
**Non-Archimedean John, Non-Archimedean Dvoretzky-Milman, Non-Archimedean
Type-Cotype and Non-Archimedean Kwapien Problems**

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Date: October 29, 2026

Abstract: We ask for non-Archimedean version of following four: (1) John Theorem, (2) Dvoretzky-Milman Theorem, (3) Type-Cotype of Banach space, (4) Kwapien Theorem.

Keywords: John Theorem, Dvoretzky Theorem, Type-Cotype of Banach space, Kwapien Theorem.

Mathematics Subject Classification (2020): 47B10, 47L20, 12J25, 46S10.

1. NON-ARCHIMEDEAN JOHN AND DVORETZKY-MILMAN PROBLEMS

Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite dimensional Banach spaces having same dimension. The **Banach-Mazur distance** between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$d_{BM}(\mathcal{X}, \mathcal{Y}) := \inf \{ \|T\| \|T^{-1}\| : T : \mathcal{X} \rightarrow \mathcal{Y} \text{ is invertible linear operator} \}.$$

For $n \in \mathbb{N}$, let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be the standard Euclidean space. Famous John's theorem says following.

Theorem 1.1. [1, 14] (*John Theorem*) *If $\mathcal{X}$ is any n -dimensional real Banach space, then*

$$d_{BM}(\mathcal{X}, (\mathbb{R}^n, \langle \cdot, \cdot \rangle)) \leq \sqrt{n}.$$

Following surprising result says that every finite dimensional Banach space has a subspace which is close (in the Banach-Mazur distance) to Euclidean space [6].

Theorem 1.2. [1, 6, 27] (*Dvoretzky-Milman Theorem*) *There is a universal number $C > 0$ satisfying the following property: If $\mathcal{X}$ is any n -dimensional real Banach space and $0 < \varepsilon < \frac{1}{3}$, then for every natural number*

$$k \leq C \frac{\varepsilon^2}{|\log \varepsilon|} \log n,$$

there exists a k -dimensional subspace $\mathcal{Y}$ of $\mathcal{X}$ such that

$$d_{BM}(\mathcal{Y}, (\mathbb{R}^k, \langle \cdot, \cdot \rangle)) < 1 + \varepsilon.$$

We refer [4, 7–10, 10–13, 23, 24, 26–28, 28, 31–33, 35] for more information on Dvoretzky-Milman theorem. Let $\mathbb{K}$ be a field. A map $|\cdot| : \mathbb{K} \rightarrow [0, \infty)$ is said to be a **non-Archimedean valuation** if following conditions are satisfied.

- (i) If $\lambda \in \mathbb{K}$ is such that $|\lambda| = 0$, then $\lambda = 0$.
- (ii) $|\lambda\mu| = |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{K}$.
- (iii) (Ultra-triangle inequality) $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$ for all $\lambda, \mu \in \mathbb{K}$.

In this case, $\mathbb{K}$ is called as a **non-Archimedean valued field** [34]. If $\mathbb{K}$ is complete in the ultra-metric

$$d(\lambda, \mu) := |\lambda - \mu|, \quad \forall \lambda, \mu \in \mathbb{K},$$

then $\mathbb{K}$ is called as a complete non-Archimedean valued field. In the paper, we assume that $\mathbb{K}$ is complete. Let $\mathcal{X}$ be a vector space over a complete non-Archimedean valued field $\mathbb{K}$ with valuation $|\cdot|$. A map $\|\cdot\| : \mathcal{X} \rightarrow [0, \infty)$ is said to be a **non-Archimedean norm** if following conditions holds.

- (i) If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.
- (ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{K}$, for all $x \in \mathcal{X}$.
- (iii) (Ultra-norm inequality) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathcal{X}$.

In this case, $\mathcal{X}$ is called as a **non-Archimedean linear space** (NALS) [29]. If $\mathcal{X}$ is complete in the ultra-metric

$$d(x, y) := \|x - y\|, \quad \forall x, y \in \mathcal{X},$$

then $\mathcal{X}$ is called as a non-Archimedean Banach space. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite dimensional non-Archimedean Banach spaces having same dimension. The **non-Archimedean Banach-Mazur distance** between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$d_{BM}(\mathcal{X}, \mathcal{Y}) := \inf\{\|T\| \|T^{-1}\| : T : \mathcal{X} \rightarrow \mathcal{Y} \text{ is invertible linear operator}\}.$$

Given $n \in \mathbb{N}$ and a complete non-Archimedean valued field $\mathbb{K}$, the standard vector space $\mathbb{K}^n$ is a non-Archimedean Banach space equipped with norm

$$\|(\lambda_j)_{j=1}^n\| := \max_{1 \leq j \leq n} |\lambda_j|, \quad \forall (\lambda_j)_{j=1}^n \in \mathbb{K}^n.$$

Problem 1.3. (Non-Archimedean John Problem) Let $\mathcal{K}$ be the set of all complete non-Archimedean valued fields. What is the best function $\Psi : \mathcal{K} \times \mathbb{N} \rightarrow (0, \infty)$ satisfying the following property: If $\mathcal{X}$ is any n -dimensional non-Archimedean Banach space over $\mathbb{K}$, then

$$d_{BM}(\mathcal{X}, (\mathbb{K}^n, \langle \cdot, \cdot \rangle)) \leq \Psi(\mathbb{K}, n).$$

Problem 1.4. (Non-Archimedean Dvoretzky-Milman Problem) Let $\mathcal{K}$ be the set of all complete non-Archimedean valued fields. What is the best function $\Psi : \mathcal{K} \times (0, \infty) \times \mathbb{N} \rightarrow (0, \infty)$ satisfying the following property: If $\mathcal{X}$ is any n -dimensional non-Archimedean Banach space over a complete non-Archimedean valued field $\mathbb{K}$ and $\varepsilon > 0$, then for every natural number

$$k \leq \Psi(\mathbb{K}, \varepsilon, n),$$

there exists a k -dimensional complete subspace $\mathcal{Y}$ of $\mathcal{X}$ such that

$$d_{BM}(\mathcal{Y}, (\mathbb{K}^k, \|\cdot\|)) < 1 + \varepsilon.$$

2. NON-ARCHIMEDEAN TYPE-COTYPE AND KWAPIEN PROBLEMS

Let $\mathcal{H}$ be a Hilbert space and $n \in \mathbb{N}$. Then for $h_1, \dots, h_n \in \mathcal{H}$, we have the generalized parallelogram identity

$$(1) \quad \frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} \left\| \sum_{j=1}^n \varepsilon_j h_j \right\|^2 = \sum_{j=1}^n \|h_j\|^2.$$

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On the other hand, celebrated Jordan-von Neumann result says that Equality (1) characterizes the Hilbert space [17]. Equality (1) motivated the definition of Type and Cotype for Banach spaces.

Definition 2.1. [1] Let $1 \leq p \leq 2$. A Banach space $\mathcal{X}$ is said to be of **(Rademacher) Type p** if there exists $T_p(\mathcal{X}) > 0$ such that

$$\left(\frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p \right)^{\frac{1}{p}} \leq T_p(\mathcal{X}) \left(\sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}}, \quad \forall x_1, \dots, x_n \in \mathcal{X}, \quad \forall n \in \mathbb{N}.$$

Definition 2.2. [1] Let $2 \leq q < \infty$. A Banach space $\mathcal{X}$ is said to be of **(Rademacher) Cotype q** if there exists $C_q(\mathcal{X}) > 0$ such that

$$\left(\sum_{j=1}^n \|x_j\|^q \right)^{\frac{1}{q}} \leq C_q(\mathcal{X}) \left(\frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^q \right)^{\frac{1}{q}}, \quad \forall x_1, \dots, x_n \in \mathcal{X}, \quad \forall n \in \mathbb{N}.$$

There is a vast literature on Type-Cotype of Banach spaces, see [1, 5, 15, 16, 20–22, 25, 30, 36]. In 1972, Kwapien proved the following breakthrough result using Type-Cotype notions [19].

Theorem 2.3. [19, 37] **(Kwapien Theorem)** A Banach space $\mathcal{X}$ has Type 2 and Cotype 2 if and only if it is isomorphic to a Hilbert space.

We ask for non-Archimedean version of previous Type-Cotype and Kwapien theorem as follows. Let $\mathbb{K}$ be a complete non-Archimedean valued field. Define

$$c_0(\mathbb{N}, \mathbb{K}) := \{ \{x_n\}_{n=1}^\infty : x_n \in \mathbb{K}, \forall n \in \mathbb{N}, \lim_{n \rightarrow \infty} |x_n| = 0 \}.$$

Then $c_0(\mathbb{N}, \mathbb{K})$ is a non-Archimedean Banach space w.r.t. the norm

$$\| \{x_n\}_{n=1}^\infty \| := \max_{n \in \mathbb{N}} |x_n|, \quad \forall \{x_n\}_{n=1}^\infty \in c_0(\mathbb{N}, \mathbb{K}).$$

The space $c_0(\mathbb{N}, \mathbb{K})$ is very important (among other non-Archimedean Banach spaces) and is known as p -adic Hilbert space, see [2, 3, 18].

Problem 2.4. (Non-Archimedean Type-Cotype and Non-Archimedean Kwapien Problems)

Whether there is a way to define non-Archimedean Type and non-Archimedean Cotype for non-Archimedean Banach spaces which will characterize the standard non-Archimedean Banach spaces $c_0(\mathbb{N}, \mathbb{K})$? i.e., whether we have a non-Archimedean Kwapien theorem?

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