SYMMETRIC VOLUME DESCENT FROM U_4 AND A FOUR-DIMENSIONAL LOWER BOUND

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Abstract. We explain a symmetric deformation that decreases the volume of the four-dimensional constant-width body U_4. The emphasis is on why the deformation works, rather than the size of the decrease. Oddness preserves width; convexity requires control of the curvature radii. A positive cap on one support branch is coupled to a negative correction on the adjacent branch. Their first-order volume contributions have different orders in the localization parameter, so the favorable contribution dominates. A small motion towards the ball makes the direction feasible for positive time. In a separate part we give a spectral lower bound for all four-dimensional bodies of constant width one, including an explicit positive refinement. A short projection argument using a three-dimensional theorem of Jiang gives the stronger lower bound 0.17872203. A substantial gap between the lower and upper bounds remains.

hexagon:2610.00650v1

1. The result and the geometric idea

The body U_4 has the symmetries of a regular four-simplex [1]; write $U_4^{(1)}$ for its width-one normalization. The companion preprint [5, Theorem 1.1] establishes that it has smaller volume than the other members of the finite collection studied there. Here we show that it can be deformed, while retaining those symmetries and constant width, to have smaller volume.

Theorem 1.1 (Symmetric local descent)**.** *There exist S_5-invariant bodies K_t of constant width one, converging to $U_4^{(1)}$ as $t \downarrow 0$, such that*

$$\mathrm{Vol}_4(K_t) < \mathrm{Vol}_4(U_4^{(1)}) \qquad (t > 0 \text{ sufficiently small}).$$

Width is a linear constraint. The support function $h(u)$ gives the signed distance from the origin to the supporting hyperplane with outward unit normal u. Width one means $h(u) + h(-u) = 1$. Thus moving one supporting plane by $\varphi(u)$ and its opposite by $-\varphi(u)$ preserves width exactly. In support-function language, the perturbation must be odd.

Date: October 29, 2026. Preprint.

Key words and phrases. Bodies of constant width, volume minimization, support functions, simplex symmetry, spherical harmonics.

Research development and supervision. The initial idea of a volume-decreasing deformation of U_4 was developed by the author and communicated to Professor Andriy Prymak on August 7, 2026. Professor Prymak subsequently examined the proposed approach and contributed further mathematical details and expanded, AI-assisted expositions of the argument. The present manuscript was developed through this exchange and the author's subsequent investigations and verification.

Affiliations. China University of Geosciences (Wuhan), China; University of Manitoba, Canada (Mitacs Globalink Research Intern, 2026).

Acknowledgements. Most of this work was completed during the author's 2026 Mitacs Globalink Research Internship at the University of Manitoba. The author gratefully acknowledges Mitacs for its support and thanks Professor Andriy Prymak for his supervision, mathematical contributions, and generous guidance in developing more effective research and learning methods.

AI assistance and manuscript preparation. The manuscript combines human-written material with AI-assisted drafting, organization, revision, and editorial refinement. OpenAI's *GPT-5.6 Sol, GPT-6 Sol, and Astra* assisted with selected proof steps, auxiliary checks, code development, and manuscript preparation. Astra was used through Codex; Claude provided limited editorial assistance with the LaTeX manuscript. The author takes responsibility for the mathematical content and conclusions.

Verification: all parts checked and understood. The author has checked and understood all parts of this manuscript, including the mathematical arguments and supporting calculations. This statement describes the author's own review and does not assert formal verification or independent peer review.

Convexity is the constraint that matters. An odd function need not describe a convex body. The matrix $A_h = \nabla^2 h + hI$ describes its curvature radii and must satisfy $0 \preceq A_h \preceq I$. Here the spectral endpoints are the eigenvalues 0 and 1. At U_4 some radii are already at an endpoint. The perturbation must move these radii in the permissible direction. This is why an arbitrary asymmetric bump, or a bump on just one branch, does not settle the problem.

Why two coupled pieces succeed. We place a favorable cap on a region with two positive and three negative coordinates. Matching the opposite support branch forces a compensating piece on the region with three positive and two negative coordinates. There is more room for the favorable piece. For localization parameter δ, its contribution has size at least $c\delta^{9/2}$, whereas the loss is at most $C\delta^{13/2}$. The loss-to-gain ratio is therefore $O(\delta^2)$ and tends to zero. The proof therefore needs no optimized numerical choice of parameters.

From a direction to a body. We finally add a motion of order δ^5 towards the ball. It supplies strict curvature margins while costing less than the favorable $\delta^{9/2}$ term. A compactness lemma then produces an actual support function for small positive time. This step certifies global convexity, not just the constant-width identity or convexity at sampled directions.

The lower-bound part is independent. Section 5 starts again with an arbitrary width-one body, not a perturbation of U_4. It gives

$$\mathrm{Vol}_4(K) \geq \frac{11\pi^2}{224}\left(1 - \sqrt{2/5}\right),$$

and an independent spectral refinement above 0.17818404. Corollary 5.4 then applies Jiang's three-dimensional bound [6] to all hyperplane projections and yields

$$\mathrm{Vol}_4(K) \geq \frac{4953\pi}{32000}\left(1 - \sqrt{2/5}\right) > 0.17872203.$$

The latter improvement uses an external theorem; the four-dimensional spectral proof is independent of it. Neither argument identifies a minimizer. Detailed optimization of tiny volume reductions is omitted.

2. Support functions and convexity

2.1. **Support functions, constant width, and curvature radii.** A *convex body* is a compact convex subset of $\mathbb{R}^n$ with nonempty interior. Its support function is

$$h_K(u) = \max_{x \in K} \langle x, u \rangle, \qquad u \in S^{n-1},$$

and its positively one-homogeneous extension is

$$H_K(x) = \max_{y \in K} \langle y, x \rangle, \qquad H_K(ru) = r h_K(u) \quad (r \geq 0).$$

The body can be recovered from its support function through

$$K = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_K(u)\}. \tag{2.1}$$

If $a \in \mathbb{R}^n$, then $h_{K+a}(u) = h_K(u) + \langle a, u \rangle$; thus first spherical harmonics represent translations.

The width of K in direction u is

$$w_K(u) = h_K(u) + h_K(-u).$$

Accordingly, K has constant width $d > 0$ if and only if

$$h_K(u) + h_K(-u) = d \qquad (u \in S^{n-1}), \tag{2.2}$$

or, equivalently,

$$H_K(x) + H_K(-x) = d\|x\| \qquad (x \in \mathbb{R}^n). \tag{2.3}$$

The antipodal identity is the basic constraint used throughout the paper. It gives both the regularity of constant-width support functions and the relation between their curvature-radius operators at antipodal points.

Lemma 2.1 (Regularity of constant-width support functions)**.** *Let $K \subset \mathbb{R}^n$ be a convex body of constant width d, and write $h = h_K$. Then $h_K \in C^{1,1}(S^{n-1})$. At almost every $u \in S^{n-1}$ the operator*

$$A_h(u) := \nabla^2_{S^{n-1}} h(u) + h(u) I_{T_u S^{n-1}} \tag{2.4}$$

is defined and satisfies

$$0 \preceq A_h(u) \preceq dI, \qquad A_h(-u) = dI - A_h(u), \tag{2.5}$$

after identifying $T_u S^{n-1} = T_{-u} S^{n-1} = u^\perp$.

Proof. The homogeneous support function $H = H_K$ is convex. By (2.3),

$$H(-x) = d\|x\| - H(x),$$

and $H(-\cdot)$ is also convex. Hence, in the sense of distributions,

$$0 \preceq D^2 H \preceq dD^2\|x\|.$$

On an annulus $a \le \|x\| \le b$ with $a > 0$,

$$D^2\|x\| = \frac{1}{\|x\|}\left(I - \frac{x \otimes x}{\|x\|^2}\right) \preceq \frac{1}{a} I,$$

so

$$0 \preceq D^2 H \preceq \frac{d}{a} I.$$

Thus H is locally semiconvex and semiconcave on $\mathbb{R}^n \setminus \{0\}$, whence $H \in C^{1,1}_{\mathrm{loc}}$ there. Restricting to S^{n-1} gives $h \in C^{1,1}(S^{n-1})$.

At a point of twice differentiability, the standard identity for a one-homogeneous extension gives

$$D^2 H(u)|_{u^\perp} = \nabla^2_{S^{n-1}} h(u) + h(u) I = A_h(u).$$

Restricting the distributional Hessian inequalities to $u^\perp$ yields $0 \preceq A_h(u) \preceq dI$. Differentiating (2.3) twice at u and $-u$ gives $A_h(-u) = dI - A_h(u)$ almost everywhere. □

For a smooth strictly convex body, the eigenvalues of A_h are the principal radii of curvature. Lemma 2.1 shows that the same terminology is meaningful almost everywhere for every body of constant width: the principal radii lie in $[0, d]$, and antipodal radii add to d.

We also need the support-function volume formula in precisely the nonsmooth regularity class that occurs for U_4. Let $S_{n-1}(K, \cdot)$ denote the surface-area measure of K on S^{n-1}, and let σ denote ordinary, unnormalized spherical surface measure. The Minkowski integral formula [2, Section 5.1] is

$$\mathrm{Vol}_n(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u)\, dS_{n-1}(K, u). \tag{2.6}$$

For $C^{1,1}$ support functions the measure has an almost-everywhere density.

Proposition 2.2 ($C^{1,1}$ surface-area density)**.** *If $h_K \in C^{1,1}(S^{n-1})$, then*

$$dS_{n-1}(K, u) = \det A_{h_K}(u)\, d\sigma(u), \tag{2.7}$$

where the Hessian is understood almost everywhere. Consequently,

$$\boxed{\mathrm{Vol}_n(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) \det A_{h_K}(u)\, d\sigma(u).} \tag{2.8}$$

Proof. Because $h = h_K$ is $C^{1,1}$, the reverse Gauss parametrization

$$X_h(u) = h(u)u + \nabla_{S^{n-1}} h(u) \tag{2.9}$$

is Lipschitz. Rademacher's theorem gives differentiability almost everywhere, and Aleksandrov's theorem gives twice differentiability of h almost everywhere. If $v \in T_u S^{n-1}$ is a tangent vector, differentiation of (2.9), followed by cancellation of the normal terms, gives

$$DX_h(u)v = (\nabla^2 h(u) + h(u) I)v = A_h(u)v.$$

Since $A_h(u) \succeq 0$ almost everywhere, the tangential Jacobian is

$$J_{n-1}X_h(u) = \det A_h(u).$$

Almost every boundary point of a convex body has a unique outer normal. Since differentiability of h makes each support face a singleton, X_h has multiplicity one at these regular boundary points. The exceptional boundary set has zero $(n-1)$-dimensional measure. The area formula [3] therefore accounts for collapsed normal regions as zero Jacobian and gives, for every Borel set $\omega \subset S^{n-1}$,

$$S_{n-1}(K,\omega) = \int_\omega \det A_h(u)\, d\sigma(u).$$

Substitution into (2.6) yields (2.8). This argument is the $C^{1,1}$ realization of Schneider's general measure-valued formula and does not require a globally C^2 boundary. □

3. The support geometry of U_4

We recall the support formula and curvature spectrum from [5, Section 3 and Proposition 3.1], where the latter is proved. Only the notation and statements needed for the descent argument are retained here.

3.1. **The support function on sign cells.** Identify $H_4 = \mathbf{1}_5^\perp \subset \mathbb{R}^5$ with $\mathbb{R}^4$ and let $S(H_4) \cong S^3$. For a nonempty proper subset P of $\{1,\dots,5\}$, the corresponding *open sign cell* is

$$C_P = \{u \in S(H_4) : u_i > 0 \ (i \in P),\ u_j < 0 \ (j \notin P)\}.$$

For $u \in S(H_4)$ put

$$P(u) = \{i : u_i > 0\}, \qquad k(u) = |P(u)|,$$
$$r(u)^2 = \sum_{i\in P(u)} u_i^2, \qquad s(u)^2 = 1 - r(u)^2, \qquad \mu(u) = \sum_{i\in P(u)} u_i.$$

The projection construction of Arman–Bondarenko–Prymak–Radchenko [1] gives the following width-two support function:

$$H(u) = \begin{cases} \sqrt{2}\, r(u), & r(u) \ge s(u), \\ 2 - \sqrt{2}\, s(u), & r(u) \le s(u). \end{cases} \tag{3.1}$$

Write

$$\mathcal{D} = \{u \in S(H_4) : r(u) \ge s(u)\}.$$

The branch-switch set $r = s$ and the coordinate hyperplanes have spherical measure zero.

Proposition 3.1 (Curvature-radius spectrum, [5, Proposition 3.1])**.** *On a sign cell in $\mathcal{D}$ with $k = |P| \ge 2$, let*

$$Q_H := \nabla^2_{S^3}(\sqrt{2}\, r) + \sqrt{2}\, r I.$$

Its eigenvalues on $T_u S(H_4)$ are

$$\underbrace{\alpha,\dots,\alpha}_{k-2}, \quad \beta_k, \quad \underbrace{0,\dots,0}_{4-k},$$

where

$$\alpha = \frac{\sqrt{2}}{r}, \qquad \beta_k = \frac{\sqrt{2}((5-k)r^2 + \mu^2)}{5r^3}. \tag{3.2}$$

For $k = 1$, $Q_H = 0$.

If $u \in \mathcal{D}$ had four positive coordinates, the unique negative coordinate would have absolute value μ, hence $s = \mu > r$ in the interior of the cell. Therefore

$$k(u) \in \{1,2,3\} \quad \text{for almost every } u \in \mathcal{D}, \tag{3.3}$$

and consequently $\det Q_H = 0$ almost everywhere on $\mathcal{D}$.

For the rest of the descent argument we use width one, so $h = H/2$ and $A_h = Q_H/2$ on the first branch. In particular $\det A_h = 0$ there. The zero eigenvalues describe the directions along

which the initial body has no curvature margin; the construction below is adapted to these directions.

4. A volume-decreasing deformation of U_4

The initial idea of a volume-decreasing deformation of U_4 was proposed by the author and communicated to Prymak on August 7, 2026. The expanded exposition here draws on the subsequent research notes supplied by Prymak [4], specifically the coupled cap formulas and branch matching, the endpoint curvature checks, the comparison of the positive and negative first-variation contributions, and strictification towards the ball. The expanded notes also prove a dimension-independent endpoint tangent-cone criterion and an explicit symmetric descent theorem for all $n \geq 4$; their Theorem 18 includes the four-dimensional nonminimality conclusion. Here we give a detailed four-dimensional exposition of the construction and feasibility proof, without invoking the higher-dimensional theorem as a black box.

This section proves Theorem 1.1. We first derive the endpoint curvature constraints and pair antipodal points in the volume variation. We then construct the two matching pieces, check their curvature signs, and compare their contributions. The last step adds a smaller motion towards the ball and applies compactness to obtain a globally feasible path. The geometric reasons for these steps were described in Section 1.

4.1. Local minimizers, admissible paths, and endpoint constraints.

Let $\mathcal{W}_{4,1}$ be the class of four-dimensional convex bodies of constant width one. Since translations preserve width and volume, we work modulo translations.

Definition 4.1 (Local volume minimizer)**.** A body $K \in \mathcal{W}_{4,1}$ is a local volume minimizer modulo translations if there exists $\varepsilon > 0$ such that

$$\mathrm{Vol}_4(K) \leq \mathrm{Vol}_4(L)$$

whenever $L \in \mathcal{W}_{4,1}$ and

$$\inf_{a \in \mathbb{R}^4} d_H(K, L + a) < \varepsilon.$$

Hausdorff convergence of convex bodies is equivalent to uniform convergence of their support functions. Let h be the width-one support function of U_4. For the affine paths used here, a candidate support function has the form

$$h_t = h + t\varphi \qquad (t \downarrow 0) \tag{4.1}$$

with $h, \varphi \in C^{1,1}$ and continuous second derivatives on each closed smooth piece. The width condition forces φ to be odd.

Write

$$A_h = \nabla^2 h + hI, \qquad B_\varphi = \nabla^2 \varphi + \varphi I.$$

Here B_φ is the linearized curvature-radius operator: $A_{h_t} = A_h + tB_\varphi$. If (4.1) is a path of width-one support functions, then $0 \preceq A_{h_t} \preceq I$. Let P_0 and P_1 be the orthogonal projections onto $\ker A_h$ and $\ker(I - A_h)$.

Lemma 4.2 (Necessary endpoint inequalities)**.** *Every feasible affine velocity in* (4.1) *satisfies almost everywhere*

$$P_0 B_\varphi P_0 \succeq 0, \qquad P_1 B_\varphi P_1 \preceq 0. \tag{4.2}$$

Proof. If $v \in \ker A_h(u)$, then

$$0 \leq \langle A_{h_t}(u)v, v\rangle = t\langle B_\varphi(u)v, v\rangle,$$

so dividing by $t > 0$ and letting $t \downarrow 0$ gives $\langle B_\varphi v, v\rangle \geq 0$. Similarly, for $v \in \ker(I - A_h(u))$,

$$0 \leq \langle (I - A_{h_t})v, v\rangle = -t\langle B_\varphi v, v\rangle,$$

which yields the upper inequality. □

These pointwise inequalities are only necessary; gluing across sign and branch interfaces is part of the global construction below.

4.2. **First variation of volume.** The first mixed-volume functional, evaluated on a $C^{1,1}$ velocity, is

$$\delta V_h(\varphi) := \int_{S^3} \varphi \, dS_3(U_4^{(1)}) = \int_{S^3} \varphi(u) \det A_h(u) \, d\sigma(u). \tag{4.3}$$

Here S^3 is the unit sphere in $H_4 = \mathbf{1}_5^\perp$. For a feasible affine path this equals the derivative of volume at zero, by Minkowski's first-variation formula. Before feasibility is proved we use it only as the displayed linear functional.

Let

$$\mathcal{D} = \{u \in S(H_4) : \|u_+\| \geq \|u_-\|\}$$

be the first support branch. The spectrum calculation in Proposition 3.1 shows that $\det A_h = 0$ almost everywhere on $\mathcal{D}$. Since width one gives $A_h(-u) = I - A_h(u)$ and an admissible velocity is odd,

$$\begin{aligned} \delta V_h(\varphi) &= \int_{\mathcal{D}} [\varphi(u) \det A_h(u) + \varphi(-u) \det A_h(-u)] \, d\sigma(u) \\ &= -\int_{\mathcal{D}} \varphi(u) \det(I - A_h(u)) \, d\sigma(u). \end{aligned}$$

Thus

$$\boxed{\delta V_h(\varphi) = -\int_{\mathcal{D}} \varphi \det(I - A_h) \, d\sigma.} \tag{4.4}$$

A feasible direction with a positive integral on the right therefore gives negative first volume variation.

4.3. **Leaf-adapted parametrization and a localized perturbation.** On the first branch in $H_4 = \mathbf{1}_5^\perp$, the homogeneous width-one support function is

$$\widehat{h}(x) = \frac{1}{\sqrt{2}} \|x_+\|. \tag{4.5}$$

Fix a smooth sign cell with positive coordinate set P and put $r = \|x_+\|$. Write

$$x = r(p - q), \qquad p = \frac{x_+}{r}, \qquad q = \frac{-x_-}{r}. \tag{4.6}$$

Then

$$p, q \geq 0, \qquad \|p\| = 1, \qquad \sum p_i = \sum q_j =: m, \qquad \|q\| \leq 1. \tag{4.7}$$

On the open cone over the interior of a first-branch sign cell, let L_p be a fiber of the map $x \mapsto p = x_+/\|x_+\|$. Its tangent space is

$$T_x L_p = \ker D^2 \widehat{h}(x) = \{w \in H_4 : w_P \in \operatorname{span}\{p\}\},$$

where the Hessian is restricted to H_4 and w_P is the component of w on the positive coordinate indices. Thus these fibers are the integral leaves of the Hessian kernel. At fixed r and p, variations of q satisfying $\sum q_j = m$ lie in a slice of one such leaf; the unperturbed homogeneous support function has vanishing second derivative in these directions. Here *null* refers to the support Hessian and, on the sphere, to the curvature-radius operator.

Here m is normalized: on the unit sphere it equals $\mu(u)/r(u)$, where μ is the unnormalized coordinate sum from Section 3 (denoted by $m(u)$ in [5, Section 3]). The symbol $\widehat{h}$ denotes the homogeneous width-one support, whereas H in (3.1) has width two. The branch-switch set is $\|q\| = 1$. The first branch has one, two, or three positive coordinates. The interface used below has two positive and three negative coordinates, or the antipodal sign pattern.

The reason for choosing this interface is visible in the negative-coordinate sections. Put $k = |P|$ and $\ell = 5 - k$, and suppress the zero coordinates of q. At fixed r, p (hence fixed m), write

$$q = \frac{m}{\ell} \mathbf{1}_\ell + z, \qquad z \perp \mathbf{1}_\ell, \qquad \|q\|^2 = \frac{m^2}{\ell} + \|z\|^2 \leq 1.$$

Thus the section is given by $\|z\|^2 \leq 1 - m^2/\ell$, with the coordinate conditions $q_j > 0$ still imposed. For $k = 2$, its limiting radius as $m \uparrow \sqrt{2}$ is $1/\sqrt{3}$, and the section retains nonempty

two-dimensional relative interior around $(\sqrt{2}/3)\mathbf{1}_3$. For $k = 3$, the radius tends to zero and the section contracts to the single point $(1, 1)/\sqrt{2}$. This contrast motivates placing the positive perturbation on the two-positive cells and its compensating continuation on the three-positive cells near $m = \sqrt{2}$; the weighted volume comparison is proved below.

Fix

$$c = \frac{4}{5}, \qquad 0 < \delta < \sqrt{2} - \frac{7}{5}, \qquad m_0 = \sqrt{2} - \delta, \tag{4.8}$$

and define

$$\chi_\delta(m) = (m - m_0)_+^4, \qquad F(y) = \sum_i (y_i - c)_+^4. \tag{4.9}$$

The cutoff χ_δ restricts the perturbation to $m > \sqrt{2} - \delta$, near $m = \sqrt{2}$ on the active branches. We use *cap* as shorthand for this localized perturbation. The function F is convex, permutation invariant, and

$$y \cdot \nabla F(y) - F(y) = \sum_i (y_i - c)_+^3 (3y_i + c) \geq 0. \tag{4.10}$$

The numerical-looking constant $4/5$ is chosen for a simple exact geometric reason.

Lemma 4.3 (The cutoff is inactive near coordinate interfaces)**.** *On the support of χ_δ, the functions F and DF vanish in a neighborhood of every coordinate interface of the relevant two-coordinate leaves.*

Proof. Let (a, b) be a nonnegative unit two-vector with $a + b = m$. Solving $a^2 + b^2 = 1$ and $a + b = m$ gives

$$a_{\max}(m) = \frac{m + \sqrt{2 - m^2}}{2}.$$

For $m > 1$ this is strictly decreasing, and

$$a_{\max}(7/5) = \frac{4}{5}.$$

On the support of χ_δ we have $m \geq m_0 > 7/5$, hence each coordinate is strictly below $4/5$ when the configuration approaches the coordinate boundary that would change the sign pattern. The same upper bound applies to a nonnegative two-vector of norm at most one: at fixed sum m, its largest coordinate is maximized when the norm is one. This covers the negative leaf when one of its three coordinates vanishes. On a positive three-vector boundary the norm is already one. The remaining one-positive boundary has $m = 1 < m_0$ and is excluded by χ_δ itself. The strict gap $m_0 > 7/5$ makes these statements hold on neighborhoods of the interfaces. The remaining $3 \leftrightarrow 4$ interface cannot meet the active first branch: there $q \to (m, 0)$ up to permutation, contradicting $\|q\| \leq 1$ and $m \geq m_0 > 1$. Therefore every positive-part factor $(y_i - 4/5)_+$ is identically zero on a neighborhood of that interface; the same holds for its first derivative. □

On a two-positive first-branch cell define

$$G_{2,\delta}(r(p - q)) = r\chi_\delta(m)F(q) \geq 0. \tag{4.11}$$

For a positive unit three-vector p define

$$\Gamma_\delta(p) = \frac{1}{2}\left[m\chi_\delta'(m)F(p) + \chi_\delta(m)(p \cdot \nabla F(p) - F(p))\right] \geq 0, \tag{4.12}$$

and on a three-positive cell set

$$G_{3,\delta}(r(p - q)) = r\left[-\chi_\delta(m)F(p) + \Gamma_\delta(p)(\|q\|^2 - 1)\right] \leq 0. \tag{4.13}$$

On one-positive cells set $G_\delta = 0$, and define the second support branch by oddness,

$$G_\delta(-x) = -G_\delta(x).$$

Finally put

$$\varphi_\delta = G_\delta|_{S(H_4)}. \tag{4.14}$$

4.4. **Matching across the mixed branch.** The correction $\Gamma_\delta(\|q\|^2-1)$ in (4.13) is designed exactly so that the two branch formulas match to first order.

Lemma 4.4 (Exact C^1 matching)**.** *Across every active mixed $2+3$ branch-switch interface, the two formulas for G_δ have the same value and differential.*

Proof. At a mixed branch point write the positive two-vector as y, the absolute negative three-vector as z, and set

$$r=\|y\|, \qquad R=\|z\|, \qquad S=\sum y_i=\sum z_j.$$

The branch equation is $r=R$. Define

$$\Psi(\rho)=\rho\chi_\delta(S/\rho)F(z/\rho). \tag{4.15}$$

The two-positive formula is $\Psi(r)$. On the opposite branch, oddness of (4.13) gives

$$G_{\mathrm{opp}}=\Psi(R)-\Gamma_\delta(z/R)\left(\frac{r^2}{R}-R\right). \tag{4.16}$$

At $r=R$ the correction factor vanishes, so the values agree.

Differentiate (4.15). Writing $p=z/\rho$ and $m=S/\rho$, Euler differentiation gives

$$\Psi'(\rho)=\chi_\delta(m)F(p)-m\chi_\delta'(m)F(p)-\chi_\delta(m)p\cdot\nabla F(p)=-2\Gamma_\delta(p).$$

At $r=R$,

$$d\left(\frac{r^2}{R}-R\right)=2(dr-dR).$$

Here Ψ' is the derivative in ρ with S,z fixed. When the point varies, the S,z differential terms of $\Psi(r)$ and $\Psi(R)$ agree on $r=R$ and cancel in their difference. Thus

$$d(\Psi(r)-\Psi(R))=-2\Gamma_\delta(z/R)(dr-dR).$$

The differential of the correction term in (4.16) has exactly this value; derivatives of Γ_δ are multiplied by the vanishing factor $r^2/R-R$. Hence $dG_{\mathrm{opp}}=d\Psi(r)$, as required. □

Proposition 4.5 (Regularity and symmetry)**.** *The piecewise function G_δ is positively one-homogeneous, odd, S_5-invariant, and belongs to $C^{1,1}_{\mathrm{loc}}(H_4\setminus\{0\})$. Consequently $\varphi_\delta\in C^{1,1}(S(H_4))$.*

Proof. Each formula is one-homogeneous because p,q,m are scale invariant and the only dimensional factor is r. Permutation invariance follows from the symmetric definitions of F and m, and oddness is imposed by definition. Lemma 4.4 gives C^1 matching across the active branch interface, while Lemma 4.3 makes the perturbation identically zero near coordinate interfaces. On each remaining compact piece the displayed formulas have bounded second derivatives. A finite piecewise C^1 function with uniformly bounded piecewise Hessian is $C^{1,1}$. □

4.5. **Verification of the endpoint curvature inequalities.** We now verify Lemma 4.2 for the constructed velocity. The calculation is most transparent in homogeneous leaf coordinates.

Lemma 4.6 (Endpoint feasibility of the cap)**.** *Let*

$$B_{\varphi_\delta}:=\nabla^2\varphi_\delta+\varphi_\delta I.$$

Then almost everywhere

$$P_0B_{\varphi_\delta}P_0\succeq 0, \qquad P_1B_{\varphi_\delta}P_1\preceq 0.$$

Proof. For lower-endpoint directions, the null directions of the homogeneous support Hessian are the radial direction together with the zero-sum directions supported on the negative coordinates. Indeed the Hessian of $\|x_+\|$ vanishes precisely when the positive part of a vector is parallel to x_+. Subtracting a radial vector leaves such a negative zero-sum vector. The Hessian of every one-homogeneous perturbation annihilates the radial vector, so its quadratic form on a spherical null direction is the same as on the corresponding leaf direction. On a two-positive cell, the q

dependence of (4.11) is through the convex function $F(q)$, so the corresponding homogeneous Hessian is positive semidefinite. On a three-positive cell the q-dependent correcting term is

$$r\Gamma_\delta(p)\|q\|^2,$$

whose q Hessian is $2r\Gamma_\delta(p)I \succeq 0$ by (4.12). This proves the lower endpoint inequality.

For the upper endpoint we use the explicit principal radii. On a width-one two-positive cell the unique nonzero radius, denoted by λ_2, is nonsaturated in the interior and equals

$$\lambda_2 = \frac{(3+m^2)\sqrt{1+\|q\|^2}}{5\sqrt{2}}. \tag{4.17}$$

It reaches 1 only at the mixed branch endpoint $m = \sqrt{2}$, $\|q\| = 1$. Here the positive unit vector is $p = (1,1)/\sqrt{2}$, and the saturated vector w is supported on its two coordinates with zero sum. Along $x + tw$ the negative coordinates and their sum are fixed, while $r'(0) = 0$ and $r''(0) = \|w\|^2/r$. Thus $G_{2,\delta} = \Psi(r)$, and $\Psi' = -2\Gamma_\delta$ from the matching calculation yields

$$D^2G_\delta[w,w] = -\frac{2\Gamma_\delta(q)}{r}\|w\|^2 \le 0. \tag{4.18}$$

On a three-positive cell the two positive radii are

$$\lambda_{3,1} = \frac{\sqrt{1+\|q\|^2}}{\sqrt{2}}, \qquad \lambda_{3,2} = \frac{(2+m^2)\sqrt{1+\|q\|^2}}{5\sqrt{2}}. \tag{4.19}$$

The second is at most $4/5$, so only $\lambda_{3,1}$ can saturate at 1, and it does so exactly on $\|q\| = 1$. Its saturated vector w is supported on the three positive coordinates and satisfies $\sum w_i = 0$ and $p \cdot w = 0$. To display the cancellation, put $L(p) = \chi_\delta(m)F(p)$, where $m = \sum p_i$, so that $p \cdot \nabla L - L = 2\Gamma_\delta(p)$. Along $x + tw$ at the interface,

$$r'(0) = 0, \quad r''(0) = \frac{\|w\|^2}{r}, \quad p'(0) = \frac{w}{r}, \quad p''(0) = -\frac{p\|w\|^2}{r^2}, \quad (\|q\|^2)''(0) = -\frac{2\|w\|^2}{r^2}.$$

Consequently the two terms in (4.13) have second derivatives

$$(-rL(p))'' = \frac{2\Gamma_\delta(p)\|w\|^2 - \chi_\delta(m)D^2F(p)[w,w]}{r}, \qquad \Big(r\Gamma_\delta(p)(\|q\|^2-1)\Big)'' = -\frac{2\Gamma_\delta(p)\|w\|^2}{r}.$$

Here $D^2L(p)[w,w] = \chi_\delta(m)D^2F(p)[w,w]$ because $\sum w_i = 0$. Adding gives

$$D^2G_\delta[w,w] = -\frac{\chi_\delta(m)}{r}D^2F(p)[w,w] = -\frac{12\chi_\delta(m)}{r}\sum_i (p_i - \tfrac{4}{5})_+^2 w_i^2 \le 0. \tag{4.20}$$

The endpoint inequalities hold for the continuous one-sided Hessian extensions on the closures of the smooth pieces. On first-branch mixed closures, the positive radii stay bounded away from zero, so no new lower kernel appears; the lower quadratic forms retain their signs. The only newly saturated upper directions are those checked in (4.18) and (4.20). Near coordinate-sign interfaces the cap vanishes identically, so rank changes there impose no further condition. At the cutoff $m = m_0$ and hinge thresholds $y_i = c$, the newly activated terms, including their corrections, have vanishing derivatives through order two. On one-positive cells the cap is zero. These observations also cover intersections of the interfaces. The antipodal cases follow from oddness and $A_h(-u) = I - A_h(u)$. □

4.6. **The sign of the first variation.** Define

$$I_\delta = \int_{\mathcal{D}} \varphi_\delta(u)\det(I - A_h(u))\,d\sigma(u) = I_{2,\delta} + I_{3,\delta}, \tag{4.21}$$

where the two terms come from the two-positive and three-positive cells. By construction $I_{2,\delta} \ge 0$ and $I_{3,\delta} \le 0$.

Proposition 4.7 (Asymptotic dominance of the two-positive cap)**.** *There are constants $c_2, C_3 > 0$ such that, for all sufficiently small $\delta > 0$,*

$$I_{2,\delta} \ge c_2\delta^{9/2}, \qquad |I_{3,\delta}| \le C_3\delta^{13/2}. \tag{4.22}$$

Hence $I_\delta > 0$ for all sufficiently small δ.

Proof. On a two-positive unit vector write

$$p(\theta) = (\cos\theta, \sin\theta), \qquad m(\theta) = \cos\theta + \sin\theta = \sqrt{2}\cos(\theta - \pi/4).$$

Use q_1, q_2 as independent leaf coordinates and set $q_3 = m - q_1 - q_2$. The parametrization $u = (p, -q)/\sqrt{1 + \|q\|^2}$ has spherical measure

$$d\sigma(u) = \frac{\sqrt{5}}{(1 + \|q\|^2)^2}\, d\theta\, dq_1\, dq_2.$$

For clarity, in a cell with ℓ negative coordinates write $q = (m/\ell)\mathbf{1}_\ell + z$, $\sum z_i = 0$. On the slice $\|p\| = 1$ the Gram determinant relative to $d\sigma(p)\, dz$ is $(5 - m^2)/\ell$. Radial projection to the sphere multiplies its area element by $(1 - m^2/5)^{-1/2}(1 + \|q\|^2)^{-2}$. Thus the resulting density is $\sqrt{5/\ell}(1 + \|q\|^2)^{-2} d\sigma(p)\, dz$; the displayed formula uses $dz = \sqrt{3}\, dq_1 dq_2$. Taylor expansion at $\pi/4$ gives

$$\sqrt{2} - m(\theta) = \frac{\sqrt{2}}{2}(\theta - \pi/4)^2 + O((\theta - \pi/4)^4).$$

Thus the angular set on which $m \geq \sqrt{2} - \delta/2$ has length comparable to $\sqrt{\delta}$. There

$$\chi_\delta(m) = (m - \sqrt{2} + \delta)^4 \geq (\delta/2)^4.$$

Such a negative-leaf patch exists explicitly: at $m = \sqrt{2}$ take

$$q = \left(\frac{17}{20}, \frac{\sqrt{2} - 17/20}{2}, \frac{\sqrt{2} - 17/20}{2}\right).$$

All coordinates are positive, $\|q\| < 1$, and $F(q) > 0$. A small patch around it varies continuously for nearby m. Choose such a compact interior patch of the negative leaf for which $F(q) > 0$ and which stays away from all coordinate and branch boundaries. On this fixed patch the spherical Jacobian and $\det(I - A_h)$ have positive lower bounds independent of δ. Integrating over the angular cap and this fixed leaf patch therefore yields

$$I_{2,\delta} \geq c_2 \delta^4 \sqrt{\delta} = c_2 \delta^{9/2}.$$

For the three-positive contribution, the negative leaf has two coordinates. Write

$$q_1 = \frac{m}{2} + d, \qquad q_2 = \frac{m}{2} - d, \qquad |d| \leq \eta(m) := \frac{1}{2}\sqrt{2 - m^2}.$$

Let α be the angular coordinate on the circle $\{p \in S^2 : \sum p_i = m\}$. Then $d\sigma_{S^2}(p) = dm\, d\alpha/\sqrt{3}$ and $dz = \sqrt{2}\, dd$, so the same radial calculation gives

$$d\sigma(u) = \frac{\sqrt{5/3}}{(1 + \|q\|^2)^2}\, dm\, d\alpha\, dd.$$

This density is bounded uniformly for $m_0 \leq m \leq \sqrt{2}$; the range of α has length at most 2π. A direct calculation gives

$$1 - \|q\|^2 = 2(\eta(m)^2 - d^2). \tag{4.23}$$

Let

$$x = m - m_0, \qquad y = \sqrt{2} - m, \qquad x + y = \delta.$$

On the active cap, $0 \leq x \leq \delta$. From (4.9) and (4.12),

$$\chi_\delta = O(x^4), \qquad \Gamma_\delta = O(x^3).$$

Writing $\eta = \eta(m)$, we have $\eta^2 = y(\sqrt{2} + m)/4 = O(y)$. By (4.19), $\lambda_{3,2} \leq 4/5$ and

$$1 - \lambda_{3,1} = \frac{2(\eta^2 - d^2)}{\sqrt{2}(\sqrt{2} + \sqrt{1 + \|q\|^2})}.$$

Thus $\det(I - A_h)$ is bounded above by a constant times $\eta^2 - d^2$. The bounded density just computed and the identities

$$\int_{-\eta}^{\eta} (\eta^2 - d^2)\, dd = \frac{4}{3}\eta^3, \qquad \int_{-\eta}^{\eta} (\eta^2 - d^2)^2\, dd = \frac{16}{15}\eta^5$$

give, with constants independent of small δ,

$$|I_{3,\delta}| \le C \int_0^\delta \left[x^4(\delta-x)^{3/2} + x^3(\delta-x)^{5/2}\right] dx.$$

Rescaling $x = \delta s$ gives

$$|I_{3,\delta}| \le C_3 \delta^{13/2},$$

because both beta-type integrals in $s \in [0,1]$ are finite. Since $13/2 > 9/2$, the positive two-cell contribution dominates for sufficiently small δ. □

By (4.4),

$$\boxed{\delta V_h(\varphi_\delta) = -I_\delta < 0} \tag{4.24}$$

for all sufficiently small δ.

4.7. **Strict curvature margins and construction of an admissible support path.** Endpoint feasibility is non-strict, so it is not by itself a uniform finite-t convexity certificate. We create a small spectral margin by moving slightly toward the width-one ball. Let

$$b(u) = \frac{1}{2} - h(u). \tag{4.25}$$

This function is odd, and

$$B_b = \nabla^2 b + bI = \frac{1}{2} I - A_h.$$

Define the modified perturbation direction

$$\widetilde{\varphi}_\delta = \varphi_\delta + \delta^5 \left(\frac{1}{2} - h\right). \tag{4.26}$$

The tilde denotes the modified perturbation direction, now with a strict spectral margin. On the two saturated subspaces,

$$B_{\widetilde{\varphi}_\delta}|_{\ker A_h} \succeq \frac{\delta^5}{2} I, \qquad B_{\widetilde{\varphi}_\delta}|_{\ker(I-A_h)} \preceq -\frac{\delta^5}{2} I. \tag{4.27}$$

Since $|\delta V_h(b)| \le 2\pi^2 \|b\|_\infty$, the added term changes the first variation by $O(\delta^5)$ with a constant independent of δ, whereas the negative contribution in (4.24) has magnitude at least a constant multiple of $\delta^{9/2}$. Since $5 > 9/2$,

$$\delta V_h(\widetilde{\varphi}_\delta) < 0 \tag{4.28}$$

for all sufficiently small δ.

We use one elementary compactness lemma to pass from the infinitesimal spectral inequalities to a genuine path.

Lemma 4.8 (Uniform matrix strictification)**.** *Let $A(x), C(x)$ be continuous symmetric matrices on a compact set, with $0 \preceq A(x) \preceq I$. Assume that for some $\eta > 0$,*

$$C|_{\ker A} \succeq \eta I, \qquad C|_{\ker(I-A)} \preceq -\eta I$$

at every point. Then there is $t_0 > 0$ such that

$$0 \preceq A + tC \preceq I \qquad (0 \le t \le t_0)$$

uniformly on the compact set.

Proof. Suppose the lower inequality failed for a sequence $t_j \downarrow 0$. Then there are points x_j and unit vectors v_j with

$$\langle (A(x_j) + t_j C(x_j)) v_j, v_j \rangle < 0.$$

By compactness, after passing to a subsequence, $x_j \to x$ and $v_j \to v$. Boundedness of C and $A \succeq 0$ give

$$0 \le \langle A(x_j) v_j, v_j \rangle < t_j \|C(x_j)\|, \qquad \langle C(x_j) v_j, v_j \rangle < 0.$$

Taking limits yields $v \in \ker A(x)$ and $\langle C(x) v, v \rangle \le 0$, contradicting the strict positive bound. Apply the same argument to $I - A$ and $-C$ for the upper inequality. □

For fixed δ, partition the sphere by coordinate signs, support branches, and cutoff and hinge thresholds, and take the finitely many piece closures. On each closure, A_h and B_{φ_δ} have continuous one-sided extensions: on the first branch $r = \|u_+\| \geq 1/\sqrt{2}$, and the positive-part formulas are smooth on their respective pieces; the opposite branch follows by antipodality. Lemma 4.6 covers these extensions, including newly saturated boundary directions. Thus $C = B_{\varphi_\delta} + \delta^5(\frac{1}{2}I - A_h)$ satisfies the strict endpoint conditions with $\eta = \delta^5/2$ on every closed piece. Using finitely many compact patches with local orthonormal tangent frames, apply Lemma 4.8 and take the minimum of the resulting positive times to obtain a common $t_\delta > 0$.

Theorem 4.9 (A feasible symmetric descent family)**.** *There exists $\delta_* > 0$ such that, for every $0 < \delta < \delta_*$, there is $t_\delta > 0$ for which*

$$h_{\delta,t} = h + t\widetilde{\varphi}_\delta, \qquad 0 \leq t < t_\delta, \tag{4.29}$$

is the support function of an S_5-invariant convex body $K_{\delta,t}$ of constant width one. For all sufficiently small $t > 0$,

$$\boxed{\mathrm{Vol}_4(K_{\delta,t}) < \mathrm{Vol}_4(U_4^{(1)}).} \tag{4.30}$$

Consequently $U_4^{(1)}$ is not a local volume minimizer in $\mathcal{W}_{4,1}$.

Proof. Proposition 4.5 shows that φ_δ is odd, S_5-invariant and $C^{1,1}$. The ball direction $b = \frac{1}{2} - h$ has the same properties, so (4.26) implies that $\widetilde{\varphi}_\delta$ does as well. Hence

$$h_{\delta,t}(u) + h_{\delta,t}(-u) = 1.$$

The compactness argument above gives

$$0 \preceq A_h + tB_{\widetilde{\varphi}_\delta} \preceq I$$

on every smooth piece for $0 \leq t < t_\delta$. Since $h_{\delta,t} \in C^{1,1}$, nonnegativity of the homogeneous curvature-radius operator is the support-function convexity criterion. More explicitly, the homogeneous extension has positive semidefinite Hessian almost everywhere away from the origin. Its $C^{1,1}_{\mathrm{loc}}$ regularity rules out negative singular Hessian terms at the interfaces, so its restriction to every segment avoiding the origin is convex. On a line through the origin, for a unit vector u, the extension equals $\tau h_{\delta,t}(u)$ for $\tau \geq 0$ and $-\tau h_{\delta,t}(-u)$ for $\tau \leq 0$. The right derivative minus the left derivative is $h_{\delta,t}(u) + h_{\delta,t}(-u) = 1$. Thus the restriction to these lines is also convex. Hence $h_{\delta,t}$ is the support function of a convex body. The antipodal identity gives width one, and the construction is S_5-invariant. Finally,

$$\mathrm{Vol}_4(K_{\delta,t}) = \mathrm{Vol}_4(U_4^{(1)}) + t\,\delta V_h(\widetilde{\varphi}_\delta) + o(t),$$

and the coefficient is strictly negative by (4.28). Therefore the volume is smaller for all sufficiently small positive t. □

Equivalently, the deformed body can be written directly as

$$K_{\delta,t} = \bigcap_{u \in S(H_4)} \{x \in H_4 : \langle x, u\rangle \leq h_{\delta,t}(u)\}.$$

Thus the result is not merely an abstract existence theorem: the formulas (4.11), (4.13), and (4.26) define an explicit nearby family. The choices "δ sufficiently small" and "t sufficiently small" have been justified analytically, without assigning numerical endpoints to the allowable ranges. Thus the theorem proves strict descent and global feasibility; it supplies no six-decimal volume for a specified optimized member of this family.

Remark 4.10 (One-sided and two-sided questions). The path only uses positive t. A two-sided Minkowski decomposition asks for feasibility in both directions, which is a stronger requirement. Restrictions on two-sided perturbations therefore do not rule out the one-sided descent just constructed.

5. A lower bound for all four-dimensional constant-width bodies

In this independent section, p denotes the support function of an arbitrary body K of constant width one, and $S_3(K)$ denotes its surface area. We first obtain a surface-area lower bound from the curvature constraint, then convert it to volume using an inscribed ball.

5.1. **A spectral surface-area estimate.** First suppose p is smooth and write $p = 1/2 + f$. Constant width makes f odd. A translation removes its degree-one spherical harmonic without changing $\nabla^2 f + fI$. Set

$$A = \nabla^2 f + fI, \qquad \mathcal{E}(f) = \int_{S^3} (|\nabla f|^2 - 3f^2)\, d\sigma.$$

The eigenvalues of A lie in $[-1/2, 1/2]$. Here A is the *centered* curvature matrix: the support curvature matrix from the preceding sections is $A_p = \frac{1}{2}I + A$. Throughout this section, $d\sigma$ is unnormalized spherical surface measure, so $\sigma(S^3) = 2\pi^2$. Our norm conventions are

$$\|f\|_2^2 := \int_{S^3} |f(u)|^2\, d\sigma(u), \qquad \|A\|_2^2 := \int_{S^3} \|A(u)\|_F^2\, d\sigma(u),$$

where $\|A(u)\|_F$ is the pointwise Frobenius (Hilbert–Schmidt) norm. Thus $\|A\|_2$ is a spherical L^2 norm of a matrix field.

Lemma 5.1 (Spectral identities)**.** *Let f_ℓ be the degree-ℓ spherical harmonic component of f, and $\lambda_\ell = \ell(\ell+2)$. Then*

$$\begin{aligned} S_3(K) &= \frac{\pi^2}{4} - \frac{1}{2}\mathcal{E}(f), \\ \mathcal{E}(f) &= \sum_{\ell=3,5,\dots} (\lambda_\ell - 3)\|f_\ell\|_2^2, \\ \|A\|_2^2 &= \sum_{\ell=3,5,\dots} (\lambda_\ell - 1)(\lambda_\ell - 3)\|f_\ell\|_2^2. \end{aligned}$$

Proof. Oddness gives $A(-u) = -A(u)$ after identifying antipodal tangent spaces. In the expansion of $\det(\frac{1}{2}I + A)$, the linear and cubic terms therefore integrate to zero. If $\sigma_2(A)$ is the second elementary symmetric function of its eigenvalues, then

$$S_3(K) = \frac{|S^3|}{8} + \frac{1}{2}\int_{S^3} \sigma_2(A)\, d\sigma.$$

The identity $2\sigma_2(A) = (\operatorname{tr} A)^2 - \|A\|_F^2$, integration by parts, and the integrated Bochner formula on S^3 give

$$\int |\nabla^2 f|^2 = \int (\Delta f)^2 - 2\int |\nabla f|^2, \qquad \int \sigma_2(A) = 3\int f^2 - \int |\nabla f|^2.$$

This proves the area identity since $|S^3| = 2\pi^2$. On a degree-ℓ harmonic, $-\Delta f_\ell = \lambda_\ell f_\ell$. Expanding $\|\nabla^2 f + fI\|_2^2$ gives $(\lambda_\ell^2 - 4\lambda_\ell + 3)\|f_\ell\|_2^2$. Orthogonality of distinct degrees finishes the proof. □

Since the first remaining eigenvalue is $\lambda_3 = 15$,

$$\mathcal{E}(f) \le \frac{1}{14}\|A\|_2^2 \le \frac{1}{14}\cdot\frac{3}{4}|S^3| = \frac{3\pi^2}{28}.$$

Consequently

$$S_3(K) \ge \frac{11\pi^2}{56}. \tag{5.1}$$

5.2. **From surface area to volume.** Jung's theorem places a diameter-one body in a ball of radius at most $R_J = \sqrt{2/5}$. Choose its centre as the origin. Then $p(u) \le R_J$, and width one implies $p(u) = 1 - p(-u) \ge 1 - R_J =: r_J$. Thus the concentric ball of radius r_J lies inside K. Minkowski's formula yields

$$4\,\mathrm{Vol}_4(K) = \int_{S^3} p\, dS_3(K) \ge r_J S_3(K).$$

Combining this with (5.1) gives

$$\mathrm{Vol}_4(K) \ge \frac{11\pi^2}{224}(1 - \sqrt{2/5}) = 0.1781370691\ldots. \tag{5.2}$$

The translation used here may differ from the one used to remove the degree-one harmonic; surface area and volume are translation invariant.

The smoothness restriction can be removed by averaging the support function against a smooth approximate identity on the rotation group, then mixing with a vanishing amount of the radius-1/2 ball if strict curvature is needed. These operations preserve width one and converge in Hausdorff distance. Continuity of volume and surface area proves the same bounds for every constant-width body.

5.3. **An explicit improvement of the preceding bound.** There are two sources of nonsharpness in the preceding estimate: the curvature eigenvalues need not all have maximal absolute value, and harmonics of degree at least five satisfy a stronger spectral inequality. An elementary great-circle argument shows that these two losses cannot both be too small.

Theorem 5.2 (An explicit spectral surface-area gap)**.** *Put*

$$r_J = 1 - \sqrt{2/5}, \qquad \varepsilon_S = \frac{729\pi}{4480000}.$$

Every convex body $K \subset \mathbb{R}^4$ of constant width one satisfies

$$S_3(K) \ge \frac{11\pi^2}{56} + \varepsilon_S, \tag{5.3}$$

$$\mathrm{Vol}_4(K) \ge \frac{11\pi^2}{224} r_J + \frac{r_J \varepsilon_S}{4} > 0.17818404. \tag{5.4}$$

The improvement over (5.2) exceeds $4.69 \cdot 10^{-5}$; a large gap to the upper constructions remains.

Proof. First use a smooth support $p = 1/2 + f$ and remove its degree-one part. Write $A = \nabla^2 f + fI$, so its three eigenvalues are in $[-1/2, 1/2]$. Let f_3 be the degree-three part and set

$$A_3 = \nabla^2 f_3 + f_3 I, \quad A_R = A - A_3, \quad D_{\mathrm{curv}} = \frac{3\pi^2}{2} - \|A\|_2^2 \ge 0, \quad Z = \|\operatorname{tr} A_R\|_2^2.$$

Lemma 5.1 gives, degree by degree,

$$S_3(K) - \frac{11\pi^2}{56} = \frac{D_{\mathrm{curv}}}{28} + \sum_{\ell=5,7,\ldots} \frac{(\lambda_\ell - 15)(\lambda_\ell - 3)}{28} \|f_\ell\|_2^2.$$

On the other hand, $Z = \sum_{\ell \ge 5,\ \ell \text{ odd}} (\lambda_\ell - 3)^2 \|f_\ell\|_2^2$. For $\lambda_\ell \ge 35$,

$$\frac{\lambda_\ell - 15}{28(\lambda_\ell - 3)} \ge \frac{5}{224};$$

the ratio is increasing and equality holds at degree five. Consequently

$$S_3(K) - \frac{11\pi^2}{56} \ge \frac{D_{\mathrm{curv}}}{28} + \frac{5}{224} Z. \tag{5.5}$$

Set $g = \operatorname{tr} A_3 = -12 f_3$. Then

$$\|g\|_2^2 = \frac{6}{7}\|A_3\|_2^2 \le \frac{9\pi^2}{7}.$$

There are 16 degree-three harmonics on S^3. For an orthonormal basis Y_j, the tensor $T(u) = \sum_j \nabla Y_j(u) \otimes \nabla Y_j(u)$ is unchanged by rotations fixing u, so it is a scalar multiple of the identity

on T_uS^3. Its trace is constant and its integral is $16 \cdot 15$. Since $|S^3| = 2\pi^2$ and the tangent space has dimension three,

$$T(u) = \frac{16 \cdot 15}{3|S^3|} I = \frac{40}{\pi^2} I.$$

Write $g = \sum_j c_j Y_j$. For every unit tangent vector v, Cauchy–Schwarz gives

$$|\partial_v g|^2 \le \sum_j c_j^2 \sum_j |\partial_v Y_j|^2 = \|g\|_2^2 \frac{40}{\pi^2} \le \frac{360}{7}.$$

Taking the supremum over v proves the global Lipschitz bound

$$|\nabla g| \le \sqrt{360/7} < 36/5. \tag{5.6}$$

Let $a = 9/50$, $r = 1/40$, and put

$$\Omega_a = \{u \in S^3 : |g(u)| \le a\}.$$

On every great circle g is odd under a half turn, so it has a zero and an antipodal zero. Around these two zeros, the four half-arcs of length $r = 1/40$ lie in Ω_a, because $(36/5)r = a$. These arcs do not overlap, since $r < \pi/2$. Their total circle length is $4r$. Averaging over rotations of one fixed great circle and applying Fubini gives the spherical measure estimate

$$\sigma(\Omega_a) \ge |S^3| \frac{4r}{2\pi} = 4\pi r. \tag{5.7}$$

This argument does not require a choice of a common zero for different circles.

For any three eigenvalues in $[-1/2, 1/2]$,

$$\|A\|_F^2 \le \frac{1}{2} + (\operatorname{tr} A)^2.$$

To verify it, maximize $\sum \lambda_i^2 - (\sum \lambda_i)^2 = -2\sum_{i<j} \lambda_i \lambda_j$ on the cube; it is affine in each coordinate separately and its vertex maximum is $1/2$. Choose $b = 27/100$. By Chebyshev's inequality, $|\operatorname{tr} A_R| \le b$ outside a set of measure at most Z/b^2. On the remaining portion of Ω_a,

$$\frac{3}{4} - \|A\|_F^2 \ge d := \frac{1}{4} - (a+b)^2 > 0.$$

Here d is a positive scalar gap, unrelated to the width parameter in Section 2. The deficit is nonnegative everywhere; hence

$$D_{\mathrm{curv}} + CZ \ge d\sigma(\Omega_a), \qquad d = \frac{19}{400}, \qquad C = \frac{d}{b^2} = \frac{475}{729}.$$

These rational choices obey $(5/224)/C = 729/21280 < 1/28$. Combining this inequality with (5.5) and (5.7) gives

$$S_3(K) - \frac{11\pi^2}{56} \ge \frac{5}{224C}(D_{\mathrm{curv}} + CZ) \ge \frac{5b^2}{224}(4\pi r) = \frac{729\pi}{4480000} = \varepsilon_S.$$

Combining Jung's theorem with Minkowski's volume formula, as above, gives $4\,\mathrm{Vol}_4(K) \ge r_J S_3(K)$ and proves the volume assertion. We refer to this combination as the Jung–Minkowski estimate. Rotational smoothing preserves width one and converges in the convex-body topology, so continuity of surface area and volume extends the inequalities to arbitrary constant-width bodies. The rational comparisons and the displayed decimal lower endpoint can be checked by the accompanying 192-bit Arb script `certify_lower_gap.py`; this numerical check is separate from the analytic proof just given. □

Remark 5.3 (The lower-bound constant). The exact right-hand side of (5.4) equals

$$0.1781840422363513\ldots.$$

This improves (5.2) without using a three-dimensional volume lower bound. It is a lower-bound constant, not an approximation to the unknown minimum. The argument uses the simple rational thresholds $a = 9/50$ and $b = 27/100$ without optimizing them. Its longer spectral proof records an independent refinement; numerically, the projection bound in Corollary 5.4 is stronger.

5.4. **A stronger bound from three-dimensional projections.** Jiang's Theorem 1.1 in [6] states that every three-dimensional convex body of constant width one has volume at least 1651/4000. This external result gives a short improvement of the four-dimensional bound.

Corollary 5.4 (Projection lower bound)**.** *Every convex body $K \subset \mathbb{R}^4$ of constant width one satisfies*

$$\mathrm{Vol}_4(K) \geq L_{\mathrm{proj}} := \frac{4953\pi}{32000}\left(1 - \sqrt{2/5}\right) > 0.17872203. \tag{5.8}$$

Proof. Let $E \subset \mathbb{R}^4$ be any three-dimensional linear subspace. Orthogonal projection preserves the support function in directions belonging to E: $h_{K|E}(u) = h_K(u)$ for $u \in E \cap S^3$. Hence $K|E$ has constant width one, so Jiang's theorem gives $\mathrm{Vol}_3(K|E) \geq 1651/4000$. Cauchy's surface-area formula [7, Section 1], with ordinary spherical measure, reads

$$S_3(K) = \frac{1}{\kappa_3}\int_{S^3} \mathrm{Vol}_3(K|u^\perp)\, d\sigma(u), \qquad \kappa_3 = \frac{4\pi}{3}.$$

Since $|S^3| = 2\pi^2$, it follows that $S_3(K) \geq (3\pi/2)(1651/4000)$. The Jung–Minkowski estimate $4\,\mathrm{Vol}_4(K) \geq (1 - \sqrt{2/5})S_3(K)$ from above proves the claim. □

The constant satisfies

$$0.17872203991039462 < L_{\mathrm{proj}} < 0.17872203991039463.$$

This corollary uses only the stated constant 1651/4000 from Jiang's preprint. It does not require extending that paper's contact-geometry argument to four dimensions.

6. Discussion

Theorem 4.9 proves that $U_4^{(1)}$ is not a local volume minimizer, even with all simplex symmetries retained. The proof gives a formula for the deformation and justifies a positive parameter range without quantifying its endpoints. It does not determine an optimal member of this family or an optimal symmetric body.

The spectral bound in Theorem 5.2 and the literature-dependent projection bound in Corollary 5.4 remain well below $\mathrm{Vol}_4(U_4^{(1)}) \approx 0.221836$, whose six-decimal rounding is certified in [5, equation (1.4)]. This is the volume of the original body, not a numerical evaluation of the deformed bodies. A useful quantitative descent would need a larger certified gain that justifies its proof's complexity. Such an optimization is not pursued here. Neither the descent nor the lower bounds decide whether a global minimizer retains simplex symmetry.

Code for checking only the displayed lower-bound constants is available in the computational supplement (`manuscript2/lower_bound_checks/`).

References

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