poisson and ordinary sums after the second Poisson transform. Restore the complete sums before squaring. The full finite correlation identity gives the earlier second-Poisson formula, with prefactor $U^{K+g-\ell_a-a_o}$ and its inverse roots. Retain every fixed common coprimality restriction. Estimate high-frequency tails on this genuine expression before any artificial coprime extension. The common Fourier boxes are chosen before the new remaining labels, exactly as in the earlier proof with the original weighted seminorms. All smooth weights here are the actual transformed smooth weights, whose orders are uniform on bounded polynomial ranges.

For its genuine second common support use the earlier variables b_2, p_2, g_2, t_2, V , with their actual valuations and zero branches. The nonzero summation parameters after the second Poisson transform have

$$\begin{aligned} m' &= 2a_o - K - g - g_2 - V, \\ q' &= \tilde{q} + a_o + t_2 + V, \\ M' &= m' + q' = M + J - g - g_2 + t_2 \leq M - \sigma. \end{aligned}$$

Substitute K_0 in the equality. For the decrease,

$$J - g \leq -\sigma, \quad t_2 - g_2 \leq 0 \quad \implies \quad M' \leq M - \sigma.$$

The nonzero frequency annulus is empty if its cutoff lies below the minimum exterior norm. At a bounded endpoint include the edge $0 \leq \delta_2 < \xi$ before declaring its summation range:

$$\text{nonempty} \implies m' + \delta_2 \geq 0, \quad M'_{\text{act}} \leq M' + \delta_2 < M - \sigma + \xi.$$

No inequality $d \leq A$ is used.

The exact second normalization is

$$\lambda_0 = K + g - \ell_a - a_o + p_2 - s'_0 + g_2 - t_2 - b_2, \quad \Lambda_a - \lambda_0 = M' + \Delta',$$

where $\Delta' = b_2 - p_2 + a + B_C + \ell_a \geq a + \ell_a$. The complete last-divisor allocation keeps unrestricted quotients, both actual rectangle products and all surviving prime annuli. Its paired weight and count are the original weighted measures with arbitrarily small power growth. Original fixed D is not summed again here; only the genuine new second common support is counted once. The restored ordinary sums have the radical in the original character formula, including canceled factors. Their exceptional split is made only after the final character has been formed. The established M2 theorem, rather than an induction hypothesis for the new amplified family, bounds each admissible auxiliary sum.

Length restoration. Before its coarse substitution, the earlier length defect is

$$a - d - (K_0 - K) - a_o + g + 2\theta_N.$$

Its exact common-length maximum is

$$-d - (K_0 - K) - a_o + J_+ = \max\{-d - (K_0 - K) - a_o, -c - 2a\} \leq -\min(c, d) + \delta_1.$$

Only $c, d, a, a_o \geq 0$ and $K_0 - K \geq -\delta_1$ are used. The same extra global defect D_{free} from the actual full local decomposition therefore gives the additional squared cost at most $(D_{\text{free}} - \min(c, d))_+/6 + \varepsilon_{\text{loc}}$. For the prescribed $\varepsilon_{\text{loc}} > 0$, choose the positive steps and subordinate support and mesh losses so that the reserved exponent losses, including the excess from one complete factor, sum to less than ε_{loc} . After fixing the finite prime-factor list, take U sufficiently large to absorb $\theta_N = O(1/\log U)$ in this same loss. The original amplifier estimate accounts for its own $a + \ell_a$ portion once. The unchanged coupled remaining-prime identity justifies every restoration of admissible lengths; prime exhaustion uses the cited estimate for arbitrary ordinary lengths with no remaining prime factor in the earlier order. No given Gauss estimate outside the established length range is assumed.

The total length removed by the local identities. In the origin and exceptional estimates below, put

$$\begin{aligned} u_{\text{free}} &= g_{\text{good}} + v + c_{\text{cor}}, \\ t_{\text{loc}} &= f_0 + u_{\text{free}} = w - w_f - f_1, \quad 0 \leq t_{\text{loc}} \leq w \leq W. \end{aligned} \tag{4.13}$$

This is the quantity denoted by t in the decomposition into all local prime cases of Section 3.4. Here u_{free} is the removed prime width; it is distinct from the common-support length $u = R + E_0$ in Section 4.2. The displayed identity follows from the partition $w = w_f + f_0 + f_1 + u_{\text{free}}$.

The zero-frequency term in the second Poisson formula. Its length excess in the cited Poisson formula before the local coefficient contribution is

$$A - M - d - (K_0 - K) - a_o - B_C + g - \ell_a + O(\theta_N).$$

The exact maximum and $g - J_+ \leq 2\sigma$ give

$$\begin{aligned} \mathcal{O}_{\text{ret}} &\leq A - M - \min(c, d) + \delta_1 - B_C + 2\sigma - \ell_a + O(\theta_N) \\ &\leq A - M + O(\sigma + \xi + \theta_N). \end{aligned}$$

Adding the original local coefficient contribution yields

$$\mathcal{O}_{\text{total}} \leq A_t - 1 + 4t_{\text{loc}} + O(\sigma + \xi + \theta_N).$$

This uses the existing full origin formula, without $d \leq A$.

Sparse exceptional character parameters and their means. The earlier local correlation table and natural zeros force the same prescribed exterior sixth-power-free part. All earlier supports of the fixed character factors and the additional restrictions still impose coprimality, so they cannot cancel a newly forced exponent two. The sparse count applies on the enlarged actual frequency range. Its first saving is

$$F_1 = c/6 + 5(d + K_0 - K)/6 + a/3 + \tilde{q}/6 + a_o + B_C - 5g/6 + \ell_a.$$

The original amplifier data satisfy

$$0 \leq a_o \leq a \leq 7\sigma/3, \quad \sigma \leq g - J_+ \leq 2\sigma, \quad \ell_a \geq 0.$$

Put $D_0 = d + K_0 - K \geq d - \delta_1$. The common-support bound gives

$$B_C + \tilde{q}/6 \geq (3c - 5D_0)_+/6 - 5\delta_1/6.$$

Hence

$$F_1 \geq \begin{cases} c + 2a + \tilde{q}/6 + a_o/6 + B_C + \ell_a - 5\sigma/3, & J \geq 0, \\ 2c/3 + a/3 - 5\sigma/3 - 5\delta_1/6, & J < 0. \end{cases}$$

In both cases,

$$F_1 \geq \frac{2(c+a)}{3} - \frac{22\sigma}{9} - \frac{5\delta_1}{6} > \frac{2(c+a)}{3} - 3\sigma - \frac{5\xi}{6}.$$

The extraction bound is the one for the original amplifier errors; it is not an estimate for arbitrary independent a . The second local saving $F_2 \geq 2b_2/3$ concerns only genuine new second common support, so it has no original d condition.

Apply the retained fixed-character mean to the actual centered rectangles, with the comparison factor after prime-power removal $(1/4 - L_1 - L - 5c_{\text{cor}})_+$ and its volume branch. The same character, fixed coprimality restriction and whole-product Mellin twist occur on both sides, so their leading products cancel exactly at the actual equal product. After the actual local coefficient contribution, the exceptional ceiling is $A_t - 1 + 3t_{\text{loc}}$, as in the calculation including all local prime cases. This argument does not replace the comparison factor after prime-power removal by a formal $M/4$, and it does not require a larger-width moment theorem.

Uniformity and integration. The preceding terms are precisely the earlier diagonal, sparse exceptional, finite amplifier-error terms and auxiliary ordinary sums. Their Fourier measures and paired normalizations are unchanged. Positive formal and global summation-length margins, the strict origin and exceptional margins and the margin in the coupled length constraint permit the same positive step, subsidiary edges, mesh and finite orders to be chosen in the earlier order. Actual fixed C, D and remaining labels are integrated only in the original outer measure. Thus no new D -family count or independent weighted measure is introduced. This proves the asserted one-sided Poisson argument for the stated norm without a $d \leq A$ hypothesis. $\square$

4.4 The enlarged length range

In this subsection, we combine the two Gauss estimates to obtain the sufficient length inequalities used in Section 7.

The admissible width changes as follows:

$$\begin{array}{l|l} \text{Theorem 4.1} & W < (1 - 2r)/9 \\ \text{Theorem 4.9} & W < 7(1 - 2r)/54 \end{array} \quad C_{\text{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1.$$

The common condition $C_{\text{add}} < 1$ restores the comparison product. The larger first bound supplies the extra prime-product saving used in Lemma 7.1.

Theorem 4.9 (Amplified norm under the coupled length constraints). *Retain the arithmetic data, coefficient families, summation domains and exact equal-product comparison of Theorem 4.1, with $\kappa_m = 3/4$ and the initial prime product Q_0 retained. Suppose*

$$3/8 < m_- \leq r < 1/2, \quad W \geq 0,$$

and

$$\eta = \frac{7}{9}(1 - 2r) - 6W > 0, \quad C_{\text{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1. \quad (4.14)$$

For complete, disjoint prime-sum factors with

$$m \in [m_-, r], \quad 0 \leq z \leq z_0(m) = 2(1 - 2m)/9, \quad 0 \leq w \leq W,$$

one has, for every $\epsilon > 0$,

$$\sum_k \mathcal{W}(q_k/U) |P_c(k)E(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C, \quad (4.15)$$

$$\sum_{k \in \mathcal{U}} \mathcal{W}(q_k/U) |P(k)E(k)|^2 \ll U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C. \quad (4.16)$$

The two summation domains, finite independent smooth-weight and derivative families, and order of choices are those of Theorem 4.1 and Section 2.

Proof. Use the exact classification of Theorem 3.7, with zero-length factors bounded before Poisson and every extra position classified before M2. Keep C, D, s , natural zeros and the original Fourier measure. The algebraic inequality of Corollary 3.6 gives the nonpositive normalization exponent and one $O(1/\log U)$ discrepancy. This use of the algebraic inequality requires no positive formal length; the analytic M2 applications occur only after the split below. The length identities yield

$$\frac{7}{2}z_{\text{init}} - D_{\text{free}} \geq \frac{7}{9}(1 - 2r) - 6W = \eta > 0, \quad z_{\text{init}} = 0 \implies D_{\text{free}} \leq -\eta.$$

Thus the original restoration remains valid, including the no-prime choice (4.12). The global lengths satisfy

$$\begin{aligned} A_t &\leq (14r + 2)/9 + W = 1 - 5W - \eta, \\ M_* &\geq (12W + 9\eta)/7 - o(1) > 0, \\ m_* &\geq (5W + 9\eta)/7 - o(1) > 0. \end{aligned}$$

Retain the actual outer ideal D . Lemma 4.7 replaces only its numerical exponent by $\hat{d} = \min(d_D, A_*)$, using $A_* + \theta_N$ at nominal endpoints. For the actual Gauss polynomial set

$$m_f = m_* - c_C + \hat{d} - R - E_0.$$

The complete range is

$$\begin{array}{l|l} m_f \leq 1/100 & \text{Proposition 4.6: saving greater than } 3/8 \\ m_f > 1/100 & \text{Proposition 4.8: ordinary restored sums.} \end{array}$$

The first case includes negative exponents and bounded unit frequencies. In the second, $R + E_0 \leq \min(c_C, \hat{d})$ and $\hat{d} \leq A_* + O(\theta_N)$. Neither case changes the actual frequency cutoff, outer ideal, coefficients or discarded tails. Every M2 application is on a formed ordinary sum with its full radical. In Proposition 4.8, request

$$0 < \varepsilon_{\text{loc}} < \min\{\epsilon/4, \eta/100, (1 - C_{\text{add}})/100, 1/100\}.$$

This local loss is included in the final ϵ and remains inside all the strict length and origin margins.

By (4.13), $t_{\text{loc}} \leq W$, so

$$A_t + 4t_{\text{loc}} \leq 1 - W - \eta < 1, \quad A_t + 3t_{\text{loc}} \leq 1 - 2W - \eta < 1.$$

Retain the volume reference branch and both matching fixed-character means at the exact equal product. The zero-frequency term in the first Poisson formula and high-frequency tails use their original expressions. Both one-sided estimates therefore fit the same Gauss allowance; integrate them in the same measure and count the common support once. This proves (4.15).

Reflection of the comparison product on $\mathcal{U}$ has total length exponent $C_{\text{add}} < 1$. The earlier M2 theorem applies, and, in the weighted ℓ^2 norms with weight $\mathcal{W}(q_k/U)$,

$$\|PE\|_{2, \mathcal{W}, \mathcal{U}} \leq \|P_c E\|_{2, \mathcal{W}} + \|P_{\text{cmp}} E\|_{2, \mathcal{W}, \mathcal{U}}.$$

This gives (4.16) without principal character parameters. Moreover,

$$1 - \frac{279}{7}W = \frac{36}{7}\eta + 2(1 - C_{\text{add}}) + 8(r - m_-) > 0.$$

Thus $W < 7/279$ and all support norms remain polynomially bounded. Choose steps, support edges, mesh and losses inside the positive length and origin margins, then fix the finite weight families, weighted orders and thresholds before external heights. The original strict decrease in auxiliary length is retained, uniformly in the later extracted ideals and Fourier variables. $\square$

Corollary 4.10 (The amplitude-class saving in the enlarged length range). *Under Theorem 4.9, let $a = a(\delta, x, m) > 0$ be given by Lemma 4.2. Suppose the remaining positive-amplitude prime length exceeds W with a uniform margin. Then*

$$s < 2aW, \quad 0 < \gamma < aW/2$$

are admissible squared-norm and first-moment savings, respectively, with the original order of smooth weights and derivatives. For fixed m_- and class-dependent (r, W) , the norm conditions are

$$W < \frac{7}{54}(1 - 2r), \quad W < \frac{8m_- - 3}{9}.$$

Thus $7/54$ replaces $1/9$ in the first length bound; a zero-free boundary still requires the comparison of Section 7. For uniformity on parameter families, all retained length and selection margins and the gaps $2aW - s$, $aW/2 - \gamma$ have positive lower bounds.

Proof. Apply Lemma 4.2 to the original factors, before derivative expansion. The two norm conditions are equivalent to

$$W < \frac{7}{54}(1 - 2r), \quad W < \frac{8m_- - 3}{9}.$$

Equation (4.16) and the proof of Corollary 4.3 give

$$0 < \gamma < aW/2 \implies \exists s: \quad 4\gamma < s < 2aW.$$

Choose the mesh and internal losses within the strict gaps, and use

$$\gamma < s/4.$$

The stated uniform positive gaps permit common choices of the mesh and internal losses. The same finite independent weight families and order of choices apply. $\square$

5 The approximate functional equation and its contours

In this section, we prove uniform estimates for the test functions on a compact positive strip. These estimates permit the contour displacements combined with the amplified norm in Section 7.

5.1 Smooth weights beyond the $1/10$ contour shift

In this subsection, we bound the AFE test functions on a compact positive strip. This supplies the full contour range used in Section 7.

The earlier restriction to shifts of size at most $1/10$ came from the strip chosen in the lemma on smooth weights for the approximate functional equation; the holomorphic region for the correction factors permits a larger shift of the original numerator contour.

Lemma 5.1 (Smooth weights on compact substrips). *Fix $0 < \sigma_0 < 1/2$. For a primitive nonprincipal finite-order Hecke character over the fixed totally complex field F , use the original approximate functional equation with*

$$V_w(v) = \frac{1}{2\pi i} \int_{(2)} e^{\xi^2} \left(\frac{\Gamma(w + \xi)}{\Gamma(w)} \right)^{r_F} v^{-\xi} \frac{d\xi}{\xi}, \quad T_w = (1 + |\Im w|)^{r_F}, \quad v > 0.$$

Uniformly for $\sigma_0 \leq \Re w \leq 1 - \sigma_0$ and fixed $j, N \geq 0$,

$$|(v\partial_v)^j V_w(v)| \ll_{F, \sigma_0, j, N} (1 + v/T_w)^{-N}.$$

For $w = 1/2 + \theta + it$, $0 \leq \theta \leq 1/2 - \sigma_0$, put

$$\tilde{V}_w(v) = \left(\frac{\Gamma(1-w)}{\Gamma(w)} \right)^{r_F} v^{2\theta} V_{1-w}(v).$$

After increasing the tail order by one, the same derivative bound holds for $\tilde{V}_w$. Before contour displacement use

$$H_w(v) = v^{2w-1} \left(\frac{\Gamma(1-w)}{\Gamma(w)} \right)^{r_F} V_{1-w}(v), \quad H_w(v) = v^{2it} \tilde{V}_w(v) \quad (w = 1/2 + \theta + it),$$

with the real logarithm of v . The functions V_w, H_w are holomorphic on $0 < \Re w < 1$; the real-line factorization is used only after displacement. Their dyadic expansions converge normally on compact substrips for each fixed finite annulus of character parameters.

Proof. Put $T = (1 + |\Im w|)^{r_F}$. Euler differentiation inserts powers of ξ in the defining integral. For $v \geq T$, shift to any fixed line $\Re \xi = B > 0$. Stirling's formula and the Gaussian e^{ξ^2} give

$$|(v\partial_v)^j V_w(v)| \ll_{F, \sigma_0, j, B} (T/v)^B.$$

For $v \leq T$, shift to $\Re \xi = -\sigma_0/2$. Then

$$\Re(w + \xi) \geq \sigma_0/2 > 0, \quad (v\partial_v)^j V_w(v) = \mathbf{1}_{j=0} + O_{F, \sigma_0, j}((v/T)^{\sigma_0/2}).$$

Only $\xi = 0$ is crossed. These two estimates prove the required bound for every fixed j, N .

The completed nonprincipal function is entire, so its functional equation gives the same two exact AFE halves on the enlarged strip. For $w = 1/2 + \theta + it$, Stirling gives

$$\left| \frac{\Gamma(1-w)}{\Gamma(w)} \right|^{r_F} \ll T^{-2\theta}, \quad 0 \leq 2\theta \leq 1 - 2\sigma_0 < 1.$$

Hence the normalized dual factor is bounded using

$$(v/T)^{2\theta} (1 + v/T)^{-N-1} \leq (1 + v/T)^{-N}.$$

Fixed Euler derivatives obey the same bounds. Before the contour move use the holomorphic factor v^{2w-1} ; afterwards its unitary part is included in the Mellin twist. The exact conductor and gamma factors are retained, and the large- v bounds give normal convergence of the dyadic tails on compact substrips. $\square$

Lemma 5.2 (Extended contours for the dyadic sums). *Retain the nonprincipal character family, amplitude classes, correction and reflection identities, and unamplified $M2$ with $\kappa_m = 3/4$. First fix z on its final contour $\Re z = 17/50$, then assign the amplitude classes at that fixed z . No z -contour displacement is made after this assignment. For $3/10 \leq \delta \leq 3/5$, every dyadic sum in the approximate functional equation admits each fixed classwise shift*

$$0 \leq \theta \leq \delta/2, \quad \Re w = 1/2 - \theta, \quad \theta_- = 0, \quad \theta_+ = \delta/2.$$

Keep s on its initial right-hand line and move the whole dyadic integrand before the local s -contour displacement, absolute values, conductor separation and conjugation of dual weights. Retain the prescribed positive separation margins. The $M2$ exponent changes by

$$(\theta - \theta_0)\{d(1-m) - l_y\}, \quad \theta_0 = 2\delta x/9,$$

with the same central-line moment, initial prime product and prime-sum coefficients.

Proof. Here $a = (1 + \delta)/2 = a_{\text{det}}$ is the detector parameter. The entire swept range satisfies

$$0 \leq \theta \leq \delta/2, \quad 1/5 \leq \Re w = 1/2 - \theta \leq 1/2, \quad 1/2 \leq \Re(1 - w) \leq 4/5.$$

Lemma 5.1 applies with $\sigma_0 = 1/6$. The connecting region of [Liu26b, Lemma 13.10] contains

$$\Re s \geq a + 16e, \quad \Re w \geq 1 - a - 6e, \quad \Re z \geq 17/50.$$

On the final local contours,

$$\Re w \geq 1 - a \implies \Re(s + w) \geq 1 + 16e.$$

Thus the correction holomorphy and the original unit- and nonunit-prime bounds hold throughout. For each strict tuple,

$$q_p^\theta q_p^{-(j_p-1)\theta} = q_p^{-(j_p-2)\theta} \leq 1 \quad (j_p \geq 2, \theta \geq 0);$$

nonstrict errors retain their positive e margins. If S_{fin} denotes the finite places of S , the fixed-place majorant is summable:

$$\sum_{\substack{\mathfrak{a} \text{ integral} \\ \text{supp}(\mathfrak{a}) \subset S_{\text{fin}}}} q_{\mathfrak{a}}^{-1/6} = \prod_{p \in S_{\text{fin}}} (1 - q_p^{-1/6})^{-1} < \infty.$$

The independent AFE weights and derivatives therefore remain in the original M2 families with fixed polynomial height dependence.

Keep finite character and dyadic cutoffs. With s first on its original right-hand line, perform the contour moves in this order:

unsplit numerator exact two-part AFE each holomorphic dyadic sum local s -contour	$\Re w : c_w \longrightarrow 1/2$ insert on $\Re w = 1/2$ $\Re w : 1/2 \longrightarrow 1/2 - \theta$ $\Re s \geq a + 16e.$
--	---

Here c_w is the initial right-hand line. The first move is valid because the numerator is entire, the reciprocal factor is independent of w , and the cited region contains the path. Mellin decay removes horizontal joins; locally uniform bounds permit removal of the cutoffs in the original weighted height order. Only then conjugate the dual half as a whole. Keep each actual conductor, root number and gamma ratio.

The amplitude partition depends only on the character parameter and the already fixed z ; it is unchanged during the w - and local s -contour displacements. Thus its prime coefficients remain fixed, and the change of factor is

$$q_u^{\theta-\theta_0} D^{-(\theta-\theta_0)} Y^{-(\theta-\theta_0)}.$$

With $q_u \asymp U = Z^d$, $D = U^m$ and $Y = Z^{l_y}$, its exponent change is the one stated, with the bounded annular ratios in the original smooth weights. $\square$

The wider contour range is combined with the amplified norm in Section 7. Section 8 verifies the resulting comparison inequalities and obtains the endpoint B_* .

6 Amplitude-class counts and short summation ranges

In this section, we sharpen the amplitude-dependent class count and separate its parameter from that of the numerator moment. We also record the lowest-amplitude estimate needed for the comparison in Section 8.

6.1 Amplitude-dependent bounds for small-amplitude classes

In this subsection, we sharpen the small-amplitude count by retaining its amplitude dependence. The improvement is used in the short-range comparisons of Section 8.

The earlier count is retained at the admissible value $\kappa = 3/4$. This section sharpens a scalar consequence of that count; it introduces no new arithmetic moment hypothesis.

Lemma 6.1 (Sharp linear amplitude correction). *For $0 \leq \delta \leq 5/6$ and $0 \leq x \leq 1/2$, put*

$$A = 5/6 - \delta, \quad D = 3 - 17x/9, \quad P = (2 - 8x/9)(1 - x), \quad J = AD + \delta P, \quad R = 1 - \delta + \frac{A\delta P}{2J}.$$

Then $J > 0$ and

$$R \leq 1 - \frac{2\delta}{3} - \frac{22}{81}\delta x.$$

The coefficient $22/81$ is the largest constant α for which $R \leq 1 - 2\delta/3 - \alpha\delta x$ holds on this whole rectangle.

Proof. Since $D, P > 0$,

$$J = \begin{cases} (5/6)D > 0, & \delta = 0, \\ (5/6 - \delta)D + \delta P > 0, & 0 < \delta < 5/6, \\ (5/6)P > 0, & \delta = 5/6. \end{cases}$$

Put $y = 1/2 - x$. The exact identity

$$2J - 3AP - \frac{44}{27}xJ = \frac{250x^2}{729} + \frac{\delta}{243}(352y^3 + 948y^2 + 984y + 199)$$

is nonnegative. Since

$$(1 - R) - \frac{2\delta}{3} = \frac{\delta}{6J}(2J - 3AP),$$

the claimed inequality follows, including $\delta = 0$ by continuity. For optimality take $x > 0$ and let $\delta \downarrow 0$:

$$\lim_{\delta \downarrow 0} \frac{(1 - R) - 2\delta/3}{\delta x} = \frac{2D - 3P}{6Dx} = \frac{22 - 12x}{81 - 51x}.$$

Letting $x \downarrow 0$ forces $\alpha \leq 22/81$. This proves optimality only for the stated uniform linear count bound. $\square$

Proposition 6.2 (An improved comparison for small amplitudes). *Use the earlier refined full exponent and the small-amplitude shift $\theta = g \min\{\ell/d, 1/(6\kappa)\}$, with $g = \delta x$ and $\kappa = 3/4$. Suppose the balanced geometry has $0 < \ell/h < 2/9$, $h < 1$ and the earlier monotonicity hypotheses hold. Set*

$$C_0 = -1/4 + b/6 + 5\ell/4, \quad Q = \ell(1/2 + 2\ell/h) - 11h/54.$$

For $0 \leq \delta \leq 1/8$, the exact endpoint obeys

$$E_{\text{small}}(\delta, x) \leq C_0 + \delta\{(1 - h)/2 + Qx\} \leq C_0 + \frac{1 - h + \max\{0, Q\}}{16}.$$

In particular if $Q \leq 0$, the sufficient condition replacing the earlier argument's coarse $C_0 + A_{\text{sm}}/8 < -1/250$ is simply

$$C_0 + \frac{1 - h}{16} < -\frac{1}{250}.$$

The same original $1/250$ margin is retained.

Proof. Lemma 6.1 gives

$$\begin{aligned} E_{\text{small}} &= C_0 + \frac{\delta}{2} + \ell\delta x(1/2 + 2\ell/h) - \frac{3h}{4}(1 - R) \\ &\leq C_0 + \delta \left\{ \frac{1-h}{2} + Qx \right\} \\ &\leq C_0 + \frac{1-h + \max(0, Q)}{16}. \end{aligned}$$

The last step uses $0 \leq x \leq 1/2$, $0 \leq \delta \leq 1/8$ and $h < 1$. Here $\zeta > 0$ is the buffer extending the endpoint range from $d \leq h$ to $d \leq h + \zeta$, chosen with $2\zeta < 1/1000$. The earlier monotonicity gives this extension. Under the stated $-1/250$ condition,

$$E_{\text{small}} + 2\zeta < -\frac{1}{250} + \frac{1}{1000} = -\frac{3}{1000}.$$

The original count, shift and height argument are unchanged. $\square$

6.2 The lowest-amplitude classes

In this subsection, we verify the sufficient lowest-amplitude margin required by the parameters of Section 8.2.

The original lowest-amplitude estimate is unchanged. Only its sufficient numerical margin is revised, and the change is propagated through its earlier proof rather than asserted to retain every earlier threshold.

Lemma 6.3 (Lowest-amplitude estimate with a $1/125$ threshold). *Retain the earlier direct, principal, detector, correction, moment, height and contour estimates, with their order of choices for nonprincipal characters. Suppose*

$$C_0 + (1/2 + 3\ell/2)/50 < -1/125, \quad 2\zeta < 1/1000, \quad 0 < \varepsilon_{\text{cls}} \leq 1/1000.$$

Then the lowest-amplitude contribution has exponent

$$\mathcal{E}_{\text{low}}(d) < -7/1000 < -\varepsilon_{\text{cls}} \quad (d \leq h + \zeta),$$

before the prescribed subsidiary losses. It therefore satisfies the same character-sum remainder estimate with those losses chosen inside this strict margin.

Proof. Write $\mathcal{E}_{\text{low}}$ for the exponent in the cited lowest-amplitude estimate, and $\bar{g}$ for the weighted mean prime-sum amplitude in that class. Then

$$R = 1, \quad \bar{g} \leq 1/100, \quad \mathcal{E}_{\text{low}}(h) \leq C_0 + (1/2 + 3\ell/2)/50 < -1/125.$$

Its extension to $d \leq h + \zeta$ costs at most 2ζ . Therefore

$$\mathcal{E}_{\text{low}}(d) < -\frac{1}{125} + 2\zeta < -\frac{1}{125} + \frac{1}{1000} = -\frac{7}{1000} < -\varepsilon_{\text{cls}}.$$

This is the proof of [Liu26b, Proposition 14.6] with the weaker sufficient threshold and the same classification and contour identities. Choose the remaining nonprincipal losses inside the strict margin. The principal, small-norm and height/trace estimates retain their original hypotheses. $\square$

6.3 Separate parameters for the class count and the numerator moment

In this subsection, we apply the class count with a separately established parameter. The numerator moment remains fixed, as required in Section 4.

Retain the enlarged original norm at $\kappa_m = 3/4$, justified by 7/8. A separately proved global bound supplies the count parameter κ_c below; the numerator norm, selected initial prime product, length constraints, amplitude bounds and Gauss restoration remain fixed.

Proposition 6.4 (The class-count parameter from an earlier zero-free bound). *Suppose an already proved global bound is $\beta_*(F) \leq B_{\text{seed}}$. Fix*

$$\frac{13}{18} \leq \kappa_c \leq \frac{3}{4}, \quad 2B_{\text{seed}} - 1 \leq \kappa_c.$$

For the original detected amplitude classes with $0 < \delta \leq 5/6$, $0 \leq x \leq 1/2$, and available length of the original prime product $\mathcal{L} > 1/5$, the earlier count is

$$\#\mathcal{B} \ll U^{R_{\kappa_c}(\delta, x) + \epsilon} (1 + H)^C,$$

where

$$\begin{aligned} c_\kappa &= \frac{1}{3\kappa}, \\ D_\kappa &= 3 - (1 + 2c_\kappa)x, \\ P_\kappa &= (2 - 2c_\kappa x)(1 - x), \\ J_\kappa &= (5/6 - \delta)D_\kappa + \delta P_\kappa, \\ R_\kappa &= 1 - \delta + \frac{(5/6 - \delta)\delta P_\kappa}{2J_\kappa}. \end{aligned}$$

The numerator moment parameter may simultaneously remain $\kappa_m = 3/4$, justified by the earlier 7/8 theorem.

Proof. Use the three counts of [Liu26b, Section 13.4]. Their parameter is

$$\tilde{c} = 2c_{\kappa_c} = \frac{2}{3\kappa_c}, \quad \frac{13}{18} \leq \kappa_c \leq 1 \iff \frac{2}{3} \leq \tilde{c} \leq \frac{12}{13}.$$

The crossing lengths satisfy

estimate	requested prime length	crossing upper bound
M1	$(1 - r)/2$	$5/26 < 1/5$
M2	$(1 - 2m)/(6\kappa_c)$	$\tilde{c}/12 + o(1) \leq 1/13 + o(1) < 1/5$.

Thus $\mathcal{L} > 1/5$ permits the original decrements and mesh. Keep the original finite-group coefficients, detector weights and fourth-power amplitude bound in M2. The count-side premise is

$$\beta_*(F) \leq B_{\text{seed}} \leq (1 + \kappa_c)/2,$$

while the numerator uses $\kappa_m = 3/4$. Balancing the three counts gives R_{κ_c} by [loc. cit., equation (14.3)]. The later sufficient restriction 7497/10000 does not enter this count calculation. $\square$

Lemma 6.5 (Monotonicity of the count). *For the preceding domain, $J_\kappa > 0$ and*

$$\frac{\partial R_\kappa}{\partial \kappa} = \frac{\delta x (5/6 - \delta)^2 (1 - x)^2}{3\kappa^2 J_\kappa^2} \geq 0. \tag{6.1}$$

In particular $R_{\kappa_c} \leq R_{3/4}$ whenever $\kappa_c \leq 3/4$.

Proof. Both D_κ and P_κ are positive on $13/18 \leq \kappa \leq 3/4$, $0 \leq x \leq 1/2$. At $\delta = 5/6$, $J_\kappa = \delta P_\kappa > 0$; elsewhere it is a positive sum. Differentiate the rational expression for R . Since

$$(\partial_\kappa P_\kappa)D_\kappa - P_\kappa(\partial_\kappa D_\kappa) = \frac{2x(1-x)^2}{3\kappa^2},$$

with $A = 5/6 - \delta$ one obtains

$$\partial_\kappa R = \frac{A\delta}{2J_\kappa^2} \{(\partial_\kappa P_\kappa)J_\kappa - P_\kappa \partial_\kappa J_\kappa\} = \frac{A^2 \delta x(1-x)^2}{3\kappa^2 J_\kappa^2} \geq 0.$$

This is (6.1). □

Corollary 6.6 (Separate parameters in the numerator estimate). *Use the count of Proposition 6.4 and retain*

$$\kappa_m = 3/4, \quad \theta_0 = \frac{\delta x}{6\kappa_m}, \quad z_0(m) = \frac{1-2m}{6\kappa_m}, \quad R = R_{\kappa_c}.$$

The numerator exponent and its amplified version are

$$N = \frac{3R_{\kappa_c} + 1}{4} - \frac{\theta_0}{2}, \quad N_{\text{amp}} = N - \gamma.$$

The character-sum comparison at its endpoint is

$$E_{\text{dec}}(\delta, x) = C_0 + \frac{\delta}{2} + \delta x \left\{ \ell + \frac{h/2 - l_y}{6\kappa_m} \right\} - \frac{3h}{4}(1 - R_{\kappa_c}),$$

$$C_0 = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4}, \quad E_{\text{dec}} \leq E_{b,\ell,3/4}.$$

The earlier small-amplitude comparisons using $R \leq 1 - 2\delta/3$ remain valid with the smaller count R_{κ_c} .

Proof. Proposition 14.3 of [Liu26b] accepts the class-count exponent separately from the numerator moment parameter. Lemma 6.5 gives

$$R_{\kappa_c} \leq R_{3/4}, \quad R_{\kappa_c} \leq (6\kappa_m - 2\delta x)\mathcal{L} \quad \text{whenever} \quad R_{3/4} \leq (6\kappa_m - 2\delta x)\mathcal{L}.$$

Keep $\kappa_m = 3/4$, θ_0 and z_0 fixed. In the earlier full exponent [*loc. cit.*, equation (14.7)], substitution gives

$$N = \frac{3R_{\kappa_c} + 1}{4} - \frac{\theta_0}{2},$$

$$E_{\text{dec}} - E_{b,\ell,3/4} = \frac{3h}{4}(R_{\kappa_c} - R_{3/4}) \leq 0.$$

Both AFE halves, all independent derivatives and the enlarged norm retain their original moment-side parameters and length constraints. The coarse count inequalities therefore remain valid. □

Corollary 6.7 (Available prime-product length and the separated parameters). *Under Proposition 6.4, retain $\kappa_m = 3/4$. Suppose*

$$\frac{7}{20} \leq \delta \leq \frac{3}{5}, \quad \frac{2}{5} \leq x \leq \frac{1}{2}, \quad \mathcal{L} \geq \frac{21}{100}.$$

Then the prime-length condition for the numerator estimate has the uniform margin

$$(6\kappa_m - 2\delta x)\mathcal{L} - R_{\kappa_c}(\delta, x) \geq \frac{24683}{162000} > 0. \tag{6.2}$$

The unamplified affine slope in the scale exponent d of Section 2 satisfies

$$S = \frac{3R_{\kappa_c} + 1}{4} + \frac{\theta_0}{2} - \frac{17}{50} \leq \frac{52}{75} < 1. \tag{6.3}$$

Proof. By Lemmas 6.5 and 6.1,

$$R_{\kappa_c} \leq R_{3/4} \leq 1 - \frac{2\delta}{3} - \frac{22}{81}\delta x \leq 1.$$

Since $6\kappa_m - 2\delta x > 0$, the left side of (6.2) is at least

$$\left(\frac{9}{2} - 2\delta x\right) \frac{21}{100} - 1 + \frac{2\delta}{3} + \frac{22}{81}\delta x = -\frac{11}{200} + \delta \left(\frac{2}{3} - \frac{601x}{4050}\right).$$

The last expression increases with δ and decreases with x on the displayed rectangle. At $(\delta, x) = (7/20, 1/2)$ it equals 24683/162000. Finally, $\theta_0 = 2\delta x/9 \leq 1/15$, so

$$S \leq 1 + \frac{1}{30} - \frac{17}{50} = \frac{52}{75}.$$

□

The sufficient restriction $\kappa \geq 7497/10000$ in the earlier numerator comparison is thus not imposed on κ_c . The numerator moment, initial prime product and contour parameter continue to use $\kappa_m = 3/4 > 7497/10000$. The class count uses its full proved interval, while (6.2)–(6.3) verify these two compatibility conditions directly on the class rectangle where the new estimate is applied. The complementary amplitude-class and parameter-range comparisons are treated separately in Section 8.

7 Adding the moment and contour savings

In this section, we apply the norm of Section 4 on the contours of Section 5. A prime-product length chosen separately for each dyadic sum gives a common saving over all ordinary lengths; Section 8 uses that saving in the full comparison.

7.1 Combining the moment bound and contour displacement

In this subsection, we apply the selected moment on the displaced contour and minimize the resulting gain over ordinary lengths. Section 8.2 uses this common gain.

Theorem 4.9 is uniform over the independent AFE weights on every contour of Lemma 5.2. That lemma moves the whole integrand before absolute values. The selected-moment saving therefore remains available on the displaced contour with the same initial prime product and original coefficients.

Lemma 7.1 (The combined exponent on a dyadic sum). *Retain Theorem 4.9, the class-count exponent R , the initial prime product Q_0 , and a disjoint product with selected first-moment saving $\gamma > 0$. Put*

$$g = \delta x, \quad \kappa_m = 3/4, \quad \theta_0 = \frac{g}{6\kappa_m} = \frac{2\delta x}{9}, \quad z_0(m) = \frac{1 - 2m}{6\kappa_m}, \quad \mathcal{K} = \frac{3R + 1}{4}.$$

For a dyadic sum of length $D = U^m$, $m < 1/2$, and every admissible shift $0 \leq \theta \leq \delta/2$, the full comparison exponent is

$$\mathcal{E}_0(d) + (\theta - \theta_0)\{d(1 - m) - l_y\} - d\gamma, \tag{7.1}$$

where $\mathcal{E}_0(d)$ is the unamplified reference exponent. This holds for both halves of the approximate functional equation, with the actual conductors, fixed-place factors and simultaneous strict-error parameters retained. Neither the choice nor the length constraint of Q_0 changes.

Proof. Lemmas 5.1–5.2 keep the two AFE weights and their independent derivatives in the norm family of Theorem 4.9. Fix the original amplitude class, initial prime product and extra product before differentiating these weights. Their lower bounds are independent of the contour displacement.

The selected fourth moment saves $2gz_0 + 4\gamma$ in its exponent. Hölder with the class count gives the first-moment exponent

$$\begin{aligned} \frac{3R}{4} + \frac{1 - 2gz_0 - 4\gamma}{4} - \theta m &= \mathcal{K} - \frac{gz_0}{2} - \gamma - \theta m \\ &= \mathcal{K} - \frac{\theta_0}{2} - (\theta - \theta_0)m - \gamma, \quad \frac{gz_0}{2} = \frac{\theta_0(1 - 2m)}{2}. \end{aligned}$$

Arbitrarily small exponent losses and the fixed seminorm and height factors are understood. For $U = Z^d$, $D = U^m$ and $Y = Z^{l_y}$, the functional equation, AFE coefficients and outer Mellin factor therefore give the relative factor

$$U^{\theta - \theta_0} D^{-(\theta - \theta_0)} Y^{-(\theta - \theta_0)} U^{-\gamma} = Z^{(\theta - \theta_0)\{d(1-m) - l_y\} - d\gamma}.$$

This proves (7.1) for the first half.

For the dual half, move its original holomorphic expression first, then conjugate it as a whole. The normalized dual weight in Lemma 5.1 gives the same scalar $D^{-\theta}$ and aligns its character with the original prime coefficients. Retain each actual conductor. For the finite prime set $\mathcal{P}_{\text{str}}$ of a simultaneous choice of parameters for the strict-error terms, the entire conductor contribution is bounded by

$$\prod_{\mathfrak{p} \in \mathcal{P}_{\text{str}}} (q_{\mathfrak{p}}^{\theta} q_{\mathfrak{p}}^{-(j_{\mathfrak{p}}-1)\theta}) = \prod_{\mathfrak{p} \in \mathcal{P}_{\text{str}}} q_{\mathfrak{p}}^{-(j_{\mathfrak{p}}-2)\theta} \leq 1 \quad (j_{\mathfrak{p}} \geq 2, \theta \geq 0).$$

Fixed-place factors have finite absolute mass on the enlarged strip. All contour displacements precede the local s -contour displacement, absolute values and conductor separation. The same positive norm controls the later derivatives; no amplitude lower bound is required of a differentiated weight. $\square$

Lemma 7.2 (An ideal-counting bound for short sums). *Suppose the count parameter is at most $3/4$, so its count exponent is bounded by the sharp count of Lemma 6.1. On $3/10 \leq \delta \leq 3/5$, $2/5 \leq x \leq 1/2$, put $N = (3R + 1)/4 - \theta_0/2$ and $m_{\text{cut}} = 3/40$. The reference estimate ideal-count exponent for $m \leq m_{\text{cut}}$ is at most*

$$R + (1/2 - \theta_0)m_{\text{cut}} \leq N - \frac{23}{2700}.$$

For $m \geq m_{\text{cut}}$, the initial prime product required by the unamplified M2 estimate has length at most $z_0(m_{\text{cut}}) = 17/90 < 21/100$.

Proof. The ideal-count expression increases with m , since $\theta_0 < 1/2$. The sharp count bound gives

$$\begin{aligned} N - \{R + (1/2 - \theta_0)m_{\text{cut}}\} &= \frac{1 - R}{4} - \frac{3}{80} - \frac{17\theta_0}{40} \\ &\geq \frac{\delta}{6} - \frac{3}{80} - \frac{43\delta x}{1620} \\ &\geq \frac{497\delta}{3240} - \frac{3}{80} \geq \frac{23}{2700}. \end{aligned}$$

Also

$$z_0(3/40) = \frac{2}{9} \left(1 - \frac{6}{40}\right) = \frac{17}{90}.$$

At exhaustion, the earlier selection rule and $z_0 \leq \mathcal{L}$ give total original gain at least $g\mathcal{L} \geq gz_0$. $\square$

Lemma 7.3 (A combined bound with variable prime-product length). *Retain the balanced lengths and assume*

$$\frac{57}{100} \leq \lambda = \frac{l_y}{h} \leq \frac{59}{100}, \quad \frac{3}{10} \leq \delta \leq \frac{3}{5}, \quad \frac{2}{5} \leq x \leq \frac{1}{2}.$$

At every scale under consideration, assume

$$\mathcal{L} = \ell/d \geq 21/100.$$

Put $m_0 = 1 - \lambda$, $m_c = 25/62$, and

$$W_{\text{cap}}(m) = \min \left\{ \frac{7}{54}(1 - 2m), \frac{8m - 3}{9} \right\}, \quad 3/8 \leq m \leq 1/2.$$

The remaining positive-amplitude factors have mean at least $a = \delta(189x - 25)/139$. Using $\theta = 0$ for $m \leq m_0$ and $\theta = \delta/2$ for $m \geq m_0$, the limiting gain divided by d at $d = h$ is

$$F(m) = \frac{aW_{\text{cap}}(m)}{2} + \begin{cases} \theta_0(m_0 - m), & m \leq m_0, \\ (\delta/2 - \theta_0)(m - m_0), & m \geq m_0. \end{cases}$$

Consequently,

$$\begin{aligned} \Gamma(\delta, x) &= \min_{3/8 \leq m \leq 1/2} F(m) = \min\{\Gamma_L, \Gamma_N\}, \\ \Gamma_L &= \theta_0(5/8 - \lambda), \quad \Gamma_N = \frac{7a(2\lambda - 1)}{108}. \end{aligned} \tag{7.2}$$

Here $\Gamma_L = F(3/8)$ and $\Gamma_N = F(m_0)$ are the two candidate minimum values. Moreover,

$$\lambda \leq 11/19 \implies \Gamma_L \geq \Gamma_N, \quad \Gamma = \Gamma_N.$$

At $\lambda = 11/19$,

$$\begin{aligned} b &= 13\ell/2 - 1, \quad h = 19\ell/4, \quad l_y = 11\ell/4, \\ \Gamma_N &= \frac{7\delta(189x - 25)}{95076}, \quad \Gamma_N|_{x=1/2} = 7\delta/1368. \end{aligned}$$

These are limiting lengths and gains; the norm theorem is applied only with the strict margins of Proposition 7.4.

Proof. The largest initial prime product at $m = 3/8$ has width $1/18$. Lemma 4.2 with $L_0 = 21/100$ therefore gives

$$a \geq \delta \frac{(21/100)x - 1/36}{21/100 - 1/18} = \delta \frac{189x - 25}{139}.$$

The limiting width uses the branch for restoring the comparison product below m_c and the branch determined by the coupled length constraint above it. One has $3/8 < m_c < m_0 < 1/2$. The three successive slopes of F are

$$\frac{4a}{9} - \theta_0 > 0, \quad -\frac{7a}{54} - \theta_0 < 0, \quad \frac{\delta}{2} - \theta_0 - \frac{7a}{54} > 0.$$

The first sign follows from $239x - 50 > 0$; the last holds uniformly on the stated x interval, and the middle is immediate. Thus the only possible minimum points are $m = 3/8$ and $m = m_0$. Their gains are respectively

$$\Gamma_L = \theta_0(m_0 - 3/8), \quad \Gamma_N = \frac{7a(1 - 2m_0)}{108}.$$

These are (7.2). When $\lambda \leq 11/19$, put $y = 1/2 - x \in [0, 1/10]$ and $z = 11/19 - \lambda \geq 0$. The exact identity

$$\frac{\Gamma_L - \Gamma_N}{\delta} = \frac{175y}{47538} + z \left(\frac{19}{108} - \frac{997y}{2502} \right) \geq 0$$

gives the stronger assertion. The right endpoint $m = 1/2$ cannot be smaller, by the last positive slope. $\square$

Proposition 7.4 (Admissible lengths and contours with a uniform saving). *Under Lemma 7.3, suppose*

$$E(\delta, x) - h\Gamma(\delta, x) \leq -\Delta_{\text{cmp}} < 0$$

uniformly on the stated class rectangle. Retain strict margins for the complementary comparisons, geometric inequalities, short-sum estimate and earlier parameter ranges, and decrease Δ_{cmp} if necessary to lie inside all these margins. Choose $\zeta > 0$ also so that

$$\ell/(h + \zeta) \geq 21/100.$$

Then admissible prime-product lengths and contours give

$$\mathcal{E}(d) \leq -\Delta_{\text{cmp}} + C\varepsilon + 2\zeta + \epsilon_{\text{rem}} < 0 \quad (0 < d \leq h + \zeta),$$

after choosing ε, ζ and the remaining losses with $C\varepsilon + 2\zeta + \epsilon_{\text{rem}} < \Delta_{\text{cmp}}$. The coefficient families and order of analytic operations are unchanged.

Proof. Write $E - h\Gamma \leq -\Delta_{\text{cmp}}$ for the fixed comparison margin. Choose $0 < \varepsilon < 1/1000$ and set

$$m_L = 3/8 + \varepsilon, \quad W_\varepsilon(m) = (1 - \varepsilon)W_{\text{cap}}(m), \quad m_L \leq m \leq 9/20.$$

Apply the norm with $m_- = r = m$. Since $W_{\text{cap}}(m) \geq 8\varepsilon/9$,

$$\begin{aligned} \frac{7}{9}(1 - 2m) - 6W_\varepsilon(m) &\geq 6\varepsilon W_{\text{cap}}(m) \geq 16\varepsilon^2/3, \\ 1 - \left(\frac{5}{2} - 4m + \frac{9}{2}W_\varepsilon(m)\right) &\geq \frac{9}{2}\varepsilon W_{\text{cap}}(m) \geq 4\varepsilon^2. \end{aligned}$$

Every selected product fits, because

$$W_{\text{cap}}(m) \leq 7/279 < 2(2/5)(21/100) - 1/18.$$

Use ideal counting below $3/40$, central unamplified M2 up to m_L , and outer unamplified M2 above $9/20$. The selected and lower-edge losses are at most $a\varepsilon W_{\text{cap}}/2$ and $\theta_0\varepsilon$. At the upper endpoint the excess over Γ_N is at least $7\delta/2160$: with $y = 1/2 - x \geq 0$, $z = \lambda - 57/100 \geq 0$,

$$\begin{aligned} (1/2 - 2x/9)(\lambda - 11/20) - \frac{7(189x - 25)(2\lambda - 1)}{108 \cdot 139} - \frac{7}{2160} \\ = \frac{997}{2502}yz + \frac{4199}{250200}y + \frac{35}{108}z \geq 0. \end{aligned} \tag{7.3}$$

This holds on the full stated rectangle and λ interval.

Lemma 7.1 applies on both sides of m_0 . For each fixed selected dyadic sum, $S > 1/5$ and $\gamma < 1/250$ give

$$\begin{aligned} S - \theta_0(1 - m) - \gamma &> \frac{1}{5} - \frac{1}{15} \frac{5}{8} - \frac{1}{250} > \frac{3}{20}, \\ S + (\delta/2 - \theta_0)(1 - m) - \gamma &> \frac{1}{5} - \frac{1}{250} > \frac{19}{100}. \end{aligned}$$

Both slopes are below 2, so

$$d \leq h + \zeta \implies \mathcal{E}(d) \leq \mathcal{E}(h) + 2\zeta.$$

At m_0 apply this separately to both contours; their slopes need not match. The unamplified ranges use their finite endpoint bounds, the dominating affine extension and normally convergent tails. Smaller parameter norms use the earlier comparison.

All losses can be chosen to satisfy

$$C\varepsilon + 2\zeta + \epsilon_{\text{rem}} < \Delta_{\text{cmp}},$$

where C depends on the fixed compact data and ϵ_{rem} collects the remaining positive exponent losses, including the local loss of Proposition 4.8. Choose the step, mesh, finite allocation bound and weighted orders before later Fourier labels and external heights. Original prime lower bounds precede differentiation. Count the logarithmically many dyadic sums inside the reserved loss and remove tails in the original order. This proves the full parameter-dependent estimate without continuity across a fixed lower band in d . $\square$

8 The algebraic boundary and the successive count bounds

In this section, we first obtain a strict rational bound from the earlier 7/8 result. We then prove the algebraic endpoint by successive count improvements, and determine the optimum of the comparison system. Section 9 gives the resulting theorem over arbitrary fields.

8.1 An explicit rational bound

In this subsection, we obtain the rational bound that starts the successive count improvements in Section 8.2.

Here the ideal-count cutoff is $m_{\text{seed}} = 1/10$. The final comparison of Section 7.1 uses the smaller cutoff $m_{\text{cut}} = 3/40$ in order to obtain its uniform 23/2700 margin on the larger class rectangle. Both fit the available prime length:

$$z_0(1/10) = 8/45 < 21/100, \quad z_0(3/40) = 17/90 < 21/100.$$

Appendix A verifies their separate complete-sum inequalities.

Before taking an algebraic limit, there is an explicit interior example using the new split according to the direct-estimate range and the original theta detector. Keep $\kappa_c = \kappa_m = 3/4$ and set

$$\begin{aligned} B_{\text{seed}} &= \frac{437279}{500000} = .874558, & b &= \frac{167}{2500}, & \ell &= \frac{63163}{375000}, & h &= \frac{196513}{250000}, & l_y &= \frac{336887}{750000}, \\ m_L &= \frac{393}{1000}, & m_R &= \frac{1097}{2500}, & W &= \frac{317}{20000}, & \varepsilon_c &= 10^{-4}, & \varepsilon_{\text{amp}} &= 10^{-6}, & \varepsilon_{\text{cmp}} &= 10^{-7}. \end{aligned}$$

The actual norm admissibility conditions are

$$\frac{7}{9}(1 - 2m_R) - 6W = \frac{1}{10000}, \quad 1 - \left(\frac{5}{2} - 4m_L + \frac{9}{2}W \right) = \frac{27}{40000}.$$

Here

$$1 - 2m_R - 9W = -81/4000 < 0.$$

Theorem 4.9 therefore uses the direct and complementary Poisson estimates, with the actual outer D retained, in place of that earlier sufficient length condition.

Use

$$\mathcal{R} = [7/20, 3/5] \times [9/20, 1/2], \quad \theta_0 = 2\delta x/9, \quad \theta_- = \varepsilon_c, \quad \theta_+ = \delta/2 - \varepsilon_c.$$

Put $t_- = \theta_0 - \varepsilon_c$ and $t_+ = \delta/2 - \varepsilon_c - \theta_0$. With $z_L = 2(1 - 2m_L)/9$, put

$$a = \delta \left\{ \frac{1}{2} - \frac{(1/2 - x)(21/100)}{21/100 - z_L} \right\}, \quad \gamma = aW/2 - \varepsilon_{\text{amp}}.$$

The exact calculation in Appendix A checks 39 strict rational-function signs for these parameters. These include available length of the additional prime product and restoration of the comparison product; the complete left, middle and right comparisons on $\mathcal{R}$; the earlier comparison on its three complementary rectangles; both endpoints of every near- h affine comparison exponent; the short ideal-count comparison; and all remaining comparison geometry. The original lowest-amplitude estimate 1/100 and 1/20 margins for parameters with small norm hold, while the amplitude-sensitive small-amplitude calculation preserves the original 1/250 threshold.

Theorem 8.1 (An explicit boundary from the enlarged norm). *The finite-order Hecke family over the fixed field F satisfies*

$$\beta_*(F) \leq \frac{437279}{500000} = 0.874558.$$

Proof. Theorems 4.9 and Lemma 5.2 supply the analytic estimates. The completed Gauss sums are covered by

$$\begin{cases} m_f \leq 1/100 : & \text{Proposition 4.6,} \\ m_f > 1/100 : & \text{Proposition 4.8.} \end{cases}$$

For these rational parameters,

$$0 < \zeta \leq 1/2000 \implies \frac{\ell}{h + \zeta} \geq \frac{63163}{294957} > \frac{21}{100}.$$

Thus Lemma 4.2 applies with $L_0 = 21/100$. Use ideal counting for $m \leq 1/10$, unamplified M2 on the two outer ranges, and the selected norm on $[m_L, m_R]$. Write $\mathcal{E}_{\text{chosen}}$ for the full corrected exponent. The exact checks of Appendix A give

$$\begin{aligned} \mathcal{E}_{\text{chosen}}(h) &< -\varepsilon_{\text{cmp}}, & 0 < \partial_d \mathcal{E}_{\text{chosen}} &< 2, & S &> 1/5, \\ \frac{1}{5 \cdot 80} - \frac{1}{500} &> \varepsilon_{\text{cmp}}, & 2\zeta &< \varepsilon_{\text{cmp}}/2. \end{aligned}$$

Hence the two endpoints of the near- h band and $d \leq h - 1/80$ have strict negative exponents. Apply prime lower bounds before derivatives, retaining both actual-conductor AFE halves and strict tuples. Choose all subsidiary losses within these strict margins. The earlier direct estimate and nonvanishing principal correction then give

$$\beta_*(F) \leq B(\ell) = B_{\text{seed}}.$$

The 7/8 bound justifies both moment parameters used in this interior construction. $\square$

8.2 The algebraic endpoint

In this subsection, we determine the algebraic limiting parameters and prove the resulting bound. The field descent is given in Section 9.

The amplified norm and contour displacement apply to the same ordinary dyadic sum. We insert their sum in the full comparison of Section 2.

Definition 8.2 (The combined algebraic endpoint). Let r_* be the unique root in $(9/20, 23/50)$ of

$$218582r^3 - 376469r^2 - 11376r + 61560 = 0.$$

Put

$$\ell_* = \frac{3410r_*^2 - 7127r_* + 2520}{2286r_*^2 - 5011r_* + 1800}, \quad \kappa_* = \frac{5}{6} - \frac{\ell_*}{2}, \quad B_* = \frac{11}{12} - \frac{\ell_*}{4}.$$

To define b_* , use independent variables r, k and

$$\begin{aligned} n_b &= -6858kr^2 + 15033kr - 5400k + 600r^2 - 1837r + 720, \\ n_e &= -702kr^2 + 34995kr - 40500k - 408r^2 - 3775r + 5400, \\ n_0 &= 4806kr^2 - 10887kr + 8100k - 696r^2 + 1403r - 1080. \end{aligned}$$

Set

$$b_* = -\frac{\ell_* \partial_r n_e(r_*, \kappa_*) + \partial_r n_0(r_*, \kappa_*)}{\partial_r n_b(r_*, \kappa_*)},$$

where the partial derivatives hold k fixed. Numerically,

$$\begin{aligned} r_* &= 0.45057074672406073136\dots, \\ \ell_* &= 0.16879343589254482845\dots, \\ b_* &= 0.09179235229895088958\dots, \\ \kappa_* &= 0.74893661538706091910\dots, \\ B_* &= 0.87446830769353045955\dots \end{aligned}$$

In particular $\kappa_* = 2B_* - 1$ exactly.

Lemma 8.3 (The comparison inequality and its strict margins). *Put*

$$Q_{\kappa,b,\ell}(\delta, x) = E_\kappa(\delta, x) - \frac{7}{108}a(\delta, x)(2l_y - h), \quad \mathcal{R}_* = [7/20, 3/5] \times [2/5, 1/2].$$

For the parameters of Definition 8.2,

$$\begin{aligned} Q_{\kappa_*, b_*, \ell_*} &\leq 0 \quad \text{on } \mathcal{R}_*, \\ Q_{\kappa_*, b_*, \ell_*}(\delta, x) &= 0 \iff (\delta, x) = (r_*, 1/2). \end{aligned}$$

Uniformly on $\mathcal{R}_*$ and

$$\begin{aligned} .7489 &\leq \kappa \leq .7495, & .16876 &\leq \ell \leq .16880, \\ .0917923522 &\leq b \leq .0917923524, \end{aligned}$$

one has

$$\partial_\ell Q > 3/4, \quad 0 \leq \partial_\kappa Q < 1/100.$$

The remaining geometric, complementary-class, ordinary-sum and short-sum comparisons have uniform strict margins on this family.

Proof. Write $\mathcal{R}_* = [7/20, 3/5] \times [2/5, 1/2]$ and use the intervals I_b, I_ℓ, I_κ of Appendix A.3. At $x = 1/2$,

$$Q_{k,b,\ell}(\delta, 1/2) = \frac{bn_b(\delta, k) + \ell n_e(\delta, k) + n_0(\delta, k)}{432(54k\delta - 75k - 6\delta + 10)}.$$

Polynomial division by (1.1) gives

$$n_b(r_*, \kappa_*) = 0, \quad \ell_* n_e(r_*, \kappa_*) + n_0(r_*, \kappa_*) = 0.$$

Put $N_*(\delta) = b_* n_b(\delta, \kappa_*) + \ell_* n_e(\delta, \kappa_*) + n_0(\delta, \kappa_*)$. Definition 8.2 and the exact interval signs give

$$\begin{aligned} N_*(r_*) &= N'_*(r_*) = 0, & N_*(\delta) &= A_Q(\delta - r_*)^2, \\ A_Q &= (-6858\kappa_* + 600)b_* + (-702\kappa_* - 408)\ell_* \\ &\quad + 4806\kappa_* - 696 > 0, \\ 54k\delta - 75k - 6\delta + 10 &< 0 \quad (\delta \in [7/20, 3/5], k \in I_\kappa). \end{aligned}$$

Hence $Q_*(\delta, 1/2) \leq 0$, with equality only at $\delta = r_*$. The exact root isolation is

$$\begin{aligned} .450570746724060730 &< r_* < .450570746724060733, \\ .87446830769352 &< B_* < .87446830769355. \end{aligned}$$

Appendix A proves the following rational signs:

quantity	parameter domain
$\partial_x Q > 0$	$\mathcal{R}_* \times [.7489366153, .7489366155] \times I_b \times I_\ell$
$\partial_\ell Q > 3/4$	$\mathcal{R}_* \times \{.7489\} \times I_b \times I_\ell$

Here each cleared numerator is affine in (b, ℓ) ; its four corners suffice. Exact Bernstein coefficients and rational subdivisions retain the full κ coordinate. In particular,

$$x < 1/2 \implies Q_*(\delta, x) < Q_*(\delta, 1/2) \leq 0.$$

For the larger count interval,

$$\partial_\kappa \partial_\ell Q = \frac{9}{8} \partial_\kappa R \geq 0 \implies \partial_\ell Q > 3/4,$$

and

$$0 \leq \partial_\kappa Q = \frac{h(5/6 - \delta)^2 \delta x(1-x)^2}{4\kappa^2 J^2} < \frac{25600}{2686321} < \frac{1}{100},$$

$$h \leq \frac{1 + .0917923524 + 3(.16880)}{2} < \frac{4}{5}.$$

For completeness, with $\mathcal{G} = [1/8, 5/6] \times [0, 1/2]$, the complementary rectangles are

$$\begin{aligned} \mathcal{C}_1 &= [1/8, 5/6] \times [0, 2/5], \\ \mathcal{C}_2 &= [1/8, 7/20] \times [2/5, 1/2], \\ \mathcal{C}_3 &= [3/5, 5/6] \times [2/5, 1/2]. \end{aligned}$$

The same exact tests, at the adverse count endpoints, give

$$\left. \begin{array}{c} \mathcal{G} \\ \bigcup_i \mathcal{C}_i \\ \mathcal{R}_* \end{array} \right| \begin{array}{l} E < 1/500 \\ E < -10^{-5} \\ S > 1/5, \quad E + h\Delta_{\text{short}} < -10^{-5}. \end{array}$$

The ratio $\ell/(h + \zeta) = 2\ell/(1 + b + 3\ell + 2\zeta)$ increases with ℓ and decreases with b, ζ . On $I_b \times I_\ell$,

$$0 < \zeta \leq 1/2000 \implies \frac{\ell}{h + \zeta} \geq \frac{843800000}{3997680881} > \frac{21}{100}.$$

This verifies the prime-length hypothesis of Lemma 7.3 and Proposition 7.4 throughout the parameter family. The ordinary-gain and prime-length conditions are

$$\begin{aligned} a > 0, \quad 4a/9 - \theta_0 > 0, \quad \delta/2 - \theta_0 - 7a/54 > 0, \quad \Gamma_L > \Gamma_N, \\ 2x(21/100) - 1/18 > 7/279, \quad 57/100 < l_y/h < 59/100, \quad l_y/h < 11/19. \end{aligned}$$

For $m_{\text{cut}} = 3/40$,

$$N - \{R + (1/2 - \theta_0)m_{\text{cut}}\} \geq \delta(1/6 - 43x/1620) - 3/80 \geq 23/2700, \quad z_0(m_{\text{cut}}) = 17/90 < 21/100.$$

The last lower bound holds already for $\delta \geq 3/10$. The remaining geometric, principal and small-amplitude conditions are the explicit inequalities of Appendix A, using the 1/125 threshold of Lemma 6.3. Their analytic application is Lemma 7.1 and Proposition 7.4. $\square$

Theorem 8.4 (The algebraic zero-free boundary). *Retain the established inputs of Section 2, the amplified norm of Theorem 4.9 and the contours of Lemma 5.2. The 7/8 bound and Theorem 8.1 give*

$$\beta_*(F) \leq B_{\text{seed}} = .874558 \implies \boxed{\beta_*(F) \leq B_* = 0.87446830769353045955\dots}$$

Proof. Put $\kappa_0 = 187279/250000 = 2B_{\text{seed}} - 1$ and fix $b = b_*$. For $\kappa \in (\kappa_*, \kappa_0]$, set

$$\ell(\kappa) = \ell_* - \frac{\kappa - \kappa_*}{25}.$$

This segment lies in the family of Lemma 8.3, and

$$\frac{d}{d\kappa} Q_{\kappa, b_*, \ell(\kappa)} = \partial_\kappa Q - \frac{1}{25} \partial_\ell Q < -\frac{1}{100}.$$

Together with $Q_{\kappa_*, b_*, \ell_*} \leq 0$, this gives, uniformly on the complete class rectangle,

$$Q_{\kappa, b_*, \ell(\kappa)} < -\nu_\kappa, \quad \nu_\kappa = \frac{\kappa - \kappa_*}{100} > 0.$$

Let ϵ_{rem} collect the remaining positive exponent losses. Choose the losses, in the order of Proposition 7.4, so that

$$0 < \varepsilon < 1/1000, \quad C\varepsilon + 2\zeta + \epsilon_{\text{rem}} < \nu_\kappa.$$

Put $m_a = 3/8 + \varepsilon$, $m_b = 9/20$. Here C depends only on the fixed compact data. The choice of lengths and contours in Proposition 7.4 uses

ordinary length exponent	estimate
$m < 3/40$	ideal counting
$3/40 \leq m < m_a$	unamplified M2 on the central contour
$m_a \leq m \leq m_b$	selected moment and contour saving
$m > m_b$	unamplified M2 on the outer contour.

On the selected interval take $m_- = r = m$ and $W(m) = (1 - \varepsilon)W_{\text{cap}}(m)$. Then

$$\frac{7}{9}(1 - 2m) - 6W(m) \geq \frac{7}{90}\varepsilon > 0, \quad 1 - \left(\frac{5}{2} - 4m + \frac{9}{2}W(m)\right) \geq 4\varepsilon^2 > 0.$$

The ideal gain changes by $O(\varepsilon)$; the outer range has the strict margin of Proposition 7.4. On each selected dyadic sum, the slopes of the full comparison exponent satisfy

$$\begin{aligned} S - \theta_0(1 - m) - \gamma &> \frac{1}{5} - \frac{1}{15} \frac{37}{40} - \frac{3}{10} \frac{7}{279} \frac{1}{2} > \frac{1}{8}, \\ S + (\delta/2 - \theta_0)(1 - m) - \gamma &> \frac{1}{5} - \frac{3}{10} \frac{7}{279} \frac{1}{2} > \frac{1}{8}. \end{aligned}$$

Both slopes are below 2. Hence the extension to $d \leq h + \zeta$ costs at most 2ζ . Choose ε , then the mesh, smooth-weight, height and ζ losses inside ν_κ . Apply the norm before differentiation and retain both actual-conductor AFE halves. Proposition 7.4 handles the complementary length ranges and tails; the earlier estimates apply for $d \leq h - 1/80$. The lowest-amplitude, principal and direct comparisons retain their checked hypotheses.

The continuation argument now gives

$$\beta_*(F) \leq B(\kappa) = B_* + \frac{\kappa - \kappa_*}{100}, \quad 2B(\kappa) - 1 - \kappa_* = \frac{\kappa - \kappa_*}{50}.$$

Starting from $B_0 = B_{\text{seed}}$ and setting $\kappa_n = 2B_n - 1$,

$$\beta_*(F) \leq B_n \implies \beta_*(F) \leq B_{n+1}, \quad B_n - B_* = 50^{-n}(B_{\text{seed}} - B_*) \longrightarrow 0.$$

Every finite step has strictly positive margins and its own threshold, while $\kappa_m = 3/4$ remains justified by $7/8$. Taking the infimum over n proves the theorem. $\square$

Theorem 8.5 (Optimality under the coupled norm length constraints). *Fix the detector, class count, balanced-length hypotheses and direct theta estimate used above, and the norm of Theorem 4.9. On each amplitude class, allow dyadic-sum-dependent prime lengths and contours satisfying*

$$\begin{aligned} \kappa_m = 3/4, \quad 0 \leq W(m) \leq \frac{7}{54}(1 - 2m), \quad 0 \leq \theta(m) \leq \delta/2, \\ 1/6 \leq \ell \leq 1/5, \quad B(\ell) = 11/12 - \ell/4. \end{aligned}$$

Every valid sequence of count improvements in this comparison system, starting at $B_0 = B_{\text{seed}}$, with

$$13/18 \leq \kappa_n \leq 3/4, \quad \kappa_n \geq 2B_n - 1,$$

satisfies

$$B_n \geq B_* \quad (n \geq 0), \quad \liminf_{n \rightarrow \infty} B_n \geq B_*.$$

Theorem 8.4 attains B_* as a limit. This optimality keeps the detector, moment estimates, moment parameter and admissible norm lengths fixed.

Proof. The retained geometry gives

$$0 < b < \frac{3(1-3\ell)}{8} \leq \frac{3}{16}, \quad m_0 = \frac{4\ell}{1+b+3\ell} > \frac{32}{81} > \frac{3}{8}, \quad \frac{1}{2} - m_0 = \frac{1+b-5\ell}{2(1+b+3\ell)} > 0.$$

At the critical class $(\delta, x) = (r_*, 1/2)$,

$$h(1-m_0) - l_y = 0, \quad \gamma \leq \frac{\delta W}{4}, \quad h\gamma \leq \frac{7\delta}{216}(2l_y - h).$$

The ideal-count alternative is also weaker at this length. Put

$$\theta_0 = r_*/9, \quad N = (3R+1)/4 - \theta_0/2, \quad I = R + (1/2 - \theta_0)m_0.$$

Since $R \geq 1 - r_*$ and $r_* < 23/50$,

$$\begin{aligned} I - N &= \frac{R-1}{4} + \frac{m_0}{2} + \theta_0(1/2 - m_0) \\ &> \frac{16}{81} - \frac{23}{200} = \frac{1337}{16200} > \frac{2}{25}. \end{aligned}$$

At m_0 the full contour contribution vanishes for this alternative as well. The unamplified moment omits the nonnegative selected saving. Thus neither alternative improves the selected comparison, and $Q_{\kappa,b,\ell}(r_*, 1/2) \leq 0$ is necessary. At $\kappa = \kappa_*$ the coefficient of b vanishes, and

$$Q_{\kappa_*,b,\ell}(r_*, 1/2) = q_*(\ell - \ell_*), \quad q_* > 0.$$

Since $\partial_\kappa Q = (3h/4)\partial_\kappa R \geq 0$,

$$\kappa \geq \kappa_*, \quad \ell > \ell_* \implies Q_{\kappa,b,\ell}(r_*, 1/2) > 0.$$

Every preceding bound $X \geq B_*$ permits only $\kappa \geq 2X - 1 \geq \kappa_*$. No finite permitted iteration can cross B_* ; neither can a limit of such bounds. The constructive sequence of Theorem 8.4 reaches exactly this endpoint. $\square$

9 Arbitrary number fields and proof of the main theorem

In this section, we verify that the new estimates apply over every fixed field containing μ_{12} . We then descend from $K(\zeta_{12})$ to K and complete the proof of Theorem 1.1.

9.1 The estimates over fields containing the twelfth roots of unity

In this subsection, we check the new norm and contour estimates over a fixed field containing μ_{12} . Section 9.2 then gives the arbitrary-field theorem.

The general-field arithmetic is supplied by [Liu26b, Part III]. We verify that every new estimate above uses only that arithmetic and fixed-field estimates. The descent is proved in Section 9.2.

Proposition 9.1 (The combined comparison over number fields). *For every fixed $F \supset \mu_{12}$, the norm, smooth-weight, contour and combined-comparison estimates used in Theorem 8.4 hold with the same numerical exponents and admissibility inequalities. Consequently,*

$$\beta_*(F) \leq B_* = 0.87446830769353045955 \dots,$$

with B_* as in Definition 8.2. Constants, derivative orders and thresholds may depend on F and its fixed arithmetic data.

Proof. Fix $F \supset \mu_{12}$ and the arithmetic data of [Liu26b, Theorem 18.2]. Enlarge S to contain the places above 6, the different and the fixed exceptional local data. All constants below may depend on these fixed choices.

Local arithmetic and summation lengths. For $\mathfrak{p} \notin S$,

$$\mu_{12} \hookrightarrow (\mathcal{O}_F/\mathfrak{p})^\times, \quad 12 \mid (q_{\mathfrak{p}} - 1), \quad \left(\frac{x}{\mathfrak{p}}\right)_n \equiv x^{(q_{\mathfrak{p}}-1)/n} \pmod{\mathfrak{p}} \quad (n = 2, 3, 6),$$

for $x \notin \mathfrak{p}$, with value zero at $x \in \mathfrak{p}$. Thus the full local Gauss identities and natural zeros used in Sections 3–4 are unchanged. Put $F_0 = \mathbb{Q}(\omega)$, where $\omega = \zeta_3 = e^{2\pi i/3}$. Write Λ_F for the unitary Eisenstein coefficient character denoted by Λ in [Liu26b, Lemma 4.2]. It is given by

$$\Lambda_{F_0}(\mathfrak{a}) = \frac{j(\pi_+(\mathfrak{a}))}{|j(\pi_+(\mathfrak{a}))|}, \quad \Lambda_F = \Lambda_{F_0} \circ N_{F/F_0}.$$

Here $\mathfrak{a}$ is an integral F_0 -ideal prime to 3, $\pi_+(\mathfrak{a}) \equiv 1 \pmod{3}$ is its primary generator, and $j : F_0 \hookrightarrow \mathbb{C}$ is the fixed embedding used for the cubic residue character and Jacobi sums. The map N_{F/F_0} is the ideal norm. The character Λ_F need not have finite order. The target-dependent character of the theta identity is $\overline{\Lambda_F} \overline{G_0} \eta$, where G_0 is the finite-order coefficient character in that lemma. Neither this norm construction nor the local identities require F/F_0 to be normal. The fixed unit and ray-class sets contribute constants. If τ_F counts ideal divisors, then, for each fixed A ,

$$\tau_F(\mathfrak{a})^A \ll_{F,A,\epsilon} (N\mathfrak{a})^\epsilon.$$

Apply the proofs of Theorems 4.1 and 4.9 using these identities and the transferred Poisson and M2 estimates. Keep the actual frequency domain in Proposition 4.6, and the actual outer ideal and Fourier measure in Proposition 4.8. Their length conditions remain

$$\eta = \frac{7}{9}(1 - 2r) - 6W > 0, \quad C_{\text{add}} = \frac{5}{2} - 4m_- + \frac{9}{2}W < 1.$$

Write $\mathcal{N}_{F,c}$ and $\mathcal{N}_{F,\mathcal{M}}$ for the left sides of (4.15)–(4.16). Then

$$\max\{\mathcal{N}_{F,c}, \mathcal{N}_{F,\mathcal{M}}\} \ll_{F,\epsilon} U^{1+\epsilon} \mathfrak{S}_J(\mathcal{W})(1+H)^C,$$

with the same nonexceptional character parameters, centered means and exact equal-product comparison. No additional principal character parameters occur.

Smooth weights and contours. The field F is totally complex, and its finite-order characters have trivial infinity type. With $r_F = [F : \mathbb{Q}]/2$, the gamma ratios in Lemma 5.1 are

$$\left(\frac{\Gamma(w + \xi)}{\Gamma(w)}\right)^{r_F}, \quad \left(\frac{\Gamma(1 - w)}{\Gamma(w)}\right)^{r_F}.$$

For fixed j, N and $1/6 \leq \Re w \leq 5/6$, that lemma gives

$$|(v\partial_v)^j V_w(v)| \ll_{F,j,N} \left(1 + \frac{v}{(1 + |\Im w|)^{r_F}}\right)^{-N}.$$

The original and reflected contours satisfy

$$\frac{1}{5} \leq \Re w \leq \frac{1}{2}, \quad \frac{1}{2} \leq \Re(1-w) \leq \frac{4}{5}, \quad q_p^{-(j_p-2)\theta} \leq 1 \quad (j_p \geq 2, \theta \geq 0).$$

Retain the actual conductors and fixed- S factors. The M2 estimate for arbitrary bounded length exponents of the transfer theorem handles the degree-dependent long AFE sums; the short and selected ranges keep their original numerical cutoffs. Thus the dependence on r_F changes constants and height powers, rather than powers of U .

Selection and successive bounds. Prime ideals in the fixed ray classes supply the original ranges. After fixing the field data, the same selection and mesh choices give

$$\theta_0 = \frac{2\delta x}{9}, \quad \mathcal{E}_{\theta, \gamma} - \mathcal{E}_{\theta_0, 0} = (\theta - \theta_0)\{d(1-m) - l_y\} - d\gamma.$$

The scalar estimates of Section 8 therefore remain unchanged. The transferred 7/8 theorem supplies $\kappa_m = 3/4$; Theorem 8.1, applied over F , gives $B_0 = B_{\text{seed}}$. The finite-step implication is

$$\beta_*(F) \leq B_n, \quad \kappa_n = 2B_n - 1 \implies \beta_*(F) \leq B_{n+1} = B_* + \frac{\kappa_n - \kappa_*}{100}.$$

Since $\kappa_* = 2B_* - 1$,

$$B_n - B_* = \frac{B_{\text{seed}} - B_*}{50^n} \longrightarrow 0, \quad \beta_*(F) \leq \inf_n B_n = B_*.$$

Each finite step uses its own strictly positive margins and threshold; no threshold uniform in n or F is required. $\square$

9.2 Descent to an arbitrary number field

In this subsection, we descend through $K(\zeta_{12})$ using abelian factorization. This completes the proof of Theorem 1.1.

Theorem 9.2 (The zero-free half-plane over every number field). *For every number field K and every finite-order Hecke character χ of K , the full Hecke L -function has no zeros in*

$$\boxed{\Re s > B_* = 0.87446830769353045955 \dots}$$

The principal pole at $s = 1$ is permitted. In particular, for every number field K ,

$$\zeta_K(s) \neq 0 \quad (\Re s > B_*, s \neq 1).$$

The numerical boundary is independent of K and χ . No assertion is made on the line $\Re s = B_$.*

Proof. For primitive χ of order d , let M/K be its cyclic class-field extension and put $F = K(\zeta_{12})$, $E = MF$. Restriction gives

$$\text{Gal}(E/F) \hookrightarrow \text{Gal}(M/K), \quad \text{Gal}(E/M) \hookrightarrow (\mathbb{Z}/12\mathbb{Z})^\times.$$

For a finite abelian group G , write $\widehat{G} = \text{Hom}(G, \mathbb{C}^\times)$. Both extensions are abelian. Proposition 9.1 and class-field factorization [Mil20], as in [Liu26b, Proposition 19.1], give

$$\begin{aligned} \zeta_E(s) &= \prod_{\psi \in \widehat{\text{Gal}(E/F)}} L_F(s, \psi) \\ &= \zeta_M(s) \prod_{\substack{\omega \in \widehat{\text{Gal}(E/M)} \\ \omega \neq 1}} L_M(s, \omega), \\ \zeta_M(s) &= \prod_{j=0}^{d-1} L_K(s, \chi^j). \end{aligned}$$

For $\Re s > B_*$, $s \neq 1$, the first product is nonzero and every complementary nonprincipal factor is entire. Hence

$$\zeta_E(s) \neq 0 \implies \zeta_M(s) \neq 0 \implies L_K(s, \chi) \neq 0.$$

At $s = 1$, for $d > 1$,

$$\prod_{j=1}^{d-1} L_K(1, \chi^j) = \frac{\text{Res}_{s=1} \zeta_M(s)}{\text{Res}_{s=1} \zeta_K(s)} > 0.$$

Thus each nonprincipal factor is nonzero; $d = 1$ permits the principal pole. An omitted Euler factor satisfies

$$1 - \alpha q_p^{-s} = 0, \quad |\alpha| = 1 \implies \Re s = 0.$$

If $\alpha = 0$ the factor is one. This handles imprimitive functions. The full factorizations include ramified places and finite-order sign characters at real places of K . $\square$

Theorem 1.1 is Theorem 9.2.

Corollary 9.3 (Optimality within the stated comparison system). *For every fixed $F \supset \mu_{12}$, the comparison system of Theorem 8.5 has the same optimal boundary B_* . The descent of Theorem 9.2 preserves the strict half-plane $\Re s > B_*$. The optimality concerns only this comparison system.*

Proof. For every fixed $F \supset \mu_{12}$,

$$h(1 - m_0) - l_y = 0, \quad h\gamma \leq \frac{7\delta}{216}(2l_y - h).$$

Proposition 9.1 preserves these quantities and the critical class; Theorem 8.5 gives the same scalar optimum. The descent in Theorem 9.2 preserves the strict half-plane exactly. $\square$

A Exact rational inequalities in the comparison

In this appendix, we give the rational computation used in Section 8. The method proves inequalities on complete closed rectangles; it does not replace any of the analytic estimates in Sections 3–7.

A.1 Polynomial inequalities on a rectangle

In this subsection, we state the coefficient criterion used for the rational functions in Section 8.1. The same criterion, with one more variable, applies to the parameter intervals in Section 8.2.

For a polynomial p on a rational rectangle, make the affine change of variables to $[0, 1]^n$ and write

$$p(t) = \sum_{\alpha \leq d} a_\alpha t^\alpha = \sum_{\beta \leq d} b_\beta \prod_{j=1}^n \binom{d_j}{\beta_j} t_j^{\beta_j} (1 - t_j)^{d_j - \beta_j}.$$

The exact conversion is

$$b_\beta = \sum_{\alpha \leq \beta} a_\alpha \prod_{j=1}^n \frac{\binom{\beta_j}{\alpha_j}}{\binom{d_j}{\alpha_j}}. \quad (\text{A.1})$$

For $t \in [0, 1]^n$, the basis functions satisfy

$$B_\beta(t) := \prod_{j=1}^n \binom{d_j}{\beta_j} t_j^{\beta_j} (1 - t_j)^{d_j - \beta_j} \geq 0, \quad \sum_{\beta \leq d} B_\beta(t) = 1.$$

Consequently

$$p(t) \geq \min_{\beta} b_{\beta}. \quad (\text{A.2})$$

When necessary, bisect one coordinate at a rational midpoint and repeat (A.1). A finite covering by boxes with positive coefficient minima proves positivity on their union. For a rational function p/q , perform this calculation on both p and q , after fixing their common sign. All operations are rational. For the parameters (b, ℓ) , the cleared numerators are affine, so their four corners suffice. Dependence on κ is retained as a polynomial coordinate, not replaced by its endpoints unless monotonicity has been proved.

A.2 The strict rational starting bound

In this subsection, we specify the inequalities proving Theorem 8.1. The functions E, S, R, θ_0 are those of Sections 2 and 6, evaluated at the rational parameters of Section 8.1.

Put $\varepsilon_{\text{cmp}} = 10^{-7}$, $\zeta = 10^{-8}$, $m_{\text{seed}} = 1/10$, $N = (3R + 1)/4 - \theta_0/2$, and

$$\Delta_{\text{short}} = R + (1/2 - \theta_0)m_{\text{seed}} - N, \quad I = [h - 1/80, h + \zeta].$$

Use $\mathcal{R} = [7/20, 3/5] \times [9/20, 1/2]$ and $\mathcal{G} = [1/8, 5/6] \times [0, 1/2]$. Its three complementary rectangles are

$$\begin{aligned} \mathcal{C}_1 &= [1/8, 5/6] \times [0, 9/20], \\ \mathcal{C}_2 &= [1/8, 7/20] \times [9/20, 1/2], \\ \mathcal{C}_3 &= [3/5, 5/6] \times [9/20, 1/2]. \end{aligned}$$

On $\mathcal{G}$, one has $J > 0$ and $E < 1/500$; on each $\mathcal{C}_i$, one has $E < -\varepsilon_{\text{cmp}}$. On $\mathcal{R}$, one has

$$S > 1/5, \quad a > 0, \quad \gamma > 0, \quad 2x(21/100) - z_L - W > 0, \quad (\text{A.3})$$

$$E - t_- \{h(1 - m_L) - l_y\} < -\varepsilon_{\text{cmp}}, \quad E - h\gamma < -\varepsilon_{\text{cmp}}, \quad (\text{A.4})$$

$$E + t_+ \{h(1 - m_R) - l_y\} < -\varepsilon_{\text{cmp}}, \quad E + h\Delta_{\text{short}} < -\varepsilon_{\text{cmp}}. \quad (\text{A.5})$$

For d equal to either endpoint of I , each of

$$\begin{aligned} E + (d - h)S + d\Delta_{\text{short}}, \quad E + (d - h)S - t_- \{d(1 - m_L) - l_y\}, \\ E + (d - h)S - d\gamma, \quad E + (d - h)S + t_+ \{d(1 - m_R) - l_y\} \end{aligned}$$

is less than $-\varepsilon_{\text{cmp}}/2$. Their slopes are positive and less than two on the ordinary left, middle and right ranges. Further,

$$\frac{1}{5} \frac{1}{80} - \frac{1}{500} > \varepsilon_{\text{cmp}}, \quad 2\zeta < \varepsilon_{\text{cmp}}/2,$$

so parameters with smaller norm and the upper buffer in d have the required strict bounds. The elementary remaining inequalities are

$$\begin{aligned} b > 0, \quad \frac{1}{6} < \ell < \frac{1}{5}, \quad l_x > \ell, \quad l_y - \ell > \frac{11b}{6}, \quad \frac{9}{2}\ell > \frac{3}{4}, \\ \frac{21}{100} < \frac{\ell}{h + \zeta}, \quad \frac{\ell}{h - 1/80} < \frac{11}{50}, \quad \frac{\ell}{h - 1/80} < \frac{2}{9}, \\ h < 2l_y, \quad l_y < \frac{1}{2}, \quad B > \frac{437}{500}, \\ Q_{\text{sm}} := \ell(1/2 + 2\ell/h) - 11h/54 < 0, \\ C_0 + (1 - h)/16 < -1/250, \\ C_0 + (1/2 + 3\ell/2)/50 < -1/100, \\ 13h/75 - l_y/2 + 1/50 < -1/20. \end{aligned}$$

These inequalities are obtained by (A.1)–(A.2) after substitution of the displayed rational parameters. The accompanying rational computations retain every polynomial coefficient and the finite subdivisions. They include the conductor-length inequality $(9/2 - 2\delta x)(21/100) > R$ and the two interior contour inequalities $t_- > 0$, $t_+ > 0$. Thus the signs apply to the full comparison exponent, rather than only its value at $d = h$.

A.3 The algebraic parameter interval

In this subsection, we describe the calculation used in Lemma 8.3. The algebraic identities at the critical class are proved by polynomial division by (1.1); the remaining signs are rational inequalities on the intervals below.

Use

$$\begin{aligned} I_b &= [0.0917923522, 0.0917923524], \\ I_\ell &= [0.16876, 0.16880], \quad I_\kappa = [0.7489, 0.7495], \\ \mathcal{R}_* &= [7/20, 3/5] \times [2/5, 1/2]. \end{aligned}$$

Every decimal endpoint here is an exact terminating decimal. The critical root is isolated in

$$0.450570746724060730 < r_* < 0.450570746724060733.$$

For $Q = E_\kappa - 7a(2l_y - h)/108$, the coefficient test gives

$$\partial_x Q > 0 \quad \text{on } \mathcal{R}_*$$

for $\kappa \in [0.7489366153, 0.7489366155]$ and $(b, \ell) \in I_b \times I_\ell$. It also gives $\partial_\ell Q > 3/4$ at $\kappa = 0.7489$; the identity

$$\partial_\kappa \partial_\ell Q = \frac{9}{8} \partial_\kappa R \geq 0$$

extends this inequality to I_κ . The formula in Lemma 8.3 gives $0 \leq \partial_\kappa Q < 1/100$ throughout the larger interval.

Use the complementary rectangles of Lemma 8.3, with amplitude endpoint $2/5$. Lemma 6.5 gives

$$\partial_\kappa E = \frac{3h}{4} \partial_\kappa R \geq 0, \quad \partial_\kappa S = \frac{3}{4} \partial_\kappa R \geq 0.$$

At the adverse count endpoints, the exact test gives

domain	count parameter	bound
$\mathcal{G}$	$\max I_\kappa$	$E < 1/500$
$\bigcup_i \mathcal{C}_i$	$\max I_\kappa$	$E < -10^{-5}$
$\mathcal{R}_*$	$\min I_\kappa$	$S > 1/5$.

The common ordinary gain satisfies

$$\begin{aligned} a &> 0, \quad 4a/9 - \theta_0 > 0, \quad \delta/2 - \theta_0 - 7a/54 > 0, \\ \theta_0(5h/8 - l_y) &> 7a(2l_y - h)/108, \\ (\delta/2 - \theta_0)(l_y - 11h/20) &> 7a(2l_y - h)/108, \\ 2x(21/100) - 1/18 &> 7/279. \end{aligned}$$

The short comparison uses $m_{\text{cut}} = 3/40$ and $\Delta_{\text{short}} = R + (1/2 - \theta_0)m_{\text{cut}} - N$:

$$E + h\Delta_{\text{short}} < -10^{-5}.$$

The geometric inequalities of the previous subsection persist, with the threshold for the lowest-amplitude estimate replaced by $1/125$. Finally

$$\frac{57}{100} < \frac{l_y}{h} < \frac{59}{100}, \quad l_y/h < 11/19, \quad 25/62 < 1 - l_y/h < 9/20.$$

The two bounds on l_y/h and the upper bound on $1 - l_y/h$ are checked by exact rational interval arithmetic on $I_b \times I_\ell$. The quantitative outer margin is the identity (7.3); the independent rational calculation also checks it on the full parameter box.

These are precisely the strict inequalities used in the construction; no sign is inferred from a sampled maximum. The exact rational coefficient calculations and an independent rational Taylor expansion calculation are included with the accompanying source files.

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