

COLLISION GEOMETRY AND HOLONOMY OF FINITE BLASCHKE MODEL SPACES

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ABSTRACT. We investigate the rank- n Hermitian vector bundle of finite Blaschke model spaces over the coefficient domain of monic polynomials whose zeros lie entirely in the open unit disk. By introducing a global anti-holomorphic rational frame, we show that both the bundle and its orthogonal-projection connection extend real-analytically across all internal root-collision strata. At the maximally coincident divisor, we compute the full curvature tensor explicitly; the successive differences of its matrix coefficients yield the standard matrix units, which implies that both the restricted and full holonomy groups coincide with the full unitary group $U(n)$. Furthermore, the induced metric on the determinant line bundle is shown to agree with the diagonal restriction of the Szegő kernel of the symmetrized polydisc, and its power-sum Kähler potential endows the coefficient domain with a complete Kähler metric. By contrast, once the Clark phase circle is adjoined to the base, the Clark spectral lines together with the canonical model-space conjugation single out a unique compatible unitary connection. This real-spectral connection is flat, extends real-analytically across internal collisions, and exhibits a finite signed cyclic holonomy group. We also derive a weighted residue formula for the complex-bilinear pairing induced by the canonical conjugation, and analyze explicit low-dimensional subfamilies that illustrate the sharp geometric contrast between the curved projection transport of the moving model space and the flat parallel transport of its real spectral lines.

1. INTRODUCTION

A finite Blaschke product B of degree n determines an n -dimensional model space

$$K_B = H^2(\mathbb{D}) \ominus BH^2(\mathbb{D}).$$

Its reproducing kernel, compressed shift, Clark unitary perturbations, and canonical conjugation are classical objects in operator-theoretic function theory. As the zeros of B vary, these spaces assemble into a smooth family of finite-dimensional subspaces of the fixed ambient Hardy space $H^2(\mathbb{D})$. The differential geometry of this moving family depends crucially on the choice of connection. The primary objective of this paper is to compare the natural connection induced by orthogonal projection with a flat connection governed by the real spectral lines of the Clark unitary perturbations.

Root collisions play a central role in this comparison. When the zeros $\alpha_1, \dots, \alpha_n$ of B are pairwise distinct, the Cauchy–Szegő kernels $(1 - \bar{\alpha}_j z)^{-1}$ form a natural basis for K_B , which degenerates as roots coalesce. Working instead in polynomial coefficient coordinates, we construct a global rational frame that remains linearly independent across the entire discriminant locus. The maximally coincident divisor thus serves as a smooth interior point at which the full curvature tensor can be evaluated explicitly.

Our main results reveal a sharp dichotomy between the holonomy groups of these two connections. The natural projection connection possesses full unitary holonomy $U(n)$, a phenomenon already detected infinitesimally by its curvature at a single maximally coincident divisor. On the other hand, after adjoining the Clark phase circle to the parameter space, the Clark spectral lines together with the canonical conjugation induce a discrete reduction of the unitary structure group. The resulting connection is flat and exhibits an explicitly

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computable signed cyclic monodromy. Notably, this contrast arises between two geometrically natural connections on the same topologically trivial Hermitian bundle, rather than from any change in fiber rank or bundle topology.

1.1. Parameter space and main results. For an integer $n \geq 1$, let

$$(1.1) \quad \mathcal{B}_n = \left\{ p = (p_0, \dots, p_{n-1}) \in \mathbb{C}^n : P_p(z) = z^n + \sum_{j=0}^{n-1} p_j z^j \text{ has all its zeros in } \mathbb{D} \right\}.$$

The domain $\mathcal{B}_n$ provides a smooth polynomial-coefficient realization of the n th symmetric power $\text{Sym}^n(\mathbb{D})$, commonly known in several complex variables as the symmetrized polydisc. Associated with each $p \in \mathcal{B}_n$ are the reflected polynomial and the normalized finite Blaschke product

$$(1.2) \quad Q_p(z) = z^n \overline{P_p(1/\bar{z})} = 1 + \sum_{j=0}^{n-1} \bar{p}_j z^{n-j}, \quad B_p = \frac{P_p}{Q_p}.$$

This normalization fixes the unimodular constant factor of B_p coherently across the entire parameter space. Let $\mathcal{E}_p = K_{B_p}$, and denote by Π_p the orthogonal projection of $H^2(\mathbb{D})$ onto $\mathcal{E}_p$.

Throughout this paper, inner products are taken to be anti-linear in the first argument. Connections are written in local frames as $\nabla = d + A$, with curvature two-form $F = dA + A \wedge A$. The natural projection connection on the bundle $\mathcal{E} \rightarrow \mathcal{B}_n$ is defined by $\nabla^{\text{nat}} = \Pi d$.

Theorem A (The collision family and its holonomy). The model spaces $\mathcal{E}_p$ form a topologically trivial, rank- n real-analytic Hermitian vector bundle $\mathcal{E}$ over $\mathcal{B}_n$, trivialized globally by the anti-holomorphic rational frame

$$(1.3) \quad f_r(p; z) = \frac{z^r}{Q_p(z)}, \quad 0 \leq r < n.$$

The orthogonal projections Π_p depend real-analytically on p in the operator norm across all root-collision strata. At the origin $p = 0$, relative to the standard orthonormal basis $1, z, \dots, z^{n-1}$, the curvature of ∇^{nat} is given by

$$(1.4) \quad F^{\text{nat}}|_0 = \sum_{i,j=0}^{n-1} F_{i\bar{j}} dp_i \wedge d\bar{p}_j, \quad F_{i\bar{j}} = \sum_{q=0}^{n-1-\max(i,j)} E_{i+q,j+q},$$

where E_{ij} denotes the standard matrix unit. Under the convention that out-of-range coefficients vanish, these curvature matrices satisfy the difference identity

$$(1.5) \quad F_{i\bar{j}} - F_{i+1,\bar{j}+1} = E_{ij}.$$

Consequently, at every base point $p \in \mathcal{B}_n$, the restricted and full holonomy groups coincide with the full unitary group of the fiber:

$$\text{Hol}_p^0(\nabla^{\text{nat}}) = \text{Hol}_p(\nabla^{\text{nat}}) = U(\mathcal{E}_p).$$

More locally, the restriction of ∇^{nat} to any connected open neighborhood of 0 also has restricted holonomy $U(n)$.

While the full curvature is genuinely matrix-valued, its trace admits a transparent scalar description in terms of reproducing kernels. Let $H_p = (\langle f_r(p), f_s(p) \rangle)_{0 \leq r,s < n}$ be the Gram matrix of the rational frame, and let $s_m(p)$ denote the m th power sum of the zeros of P_p , counted with multiplicity.

Theorem B (Determinant geometry). If $\alpha_1, \dots, \alpha_n$ are the zeros of P_p repeated according to multiplicity, then the determinant of the Gram matrix is given by

$$(1.6) \quad \det H_p = \prod_{a,b=1}^n (1 - \alpha_a \bar{\alpha}_b)^{-1}.$$

Its logarithmic potential

$$(1.7) \quad \Phi(p) = \log \det H_p = \sum_{m \geq 1} \frac{|s_m(p)|^2}{m}$$

satisfies $\mathrm{tr} F^{\mathrm{nat}} = \partial \bar{\partial} \Phi$. Endowed with the Riemannian length element and Kähler form

$$(1.8) \quad \|v\|_g^2 = \sum_{m \geq 1} \frac{|ds_m(v)|^2}{m}, \quad \omega_g = \frac{i}{2} \partial \bar{\partial} \Phi,$$

$(\mathcal{B}_n, g)$ is a complete Kähler manifold. Moreover, the product in (1.6) coincides with the diagonal restriction of the Szegő kernel of the symmetrized polydisc under the normalization $\|1\| = 1$.

The factor of $1/2$ in (1.8) adheres to the standard Euclidean length convention: for the model potential $|z|^2$, the induced metric has length element $|dz|$ and Kähler form $(i/2)dz \wedge d\bar{z}$.

Theorem C (The real-spectral connection). Let $\mathcal{E}$ be pulled back to the extended parameter space $\hat{\mathcal{B}}_n = \mathcal{B}_n \times \mathbb{T}$, and equip each fiber over (p, λ) with the spectral lines of the Clark unitary operator $U_{p, \lambda}$ and the canonical conjugation C_p of K_{B_p} . There exists a unique unitary connection ∇^{sp} on the pullback bundle that preserves both the canonical conjugation and every locally labelled spectral-line projection. This connection is flat and extends real-analytically across all internal root collisions. In a suitable conjugation-real orthonormal frame, the holonomy along a positively oriented phase loop is generated by the signed cyclic shift

$$(1.9) \quad C_\epsilon e_j = e_{j+1} \quad (0 \leq j < n-1), \quad C_\epsilon e_{n-1} = \epsilon e_0, \quad \epsilon = (-1)^{n-1}.$$

The holonomy group of ∇^{sp} is cyclic of order n when n is odd and of order $2n$ when n is even. In addition, the induced connection on the determinant line bundle has trivial holonomy, and every loop in the divisor space $\mathcal{B}_n$ at a fixed Clark phase has trivial monodromy.

In summary, whereas the determinant metric captures only the scalar trace of the natural curvature, the full matrix-valued curvature tensor at the maximal collision stratum forces the holonomy of ∇^{nat} to be the entire unitary group $U(n)$. By contrast, the construction of the flat connection ∇^{sp} relies on auxiliary boundary spectral data and the canonical real structure; its flatness represents an alternative geometric transport rather than a gauge trivialization of ∇^{nat} .

1.2. Relation to earlier work. Our analysis builds on the classical theory of model spaces and Clark unitary perturbations developed in [2, 5, 7, 10]. The canonical conjugation on K_B and conjugation-real Clark bases are treated in detail in [4, 9]. More broadly, the interplay between parameterized operator families and Hermitian bundle geometry goes back to the foundational work of Cowen and Douglas [3]. A distinctive feature of our approach is that the base manifold $\mathcal{B}_n$ parametrizes all degree- n interior divisors simultaneously, realizing the root-collision discriminant as an interior analytic subset of the smooth polynomial coefficient domain.

The scalar product identity (1.6) recovers the explicit Szegő kernel formula for the symmetrized polydisc obtained by Misra, Shyam Roy, and Zhang [8, Theorem 3.1]. Here we reinterpret this kernel geometrically as the Plücker Gram metric on the determinant line bundle $\det \mathcal{E}$ and establish global metric completeness directly via the power-sum immersion. To pass from scalar trace identities to the full unitary holonomy of Theorem A, we combine the explicit matrix-unit decomposition (1.5) at the origin with the Ambrose–Singer holonomy theorem [1].

The study of cyclic eigenvalue transport in one-parameter families of rank-one unitary perturbations has a direct antecedent in the work of Tanaka and Miyamoto [11]. In our setting, incorporating the canonical model-space conjugation fixes the phases of the Clark eigenlines canonically across the entire divisor family. This rigidification determines the precise

parity sign $\epsilon = (-1)^{n-1}$ in the holonomy generator, guarantees real-analytic extension across root collisions, and enables a direct comparison with the curved projection connection ∇^{nat} . Standard Clark perturbation formulas and the qualitative cycling of boundary eigenvalues are used here as established background.

The paper is organized as follows. Section 2 constructs the model-space bundle over $\mathcal{B}_n$ and establishes uniform operator-norm bounds across the collision locus. Section 3 computes the full curvature tensor at the origin and proves the $U(n)$ holonomy theorem. Section 4 examines the determinant line bundle, identifies the Szegő kernel potential, and proves Kähler completeness. Sections 5 and 6 analyze the canonical conjugation, the weighted residue pairing, and the flat real-spectral connection. Finally, Section 7 contrasts the two connections along explicit one- and two-parameter subfamilies.

2. THE MODEL-SPACE BUNDLE ACROSS THE DISCRIMINANT

2.1. Coefficient coordinates. Assigning to a monic polynomial its tuple of coefficients identifies the symmetric product $\text{Sym}^n(\mathbb{D})$ of unordered n -tuples in $\mathbb{D}$ with the domain $\mathcal{B}_n \subset \mathbb{C}^n$. Because the roots of a polynomial depend continuously on its coefficients, $\mathcal{B}_n$ is an open subset of $\mathbb{C}^n$. It is also bounded: expressing the coefficients via elementary symmetric polynomials in the roots yields the uniform estimate $|p_j| \leq \binom{n}{j}$. Although the root-collision discriminant stratifies $\mathcal{B}_n$ into loci of varying multiplicity, the ambient coefficient domain itself is a smooth complex manifold.

Lemma 2.1. *The domain $\mathcal{B}_n$ is contractible. Specifically, the map*

$$(2.1) \quad (p, t) \longmapsto (t^n p_0, t^{n-1} p_1, \dots, t p_{n-1}), \quad 0 \leq t \leq 1,$$

defines a strong deformation retraction of $\mathcal{B}_n$ onto the origin 0.

Proof. For $t > 0$, the rescaled tuple corresponds to the monic polynomial $t^n P_p(z/t)$, whose roots are $t\alpha_a \in \mathbb{D}$; at $t = 0$, it reduces to the monomial z^n . The map (2.1) is manifestly continuous, preserves $\mathcal{B}_n$ for all $t \in [0, 1]$, and leaves the origin fixed. $\square$

Throughout the paper, all differential operations on the base are performed with respect to the global complex coordinates $(p_0, \dots, p_{n-1})$. Root coordinates $(\alpha_1, \dots, \alpha_n)$ are employed solely to express symmetric algebraic identities and to parameterize specific low-dimensional examples, rather than as local coordinates near points of higher multiplicity.

2.2. A global rational frame. We equip the Hardy space $H^2(\mathbb{D})$ with its standard norm

$$\|\sum_{r \geq 0} a_r z^r\|_{H^2}^2 = \sum_{r \geq 0} |a_r|^2,$$

corresponding on the unit circle $\mathbb{T}$ to the normalized arc-length measure $d\theta/(2\pi)$. Point evaluation at $a \in \mathbb{D}$ is reproduced by the Cauchy–Szegő kernel $k_a(z) = (1 - \bar{a}z)^{-1}$, so that $\langle k_a, f \rangle = f(a)$ for all $f \in H^2(\mathbb{D})$. Because multiplication M_{B_p} by the inner function B_p is an isometry on $H^2(\mathbb{D})$, the orthogonal projection onto the model space $\mathcal{E}_p = K_{B_p}$ and the associated reproducing kernel take the classical forms

$$(2.2) \quad \Pi_p = \text{I} - M_{B_p} M_{B_p}^*, \quad K_p(z, w) = \frac{1 - B_p(z) \overline{B_p(w)}}{1 - z\bar{w}}.$$

Lemma 2.2. *For every parameter $p \in \mathcal{B}_n$, regardless of root multiplicities, the model space admits the rational description*

$$(2.3) \quad K_{B_p} = \left\{ \frac{A}{Q_p} : A \in \mathbb{C}[z], \deg A < n \right\}.$$

Proof. Since all zeros of P_p lie in $\mathbb{D}$, the reflected polynomial $Q_p(z) = \prod_{a=1}^n (1 - \bar{\alpha}_a z)$ vanishes nowhere on the closed unit disk $\bar{\mathbb{D}}$. On the unit circle $\mathbb{T}$, the identity $B_p = P_p/Q_p$ and the relation $\overline{Q_p(\zeta)} = \zeta^{-n} P_p(\zeta)$ for $|\zeta| = 1$ imply that

$$\overline{B_p} \frac{A}{Q_p} = \frac{A}{P_p} \quad \text{on } \mathbb{T}.$$

Whenever $\deg A < n$, the rational function A/P_p is holomorphic on $\mathbb{C} \setminus \mathbb{D}$ and vanishes at infinity, so its Laurent expansion on $\mathbb{T}$ contains only strictly negative powers of z . Consequently, A/Q_p belongs to $H^2(\mathbb{D}) \ominus B_p H^2(\mathbb{D}) = K_{B_p}$. Since $\deg A < n$, the right-hand side of (2.3) is an n -dimensional subspace of K_{B_p} . On the other hand, inductive application of the orthogonal decomposition

$$K_{B_1 B_2} = K_{B_1} \oplus B_1 K_{B_2}$$

to the n linear Blaschke factors of B_p shows that $\dim K_{B_p} = n$ for arbitrary root multiplicities, completing the proof. $\square$

Associated with the rational basis elements $f_r(p; z) = z^r/Q_p(z)$ ($0 \leq r < n$) from (1.3), we define the synthesis (frame) operator and its Gram matrix by

$$(2.4) \quad F_p : \mathbb{C}^n \longrightarrow H^2(\mathbb{D}), \quad F_p e_r = f_r(p), \quad H_p = F_p^* F_p.$$

By Lemma 2.2, F_p is injective, H_p is strictly positive definite, and the orthogonal projection onto $\mathcal{E}_p$ is given by

$$(2.5) \quad \Pi_p = F_p H_p^{-1} F_p^*.$$

Proposition 2.3. *The operator-valued map $p \mapsto F_p$ is anti-holomorphic on $\mathcal{B}_n$ with respect to the operator norm. Consequently, the maps $p \mapsto H_p$ and $p \mapsto \Pi_p$ are real-analytic on $\mathcal{B}_n$. In particular, the basis (1.3) provides a global anti-holomorphic trivialization of $\mathcal{E}$; equivalently, $\mathcal{E}$ carries the structure of a trivial holomorphic vector bundle over the conjugate complex manifold $\overline{\mathcal{B}_n}$.*

Proof. Fix a base point $p^0 \in \mathcal{B}_n$. Since Q_{p^0} has no zeros on the compact set $\bar{\mathbb{D}}$, $|Q_{p^0}(z)|$ is bounded below by a positive constant on $\bar{\mathbb{D}}$. For p in a sufficiently small neighborhood of p^0 , the reciprocal $Q_p(z)^{-1}$ expands into a Neumann series in $H^\infty(\mathbb{D})$ that converges uniformly on $\bar{\mathbb{D}}$ and depends holomorphically on the conjugate coordinates $\bar{p}$. This establishes the anti-holomorphic dependence of each frame vector $f_r(p)$ in $H^2(\mathbb{D})$, and hence of the finite-rank operator F_p in the operator norm. It follows immediately that $H_p = F_p^* F_p$ is real-analytic and positive definite. Combining (2.5) with the real analyticity of matrix inversion on the general linear group yields the real analyticity of Π_p . $\square$

2.3. Confluent kernels and uniform bounds. When a root $a \in \mathbb{D}$ of P_p has multiplicity $m \geq 1$, the associated reproducing derivative (jet) kernels are given by

$$(2.6) \quad k_{a,r}(z) = \partial_a^r k_a(z) = \frac{r! z^r}{(1 - \bar{a}z)^{r+1}}, \quad 0 \leq r < m,$$

which reproduce higher derivatives via $\langle k_{a,r}, f \rangle = f^{(r)}(a)$. Taking the union of these jet kernels over all distinct roots of P_p yields n linearly independent functions that span K_{B_p} : indeed, any $f \in H^2(\mathbb{D})$ orthogonal to all of them vanishes to the prescribed order at each root of P_p , hence is divisible by B_p in $H^2(\mathbb{D})$. From this perspective, the collapse of the standard point-evaluation basis at a root collision is compensated by the emergence of confluent jet kernels, preserving the fiber dimension n .

At a completely coincident divisor with $P_p(z) = (z - a)^n$, the Gram–Schmidt process applied to the confluent kernels produces the classical Malmquist–Takenaka orthonormal basis

$$(2.7) \quad u_r(z) = \frac{\sqrt{1 - |a|^2}}{1 - \bar{a}z} \left(\frac{z - a}{1 - \bar{a}z} \right)^r, \quad 0 \leq r < n.$$

For computing parameter derivatives across strata of varying multiplicity, however, the global rational frame (1.3) is far more tractable.

Proposition 2.4. *Fix $r_0 \in (0, 1)$ and consider the compact set of parameters $p \in \mathcal{B}_n$ whose roots satisfy $|\alpha_a| \leq r_0$. On this set, every partial derivative of Π_p of fixed order with respect to the coefficient coordinates $(p, \bar{p})$ is uniformly bounded in the operator norm by a constant depending only on n , r_0 , and the order of differentiation.*

Proof. Whenever $|\alpha_a| \leq r_0$ for all a , the boundary values of Q_p on $\mathbb{T}$ satisfy the uniform two-sided estimate

$$(1 - r_0)^n \leq |Q_p(\zeta)| \leq (1 + r_0)^n, \quad \zeta \in \mathbb{T}.$$

For any coefficient vector $c = (c_0, \dots, c_{n-1}) \in \mathbb{C}^n$, comparing the squared $L^2(\mathbb{T})$ norm of $F_p c = (\sum c_r z^r)/Q_p$ with the squared Hardy norm $|c|^2$ of its numerator yields the two-sided operator inequality

$$(2.8) \quad (1 + r_0)^{-2n} \mathbf{I} \leq H_p \leq (1 - r_0)^{-2n} \mathbf{I}.$$

Furthermore, differentiating $f_r(p; z) = z^r/Q_p(z)$ with respect to the conjugate coordinates gives

$$(2.9) \quad \partial_{\bar{p}_{j_1}} \cdots \partial_{\bar{p}_{j_m}} f_r(p; z) = (-1)^m m! \frac{z^{r+mn-j_1-\cdots-j_m}}{Q_p(z)^{m+1}},$$

whose $H^2(\mathbb{D})$ norm is bounded above by $m!(1 - r_0)^{-n(m+1)}$. Substituting (2.8) and (2.9) into the projection formula (2.5) and applying the standard Leibniz rule for the derivatives of H_p^{-1} provides uniform operator-norm bounds of arbitrary order, completely free of singular Vandermonde denominators $(\alpha_a - \alpha_b)^{-1}$. $\square$

3. CURVATURE AND FULL UNITARY HOLONOMY

3.1. The projection connection. For any smooth section $s(p) \in \mathcal{E}_p$, the natural connection is defined by $\nabla^{\text{nat}} s = \Pi_p ds$. Since the ambient Hilbert space inner product on $H^2(\mathbb{D})$ is constant, ∇^{nat} is automatically compatible with the induced Hermitian metric on $\mathcal{E}$. Differentiating the idempotent relation $\Pi_p^2 = \Pi_p$ yields $\Pi_p(d\Pi_p)\Pi_p = 0$, from which one obtains the standard projection curvature formula

$$(3.1) \quad F^{\text{nat}} = \Pi_p d\Pi_p \wedge d\Pi_p \Pi_p.$$

Here all differentials are understood in the norm topology of bounded operators on $H^2(\mathbb{D})$.

Expressed in the global rational frame F , the connection one-form of ∇^{nat} takes the form

$$(3.2) \quad A = H^{-1} F^* dF = H^{-1} \bar{\partial} H.$$

Indeed, because F_p is anti-holomorphic in p , its adjoint F_p^* is holomorphic, which implies $\bar{\partial} F^* = 0$ and $dF = \bar{\partial} F$. Hence $F^* dF = F^* \bar{\partial} F = \bar{\partial}(F^* F) = \bar{\partial} H$. In particular, A is a matrix-valued $(0, 1)$ -form, its $(0, 2)$ curvature component vanishes by the Maurer–Cartan identity $\bar{\partial}(H^{-1} \bar{\partial} H) + (H^{-1} \bar{\partial} H) \wedge (H^{-1} \bar{\partial} H) = 0$, and $F^{\text{nat}} = \partial A$ is purely of type $(1, 1)$.

To express the curvature in terms of the second fundamental form, introduce the normal-derivative operators

$$(3.3) \quad N_j = (1 - \Pi_p) \partial_{\bar{p}_j} F_p : \mathbb{C}^n \longrightarrow \mathcal{E}_p^\perp.$$

In the rational frame, the $(1, 1)$ curvature components are given by the second fundamental form identity

$$(3.4) \quad (F^{\text{nat}})_{i\bar{j}} = H_p^{-1} N_i^* N_j.$$

Indeed, differentiating (3.2) with respect to p_i and using the relations $\partial_{p_i} F_p = 0$, $\partial_{p_i} F_p^* = (\partial_{\bar{p}_i} F_p)^*$, and $F_p H_p^{-1} F_p^* = \Pi_p$, one computes directly that

$$\partial_{p_i} (H_p^{-1} \partial_{\bar{p}_j} H_p) = H_p^{-1} (\partial_{\bar{p}_i} F_p)^* (1 - F_p H_p^{-1} F_p^*) \partial_{\bar{p}_j} F_p = H_p^{-1} N_i^* N_j.$$

At any point where the frame is orthonormal, the prefactor H_p^{-1} reduces to the identity.

3.2. Curvature at the maximally coincident divisor.

Theorem 3.1. *At the maximally coincident divisor $p = 0$, the curvature of ∇^{nat} in the monomial basis $f_r(0; z) = z^r$ is given by (1.4) and satisfies the difference identity (1.5).*

Proof. At $p = 0$, we have $Q_0(z) = 1$, $B_0(z) = z^n$, and the frame elements $f_r(0; z) = z^r$ ($0 \leq r < n$) form an orthonormal basis of $\mathcal{E}_0 = \text{span}_{\mathbb{C}}\{1, z, \dots, z^{n-1}\}$, so $H_0 = \text{I}$. Differentiating $f_r(p; z) = z^r(1 + \sum_{j=0}^{n-1} \bar{p}_j z^{n-j})^{-1}$ at $p = 0$ gives

$$\partial_{\bar{p}_j} f_r|_0 = -z^{n+r-j}.$$

Projecting onto the orthogonal complement $\mathcal{E}_0^\perp = z^n H^2(\mathbb{D})$, we find that the normal component acts on the standard basis vectors $e_r \in \mathbb{C}^n$ by

$$(3.5) \quad N_j e_r = \begin{cases} -z^{n+r-j}, & r \geq j, \\ 0, & r < j. \end{cases}$$

Taking inner products in $H^2(\mathbb{D})$, we see that $N_i^* N_j e_r$ is nonzero if and only if $r \geq j$ and $0 \leq i + r - j < n$, in which case

$$N_i^* N_j e_r = \begin{cases} e_{i+r-j}, & r \geq j \text{ and } i + r - j < n, \\ 0, & \text{otherwise.} \end{cases}$$

Setting $r = j + q$ with $0 \leq q \leq n - 1 - \max(i, j)$ yields the explicit representation

$$(3.6) \quad F_{i\bar{j}} = N_i^* N_j = \sum_{q=0}^{n-1-\max(i,j)} E_{i+q, j+q}.$$

Shifting both indices i and j upward by one removes precisely the $q = 0$ summand E_{ij} from the sum, which establishes

$$(3.7) \quad F_{i\bar{j}} - F_{i+1, \overline{j+1}} = E_{ij},$$

including the boundary indices $i = n - 1$ or $j = n - 1$ where $F_{i+1, \overline{j+1}} = 0$. $\square$

Specializing (3.6) to the extreme diagonal indices and taking traces, we record the identities

$$(3.8) \quad F_{0\bar{0}} = \text{I}, \quad F_{n-1, \overline{n-1}} = E_{n-1, n-1}, \quad \text{tr } F_{i\bar{j}} = (n - i) \delta_{ij}.$$

In particular, whenever $n > 1$, the scalar trace $\text{tr } F^{\text{nat}}$ retains only a small fraction of the algebraic information encoded in the full matrix curvature F^{nat} .

3.3. Real curvature values and holonomy.

Theorem 3.2. *For every $n \geq 1$ and every base point $p \in \mathcal{B}_n$, the restricted and full holonomy groups of the natural connection are given by*

$$(3.9) \quad \text{Hol}_p^0(\nabla^{\text{nat}}) = \text{Hol}_p(\nabla^{\text{nat}}) = U(\mathcal{E}_p).$$

Moreover, if $U \subset \mathcal{B}_n$ is any connected open neighborhood of the origin 0, then the restriction of ∇^{nat} to U has restricted holonomy group $U(n)$ at every point of U .

Proof. Write the complex coordinates in real and imaginary parts as $p_j = x_j + iy_j$. Because the curvature coefficients at $p = 0$ satisfy the Hermitian symmetry $F_{i\bar{j}}^* = F_{j\bar{i}}$, evaluating the curvature two-form on pairs of real coordinate tangent vectors at the origin gives

$$(3.10) \quad F^{\text{nat}}(\partial_{x_i}, \partial_{x_j}) = F_{i\bar{j}} - F_{j\bar{i}},$$

$$(3.11) \quad F^{\text{nat}}(\partial_{x_i}, \partial_{y_j}) = -i(F_{i\bar{j}} + F_{j\bar{i}}).$$

In particular, for $i = j$, equation (3.11) gives $F^{\text{nat}}(\partial_{x_i}, \partial_{y_i}) = -2iF_{i\bar{i}}$. All of these endomorphisms are skew-Hermitian. Applying the difference operator (3.7) to (3.10) and (3.11) shows that the real linear span of the curvature values at $p = 0$ contains all elements of the form

$$E_{ij} - E_{ji}, \quad i(E_{ij} + E_{ji}) \quad (0 \leq i < j < n), \quad iE_{ii} \quad (0 \leq i < n).$$

Since these matrices constitute a standard $\mathbb{R}$ -basis for the Lie algebra $\mathfrak{u}(n)$, the curvature endomorphisms at the single point $p = 0$ already span $\mathfrak{u}(\mathcal{E}_0) \simeq \mathfrak{u}(n)$.

By the Ambrose–Singer theorem [1], the Lie algebra of the restricted holonomy group $\text{Hol}_0^0(\nabla^{\text{nat}})$ contains the real span of the curvature values at 0, while compatibility of ∇^{nat} with the Hermitian metric forces the reverse inclusion $\text{Lie Hol}_0^0(\nabla^{\text{nat}}) \subset \mathfrak{u}(\mathcal{E}_0)$. Consequently, $\text{Lie Hol}_0^0(\nabla^{\text{nat}}) = \mathfrak{u}(\mathcal{E}_0)$, and since $U(n)$ is connected, we conclude that $\text{Hol}_0^0(\nabla^{\text{nat}}) = U(\mathcal{E}_0)$. Because $\mathcal{B}_n$ is connected, parallel transport along paths conjugates the holonomy groups at different base points, extending the equality $\text{Hol}_0^0(\nabla^{\text{nat}}) = U(\mathcal{E}_p)$ to all $p \in \mathcal{B}_n$. That the full holonomy group $\text{Hol}_p(\nabla^{\text{nat}})$ also equals $U(\mathcal{E}_p)$ follows immediately, either because $\mathcal{B}_n$ is contractible (Lemma 2.1) or because $\text{Hol}_p(\nabla^{\text{nat}})$ is sandwiched between $\text{Hol}_p^0(\nabla^{\text{nat}}) = U(\mathcal{E}_p)$ and $U(\mathcal{E}_p)$. Finally, since the Lie algebra generation occurs pointwise at $p = 0$, the exact same argument applies within any connected open neighborhood U of 0. $\square$

Corollary 3.3. *The only parallel complex-linear endomorphisms of $(\mathcal{E}, \nabla^{\text{nat}})$ are constant scalar multiples of the identity. In particular, for $n > 1$, the bundle $\mathcal{E}$ admits no nonzero proper ∇^{nat} -parallel subbundle and no nontrivial parallel orthogonal direct-sum decomposition.*

Proof. Any parallel endomorphism must commute with the holonomy group $\text{Hol}_p(\nabla^{\text{nat}}) = U(\mathcal{E}_p)$ at each point p ; by Schur’s lemma for the irreducible defining representation of $U(n)$, it is a scalar multiple of the identity, and parallel transport forces this scalar to be constant across $\mathcal{B}_n$. Similarly, any parallel subbundle would define a nontrivial $U(\mathcal{E}_p)$ -invariant subspace of $\mathcal{E}_p$, which is impossible for $n > 1$. $\square$

Corollary 3.4. *For $n > 1$, the projective holonomy group of ∇^{nat} is the full projective unitary group $PU(n)$. Consequently, no twist of ∇^{nat} by a scalar imaginary-valued one-form can flatten the connection or make it preserve a complete orthogonal decomposition into line bundles.*

Proof. The canonical quotient homomorphism $U(n) \rightarrow PU(n)$ is surjective, and modifying a unitary connection by a scalar imaginary one-form $i\alpha \otimes I$ leaves the induced projective connection unchanged. Were the twisted connection flat, its restricted projective holonomy would be trivial; were it to preserve an orthogonal decomposition into n line subbundles, its restricted projective holonomy would be contained in a maximal torus of $PU(n)$. For $n > 1$, both possibilities contradict the fact that the projective holonomy group is all of $PU(n)$. $\square$

The hypothesis $n > 1$ in Corollary 3.4 is indispensable, as the projectivization of a rank-one bundle has trivial fibers. We also emphasize that the presence of full $U(n)$ holonomy on a topologically trivial bundle over a contractible domain is purely a manifestation of curvature, rather than a topological obstruction.

4. THE DETERMINANT KERNEL AND A COMPLETE KÄHLER METRIC

4.1. Cross-Gram determinants. In terms of boundary values on the unit circle, the entries of the Gram matrix H_p of the rational frame (1.3) take the Toeplitz integral form

$$(4.1) \quad (H_p)_{rs} = \int_0^{2\pi} \frac{e^{i(s-r)\theta}}{|Q_p(e^{i\theta})|^2} \frac{d\theta}{2\pi}, \quad 0 \leq r, s < n.$$

More generally, for any pair of parameters $p, q \in \mathcal{B}_n$, we define the cross-Gram matrix $H(p, q) = F_p^* F_q$.

Proposition 4.1. *Let $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ be the roots of P_p and P_q , respectively, listed with multiplicities. Then the determinant of the cross-Gram matrix is given by*

$$(4.2) \quad \det H(p, q) = \prod_{a,b=1}^n (1 - \alpha_a \bar{\beta}_b)^{-1}.$$

This identity holds throughout $\mathcal{B}_n \times \mathcal{B}_n$, across all root collisions in either variable.

Proof. Assume first that P_p has distinct roots $\alpha_1, \dots, \alpha_n$ and P_q has distinct roots $\beta_1, \dots, \beta_n$. Let R_p denote the $n \times n$ transition matrix whose a th column consists of the coefficients in the monomial basis of the degree- $(n-1)$ polynomial

$$\frac{Q_p(z)}{1 - \bar{\alpha}_a z} = \prod_{b \neq a} (1 - \bar{\alpha}_b z).$$

By construction, the tuple of point-evaluation kernels $(k_{\alpha_1}, \dots, k_{\alpha_n})$ is related to the rational frame by $(k_{\alpha_1}, \dots, k_{\alpha_n}) = F_p R_p$. A standard elementary symmetric polynomial calculation shows that $\det R_p$ equals the conjugate Vandermonde determinant $\prod_{a < b} (\bar{\alpha}_b - \bar{\alpha}_a)$. Defining R_q analogously, we can express the cross-Gram matrix of the Cauchy-Szegő kernels as

$$(\langle k_{\alpha_a}, k_{\beta_b} \rangle)_{a,b} = \left(\frac{1}{1 - \alpha_a \bar{\beta}_b} \right)_{a,b} = R_p^* H(p, q) R_q.$$

Taking determinants on both sides and applying Cauchy's determinant identity

$$\det \left(\frac{1}{1 - \alpha_a \bar{\beta}_b} \right)_{a,b} = \frac{\prod_{a < b} (\alpha_b - \alpha_a)(\bar{\beta}_b - \bar{\beta}_a)}{\prod_{a,b=1}^n (1 - \alpha_a \bar{\beta}_b)},$$

we see that the two Vandermonde factors in the numerator cancel $\overline{\det R_p} \det R_q$ identically, leaving (4.2). Since both sides of (4.2) are continuous functions on $\mathcal{B}_n \times \mathcal{B}_n$ and the open locus of distinct roots is dense, the identity extends by continuity to arbitrary root multiplicities. $\square$

Geometrically, the exterior product $\mathbf{w}(p) = f_0(p) \wedge \dots \wedge f_{n-1}(p)$ defines a global nonvanishing anti-holomorphic section of the determinant line bundle $\det \mathcal{E} = \Lambda^n \mathcal{E}$, and the induced Hermitian inner product on $\Lambda^n H^2(\mathbb{D})$ satisfies

$$(4.3) \quad \langle \mathbf{w}(p), \mathbf{w}(q) \rangle = \det H(p, q).$$

Thus (4.2) represents the scalar reproducing kernel associated with the Plücker frame of $\det \mathcal{E}$. Under the identification of $\mathcal{B}_n$ with the symmetrized polydisc via elementary symmetric functions, the right-hand side of (4.2) coincides precisely with the Szegő kernel formula established by Misra, Shyam Roy, and Zhang [8, Theorem 3.1] under the normalization $\|1\| = 1$ (with no extraneous factor of $n!$). Proposition 4.1 therefore provides a natural vector-bundle interpretation of the Szegő kernel of the symmetrized polydisc as the determinant Gram metric of the finite Blaschke model-space bundle.

4.2. The power-sum potential. Restricting (4.2) to the diagonal $p = q$ yields a strictly positive real-analytic function $\det H_p$. Expanding each logarithmic factor $-\log(1 - \alpha_a \bar{\alpha}_b)$ into its convergent power series on $\mathbb{D}$, we obtain the Kähler potential

$$(4.4) \quad \begin{aligned} \Phi(p) &= \log \det H_p = - \sum_{a,b=1}^n \log(1 - \alpha_a \bar{\alpha}_b) \\ &= \sum_{m=1}^{\infty} \frac{|s_m(p)|^2}{m}, \quad s_m(p) = \sum_{a=1}^n \alpha_a^m. \end{aligned}$$

By Newton's identities, each power sum $s_m(p)$ is a holomorphic polynomial in the coefficients $(p_0, \dots, p_{n-1})$. On any relatively compact subdomain of $\mathcal{B}_n$ where the roots satisfy $|\alpha_a| \leq r < 1$, we have the uniform bound $|s_m(p)| \leq nr^m$. For coefficient neighborhoods $U \Subset V \Subset \mathcal{B}_n$, choose $r < 1$ bounding all roots for $p \in V$. Cauchy's estimates in coefficient coordinates give $\sup_U |\partial^\nu s_m| \leq C_\nu nr^m$ for every fixed multi-index ν . Thus the series in (4.4), along with all of its partial derivatives of arbitrary order in $(p, \bar{p})$, converges locally uniformly on $\mathcal{B}_n$, including across the collision locus.

Taking the trace of the curvature $F^{\text{nat}} = \partial(H^{-1} \bar{\partial} H)$ from (3.2) and applying Jacobi's formula for the derivative of a determinant gives

$$(4.5) \quad \text{tr } F^{\text{nat}} = \partial \bar{\partial} \log \det H_p = \partial \bar{\partial} \Phi = \sum_{m \geq 1} \frac{1}{m} ds_m \wedge d\bar{s}_m.$$

Note that the positive sign in $\text{tr } F^{\text{nat}} = +\partial\bar{\partial} \log \det H_p$ reflects the anti-holomorphicity of the frame F_p ; if viewed instead as a holomorphic bundle over the conjugate manifold $\overline{\mathcal{B}_n}$, equation (3.2) recovers the standard Chern connection sign convention. To avoid ambiguity, we work with the original complex coordinates $p \in \mathcal{B}_n$ throughout.

Proposition 4.2. *The Hermitian quadratic form*

$$(4.6) \quad \|v\|_g^2 = \sum_{m \geq 1} \frac{|\text{d}s_m(v)|^2}{m}, \quad v \in T_p \mathcal{B}_n,$$

defines a smooth, real-analytic Kähler metric on $\mathcal{B}_n$, whose associated Kähler form is $\omega_g = (i/2)\partial\bar{\partial}\Phi$ under the length convention of (1.8).

Proof. That ω_g is closed, real-analytic, and positive semidefinite is immediate from (4.4) and (4.5). To verify strict positive definiteness, recall that Newton's identities express the first n power sums $(s_1, \dots, s_n)$ as a triangular polynomial transformation of the elementary symmetric coordinates $(p_{n-1}, \dots, p_0)$, with nonzero constant diagonal entries in its Jacobian matrix. Consequently, if $\|v\|_g = 0$ for some tangent vector $v \in T_p \mathcal{B}_n$, then $\text{d}s_1(v) = \dots = \text{d}s_n(v) = 0$, which forces $v = 0$. $\square$

At the origin $p = 0$, Newton's identities yield the linearization

$$(\partial_{p_i} s_m)(0) = -m\delta_{m,n-i} \quad (1 \leq m \leq n), \quad (\partial_{p_i} s_m)(0) = 0 \quad (m > n),$$

so that the metric tensor at the maximally coincident divisor is diagonal and nondegenerate:

$$(4.7) \quad g_{i\bar{j}}(0) = (n-i)\delta_{ij}, \quad 0 \leq i, j < n.$$

By contrast, if one pulls g back to the branched cover $\mathbb{D}^n$ of ordered roots $(\alpha_1, \dots, \alpha_n)$, the pulled-back metric degenerates along the collision diagonals precisely because the quotient map $\mathbb{D}^n \rightarrow \text{Sym}^n(\mathbb{D}) \simeq \mathcal{B}_n$ is ramified there.

4.3. Completeness.

Theorem 4.3. *The Kähler metric g defined in (4.6) is complete on $\mathcal{B}_n$.*

Proof. Consider the map into the sequence space $\ell^2(\mathbb{N})$ given by

$$(4.8) \quad \mathcal{S} : \mathcal{B}_n \longrightarrow \ell^2, \quad \mathcal{S}(p) = \left(\frac{s_m(p)}{\sqrt{m}} \right)_{m \geq 1}.$$

The local uniform convergence estimates established after (4.4) show that $\mathcal{S}$ is a well-defined holomorphic map into ℓ^2 . By the triangular invertibility of Newton's identities on the first n coordinates $(s_1, \dots, s_n)$, $\mathcal{S}$ is an injective holomorphic immersion whose inverse on the image $\mathcal{S}(\mathcal{B}_n)$ is continuous (indeed, polynomial in the first n components). By definition (4.6), the Kähler metric g is precisely the Riemannian pullback of the flat Hilbert-space metric on ℓ^2 via $\mathcal{S}$.

Next, we verify that the squared norm $\Phi(p) = \|\mathcal{S}(p)\|_{\ell^2}^2$ is a proper exhaustion function on $\mathcal{B}_n$. Grouping off-diagonal terms in (4.4) into conjugate pairs and using the bound $|1 - \alpha_a \bar{\alpha}_b| < 2$ for $\alpha_a, \alpha_b \in \mathbb{D}$, we obtain the lower bound

$$(4.9) \quad \begin{aligned} \Phi(p) &= - \sum_{a=1}^n \log(1 - |\alpha_a|^2) - 2 \sum_{1 \leq a < b \leq n} \log |1 - \alpha_a \bar{\alpha}_b| \\ &\geq - \sum_{a=1}^n \log(1 - |\alpha_a|^2) - n(n-1) \log 2. \end{aligned}$$

It follows that $\Phi(p) \rightarrow +\infty$ whenever $\max_a |\alpha_a| \rightarrow 1$. Since $\mathcal{B}_n$ is bounded in $\mathbb{C}^n$ and the root set depends continuously on the coefficients, a subset of $\mathcal{B}_n$ is relatively compact in $\mathcal{B}_n$ if and only if $\max_a |\alpha_a|$ is bounded away from 1. Inequality (4.9) therefore implies that for every $c \in \mathbb{R}$, the sublevel set $\{p \in \mathcal{B}_n : \Phi(p) \leq c\}$ is compact.

Now let $(p_k)_{k \geq 1}$ be a Cauchy sequence in $\mathcal{B}_n$ with respect to the intrinsic Riemannian distance d_g . Because $\mathcal{S} : (\mathcal{B}_n, g) \rightarrow \ell^2$ is an isometric immersion, the ambient Hilbert-space distance is bounded by the intrinsic path distance:

$$\|\mathcal{S}(p_k) - \mathcal{S}(p_l)\|_{\ell^2} \leq d_g(p_k, p_l) \quad \forall k, l \geq 1.$$

In particular, the sequence $(\|\mathcal{S}(p_k)\|_{\ell^2})_{k \geq 1}$ is bounded, so the entire sequence (p_k) is confined to a single compact sublevel set of $\Phi = \|\mathcal{S}\|_{\ell^2}^2$. Passing to a subsequence, we obtain convergence $p_{k_j} \rightarrow p_\infty \in \mathcal{B}_n$ in the Euclidean topology of $\mathbb{C}^n$. Since g is smooth and positive definite near p_∞ , d_g is locally equivalent to the Euclidean metric, which implies $d_g(p_{k_j}, p_\infty) \rightarrow 0$. As (p_k) is d_g -Cauchy, the full sequence converges to p_∞ in $(\mathcal{B}_n, d_g)$, and completeness follows from the Hopf–Rinow theorem. $\square$

We remark that the isometric immersion $\mathcal{S} : \mathcal{B}_n \rightarrow \ell^2$ plays a decisive role in this proof: in general, a proper Kähler potential alone does not guarantee metric completeness. Here $\Phi = \|\mathcal{S}\|_{\ell^2}^2$ also gives the estimate $|\mathrm{d}\Phi(v)| \leq 2\sqrt{\Phi} \|v\|_g$, and hence $|\mathrm{d}\sqrt{1 + \Phi}(v)| \leq \|v\|_g$.

While Theorem 4.3 characterizes the scalar geometry of the determinant line bundle $\det \mathcal{E}$, it carries no information about the projectivized bundle or the off-diagonal structure of F^{nat} , for which the matrix-valued curvature analysis of Theorem 3.1 is essential.

5. CANONICAL CONJUGATION AND THE RESIDUE PAIRING

5.1. The real structure in coefficient coordinates. Every model space K_{B_p} carries a canonical conjugation (an antiunitary involution) $C_p : K_{B_p} \rightarrow K_{B_p}$, defined in terms of boundary values on $\mathbb{T}$ by

$$(5.1) \quad (C_p f)(\zeta) = B_p(\zeta) \bar{\zeta} \overline{f(\bar{\zeta})}, \quad \zeta \in \mathbb{T}.$$

This antiunitary operator is a cornerstone of the theory of complex symmetric operators and truncated Toeplitz operators on model spaces [4, 5]. Expressed in our global rational frame (1.3), C_p takes a remarkably simple, parameter-independent algebraic form.

Proposition 5.1. *For any polynomial $A(z) = \sum_{r=0}^{n-1} a_r z^r$ of degree less than n , define its degree- $(n-1)$ reflection by*

$$A^\natural(z) = z^{n-1} \overline{A(1/\bar{z})} = \sum_{r=0}^{n-1} \bar{a}_{n-1-r} z^r.$$

Then the canonical conjugation acts on the rational presentation (2.3) via

$$(5.2) \quad C_p(A/Q_p) = A^\natural/Q_p.$$

In particular, C_p is an antiunitary involution satisfying $C_p f_r(p) = f_{n-1-r}(p)$ for all $0 \leq r < n$ and all $p \in \mathcal{B}_n$, and the family $p \mapsto C_p$ defines a real-analytic anti-linear bundle automorphism of $\mathcal{E}$.

Proof. For $|z| = 1$, the identity $\overline{Q_p(z)} = z^{-n} P_p(z)$ implies that

$$C_p(A/Q_p)(z) = \frac{P_p(z)}{Q_p(z)} z^{-1} \frac{\overline{A(z)}}{z^{-n} P_p(z)} = \frac{z^{n-1} \overline{A(1/\bar{z})}}{Q_p(z)} = \frac{A^\natural(z)}{Q_p(z)}.$$

This verifies (5.2), from which $C_p^2 = \mathrm{I}$ and $C_p f_r(p) = f_{n-1-r}(p)$ follow immediately. Since $|B_p(\zeta) \bar{\zeta}| = 1$ almost everywhere on $\mathbb{T}$, C_p is isometric on $L^2(\mathbb{T})$ and hence antiunitary on $\mathcal{E}_p$. Finally, because C_p acts on the global real-analytic frame $(f_0, \dots, f_{n-1})$ by a constant anti-linear permutation of coordinates, C is a real-analytic bundle map on $\mathcal{B}_n$. $\square$

Let J_n denote the $n \times n$ anti-diagonal reversal matrix with entries $(J_n)_{rs} = \delta_{r, n-1-s}$. In the coordinates of the rational frame $\mathbf{F}_p$, the conjugation C_p acts by $a \mapsto J_n \bar{a}$, and the antiunitarity relation $\langle C_p f, C_p h \rangle = \langle h, f \rangle$ is equivalent to the persymmetry of the Gram matrix:

$$(5.3) \quad J_n \bar{H}_p J_n = H_p.$$

This identity is also directly visible from the Hermitian Toeplitz integral representation (4.1).

5.2. The weighted residue formula. Combining the inner product with the canonical conjugation produces a canonical complex-bilinear pairing $g_p : \mathcal{E}_p \times \mathcal{E}_p \rightarrow \mathbb{C}$, defined by

$$(5.4) \quad g_p(f, h) = \langle C_p f, h \rangle.$$

Because C_p is an antiunitary involution, g_p is symmetric and nondegenerate for every $p \in \mathcal{B}_n$.

Proposition 5.2. *For any polynomials $A, D \in \mathbb{C}[z]$ with $\deg A, \deg D < n$, the bilinear form (5.4) admits the contour integral representation*

$$(5.5) \quad g_p(A/Q_p, D/Q_p) = \frac{1}{2\pi i} \int_{|z|=1} \frac{A(z)D(z)}{P_p(z)Q_p(z)} dz.$$

When the roots $\alpha_1, \dots, \alpha_n$ of P_p are simple, (5.5) evaluates to the weighted Lagrange interpolation sum

$$(5.6) \quad g_p(A/Q_p, D/Q_p) = \sum_{a=1}^n \frac{A(\alpha_a)D(\alpha_a)}{P'_p(\alpha_a)Q_p(\alpha_a)}.$$

At parameters with multiple roots, (5.5) evaluates via the total residue of $AD/(P_p Q_p)$ inside $\mathbb{D}$ and remains nondegenerate.

Proof. On the unit circle $|z| = 1$, we have $\overline{(C_p f)(z)} = \overline{B_p(z)} z f(z)$ and $d\theta/(2\pi) = dz/(2\pi i z)$. Consequently,

$$g_p(f, h) = \langle C_p f, h \rangle = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)h(z)}{B_p(z)} dz.$$

Substituting $f = A/Q_p$, $h = D/Q_p$, and $B_p = P_p/Q_p$ yields (5.5). Because Q_p is zero-free on $\overline{\mathbb{D}}$, the integrand is meromorphic on $\overline{\mathbb{D}}$ with poles confined to the zeros of P_p in $\mathbb{D}$, so Cauchy's residue theorem immediately yields (5.6) in the simple-root case. Nondegeneracy across all collision strata is guaranteed both by the operator definition (5.4) and by the invertibility of its Gram matrix $J_n H_p$ in the rational frame. $\square$

For simple roots, formula (5.6) is the level-zero case of the bilinear pairing formula in [4, Section 7.2]. Notice that the presence of the weight factor $Q_p(z)^{-1}$ in the integrand of (5.5) is a genuine imprint of the $H^2(\mathbb{D})$ Hilbert-space metric. To interpret this pairing algebraically within the finite-dimensional quotient algebra $\mathcal{A}_p = \mathbb{C}[z]/(P_p)$, let $\Lambda_p : \mathcal{A}_p \rightarrow \mathbb{C}$ denote the residue functional that extracts the coefficient of z^{n-1} in the unique remainder of degree less than n :

$$\Lambda_p(A) = [z^{n-1}](A \bmod P_p).$$

By Euler's algebraic residue identity, (5.5) can be rewritten intrinsically as

$$(5.7) \quad g_p(A/Q_p, D/Q_p) = \Lambda_p(ADQ_p^{-1} \bmod P_p),$$

where Q_p is invertible in $\mathcal{A}_p$ because P_p and Q_p have no common zeros. While the unweighted Frobenius pairing $(A, D) \mapsto \Lambda_p(AD \bmod P_p)$ also defines a nondegenerate symmetric bilinear form on $\mathcal{A}_p$, it differs from the Hilbert-space pairing g_p by the multiplication operator $Q_p^{-1} \bmod P_p$.

To illustrate this distinction, consider the degree-two family $P_a(z) = z(z - a)$ with $a \in \mathbb{D}$, for which $Q_a(z) = 1 - \bar{a}z$. In the rational frame (f_0, f_1) , a direct residue calculation gives

$$(5.8) \quad H_a = \frac{1}{1 - |a|^2} \begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix}, \quad J_2 H_a = \frac{1}{1 - |a|^2} \begin{pmatrix} \bar{a} & 1 \\ 1 & a \end{pmatrix},$$

whereas the unweighted Frobenius pairing $\Lambda_p(AD)$ has matrix $\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$. By contrast, for the symmetric family $P_t(z) = z^n - t$ ($t \in \mathbb{D}$), the reflected polynomial $Q_t(z) = 1 - \bar{t}z^n$ satisfies $Q_t(z) \equiv 1 - |t|^2 \pmod{P_t(z)}$. Rescaling the rational frame to the orthonormal basis

$$(5.9) \quad e_r(t; z) = \frac{\sqrt{1 - |t|^2} z^r}{1 - \bar{t}z^n}, \quad 0 \leq r < n,$$

absorbs the constant weight $1 - |t|^2$ and reduces g_t to the standard anti-diagonal pairing J_n .

Corollary 5.3. *The natural connection ∇^{nat} does not preserve the canonical conjugation C_p , i.e., $\nabla^{\text{nat}} C \neq 0$. More generally, on any connected open neighborhood U of $0 \in \mathcal{B}_n$, the bundle $(\mathcal{E}|_U, \nabla^{\text{nat}})$ admits no parallel antiunitary involution.*

Proof. If a unitary connection preserves an antiunitary involution C , then parallel transport commutes with C and its holonomy group at p is contained in the real orthogonal subgroup $O(\mathcal{E}_p, C) \simeq O(n) \subset U(n)$ of unitary operators commuting with C . By Theorem 3.2, however, the restricted holonomy group on U is the full unitary group $U(n)$, which is strictly larger than $O(n)$ for every $n \geq 1$ (including $n = 1$, where $O(1) = \{\pm 1\} \subsetneq \mathbb{T} = U(1)$). $\square$

6. CLARK SPECTRAL LINES AND THE FLAT REAL-SPECTRAL CONNECTION

6.1. The Clark unitary and its boundary kernels. Let $S_p = \Pi_p M_z|_{\mathcal{E}_p}$ denote the compressed shift operator on the model space $\mathcal{E}_p$, and introduce the defect vectors

$$k_0(z) = K_p(z, 0) = 1 - \overline{B_p(0)}B_p(z), \quad \ell_0(z) = \frac{B_p(z) - B_p(0)}{z}.$$

For each phase parameter $\lambda \in \mathbb{T}$, the classical Clark unitary operator [2, 7] is the rank-one perturbation of S_p given in our inner-product convention by

$$(6.1) \quad U_{p,\lambda} = S_p + \frac{\lambda}{1 - \lambda \overline{B_p(0)}} |k_0\rangle \langle \ell_0|,$$

where the Dirac rank-one operator acts by $|u\rangle \langle v|f = \langle v, f\rangle u$. Since B_p is rational and holomorphic across $\mathbb{T}$, for every boundary point $\zeta \in \mathbb{T}$ satisfying $B_p(\zeta) = \lambda$, the boundary reproducing kernel $k_\zeta(z) = K_p(z, \zeta)$ belongs to $\mathcal{E}_p$. Using the elementary action of the backward shift $S_p^* = M_z^*|_{\mathcal{E}_p}$ on reproducing kernels,

$$(6.2) \quad S_p^* k_w = \bar{w} k_w - \overline{B_p(w)} \ell_0, \quad \langle k_0, k_w \rangle = k_w(0) = 1 - \overline{B_p(w)} B_p(0),$$

and specializing to $w = \zeta \in \mathbb{T}$ with $B_p(\zeta) = \lambda$, one verifies directly that

$$(6.3) \quad U_{p,\lambda}^* k_\zeta = \bar{\zeta} k_\zeta, \quad \text{or equivalently} \quad U_{p,\lambda} k_\zeta = \zeta k_\zeta.$$

Our next step is to construct a global real-analytic ordering of these boundary eigenlines over the universal cover of the phase circle, valid seamlessly across all interior root collisions.

6.2. A global boundary argument and uniform separation. Because Q_p has no zeros on the simply connected closed disk $\overline{\mathbb{D}}$, it admits a unique holomorphic logarithm $\log Q_p$ on an open neighborhood of $\overline{\mathbb{D}}$ normalized by $\log Q_p(0) = 0$. For $\varphi \in \mathbb{R}$, we define the unwrapped boundary phase function

$$(6.4) \quad \begin{aligned} \Psi_p(\varphi) &= n\varphi - 2 \operatorname{Im} \log Q_p(e^{i\varphi}) \\ &= n\varphi + 2 \sum_{a=1}^n \operatorname{Arg}(1 - \alpha_a e^{-i\varphi}), \end{aligned}$$

where $\operatorname{Arg} : \{\operatorname{Re} w > 0\} \rightarrow (-\pi/2, \pi/2)$ denotes the principal branch of the argument. While the second expression decomposes Ψ_p into contributions from the individual roots, the first expression shows directly that $\Psi_p(\varphi)$ is a real-analytic function of $(p, \varphi) \in \mathcal{B}_n \times \mathbb{R}$.

Lemma 6.1. *For every $p \in \mathcal{B}_n$ and $\varphi \in \mathbb{R}$, the boundary phase function satisfies*

$$(6.5) \quad B_p(e^{i\varphi}) = e^{i\Psi_p(\varphi)}, \quad \Psi'_p(\varphi) = |B'_p(e^{i\varphi})| = \sum_{a=1}^n \frac{1 - |\alpha_a|^2}{|e^{i\varphi} - \alpha_a|^2} > 0,$$

together with the quasi-periodicity relation $\Psi_p(\varphi + 2\pi) = \Psi_p(\varphi) + 2\pi n$. Consequently, for every real phase angle $\theta \in \mathbb{R}$ and every integer $j \in \mathbb{Z}$, the phase equation

$$(6.6) \quad \Psi_p(\varphi_j) = \theta + 2\pi j$$

admits a unique real solution $\varphi_j = \varphi_j(p, \theta)$, which depends real-analytically on $(p, \theta) \in \mathcal{B}_n \times \mathbb{R}$.

Proof. On $\mathbb{T}$, the identity $B_p(e^{i\varphi}) = e^{in\varphi} \overline{Q_p(e^{i\varphi})} / Q_p(e^{i\varphi})$ yields the first relation in (6.5), and differentiating the logarithmic argument of each Blaschke factor produces the positive Poisson-kernel sum for $\Psi'_p(\varphi)$. Because $\Psi'_p(\varphi) > 0$ everywhere and $\Psi_p(\varphi + 2\pi) = \Psi_p(\varphi) + 2\pi n$, the map $\varphi \mapsto \Psi_p(\varphi)$ is a real-analytic diffeomorphism of $\mathbb{R}$ onto itself. Existence and uniqueness of $\varphi_j(p, \theta)$ are thus immediate, and global real analyticity in (p, θ) follows from the real-analytic implicit function theorem. $\square$

For $0 \leq j < n$, the points $\zeta_j(p, \theta) = e^{i\varphi_j(p, \theta)} \in \mathbb{T}$ constitute the n distinct solutions to the Clark spectral equation $B_p(\zeta) = e^{i\theta}$. Evaluating the reproducing kernel (2.2) at these boundary points and applying L'Hôpital's rule along the diagonal gives the classical Clark orthogonality and norm relations

$$(6.7) \quad \langle k_{\zeta_i}, k_{\zeta_j} \rangle = 0 \quad (i \neq j), \quad \|k_{\zeta_j}\|_{H^2}^2 = |B'_p(\zeta_j)| = \Psi'_p(\varphi_j).$$

In particular, the boundary kernels $\{k_{\zeta_0}, \dots, k_{\zeta_{n-1}}\}$ form an orthogonal eigenbasis for $U_{p, e^{i\theta}}$, confirming its unitarity and yielding the Shannon–Clark sampling identity

$$(6.8) \quad \|f\|_{H^2}^2 = \sum_{j=0}^{n-1} \frac{|f(\zeta_j)|^2}{\Psi'_p(\varphi_j)} \quad \forall f \in \mathcal{E}_p.$$

Notice that the real analyticity of the map $(p, \theta) \mapsto k_{\zeta_j(p, \theta)} \in H^2(\mathbb{D})$ can be seen directly without differentiating the difference quotient in (2.2) across the boundary diagonal $z = \zeta$. Indeed, letting $\mathbf{f}_p(w) = (f_0(p; w), \dots, f_{n-1}(p; w))^t \in \mathbb{C}^n$, the reproducing property of $\Pi_p = F_p H_p^{-1} F_p^*$ expresses k_w for any $w \in \overline{\mathbb{D}}$ in terms of the global rational frame as

$$(6.9) \quad k_w = F_p H_p^{-1} \overline{\mathbf{f}_p(w)}.$$

Because F_p , H_p^{-1} , and the evaluation map $(p, w) \mapsto \mathbf{f}_p(w)$ are all real-analytic on $\mathcal{B}_n \times \mathbb{T}$, composing (6.9) with the real-analytic root maps $w = \zeta_j(p, \theta)$ from Lemma 6.1 establishes that the boundary eigenvectors $k_{\zeta_j(p, \theta)}$ depend real-analytically on $(p, \theta) \in \mathcal{B}_n \times \mathbb{R}$ in the $H^2(\mathbb{D})$ norm, completely unaffected by internal root collisions.

Proposition 6.2. *Suppose that $p \in \mathcal{B}_n$ has all its roots in the closed disk $|\alpha_a| \leq r_0 < 1$. Then the phase derivative Ψ'_p satisfies the uniform two-sided estimate*

$$(6.10) \quad n \frac{1 - r_0}{1 + r_0} \leq \Psi'_p(\varphi) \leq n \frac{1 + r_0}{1 - r_0} \quad \forall \varphi \in \mathbb{R}.$$

Consequently, consecutive angular roots satisfy $\varphi_{j+1}(p, \theta) - \varphi_j(p, \theta) \geq \delta_0$, where

$$(6.11) \quad \delta_0 = \frac{2\pi(1 - r_0)}{n(1 + r_0)}.$$

In particular, for $n \geq 2$, any two distinct eigenvalues of $U_{p, \lambda}$ on $\mathbb{T}$ are separated by a chordal distance of at least $2 \sin(\delta_0/2) > 0$, and the sampling weights $\Psi'_p(\varphi_j)^{-1}$ in (6.8) are uniformly bounded above and below by positive constants depending only on n and r_0 .

Proof. For $|\alpha_a| \leq r_0$, each Poisson kernel in (6.5) is bounded between $(1 - r_0)/(1 + r_0)$ and $(1 + r_0)/(1 - r_0)$, which yields (6.10) upon summing over $1 \leq a \leq n$. Integrating Ψ'_p over the interval $[\varphi_j, \varphi_{j+1}]$ and using $\Psi_p(\varphi_{j+1}) - \Psi_p(\varphi_j) = 2\pi$ gives

$$2\pi = \int_{\varphi_j}^{\varphi_{j+1}} \Psi'_p(\varphi) d\varphi \leq n \frac{1 + r_0}{1 - r_0} (\varphi_{j+1} - \varphi_j),$$

which rearranges to the angular gap estimate $\varphi_{j+1} - \varphi_j \geq \delta_0$. Because both complementary arcs on $\mathbb{T}$ between any two distinct eigenvalues ζ_i and ζ_j contain at least one interval between consecutive roots, their shortest angular distance is at least $\delta_0 \leq \pi$, giving the chordal distance bound $2 \sin(\delta_0/2)$. $\square$

Proposition 6.2 underscores a fundamental structural feature: while the interior zeros α_a of B_p may collide freely inside $\mathbb{D}$, the boundary spectrum $\{\zeta_0, \dots, \zeta_{n-1}\}$ of the Clark unitary remains simple on $\mathbb{T}$, with a uniform separation bound whenever all interior roots stay in a fixed disk $|\alpha_a| \leq r_0 < 1$.

6.3. Conjugation-real spectral frames. Whenever $\zeta \in \mathbb{T}$ satisfies $B_p(\zeta) = \lambda \in \mathbb{T}$, the action of the canonical conjugation C_p on the boundary kernel k_ζ is given by

$$(6.12) \quad C_p k_\zeta = \lambda \bar{\zeta} k_\zeta.$$

Indeed, for $z \in \mathbb{T} \setminus \{\zeta\}$, definition (5.1) gives

$$(C_p k_\zeta)(z) = B_p(z) \bar{z} \frac{1 - \lambda \overline{B_p(z)}}{1 - \zeta \bar{z}} = \frac{B_p(z) - \lambda}{z - \zeta} = \lambda \bar{\zeta} \frac{1 - \bar{\lambda} B_p(z)}{1 - \bar{\zeta} z} = \lambda \bar{\zeta} k_\zeta(z),$$

and the equality extends to $\mathbb{D}$ by analytic continuation. Using the unwrapped phase $\varphi_j(p, \theta)$ from Lemma 6.1 to take a canonical square root of the phase factor $\lambda \bar{\zeta}_j = e^{i(\theta - \varphi_j)}$, we define on the universal phase cover $\mathcal{B}_n \times \mathbb{R}$ the normalized vectors

$$(6.13) \quad v_j(p, \theta) = (-1)^j e^{i(\theta - \varphi_j(p, \theta))/2} \frac{k_{\zeta_j(p, \theta)}}{\sqrt{\Psi'_p(\varphi_j(p, \theta))}}, \quad 0 \leq j < n.$$

By (6.7) and (6.12), the tuple $(v_0(p, \theta), \dots, v_{n-1}(p, \theta))$ forms a real-analytic orthonormal eigenbasis of $\mathcal{E}_p$ consisting of C_p -real vectors: $C_p v_j(p, \theta) = v_j(p, \theta)$. While modified Clark bases fixed by C_p are well known for individual inner functions [9], the unwrapped phase prescription (6.13) renders their global dependence on (p, θ) completely explicit and single-valued on $\mathcal{B}_n \times \mathbb{R}$.

6.4. Existence, uniqueness and descent. Let $\pi : \widehat{\mathcal{B}}_n = \mathcal{B}_n \times \mathbb{T} \rightarrow \mathcal{B}_n$ be the canonical projection, and denote the pulled-back bundle $\pi^* \mathcal{E}$ and conjugation $\pi^* C$ by the same symbols. Over the universal phase cover $\mathcal{B}_n \times \mathbb{R}$, we assemble the real-analytic orthonormal frame (6.13) into the isometric synthesis operator

$$V(p, \theta) : \mathbb{C}^n \longrightarrow H^2(\mathbb{D}), \quad V(p, \theta) e_j = v_j(p, \theta),$$

which satisfies $V(p, \theta) V(p, \theta)^* = \Pi_p$.

Theorem 6.3. *On the phase cover $\mathcal{B}_n \times \mathbb{R}$, the prescription*

$$(6.14) \quad \nabla^{\text{sp}} s = V d(V^* s)$$

defines a flat unitary connection on the pullback of $\mathcal{E}$. Moreover, ∇^{sp} descends to a well-defined, real-analytic flat unitary connection on $\widehat{\mathcal{B}}_n = \mathcal{B}_n \times \mathbb{T}$, and it is the unique unitary connection on $\mathcal{E} \rightarrow \widehat{\mathcal{B}}_n$ under which both the canonical conjugation C and all locally labelled Clark spectral-line projections $\Pi_j = |v_j\rangle\langle v_j|$ are parallel.

Proof. By definition (6.14), the frame $V = (v_0, \dots, v_{n-1})$ is parallel with respect to ∇^{sp} , so ∇^{sp} reduces to the exterior differential d in the coordinates of V . Because $V(p, \theta)$ is an orthonormal frame of C_p -real eigenvectors at every point, ∇^{sp} is flat, unitary, and preserves both C and each spectral-line projection $\Pi_j = |v_j\rangle\langle v_j|$.

To establish uniqueness, suppose $\tilde{\nabla} = \nabla^{\text{sp}} + \Omega$ is another unitary connection on a local chart that preserves each line projection Π_j and the conjugation C . Because $\tilde{\nabla} \Pi_j = 0$, the connection one-form Ω in the frame V must be diagonal: $\Omega = \text{diag}(\omega_0, \dots, \omega_{n-1})$. Compatibility with the Hermitian metric requires each scalar one-form ω_j to be purely imaginary, whereas compatibility with the real structure ($\tilde{\nabla} C = 0$, with C acting as complex conjugation in the C -real frame V) forces each ω_j to be real-valued. Hence $\omega_j = 0$ for all j , proving local uniqueness.

To verify that ∇^{sp} descends from $\mathcal{B}_n \times \mathbb{R}$ to $\hat{\mathcal{B}}_n = \mathcal{B}_n \times \mathbb{T}$, observe from the defining equation (6.6) and the quasi-periodicity $\Psi_p(\varphi + 2\pi) = \Psi_p(\varphi) + 2\pi n$ that shifting $\theta \mapsto \theta + 2\pi$ shifts the unwrapped roots according to

$$\varphi_j(p, \theta + 2\pi) = \varphi_{j+1}(p, \theta) \quad (0 \leq j < n-1), \quad \varphi_{n-1}(p, \theta + 2\pi) = \varphi_0(p, \theta) + 2\pi.$$

Substituting these shifts into the definition (6.13) of $v_j(p, \theta)$ yields the deck transformation formulas

$$(6.15) \quad \begin{aligned} v_j(p, \theta + 2\pi) &= v_{j+1}(p, \theta), & 0 \leq j < n-1, \\ v_{n-1}(p, \theta + 2\pi) &= (-1)^{n-1} v_0(p, \theta). \end{aligned}$$

In matrix notation, $V(p, \theta + 2\pi) = V(p, \theta)C_\epsilon$, where $C_\epsilon \in U(n)$ is the constant signed cyclic permutation matrix defined in (1.9). Because the transition matrix C_ϵ is constant, $d(C_\epsilon^* V^* s) = C_\epsilon^* d(V^* s)$, ensuring that $\nabla^{\text{sp}} s = V d(V^* s)$ is invariant under $\theta \mapsto \theta + 2\pi$ and thus descends to $\hat{\mathcal{B}}_n$. $\square$

From a principal-bundle viewpoint, equipping each fiber $\mathcal{E}_p$ with an unordered n -tuple of mutually orthogonal lines together with a real structure C_p reduces the structure group from $U(n)$ to the finite hyperoctahedral group of signed permutation matrices. Theorem 6.3 states that ∇^{sp} is the canonical flat connection induced by this discrete reduction of structure group, and shows that it extends real-analytically across the interior collision discriminant with deck transformation C_ϵ . Note that requiring only unitarity and preservation of the spectral lines Π_j (without C_p) would reduce the structure group only to the normalizer of the maximal torus $\mathbb{T}^n \subset U(n)$, leaving an n -dimensional torus of ambiguous imaginary connection one-forms $\text{diag}(\omega_0, \dots, \omega_{n-1})$.

6.5. The holonomy representation.

Theorem 6.4. *With respect to the conjugation-real frame $V(p, 0)$ at a base point $(p, 1) \in \mathcal{B}_n \times \mathbb{T}$, the holonomy representation of the flat connection ∇^{sp} is given by*

$$(6.16) \quad \rho : \pi_1(\mathcal{B}_n \times \mathbb{T}) \simeq \mathbb{Z} \longrightarrow U(n), \quad \rho(1) = C_\epsilon, \quad \epsilon = (-1)^{n-1}.$$

The monodromy generator C_ϵ satisfies the algebraic identities

$$(6.17) \quad C_\epsilon^n = \epsilon I, \quad \det C_\epsilon = 1, \quad \det(\mu I - C_\epsilon) = \mu^n - \epsilon.$$

Consequently, the holonomy group $\text{Hol}(\nabla^{\text{sp}})$ is cyclic of order n when n is odd and of order $2n$ when n is even; in particular, for $n = 1$ the holonomy is trivial.

Proof. Since $\mathcal{B}_n$ is contractible (Lemma 2.1), the fundamental group $\pi_1(\mathcal{B}_n \times \mathbb{T}, (p, 1))$ is infinite cyclic, generated by the positively oriented phase loop $\theta \in [0, 2\pi]$ at fixed p . Any ∇^{sp} -parallel section along this loop has constant coordinates in the frame $V(p, \theta)$, so transporting a vector from $\theta = 0$ to $\theta = 2\pi$ multiplies its coordinates in the initial frame $V(p, 0)$ by the transition matrix C_ϵ from (6.15). Iterating the cyclic shift n times accumulates the single wrap sign $\epsilon = (-1)^{n-1}$, giving $C_\epsilon^n = \epsilon I$ and $\det(\mu I - C_\epsilon) = \mu^n - \epsilon$. Finally, since the underlying n -cycle permutation has parity $(-1)^{n-1}$, multiplying its final entry by $\epsilon = (-1)^{n-1}$ yields $\det C_\epsilon = (-1)^{n-1} \epsilon = 1$. The order of C_ϵ is immediate from $C_\epsilon^n = (-1)^{n-1} I$. $\square$

Corollary 6.5. *At any fixed Clark phase $\lambda \in \mathbb{T}$, every closed loop in the divisor space $\mathcal{B}_n$ has trivial ∇^{sp} -holonomy. In particular, any loop confined to the configuration space of distinct roots that represents a nontrivial braid of the interior zeros $\alpha_1, \dots, \alpha_n$ induces the identity holonomy transformation under ∇^{sp} .*

Proof. Fix a real lift $\theta \in \mathbb{R}$ with $e^{i\theta} = \lambda$. Over the slice $\mathcal{B}_n \times \{\theta\} \simeq \mathcal{B}_n$, the global section $p \mapsto V(p, \theta)$ provides a single-valued parallel frame across all of $\mathcal{B}_n$. Consequently, parallel transport along any closed loop in $\mathcal{B}_n \times \{\lambda\}$ is trivial; topologically, any braid loop in the distinct-root locus can be contracted across the collision discriminant inside $\mathcal{B}_n$, across which ∇^{sp} remains flat and regular. $\square$

Corollary 6.5 highlights a crucial conceptual distinction between three types of transport: the braiding of labelled interior roots in $\mathbb{D}$, the continuous $U(n)$ holonomy of ∇^{nat} driven by interior curvature on $\mathcal{B}_n$, and the discrete cyclic monodromy of ∇^{sp} driven by the boundary Clark phase $\lambda \in \mathbb{T}$. While cyclic permutation of eigenvalues along the phase circle is a classical feature of rank-one unitary perturbations [11], Theorem 6.4 shows that the canonical conjugation C_p rigidifies the eigenline phases globally and forces the signed monodromy law $C_\epsilon^n = (-1)^{n-1}I$ across the entire collision family.

6.6. Kato transport at fixed divisor. When the interior divisor $p \in \mathcal{B}_n$ is held fixed, the ambient subspace $\mathcal{E}_p \subset H^2(\mathbb{D})$ and the conjugation C_p are stationary, while the orthonormal eigenbasis $v_j(p, \theta)$ rotates inside $\mathcal{E}_p$ as θ varies. Because each vector $v_j(p, \theta)$ has unit norm and satisfies $C_p v_j = v_j$, differentiating $\langle v_j, v_j \rangle = 1$ and using the antiunitarity of C_p gives

$$(6.18) \quad \langle v_j, \partial_\theta v_j \rangle = \langle C_p \partial_\theta v_j, C_p v_j \rangle = \langle \partial_\theta v_j, v_j \rangle = -\langle v_j, \partial_\theta v_j \rangle, \quad \text{so} \quad \langle v_j, \partial_\theta v_j \rangle = 0.$$

In terms of the rank-one spectral projections $\Pi_j(p, \theta) = |v_j(p, \theta)\rangle\langle v_j(p, \theta)|$, let us introduce the Kato adiabatic generator [6]

$$(6.19) \quad \mathcal{K}_\theta = \sum_{j=0}^{n-1} (\partial_\theta \Pi_j) \Pi_j : \mathcal{E}_p \longrightarrow \mathcal{E}_p.$$

Using (6.18) and the completeness relation $\sum_j \Pi_j = I_{\mathcal{E}_p}$, one checks directly that $\mathcal{K}_\theta v_j = \partial_\theta v_j$, so that

$$\mathcal{K}_\theta = (\partial_\theta V) V^*, \quad \mathcal{K}_\theta^* = -\mathcal{K}_\theta, \quad \partial_\theta \Pi_j = [\mathcal{K}_\theta, \Pi_j].$$

Consequently, a section $s(\theta) = V(p, \theta)c(\theta) \in \mathcal{E}_p$ is ∇^{sp} -parallel along the phase circle ($c'(\theta) = 0$) if and only if it satisfies the Kato transport equation $\partial_\theta s = \mathcal{K}_\theta s$. By contrast, the pullback $\pi^* \nabla^{\text{nat}}$ along a fixed-divisor phase circle simply coincides with the ambient derivative $\partial_\theta s$ in the stationary subspace $\mathcal{E}_p$, and therefore has trivial holonomy around $\mathbb{T}$.

At the maximally coincident divisor $p = 0$, where $B_0(z) = z^n$ and $\varphi_j(0, \theta) = (\theta + 2\pi j)/n$, the conjugation-real Clark basis (6.13) is given explicitly by the discrete Fourier basis

$$v_j(0, \theta; z) = \frac{(-1)^j}{\sqrt{n}} e^{i\theta/2} \sum_{r=0}^{n-1} e^{-i(r+1/2)(\theta+2\pi j)/n} z^r.$$

Introducing the centered diagonal generator

$$(6.20) \quad R_n = \text{diag} \left(\frac{r - (n-1)/2}{n} \right)_{r=0}^{n-1},$$

we see that in the stationary monomial basis $(1, z, \dots, z^{n-1})$ of $\mathcal{E}_0$, the frame evolution and Kato generator take the constant-coefficient forms

$$(6.21) \quad V(0, \theta) = e^{-i\theta R_n} V(0, 0), \quad \mathcal{K}_\theta = -iR_n.$$

In the fixed monomial frame of $\mathcal{E}_0$, the connection form of ∇^{sp} along $\{0\} \times \mathbb{T}$ is therefore $iR_n d\theta$, and integrating from $\theta = 0$ to 2π yields the holonomy operator $e^{-2\pi i R_n}$, which is unitarily conjugate via $V(0, 0)$ to C_ϵ . Notice that the centering shift by $(n-1)/2$ in (6.20) ensures $\text{tr } R_n = 0$ (and hence $\det e^{-2\pi i R_n} = 1$) and originates directly from the phase factor $e^{i(\theta-\varphi_j)/2}$ dictated by the canonical conjugation C_0 .

Finally, we note that while ∇^{sp} makes every spectral projection Π_j parallel, the Clark unitary operator $U_{p, e^{i\theta}} = \sum_{j=0}^{n-1} \zeta_j \Pi_j$ itself is not parallel; rather, its covariant derivative records the motion of the boundary eigenvalues:

$$(6.22) \quad U_{p, e^{i\theta}} = \sum_{j=0}^{n-1} \zeta_j \Pi_j \implies \nabla^{\text{sp}} U = \sum_{j=0}^{n-1} d\zeta_j \Pi_j.$$

Thus the parallel transport of the eigenlines and the drift of the eigenvalues along $\mathbb{T}$ are cleanly decoupled.

7. COMPARISON AND EXPLICIT SUBFAMILIES

7.1. The difference between the two connections. Over the universal phase cover $\mathcal{B}_n \times \mathbb{R}$, expressing both connections in the conjugation-real orthonormal frame $V(p, \theta)$ yields

$$(7.1) \quad \pi^* \nabla^{\text{nat}} = d + \mathcal{A}, \quad \mathcal{A} = V^* dV, \quad \nabla^{\text{sp}} = d.$$

Their difference $\pi^* \nabla^{\text{nat}} - \nabla^{\text{sp}} = \mathcal{A}$ is a globally defined skew-Hermitian endomorphism-valued one-form, which descends to $\hat{\mathcal{B}}_n = \mathcal{B}_n \times \mathbb{T}$ because the deck transformation matrix C_ϵ is constant: $C_\epsilon^*(V^* dV)C_\epsilon = (VC_\epsilon)^* d(VC_\epsilon)$.

Proposition 7.1. *On $\hat{\mathcal{B}}_n = \mathcal{B}_n \times \mathbb{T}$, the connections $\pi^* \nabla^{\text{nat}}$ and ∇^{sp} are not unitarily gauge equivalent. Specifically, $\pi^* \nabla^{\text{nat}}$ has full holonomy group $U(n)$ and nonvanishing determinant curvature, whereas ∇^{sp} is flat, has the finite cyclic holonomy group of Theorem 6.4, and induces a flat connection with trivial holonomy on $\det \mathcal{E}$.*

Proof. Under any unitary gauge transformation of $\mathcal{E}$, the curvature two-form transforms by conjugation. At the origin $p = 0$, Theorem 3.1 shows that $(F^{\text{nat}})_{00} = I \neq 0$, whereas $F^{\text{sp}} = 0$ identically. The claims regarding $\det \mathcal{E}$ follow from $\text{tr } F^{\text{nat}} = \partial \bar{\partial} \Phi \neq 0$ and $\det C_\epsilon = 1$, while the equality $\text{Hol}(\pi^* \nabla^{\text{nat}}) = U(n)$ holds because loops in $\mathcal{B}_n$ at any fixed phase $\lambda \in \mathbb{T}$ already generate $U(n)$ by Theorem 3.2. $\square$

To isolate the role of the real structure C_p from that of the orthogonal eigenline decomposition $\mathcal{E}_p = \bigoplus_{j=0}^{n-1} \text{ran } \Pi_j$, it is instructive to consider the intermediate adiabatic (or Berry-type) splitting connection defined by projecting $\pi^* \nabla^{\text{nat}}$ diagonally onto the Clark eigenlines:

$$(7.2) \quad \nabla^{\text{split}} = \sum_{j=0}^{n-1} \Pi_j (\pi^* \nabla^{\text{nat}}) \Pi_j.$$

By construction, ∇^{split} is a unitary connection on $\hat{\mathcal{B}}_n$ that preserves each spectral line Π_j . In the frame V , its connection matrix is simply the diagonal part $\text{diag}(\mathcal{A}_{00}, \dots, \mathcal{A}_{n-1, n-1})$ of $\mathcal{A} = V^* dV$. In general, however, $\mathcal{A}_{jj} = \langle v_j, dv_j \rangle$ does not vanish in the divisor directions dp , so ∇^{split} fails to preserve the canonical conjugation C_p , and the individual eigenlines carry nonvanishing, generically distinct Berry curvatures $d\mathcal{A}_{jj}$ (see Subsection 7.5 below). The uniqueness statement in Theorem 6.3 shows that imposing parallelity of C_p kills precisely these diagonal imaginary one-forms $\mathcal{A}_{jj}$ and forces the connection to be flat.

7.2. A symmetric collision disk. Consider the one-parameter symmetric subfamily

$$P_t(z) = z^n - t, \quad t \in \mathbb{D},$$

for which the rescaled rational basis $e_r(t; z) = \sqrt{1 - |t|^2} z^r / (1 - \bar{t} z^n)$ from (5.9) is orthonormal. Expanding $e_r(t; z) = \sqrt{1 - |t|^2} \sum_{k \geq 0} \bar{t}^k z^{kn+r}$ and computing inner products $\langle e_r, de_s \rangle$ in $H^2(\mathbb{D})$ yields the diagonal connection and scalar curvature forms

$$(7.3) \quad A_{rs}^{\text{nat}} = \delta_{rs} \frac{t d\bar{t} - \bar{t} dt}{2(1 - |t|^2)}, \quad F^{\text{nat}} = \frac{dt \wedge d\bar{t}}{(1 - |t|^2)^2} I_n.$$

The restricted determinant potential and induced Kähler metric on the t -disk are multiples of the hyperbolic Poincaré metric:

$$\Phi(t) = -n \log(1 - |t|^2), \quad \|v\|_g^2 = \frac{n|v|^2}{(1 - |t|^2)^2}.$$

Integrating A^{nat} along the positively oriented circle $t = re^{i\vartheta}$ ($0 \leq \vartheta \leq 2\pi$) shows that ∇^{nat} -parallel transport around the circle is the scalar unitary multiplication

$$(7.4) \quad \exp\left(\frac{2\pi i r^2}{1 - r^2}\right) I_n.$$

Notice that whenever $r^2/(1-r^2) \in \mathbb{Z}$, this specific loop has trivial holonomy even though the curvature (7.3) is strictly positive everywhere.

In this symmetric subfamily, if one varies the Clark phase parameter along the section $\lambda : \mathbb{D} \rightarrow \mathbb{T}$ given by

$$(7.5) \quad \lambda(t) = \frac{1-t}{1-\bar{t}},$$

then the solutions to $B_t(\zeta) = \lambda(t)$ remain stationary at the n th roots of unity $\zeta_j = e^{2\pi i j/n}$. Along the graph of $\lambda(t)$, the conjugation-real Clark basis $v_j(t, \theta(t))$ differs from the orthonormal monomial frame $e_r(t)$ by a constant unitary discrete Fourier matrix. Indeed, choosing $\theta(t) = 2 \operatorname{Arg}(1-t)$ gives $\varphi_j = 2\pi j/n$, and the normalized boundary kernel satisfies

$$(7.6) \quad \frac{k_{\zeta_j}}{\|k_{\zeta_j}\|} = \frac{1-\bar{t}}{|1-t|} \frac{1}{\sqrt{n}} \sum_{r=0}^{n-1} \bar{\zeta}_j^r e_r(t).$$

The phase $(1-\bar{t})/|1-t|$ is cancelled by $e^{i\theta(t)/2} = (1-t)/|1-t|$ in (6.13); the remaining factors are independent of t . Consequently, in the frame $e_r(t)$, the connection form of ∇^{sp} vanishes identically along the section (7.5). Moreover, because $\operatorname{Re}(1-t) > 0$ for all $t \in \mathbb{D}$, the phase map (7.5) has winding number zero around any closed t -loop, so the ∇^{sp} -holonomy around $t = re^{i\theta}$ is the identity, in sharp contrast to the area-dependent phase (7.4) of ∇^{nat} .

7.3. A disk contained in the maximal collision stratum. Consider next the one-parameter family of completely coincident divisors

$$P_a(z) = (z-a)^n, \quad a \in \mathbb{D},$$

which embeds $\mathbb{D}$ onto the highest-multiplicity stratum of the collision discriminant in $\mathcal{B}_n$. Along this disk, Proposition 4.1 gives

$$(7.7) \quad \Phi(a) = -n^2 \log(1-|a|^2), \quad \operatorname{tr} F^{\text{nat}} = \frac{n^2 da \wedge d\bar{a}}{(1-|a|^2)^2}.$$

At $a = 0$, the embedding $a \mapsto p(a)$ has differential $dp_{n-1} = -n da$ and $dp_i = 0$ for $i < n-1$. Pulling back the curvature formula (1.4) therefore yields

$$(7.8) \quad F^{\text{nat}}|_{a=0} = n^2 E_{n-1, n-1} da \wedge d\bar{a}.$$

Thus, for $n > 1$, the curvature restricted to the maximal collision stratum is a rank-one matrix at $a = 0$, in contrast to the scalar identity curvature (7.3) of the transverse symmetric disk $P_t(z) = z^n - t$ passing through the very same point $p = 0$. This demonstrates that the algebraic structure of the restricted curvature at a collision point depends on the tangent direction of the subfamily rather than on the multiplicity of the divisor alone.

7.4. Noncommuting directions in degree two. For $n = 2$, parameterize $\mathcal{B}_2$ by

$$P_{q,t}(z) = z^2 - qz - t, \quad (p_0, p_1) = (-t, -q).$$

At the origin $(q, t) = (0, 0)$, Theorem 3.1 gives the four curvature coefficient matrices

$$(7.9) \quad \begin{aligned} F_{q\bar{q}} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & F_{q\bar{t}} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ F_{t\bar{q}} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & F_{t\bar{t}} &= I_2. \end{aligned}$$

In particular, the curvature components in the q - and mixed (q, t) -directions fail to commute:

$$[F_{q\bar{q}}, F_{q\bar{t}}] = F_{q\bar{t}} \neq 0.$$

Restricting to the symmetric slice $q = 0$ retains only the central coefficient $F_{t\bar{t}} = I_2$, obscuring the off-diagonal generators $F_{q\bar{t}}$ and $F_{t\bar{q}}$. As shown in the proof of Theorem 3.2, taking real and imaginary tangent combinations in the (q, t) -plane at $(0, 0)$ generates the full nonabelian Lie algebra $\mathfrak{u}(2)$ via infinitesimal holonomy loops around coordinate rectangles.

7.5. Unequal curvatures of two spectral lines. To see that the diagonal splitting connection ∇^{split} of (7.2) is genuinely curved even at a fixed Clark phase, consider the degree-two family with one root pinned at the origin:

$$(7.10) \quad B_a(z) = z \frac{z - a}{1 - \bar{a}z}, \quad a = x + iy \in \mathbb{D},$$

at the fixed phase $\lambda = 1$. The Malmquist–Takenaka basis

$$e_0(z) = 1, \quad e_1(a; z) = \frac{\sqrt{1 - |a|^2} z}{1 - \bar{a}z}$$

provides a global orthonormal frame for $\mathcal{E}_a = K_{B_a}$, in which the natural connection one-form is $\text{diag}(0, A_a)$ with

$$(7.11) \quad A_a = \langle e_1, de_1 \rangle = \frac{ad\bar{a} - \bar{a}da}{2(1 - |a|^2)} = \frac{i(ydx - xdy)}{1 - x^2 - y^2}.$$

For $\zeta \in \mathbb{T}$, the spectral equation $B_a(\zeta) = 1$ simplifies to $\zeta^2 - 2iy\zeta - 1 = 0$, whose roots are

$$\zeta_{\pm} = iy \pm \sqrt{1 - y^2}.$$

Setting $h = \sqrt{1 - y^2}$, one finds that the boundary kernels $k_{\zeta_{\pm}}$ have coordinate vectors in the orthonormal frame (e_0, e_1) proportional to the real-valued tuples

$$\left(1, \frac{\sqrt{1 - |a|^2}}{\pm h - x}\right)^t.$$

Normalizing these real coordinate vectors to unit length, the squared weight of the second component is easily computed from $(\pm h - x)^2 + (1 - x^2 - y^2) = 2h(h \mp x)$ to be

$$(7.12) \quad w_{\pm}(x, y) = \frac{1 - x^2 - y^2}{2h(h \mp x)} = \frac{1}{2} \pm \frac{x}{2\sqrt{1 - y^2}}.$$

Because the normalized coefficient vectors in the frame (e_0, e_1) are real-valued, their exterior differentials are orthogonal to themselves, so the connection one-forms and curvatures induced on the two eigenlines $\Pi_{\pm}$ by the splitting connection (7.2) are simply

$$A_{\pm} = w_{\pm}A_a, \quad F_{\pm} = dA_{\pm} = d(w_{\pm}A_a).$$

Evaluating $F_{\pm} = dw_{\pm} \wedge A_a + w_{\pm}dA_a$ at the base point $a = 1/2$ ($x = 1/2, y = 0$) yields the distinct line curvatures

$$(7.13) \quad F_+|_{a=1/2} = -3i dx \wedge dy, \quad F_-|_{a=1/2} = -\frac{5i}{9} dx \wedge dy,$$

whose sum recovers the total trace curvature $dA_a|_{a=1/2} = -(32i/9) dx \wedge dy$. Notice that multiplying the normalized eigenframe vectors by unimodular phases to make them C_a -real alters $A_{\pm}$ only by exact imaginary one-forms and therefore leaves the line curvatures $F_{\pm}$ unchanged. Equation (7.13) thus proves two facts simultaneously: first, simply discarding the off-diagonal terms of ∇^{nat} in the Clark eigenbasis does not yield the flat connection ∇^{sp} ; second, because $F_+ \neq F_-$, no twist of ∇^{split} by a scalar one-form can flatten both spectral lines at once.

7.6. Internal collisions and boundary degeneration. Finally, to contrast internal root collisions with boundary degeneration, consider the rank-one case $n = 1$ along the radial segment $a \in (0, 1)$, where $\mathcal{E}_a$ is spanned by the unit vector

$$e_a(z) = \frac{\sqrt{1 - a^2}}{1 - az} = \sqrt{1 - a^2} \sum_{k=0}^{\infty} a^k z^k.$$

As $a \uparrow 1$, each Taylor coefficient of e_a tends to zero, so $e_a \rightharpoonup 0$ weakly in $H^2(\mathbb{D})$. Consequently, the rank-one projections $\Pi_a = |e_a\rangle\langle e_a|$ converge to 0 in the strong operator topology, while

$\|\Pi_a\| = 1$ prevents norm convergence. Pointwise on $\mathbb{D}$, the Blaschke factor $(z - a)/(1 - az)$ converges to the unimodular constant -1 , whose associated model space is $\{0\}$.

Geometrically, this rank drop at the boundary $\partial\mathcal{B}_n$ is completely separated from internal collisions by the completeness of the Kähler metric g : under the length convention (1.8), the radial ray $a \in [0, 1)$ has infinite Riemannian length

$$\int_0^1 \frac{da}{1 - a^2} = +\infty.$$

Thus internal repeated-root divisors lie at finite distance as smooth interior points of $(\mathcal{B}_n, g)$, whereas boundary degenerations where roots escape to $\mathbb{T}$ and the fiber dimension drops lie at infinite metric distance.

8. CONCLUDING REMARKS

Working in polynomial coefficient coordinates cleanly decouples genuine geometric phenomena at root collisions from artificial coordinate singularities of point-evaluation bases. Through the global anti-holomorphic rational frame (1.3), the rank- n bundle of finite Blaschke model spaces extends real-analytically across all strata of the collision discriminant, allowing the full curvature tensor of the natural projection connection to be evaluated explicitly at the maximally coincident divisor $p = 0$. The resulting matrix-unit differences generate the entire unitary Lie algebra $\mathfrak{u}(n)$ at a single point and force both the restricted and full holonomy groups to be $U(n)$.

Our analysis also clarifies the relationship between scalar kernel invariants and full vector-bundle geometry. On the determinant line bundle $\det \mathcal{E}$, the Plücker Gram metric coincides with the Szegő kernel of the symmetrized polydisc, and its power-sum Kähler potential makes the coefficient domain a complete Kähler manifold. Nevertheless, for $n > 1$, the scalar trace curvature captures only the diagonal trace of the second fundamental form; it is the full matrix-valued curvature that obstructs the existence of parallel subbundles and rules out projective flatness.

Adjoining the Clark phase circle $\mathbb{T}$ and incorporating the canonical conjugation C_p leads to a fundamentally different geometric structure on the same underlying bundle. The conjugation-real Clark eigenlines reduce the unitary structure group to a discrete signed permutation group, singling out a unique compatible unitary connection ∇^{sp} that is flat across the entire extended parameter space $\mathcal{B}_n \times \mathbb{T}$ and has a finite cyclic holonomy group of order n or $2n$. Because ∇^{sp} remains regular across the collision locus, closed braid loops of interior zeros at fixed Clark phase have trivial monodromy, and the contrast between the curved projection connection and the flat real-spectral connection is already fully visible in explicit low-degree subfamilies.

APPENDIX A. CONVENTIONS AND COMPUTATIONAL VERIFICATION

For ease of reference, we summarize our sign and normalization conventions. The Hardy inner product $\langle \cdot, \cdot \rangle$ is anti-linear in the first argument and linear in the second. For the anti-holomorphic frame operator $F_p : \mathbb{C}^n \rightarrow H^2(\mathbb{D})$ with Gram matrix $H_p = F_p^* F_p$, the connection one-form of ∇^{nat} is $A = H^{-1} \bar{\partial} H$, whereas in any orthonormal frame V the connection one-form is $V^* dV$. In the non-orthonormal frame F_p , the matrix $A = H^{-1} \bar{\partial} H$ is compatible with the metric H ($dH = HA + A^* H$) rather than skew-Hermitian. In complex coordinates $p = x + iy$, the area element satisfies

$$dp \wedge d\bar{p} = -2i dx \wedge dy,$$

so the symmetry $F_{i\bar{j}}^* = F_{j\bar{i}}$ in an orthonormal frame ensures that $F = \sum_{i,j} F_{i\bar{j}} dp_i \wedge d\bar{p}_j$ evaluates on real tangent vectors to skew-Hermitian matrices in $\mathfrak{u}(n)$. In the rational frame the corresponding relation is $F_{i\bar{j}}^* H = H F_{j\bar{i}}$; mixed coefficients need not themselves be Hermitian.

The Kähler form is normalized by $\omega_g = (i/2)\partial\bar{\partial}\Phi$, which matches the Riemannian length element $\|v\|_g^2 = \sum_{m \geq 1} m^{-1} |ds_m(v)|^2$ in (1.8).

As an independent consistency check, symbolic and numerical routines were used to verify the finite-dimensional curvature matrices (1.4) and their real Lie-algebraic span for low degrees, the cross-Gram determinant identity (4.2) at confluent root configurations, and the weighted residue pairing (5.5). Additional tests compared the operator-projection curvature (3.1) at generic points against the second fundamental form formula (3.4), verified the agreement of the Clark unitary matrix constructed via boundary quadrature with its rational-frame representation, and checked the symbolic derivation of the unequal line curvatures in (7.13). All analytical results—including real analyticity across the discriminant, full $U(n)$ holonomy, Kähler completeness, and signed cyclic monodromy—are established rigorously in the main text; the computational scripts serve purely as independent verification of the explicit coordinate formulas.

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