

A bridge between analysis on local fields and quantum groups: the noncompact case

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Abstract

Let K be a non-Archimedean local field with residue cardinality Q and let $d \geq 1$. We establish a radial correspondence between analysis on K^d and Woronowicz's quantum group $E_r(2)$, with $r^2 = Q^{-d}$. For every $\alpha > 0$, the radial Taibleson operator of order α is unitarily equivalent, through a unitary commuting with dilations, to a bilateral birth–death operator with Hahn–Exton q -Bessel eigenfunctions. Their Dirichlet forms are equivalent on the same weighted sequence space, with optimal comparison constants. For $\alpha = d$, the bilateral operator is the radial part of a self-adjoint Casimir on the quantum plane $E_r(2)/\mathbb{T}$, whose heat semigroup is covariant and completely positive. We further obtain these quantum-plane dynamics by contraction of the compact Casimir dynamics on the Podleś sphere, with Mosco convergence of the forms and pointwise norm convergence of the compressed heat maps.

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1 Introduction

Let K be a non-Archimedean local field with residue cardinality Q , ring of integers $\mathcal{O}$ and uniformizer π , and let $d \geq 1$ be an integer. Normalize the additive Haar measure of K^d by $\mu(\mathcal{O}^d) = 1$, use the maximum norm $\|\cdot\|_K$ on K^d , and put $\tau \stackrel{\text{def}}{=} Q^{-d}$ and $r \stackrel{\text{def}}{=} \tau^{\frac{1}{2}}$. Aref'eva and Volovich observed that the radial Haar distribution of the unit ball $\mathcal{O}^d$ coincides with the distribution of $\gamma^* \gamma$ under the Haar state of $\text{SU}_r(2)$ [AV91, Section 6.1]. In [Arh26] we showed that this coincidence extends to an equivalence of energies between the radial Taibleson operators on $\mathcal{O}^d$ and the zonal Casimir operator of $\text{SU}_r(2)$, and that the corresponding quantum dynamics live on the standard Podleś sphere. The unit ball has total mass one, whereas the whole space K^d has infinite Haar measure. The purpose of this paper is to show that the whole space corresponds in the same way to the locally compact quantum group $E_r(2)$ of Woronowicz [Wor91], [Wor92].

Recall that $E_r(2)$ is generated by a unitary v and a normal operator n satisfying $vnv^* = rn$, whose spectrum is $\{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| \in r^{\mathbb{Z}}\}$. The volume observable $x \mapsto \|x\|_K^d$ takes the values τ^m , $m \in \mathbb{Z}$, on $K^d \setminus \{0\}$, and the Haar weight h of $E_r(2)$ computed by Pal [Pal96] gives, for any Borel function $F: \tau^{\mathbb{Z}} \rightarrow [0, \infty]$,

$$(1.1) \quad \int_{K^d} F(\|x\|_K^d) dx = (1 - \tau) \sum_{m \in \mathbb{Z}} \tau^m F(\tau^m) = (1 - \tau) h(F(n^* n)),$$

see Proposition 5.1. The restriction of (1.1) to the unit ball, that is to the spectral projection $\mathbf{1}_{[0,1]}(n^* n)$, is the identity of Aref'eva and Volovich. Moreover, the automorphism $x \mapsto vxv^*$, which is used by Woronowicz to obtain $E_r(2)$ from $\text{SU}_r(2)$ by contraction [Wor92], corresponds to the dilation $x \mapsto \pi x$ of K^d , and $h(vxv^*) = \tau^{-1}h(x)$ corresponds to $\mu(\pi^{-1}A) = \tau^{-1}\mu(A)$. Thus the quantum group $E_r(2)$ carries both the infinite volume of K^d and its discrete group of dilations. In a different direction, Zúñiga-Galindo realized q -oscillator algebras on $L^2(K)$ with the multiplication by $|x|_K$ and a Jackson-type difference operator built from $x \mapsto \pi^{\pm 1}x$ [Zun22].

The analytic input is the Taibleson operator D^α of order $\alpha > 0$ on K^d , that is the Fourier multiplier with symbol $\|\xi\|_K^\alpha$. Its restriction D_{rad}^α to radial functions acts on the weighted sequence space $\ell_{\mathbb{Z}}^2(w)$, $w_m \stackrel{\text{def}}{=} (1 - \tau)\tau^m$. It is diagonal in a basis of Haar wavelets $(g_m)_{m \in \mathbb{Z}}$,

with eigenvalues ρ^{m+1} , where $\rho \stackrel{\text{def}}{=} Q^\alpha$ (Proposition 2.1). On the other side, for $0 < \eta < 1$ we consider the bilateral birth–death operator

$$(1.2) \quad (L_\eta^\infty f)(k) = \tau \eta^{-k-1} (f(k) - f(k+1)) + \eta^{-k} (f(k) - f(k-1)), \quad k \in \mathbb{Z},$$

which is symmetric on $\ell_{\mathbb{Z}}^2(w)$. It is obtained from the little η -Jacobi operator L_η of [Arh26, (1.3)] on $\ell_{\mathbb{N}_0}^2(w)$ by a translation of N steps towards the origin of the ball and a rescaling by η^N , in the limit $N \rightarrow \infty$. For $\eta = \tau$, the operator $(L_\tau^\infty f)(k) = \tau^{-k} (2f(k) - f(k+1) - f(k-1))$ is a second order τ -difference operator whose eigenfunctions are Hahn–Exton q -Bessel functions of order 0 in base $\tau = r^2$. These functions appear as matrix coefficients of the irreducible representations of the quantum group of plane motions [VK89], [Koe94], see also [KS92, Section 1].

Theorem 1.1 (Energy bridge on K^d) *Let $\alpha > 0$ and put $\eta \stackrel{\text{def}}{=} Q^{-\alpha}$ and $\sigma_\alpha \stackrel{\text{def}}{=} Q^{-\frac{d+\alpha}{2}}$.*

1. *The closed forms $\mathcal{E}_D$ and $\mathcal{E}_\eta^\infty$ of D_{rad}^α and L_η^∞ have the same domain*

$$\text{dom } \mathcal{E}_D = \text{dom } \mathcal{E}_\eta^\infty = \left\{ f \in \ell_{\mathbb{Z}}^2(w) : \sum_{m \in \mathbb{Z}} Q^{(\alpha-d)m} |f(m+1) - f(m)|^2 < \infty \right\},$$

and for any f in this space we have

$$(1.3) \quad (1 + \sigma_\alpha)^{-2} \mathcal{E}_\eta^\infty(f) \leq \mathcal{E}_D(f) \leq (1 - \sigma_\alpha)^{-2} \mathcal{E}_\eta^\infty(f).$$

The constants are optimal.

2. *The operators D_{rad}^α and L_η^∞ have the same spectrum $\{Q^{\alpha m} : m \in \mathbb{Z}\} \cup \{0\}$. Their spectral measures are purely atomic, every nonzero spectral point is a simple eigenvalue, and 0 belongs to the continuous spectrum. There exists a unitary operator W on $\ell_{\mathbb{Z}}^2(w)$ which commutes with the dilation of K^d and satisfies $W D_{\text{rad}}^\alpha W^* = L_\eta^\infty$. The eigenvalue $Q^{\alpha m}$ of L_η^∞ has the eigenfunction $k \mapsto {}_1\varphi_1(0; \tau; \eta, \eta^{k+1-m})$, which is a Hahn–Exton q -Bessel function of order $\frac{d}{\alpha} - 1$ in base η .*
3. *Let $\vartheta > 0$. Then $\text{dom } \mathcal{E}_D \subset \text{dom}(L_{Q^{-\vartheta}}^\infty)^{\frac{\alpha}{2\vartheta}}$, and the closed forms of D_{rad}^α and of $(L_{Q^{-\vartheta}}^\infty)^{\frac{\alpha}{2\vartheta}}$ are equivalent if and only if $\alpha < d + 2\vartheta$.*

These assertions are proved in Theorem 4.2, Theorem 3.6 and Theorem 4.6. For $\alpha = d$, the first part gives the constants $(1 + \tau)^{-2}$ and $(1 - \tau)^{-2}$, and the energy of D_{rad}^d is $1 - \tau$ times the squared $\ell_{\mathbb{Z}}^2$ -norm of $(I - \tau S)^{-1}$ applied to the discrete gradient, where S is the shift (Remark 4.3). On the unit ball, the corresponding result [Arh26, Theorem 1.1] compares the regional Taibleson operator with L_η , and their eigenvalues differ by a bounded sequence [Arh26, (1.5)]. On the whole space, this bounded error disappears and the two operators become unitarily equivalent. They are nevertheless different operators: their eigenfunctions are step functions on one side and q -Bessel functions on the other, and they have no common eigenvector (Theorem 4.2). The third part has the same threshold $\alpha < d + 2\vartheta$ as on the unit ball [Arh26, Theorem 1.1]; for $\vartheta = d$ it shows that the powers $(L_\tau^\infty)^{\frac{\alpha}{2}}$ reproduce the Taibleson energies of the orders $\alpha < 3d$.

The proofs of the first two parts rest on two simple mechanisms. First, the coefficients of a radial function in the wavelet basis are, up to normalization, the differences between its value on a sphere and its mean on the next ball. This gives a martingale expression of $\mathcal{E}_D$ (Proposition 2.3), in which the mean on the ball is the image of the discrete gradient under a weighted Hardy operator $(I - \tau S)^{-1}$. On the weighted space attached to $\mathcal{E}_\eta^\infty$, the operator τS is σ_α times a unitary bilateral shift, which gives (1.3) and its optimality. Secondly, both D_{rad}^α and

L_η^∞ are covariant under the dilation U of K^d , in the sense that $U^{-1}AU = \rho^{-1}A$. Their spectra are therefore invariant under multiplication by ρ . To compute the spectrum of L_η^∞ , we prove that the rescaled little η -Jacobi operators converge to L_η^∞ in the strong resolvent sense (Theorem 3.6). The convergence of spectral projections and the covariance then give simple eigenvalues η^z , $z \in \mathbb{Z}$. The eigenfunctions are identified with Hahn–Exton q -Bessel functions by a Wronskian argument at the origin. This also recovers the orthogonality relation of Koornwinder and Swarttouw [KS92, Proposition 2.6] for these parameters, and gives the completeness of the corresponding system. The third part uses the dilation covariance once more: the transition coefficients between the two eigenbases depend only on the sum of the indices, so that the comparison of the forms reduces to a convolution operator on $\mathbb{Z}$ (Theorem 4.6).

On the quantum side, we work in the concrete model of $E_r(2)$ on $\ell_\mathbb{Z}^2 \otimes L^2(\mathbb{T})$, where $L^\infty(E_r(2)) = B(\ell_\mathbb{Z}^2) \otimes L^\infty(\mathbb{T})$ and h is a weighted sum of diagonal entries. The spectral projections $P_m = \mathbf{1}_{[0, \tau^m]}(n^*n)$, $m \in \mathbb{Z}$, correspond to the balls $\pi^m \mathcal{O}^d$. The tail expectations of [Arh26, Section 7.1] have a bilateral version, which gives the following result, proved in Theorem 5.5. Here J denotes the normal $*$ -isomorphism $F(\|\cdot\|_K^d) \mapsto F(n^*n)$ from the radial bounded functions onto the von Neumann algebra generated by n^*n .

Theorem 1.2 (Lifting to $E_r(2)$) *Suppose that $r^2 = Q^{-d}$ and let $\alpha > 0$. There exists a semigroup $(\mathcal{T}_t)_{t \geq 0}$ of normal unital completely positive maps on $L^\infty(E_r(2))$ which preserves the Haar weight, is GNS-symmetric and KMS-symmetric, commutes with the modular group of the Haar weight, is point- σ -weakly continuous at zero, and satisfies $\mathcal{T}_t J = J e^{-tD_{\text{rad}}^\alpha}$ for any $t \geq 0$.*

As in [Arh26, Theorem 7.2], this lifting is a probability mixture of conditional expectations, and does not use the coproduct of $E_r(2)$. A different construction, using the coproduct, gives the covariant Casimir dynamics on the quantum plane in Section 7. It has radial generator L_τ^∞ , rather than the Taibleson generator in Theorem 1.2.

Finally, we explain how the whole space appears in the random walk of [Arh26, Section 6]. Recall that the random walk on the dual of $SU_r(2)$ driven by the fundamental representation has a diagonal part, computed by Biane [Bia06, Section 4], which is a Markov chain on $\{(n, k) \in \mathbb{N}_0^2 : k \leq n\}$ whose second coordinate converges to a limit k_∞ . The harmonic measures $\nu_{(n,k)}(j) = \mathbf{P}_{(n,k)}(k_\infty = j)$ are described in [Arh26, Theorem 6.13] by an entire function ϱ_∞ and a family of polynomials ϱ_d , and ϱ_∞ is a Jacobi theta function [Arh26, Corollary 6.19]. The horizontal coordinate k is the radial coordinate of $\mathcal{O}^d$, the origin corresponding to $k \rightarrow \infty$. Translating the horizontal coordinate towards the origin is the analogue, for the walk, of the passage from $\mathcal{O}^d$ to K^d .

Theorem 1.3 (Rescaled Poisson kernel) *Let $0 < \tau < 1$ and put $\lambda(\theta) \stackrel{\text{def}}{=} \frac{2\tau(1-\cos \theta)}{1-\tau^2}$ for $\theta \in \mathbb{R}$. For any $d \in \mathbb{N}_0$ and any $l \in \mathbb{Z}$, we have*

$$\lim_{k \rightarrow \infty} \nu_{(d+k,k)}(k+l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\varrho_d}{\varrho_\infty}(\lambda(\theta)) e^{il\theta} d\theta.$$

For $d = 0$, the limit is equal to $(\tau; \tau)_\infty^2 \sum_{m \geq 0} \frac{\tau^{|l|+2m}}{(\tau; \tau)_m (\tau; \tau)_{m+|l|}}$, the convergence holds at the rate $O(\tau^k)$ uniformly in l , and

$$\sum_{l \in \mathbb{Z}} \left(\lim_{k \rightarrow \infty} \nu_{(k,k)}(k+l) \right) e^{il\theta} = \frac{(\tau; \tau)_\infty^2}{|(\tau e^{i\theta}; \tau)_\infty|^2}, \quad \theta \in \mathbb{R}.$$

This is proved in Theorem 6.1. The limit operator is the translation-invariant horizontal network on $\mathbb{Z}$ with constant conductances, which is, up to a constant factor, the network of

L_τ^∞ (Remark 6.2). The spectral variable z of [Arh26, (6.18)], defined by $z + z^{-1} = 2 - \frac{(1-\tau^2)\lambda}{\tau}$, becomes the Fourier variable $e^{i\theta}$.

The correspondences in this paper have several distinct levels. The Haar identity is an identification of radial measures, and the energy bridge is an equivalence of closed forms with a common domain. In the spectral theorem the two radial operators are, more strongly, unitarily equivalent, whereas the Taibleson lifting and the quotient Casimir dynamics are different completely positive constructions. The contraction results in Section 8 concern strong resolvent convergence on fixed angular bands, Mosco convergence of the full forms, and point-norm convergence of compressed completely positive maps on the quotient algebra. They do not assert global operator-norm convergence of the Hilbert-space operators.

Approach of the paper The paper follows the structure of [Arh26], and we refer to [Arh26] for the compact case. The two new phenomena are the following. Firstly, the whole space is invariant under dilations, and every operator we consider is covariant under the corresponding unitary U ; this removes the bounded spectral errors of the compact case, and reduces several comparisons to convolution operators on $\mathbb{Z}$. Secondly, the boundary of the unit ball disappears, but the origin remains: at the origin, the bilateral operator L_η^∞ is in the limit circle case as soon as $\alpha > \frac{d}{2}$, and the relevant self-adjoint realization is the one of its closed form, which corresponds to a Neumann condition at the origin (Proposition 3.1). At the infinity of K^d , the eigenfunctions decay faster than any exponential (Lemma 4.4).

Structure of the paper Section 2 gives the radial analysis of the Taibleson operator on K^d and the martingale formula for its energy. Section 3 studies the bilateral operators L_η^∞ , their spectrum and their eigenfunctions. Section 4 proves Theorem 1.1. Section 5 recalls the quantum group $E_r(2)$, proves (1.1) and Theorem 1.2. Section 6 proves Theorem 1.3. Section 7 identifies the quantum plane and the closed Casimir and constructs its covariant completely positive heat semigroup. Section 8 proves the nonradial contraction, the Mosco limit and point-norm convergence of the quantum Markov dynamics.

2 Radial analysis on the whole space

2.1 Parameters and radial functions

We use the conventions of [Arh26, Section 2.1]. Let K be a non-Archimedean local field, $\mathcal{O}$ its ring of integers, π a uniformizer and $Q \stackrel{\text{def}}{=} \#(\mathcal{O}/\pi\mathcal{O})$, with $|\pi|_K = Q^{-1}$. On K^d we use the maximum norm and the additive Haar measure dx normalized by $\mu(\mathcal{O}^d) = 1$, and we fix a nontrivial additive character χ of K whose annihilator of $\mathcal{O}$ is $\mathcal{O}$, so that this measure is self-dual for the Fourier transform $\widehat{f}(\xi) \stackrel{\text{def}}{=} \int f(x)\overline{\chi(x \cdot \xi)} dx$, see [Tai75, Chapters I and II]. Throughout the paper, $\alpha > 0$ and

$$(2.1) \quad \tau \stackrel{\text{def}}{=} Q^{-d}, \quad \rho \stackrel{\text{def}}{=} Q^\alpha, \quad \eta \stackrel{\text{def}}{=} Q^{-\alpha} = \rho^{-1}, \quad r \stackrel{\text{def}}{=} \tau^{\frac{1}{2}}, \quad c \stackrel{\text{def}}{=} \frac{\rho - 1}{1 - \frac{\tau}{\rho}}, \quad \sigma_\alpha \stackrel{\text{def}}{=} Q^{-\frac{d+\alpha}{2}} = \left(\frac{\tau}{\rho}\right)^{\frac{1}{2}}.$$

In Section 3, the letters τ and η also denote arbitrary numbers of $(0, 1)$. For any integer $m \in \mathbb{Z}$ we put $B_m \stackrel{\text{def}}{=} \pi^m \mathcal{O}^d$ and $S_m \stackrel{\text{def}}{=} B_m \setminus B_{m+1}$. Then

$$(2.2) \quad \mu(B_m) = \tau^m, \quad \mu(S_m) = w_m \stackrel{\text{def}}{=} (1 - \tau)\tau^m, \quad \widehat{\mathbf{1}_{B_m}} = \tau^m \mathbf{1}_{B_{-m}}.$$

A radial function on K^d is a function of $\|x\|_K$, equivalently a function which is constant on each sphere S_m ; the origin has measure zero. By [Arh26, Section 2.1], radially is equivalent to the invariance under $\mathrm{GL}_d(\mathcal{O})$. We identify a radial function f with the sequence $(f_m)_{m \in \mathbb{Z}}$ of its values on the spheres, and we also write $f(m) = f_m$. In this way, the space $L^2_{\mathrm{rad}}(K^d)$ of radial square-integrable functions is the weighted sequence space $\ell^2_{\mathbb{Z}}(w)$, with the inner product $\langle f, g \rangle = \sum_{m \in \mathbb{Z}} w_m f_m \overline{g_m}$, linear in the first variable. Since $\sum_{m \leq 0} w_m = \infty$, the only constant sequence in $\ell^2_{\mathbb{Z}}(w)$ is zero. For any function $f: \mathbb{Z} \rightarrow \mathbb{C}$, we put

$$(\Delta f)_m \stackrel{\mathrm{def}}{=} f_{m+1} - f_m, \quad m \in \mathbb{Z}.$$

For any $f \in \ell^2_{\mathbb{Z}}(w)$ and any $m \in \mathbb{Z}$, the mean of f on the ball B_m is

$$(2.3) \quad m_m(f) \stackrel{\mathrm{def}}{=} \frac{1}{\mu(B_m)} \int_{B_m} f = (1 - \tau) \sum_{i \geq 0} \tau^i f_{m+i}.$$

The series converges absolutely: indeed $w_k |f_k|^2 \leq \|f\|^2$ gives $|f_k| \leq (1 - \tau)^{-\frac{1}{2}} \tau^{-\frac{k}{2}} \|f\|$, hence $\tau^i |f_{m+i}| \leq (1 - \tau)^{-\frac{1}{2}} \tau^{-\frac{m}{2}} \tau^{\frac{i}{2}} \|f\|$.

We also use the dilation. The operator U defined on $\ell^2_{\mathbb{Z}}(w)$ by

$$(2.4) \quad (Uf)_m \stackrel{\mathrm{def}}{=} \tau^{\frac{1}{2}} f_{m+1}, \quad m \in \mathbb{Z},$$

is unitary, since $\sum_m w_m \tau |f_{m+1}|^2 = \sum_m w_{m+1} |f_{m+1}|^2$. Since $\pi S_m = S_{m+1}$, it corresponds to the unitary $F \mapsto \tau^{\frac{1}{2}} F(\pi \cdot)$ of $L^2(K^d)$ restricted to radial functions.

2.2 The radial Taibleson operator

The Taibleson operator D^α of order α is the nonnegative self-adjoint Fourier multiplier $D^\alpha = \mathcal{F}^{-1} M_{\|\xi\|_K^\alpha} \mathcal{F}$ on $L^2(K^d)$. On the Bruhat–Schwartz functions, it is given by

$$(2.5) \quad D^\alpha f(x) = c \int_{K^d} \frac{f(x) - f(y)}{\|x - y\|_K^{d+\alpha}} dy,$$

with the constant c of (2.1), see [Arh26, Section 2.2], [AK26, (1.2)–(1.4)] and [Koc01, Section 2.2]. For any integer $m \in \mathbb{Z}$, we define the Haar wavelet

$$(2.6) \quad g_m \stackrel{\mathrm{def}}{=} \frac{\mathbf{1}_{B_{m+1}} - \tau \mathbf{1}_{B_m}}{s_m}, \quad s_m \stackrel{\mathrm{def}}{=} ((1 - \tau) \tau^{m+1})^{\frac{1}{2}}.$$

It takes the values 0 on S_k for $k < m$, $-\frac{\tau}{s_m}$ on S_m and $\frac{1-\tau}{s_m}$ on B_{m+1} . These are the functions g_n of [Arh26, (2.8)], which are also defined there for $n \in \mathbb{Z}$.

Proposition 2.1 *The family $(g_m)_{m \in \mathbb{Z}}$ is an orthonormal basis of $\ell^2_{\mathbb{Z}}(w)$. For any $m \in \mathbb{Z}$, the function g_m belongs to $\mathrm{dom} D^\alpha$ and $D^\alpha g_m = \rho^{m+1} g_m$. The closed subspace $L^2_{\mathrm{rad}}(K^d)$ reduces D^α , and the part D^α_{rad} of D^α in this subspace is the nonnegative self-adjoint operator*

$$D^\alpha_{\mathrm{rad}} f = \sum_{m \in \mathbb{Z}} \rho^{m+1} \langle f, g_m \rangle g_m, \quad \mathrm{dom} D^\alpha_{\mathrm{rad}} = \left\{ f \in \ell^2_{\mathbb{Z}}(w) : \sum_{m \in \mathbb{Z}} \rho^{2(m+1)} |\langle f, g_m \rangle|^2 < \infty \right\}.$$

Its spectrum is $\{\rho^m : m \in \mathbb{Z}\} \cup \{0\}$, every ρ^m is a simple eigenvalue and 0 is not an eigenvalue. Finally, $U D^\alpha_{\mathrm{rad}} U^{-1} = \rho D^\alpha_{\mathrm{rad}}$ and $U g_m = g_{m-1}$ for any $m \in \mathbb{Z}$.

Proof : By (2.2), the Fourier transform of the numerator of g_m is $\tau^{m+1}\mathbf{1}_{B_{-m-1}} - \tau\tau^m\mathbf{1}_{B_{-m}} = \tau^{m+1}\mathbf{1}_{S_{-m-1}}$, on which the symbol $\|\xi\|_K^\alpha$ is constant, equal to $Q^{\alpha(m+1)} = \rho^{m+1}$. Hence $g_m \in \text{dom } D^\alpha$ and $D^\alpha g_m = \rho^{m+1}g_m$. The numerator of g_m has integral zero and squared norm $(1-\tau)\tau^{m+1}$, and for $m < m'$ the function g_m is constant on $B_{m+1} \supset B_{m'}$, which contains the support of $g_{m'}$. Hence the family is orthonormal.

We show that it is complete. Let $f \in \ell_{\mathbb{Z}}^2(w)$. A direct computation gives

$$\begin{aligned} \langle f, g_m \rangle &= \frac{1}{s_m} \left[(1-\tau) \sum_{k \geq m+1} w_k f_k - \tau w_m f_m \right] = \frac{(1-\tau)\tau^{m+1}}{s_m} (m_{m+1}(f) - f_m) \\ (2.7) \quad &= s_m (m_{m+1}(f) - f_m), \end{aligned}$$

where we used $\sum_{k \geq m+1} w_k f_k = \tau^{m+1} m_{m+1}(f)$ by (2.3). Moreover, the absolute convergence of the series (2.3) gives, for any $m \in \mathbb{Z}$,

$$(2.8) \quad m_{m+1}(f) - f_m = (1-\tau) \sum_{i \geq 1} \tau^{i-1} f_{m+i} - f_m = \sum_{i \geq 0} \tau^i (\Delta f)_{m+i}.$$

Indeed, the series $\sum_{i \geq 0} \tau^i (f_{m+i+1} - f_{m+i})$ is the difference of two absolutely convergent series, and $\sum_{i \geq 0} \tau^i f_{m+i+1} - \sum_{i \geq 0} \tau^i f_{m+i} = \sum_{i \geq 1} (\tau^{i-1} - \tau^i) f_{m+i} - f_m$. Suppose now that f is orthogonal to every g_m . Put $h_m \stackrel{\text{def}}{=} m_{m+1}(f) - f_m$. Then $h = 0$ by (2.7). On the other hand, (2.8) gives $h_m - \tau h_{m+1} = (\Delta f)_m$ for any $m \in \mathbb{Z}$. Hence $\Delta f = 0$, so that f is constant, and $f = 0$. This proves the completeness.

Each g_m is an eigenvector of D^α . Hence, for any Borel set $E \subset \mathbb{R}$, the spectral projection $\mathbf{1}_E(D^\alpha)$ maps g_m to $\mathbf{1}_E(\rho^{m+1})g_m$, so that it leaves $L_{\text{rad}}^2(K^d)$ invariant. Consequently, this subspace reduces D^α , and the description of D_{rad}^α , of its domain and of its spectrum follows from the spectral theorem for the diagonal operator in the orthonormal basis (g_m) . Since the eigenvalues ρ^{m+1} are pairwise distinct and $\rho^{m+1} \rightarrow 0$ as $m \rightarrow -\infty$, the spectrum is $\{\rho^m : m \in \mathbb{Z}\} \cup \{0\}$, and 0 is not an eigenvalue since (g_m) is complete. Finally, for any $k \in \mathbb{Z}$,

$$(Ug_m)_k = \tau^{\frac{1}{2}} g_m(k+1) = \frac{\tau^{\frac{1}{2}}}{s_m} (\mathbf{1}_{k \geq m} - \tau \mathbf{1}_{k \geq m-1}) = \frac{\tau^{\frac{1}{2}} s_{m-1}}{s_m} g_{m-1}(k) = g_{m-1}(k),$$

since $s_{m-1} = \tau^{-\frac{1}{2}} s_m$. Hence $UD_{\text{rad}}^\alpha U^{-1} g_{m-1} = UD_{\text{rad}}^\alpha g_m = \rho^{m+1} g_{m-1} = \rho D_{\text{rad}}^\alpha g_{m-1}$, and the equality $UD_{\text{rad}}^\alpha U^{-1} = \rho D_{\text{rad}}^\alpha$ of self-adjoint operators follows from the spectral decomposition. $\blacksquare$

Remark 2.2 The functions g_m are the radial p -adic wavelets of Kozyrev [Koz02], and the spectral decomposition of isotropic Markov generators on ultrametric spaces in such bases is studied in detail in [BGP+14]. On the unit ball, the analogous statement is [Arh26, Theorem 2.3], where the eigenvalues of the regional operator are $\rho^{m+1} - \kappa$ and the constants form an additional eigenspace.

2.3 A martingale formula for the energy

We denote by $\mathcal{E}_D$ the closed form of D_{rad}^α , that is $\mathcal{E}_D(f) = \left\| (D_{\text{rad}}^\alpha)^{\frac{1}{2}} f \right\|^2 = \sum_m \rho^{m+1} |\langle f, g_m \rangle|^2$ on $\text{dom } \mathcal{E}_D = \text{dom}(D_{\text{rad}}^\alpha)^{\frac{1}{2}}$. We extend $\mathcal{E}_D$ to $\ell_{\mathbb{Z}}^2(w)$ by the same series, with values in $[0, \infty]$. We introduce the weight

$$(2.9) \quad \varpi_m \stackrel{\text{def}}{=} (\rho\tau)^{m+1} = Q^{(\alpha-d)(m+1)}, \quad m \in \mathbb{Z},$$

and the shift S defined on sequences by $(Su)_m \stackrel{\text{def}}{=} u_{m+1}$.

Proposition 2.3 *For any $f \in \ell_{\mathbb{Z}}^2(w)$, we have in $[0, \infty]$*

$$(2.10) \quad \mathcal{E}_D(f) = (1 - \tau) \sum_{m \in \mathbb{Z}} \varpi_m |f_m - m_{m+1}(f)|^2 = (1 - \tau) \sum_{m \in \mathbb{Z}} \varpi_m \left| \sum_{i \geq 0} \tau^i (\Delta f)_{m+i} \right|^2.$$

Proof : By (2.7), we have $\rho^{m+1} |\langle f, g_m \rangle|^2 = \rho^{m+1} (1 - \tau) \tau^{m+1} |m_{m+1}(f) - f_m|^2 = (1 - \tau) \varpi_m |m_{m+1}(f) - f_m|^2$. Summing over m gives the first equality, and (2.8) gives the second one. $\blacksquare$

Remark 2.4 The form $\mathcal{E}_D$ also has the jump-kernel expression

$$\mathcal{E}_D(f) = \frac{c}{2} \sum_{m \neq k} w_m w_k Q^{(d+\alpha) \min(m, k)} |f_m - f_k|^2 = c(1 - \tau)^2 \sum_{j \geq 1} \tau^j \sum_{m \in \mathbb{Z}} (\rho \tau)^m |f_{m+j} - f_m|^2,$$

for any $f \in \mathcal{D}$, that is for any radial function vanishing near infinity and constant near the origin, see Section 3. This is the bilateral version of [Arh26, (2.11)]. Indeed, such a function f is a Bruhat–Schwartz function, and the kernel formula (2.5) and a symmetrization give $\langle D^\alpha f, f \rangle = \frac{c}{2} \iint \frac{|f(x) - f(y)|^2}{\|x - y\|_K^{d+\alpha}} dx dy$, the integrand being zero near the diagonal. For $x \in S_m$ and $y \in S_k$ with $m \neq k$, we have $\|x - y\|_K = Q^{-\min(m, k)}$, and the spheres S_m do not contribute to their own interaction. The second expression follows from $w_m w_k Q^{(d+\alpha)m} = (1 - \tau)^2 \rho^m \tau^k$ for $m < k$. We will only use (2.10).

3 Bilateral birth–death operators and Hahn–Exton q -Bessel functions

In this section, $0 < \tau < 1$ and $0 < \eta < 1$ are arbitrary, and $w_k = (1 - \tau) \tau^k$ for $k \in \mathbb{Z}$. We consider the Hilbert space $\ell_{\mathbb{Z}}^2(w)$. We use the notation $(a; \eta)_n \stackrel{\text{def}}{=} \prod_{i=0}^{n-1} (1 - a\eta^i)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$, and the basic hypergeometric series

$$(3.1) \quad {}_1\varphi_1(0; b; \eta, x) \stackrel{\text{def}}{=} \sum_{j \geq 0} \frac{(-1)^j \eta^{\frac{j(j-1)}{2}}}{(\eta; \eta)_j (b; \eta)_j} x^j, \quad x \in \mathbb{C},$$

of [GR04, Section 1.2], see also [KS92, (2.1)] and [DLMF, (17.4.1)], which defines an entire function of x for any $b \in \mathbb{C} \setminus \{\eta^{-i} : i \in \mathbb{N}_0\}$.

3.1 The closed form and the operator

For any $k \in \mathbb{Z}$, put

$$(3.2) \quad \beta_k \stackrel{\text{def}}{=} \tau \eta^{-k-1}, \quad \delta_k \stackrel{\text{def}}{=} \eta^{-k}, \quad c_k \stackrel{\text{def}}{=} w_k \beta_k = w_{k+1} \delta_{k+1} = (1 - \tau) \left(\frac{\tau}{\eta} \right)^{k+1}.$$

For any function $f: \mathbb{Z} \rightarrow \mathbb{C}$, we define the function $Lf: \mathbb{Z} \rightarrow \mathbb{C}$ by

$$(3.3) \quad (Lf)(k) \stackrel{\text{def}}{=} \beta_k (f(k) - f(k+1)) + \delta_k (f(k) - f(k-1)), \quad k \in \mathbb{Z}.$$

Equivalently, $w_k(Lf)(k) = c_{k-1}(\Delta f)_{k-1} - c_k(\Delta f)_k$. Consequently, for any functions $f, g: \mathbb{Z} \rightarrow \mathbb{C}$ such that g has finite support, we have the finite summation by parts

$$(3.4) \quad \sum_{k \in \mathbb{Z}} w_k(Lg)(k) \overline{f(k)} = \sum_{k \in \mathbb{Z}} c_k(\Delta g)_k \overline{(\Delta f)_k} = \sum_{k \in \mathbb{Z}} w_k g(k) \overline{(Lf)(k)}.$$

For any $f \in \ell_{\mathbb{Z}}^2(w)$, we put

$$(3.5) \quad \mathcal{E}_{\eta}^{\infty}(f) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} c_k |f(k+1) - f(k)|^2 \in [0, \infty], \quad \text{dom } \mathcal{E}_{\eta}^{\infty} \stackrel{\text{def}}{=} \{f \in \ell_{\mathbb{Z}}^2(w) : \mathcal{E}_{\eta}^{\infty}(f) < \infty\},$$

and we denote by $\mathcal{E}_{\eta}^{\infty}(f, g)$ the associated sesquilinear form. Finally, let $\mathcal{D}$ be the space of functions $f: \mathbb{Z} \rightarrow \mathbb{C}$ for which there exists an integer $M \geq 0$ such that $f(k) = 0$ for $k \leq -M$ and f is constant on $\{k \geq M\}$. In the picture of Section 2, the elements of $\mathcal{D}$ are the radial functions which vanish near infinity and are constant near the origin. For any $j \in \mathbb{Z}$, the function $\mathbf{1}_{\geq j} \stackrel{\text{def}}{=} \mathbf{1}_{\{k \geq j\}}$ belongs to $\mathcal{D}$.

Proposition 3.1 1. The space $\mathcal{D}$ is dense in $\ell_{\mathbb{Z}}^2(w)$ and contained in $\text{dom } \mathcal{E}_{\eta}^{\infty}$. For any $g \in \mathcal{D}$, the function Lg has finite support, and for any $f \in \text{dom } \mathcal{E}_{\eta}^{\infty}$ we have

$$(3.6) \quad \mathcal{E}_{\eta}^{\infty}(f, g) = \langle f, Lg \rangle_{\ell_{\mathbb{Z}}^2(w)}.$$

2. The form $\mathcal{E}_{\eta}^{\infty}$ is closed and $\mathcal{D}$ is a core for it.
3. The restriction of L to $\mathcal{D}$ is an essentially self-adjoint nonnegative operator on $\ell_{\mathbb{Z}}^2(w)$. Its closure, which we denote by L_{η}^{∞} , is the nonnegative self-adjoint operator associated with the closed form $\mathcal{E}_{\eta}^{\infty}$.
4. For any $f \in \text{dom } L_{\eta}^{\infty}$, we have $L_{\eta}^{\infty} f = Lf$ and $\lim_{K \rightarrow +\infty} c_K(f(K+1) - f(K)) = 0$.
5. The kernel of L_{η}^{∞} is $\{0\}$.
6. The unitary U of (2.4) satisfies $U\mathcal{D} = \mathcal{D}$, $\mathcal{E}_{\eta}^{\infty}(Uf) = \eta \mathcal{E}_{\eta}^{\infty}(f)$ for any $f \in \ell_{\mathbb{Z}}^2(w)$, and $U^{-1}L_{\eta}^{\infty}U = \eta L_{\eta}^{\infty}$.

Proof : 1. The finitely supported functions belong to $\mathcal{D}$ and are dense in $\ell_{\mathbb{Z}}^2(w)$. Let $g \in \mathcal{D}$ and M as in the definition. The function Δg vanishes outside $[-M-1, M-1]$, hence $\mathcal{E}_{\eta}^{\infty}(g) < \infty$, and Lg vanishes outside $[-M-1, M]$ since $w_k(Lg)(k) = c_{k-1}(\Delta g)_{k-1} - c_k(\Delta g)_k$. Let $f \in \text{dom } \mathcal{E}_{\eta}^{\infty}$. Let $a \leq -M-2$ and $b \geq M+1$. A summation by parts gives

$$\sum_{k=a}^{b-1} c_k(\Delta f)_k \overline{(\Delta g)_k} = \sum_{k=a+1}^{b-1} f(k) \overline{(c_{k-1}(\Delta g)_{k-1} - c_k(\Delta g)_k)} + c_{b-1} f(b) \overline{(\Delta g)_{b-1}} - c_a f(a) \overline{(\Delta g)_a}.$$

The boundary terms vanish, since $(\Delta g)_a = (\Delta g)_{b-1} = 0$. The left-hand side is $\mathcal{E}_{\eta}^{\infty}(f, g)$ and the right-hand side is $\sum_k w_k f(k) \overline{(Lg)(k)} = \langle f, Lg \rangle$, since all the terms vanish outside $[a, b]$. This proves (3.6).

2. Let (f_i) be a sequence of $\text{dom } \mathcal{E}_{\eta}^{\infty}$ converging to f in $\ell_{\mathbb{Z}}^2(w)$ such that $\mathcal{E}_{\eta}^{\infty}(f_i - f_j) \rightarrow 0$ as $i, j \rightarrow \infty$. Convergence in $\ell_{\mathbb{Z}}^2(w)$ implies pointwise convergence, so that Fatou's lemma gives $\mathcal{E}_{\eta}^{\infty}(f - f_j) \leq \liminf_i \mathcal{E}_{\eta}^{\infty}(f_i - f_j)$, which tends to zero as $j \rightarrow \infty$. Hence the form is closed.

Now, let $f \in \text{dom } \mathcal{E}_{\eta}^{\infty}$. For integers $M, N \geq 1$, we define $f^{(M, N)} \in \mathcal{D}$ by $f^{(M, N)}(k) = 0$ if $k < -M$, $f^{(M, N)}(k) = f(k)$ if $-M \leq k \leq N$ and $f^{(M, N)}(k) = f(N)$ if $k > N$. The difference

$u = f - f^{(M,N)}$ satisfies $(\Delta u)_k = (\Delta f)_k$ for $k \leq -M-2$ and $k \geq N$, $(\Delta u)_{-M-1} = -f(-M-1)$, and $(\Delta u)_k = 0$ otherwise. Hence

$$\mathcal{E}_\eta^\infty(u) = \sum_{k \leq -M-2} c_k |(\Delta f)_k|^2 + c_{-M-1} |f(-M-1)|^2 + \sum_{k \geq N} c_k |(\Delta f)_k|^2.$$

The first and third terms are tails of a convergent series. For the second one, we have $c_{-M-1} |f(-M-1)|^2 = \beta_{-M-1} w_{-M-1} |f(-M-1)|^2 \leq \tau \eta^M \|f\|^2$, which tends to zero. On the other hand, $\|u\|^2 = \sum_{k < -M} w_k |f(k)|^2 + \sum_{k > N} w_k |f(k) - f(N)|^2$. The first sum tends to zero. For the second one, the Cauchy-Schwarz inequality gives, for any $k > N$,

$$|f(k) - f(N)|^2 \leq \left(\sum_{i=N}^{k-1} c_i |(\Delta f)_i|^2 \right) \left(\sum_{i=N}^{k-1} c_i^{-1} \right) \leq \varepsilon_N \sum_{i=N}^{k-1} c_i^{-1}, \quad \varepsilon_N \stackrel{\text{def}}{=} \sum_{i \geq N} c_i |(\Delta f)_i|^2.$$

Tonelli's theorem gives the exact estimate

$$\sum_{k > N} w_k \sum_{i=N}^{k-1} c_i^{-1} = \sum_{i=N}^{\infty} c_i^{-1} \sum_{k > i} w_k = \frac{1}{1-\tau} \sum_{i=N}^{\infty} \eta^{i+1} = \frac{\eta^{N+1}}{(1-\tau)(1-\eta)}.$$

Consequently $\sum_{k > N} w_k |f(k) - f(N)|^2 \leq \varepsilon_N \eta^{N+1} / ((1-\tau)(1-\eta)) \rightarrow 0$. Letting M and N tend to infinity, we conclude that $f^{(M,N)} \rightarrow f$ for the norm $(\mathcal{E}_\eta^\infty(\cdot) + \|\cdot\|^2)^{\frac{1}{2}}$, which proves that $\mathcal{D}$ is a core.

3. By (3.6), for any $f, g \in \mathcal{D}$ we have $\langle f, Lg \rangle = \mathcal{E}_\eta^\infty(f, g) = \overline{\mathcal{E}_\eta^\infty(g, f)} = \langle Lf, g \rangle$, and $\langle Lf, f \rangle = \mathcal{E}_\eta^\infty(f) \geq 0$. Hence the restriction $L|_{\mathcal{D}}$ is symmetric and nonnegative. We show that the range of $L|_{\mathcal{D}} + I$ is dense. Let $h \in \ell_{\mathbb{Z}}^2(w)$ be orthogonal to $(L + I)\mathcal{D}$. Since the coefficients of L are real, the real and imaginary parts of h have the same property, and we can assume that h is real-valued. Applying the orthogonality to $\mathbf{1}_{\{j\}} \in \mathcal{D}$ and using (3.4), we obtain $w_j((L + I)h)(j) = 0$, that is

$$(3.7) \quad (Lh)(k) = -h(k), \quad k \in \mathbb{Z}.$$

Applying the orthogonality to $\mathbf{1}_{\geq j}$, and using $L\mathbf{1}_{\geq j} = -\beta_{j-1}\mathbf{1}_{\{j-1\}} + \delta_j\mathbf{1}_{\{j\}}$ together with $w_{j-1}\beta_{j-1} = w_j\delta_j = c_{j-1}$, we obtain $\sum_{k \geq j} w_k h(k) + c_{j-1}(h(j) - h(j-1)) = 0$, that is

$$(3.8) \quad c_{j-1}(\Delta h)_{j-1} = -R_j, \quad R_j \stackrel{\text{def}}{=} \sum_{k \geq j} w_k h(k), \quad j \in \mathbb{Z}.$$

By the Cauchy-Schwarz inequality, $|R_j| \leq (\sum_{k \geq j} w_k)^{\frac{1}{2}} \|h\| = \tau^{\frac{j}{2}} \|h\|$. Moreover, $w_k |h(k)|^2 \rightarrow 0$ as $k \rightarrow \pm\infty$, that is $|h(k)| = o(\tau^{-\frac{k}{2}})$. Let $a < b$ be integers. A summation by parts using (3.8) and $R_k - R_{k+1} = w_k h(k)$ gives

$$\sum_{k=a}^b c_k (\Delta h)_k^2 \stackrel{(3.8)}{=} - \sum_{k=a}^b R_{k+1} (h(k+1) - h(k)) = R_{a+1} h(a) - R_{b+1} h(b+1) - \sum_{k=a+1}^b w_k h(k)^2.$$

Since $|R_{b+1} h(b+1)| = O(\tau^{\frac{b}{2}}) o(\tau^{-\frac{b}{2}}) \rightarrow 0$ as $b \rightarrow +\infty$, we obtain

$$\sum_{k \geq a} c_k (\Delta h)_k^2 + \sum_{k > a} w_k h(k)^2 = R_{a+1} h(a), \quad a \in \mathbb{Z}.$$

The left-hand side is nonnegative and nonincreasing in a , whereas the right-hand side is $O(\tau^{\frac{a}{2}})o(\tau^{-\frac{a}{2}}) = o(1)$ as $a \rightarrow -\infty$. Hence the left-hand side vanishes for any a , and $h = 0$. Consequently, the range of $L|_{\mathcal{D}} + I$ is dense.

Let $\bar{L}$ be the closure of $L|_{\mathcal{D}}$. It is symmetric and $\|(\bar{L} + I)f\| \geq \|f\|$ for $f \in \text{dom } \bar{L}$, so that the range of $\bar{L} + I$ is closed, hence equal to $\ell_{\mathbb{Z}}^2(w)$. If $g \in \text{dom } \bar{L}^*$, there exists $f \in \text{dom } \bar{L}$ with $(\bar{L} + I)f = (\bar{L}^* + I)g$, and $g - f$ belongs to $\text{Ker}(\bar{L}^* + I) = \text{Ran}(\bar{L} + I)^\perp = \{0\}$. Hence $\bar{L}^* = \bar{L}$. The closed form of the nonnegative self-adjoint operator $\bar{L}$ is the closure of the form $f \mapsto \langle \bar{L}f, f \rangle$ on the core $\mathcal{D}$, which is the restriction of $\mathcal{E}_\eta^\infty$ to $\mathcal{D}$ by (3.6). Since $\mathcal{D}$ is a core for the closed form $\mathcal{E}_\eta^\infty$ by part 2, the closed form of $\bar{L}$ is $\mathcal{E}_\eta^\infty$. This proves part 3.

4. Let $f \in \text{dom } L_\eta^\infty$. For any $j \in \mathbb{Z}$, we have $\mathbf{1}_{\{j\}} \in \mathcal{D}$, hence $w_j(L_\eta^\infty f)(j) = \langle L_\eta^\infty f, \mathbf{1}_{\{j\}} \rangle = \langle f, L\mathbf{1}_{\{j\}} \rangle = w_j(Lf)(j)$ by self-adjointness and (3.4). Hence $L_\eta^\infty f = Lf$. Moreover, $\langle f, L\mathbf{1}_{\geq j} \rangle = c_{j-1}(f(j) - f(j-1))$ by the computation of part 3, whereas

$$|\langle L_\eta^\infty f, \mathbf{1}_{\geq j} \rangle| \leq \left(\sum_{k \geq j} w_k \right)^{\frac{1}{2}} \|L_\eta^\infty f\| = \tau^{\frac{j}{2}} \|L_\eta^\infty f\|.$$

Since these two quantities are equal, we obtain $c_{j-1}(f(j) - f(j-1)) \rightarrow 0$ as $j \rightarrow +\infty$.

5. If $L_\eta^\infty f = 0$, then $\mathcal{E}_\eta^\infty(f) = 0$, so that $\Delta f = 0$ and f is constant. Since $f \in \ell_{\mathbb{Z}}^2(w)$, we obtain $f = 0$.

6. The first assertion is clear. For any $f \in \ell_{\mathbb{Z}}^2(w)$, we have

$$\mathcal{E}_\eta^\infty(Uf) = \tau \sum_{k \in \mathbb{Z}} c_k |f(k+2) - f(k+1)|^2 = \tau \sum_{k \in \mathbb{Z}} c_{k-1} |(\Delta f)_k|^2 \stackrel{(3.2)}{=} \eta \mathcal{E}_\eta^\infty(f).$$

Since U is unitary, it follows that the closed form of $U^{-1}L_\eta^\infty U$ is $\eta \mathcal{E}_\eta^\infty$, which is the closed form of ηL_η^∞ . $\blacksquare$

Remark 3.2 The solutions of $Lf = 0$ are the functions f such that $c_k(\Delta f)_k$ does not depend on k . At $-\infty$, which corresponds to the infinity of K^d , the operator (3.3) is in the limit point case, in the sense of the Weyl alternative for Jacobi operators [Tes00, Chapter 2], since the constant function 1 is a solution which is not square-summable near $-\infty$. At $+\infty$, which corresponds to the origin of K^d , a nonconstant solution satisfies $(\Delta f)_k = Cc_k^{-1}$ with $C \neq 0$, and it belongs to $\ell_{\mathbb{Z}}^2(w)$ near $+\infty$ if and only if $\eta^2 < \tau$. If $\tau = Q^{-d}$ and $\eta = Q^{-\alpha}$, this means $\alpha > \frac{d}{2}$. In this case, both solutions are square-summable near the origin, and a boundary condition at the origin is needed. The constants near the origin, which belong to $\mathcal{D}$, select the realization of Proposition 3.1, which satisfies the Neumann condition of part 4. Moreover, a nonconstant solution has finite energy near $+\infty$ if and only if $\eta < \tau$, that is $\alpha > d$.

3.2 Eigenfunctions

Proposition 3.3 *Let $\lambda \in \mathbb{C}$ and define $f_\lambda: \mathbb{Z} \rightarrow \mathbb{C}$ by*

$$(3.9) \quad f_\lambda(k) \stackrel{\text{def}}{=} {}_1\varphi_1(0; \tau; \eta, \lambda\eta^{k+1}), \quad k \in \mathbb{Z}.$$

1. *We have $Lf_\lambda = \lambda f_\lambda$. Moreover $f_\lambda(k) \rightarrow 1$ and $c_k(\Delta f_\lambda)_k = O(\tau^k)$ as $k \rightarrow +\infty$.*
2. *Let $g: \mathbb{Z} \rightarrow \mathbb{C}$ be a function such that $Lg = \lambda g$, $c_k(\Delta g)_k \rightarrow 0$ as $k \rightarrow +\infty$, and $g(k) = o(\tau^{-\frac{k}{2}})$ as $k \rightarrow +\infty$. Then g is a multiple of f_λ .*

Proof : 1. Put $y \stackrel{\text{def}}{=} \eta^k$. For any integer $j \geq 0$, the function $k \mapsto y^j = \eta^{jk}$ satisfies

$$L(y^j) = \tau \eta^{-k-1} \eta^{jk} (1 - \eta^j) + \eta^{-k} \eta^{jk} (1 - \eta^{-j}) = -\eta^{-j} (1 - \eta^j) (1 - \tau \eta^{j-1}) y^{j-1},$$

which vanishes for $j = 0$. Write $f_\lambda(k) = \sum_{j \geq 0} a_j (\lambda \eta)^j y^j$ with $a_j = \frac{(-1)^j \eta^{j(j-1)/2}}{(\eta; \eta)_j (\tau; \eta)_j}$. The series converges absolutely for any y , and so does the series obtained by applying L termwise, since L involves only the values at $k-1$, k and $k+1$. Hence

$$(Lf_\lambda)(k) = - \sum_{j \geq 1} a_j (\lambda \eta)^j \eta^{-j} (1 - \eta^j) (1 - \tau \eta^{j-1}) y^{j-1} = \lambda \sum_{j \geq 1} a_{j-1} (\lambda \eta)^{j-1} y^{j-1} = \lambda f_\lambda(k),$$

where we used $a_j (1 - \eta^j) (1 - \tau \eta^{j-1}) = -\eta^{j-1} a_{j-1}$. As $k \rightarrow +\infty$, we have $y \rightarrow 0$, hence $f_\lambda(k) = 1 + O(\eta^k)$ and $(\Delta f_\lambda)_k = O(\eta^k)$, so that $c_k(\Delta f_\lambda)_k = O((\frac{\tau}{\eta})^k \eta^k) = O(\tau^k)$.

2. For two solutions f, g of $Lf = \lambda f$ and $Lg = \lambda g$, the Wronskian (compare [Tes00, Chapter 1]) $W(k) \stackrel{\text{def}}{=} c_k(g(k)(\Delta f)_k - f(k)(\Delta g)_k)$ does not depend on k . Indeed, multiplying the equality $c_{k-1}(\Delta f)_{k-1} - c_k(\Delta f)_k = \lambda w_k f(k)$ by $g(k)$, the analogous equality for g by $f(k)$, and subtracting, we obtain $W(k) = c_{k-1}(g(k)(\Delta f)_{k-1} - f(k)(\Delta g)_{k-1})$, and $g(k)(\Delta f)_{k-1} - f(k)(\Delta g)_{k-1} = g(k-1)f(k) - f(k-1)g(k) = g(k-1)(\Delta f)_{k-1} - f(k-1)(\Delta g)_{k-1}$. Take $f = f_\lambda$. By part 1 and the assumptions on g , we have $g(k)c_k(\Delta f)_k = o(\tau^{-\frac{k}{2}})O(\tau^k) \rightarrow 0$ and $f(k)c_k(\Delta g)_k \rightarrow 0$ as $k \rightarrow +\infty$. Hence $W = 0$, that is $g(k)f(k+1) = f(k)g(k+1)$ for any k . Since $f(k) \rightarrow 1$, there exists k_0 such that $f(k) \neq 0$ for $k \geq k_0$, and $g = Cf$ on $\{k \geq k_0\}$ for some $C \in \mathbb{C}$. Finally, the equation $Lg = \lambda g$ determines $g(k-1)$ from $g(k)$ and $g(k+1)$, since $\delta_k \neq 0$. Hence $g = Cf$ on $\mathbb{Z}$. $\blacksquare$

For any $z \in \mathbb{Z}$, we put $f_z \stackrel{\text{def}}{=} f_{\eta^z}$, that is $f_z(k) = {}_1\varphi_1(0; \tau; \eta, \eta^{z+k+1})$. By definition, we have

$$(3.10) \quad f_z(k+1) = f_{z+1}(k), \quad \text{that is} \quad Uf_z = \tau^{\frac{1}{2}} f_{z+1}, \quad z, k \in \mathbb{Z},$$

where the second equality holds as soon as $f_z \in \ell_{\mathbb{Z}}^2(w)$.

Remark 3.4 Let $\nu > -1$ be defined by $\tau = \eta^{\nu+1}$. Recall that the Hahn–Exton q -Bessel function of order ν in base η is $J_\nu(x; \eta) = x^\nu \frac{(\eta^{\nu+1}; \eta)_\infty}{(\eta; \eta)_\infty} {}_1\varphi_1(0; \eta^{\nu+1}; \eta, \eta x^2)$, see [KS92, (3.3)] for the corresponding normalization in base q^2 . Hence

$$f_z(k) = \frac{(\eta; \eta)_\infty}{(\tau; \eta)_\infty} x^{-\nu} J_\nu(x; \eta), \quad x = \eta^{\frac{z+k}{2}}.$$

If $\tau = Q^{-d}$ and $\eta = Q^{-\alpha}$, then $\nu = \frac{d}{\alpha} - 1$. In particular, for $\alpha = d$ the eigenfunctions are the Hahn–Exton q -Bessel functions of order 0 in base $\tau = r^2$. Koornwinder and Swarttouw proved the orthogonality relation [KS92, Proposition 2.6, (2.11)], which reads, after replacing their parameters z and q by $\tau^{\frac{1}{2}}$ and η ,

$$(3.11) \quad \sum_{k \in \mathbb{Z}} \tau^k f_z(k) f_{z'}(k) = \delta_{zz'} \tau^{-z} \frac{(\eta; \eta)_\infty^2}{(\tau; \eta)_\infty^2}, \quad z, z' \in \mathbb{Z}.$$

The orthogonality and completeness are already contained in [KS92, Proposition 2.6 and Section 3]: after normalization, the kernel in (3.11) is symmetric in the lattice and spectral indices, and the corresponding transform has the same inverse. Theorem 3.6 gives an independent operator-theoretic proof, up to the value of the norm, and identifies this transform with the particular self-adjoint realization of Proposition 3.1.

3.3 The spectrum by a limit transition

We now relate L_η^∞ to the little η -Jacobi operator L_η of [Arh26, Section 3]. Recall that L_η is the nonnegative self-adjoint operator on $\ell_{\mathbb{N}_0}^2(w)$ associated with the closed form $\mathcal{E}_\eta(f) = \sum_{k \geq 0} w_k \beta_k^\eta |f(k+1) - f(k)|^2$, where $\beta_k^\eta = \tau \eta^{-k-1} (1 - \eta^{k+1})$ and $\delta_k^\eta = \eta^{-k} (1 - \eta^k)$, and that L_η acts by $(L_\eta f)(k) = \beta_k^\eta (f(k) - f(k+1)) + \delta_k^\eta (f(k) - f(k-1))$ on the space of polynomials in η^k [Arh26, (3.2) and Proposition 3.1]. By [Arh26, Proposition 3.1], the spectrum of L_η consists of the simple eigenvalues

$$(3.12) \quad \Lambda_n(\eta) = \eta^{-n} (1 - \eta^n) (1 - \tau \eta^n), \quad n \in \mathbb{N}_0,$$

which form a strictly increasing sequence, and the corresponding eigenvectors are the little η -Jacobi polynomials $p_n(\eta^k; \tau \eta^{-1}, 1; \eta)$. For any integer $N \geq 0$, we define the isometry $V_N: \ell_{\mathbb{N}_0}^2(w) \rightarrow \ell_{\mathbb{Z}}^2(w)$ by

$$(V_N f)(k) \stackrel{\text{def}}{=} \tau^{\frac{N}{2}} f(k+N) \quad \text{if } k \geq -N, \quad (V_N f)(k) \stackrel{\text{def}}{=} 0 \quad \text{if } k < -N.$$

Its range is the space H_N of functions vanishing on $\{k < -N\}$. We put

$$(3.13) \quad \tilde{A}_N \stackrel{\text{def}}{=} \eta^N V_N L_\eta V_N^* \oplus 0_{H_N^\perp},$$

which is a nonnegative self-adjoint operator on $\ell_{\mathbb{Z}}^2(w)$. Its restriction to H_N is unitarily equivalent to $\eta^N L_\eta$, so that its positive spectral points are the simple eigenvalues $\eta^N \Lambda_n(\eta)$, $n \geq 1$, whereas 0 is an eigenvalue of infinite multiplicity because of the zero summand. Under the identification of Section 2, V_N corresponds to the passage from $\mathcal{O}^d$ to the ball $B_{-N} = \pi^{-N} \mathcal{O}^d$, followed by the dilation which preserves the L^2 -norm.

Lemma 3.5 *Let $g \in \mathcal{D}$ and let M be as in the definition of $\mathcal{D}$. For any integer $N > M$, the function g belongs to $\text{dom } \tilde{A}_N$, and*

$$((\tilde{A}_N - L)g)(k) = -\eta^N \left[\tau(g(k) - g(k+1)) + (g(k) - g(k-1)) \right], \quad k \in \mathbb{Z}.$$

In particular, $\|(\tilde{A}_N - L_\eta^\infty)g\| \rightarrow 0$ as $N \rightarrow \infty$.

Proof : Since $N > M$, we have $g \in H_N$ and the function $u = V_N^* g$, given by $u(m) = \tau^{-\frac{N}{2}} g(m-N)$, is constant on $\{m \geq M+N\}$. We first show that $u \in \text{dom } L_\eta$ and that $L_\eta u$ is given by the formula of L_η . The function u belongs to $\text{dom } \mathcal{E}_\eta$, since Δu has finite support. For any $f \in \text{dom } \mathcal{E}_\eta$ and any integer $b \geq M+N+1$, the summation by parts [Arh26, (3.4)] gives

$$\sum_{k=0}^{b-1} w_k \beta_k^\eta (\Delta f)_k \overline{(\Delta u)_k} = \sum_{k=0}^{b-1} w_k f(k) \overline{(L_\eta u)(k)} + w_{b-1} \beta_{b-1}^\eta f(b) \overline{(\Delta u)_{b-1}},$$

where $L_\eta u$ denotes the formal expression. The boundary term vanishes since $(\Delta u)_{b-1} = 0$, and $L_\eta u$ vanishes on $\{k \geq M+N+1\}$. Hence $\mathcal{E}_\eta(f, u) = \langle f, L_\eta u \rangle$ for any $f \in \text{dom } \mathcal{E}_\eta$. By the definition of the operator associated with a closed form, $u \in \text{dom } L_\eta$ and $L_\eta u$ is the formal expression. Consequently, $g = V_N u \in \text{dom } \tilde{A}_N$. For $k \geq -N$, we obtain

$$\begin{aligned} (\tilde{A}_N g)(k) &= \eta^N \tau^{\frac{N}{2}} (L_\eta u)(k+N) \\ &= \tau \eta^{-k-1} (1 - \eta^{k+N+1}) (g(k) - g(k+1)) + \eta^{-k} (1 - \eta^{k+N}) (g(k) - g(k-1)), \end{aligned}$$

and $(\tilde{A}_N g)(k) = 0$ for $k < -N$. Comparing with (3.3), and using $g(k) = 0$ for $k \leq -M$, we obtain the formula. The right-hand side vanishes outside $[-M-1, M+1]$ and is bounded by $4\eta^N \max |g|$. Finally, $L_\eta^\infty g = Lg$ by Proposition 3.1. $\blacksquare$

We say that a sequence (A_N) of self-adjoint operators converges to a self-adjoint operator A in the strong resolvent sense if $(A_N - z)^{-1} \rightarrow (A - z)^{-1}$ strongly for any $z \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 3.6 *Let $0 < \tau < 1$ and $0 < \eta < 1$.*

1. *The operators $\tilde{A}_N$ converge to L_η^∞ in the strong resolvent sense as $N \rightarrow \infty$.*
2. *The spectrum of L_η^∞ is $\{\eta^z : z \in \mathbb{Z}\} \cup \{0\}$. Its spectral measures are purely atomic, each η^z is a simple eigenvalue, and 0 belongs to the continuous spectrum.*
3. *For any $z \in \mathbb{Z}$, the function f_z belongs to $\ell_{\mathbb{Z}}^2(w)$ and spans the eigenspace of L_η^∞ for the eigenvalue η^z . The family $(\phi_z)_{z \in \mathbb{Z}}$, where $\phi_z \stackrel{\text{def}}{=} \|f_z\|^{-1} f_z$, is an orthonormal basis of $\ell_{\mathbb{Z}}^2(w)$, and $U\phi_z = \phi_{z+1}$.*
4. *For any $z \in \mathbb{Z}$, let e_n^η be a normalized eigenvector of L_η for the eigenvalue $\Lambda_n(\eta)$. Then the orthogonal projection onto $\mathbb{C}V_N e_{N-z}^\eta$ converges strongly to the orthogonal projection onto $\mathbb{C}\phi_z$ as $N \rightarrow \infty$.*

Proof : 1. Let $z \in \mathbb{C} \setminus \mathbb{R}$. For any $g \in \mathcal{D}$ and any integer N large enough, Lemma 3.5 gives $g \in \text{dom } \tilde{A}_N$, hence

$$(\tilde{A}_N - z)^{-1}(L_\eta^\infty - z)g - (L_\eta^\infty - z)^{-1}(L_\eta^\infty - z)g = (\tilde{A}_N - z)^{-1}(L_\eta^\infty - \tilde{A}_N)g,$$

whose norm is bounded by $|\text{Im } z|^{-1} \|(L_\eta^\infty - \tilde{A}_N)g\|$, which tends to zero. Since $\mathcal{D}$ is a core for L_η^∞ by Proposition 3.1, the space $(L_\eta^\infty - z)\mathcal{D}$ is dense in $\ell_{\mathbb{Z}}^2(w)$. Since $\|(\tilde{A}_N - z)^{-1}\| \leq |\text{Im } z|^{-1}$ for any N , we conclude that $(\tilde{A}_N - z)^{-1} \rightarrow (L_\eta^\infty - z)^{-1}$ strongly.

2. Let $0 < a < b$ be real numbers which do not belong to $\eta^{\mathbb{Z}}$ and which are not eigenvalues of L_η^∞ , and put $Z_{a,b} \stackrel{\text{def}}{=} \{z \in \mathbb{Z} : a < \eta^z < b\}$, which is finite. We claim that, for N large enough, the eigenvalues of $\tilde{A}_N$ in (a, b) are the numbers $\eta^N \Lambda_{N-z}(\eta)$, $z \in Z_{a,b}$, each of multiplicity one. Indeed, for any fixed $z \in \mathbb{Z}$, we have

$$(3.14) \quad \eta^N \Lambda_{N-z}(\eta) = \eta^z (1 - \eta^{N-z})(1 - \tau \eta^{N-z}) \xrightarrow{N \rightarrow \infty} \eta^z.$$

Let z_+ be the largest integer such that $\eta^{z_+} \geq b$, and z_- the smallest integer such that $\eta^{z_-} \leq a$. By (3.14), for N large enough we have $\eta^N \Lambda_{N-z_+}(\eta) > b$, $\eta^N \Lambda_{N-z_-}(\eta) < a$ and $\eta^N \Lambda_{N-z}(\eta) \in (a, b)$ for any $z \in Z_{a,b}$. Since $n \mapsto \eta^N \Lambda_n(\eta)$ is increasing, the eigenvalues $\eta^N \Lambda_n(\eta)$ with $n \geq N - z_+$ are larger than b and those with $n \leq N - z_-$ are smaller than a . This proves the claim, since $Z_{a,b} = \{z : z_+ < z < z_-\}$. Consequently, the spectral projection $\mathbf{1}_{(a,b)}(\tilde{A}_N)$ has rank $\text{card } Z_{a,b}$ for N large enough. By part 1 and [RS80, Theorem VIII.24], we have $\mathbf{1}_{(a,b)}(\tilde{A}_N) \rightarrow \mathbf{1}_{(a,b)}(L_\eta^\infty)$ strongly. A strong limit of orthogonal projections of rank at most p has rank at most p : if $u_1, \dots, u_{p+1}$ were orthonormal vectors in the range of the limit P , the Gram matrices of the vectors $P_N u_1, \dots, P_N u_{p+1}$ would be singular and would converge to the identity matrix. Hence $\text{rank } \mathbf{1}_{(a,b)}(L_\eta^\infty) \leq \text{card } Z_{a,b}$.

Now, let $\lambda \in (0, \infty) \setminus \eta^{\mathbb{Z}}$. Since the set of eigenvalues of L_η^∞ is countable, we can choose $a < \lambda < b$ as above with $[a, b] \cap \eta^{\mathbb{Z}} = \emptyset$. Then $Z_{a,b} = \emptyset$ and $\mathbf{1}_{(a,b)}(L_\eta^\infty) = 0$, so that λ does

not belong to the spectrum. Since $L_\eta^\infty \geq 0$, the spectrum is contained in $\eta^\mathbb{Z} \cup \{0\}$. In the same way, for any $z \in \mathbb{Z}$, choosing (a, b) with $Z_{a,b} = \{z\}$ shows that $\mathbf{1}_{\{\eta^z\}}(L_\eta^\infty)$ has rank at most one. Since 0 is not an eigenvalue by Proposition 3.1, the spectral measure of L_η^∞ is carried by the countable set $\eta^\mathbb{Z}$, hence $L_\eta^\infty = \sum_z \eta^z \mathbf{1}_{\{\eta^z\}}(L_\eta^\infty)$ and the spectral measures are purely atomic. By part 6 of Proposition 3.1, if $L_\eta^\infty f = \lambda f$, then $L_\eta^\infty Uf = \eta\lambda Uf$, so that U maps the eigenspace of λ onto the eigenspace of $\eta\lambda$. Hence all the eigenspaces $\text{Ker}(L_\eta^\infty - \eta^z)$, $z \in \mathbb{Z}$, have the same dimension, which is at most one. This dimension is not zero, since $L_\eta^\infty \neq 0$. This proves part 2.

3. Let ϕ be a nonzero eigenvector for the eigenvalue η^z . By part 4 of Proposition 3.1, we have $L\phi = \eta^z \phi$ and $c_k(\Delta\phi)_k \rightarrow 0$ as $k \rightarrow +\infty$. Moreover $w_k|\phi(k)|^2 \leq \|\phi\|^2$, hence $\phi(k) = O(\tau^{-\frac{k}{2}})$, and in fact $\phi(k) = o(\tau^{-\frac{k}{2}})$ since $w_k|\phi(k)|^2 \rightarrow 0$. By part 2 of Proposition 3.3, ϕ is a nonzero multiple of f_z . Hence $f_z \in \ell_\mathbb{Z}^2(w)$, and f_z spans the eigenspace. The family (ϕ_z) is orthonormal since the eigenvalues are distinct, and complete since the spectral measures are supported on the eigenvalues. Finally, (3.10) gives $\|f_{z+1}\| = \tau^{-\frac{1}{2}}\|f_z\|$ and $U\phi_z = \phi_{z+1}$.

4. Choose $0 < a < b$ as in part 2 with $Z_{a,b} = \{z\}$. For N large enough, $\mathbf{1}_{(a,b)}(\tilde{A}_N)$ is the orthogonal projection onto $\mathbb{C}V_N e_{N-z}^\eta$, and $\mathbf{1}_{(a,b)}(L_\eta^\infty)$ is the orthogonal projection onto $\mathbb{C}\phi_z$. The assertion follows from the convergence of part 2. $\blacksquare$

Remark 3.7 Part 4 is a Hilbert space version of the limit transition from little q -Jacobi polynomials to Hahn–Exton q -Bessel functions [KS92, Proposition A.1], which states that ${}_2\varphi_1(q^{-n}, q^{n+1}ab; qa; q, q^n x) \rightarrow {}_1\varphi_1(0; qa; q, x)$ uniformly on compact sets. With $q = \eta$, $a = \tau\eta^{-1}$ and $b = 1$, the eigenvector $V_N e_{N-z}^\eta$ is proportional to $k \mapsto p_{N-z}(\eta^{k+N}; \tau\eta^{-1}, 1; \eta) = {}_2\varphi_1(\eta^{-(N-z)}, \tau\eta^{N-z}; \tau; \eta, \eta^{N-z}\eta^{k+z+1})$, which converges pointwise to $f_z(k)$. Theorem 3.6 upgrades this to the convergence of the corresponding operators and spectral projections.

4 The energy bridge on the whole space

We return to the local field, so that $\tau = Q^{-d}$, and we fix $\alpha > 0$, $\rho = Q^\alpha$ and $\eta = Q^{-\alpha}$. Then $\frac{\tau}{\eta} = \rho\tau$, and (3.2) gives $c_k = (1 - \tau)\varpi_k$, where ϖ is the weight (2.9). We denote by $\ell_\mathbb{Z}^2(\varpi)$ the weighted sequence space with norm $\|u\|_\varpi = (\sum_m \varpi_m |u_m|^2)^{\frac{1}{2}}$.

4.1 Optimal constants

Lemma 4.1 *The shift S is bounded on $\ell_\mathbb{Z}^2(\varpi)$, and $\tau S = \sigma_\alpha \tilde{S}$, where $\tilde{S}$ is a unitary operator on $\ell_\mathbb{Z}^2(\varpi)$ whose spectrum is the unit circle and which has no eigenvalue. Consequently, $I - \tau S$ is invertible on $\ell_\mathbb{Z}^2(\varpi)$, $(I - \tau S)^{-1} = \sum_{i \geq 0} \tau^i S^i$, and for any $u \in \ell_\mathbb{Z}^2(\varpi)$,*

$$(4.1) \quad (1 + \sigma_\alpha)^{-1} \|u\|_\varpi \leq \|(I - \tau S)^{-1} u\|_\varpi \leq (1 - \sigma_\alpha)^{-1} \|u\|_\varpi.$$

Both constants are optimal, also when u runs over the finitely supported sequences.

Proof : Put $\tilde{S} \stackrel{\text{def}}{=} (\rho\tau)^{\frac{1}{2}} S$. For any $u \in \ell_\mathbb{Z}^2(\varpi)$, we have $\|\tilde{S}u\|_\varpi^2 = \sum_m \varpi_m \rho\tau |u_{m+1}|^2 = \sum_m \varpi_{m+1} |u_{m+1}|^2 = \|u\|_\varpi^2$, and $\tilde{S}$ is surjective. Hence $\tilde{S}$ is unitary. Moreover $\tau(\rho\tau)^{-\frac{1}{2}} = (\frac{\tau}{\rho})^{\frac{1}{2}} = \sigma_\alpha$. The unitary $u \mapsto (\varpi_m^{\frac{1}{2}} u_m)_m$ from $\ell_\mathbb{Z}^2(\varpi)$ onto $\ell_\mathbb{Z}^2$ conjugates $\tilde{S}$ to the bilateral shift, which is unitarily equivalent by the Fourier transform to the multiplication by $e^{-i\theta}$ on $L^2(\mathbb{T})$. Its spectrum is the unit circle and it has no eigenvalue. Since $\sigma_\alpha < 1$, the Neumann series gives the invertibility of $I - \tau S = I - \sigma_\alpha \tilde{S}$ and the formula for its inverse. For any unit vector v , we have $1 - \sigma_\alpha \leq \|(I - \sigma_\alpha \tilde{S})v\|_\varpi \leq 1 + \sigma_\alpha$, which gives (4.1). For the optimality,

let $\zeta \in \{-1, 1\}$. Under the Fourier transform, the function $\theta \mapsto |1 - \sigma_\alpha e^{-i\theta}|^{-1}$ is continuous on $\mathbb{T}$ and equals $|1 - \sigma_\alpha \zeta|^{-1}$ at the point $e^{-i\theta} = \zeta$. Choosing trigonometric polynomials whose squared moduli concentrate at this point, we obtain finitely supported unit vectors v_j such that $\|(I - \sigma_\alpha \tilde{S})^{-1} v_j\|_\varpi \rightarrow |1 - \sigma_\alpha \zeta|^{-1}$. The cases $\zeta = 1$ and $\zeta = -1$ give the optimality of the two constants. $\blacksquare$

Theorem 4.2 *Let $\alpha > 0$ and $\eta = Q^{-\alpha}$.*

1. *For any $f \in \ell_{\mathbb{Z}}^2(w)$, we have $f \in \text{dom } \mathcal{E}_D$ if and only if $\Delta f \in \ell_{\mathbb{Z}}^2(\varpi)$, if and only if $f \in \text{dom } \mathcal{E}_\eta^\infty$. In this case,*

$$(4.2) \quad \mathcal{E}_D(f) = (1 - \tau) \|(I - \tau S)^{-1} \Delta f\|_\varpi^2, \quad \mathcal{E}_\eta^\infty(f) = (1 - \tau) \|\Delta f\|_\varpi^2.$$

2. *For any $f \in \text{dom } \mathcal{E}_D$, we have*

$$(1 + \sigma_\alpha)^{-2} \mathcal{E}_\eta^\infty(f) \leq \mathcal{E}_D(f) \leq (1 - \sigma_\alpha)^{-2} \mathcal{E}_\eta^\infty(f),$$

and these constants are optimal.

3. *The operators D_{rad}^α and L_η^∞ have the same spectrum $\{\rho^m : m \in \mathbb{Z}\} \cup \{0\}$, their spectral measures are purely atomic, every ρ^m is a simple eigenvalue of both operators, and 0 belongs to the continuous spectrum. The unitary operator W on $\ell_{\mathbb{Z}}^2(w)$ defined by $Wg_m \stackrel{\text{def}}{=} \phi_{-m-1}$, $m \in \mathbb{Z}$, satisfies $WD_{\text{rad}}^\alpha W^* = L_\eta^\infty$ and $WU = UW$.*
4. *The operators D_{rad}^α and L_η^∞ have no common eigenvector.*

Proof : 1. Let $f \in \ell_{\mathbb{Z}}^2(w)$ and put $h_m \stackrel{\text{def}}{=} m_{m+1}(f) - f_m$ and $u \stackrel{\text{def}}{=} \Delta f$. By (2.8), $h_m = \sum_{i \geq 0} \tau^i u_{m+i}$, the series being convergent, and $h_m - \tau h_{m+1} = u_m$ for any m , that is $u = (I - \tau S)h$. By Proposition 2.3, we have $\mathcal{E}_D(f) = (1 - \tau) \|h\|_\varpi^2$ in $[0, \infty]$. If $h \in \ell_{\mathbb{Z}}^2(\varpi)$, then $u = (I - \tau S)h \in \ell_{\mathbb{Z}}^2(\varpi)$ by Lemma 4.1. Conversely, if $u \in \ell_{\mathbb{Z}}^2(\varpi)$, then $(I - \tau S)^{-1}u = \sum_{i \geq 0} \tau^i S^i u$ converges in $\ell_{\mathbb{Z}}^2(\varpi)$, hence coordinatewise, and its coordinates are $\sum_{i \geq 0} \tau^i u_{m+i} = h_m$, so that $h = (I - \tau S)^{-1}u \in \ell_{\mathbb{Z}}^2(\varpi)$. Since $\mathcal{E}_\eta^\infty(f) = \sum_k c_k |u_k|^2 = (1 - \tau) \|u\|_\varpi^2$, this proves part 1.

2. The inequalities follow from (4.2) and (4.1). Let v be a finitely supported sequence. The function f defined by $f_m \stackrel{\text{def}}{=} \sum_{i < m} v_i$ vanishes near $-\infty$ and is constant near $+\infty$, hence belongs to $\mathcal{D} \subset \ell_{\mathbb{Z}}^2(w)$, and $\Delta f = v$. Thus every finitely supported sequence is the gradient of an element of $\text{dom } \mathcal{E}_D$, and the optimality follows from the optimality in Lemma 4.1.

3. By Proposition 2.1, D_{rad}^α is diagonal in the orthonormal basis (g_m) with eigenvalues ρ^{m+1} . By Theorem 3.6, L_η^∞ is diagonal in the orthonormal basis (ϕ_z) with eigenvalues $\eta^z = \rho^{-z}$. Hence W is unitary and maps the eigenvector g_m of D_{rad}^α for ρ^{m+1} to the eigenvector ϕ_{-m-1} of L_η^∞ for the same eigenvalue, which gives $WD_{\text{rad}}^\alpha W^* = L_\eta^\infty$. Finally, $Ug_m = g_{m-1}$ and $U\phi_z = \phi_{z+1}$ give $WUg_m = \phi_{-m} = UWg_m$ for any m .

4. Since the eigenvalues are simple, a common eigenvector would be a multiple of some g_m and of some ϕ_z . The function g_m is constant on $\{k \geq m+1\}$. If ϕ_z were constant there, then $(Lf_z)(k) = 0$ for $k \geq m+2$, and $\eta^z f_z(k) = 0$ for $k \geq m+2$, which contradicts $f_z(k) \rightarrow 1$ as $k \rightarrow +\infty$ (Proposition 3.3). $\blacksquare$

Remark 4.3 For $\alpha = d$, we have $\varpi = 1$, $\sigma_d = \tau$, $c_k = 1 - \tau$, and $L_\tau^\infty f(k) = \tau^{-k}(2f(k) - f(k+1) - f(k-1))$. In this case, the forms are $\mathcal{E}_\tau^\infty(f) = (1 - \tau) \|\Delta f\|_{\ell_{\mathbb{Z}}^2}^2$ and $\mathcal{E}_D(f) =$

$(1 - \tau) \|(I - \tau S)^{-1} \Delta f\|_{\ell_{\mathbb{Z}}^2}^2$. By the Fourier transform on $\ell_{\mathbb{Z}}^2$, the ratio of the two forms is described by the symbol $|1 - \tau e^{i\theta}|^{-2}$, whose extreme values are $(1 + \tau)^{-2}$ and $(1 - \tau)^{-2}$. On the unit ball, [Arh26, Theorem 4.1] gives the constants $c_1 = \frac{1-\tau}{1+\tau}$ and $c_2 = (1 - \tau)^{-3}$ for the regional operator R_{rad}^d and L_τ . The same martingale argument applies on the unit ball: with (2.7) for $m \geq 0$ and the eigenvalues $\rho^{m+1} - \kappa$ of R_{rad}^α , where $\kappa = \frac{1-\tau}{1-\tau\rho^{-1}}$ [Arh26, (2.1)], one obtains for any $\alpha > 0$ the constants $c'_1 = \frac{\rho-1}{\rho-\tau}(1+\sigma_\alpha)^{-2}$ and $c'_2 = \frac{\rho}{\rho-1}(1-\sigma_\alpha)^{-2}$, by the bounds $1 - \frac{\kappa}{\rho} \leq 1 - \kappa\rho^{-m-1} \leq 1$ and $1 - \eta \leq 1 - \eta^{m+1} \leq 1$ and the contraction $\|\tau S\|_{\ell_{\mathbb{N}_0}^2(\varpi) \rightarrow \ell_{\mathbb{N}_0}^2(\varpi)} \leq \sigma_\alpha$ of the backward shift. These constants are neither better nor worse than those of [Arh26, Theorem 4.1] in general: for $\alpha = d$ we obtain $c'_1 = (1 + \tau)^{-3}$ and $c'_2 = (1 - \tau)^{-3}$, whereas c'_2 remains bounded as $\alpha \rightarrow d^-$, unlike the constant of [Arh26, (4.3)].

4.2 Powers of the bilateral operators

In this subsection, we fix $\vartheta > 0$, we put $\eta' \stackrel{\text{def}}{=} Q^{-\vartheta}$ and $s \stackrel{\text{def}}{=} \frac{\alpha}{\vartheta}$, and we denote by $(\phi_z)_{z \in \mathbb{Z}}$ the orthonormal basis of eigenvectors of $L_{\eta'}^\infty$ given by Theorem 3.6. Then $(L_{\eta'}^\infty)^s \phi_z = \eta'^{zs} \phi_z = \rho^{-z} \phi_z$. Hence $(L_{\eta'}^\infty)^s$ is unitarily equivalent to D_{rad}^α for any $\vartheta > 0$, and the question is the comparison of the closed forms.

Lemma 4.4 *Let $\lambda > 0$ and let $\phi \in \ell_{\mathbb{Z}}^2(w)$ satisfy $L\phi = \lambda\phi$, where L is the operator (3.3) with parameters τ and η' . For any $0 < \theta < 1$, we have $\phi(k) = O(\theta^{|k|})$ as $k \rightarrow -\infty$.*

Proof : We can assume that $\phi \neq 0$. Dividing the equation $(L\phi)(k) = \lambda\phi(k)$ by $\delta_k = \eta'^{-k}$, we obtain

$$(4.3) \quad \phi(k-1) = -A_k \phi(k) - \frac{\tau}{\eta'} \phi(k+1), \quad A_k \stackrel{\text{def}}{=} \lambda \eta'^k - 1 - \frac{\tau}{\eta'}, \quad k \in \mathbb{Z}.$$

Since $A_k \rightarrow +\infty$ as $k \rightarrow -\infty$, there exists k_0 such that $A_k \geq 2 + \frac{\tau}{\eta'}$ for any $k \leq k_0$. Suppose that $|\phi(k)| \geq |\phi(k+1)|$ for some $k \leq k_0$. Then (4.3) gives $|\phi(k-1)| \geq (A_k - \frac{\tau}{\eta'}) |\phi(k)| \geq 2 |\phi(k)|$, and by induction $|\phi(j)| \geq 2^{k-j} |\phi(k)|$ for any $j \leq k$. Since $w_j \rightarrow \infty$ as $j \rightarrow -\infty$ and $\phi \in \ell_{\mathbb{Z}}^2(w)$, we have $\phi(j) \rightarrow 0$, hence $\phi(k) = 0$, and then $\phi(k+1) = 0$. By the recurrence, this gives $\phi = 0$, which is excluded. Hence $|\phi(k)| < |\phi(k+1)|$ for any $k \leq k_0$, and (4.3) gives

$$A_k |\phi(k)| \leq |\phi(k-1)| + \frac{\tau}{\eta'} |\phi(k+1)| \leq \left(1 + \frac{\tau}{\eta'}\right) |\phi(k+1)|, \quad k \leq k_0.$$

Since $A_k^{-1}(1 + \frac{\tau}{\eta'}) \rightarrow 0$ as $k \rightarrow -\infty$, the ratio $\frac{|\phi(k)|}{|\phi(k+1)|}$ is eventually smaller than θ , which proves the lemma. $\blacksquare$

By the dilation covariance, the transition coefficients between the two eigenbases depend only on the sum of the indices. Indeed, $Ug_n = g_{n-1}$ and $U\phi_z = \phi_{z+1}$ give $\langle \phi_{z+1}, g_{n-1} \rangle = \langle U\phi_z, Ug_n \rangle = \langle \phi_z, g_n \rangle$. Hence there exists a real sequence $(t(p))_{p \in \mathbb{Z}}$ such that

$$(4.4) \quad \langle \phi_z, g_n \rangle = t(n+z), \quad n, z \in \mathbb{Z}, \quad \sum_{p \in \mathbb{Z}} t(p)^2 = \sum_{n \in \mathbb{Z}} |\langle \phi_0, g_n \rangle|^2 = 1.$$

Lemma 4.5 *1. There exist constants $0 < C_1 \leq C_2$ such that $C_1 \tau^{\frac{p}{2}} \eta'^p \leq |t(p)| \leq C_2 \tau^{\frac{p}{2}} \eta'^p$ for any integer p large enough.*

2. For any $0 < \theta < 1$, we have $t(p) = O(\theta^{|p|})$ as $p \rightarrow -\infty$.

Proof : 1. By (2.7), we have $t(p) = \langle \phi_0, g_p \rangle = s_p(m_{p+1}(\phi_0) - \phi_0(p))$, where $\phi_0 = \|f_0\|^{-1} f_0$ and $f_0(k) = {}_1\varphi_1(0; \tau; \eta', \eta'^{k+1})$. Write $f_0(k) = \sum_{j \geq 0} a_j \eta'^{j(k+1)}$ with $a_j = \frac{(-1)^j \eta'^{j(j-1)/2}}{(\eta'; \eta')_j (\tau; \eta')_j}$. For any $j \geq 0$, the function $k \mapsto \eta'^{jk}$ satisfies $m_{p+1}(\eta'^{j \cdot}) = (1 - \tau) \sum_{i \geq 1} \tau^{i-1} \eta'^{j(p+i)} = \frac{(1-\tau) \eta'^{j(p+1)}}{1 - \tau \eta'^j}$. Since the double series converges absolutely, we obtain

$$m_{p+1}(f_0) - f_0(p) = \sum_{j \geq 1} a_j \eta'^{j(p+1)} \left(\frac{(1-\tau) \eta'^j}{1 - \tau \eta'^j} - 1 \right) = \sum_{j \geq 1} a_j \eta'^{j(p+1)} \frac{\eta'^j - 1}{1 - \tau \eta'^j}.$$

The term $j = 1$ is $\frac{\eta'^{p+1}}{(1-\tau)(1-\tau \eta')}$, and the sum of the other terms is $O(\eta'^{2p})$ as $p \rightarrow +\infty$. Since $s_p = ((1-\tau)\tau^{p+1})^{\frac{1}{2}}$ and $\phi_0 = \|f_0\|^{-1} f_0$, this proves part 1.

2. Let $p \in \mathbb{Z}$. Since $g_p \in \mathcal{D} \subset \text{dom } L_{\eta'}^\infty$ and $L_{\eta'}^\infty \phi_0 = \phi_0$, we have $t(p) = \langle L_{\eta'}^\infty \phi_0, g_p \rangle = \langle \phi_0, Lg_p \rangle$. The function Δg_p is supported in $\{p-1, p\}$, with $|(\Delta g_p)_{p-1}| = \frac{\tau}{s_p}$ and $(\Delta g_p)_p = \frac{1}{s_p}$, so that $w_k(Lg_p)(k) = c_{k-1}(\Delta g_p)_{k-1} - c_k(\Delta g_p)_k$ is supported in $\{p-1, p, p+1\}$ and bounded by $\frac{2 \max(c_{p-1}, c_p)}{s_p}$. Here $c_k = (1-\tau)(\frac{\tau}{\eta'})^{k+1}$. Hence there exists a constant C , independent of p , such that

$$|t(p)| \leq \frac{6 \max(c_{p-1}, c_p)}{s_p} \max_{|i| \leq 1} |\phi_0(p+i)| \leq C \left(\frac{\tau^{\frac{1}{2}}}{\eta'} \right)^p \max_{|i| \leq 1} |\phi_0(p+i)|.$$

By Lemma 4.4, ϕ_0 decays faster than any exponential at $-\infty$, which proves part 2. $\blacksquare$

Theorem 4.6 Let $\alpha > 0$ and $\vartheta > 0$, and put $s \stackrel{\text{def}}{=} \frac{\alpha}{\vartheta}$.

1. There exists a constant $C > 0$ such that $\left\| (L_{Q^{-\vartheta}}^\infty)^{\frac{s}{2}} f \right\|^2 \leq C \mathcal{E}_D(f)$ for any $f \in \text{dom } \mathcal{E}_D$.
In particular $\text{dom } \mathcal{E}_D \subset \text{dom}(L_{Q^{-\vartheta}}^\infty)^{\frac{s}{2}}$.
2. The closed forms of D_{rad}^α and of $(L_{Q^{-\vartheta}}^\infty)^s$ are equivalent if and only if $\alpha < d + 2\vartheta$.
3. If $\alpha \geq d + 2\vartheta$, then no eigenvector of $L_{Q^{-\vartheta}}^\infty$ belongs to $\text{dom } \mathcal{E}_D$.

Proof : Let $f \in \ell_{\mathbb{Z}}^2(w)$ and put $a_z \stackrel{\text{def}}{=} \langle f, \phi_z \rangle$ and $c_n \stackrel{\text{def}}{=} \langle f, g_n \rangle$. Then $\left\| (L_{\eta'}^\infty)^{\frac{s}{2}} f \right\|^2 = \sum_z \rho^{-z} |a_z|^2$ and $\mathcal{E}_D(f) = \sum_n \rho^{n+1} |c_n|^2$ in $[0, \infty]$. Since (g_n) and (ϕ_z) are orthonormal bases and the coefficients (4.4) are real, Parseval's identity gives

$$c_n = \sum_{z \in \mathbb{Z}} a_z t(n+z), \quad a_z = \sum_{n \in \mathbb{Z}} c_n t(n+z),$$

the series being absolutely convergent by the Cauchy-Schwarz inequality and (4.4). Put $B_m \stackrel{\text{def}}{=} \rho^{\frac{m+1}{2}} a_{-m-1}$ and $C_n \stackrel{\text{def}}{=} \rho^{\frac{n+1}{2}} c_n$, so that $\sum_m |B_m|^2 = \left\| (L_{\eta'}^\infty)^{\frac{s}{2}} f \right\|^2$ and $\sum_n |C_n|^2 = \mathcal{E}_D(f)$. A direct computation gives

$$(4.5) \quad C_n = \sum_{m \in \mathbb{Z}} \kappa_1(n-m) B_m, \quad B_m = \sum_{n \in \mathbb{Z}} \kappa_2(m-n) C_n,$$

where $\kappa_1(j) \stackrel{\text{def}}{=} \rho^{\frac{j}{2}} t(j-1)$ and $\kappa_2(j) \stackrel{\text{def}}{=} \rho^{\frac{j}{2}} t(-j-1)$.

1. By part 2 of Lemma 4.5, the sequence $(\kappa_2(j))_{j \geq 0}$ is summable. Since $|t| \leq 1$, the sequence $(\kappa_2(j))_{j < 0}$ is dominated by $\rho^{\frac{j}{2}}$, hence summable. By Young's inequality, the convolution by κ_2

is bounded on $\ell_{\mathbb{Z}}^2$ with norm at most $\|\kappa_2\|_{\ell_{\mathbb{Z}}^1}$. If $f \in \text{dom } \mathcal{E}_D$, then $C \in \ell_{\mathbb{Z}}^2$, so that $B \in \ell_{\mathbb{Z}}^2$ by (4.5) and $\|(L_{\eta'}^\infty)^{\frac{s}{2}} f\|^2 \leq \|\kappa_2\|_{\ell_{\mathbb{Z}}^1}^2 \mathcal{E}_D(f)$.

2. and 3. Suppose that $\alpha < d + 2\vartheta$. By part 1 of Lemma 4.5, we have $|\kappa_1(j)| \leq C\rho^{\frac{j}{2}}\tau^{\frac{j}{2}}\eta'^j = CQ^{\frac{j(\alpha-d-2\vartheta)}{2}}$ for j large, and $|\kappa_1(j)| \leq \rho^{\frac{j}{2}}$ for $j < 0$. Hence $\kappa_1 \in \ell_{\mathbb{Z}}^1$, and (4.5) gives $\mathcal{E}_D(f) \leq \|\kappa_1\|_{\ell_{\mathbb{Z}}^1}^2 \|(L_{\eta'}^\infty)^{\frac{s}{2}} f\|^2$ for any $f \in \text{dom}(L_{\eta'}^\infty)^{\frac{s}{2}}$. With part 1, the two closed forms have the same domain and are equivalent. Suppose now that $\alpha \geq d + 2\vartheta$ and let $z \in \mathbb{Z}$. For $f = \phi_z$, we have $c_n = t(n+z)$, hence by part 1 of Lemma 4.5

$$\mathcal{E}_D(\phi_z) \geq \sum_{n \geq n_0} \rho^{n+1} C_1^2 \tau^{n+z} \eta'^{2(n+z)} = C \sum_{n \geq n_0} Q^{n(\alpha-d-2\vartheta)} = \infty$$

for some n_0 and some constant $C > 0$. Since $\phi_z \in \text{dom}(L_{\eta'}^\infty)^{\frac{s}{2}}$, the forms are not equivalent. This proves parts 2 and 3. $\blacksquare$

Remark 4.7 The threshold $\alpha < d + 2\vartheta$ and the inclusion of part 1 are the same as on the unit ball [Arh26, Theorem 1.1 and Theorem 4.6]. On the unit ball, the threshold is attained by the polynomial $k \mapsto \eta'^k$ of degree one [Arh26, Proposition 4.7]; on the whole space, it is attained by every eigenvector of $L_{\eta'}^\infty$, and it is governed by the behaviour of the transition coefficients near the origin of K^d , since they decay faster than any exponential towards the infinity of K^d . For $\vartheta = d$, the powers $(L_\tau^\infty)^{\frac{s}{2}}$ have closed forms equivalent to the Taibleson forms of the orders $\alpha < 3d$.

5 The quantum group $E_r(2)$

5.1 The concrete model and the Haar weight

Let $0 < r < 1$ and put $\tau = r^2$. We recall the quantum group $E_r(2)$ of Woronowicz [Wor91], in the concrete realization of [Wor92] and [Pal96]. Its Haar weight was obtained by Baa [Baa95], and computed in this realization by Pal [Pal96]. Let $(e_{ij})_{i,j \in \mathbb{Z}}$ be the canonical orthonormal basis of $\ell_{\mathbb{Z} \times \mathbb{Z}}^2$. Consider the unitary operator v and the closed normal operator n defined by

$$(5.1) \quad ve_{ij} = e_{i-1,j}, \quad ne_{ij} = r^i e_{i,j+1}, \quad i, j \in \mathbb{Z}.$$

Then $vnv^* = rn$, and the spectrum of n is $\overline{\mathbb{C}_r} = \{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| \in r^{\mathbb{Z}}\}$. The C^* -algebra $C_0(E_r(2))$ is the norm closed linear span of the operators $v^k f(n)$, $k \in \mathbb{Z}$, $f \in C_0(\overline{\mathbb{C}_r})$, and the comultiplication is determined by $\Delta(v) = v \otimes v$ and $\Delta(n) = v \otimes n + n \otimes v^*$, where $+$ denotes the closure of the sum [Wor91], [Wor92]. We denote by $L^\infty(E_r(2))$ the von Neumann algebra generated by $C_0(E_r(2))$. For any $l \in \mathbb{Z}$, the automorphism $\tau_l \stackrel{\text{def}}{=} \text{Ad}(v^l)$ satisfies $\tau_l(n) = r^l n$, and Woronowicz obtains the comultiplication of $E_r(2)$ from the one of $\text{SU}_r(2)$ by conjugating with these automorphisms [Wor92].

The Fourier transform in the second variable gives a unitary operator from $\ell_{\mathbb{Z} \times \mathbb{Z}}^2$ onto $\ell_{\mathbb{Z}}^2 \otimes L^2(\mathbb{T})$, $e_{ij} \mapsto e_i \otimes z^j$, where $\mathbb{T}$ carries its normalized Haar measure m and z denotes the identity function of $\mathbb{T}$. In this picture, $v = v_0 \otimes 1$ with $v_0 e_i = e_{i-1}$, and n is the product of the commuting operators $\text{diag}(r^i)_{i \in \mathbb{Z}} \otimes 1$ and $1 \otimes M_z$. We identify $B(\ell_{\mathbb{Z}}^2) \overline{\otimes} L^\infty(\mathbb{T})$ with the von Neumann algebra of bounded matrices $x = (x_{ij})_{i,j \in \mathbb{Z}}$ with entries in $L^\infty(\mathbb{T})$, acting on $\ell_{\mathbb{Z}}^2 \otimes L^2(\mathbb{T})$.

Proposition 5.1 *Suppose that $0 < r < 1$ and $\tau = r^2$.*

1. *We have $L^\infty(E_r(2)) = B(\ell_{\mathbb{Z}}^2) \overline{\otimes} L^\infty(\mathbb{T})$.*

2. The formula

$$(5.2) \quad \varphi(x) \stackrel{\text{def}}{=} (1 - \tau) \sum_{i \in \mathbb{Z}} \tau^i \int_{\mathbb{T}} x_{ii} \, dm, \quad x \in L^\infty(E_r(2))_+,$$

defines a normal semifinite faithful weight φ , and $\varphi = (1 - \tau)h$, where h is the Haar weight of $E_r(2)$ computed by Pal [Pal96, (2.1) and Theorem 2.1]. Moreover $\varphi \circ \tau_l = \tau^{-l}\varphi$ for any $l \in \mathbb{Z}$.

3. Put $A \stackrel{\text{def}}{=} n^*n$. Then $A = \text{diag}(\tau^i)_{i \in \mathbb{Z}} \otimes 1$, and for any Borel function $F: \tau^\mathbb{Z} \rightarrow [0, \infty]$, we have

$$\varphi(F(A)) = (1 - \tau) \sum_{i \in \mathbb{Z}} \tau^i F(\tau^i).$$

If $\tau = Q^{-d}$, this is equal to $\int_{K^d} F(\|x\|_K^d) \, dx$.

4. Suppose that $\tau = Q^{-d}$. The map $J: L_{\text{rad}}^\infty(K^d) \rightarrow W^*(A)$ defined by $J(F(\|\cdot\|_K^d)) = F(A)$ is a normal $*$ -isomorphism such that $\varphi \circ J = \int_{K^d} \cdot \, dx$ on positive elements. For any $f \in L_{\text{rad}}^\infty(K^d)$, we have $vJ(f)v^* = J(f(\pi \cdot))$. Finally, for any integer $m \in \mathbb{Z}$, the spectral projection $P_m \stackrel{\text{def}}{=} \mathbf{1}_{[0, \tau^m]}(A)$ is equal to $J(\mathbf{1}_{B_m})$, and $\varphi(P_m) = \tau^m$.

Proof : 1. In the picture $\ell_\mathbb{Z}^2 \otimes L^2(\mathbb{T})$, the operator $f(n)$ acts on $e_i \otimes \xi$ by $e_i \otimes f(r^i z)\xi$. Hence $v^k f(n)$ is a bounded matrix with entries in $L^\infty(\mathbb{T})$, and $C_0(E_r(2)) \subset B(\ell_\mathbb{Z}^2) \overline{\otimes} L^\infty(\mathbb{T})$. Conversely, the von Neumann algebra generated by $C_0(E_r(2))$ contains the spectral projections of $|n| = \text{diag}(r^i) \otimes 1$, hence the diagonal matrix units $e_{ii} \otimes 1$, the powers of v , hence all the matrix units $e_{ij} \otimes 1$, and the phase $1 \otimes M_z$ of n . These operators generate $B(\ell_\mathbb{Z}^2) \overline{\otimes} L^\infty(\mathbb{T})$.

2. In the picture $\ell_{\mathbb{Z} \times \mathbb{Z}}^2$, we have $\int_{\mathbb{T}} x_{ii} \, dm = \langle x e_{i0}, e_{i0} \rangle$. Hence

$$\varphi(x) = (1 - \tau) \sum_{i \in \mathbb{Z}} r^{2i} \langle x e_{i0}, e_{i0} \rangle,$$

which is $1 - \tau$ times the weight of [Pal96, (2.1)]. By [Pal96, Theorem 2.1], this weight is left and right invariant, and the left Haar weight of a locally compact quantum group is unique up to a positive scalar [KV00].

3. We have $A e_{ij} = r^{2i} e_{ij}$, which gives $A = \text{diag}(\tau^i) \otimes 1$ and $F(A) = \text{diag}(F(\tau^i)) \otimes 1$. The formula for $\varphi(F(A))$ follows from (5.2), and the last assertion follows from (2.2), since $\|x\|_K^d = \tau^i$ on S_i .

4. The radial algebra $L_{\text{rad}}^\infty(K^d)$ is identified with $\ell_\mathbb{Z}^\infty$, the origin having measure zero, and J is the map $f \mapsto \text{diag}(f_i) \otimes 1$ onto $W^*(A) = \{\text{diag}(c_i) \otimes 1 : c \in \ell_\mathbb{Z}^\infty\}$, which is a normal $*$ -isomorphism. The weight identity follows from part 3. Since $v(\text{diag}(c_i) \otimes 1)v^* = \text{diag}(c_{i+1}) \otimes 1$ and $\pi S_i = S_{i+1}$, we obtain $vJ(f)v^* = J(f(\pi \cdot))$. Finally $\mathbf{1}_{[0, \tau^m]}(\tau^i) = 1$ if and only if $i \geq m$, which gives $P_m = J(\mathbf{1}_{B_m})$, and $\varphi(P_m) = (1 - \tau) \sum_{i \geq m} \tau^i = \tau^m$. ■

Remark 5.2 The restriction of φ to the corner $P_0 L^\infty(E_r(2)) P_0$ is a normal state, and its restriction to $W^*(A) P_0$ is the law $(1 - \tau) \sum_{i \geq 0} \tau^i \delta_{\tau^i}$ of Aref'eva and Volovich, which is the radial Haar distribution of $\mathcal{O}^d$ [Arh26, (1.1)]. In this sense, the unit ball corresponds to the spectral projection P_0 , and the balls $B_m = \pi^m \mathcal{O}^d$ correspond to the projections $P_m = \tau_{-m}(P_0)$. The automorphisms $\tau_l = \text{Ad}(v^l)$, which Woronowicz uses to contract $\text{SU}_r(2)$ to $E_r(2)$ [Wor92], implement the dilations $x \mapsto \pi^l x$ on the radial algebra. The passage from $L^\infty(\mathcal{O}^d)$ to $L^\infty(K^d)$ by the rescaled balls $\pi^{-N} \mathcal{O}^d$, $N \rightarrow \infty$, is the radial counterpart of this contraction; we do not use the contraction procedure itself.

Remark 5.3 By Remark 3.4, for $\alpha = d$ the eigenfunctions of L_τ^∞ are Hahn–Exton q -Bessel functions of order 0 in base $\tau = r^2$. Vaksman and Korogodskii interpreted the Hahn–Exton q -Bessel functions as matrix coefficients of irreducible representations of the quantum group of plane motions [VK89], see [KS92, Section 1] and [Koe94]. Section 7 identifies L_τ^∞ with the radial part of a closed Casimir on the homogeneous space $E_r(2)/\mathbb{T}$. This is a quotient-space assertion, not a construction of an ordinary convolution heat semigroup on the whole group; see Theorem 7.17.

5.2 Tail expectations and an exact lifting

In this subsection we put $\mathcal{M} \stackrel{\text{def}}{=} L^\infty(E_r(2)) = B(\ell_\mathbb{Z}^2) \overline{\otimes} L^\infty(\mathbb{T})$, we use the weight φ of (5.2), the projections $P_m = \sum_{i \geq m} e_{ii} \otimes 1$ and the positive invertible operator $D \stackrel{\text{def}}{=} (1 - \tau) \text{diag}(\tau^i)_{i \in \mathbb{Z}} \otimes 1$, which is affiliated with $\mathcal{M}$ and commutes with every P_m . We denote by $\mathfrak{S}$ the Hilbert space of matrices $\xi = (\xi_{ij})_{i,j \in \mathbb{Z}}$ with entries in $L^2(\mathbb{T})$ such that $\|\xi\|_\mathfrak{S}^2 = \sum_{i,j} \|\xi_{ij}\|_{L^2(\mathbb{T})}^2 < \infty$, with the inner product $\langle \xi, \zeta \rangle_\mathfrak{S} = \sum_{i,j} \int_{\mathbb{T}} \xi_{ij} \overline{\zeta_{ij}} dm$. Left and right multiplications by elements of $\mathcal{M}$ act boundedly on $\mathfrak{S}$. Let $\mathfrak{n}_\varphi = \{x \in \mathcal{M} : \varphi(x^*x) < \infty\}$. For any $x \in \mathfrak{n}_\varphi$, the i -th column of $x D^{\frac{1}{2}}$ is $((1 - \tau)\tau^i)^{\frac{1}{2}}$ times the i -th column of x , so that $x D^{\frac{1}{2}} \in \mathfrak{S}$ and

$$\|x D^{\frac{1}{2}}\|_\mathfrak{S}^2 = (1 - \tau) \sum_i \tau^i \sum_k \int_{\mathbb{T}} |x_{ki}|^2 dm = \varphi(x^*x).$$

The finite matrices with entries in $L^\infty(\mathbb{T})$ belong to $\mathfrak{n}_\varphi$, and their images are dense in $\mathfrak{S}$. Hence $\Lambda(x) \stackrel{\text{def}}{=} x D^{\frac{1}{2}}$ realizes the GNS construction of $(\mathcal{M}, \varphi)$ on $L^2(\mathcal{M}, \varphi) = \mathfrak{S}$. In the same way, $x \mapsto D^{\frac{1}{4}} x D^{\frac{1}{4}}$ realizes the symmetric embedding of $\mathcal{M}$ in $\mathfrak{S}$ on the elements for which it is defined. For any $m \in \mathbb{Z}$, we define

$$(5.3) \quad \Phi_m(x) \stackrel{\text{def}}{=} (1 - P_m)x(1 - P_m) + \tau^{-m} \varphi(P_m x P_m) P_m, \quad \mathcal{M}_m \stackrel{\text{def}}{=} (1 - P_m) \mathcal{M} (1 - P_m) \oplus \mathbb{C} P_m,$$

where $\varphi(P_m x P_m)$ denotes the value at x of the normal positive functional $x \mapsto (1 - \tau) \sum_{i \geq m} \tau^i \int_{\mathbb{T}} x_{ii} dm$, whose norm is $\varphi(P_m) = \tau^m$.

Lemma 5.4 1. For any $m \in \mathbb{Z}$, the map Φ_m is a normal unital completely positive conditional expectation onto $\mathcal{M}_m$, and $\varphi \circ \Phi_m = \varphi$ on $\mathcal{M}_+$. The algebras $\mathcal{M}_m$ increase with m , and $\Phi_i \Phi_j = \Phi_{\min(i,j)}$ for any $i, j \in \mathbb{Z}$.

2. For any $m \in \mathbb{Z}$ and any $x \in \mathfrak{n}_\varphi$, we have $\Phi_m(x) \in \mathfrak{n}_\varphi$ and $\Lambda(\Phi_m(x)) = \Phi_m^{(2)} \Lambda(x)$, where $\Phi_m^{(2)}$ is the orthogonal projection on $\mathfrak{S}$ defined by

$$(5.4) \quad \Phi_m^{(2)} \xi \stackrel{\text{def}}{=} (1 - P_m) \xi (1 - P_m) + \tau^{-m} \left\langle \xi, P_m D^{\frac{1}{2}} \right\rangle_\mathfrak{S} P_m D^{\frac{1}{2}}.$$

The same formula holds for the symmetric embedding. Moreover $\Phi_m^{(2)} \rightarrow I$ strongly as $m \rightarrow +\infty$ and $\Phi_m^{(2)} \rightarrow 0$ strongly as $m \rightarrow -\infty$.

3. The maps Φ_m commute with the modular automorphism group $\sigma_s^\varphi = \text{Ad}(D^{is})$, $s \in \mathbb{R}$, and with the automorphisms $\text{Ad}(1 \otimes R_\lambda)$, $\lambda \in \mathbb{T}$, where $(R_\lambda \xi)(z) = \xi(\lambda z)$ on $L^2(\mathbb{T})$.

4. Suppose that $\tau = Q^{-d}$. For any $m \in \mathbb{Z}$ and any $f \in L_{\text{rad}}^\infty(K^d)$, we have $\Phi_m J(f) = J(\mathbb{E}_m f)$, where $\mathbb{E}_m f$ is the radial function equal to f on $K^d \setminus B_m$ and to the mean $\mathfrak{m}_m(f)$ on B_m .

Proof : 1. The compression by $1 - P_m$ is normal and completely positive, and $x \mapsto \tau^{-m}\varphi(P_m x P_m)$ is a normal state on $\mathcal{M}$ since $\varphi(P_m) = \tau^m$. Hence Φ_m is normal and unital completely positive. It is the identity on $\mathcal{M}_m$ and $\mathcal{M}_m$ -bimodular, since P_m commutes with $\mathcal{M}_m$ and $(1 - P_m)yP_m = 0$ for $y \in \mathcal{M}_m$. For any $x \in \mathcal{M}_+$, the diagonal entries of x with index $i < m$ are those of $(1 - P_m)x(1 - P_m)$, and those with index $i \geq m$ are those of $P_m x P_m$. Hence $\varphi(x) = \varphi((1 - P_m)x(1 - P_m)) + \varphi(P_m x P_m) = \varphi(\Phi_m(x))$ in $[0, \infty]$. The decomposition $P_m = e_{mm} \otimes 1 + P_{m+1}$ shows that every element $y = (1 - P_m)y(1 - P_m) + cP_m$ of $\mathcal{M}_m$ satisfies $(1 - P_{m+1})yP_{m+1} = 0 = P_{m+1}y(1 - P_{m+1})$ and $P_{m+1}yP_{m+1} = cP_{m+1}$, so that $\mathcal{M}_m \subset \mathcal{M}_{m+1}$. If $i \geq j$, then $\Phi_i \Phi_j = \Phi_j$ since $\Phi_j(x) \in \mathcal{M}_j \subset \mathcal{M}_i$. If $i \leq j$, then $(1 - P_i)\Phi_j(x)(1 - P_i) = (1 - P_i)x(1 - P_i)$, since $(1 - P_i)(1 - P_j) = 1 - P_i$ and $(1 - P_i)P_j = 0$, and

$$\varphi(P_i \Phi_j(x) P_i) = \varphi((P_i - P_j)x(P_i - P_j)) + \tau^{-j}\varphi(P_j x P_j)\varphi(P_j) = \varphi(P_i x P_i),$$

where we used that the functional $\varphi(P_i \cdot P_i)$ only involves the diagonal entries with index at least i . Hence $\Phi_i \Phi_j = \Phi_i$.

2. Let $x \in \mathfrak{n}_\varphi$. Since D commutes with P_m , we have $(1 - P_m)x(1 - P_m)D^{\frac{1}{2}} = (1 - P_m)x D^{\frac{1}{2}}(1 - P_m)$. Moreover $P_m D^{\frac{1}{2}} \in \mathfrak{S}$ with $\|P_m D^{\frac{1}{2}}\|_{\mathfrak{S}}^2 = \varphi(P_m) = \tau^m$, and

$$\left\langle x D^{\frac{1}{2}}, P_m D^{\frac{1}{2}} \right\rangle_{\mathfrak{S}} = (1 - \tau) \sum_{i \geq m} \tau^i \int_{\mathbb{T}} x_{ii} dm = \varphi(P_m x P_m).$$

This gives $\Lambda(\Phi_m(x)) = \Phi_m^{(2)} \Lambda(x)$, and in particular $\Phi_m(x) \in \mathfrak{n}_\varphi$. The computation for the symmetric embedding is the same, since $\left\langle D^{\frac{1}{4}} x D^{\frac{1}{4}}, P_m D^{\frac{1}{2}} \right\rangle_{\mathfrak{S}} = \varphi(P_m x P_m)$ for the elements for which the left-hand side is defined. The first term of (5.4) is the orthogonal projection onto $(1 - P_m)\mathfrak{S}(1 - P_m)$, and the second one is the orthogonal projection onto the line spanned by the unit vector $\tau^{-\frac{m}{2}} P_m D^{\frac{1}{2}}$, which is orthogonal to $(1 - P_m)\mathfrak{S}(1 - P_m)$. Hence $\Phi_m^{(2)}$ is an orthogonal projection. Let $\xi \in \mathfrak{S}$ be a matrix with finitely many nonzero entries, all with indices in $[-M, M]$. For $m > M$, we have $\Phi_m^{(2)} \xi = \xi$, since $P_m \xi = \xi P_m = 0$. For $m \leq -M$, we have $(1 - P_m)\xi = 0$, hence $\Phi_m^{(2)} \xi = \tau^{-m} \left\langle \xi, P_m D^{\frac{1}{2}} \right\rangle P_m D^{\frac{1}{2}}$, whose norm is at most $\tau^{-\frac{m}{2}} \left| \left\langle \xi, P_m D^{\frac{1}{2}} \right\rangle \right| \leq \tau^{-\frac{m}{2}} \|\xi\|_{\mathfrak{S}} \left((1 - \tau) \sum_{|i| \leq M} \tau^i \right)^{\frac{1}{2}}$, which tends to zero as $m \rightarrow -\infty$ since $\tau^{-\frac{m}{2}} = \tau^{\frac{|m|}{2}}$. Since such matrices are dense and the projections are contractions, the strong limits follow.

3. Since D^{is} commutes with P_m and $\varphi \circ \text{Ad}(D^{\text{is}}) = \varphi$, the formula (5.3) gives $\Phi_m \circ \text{Ad}(D^{\text{is}}) = \text{Ad}(D^{\text{is}}) \circ \Phi_m$. The same argument applies to $\text{Ad}(1 \otimes R_\lambda)$, which fixes P_m and preserves φ . Recall that the modular group of φ is $\text{Ad}(D^{\text{is}})$, since $\varphi = (\text{Tr} \otimes m)(D \cdot)$ on $\mathcal{M}_+$ for the semifinite trace $\text{Tr} \otimes m$, see [Tak03, Chapter VIII].

4. We have $J(f) = \text{diag}(f_i) \otimes 1$, hence $(1 - P_m)J(f)(1 - P_m) = \text{diag}(f_i \mathbf{1}_{i < m}) \otimes 1$ and $\tau^{-m}\varphi(P_m J(f) P_m) = \tau^{-m}(1 - \tau) \sum_{i \geq m} \tau^i f_i = m_m(f)$ by (2.3). ■

For any $t > 0$ and any $m \in \mathbb{Z}$, we put

$$(5.5) \quad a_m(t) \stackrel{\text{def}}{=} e^{-t\rho^m} - e^{-t\rho^{m+1}} > 0.$$

Since $\rho^m \rightarrow 0$ as $m \rightarrow -\infty$ and $\rho^m \rightarrow \infty$ as $m \rightarrow +\infty$, we have $\sum_{m \in \mathbb{Z}} a_m(t) = 1$, and $\sum_{m > k} a_m(t) = e^{-t\rho^{k+1}}$ for any $k \in \mathbb{Z}$.

Theorem 5.5 *Let $0 < r < 1$, $\tau = r^2$ and $\rho > 1$. For $t > 0$, the formula*

$$(5.6) \quad \mathcal{T}_t \stackrel{\text{def}}{=} \sum_{m \in \mathbb{Z}} a_m(t) \Phi_m,$$

together with $\mathcal{T}_0 = \text{Id}$, defines a semigroup of normal unital completely positive maps on $L^\infty(E_r(2))$ such that $\varphi \circ \mathcal{T}_t = \varphi$ on positive elements. It is point- σ -weakly continuous at zero, it commutes with the modular group of φ and with the rotations of $\mathbb{T}$, and it leaves $B(\ell_{\mathbb{Z}}^2) \otimes 1$ and $W^*(A)$ invariant. The maps $\Lambda(x) \mapsto \Lambda(\mathcal{T}_t(x))$ extend to a strongly continuous semigroup of self-adjoint contractions on $L^2(E_r(2), \varphi)$, and the same holds in the symmetric embedding. If $d_k \stackrel{\text{def}}{=} \Phi_{k+1}^{(2)} - \Phi_k^{(2)}$, then $(d_k)_{k \in \mathbb{Z}}$ is a family of mutually orthogonal projections with sum I , and the negative generator of this semigroup is

$$(5.7) \quad \mathcal{L} = \sum_{k \in \mathbb{Z}} \rho^{k+1} d_k, \quad \text{dom } \mathcal{L} = \left\{ \xi \in \mathfrak{S} : \sum_{k \in \mathbb{Z}} \rho^{2(k+1)} \|d_k \xi\|_{\mathfrak{S}}^2 < \infty \right\}.$$

For any compact operator $x \in \mathcal{K}(\ell_{\mathbb{Z}}^2) \otimes 1$, we have $\|\mathcal{T}_t(x) - x\| \rightarrow 0$ as $t \rightarrow 0^+$. Finally, suppose that $\tau = Q^{-d}$ and $\rho = Q^\alpha$. Then

$$(5.8) \quad \mathcal{T}_t J = J T_t^{\text{rad}}, \quad t \geq 0,$$

where $T_t^{\text{rad}} \stackrel{\text{def}}{=} \sum_m a_m(t) \mathbb{E}_m$ is a normal Markov semigroup on $L_{\text{rad}}^\infty(K^d)$ whose restriction to $L_{\text{rad}}^\infty(K^d) \cap L^2(K^d)$ is $e^{-tD_{\text{rad}}^\alpha}$. Moreover, the isometry $J_2: \ell_{\mathbb{Z}}^2(w) \rightarrow \mathfrak{S}$ which extends $f \mapsto \Lambda(J(f))$ satisfies $\mathcal{L} J_2 = J_2 D_{\text{rad}}^\alpha$, with equality of domains on the radial subspace $J_2(\ell_{\mathbb{Z}}^2(w))$.

Proof : Since the coefficients $a_m(t)$ are positive with sum one and $\|\Phi_m\| = 1$, the series (5.6) converges in the norm of bounded maps on $\mathcal{M}$, and $\mathcal{T}_t$ is normal and unital completely positive. For $x \in \mathcal{M}_+$, the series $\sum_m a_m(t) \Phi_m(x)$ of positive elements converges in norm, and the normality of φ gives $\varphi(\mathcal{T}_t(x)) = \sum_m a_m(t) \varphi(\Phi_m(x)) = \varphi(x)$ in $[0, \infty]$, by Lemma 5.4.

By part 1 of Lemma 5.4, the projections $\Phi_m^{(2)}$ increase with m , and by part 2 they converge strongly to I as $m \rightarrow +\infty$ and to 0 as $m \rightarrow -\infty$. Hence the projections d_k are mutually orthogonal and $\sum_k d_k = I$ strongly. On the range of d_k , the projection $\Phi_m^{(2)}$ is zero for $m \leq k$ and is the identity for $m \geq k+1$. Consequently, for any $x \in \mathfrak{n}_\varphi$, the Kadison–Schwarz inequality [Cho74] gives $\mathcal{T}_t(x) \in \mathfrak{n}_\varphi$, and

$$\Lambda(\mathcal{T}_t(x)) = \sum_m a_m(t) \Phi_m^{(2)} \Lambda(x) = \sum_{k \in \mathbb{Z}} \left(\sum_{m > k} a_m(t) \right) d_k \Lambda(x) = \sum_{k \in \mathbb{Z}} e^{-t\rho^{k+1}} d_k \Lambda(x).$$

Hence the L^2 -implementation is the self-adjoint contraction $\mathcal{T}_t^{(2)} = \sum_k e^{-t\rho^{k+1}} d_k$, which is a strongly continuous semigroup with negative generator (5.7). By part 2 of Lemma 5.4, the same formula holds in the symmetric embedding. In particular, $(\mathcal{T}_t)$ is GNS-symmetric and KMS-symmetric with respect to φ , in the sense of [Cip97]. For $s, t > 0$ and $x \in \mathfrak{n}_\varphi$, we obtain $\Lambda(\mathcal{T}_s \mathcal{T}_t(x)) = \Lambda(\mathcal{T}_{s+t}(x))$, hence $\mathcal{T}_s \mathcal{T}_t(x) = \mathcal{T}_{s+t}(x)$ since Λ is injective. Since $\mathfrak{n}_\varphi$ is σ -weakly dense and the maps are normal, the semigroup law holds on $\mathcal{M}$.

Let $x \in \mathcal{M}$. As $m \rightarrow +\infty$, we have $P_m \rightarrow 0$ strongly, so that $(1 - P_m)x(1 - P_m) \rightarrow x$ strongly and $\tau^{-m} \varphi(P_m x P_m) P_m \rightarrow 0$ strongly, its scalar coefficient being bounded by $\|x\|$. Hence $\Phi_m(x) \rightarrow x$ σ -weakly. Let ω be a normal functional and $\varepsilon > 0$. Choose N such that $|\omega(\Phi_m(x)) - \omega(x)| \leq \varepsilon$ for $m > N$. Then

$$|\omega(\mathcal{T}_t(x)) - \omega(x)| \leq \sum_{m \leq N} a_m(t) 2 \|\omega\| \|x\| + \varepsilon = 2 \|\omega\| \|x\| (1 - e^{-t\rho^{N+1}}) + \varepsilon,$$

which is at most 2ε for t small enough. This proves the point- σ -weak continuity at zero. If x is compact, then $\|P_m x\| \rightarrow 0$ and $\|x P_m\| \rightarrow 0$, and $|\tau^{-m} \varphi(P_m x P_m)| \leq \sup_{i \geq m} \|x_{ii}\|_\infty \rightarrow 0$ as $m \rightarrow +\infty$; hence $\|\Phi_m(x) - x\| \rightarrow 0$, and the same splitting of the series gives $\|\mathcal{T}_t(x) - x\| \rightarrow 0$. The commutation properties follow from part 3 of Lemma 5.4. Each Φ_m maps $B(\ell_{\mathbb{Z}}^2) \otimes 1$ and $W^*(A)$ into themselves, hence so does $\mathcal{T}_t$.

Suppose now that $\tau = Q^{-d}$ and $\rho = Q^\alpha$. Part 4 of Lemma 5.4 gives (5.8). The operators $\mathbb{E}_m$ are normal conditional expectations on $L_{\text{rad}}^\infty(K^d)$, so that (T_t^{rad}) is a family of normal Markov operators, and the semigroup law follows from (5.8) and the injectivity of J . For any $k \in \mathbb{Z}$, the wavelet g_k of (2.6) is constant on B_{k+1} and has mean zero on B_k . Hence $\mathbb{E}_m g_k = g_k$ if $m \geq k+1$ and $\mathbb{E}_m g_k = 0$ if $m \leq k$, since the support of g_k is contained in $B_k \subset B_m$ in the second case. Consequently $T_t^{\text{rad}} g_k = \sum_{m > k} a_m(t) g_k = e^{-t\rho^{k+1}} g_k = e^{-tD_{\text{rad}}^\alpha} g_k$ by Proposition 2.1. The operators $\mathbb{E}_m$ are contractions on $L^2(K^d)$, and the finite linear combinations of the g_k are dense in $\ell_{\mathbb{Z}}^2(w)$. The bounded radial functions in $L^2(K^d)$ are approximated in $L^2(K^d)$ by such combinations, so that $T_t^{\text{rad}} = e^{-tD_{\text{rad}}^\alpha}$ on $L_{\text{rad}}^\infty(K^d) \cap L^2(K^d)$. Finally, J_2 is an isometry since $\varphi(J(f)^* J(f)) = \sum_i w_i |f_i|^2$, and $\Phi_m J = J \mathbb{E}_m$ shows that $J_2 g_k$ belongs to the range of d_k . Hence $\mathcal{L} J_2 g_k = \rho^{k+1} J_2 g_k$, and (5.7) gives the equality of domains on the radial subspace. ■

Remark 5.6 Theorem 5.5 is the bilateral version of [Arh26, Theorem 7.2]; a first version of this construction appeared in a preliminary version of [Arh26]. The two new points are the two endpoints of the filtration: on the unit ball Φ_0 is the Haar state, whereas here the filtration $(\mathcal{M}_m)_{m \in \mathbb{Z}}$ has no smallest element, and $\Phi_m^{(2)} \rightarrow 0$ as $m \rightarrow -\infty$ because the weight is infinite. The lifting does not use the comultiplication of $E_r(2)$. The restriction to $B(\ell_{\mathbb{Z}}^2) \otimes 1$ is a semigroup on $B(\ell_{\mathbb{Z}}^2)$ preserving the weight $(1-\tau) \text{Tr}(\text{diag}(\tau^i) \cdot)$, which is the bilateral analogue of the Gibbs state $\text{Tr}((1-\tau)\tau^N \cdot)$ on $B(\ell_{\mathbb{N}_0}^2) = L^\infty(S_r^2)$ of [Arh26, Proposition 5.5].

6 From the unit ball to the whole space: the Poisson kernel

In this section $0 < \tau < 1$ is arbitrary, and we use the notation of [Arh26, Section 6]. The diagonal chain, whose transition probabilities are due to Biane [Bia06, Section 4], is a Markov chain on $\{(n, k) \in \mathbb{N}_0^2 : k \leq n\}$, its second coordinate converges almost surely to a limit k_∞ , and $\nu_{(n, k)}(j) = \mathbf{P}_{(n, k)}(k_\infty = j)$. In the coordinates $(d, k) = (n - k, k)$, it is the random walk of the Cartesian product of two networks on $\mathbb{N}_0$, with conductances

$$\mathfrak{a}_d = \frac{1 - \tau^{d+1}}{(1 - \tau^2)\tau^d}, \quad \mathfrak{b}_k = \frac{\tau(1 - \tau^{k+1})}{1 - \tau^2},$$

by [Arh26, Proposition 6.4]. Let $(\delta_k)_{k \in \mathbb{N}_0}$ be the canonical basis of $\ell_{\mathbb{N}_0}^2$ with the counting measure, and let B be the bounded operator on $\ell_{\mathbb{N}_0}^2$ defined by $(Bf)(k) = \mathfrak{b}_k(f(k) - f(k+1)) + \mathfrak{b}_{k-1}(f(k) - f(k-1))$, with $\mathfrak{b}_{-1} = 0$. Recall that $0 \leq B \leq 4 \sup_k \mathfrak{b}_k = \frac{4\tau}{1-\tau^2}$. Let ϱ_d , $d \in \mathbb{N}_0$, be the polynomials defined by $\varrho_0 = 1$ and [Arh26, (6.15)], and $\varrho_\infty = \lim_d \varrho_d$. By [Arh26, Theorem 6.13], we have $\varrho_\infty \geq 1$ on $[0, \infty)$ and, for any $d, k, j \in \mathbb{N}_0$,

$$(6.1) \quad \nu_{(d+k, k)}(j) = \left\langle \frac{\varrho_d}{\varrho_\infty} (B) \delta_j, \delta_k \right\rangle_{\ell_{\mathbb{N}_0}^2}.$$

By [Arh26, Theorem 6.15], for any $\lambda \in \mathbb{C}$ and any $z \in \mathbb{C} \setminus \{0\}$ such that $z + z^{-1} = 2 - \frac{(1-\tau^2)\lambda}{\tau}$, we have

$$(6.2) \quad \varrho_\infty(\lambda) = \frac{(\tau z, \tau z^{-1}; \tau)_\infty}{(\tau; \tau)_\infty^2}.$$

As $k \rightarrow \infty$, the conductance $\mathfrak{b}_k$ converges to $\mathfrak{b}_\infty \stackrel{\text{def}}{=} \frac{\tau}{1-\tau^2}$. We denote by $\mathbf{B}_\infty$ the bounded operator on $\ell_{\mathbb{Z}}^2$ defined by $(\mathbf{B}_\infty f)(k) = \mathfrak{b}_\infty(2f(k) - f(k+1) - f(k-1))$. Under the Fourier transform $\ell_{\mathbb{Z}}^2 \rightarrow L^2(\mathbb{T})$, $\delta_k \mapsto e^{ik\theta}$, it becomes the multiplication by

$$(6.3) \quad \lambda(\theta) \stackrel{\text{def}}{=} \frac{2\tau(1 - \cos \theta)}{1 - \tau^2}, \quad \theta \in \mathbb{R},$$

and $z = e^{i\theta}$ satisfies $z + z^{-1} = 2 - \frac{(1-\tau^2)\lambda(\theta)}{\tau}$.

Theorem 6.1 *Let $0 < \tau < 1$.*

1. *For any $d \in \mathbb{N}_0$ and any $l \in \mathbb{Z}$, we have*

$$(6.4) \quad \lim_{k \rightarrow \infty} \nu_{(d+k,k)}(k+l) = \nu_d^{\mathbb{Z}}(l) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\varrho_d}{\varrho_\infty}(\lambda(\theta)) e^{il\theta} d\theta.$$

Moreover $\nu_d^{\mathbb{Z}}(l) = \nu_d^{\mathbb{Z}}(-l) \geq 0$ and $\sum_{l \in \mathbb{Z}} \nu_d^{\mathbb{Z}}(l) = 1$.

2. *For $d = 0$, we have for any $k \in \mathbb{N}_0$ and any integer $l \geq 0$*

$$(6.5) \quad \nu_{(k,k)}(k+l) = (\tau; \tau)_k (\tau; \tau)_{k+l} \sum_{m=0}^k \frac{\tau^{l+2m} (1 - \tau^{k-m+1})}{(\tau; \tau)_m (\tau; \tau)_{l+m}},$$

and for any $l \in \mathbb{Z}$,

$$(6.6) \quad \nu_0^{\mathbb{Z}}(l) = (\tau; \tau)_\infty^2 \sum_{m \geq 0} \frac{\tau^{|l|+2m}}{(\tau; \tau)_m (\tau; \tau)_{m+|l|}}, \quad \sum_{l \in \mathbb{Z}} \nu_0^{\mathbb{Z}}(l) e^{il\theta} = \frac{(\tau; \tau)_\infty^2}{|(\tau e^{i\theta}; \tau)_\infty|^2}, \quad \theta \in \mathbb{R}.$$

Finally, there exists a constant C_τ such that $|\nu_{(k,k)}(k+l) - \nu_0^{\mathbb{Z}}(l)| \leq C_\tau \tau^k$ for any $k \in \mathbb{N}_0$ and any $l \in \mathbb{Z}$ with $k+l \geq 0$.

Proof : 1. For any integer $N \geq 0$, let $W_N: \ell_{\mathbb{N}_0}^2 \rightarrow \ell_{\mathbb{Z}}^2$ be the isometry defined by $(W_N f)(k) = f(k+N)$ if $k \geq -N$ and $(W_N f)(k) = 0$ otherwise, and put $\mathbf{B}_N \stackrel{\text{def}}{=} W_N \mathbf{B} W_N^*$. Then $0 \leq \mathbf{B}_N \leq \frac{4\tau}{1-\tau^2}$, and $0 \leq \mathbf{B}_\infty \leq \frac{4\tau}{1-\tau^2}$. For any finitely supported $u \in \ell_{\mathbb{Z}}^2$ and any N large enough, we have, for any $k \in \mathbb{Z}$,

$$((\mathbf{B}_N - \mathbf{B}_\infty)u)(k) = (\mathfrak{b}_{k+N} - \mathfrak{b}_\infty)(u(k) - u(k+1)) + (\mathfrak{b}_{k+N-1} - \mathfrak{b}_\infty)(u(k) - u(k-1)),$$

which vanishes outside a finite set independent of N , and whose coefficients are $O(\tau^N)$. Hence $\mathbf{B}_N u \rightarrow \mathbf{B}_\infty u$. Since the operators $\mathbf{B}_N$ are uniformly bounded, $\mathbf{B}_N \rightarrow \mathbf{B}_\infty$ strongly, and by induction $p(\mathbf{B}_N) \rightarrow p(\mathbf{B}_\infty)$ strongly for any polynomial p . Let g be a continuous function on the interval $I = [0, \frac{4\tau}{1-\tau^2}]$, which contains the spectra of all these operators. For any $\varepsilon > 0$, the Weierstrass theorem gives a polynomial p with $\sup_I |g-p| \leq \varepsilon$, hence $\|g(\mathbf{B}_N) - p(\mathbf{B}_N)\| \leq \varepsilon$ and $\|g(\mathbf{B}_\infty) - p(\mathbf{B}_\infty)\| \leq \varepsilon$. Consequently $g(\mathbf{B}_N) \rightarrow g(\mathbf{B}_\infty)$ strongly. Since $\mathbf{B}_N = W_N \mathbf{B} W_N^* \oplus 0$ with respect to the decomposition $\ell_{\mathbb{Z}}^2 = \text{Ran } W_N \oplus (\text{Ran } W_N)^\perp$, we have $g(\mathbf{B}_N) = W_N g(\mathbf{B}) W_N^* + g(0)(I - W_N W_N^*)$. For $N \geq \max(0, -l)$, this gives $\langle g(\mathbf{B}_N) \delta_l, \delta_0 \rangle_{\ell_{\mathbb{Z}}^2} = \langle g(\mathbf{B}) \delta_{N+l}, \delta_N \rangle_{\ell_{\mathbb{N}_0}^2}$. We apply this to $g = \frac{\varrho_d}{\varrho_\infty}$, which is continuous on I since $\varrho_\infty \geq 1$ there. By (6.1), we obtain

$$\lim_{N \rightarrow \infty} \nu_{(d+N,N)}(N+l) = \langle g(\mathbf{B}_\infty) \delta_l, \delta_0 \rangle_{\ell_{\mathbb{Z}}^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\lambda(\theta)) e^{il\theta} d\theta.$$

Since λ is even, $\nu_d^{\mathbb{Z}}(l) = \nu_d^{\mathbb{Z}}(-l)$, and $\nu_d^{\mathbb{Z}}(l) \geq 0$ as a limit of probabilities. The function $\theta \mapsto g(\lambda(\theta))$ is real analytic on $\mathbb{T}$, hence its Fourier coefficients are summable and $\sum_l \nu_d^{\mathbb{Z}}(l) = g(\lambda(0)) = \frac{\varrho_d(0)}{\varrho_\infty(0)} = 1$, since $\varrho_d(0) = 1$ for any $d \in \mathbb{N}_0 \cup \{\infty\}$.

2. Let $k \in \mathbb{N}_0$ and $l \geq 0$. We write $\begin{bmatrix} m \\ n \end{bmatrix}_\tau = \frac{(\tau; \tau)_m}{(\tau; \tau)_n (\tau; \tau)_{m-n}}$, $0 \leq n \leq m$, for the Gaussian binomial coefficients in the base τ [DLMF, (17.2.27)]. By [Arh26, (6.30)], we have

$$\nu_{(k,k)}(k+l) = \sum_{n=0}^k \begin{bmatrix} k+l \\ n \end{bmatrix}_\tau \begin{bmatrix} k \\ n \end{bmatrix}_\tau (\tau; \tau)_n (\tau; \tau)_{n+1} \tau^{2k+l-2n}.$$

We put $n \stackrel{\text{def}}{=} k-m$ and use

$$\begin{bmatrix} k+l \\ k-m \end{bmatrix}_\tau \begin{bmatrix} k \\ k-m \end{bmatrix}_\tau (\tau; \tau)_{k-m} (\tau; \tau)_{k-m+1} = \frac{(\tau; \tau)_{k+l} (\tau; \tau)_k (1 - \tau^{k-m+1})}{(\tau; \tau)_{l+m} (\tau; \tau)_m},$$

which gives (6.5). For $d = 0$, we have $g(\lambda(\theta)) = \frac{(\tau; \tau)_\infty^2}{|(\tau e^{i\theta}; \tau)_\infty|^2}$ by (6.2) with $z = e^{i\theta}$. By the q -binomial theorem [GR04, Section 1.3], in the form [DLMF, (17.5.4)], we have $\frac{1}{(\tau e^{i\theta}; \tau)_\infty} = \sum_{a \geq 0} \frac{\tau^a e^{ia\theta}}{(\tau; \tau)_a}$, the series converging absolutely and uniformly in θ . Multiplying by the complex conjugate and collecting the coefficient of $e^{il\theta}$ gives (6.6).

It remains to estimate the rate. Let $l \geq 0$ and put $S_l \stackrel{\text{def}}{=} \sum_{m \geq 0} \frac{\tau^{l+2m}}{(\tau; \tau)_m (\tau; \tau)_{l+m}}$, so that $S_l \leq \frac{\tau^l}{(\tau; \tau)_\infty^2 (1 - \tau^2)}$. By (6.5) and (6.6), we have

$$\begin{aligned} \nu_{(k,k)}(k+l) - \nu_0^{\mathbb{Z}}(l) &= ((\tau; \tau)_k (\tau; \tau)_{k+l} - (\tau; \tau)_\infty^2) S_l \\ &\quad - (\tau; \tau)_k (\tau; \tau)_{k+l} \left[\sum_{m=0}^k \frac{\tau^{l+2m} \tau^{k-m+1}}{(\tau; \tau)_m (\tau; \tau)_{l+m}} + \sum_{m > k} \frac{\tau^{l+2m}}{(\tau; \tau)_m (\tau; \tau)_{l+m}} \right]. \end{aligned}$$

For any $j \geq k$, we have $0 \leq (\tau; \tau)_j - (\tau; \tau)_\infty \leq 1 - \prod_{i > j} (1 - \tau^i) \leq \sum_{i > j} \tau^i \leq \frac{\tau^{k+1}}{1 - \tau}$, so that the first term is bounded by $\frac{2\tau^{k+1}}{1 - \tau} S_l$. The bracket is bounded by $(\tau; \tau)_\infty^{-2} \tau^l \left(\frac{\tau^{k+1}}{1 - \tau} + \frac{\tau^{2k+2}}{1 - \tau^2} \right)$. Hence there exists a constant C_τ such that $|\nu_{(k,k)}(k+l) - \nu_0^{\mathbb{Z}}(l)| \leq C_\tau \tau^{k+l}$ for any $k, l \geq 0$. Now, let $l < 0$ with $k' \stackrel{\text{def}}{=} k+l \geq 0$. The symmetry $\nu_{(k,k)}(j) = \nu_{(j,j)}(k)$ of [Arh26, Theorem 6.13] and the evenness of $\nu_0^{\mathbb{Z}}$ give

$$|\nu_{(k,k)}(k+l) - \nu_0^{\mathbb{Z}}(l)| = |\nu_{(k',k')}(k' + |l|) - \nu_0^{\mathbb{Z}}(|l|)| \leq C_\tau \tau^{k'+|l|} = C_\tau \tau^k.$$

This proves the last assertion. ■

Remark 6.2 The limit operator B_∞ is the operator of the network on $\mathbb{Z}$ with constant conductances $\mathfrak{b}_\infty$, with respect to the counting measure. For $\alpha = d$, the network of L_τ^∞ has the constant conductances $c_k = 1 - \tau$, see Remark 4.3, but L_τ^∞ acts on $\ell_{\mathbb{Z}}^2(w)$ instead of $\ell_{\mathbb{Z}}^2$. This is the bilateral analogue of the relation $\langle Bf, f \rangle = \frac{c_\tau}{2} \mathcal{E}_\tau(f)$ of [Arh26, Section 6.3]. By [Arh26, Theorem 6.22], $\frac{\varrho_d(\lambda)}{\varrho_\infty(\lambda)}$ is the Laplace transform of the explosion time ζ of the vertical network started at d , and the diagonal chain is the jump chain of a pair (X, Y) of independent continuous-time chains. Formally, (6.4) reads $\nu_d^{\mathbb{Z}}(l) = \mathbb{E}_d[(e^{-\zeta B_\infty})_{0l}]$, which is the law of $Y_\zeta - Y_0$ for the simple random walk Y on $\mathbb{Z}$ with jump rate $\mathfrak{b}_\infty$ to each neighbour, run up to the independent time ζ . We do not pursue the probabilistic interpretation of the limit as a Poisson kernel of a quantum random walk.

7 The quantum plane and its covariant Casimir

The following construction identifies the quantum homogeneous space and the self-adjoint Casimir underlying the bilateral radial operator. The algebraic Casimir is classical [BCGST96, AD99]; the purpose here is its precise analytic realization, its covariant completely positive dynamics, and the distinction between the homogeneous space and the whole quantum group.

7.1 Conventions and main conclusions

Fix $0 < r < 1$ and put $\tau = r^2$. Let $H = \ell^2(\mathbb{Z})$, with basis $(e_k)_{k \in \mathbb{Z}}$, and write $Ve_k = e_{k-1}$. On $H \otimes L^2(\mathbb{T})$, set

$$(7.1) \quad v = V \otimes 1, \quad n(e_k \otimes f) = r^k e_k \otimes zf, \quad A = n^* n = \text{diag}(\tau^k) \otimes 1.$$

Here and below an unbounded generator is understood as an element affiliated with the C^* -algebra in the sense of [Wor91b]. Woronowicz's group has

$$(7.2) \quad unv^* = rn, \quad \Delta(v) = v \otimes v, \quad \Delta(n) = v \otimes n + n \otimes v^*,$$

where the last sum is closed in the sense of the quantum-group construction [Wor91, Wor92]. Its reduced C^* -algebra $\mathcal{A} = C_0(E_r(2))$ is the closed span of $v^j f(n)$ with $j \in \mathbb{Z}$ and $f \in C_0(\mathbb{C}_r)$, where $\mathbb{C}_r = \{0\} \cup \{r^k z : k \in \mathbb{Z}, z \in \mathbb{T}\}$. We use the normal semifinite faithful Haar weight

$$(7.3) \quad \varphi(x) = (1 - \tau) \sum_{k \in \mathbb{Z}} \tau^k \int_{\mathbb{T}} x_{kk}(z) dm(z), \quad M = \mathcal{A}'' = B(H) \overline{\otimes} L^\infty(\mathbb{T}).$$

It is $(1 - \tau)$ times Pal's weight [Pal96, Theorem 2.1]. This weight is both left and right invariant. Its nonraciality must not be confused with nonunimodularity. Products such as $\text{Tr}(Ax)$ for $x \geq 0$ abbreviate the extended positive expression $\text{Tr}(A^{\frac{1}{2}} x A^{\frac{1}{2}})$. Inner products are linear in the first variable.

On $H_w = \ell^2(\mathbb{Z}, w)$, $w_k = (1 - \tau)\tau^k$, let $L = L_\tau^\infty$ be the operator associated with

$$(7.4) \quad \mathcal{E}_0(f) = (1 - \tau) \sum_{k \in \mathbb{Z}} |f(k+1) - f(k)|^2, \quad \text{dom } \mathcal{E}_0 = \{f \in H_w : \mathcal{E}_0(f) < \infty\}.$$

Thus

$$(7.5) \quad (Lf)(k) = \tau^{-k} (2f(k) - f(k+1) - f(k-1)).$$

We use the self-adjoint realization of Proposition 3.1, not an arbitrary extension of the difference expression on finitely supported functions. Its operator core consists of sequences vanishing near $-\infty$ and constant near $+\infty$; at $+\infty$ its flux satisfies $\Delta f(k) \rightarrow 0$.

The principal conclusions are Theorems 7.1, 7.3, 7.14, 7.17 and 7.20. In particular, the answer to the intrinsic radial-Casimir problem is affirmative for the realization specified above. The answer to the stronger question about an ordinary convolution semigroup on the whole group starting at the counit is negative.

7.2 The quotient by the right torus

The quotient homomorphism $\pi: \mathcal{A} \rightarrow C(\mathbb{T})$ is determined by $\pi(v) = u$ in multiplier algebras and $\pi(n) = 0$. It is the quantum subgroup of characters described by Woronowicz [Wor91]. For $t \in \mathbb{T}$ put $\chi_t \stackrel{\text{def}}{=} \text{ev}_t \pi$. The left and right translations are

$$(7.6) \quad R_t = (\text{Id} \otimes \chi_t) \Delta : v \mapsto tv, \quad n \mapsto t^{-1}n; \quad L_t = (\chi_t \otimes \text{Id}) \Delta : v \mapsto tv, \quad n \mapsto tn.$$

They commute and preserve φ . Write $E_R = \int_{\mathbb{T}} R_t dm(t)$, $E_L = \int_{\mathbb{T}} L_t dm(t)$ and $E = E_L E_R$ for their normal averages. Let $P_j = \mathbf{1}_{[0, \tau^j]}(A)$, so that $P_j = \sum_{k \geq j} e_{kk} \otimes 1$ and $\varphi(P_j) = \tau^j$.

Theorem 7.1 *Define*

$$(7.7) \quad W(e_k \otimes f) = e_k \otimes z^{-k} f, \quad \Theta(x) = W(x \otimes 1)W^*.$$

Then the right fixed-point algebras are

$$(7.8) \quad N = M^R = \Theta(B(H)), \quad \mathcal{P} = \mathcal{A}^R = \Theta(\mathcal{K}(H) + \mathbb{C}P_0).$$

Under this identification, the weight is $\psi(x) = (1 - \tau) \text{Tr}(Ax)$, the radial algebra is the diagonal MASA, and $E_L|_N$ is its weight-preserving conditional expectation. The restriction $\delta = \Delta|_N$ is a normal left coaction, and its restriction to $\mathcal{P}$ is a continuous C^ -coaction. The affiliated generator $\zeta = vn$ belongs to the quotient and satisfies*

$$(7.9) \quad \zeta = \Theta(Z), \quad Z = V \text{diag}(r^k), \quad \zeta^* \zeta = A, \quad \zeta \zeta^* = \tau A, \quad \delta(\zeta) = v^2 \otimes \zeta + \zeta \otimes 1.$$

The origin character of $\mathcal{P}$ is $\varepsilon_0(\Theta(K + cP_0)) = c$. It is the restriction of the counit of $\mathcal{A}$, and $(\text{Id} \otimes \varepsilon_0)\delta(x) = x$ for $x \in \mathcal{P}$.

Proof : In the concrete model, $R_t(e_{ij} \otimes z^a) = t^{j-i-a} e_{ij} \otimes z^a$. Therefore a fixed matrix has entries $c_{ij} z^{j-i}$. Since $\Theta(e_{ij}) = e_{ij} \otimes z^{j-i}$, conjugation by W^* proves the von Neumann assertion, including boundedness of the scalar matrix (c_{ij}) . Formula (7.3) proves the weight identity. The left action multiplies this matrix unit by $t^{2(j-i)}$, hence its fixed algebra is diagonal.

For the C^* -assertion, $\mathcal{J} = \mathcal{K}(H) \otimes C(\mathbb{T})$ is an ideal of $\mathcal{A}$. Indeed all its elementary matrix units times trigonometric monomials belong to $\mathcal{A}$, and multiplying by v or by bounded functions of n preserves it. For $f \in C_0(\mathbb{C}_r)$, the diagonal matrix $f(n) - f(0)P_0$ has entries whose uniform norms tend to zero at both ends, hence belongs to $\mathcal{J}$. In $\mathcal{A}/\mathcal{J}$, the image of P_0 is a unit and the image of vP_0 is unitary. Its spectrum is $\mathbb{T}$, since the right action rotates it. Thus $\mathcal{A}/\mathcal{J} = C(\mathbb{T})$, with the quotient map π . A fixed element has constant image c , so subtracting cP_0 gives an element of $\mathcal{J}^R = \Theta(\mathcal{K}(H))$. The reverse inclusion is immediate. Notice that P_0 has infinite rank and infinite corank: the algebra in (7.8) is not $\mathcal{K}(H) + \mathbb{C}1$.

Coassociativity gives $\Delta E_R = (\text{Id} \otimes E_R)\Delta$, hence $\Delta(N) \subset M \overline{\otimes} N$. At the C^* -level, the compact averaging and the cancellation property of $\mathcal{A}$ give $\overline{\text{span}}\{\Delta(x)(a \otimes 1) : x \in \mathcal{P}, a \in \mathcal{A}\} = \mathcal{A} \otimes \mathcal{P}$: apply $\text{Id} \otimes E_R$ to the corresponding cancellation identity for $\mathcal{A}$. The inclusion is nondegenerate, since P_j , $j \rightarrow -\infty$, is also an approximate identity for $\mathcal{A}$. This proves the continuous quotient coaction. For continuous coactions and quotients of locally compact quantum groups by closed quantum subgroups, see [BS93] and [Vae05].

The product vn is right invariant by (7.6); its polar part is $V \otimes z$ and its modulus is $A^{\frac{1}{2}}$. These are exactly the images of the polar data of Z under Θ . Its bounded transform multiplies P_0 on either side into compact operators, and $(1 + A)^{-1} \in \mathcal{K}(H) + \mathbb{C}P_0$ has dense ideal range there, which verifies affiliation with the quotient algebra. The relations in (7.9) follow from (7.2). The spectral projections of A and the polar part generate N . Finally, restriction of the counit sends P_0 to 1 and all compact matrix units to 0, giving ε_0 and the last identity. ■

Remark 7.2 The spatial twist is indispensable. Except on its diagonal, $B(H) \otimes 1$ is not the right-torus fixed algebra in the original representation. The distinction between $\mathcal{K}(H) + \mathbb{C}P_0$ and a unitization is equally important for the Feller question.

7.3 A closed quantum-plane Casimir

We now suppress Θ and work on $B(H)$ with $\psi(x) = (1 - \tau) \text{Tr}(Ax)$, $A = \text{diag}(\tau^k)$. Its GNS norm is

$$(7.10) \quad \|x\|_{2,\psi}^2 = (1 - \tau) \sum_{i,j \in \mathbb{Z}} \tau^j |x_{ij}|^2.$$

The angular sector $\mathcal{H}_\ell$ consists of matrices $x_{k+\ell,k} = f(k)$ and is identified with H_w . Put $D \stackrel{\text{def}}{=} (Z^*)^{-1} = VA^{-\frac{1}{2}}$ on its natural dense domain. On finite matrices define

$$(7.11) \quad \mathcal{C}_{\text{fin}}(x) = A^{-1}x + xA^{-1} - D^*xD - \tau DxD^*.$$

Every term is a finite matrix. In particular,

$$(7.12) \quad \mathcal{C}_{\text{fin}}(e_{ij}) = (\tau^{-i} + \tau^{-j})e_{ij} - r^{-i-j-2}e_{i+1,j+1} - r^{2-i-j}e_{i-1,j-1}.$$

Theorem 7.3 *The positive quadratic form of (7.11) is closable. Its closure is the direct sum over $\ell \in \mathbb{Z}$ of*

$$(7.13) \quad \mathcal{E}_\ell(f) = (1 - \tau) \left(r^{-\ell} \sum_k |f(k+1) - f(k)|^2 + (1 - r^{-\ell})^2 \sum_k |f(k)|^2 \right).$$

For $\ell = 0$ the second term is omitted, including from the domain condition. For $\ell \neq 0$ the domain is $H_w \cap \ell^2(\mathbb{Z})$; for $\ell = 0$ it is (7.4). Its associated positive self-adjoint operator $\mathcal{C}$ acts sectorwise by

$$(7.14) \quad (\mathcal{C}_\ell f)(k) = \tau^{-k} \left((1 + r^{-2\ell})f(k) - r^{-\ell}(f(k+1) + f(k-1)) \right).$$

In particular $\mathcal{C}_0 = L_\tau^\infty$ as self-adjoint operators, with equality of domains. This is a realization of the normalized algebraic quantum-plane Casimir: in the conventions of [BCGST96, Section 4],

$$(7.15) \quad \mathcal{C}_{\text{alg}} = (1 - \tau)^2 \lambda(P_+ P_-).$$

Here (7.15) means equality of algebraic actions on ordered coordinate monomials, not the assertion that those unbounded monomials belong to the GNS space.

Proof : For a finite matrix in sector ℓ , substitution into (7.11) gives (7.14). Multiplying by $w_k f(k)$ and summing gives (7.13). Each form is closed by Fatou's lemma and coordinatewise convergence. When $\ell \neq 0$, its norm is equivalent to $\|f\|_{H_w} + \|f\|_{\ell^2}$, so finite truncations are a form core.

For $\ell = 0$, the form is exactly (7.4). The core from Proposition 3.1 is a form core. Each member of that core, constant c after some index, can in turn be approximated by finitely supported sequences: retain it through a large positive index, then decrease linearly from c to 0 over J edges. The extra gradient energy is $(1 - \tau)|c|^2/J$, and the H_w error tends to zero by moving this taper to $+\infty$. Thus finitely supported sequences are a form core also at $\ell = 0$. Truncating the angular sum proves that finite matrices are a core of the direct sum form. Summation by parts against finite matrices gives the stated operator expressions.

The algebraic coordinate relations are $ZZ^* = \tau Z^*Z$ and the coaction is the one in (7.9). They match the quantum-plane relations of [BCGST96, Proposition 3.7] under $q = r$, $z_{\text{there}} = Z$,

$v_{\text{there}} = v^2$. This is a dictionary for the coordinate action; it does not identify two global quantum-group conventions without a covering qualification. Formula (4.2) of that source gives

$$(7.16) \quad \lambda(P_+P_-)(Z^{*j}Z^m) = -r^{-j-m}[j]_r[m]_r Z^{*(j-1)}Z^{m-1}.$$

Direct multiplication on the common finite-vector domain, using $D = (Z^*)^{-1}$, gives

$$(7.17) \quad \mathcal{C}_{\text{alg}}(Z^{*j}Z^m) = -\tau^{1-j-m}(1-\tau^j)(1-\tau^m)Z^{*(j-1)}Z^{m-1},$$

with value zero if $jm = 0$. Since $[j]_r = r^{1-j}(1-\tau^j)/(1-\tau)$, this is (7.15). The positive realization has already been defined by the closed form, so no self-adjointness of the coordinate polynomial representation is being assumed. ■

Remark 7.4 There is also a coordinate-commutator formula. For any finite matrix x , we have

$$(7.18) \quad \mathcal{E}(x) = \frac{1}{2}\psi([D, x]^*[D, x] + \tau[D^*, x]^*[D^*, x]).$$

To verify it, put $G \stackrel{\text{def}}{=} -\mathcal{C}_{\text{fin}}$. Direct expansion gives $G(x^*x) - G(x)^*x - x^*G(x) = [D, x]^*[D, x] + \tau[D^*, x]^*[D^*, x]$. Furthermore $\psi(G(y)) = 0$ for every finite matrix y , as follows immediately from (7.12). Symmetry then gives (7.18). The full form is its closure; this does not assert that the two commutators are everywhere defined on bounded operators.

Remark 7.5 At $\ell = 0$ the difference expression is limit circle at $+\infty$. Finitely supported sequences are a *form* core, not an operator core. The operator core includes tails constant near $+\infty$, and the selected boundary condition is

$$(7.19) \quad (1-\tau)\Delta f(k) = \sum_{j>k} w_j(Lf)(j), \quad \Delta f(k) \rightarrow 0 \quad (k \rightarrow +\infty).$$

This is precisely the realization of Proposition 3.1. A formal Casimir calculation alone would not determine this extension.

7.3.1 Spectrum and a norm-density statement

For $p, \ell \in \mathbb{Z}$ set

$$(7.20) \quad f_{p,\ell}(k) = r^{|\ell|k} {}_1\phi_1(0; \tau^{|\ell|+1}; \tau, \tau^{p+\max(\ell,0)+k+1}).$$

Proposition 7.6 *For each ℓ , the functions $f_{p,\ell}$ are a complete orthogonal eigenfamily for $\mathcal{C}_\ell$, with eigenvalues τ^p , $p \in \mathbb{Z}$. Each is a simple eigenvalue in that sector. The full operator $\mathcal{C}$ has these eigenvalues with countably infinite multiplicity; 0 is in its spectrum but is not an eigenvalue.*

Proof : Let $s \geq 1$ and put $\theta \stackrel{\text{def}}{=} \tau^{s+1}$. The multiplication $h(k) \mapsto r^{sk}h(k)$, with a constant normalization, is a unitary from $\ell^2(\mathbb{Z}, (1-\theta)\theta^k)$ to H_w . A direct calculation conjugates $\mathcal{C}_s$ to $\tau^{-s}L_{\theta,\tau}^\infty$, where the two subscripts specify the weight parameter and the base in Section 3. The forms agree first on finite supports and then by closure. Indeed the tail conductances of $L_{\theta,\tau}^\infty$ are proportional to τ^{sk} , so its constant-tail core is approximable in form norm by finite supports. Moreover $\mathcal{C}_{-s} = \tau^s\mathcal{C}_s$. Theorem 3.6 gives (7.20), completeness and simplicity. The same theorem with both parameters equal to τ treats $\ell = 0$. The remaining assertions follow from the orthogonal angular decomposition. ■

Lemma 7.7 *Any $f \in \text{dom } L$ vanishes at $-\infty$ and has a finite limit at $+\infty$. Moreover*

$$(7.21) \quad \|f\|_\infty \leq \frac{\|f\|_{H_w}}{\sqrt{1-\tau}} + \frac{r}{(1-\tau)(1-r)} \|Lf\|_{H_w}.$$

Finite spectral sums of L approximate its constant-tail core in the uniform norm.

Proof : For $k \leq 0$, the coordinate bound gives $|f(k)| \leq (1-\tau)^{-\frac{1}{2}} \tau^{-k/2} \|f\|_{H_w}$, which also proves the limit at $-\infty$. Formula (7.19) and Cauchy–Schwarz give $|\Delta f(k)| \leq \tau^{(k+1)/2} \|Lf\|_{H_w} / (1-\tau)$. The sum over $k \geq 0$ converges and yields (7.21) by telescoping from $f(0)$. The last assertion follows from the spectral theorem and convergence of finite spectral sums in graph norm. ■

Corollary 7.8 *The eigenoperators obtained from (7.20) in all angular sectors linearly span a norm-dense subspace of $\mathcal{K}(H) + \mathbb{C}P_0$.*

Proof : For $\ell \neq 0$, form convergence of spectral sums implies convergence in $\ell^2(\mathbb{Z})$ by (7.13), hence in the operator norm of a single matrix band. This approximates every off-diagonal matrix unit. For $\ell = 0$, graph convergence and Lemma 7.7 approximate each diagonal matrix unit and P_0 . These elements span a dense subspace of the asserted algebra. The eigenoperators themselves belong to it: in nonzero sectors the coefficients vanish at both ends, and in the radial sector the coefficients vanish at $-\infty$ and tend to 1 at $+\infty$. ■

7.4 Actual unitary representations and the spherical transform

Algebraic equivariance is not yet equivariance for the locally compact quantum group. We now use the representations of Woronowicz, in Pal’s conventions [Pal96, Section 3, especially (3.4)]. For $p \in \mathbb{Z}$ there is an irreducible unitary corepresentation $u^{(p)}$ on $\ell^2(\mathbb{Z})$ with

$$(7.22) \quad u_{ab}^{(p)} = v^{a+b} f_{p-b+1, a-b}(n), \quad f_{s,a}(z) = \int_{\mathbb{T}} F_r(r^s z u) u^{-a} dm(u),$$

where

$$(7.23) \quad F_r(z) = \prod_{h \geq 0} \frac{1 + \tau^h \bar{z}}{1 + \tau^h z}$$

on $\mathbb{C}_r$, with its continuous values at the removable exceptional points. These coefficients belong to $\mathcal{A} \cap \mathfrak{n}_\varphi$. There are also half-integral families, with right-torus weights $2b+1$; they have no fixed column and are not needed on the plane. For the integral family, $R_t(u_{ab}^{(p)}) = t^{2b} u_{ab}^{(p)}$ and $L_t(u_{ab}^{(p)}) = t^{2a} u_{ab}^{(p)}$.

Lemma 7.9 *Put $F_a(s) \stackrel{\text{def}}{=} \int_{\mathbb{T}} F_r(r^s u) u^{-a} dm(u)$ for integers a, s . Then*

$$(7.24) \quad F_a(s) = \frac{(-1)^a r^{sa}}{(\tau; \tau)_a} {}_1\phi_1(0; \tau^{a+1}; \tau, \tau^s) \quad (a \geq 0),$$

and, for $b \geq 1$,

$$(7.25) \quad F_{-b}(s) = \frac{r^{sb+b(b-1)}}{(\tau; \tau)_b} {}_1\phi_1(0; \tau^{b+1}; \tau, \tau^{s+b}).$$

The formulas hold also when $s \leq 0$.

Proof : For $s > 0$, expand the numerator of (7.23) by Euler's identity $(x; \tau)_\infty = \sum_{n \geq 0} \frac{(-1)^n \tau^{n(n-1)/2}}{(\tau; \tau)_n} x^n$ [DLMF, (17.5.1)] and its denominator by the q -binomial theorem in the form $\frac{1}{(x; \tau)_\infty} = \sum_{n \geq 0} \frac{x^n}{(\tau; \tau)_n}$, $|x| < 1$ [DLMF, (17.5.4)], see also [GR04, Section 1.3] and [KS92, (1.6) and (1.7)]. Both expansions are absolutely convergent on the unit circle. Collecting the coefficient of u^a gives (7.24) and (7.25). For completeness, the continuation to integral $s \leq 0$ is not a formal use of a divergent Laurent expansion. Introduce the entire symmetric kernel

$$(7.26) \quad \Xi(w, z) = \sum_{j \geq 0} \frac{(-1)^j \tau^{j(j-1)/2} (w\tau^j; \tau)_\infty}{(\tau; \tau)_j} z^j = (w; \tau)_\infty \phi_1(0; w; \tau, z).$$

It is symmetric in w, z : expanding the infinite product by Euler's identity [DLMF, (17.5.1)] gives an absolutely locally uniformly convergent double series symmetric in its two indices. This is also [KS92, Proposition 2.1]. For $n \geq 1$, the first n terms vanish at $w = \tau^{1-n}$, giving

$$(7.27) \quad \Xi(\tau^{1-n}, z) = (-1)^n \tau^{n(n-1)/2} z^n \Xi(\tau^{n+1}, \tau^n z).$$

Both formulas of the lemma are equivalently $F_a(s) = (-1)^a r^{sa} \Xi(\tau^{a+1}, \tau^s) / (\tau; \tau)_\infty$. On the other hand, cancellation in the product gives, for any integral $s \leq 0$,

$$(7.28) \quad F_r(r^s u) = u^{s-1} F_r(r^{2-s} u), \quad F_a(s) = F_{a-s+1}(2-s).$$

The product identity holds off the exceptional points and extends there by continuity. To check the same reflection for the expression in Ξ , put $n \stackrel{\text{def}}{=} 1-s$, use symmetry, and then (7.27). The resulting factor is $(-1)^{a+n} r^{(n+1)(a+n)}$, and the remaining kernel is $\Xi(\tau^{a+n+1}, \tau^{n+1})$. This is precisely the expression for $F_{a-s+1}(2-s)$, where $2-s > 0$. Finally (7.27) with $n = b$ gives (7.25). Thus all lattice radii are covered. $\blacksquare$

Theorem 7.10 *The right-fixed coefficients $u_{a0}^{(p)}$ belong to $\mathcal{P}$, are eigenvectors of $\mathcal{C}$ with eigenvalue τ^p , and are nonzero multiples of the eigenoperators in Proposition 7.6. More precisely, in sector $\ell = -a$ their scalar coefficient is $F_a(p+1+k)$. In particular*

$$(7.29) \quad u_{00}^{(p)} = F_p(A), \quad F_p(\tau^k) = {}_1\phi_1(0; \tau; \tau, \tau^{p+k+1}), \quad F_p(0) = 1,$$

where F_p vanishes at infinity and $|F_p| \leq 1$ on $X = \{0\} \cup \tau^\mathbb{Z}$. The span of $u_{a0}^{(p)}$, $a, p \in \mathbb{Z}$, is norm dense in $\mathcal{P}$ and dense in its GNS space. Their coproducts satisfy

$$(7.30) \quad \delta(u_{a0}^{(p)}) = \sum_{b \in \mathbb{Z}} u_{ab}^{(p)} \otimes u_{b0}^{(p)}.$$

The GNS spectrum of $\mathcal{C}$ is therefore the spherical Plancherel spectrum of these genuine quantum-group representations.

Proof : In (7.22), take $b = 0$. Since $f_{p+1,a}(r^k z) = z^a F_a(p+1+k)$, the coefficient $u_{a0}^{(p)}$ lies in sector $-a$ under Θ^{-1} and has the stated scalar entries. For $a \geq 0$, (7.24) makes it $(-1)^a r^{(p+1)a} (\tau; \tau)_a^{-1} f_{p,-a}$. For $a = -b < 0$, (7.25) makes it $r^{b(p+b)} (\tau; \tau)_b^{-1} f_{p,b}$. Proposition 7.6 now proves the eigenvalue and completeness assertions; Corollary 7.8 proves norm density. Equation (7.29) is the case $a = 0$. The value at 0 follows from the power series, and the bound follows because it is a coefficient of a unitary corepresentation. Equation (7.30) is the

corepresentation identity, with the sum understood in the usual weak/strict coefficient sense. As a normalization check, Pal's orthogonality gives

$$(7.31) \quad \|u_{a0}^{(p)}\|_{2,\varphi}^2 = (1-\tau)r^{2(a-p)}, \quad \|F_p\|_{H_w}^2 = (1-\tau)\tau^{-p}.$$

■

Remark 7.11 The Fourier-coefficient calculation is the step which connects the analytic closure to the integrated quantum-group action. An equality of formal q -difference expressions or of sets of eigenvalues would not establish (7.30) for the analytic eigenoperators.

7.5 The radial heat process and its entrance law

We write $T_t = e^{-tL}$ and regard $X = \{0\} \cup \tau\mathbb{Z}$ as a locally compact space. The point 0 corresponds to $k \rightarrow +\infty$, whereas spatial infinity corresponds to $k \rightarrow -\infty$. Thus $C_0(X)$ consists of sequences vanishing at $-\infty$ and converging at $+\infty$.

Proposition 7.12 *The operator L generates a conservative normal Markov semigroup on $\ell_{\mathbb{Z}}^{\infty}$, which restricts to a strongly continuous positive contraction semigroup on $C_0(X)$. There are probability measures μ_t on X , with $\mu_0 = \delta_0$, such that*

$$(7.32) \quad \mu_t(f) = (T_t f)(0), \quad \int_X F_p d\mu_t = e^{-t\tau^p}, \quad p \in \mathbb{Z}.$$

For every $t > 0$, $\mu_t(\{0\}) = 0$, and the transition rows $P_t(k, \cdot)$ converge in $\ell_{\mathbb{Z}}^1$ to μ_t as $k \rightarrow +\infty$. Moreover

$$(7.33) \quad \mu_{\tau^j t}(k+j) = \mu_t(k) \quad (j, k \in \mathbb{Z}, t > 0).$$

Proof : The form (7.4) is a Dirichlet form [FOT11, Section 1.4]. The associated minimal continuous-time chain [Nor97, Chapter 2] has rates τ^{-k} to each neighbour. Its embedded jump chain is the recurrent simple symmetric walk on $\mathbb{Z}$. It visits 0 infinitely often, and the independent holding times there have exponential rate 2; their sum is infinite almost surely. The process is therefore nonexplosive and conservative. Finite supports are a form core by the proof of Theorem 7.3, so this minimal process realizes exactly L , rather than a different extension at $+\infty$.

For a bounded constant-tail core function g , Lg is finitely supported. The stopped Dynkin formula and nonexplosion give $\|T_t g - g\|_{\infty} \leq t\|Lg\|_{\infty}$. For $t > 0$, spectral calculus gives $T_t g \in \text{dom } L$, so Lemma 7.7 implies $T_t g \in C_0(X)$. Such core functions are uniformly dense in $C_0(X)$, hence contractivity proves the Feller assertions. The functional at 0 is initially a subprobability measure by the Riesz theorem.

We check that it has mass one. For the coordinate function $a(k) = \tau^k$, the formal Markov generator $-L$ satisfies $(-La)(k) = c_* = \tau + \tau^{-1} - 2$. Apply Dynkin's formula up to exit from a finite interval and then Fatou's lemma. This yields $\mathbb{E}_k[a(X_t)] \leq \tau^k + c_* t$. For $k \geq 0$, the probability of $X_t < -M$ is at most $(1 + c_* t)\tau^M$. Since $\mathbf{1}_{k \geq -M} \in C_0(X)$, with value one at the origin, its entrance expectation is at least $1 - (1 + c_* t)\tau^M$. Letting $M \rightarrow \infty$ proves mass one.

The eigenfunction identity in Theorem 7.10, first in H_w and then on $C_0(X)$, gives (7.32). As $p \rightarrow -\infty$, $F_p(\tau^k) = F_0(\tau^{p+k}) \rightarrow 0$ for each fixed k , whereas $F_p(0) = 1$ and $|F_p| \leq 1$. Dominated convergence in (7.32) gives $\mu_t(\{0\}) = 0$ for $t > 0$. For each lattice point its indicator belongs to $C_0(X)$, so Feller continuity gives convergence of individual transition probabilities. As both the rows and the limit have mass one on $\mathbb{Z}$, Scheffe's argument gives ℓ^1 convergence.

Finally, the shift $Qf(k) = f(k+1)$ satisfies $QLQ^{-1} = \tau^{-1}L$. Consequently $QT_t = T_{t/\tau}Q$. Evaluate at the origin, which is fixed by the shift, and iterate to obtain (7.33). ■

Remark 7.13 For $t > 0$ the entrance probabilities have the absolutely convergent formula

$$(7.34) \quad \mu_t(k) = \tau^k \sum_{p \in \mathbb{Z}} \tau^p e^{-t\tau^p} {}_1\phi_1(0; \tau; \tau, \tau^{p+k+1}).$$

Indeed, expand $\mathbf{1}_{\{k\}}$ in the eigenbasis using (7.31). The resulting series for $T_t \mathbf{1}_{\{k\}}$ converges uniformly, since $\sum_p \tau^p e^{-t\tau^p} < \infty$ and $|F_p| \leq 1$. Evaluation at the origin gives (7.34). Positivity is justified by the process above, not by the signs in this spectral sum.

7.6 Covariant completely positive heat dynamics

Let $\eta_{\mathbb{T}} = h_{\mathbb{T}}\pi$ be the Haar idempotent of the torus. It is a state of $\mathcal{A}$, not a normal state of M . For $t > 0$, define a normal state of M by

$$(7.35) \quad \omega_t(x) = \mu_t(Ex) = \sum_{k \in \mathbb{Z}} \mu_t(k) \int_{\mathbb{T}} x_{kk}(z) dm(z), \quad \omega_0 = \eta_{\mathbb{T}}.$$

The expression at $t = 0$ is used only on $\mathcal{A}$ and its multiplier algebra.

Theorem 7.14 *The maps*

$$(7.36) \quad S_t = (\text{Id} \otimes \omega_t)\Delta|_N \quad (t > 0), \quad S_0 = \text{Id}_N,$$

form a normal unital completely positive semigroup on N . They preserve $\varphi|_N$, commute with its modular group, and are GNS- and KMS-symmetric. Their GNS implementation is $e^{-t\mathcal{C}}$, and

$$(7.37) \quad \delta S_t = (\text{Id} \otimes S_t)\delta, \quad S_t(F(A)) = (T_t F)(A).$$

The semigroup preserves $\mathcal{P}$ and is point-norm continuous there, and it is point- σ -weakly continuous on N . Its action on the actual representation coefficients is

$$(7.38) \quad S_t(u_{a0}^{(p)}) = e^{-t\tau^p} u_{a0}^{(p)}.$$

*The states in (7.35) satisfy $\omega_s * \omega_t = \omega_{s+t}$ on $\mathcal{A}$ and converge weakly to $\eta_{\mathbb{T}}$ at zero. The corresponding convolutions on the whole group satisfy*

$$(7.39) \quad (\text{Id} \otimes \omega_t)\Delta = S_t E_R, \quad (\text{Id} \otimes \omega_0)\Delta = E_R,$$

not the identity on $\mathcal{A}$. Thus this is an idempotent-started convolution semigroup, not an ordinary convolution semigroup starting at the counit.

Proof : Each $\omega_t, t > 0$, is normal and bi-torus invariant. Therefore the normal unital completely positive convolution $R_{\omega_t} = (\text{Id} \otimes \omega_t)\Delta$ has range in N and satisfies $R_{\omega_t} E_R = E_R R_{\omega_t} = R_{\omega_t}$. It preserves Haar by its invariance; this is also the general convolution fact in [SV19, Theorem 2.1], with attention to the opposite convention for left/right names. In particular its GNS implementation is a contraction.

Using the torus weights and (7.32), $\omega_t(u_{b0}^{(p)}) = \delta_{b0} e^{-t\tau^p}$. Slice (7.30) by this normal functional. This proves (7.38). The complete orthogonal eigenbasis of Theorem 7.10 implies that

the GNS implementation is $e^{-t\mathcal{E}}$. Corollary 7.8, together with contractivity, shows directly that S_t preserves $\mathcal{P}$, satisfies the semigroup law there and is point-norm continuous. Normality and σ -weak density extend the semigroup identity to N .

Coassociativity gives (7.37) for $t > 0$ directly from (7.36), and for $t = 0$ trivially. The radial identity follows first on the norm-dense radial spectral span and then normally. The restriction to $\mathcal{P}$ is nondegenerate in the strict sense. Indeed $S_t(P_j) \rightarrow 1$ strongly as $j \rightarrow -\infty$, and the moment bound in Proposition 7.12 gives $\|(1 - S_t(P_j))P_0\| \leq (1 + c_*t)\tau^{-j} \rightarrow 0$. Together with norm convergence after multiplication by compact operators, this gives strict convergence [Lan95, Chapter 2] on $\mathcal{K}(H) + \mathbb{C}P_0$ and the unital strict multiplier extension. The modular group is $\sigma_s^\psi(x) = A^{is}xA^{-is}$, which multiplies sector ℓ by $\tau^{is\ell}$. Hence it commutes with S_t . The GNS implementation is self-adjoint. Put $D_\psi \stackrel{\text{def}}{=} (1 - \tau)A$. The symmetric embedding $x \mapsto D_\psi^{1/4}x D_\psi^{1/4}$ differs from the GNS embedding $x \mapsto x D_\psi^{1/2}$ on sector ℓ by the scalar $\tau^{\ell/4}$. The same self-adjoint sector operators therefore implement S_t there, giving KMS symmetry.

For clarity, point- σ -weak continuity on all of N does not follow merely from norm density of $\mathcal{P}$ in its own norm. Let $x \in B(H)$ and let Q be a finite-rank coordinate projection containing the vectors on which a matrix entry is tested. Complete positivity and the Schwarz inequality bound the entries of $S_t(x - QxQ)$ by constants times square roots of entries of $S_t(1 - Q)$. Since $S_t(Q) \rightarrow Q$ in norm, these errors tend to zero. Also $QxQ \in \mathcal{P}$ and $S_t(QxQ) \rightarrow QxQ$ in norm. Thus all matrix entries converge; uniform boundedness gives σ -weak convergence.

On $\mathcal{A}$, E takes values in $C_0(X)$, so Proposition 7.12 gives $\omega_t \rightarrow \delta_0 E = \eta_\mathbb{T}$ weakly. Bi-invariance gives (7.39). Its composition law follows from $E_R|_N = \text{Id}_N$ and the semigroup law of S_t . Applying the bounded counit of $\mathcal{A}$ then proves $\omega_s * \omega_t = \omega_{s+t}$, including $s = 0$ or $t = 0$. Finally $\eta_\mathbb{T}(p_k) = 0$ for every k , while $\sum_k p_k = 1$ strongly, so $\eta_\mathbb{T}$ has no normal extension to M . ■

Proposition 7.15 *The semigroup of Theorem 7.14 is the unique covariant completely positive contraction semigroup on $\mathcal{P}$ with the specified radial restriction T_t , where the covariance identity is understood with the strict extensions to multiplier algebras. Any normal extension to N is then unique.*

Proof : For a covariant map S put $\nu \stackrel{\text{def}}{=} \varepsilon_0 S$. Apply $\text{Id} \otimes \varepsilon_0$ to the covariance identity to obtain $S = (\text{Id} \otimes \nu)\delta$. Covariance also implies commutation with left torus translations. Since ε_0 is left-torus invariant, $\nu = \nu E_L$. Hence ν is determined by the radial restriction of S . For the prescribed T_t it is $\mu_t E_L$, giving exactly (7.36). These slices can be read first in a nondegenerate multiplier realization, or verified on the coefficient span, which is norm dense by Theorem 7.10. Normal extensions coincide on the σ -weakly dense algebra $\mathcal{P}$. ■

Remark 7.16 The complete positivity in Theorem 7.14 is established by actual convolution states, not by a merely formal Lindblad expression. Conversely, after the theorem has been proved, (7.11) is an explicit infinitesimal expression of Lindblad type [Lin76] with the two unbounded jump operators D and rD^* . No general unbounded-Lindblad existence theorem is needed.

7.7 No Feller lifting to the whole quantum group

For a probability vector μ on $\mathbb{Z}$ put

$$(7.40) \quad \mathcal{F}(\mu) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} \sqrt{\mu(k-1)\mu(k)}.$$

Cauchy–Schwarz gives $\mathcal{F}(\mu) \leq 1$. Equality would make $\sqrt{\mu}$ shift invariant in $\ell^2(\mathbb{Z})$, which is impossible for a unit vector; hence the inequality is strict.

Theorem 7.17 *Let $t > 0$, and let $R: \mathcal{A} \rightarrow \mathcal{A}$ be completely positive and contractive. Suppose its restriction to the radial algebra $\{F(A) : F \in C_0(X)\}$ is T_t , under the canonical identification. Then*

$$(7.41) \quad \|R(vP_0) - vP_0\| \geq 1 - \mathcal{F}(\mu_t) > 0.$$

Consequently, for any family $(R_t)_{t>0}$ with these radial restrictions,

$$(7.42) \quad \|R_{\tau^j}(vP_0) - vP_0\| \geq \delta_\tau \stackrel{\text{def}}{=} 1 - \mathcal{F}(\mu_1) > 0 \quad (j \geq 0).$$

There is no such family point-norm converging to the identity on $\mathcal{A}$. No semigroup law, covariance, Haar invariance or normal extension is assumed in this assertion. In particular the prescribed radial heat semigroup is not the restriction of an ordinary weakly continuous convolution semigroup of states starting at the counit.

Proof : Let $R(x) = V_1^* \Pi(x) V_1$ be a Stinespring representation [Pau02, Theorem 4.1] with $\|V_1\| \leq 1$, and put $\xi_i \stackrel{\text{def}}{=} V_1(e_i \otimes 1)$. For $p_k \stackrel{\text{def}}{=} e_{kk} \otimes 1$, the radial hypothesis gives

$$(7.43) \quad \|\Pi(p_k)\xi_i\|^2 = P_t(i, k), \quad \sum_k P_t(i, k) = 1.$$

Therefore $\|\xi_i\| = 1$ and $\xi_i = \sum_k \Pi(p_k)\xi_i$ in norm, even if Π has an additional quotient component on which every p_k vanishes. Since $vP_0 p_k = p_{k-1} vP_0 p_k$ and $\|vP_0\| = 1$,

$$(7.44) \quad |\langle R(vP_0)(e_j \otimes 1), e_i \otimes 1 \rangle| \leq \sum_{k \geq 0} \sqrt{P_t(i, k-1)P_t(j, k)} \leq \sum_{k \in \mathbb{Z}} \sqrt{P_t(i, k-1)P_t(j, k)}.$$

The sums follow by approximating the two Stinespring vectors by finite sums of their orthogonal components, not by assuming vP_0 is a norm sum of matrix units. Take $i = j - 1$ and let $j \rightarrow +\infty$. By Proposition 7.12 the two probability vectors converge in ℓ^1 to μ_t . The inequality $\|\sqrt{\alpha} - \sqrt{\beta}\|_2^2 \leq \|\alpha - \beta\|_1$ shows that the last expression tends to $\mathcal{F}(\mu_t)$. The corresponding coefficient of vP_0 equals one. This proves (7.41). Scaling (7.33) leaves the affinity unchanged and proves (7.42).

Ordinary weakly continuous convolution semigroups starting at the counit yield point-norm continuous convolution semigroups on C_0 ; see [SV19, Theorem 3.2]. Applying the preceding obstruction excludes them. This does not exclude the idempotent-started family in Theorem 7.14, whose value at zero on $\mathcal{A}$ is E_R . ■

7.8 A sharp covariance threshold for Taibleson energies

This section concerns the precise class relevant to the manuscript: normal unital, Haar-weight-preserving covariant semigroups on N whose restrictions to $\mathcal{P}$ are point-norm continuous and covariant in the multiplier-algebra sense (with strict extensions of the completely positive maps). These are the hypotheses used for the counit-slicing argument. Equivalence of radial forms is always with the canonical radial identification, not after an arbitrary Hilbert-space unitary.

Lemma 7.18 *There is a constant $C_\tau \geq 1$ such that*

$$(7.45) \quad 0 \leq 1 - F_p(x) \leq C_\tau \max(1, \tau^p)(1 - F_0(x)) \quad (x \in X, p \in \mathbb{Z}).$$

Proof : The functions are real and $|F_p| \leq 1$. We first show $F_0(\tau^k) < 1$ for every finite k . By Lemma 7.9, it is the average of the continuous unit-modulus function $u \mapsto F_\tau(r^{k+1}u)$. This function is not constant. For positive integral s , the product $(-r^s u^{-1}; \tau)_\infty / (-r^s u; \tau)_\infty$ extends analytically through the unit circle and has infinitely many zeros approaching 0 in the punctured disk, with no denominator zeros there; it cannot be a constant or a monomial. For nonpositive s , (7.28) reduces nonconstancy to this fact. An average of a nonconstant continuous unit-modulus function cannot have modulus one.

At $+\infty$, the power series gives $1 - F_0(\tau^k) = \frac{\tau^{k+1}}{(1-\tau)^2} + O(\tau^{2k})$. At $-\infty$, $F_0(\tau^k) \rightarrow 0$. Strict positivity at the remaining finitely many sites yields constants $c, C > 0$ with $c \min(1, \tau^k) \leq 1 - F_0(\tau^k) \leq C \min(1, \tau^k)$ for all k . Now use $F_p(\tau^k) = F_0(\tau^{k+p})$ and $\min(1, \tau^{k+p}) \leq \max(1, \tau^p) \min(1, \tau^k)$. At the origin both sides vanish. ■

Proposition 7.19 *For every semigroup in the class specified above there are finite numbers $\Psi(p) \geq 0$, $p \in \mathbb{Z}$, such that*

$$(7.46) \quad S_t(u_{a0}^{(p)}) = e^{-t\Psi(p)} u_{a0}^{(p)}, \quad \Psi(p) \leq C_\tau \max(1, \tau^p) \Psi(0).$$

Proof : Put $\nu_t = \varepsilon_0 S_t$. As in Proposition 7.15, covariance implies $S_t = (\text{Id} \otimes \nu_t) \delta$ and $\nu_t = \nu_t E_L$. Hence $S_t(u_{a0}^{(p)}) = \nu_t(F_p(A)) u_{a0}^{(p)}$. The scalars are real, have modulus at most one, are continuous in t , equal one at zero and multiply under composition. They are therefore $e^{-t\Psi(p)}$ for finite $\Psi(p) \geq 0$. The radial restriction of ν_t is a positive measure m_t of mass at most one on X . Integrating (7.45) and adding the missing mass (the coefficient on the right is at least one) gives

$$1 - e^{-t\Psi(p)} \leq C_\tau \max(1, \tau^p) (1 - e^{-t\Psi(0)}).$$

Divide by t and let $t \rightarrow 0^+$. This argument does not need a priori conservation at the origin. Haar preservation ensures that the scalar action is also the symmetric GNS spectral action, since the columns are a complete orthogonal family. ■

Theorem 7.20 *Let K be a local field with residue cardinality Q , let $d \geq 1$, and suppose $\tau = Q^{-d}$. There exists a semigroup in the above class whose radial generator has a closed form equivalent to that of the whole-space radial Taibleson operator D_{rad}^α if and only if $0 < \alpha \leq d$. For $0 < \alpha \leq d$, one may take the Bochner subordinate of Theorem 7.14 with generator $\mathcal{C}^{\alpha/d}$. With $s = \alpha/d$, its radial form satisfies the explicit comparison*

$$(7.47) \quad (1 + \tau)^{-2s} \mathcal{E}_{L^s}(f) \leq \mathcal{E}_{D^\alpha}(f) \leq (1 - \tau)^{-2s} \mathcal{E}_{L^s}(f).$$

For $s > 1$, the radial operator L^s cannot generate a conservative Feller Markov semigroup on $C_0(X)$.

Proof : For $s = \frac{\alpha}{d} \leq 1$, $\lambda \mapsto \lambda^s$ is a Bernstein function. Subordination [SSV10, Chapters 5 and 12] preserves normal complete positivity, Haar invariance, covariance, the two symmetries and point-norm continuity on $\mathcal{P}$. Its radial generator is L^s . The equivalence with Taibleson is Theorem 4.6, with $\vartheta = d$, since $\alpha \leq d < 3d$. The sharper displayed constants follow from the optimal comparison at order d in Theorem 4.2 by the operator-monotonicity of $x \mapsto x^s$ for $0 < s \leq 1$, and the identity $D_{\text{rad}}^\alpha = (D_{\text{rad}}^d)^s$. The operator monotonicity of $x \mapsto x^s$ is the Löwner–Heinz inequality [Bha97, Theorem V.1.9]. Its proof through the representation $\lambda^s = \frac{\sin \pi s}{\pi} \int_0^\infty t^{s-1} \lambda(\lambda + t)^{-1} dt$ applies verbatim to nonnegative self-adjoint operators ordered in the sense of closed quadratic forms, since this order is equivalent to the reverse order of the resolvents [Kat95, Theorem VI.2.21].

Conversely, the finite exponents in (7.46) imply that every F_p belongs to the radial form domain. If $\alpha \geq 3d$, this contradicts Theorem 4.6(3). Suppose $d < \alpha < 3d$ and put $s \stackrel{\text{def}}{=} \frac{\alpha}{d} > 1$. Then F_0 has finite, nonzero Taibleson energy. Writing $F_p(k)$ for $F_p(\tau^k)$ in the following calculation, the dilation covariance of D_{rad}^α , together with $F_p(k) = F_0(k+p)$, yields

$$\frac{\mathcal{E}_{D^\alpha}(F_p)}{\|F_p\|_{H_w}^2} = \tau^{ps} \frac{\mathcal{E}_{D^\alpha}(F_0)}{\|F_0\|_{H_w}^2}.$$

Equivalence of forms forces $\Psi(p) \geq c\tau^{ps}$ for all p . Proposition 7.19 gives $\Psi(p) \leq C\tau^p$ for $p \leq 0$, which is impossible as $p \rightarrow -\infty$ when $s > 1$.

For the last assertion, a conservative Feller Markov realization of e^{-tL^s} would have an origin probability measure with moments $e^{-t\tau^{ps}}$ against F_p . The proof of Proposition 7.19, using these measures and Lemma 7.18, gives $\tau^{ps} \leq C_\tau \max(1, \tau^p)$, the same contradiction. This last conclusion is about the Markov/Feller property, not merely Hilbert-space self-adjointness. ■

8 Contraction of quantum Casimir dynamics

8.1 Framework and scope

Fix $0 < r < 1$ and put $\tau \stackrel{\text{def}}{=} r^2$. Throughout, scalar products are linear in the first variable. We use the compact realization from [Arh26, Section 5] and the quantum-plane realization of Theorem 7.1:

$$(8.1) \quad H_+ = \ell_{\mathbb{N}_0}^2, \quad H = \ell_{\mathbb{Z}}^2, \quad \mathcal{P}_c = \mathcal{K}(H_+) + \mathbb{C}1, \quad \mathcal{P}_e = \mathcal{K}(H) + \mathbb{C}P_0,$$

where $P_a = \sum_{k \geq a} e_{kk}$ and P_0 has infinite rank and infinite corank. The von Neumann closures are $\mathcal{B}(H_+)$ and $\mathcal{B}(H)$. Their invariant state and weight are

$$(8.2) \quad h_c(x) = (1 - \tau) \sum_{k \geq 0} \tau^k x_{kk}, \quad \psi(x) = (1 - \tau) \sum_{k \in \mathbb{Z}} \tau^k x_{kk}, \quad w_k = (1 - \tau)\tau^k.$$

Expressions involving the weight are interpreted in the extended positive sense. Write $\mathcal{H}_c = L^2(\mathcal{B}(H_+), h_c)$ and $\mathcal{H}_e = L^2(\mathcal{B}(H), \psi)$, represented by weighted matrices, with $\|x\|_{2,\psi}^2 = \sum_{i,k} w_k |x_{ik}|^2$.

The compact Casimir $\mathcal{C}_c$ is normalized by

$$(8.3) \quad \mathcal{C}_c u_{j0}^{(l)} = \Lambda_l u_{j0}^{(l)}, \quad \Lambda_l = \tau^{-l}(1 - \tau^l)(1 - \tau^{l+1}), \quad l \geq |j|, \quad l, j \in \mathbb{Z}.$$

The compact quantum Markov semigroup with this GNS implementation is denoted by S_t^c . Its existence, normal extension, covariance and Feller property are the input from [Arh26, Section 7.3]. The letter ℓ below denotes a *matrix-band index*, rather than an a priori representation label.

On the plane, a band $x_{k+\ell,k} = f(k)$ identifies with $H_w = \ell^2(\mathbb{Z}, w)$. We use the particular self-adjoint realization $\mathcal{C}_e$ of Theorem 7.3, whose band forms and expressions are

$$(8.4) \quad \mathcal{E}_{e,\ell}(f) = (1 - \tau) \left(r^{-\ell} \sum_k |f(k+1) - f(k)|^2 + (1 - r^{-\ell})^2 \sum_k |f(k)|^2 \right),$$

$$(8.5) \quad (\mathcal{C}_{e,\ell} f)(k) = \tau^{-k} \left((1 + \tau^{-\ell})f(k) - r^{-\ell}f(k+1) - r^{-\ell}f(k-1) \right).$$

At $\ell = 0$ the second term in (8.4) is omitted from both the value and the domain condition. At $\ell \neq 0$ the domain is $H_w \cap \ell^2(\mathbb{Z})$. Finite matrices are a form core; they are not an operator core in the radial band. The latter uses the zero-flux condition at $+\infty$ specified in Proposition 3.1. In particular, $\mathcal{C}_{e,0} = L_\tau^\infty$.

We also use Proposition 7.6, Proposition 7.12 and Theorem 7.14. The band eigenfunctions are

$$(8.6) \quad f_{p,\ell}(k) = r^{|\ell|k} {}_1\phi_1(0; \tau^{|\ell|+1}; \tau, \tau^{p+\max(\ell,0)+k+1}), \quad p \in \mathbb{Z},$$

with simple eigenvalues τ^p in each band. Their associated eigenoperators $X_{p,\ell}$ span a norm-dense subspace of $\mathcal{P}_e$. There is a normal, invariant-weight-preserving, covariant quantum Markov semigroup S_t^e whose GNS implementation is $e^{-t\mathcal{C}_e}$, and it restricts to a strongly continuous semigroup on $\mathcal{P}_e$. These inputs are not claimed as new results here. The construction of the plane Casimir and its relation to actual representation coefficients have to be distinguished from the classical special-function identities; see also [Pal96, KS92].

The proofs in Sections 8.2–8.6 establish the analytic and probabilistic contraction without invoking convergence of quantum-group coproducts. Section 8.7 then adds compatibility with those coproducts, using the published C^* -contraction theorem of Woronowicz in its explicit real-parameter formulation in [RR21, Theorem 6.6]. This last input is isolated, not presented as a new proof of quantum-group contraction.

8.2 Expanding corners and the invariant weights

For $N \geq 0$ define $J_N : H_+ \rightarrow H$ by $J_N e_k = e_{k-N}$, and put

$$(8.7) \quad Q_N \stackrel{\text{def}}{=} P_{-N} = J_N J_N^*, \quad \iota_N(x) \stackrel{\text{def}}{=} J_N x J_N^*, \quad \rho_N(x) \stackrel{\text{def}}{=} J_N^* x J_N.$$

Proposition 8.1 *The map ι_N is a $*$ -isomorphism from $\mathcal{P}_e$ onto $Q_N \mathcal{P}_e Q_N$. Its value at the compact unit is Q_N . The corners increase with norm-dense union in $\mathcal{P}_e$, and $Q_N \rightarrow 1$ strictly in $M(\mathcal{P}_e)$. Moreover*

$$(8.8) \quad \psi(\iota_N(x)) = \tau^{-N} h_c(x) \quad (x \geq 0).$$

Consequently

$$(8.9) \quad \mathcal{J}_N \Lambda_c(x) = \tau^{N/2} \Lambda_e(\iota_N(x))$$

defines an isometry from $\mathcal{H}_c$ to $\mathcal{H}_e$. Its range projection R_N is the weighted-matrix compression $\xi \mapsto Q_N \xi Q_N$, and $R_N \rightarrow I$ strongly.

Proof : The projection $Q_N - P_0$ is finite rank. Thus $Q_N \mathcal{P}_e Q_N = \mathcal{K}(Q_N H) + \mathbb{C} Q_N = \iota_N(\mathcal{P}_e)$. This also shows that the corners increase. They eventually contain each finite matrix and contain P_0 for every N , which proves norm density. Multiplication by Q_N converges in norm on compact operators, and $Q_N P_0 = P_0 Q_N = P_0$; hence Q_N is a strict approximate identity. Formula (8.8) follows by setting $k = j - N$ in (8.2). It implies the isometry assertion. In weighted matrix coordinates, $\mathcal{J}_N$ translates both indices by $-N$ and multiplies coefficients by $\tau^{N/2}$, so its range projection is R_N . Finite matrices give the asserted strong convergence. ■

The same formula holds for the symmetric embeddings. Indeed, if $D_c = (1 - \tau) \text{diag}(\tau^k)_{k \geq 0}$ and $D_e = (1 - \tau) \text{diag}(\tau^k)_{k \in \mathbb{Z}}$, then

$$\tau^{N/2} D_e^{\frac{1}{4}} \iota_N(x) D_e^{\frac{1}{4}} = J_N (D_c^{\frac{1}{4}} x D_c^{\frac{1}{4}}) J_N^*.$$

Thus the isometries are literal corner embeddings in the Hilbert–Schmidt standard forms. They preserve positive cones. A C^* -embedding, a GNS isometry and a state-preserving map are nevertheless different objects: the factor in (8.8) cannot be discarded.

8.3 The compact Casimir in every angular band

Put $k_\ell \stackrel{\text{def}}{=} \max(0, -\ell)$. The compact band ℓ is formed by the matrices $x_{k+\ell, k} = f(k)$ with $k \geq k_\ell$. For these indices set

$$(8.10) \quad \begin{aligned} c_{\ell, k} &= \sqrt{(1 - \tau^k)(1 - \tau^{k+\ell})}, \\ v_{\ell, k} &= 1 + \tau^{-\ell} - (1 + \tau)\tau^k - r^{-\ell}(c_{\ell, k+1} + c_{\ell, k}). \end{aligned}$$

In particular $c_{\ell, k_\ell} = 0$.

Theorem 8.2 (Angular Jacobi expression) *The compact Casimir (8.3) has the band expression*

$$(8.11) \quad \begin{aligned} (\mathcal{C}_{c, \ell} f)(k) &= (\tau^{-k}(1 + \tau^{-\ell}) - (1 + \tau)) f(k) \\ &\quad - r^{-\ell} \tau^{-k} (c_{\ell, k+1} f(k+1) + c_{\ell, k} f(k-1)). \end{aligned}$$

The term at $k-1$ is zero at the left endpoint. Its closed form is

$$(8.12) \quad \mathcal{E}_{c, \ell}(f) = (1 - \tau) \left(r^{-\ell} \sum_{k \geq k_\ell} c_{\ell, k+1} |\Delta f(k)|^2 + \sum_{k \geq k_\ell} v_{\ell, k} |f(k)|^2 \right),$$

on the maximal finite-energy domain in $\ell^2(\{k \geq k_\ell\}, w)$. Finite supports are a form core. Furthermore

$$(8.13) \quad v_{\ell, k} \geq (1 - r^{-\ell})^2,$$

with $v_{0, k} = 0$. Thus the nonzero-band form domains are precisely the counting-measure ℓ^2 spaces on the corresponding half-lines.

Proof : We first identify the representation-theoretic operator and then its closure. Let $s \geq 0$. In the concrete compact model, $b^* \gamma$ belongs to the sphere and its scalar matrix in the sphere realization raises the index with coefficient $r^k \sqrt{1 - \tau^{k+1}}$. Hence $(b^* \gamma)^s$, up to a nonzero scalar independent of k , has coefficient

$$(8.14) \quad g_s(k) = r^{sk} \sqrt{(\tau^{k+1}; \tau)_s}, \quad k \geq 0.$$

Here $(a; \tau)_s = \prod_{j=0}^{s-1} (1 - a\tau^j)$, including the empty product at $s = 0$.

The $m+1$ linearly independent elements $(b^* \gamma)^s A_c^j$, $0 \leq j \leq m$, are right-torus fixed, have left weight $-2s$ and have polynomial degree at most $2(s+m)$. The Peter–Weyl polynomial filtration and torus weights in [Arh26, Section 5.3] show that they span exactly the coefficients with fixed column, left label s and spins $s, s+1, \dots, s+m$. Orthogonality therefore identifies the spin $s+m$ coefficient with g_s times the polynomial of degree m orthogonal for the weights $w_k g_s(k)^2$. These weights are proportional to

$$(8.15) \quad \nu_s(k) = \tau^{(s+1)k} \frac{(\tau^{s+1}; \tau)_k}{(\tau; \tau)_k}.$$

Polynomials in τ^k are dense for this finite measure, and $g_s(k) > 0$, so the resulting family is complete in the band.

For completeness the needed polynomial difference calculation is explicit. On polynomials let

$$(8.16) \quad \begin{aligned} (L_s h)(k) &= B_s(k)(h(k) - h(k+1)) + D_s(k)(h(k) - h(k-1)), \\ B_s(k) &= \tau^{s-k}(1 - \tau^{s+k+1}), \quad D_s(k) = \tau^{-k}(1 - \tau^k). \end{aligned}$$

The ratio $\nu_s(k+1)/\nu_s(k)$ proves detailed balance. Summation by parts has vanishing boundary terms on polynomials. Substitution of $h(k) = \tau^{mk}$ proves triangularity with leading coefficient

$$\lambda_{s,m} = \tau^{-m}(1 - \tau^m)(1 - \tau^{2s+m+1}).$$

Consequently its orthogonal polynomial of degree m is an eigenfunction. Since

$$(8.17) \quad \Lambda_{s+m} - \Lambda_s = \tau^{-s} \lambda_{s,m},$$

the compact Casimir on the polynomial band is $M_{g_s}(\Lambda_s + \tau^{-s} L_s)M_{g_s}^{-1}$. The ratio

$$\frac{g_s(k+1)}{g_s(k)} = r^s \sqrt{\frac{1 - \tau^{k+s+1}}{1 - \tau^{k+1}}}$$

gives (8.11) for $\ell = s$. The negative band $-s$ follows by taking adjoints and replacing k by $k - s$. This operation changes the Hilbert norm by the fixed scalar r^s and intertwines the two compact Casimirs.

Multiplying (8.11) by $\overline{w_k f(k)}$ and summing gives (8.12) for finite supports. The elementary inequality $\sqrt{(1-u)(1-v)} \leq 1 - (u+v)/2$ gives the more precise bound

$$v_{\ell,k} \geq (1 - r^{-\ell})^2 + (1 + \tau)\tau^k \left(\frac{r^{-\ell} + r^\ell}{2} - 1 \right),$$

proving (8.13). All coefficients in the form are bounded for fixed ℓ , and the potential has a positive lower bound when $\ell \neq 0$. In that case the form norm is equivalent to the counting ℓ^2 norm, which proves closedness and the finite-support core property. At $\ell = 0$ the form is $(1 - \tau) \sum_{k \geq 0} (1 - \tau^{k+1}) |\Delta f(k)|^2$. Its conductances are bounded above and below by positive constants. Truncating a finite-energy sequence and retaining its last value on the remaining tail approximates it in form norm. A constant tail can then be tapered to zero over J edges at energy cost at most $(1 - \tau)|c|^2/J$, after moving the taper sufficiently far out to make its weighted Hilbert norm small. This proves the same core assertion. Closedness follows either directly by Fatou or from the gradient realization.

Finally each polynomial eigenvector $g_s p_m$ has finite energy: for $s > 0$ it decays geometrically, and for $s = 0$ its differences are $O(\tau^k)$. The finite-support summation-by-parts identity and the form-core property give the weak eigenvalue identity on the entire form domain. Thus these vectors belong to the associated operator domain with the desired eigenvalues. Their completeness identifies that self-adjoint operator with (8.3). This proves equality of the closed realizations, not merely of their formal expressions. $\blacksquare$

Corollary 8.3 (Oscillator expression) *On H_+ put $d_c e_k \stackrel{\text{def}}{=} \sqrt{\tau^{-k} - 1} e_{k-1}$, with $d_c e_0 = 0$, and $h_0 = A_c^{-1} - \frac{1+\tau}{2}$. On finite matrices,*

$$(8.18) \quad \mathcal{C}_c(x) = h_0 x + x h_0 - d_c^* x d_c - \tau d_c x d_c^*.$$

Moreover, $\tau^{\frac{N}{2}} J_N d_c J_N^*$ converges on every finite-support vector in H to $D = V A^{-\frac{1}{2}}$. The latter is the jump operator of the plane expression.

Proof : One has $d_c^* d_c = A_c^{-1} - 1$ and $\tau d_c d_c^* = A_c^{-1} - \tau$. Matrix multiplication gives (8.11). After translation the coefficient of the lowering shift is $\sqrt{\tau^{-k} - \tau^N}$ for $k \geq -N$, with zero coefficient at $-N$. It converges to $\tau^{-\frac{k}{2}}$ at every fixed k . The complete positivity of the compact semigroup is an input already proved by convolution states; it is not deduced here from this formal expression with unbounded operators. ■

8.4 Bounded band defects, resolvents and Mosco convergence

Define the nonnegative self-adjoint operator on $\mathcal{H}_e$

$$(8.19) \quad A_N = \tau^N \mathcal{J}_N \mathcal{C}_c \mathcal{J}_N^* \oplus 0_{R_N^\perp \mathcal{H}_e}.$$

Its band restriction is denoted by $A_{N,\ell}$. On H_w let $Uf(k) = rf(k+1)$. This is a bilateral unitary. The zero-extended band at $N=0$ will be denoted by $A_{0,\ell}$.

Theorem 8.4 (A bounded defect in each band) *For every fixed $\ell \in \mathbb{Z}$ there is a bounded self-adjoint operator B_ℓ on H_w such that*

$$(8.20) \quad A_{N,\ell} = \mathcal{C}_{e,\ell} + \tau^N U^N B_\ell U^{-N}, \quad \text{dom } A_{N,\ell} = \text{dom } \mathcal{C}_{e,\ell}.$$

One may take $B_\ell = A_{0,\ell} - \mathcal{C}_{e,\ell}$ on finite supports and $\|B_\ell\| \leq M_\ell$, where

$$(8.21) \quad M_\ell = \max\{1 + \tau, (1 + \tau^{-\ell})\tau^{1-k_\ell}\} + 2r^{1-\ell} \max\{\tau^{-k_\ell}, 1 + \tau^\ell\}.$$

In particular, for $\lambda > 0$ and $t \geq 0$, we have

$$(8.22) \quad \|(A_{N,\ell} + \lambda)^{-1} - (\mathcal{C}_{e,\ell} + \lambda)^{-1}\| \leq \lambda^{-2} M_\ell \tau^N,$$

$$(8.23) \quad \|e^{-tA_{N,\ell}} - e^{-t\mathcal{C}_{e,\ell}}\| \leq t M_\ell \tau^N.$$

Proof : Pass from weighted sequences to the counting orthonormal basis by $f(k) \mapsto \sqrt{w_k} f(k)$. The diagonal of $A_{0,\ell} - \mathcal{C}_{e,\ell}$ is $-(1 + \tau)$ for $k \geq k_\ell$, and $-(1 + \tau^{-\ell})\tau^{-k}$ for $k < k_\ell$. Its supremum in modulus is bounded by the first term in (8.21). The entry joining k and $k+1$ is

$$b_\ell(k) = \begin{cases} r^{-\ell-1} \tau^{-k} (1 - c_{\ell,k+1}), & k \geq k_\ell, \\ r^{-\ell-1} \tau^{-k}, & k < k_\ell. \end{cases}$$

Using $1 - \sqrt{(1-u)(1-v)} \leq u + v$ bounds the first case by $r^{1-\ell}(1 + \tau^\ell)$ and the second by $r^{1-\ell}\tau^{-k_\ell}$. The tridiagonal Schur bound proves boundedness with (8.21). These calculations include the edge crossing the artificial left endpoint.

Finite supports are a form core for both operators: for $A_{0,\ell}$ combine the compact form core with finite supports in its zero summand; for $\mathcal{C}_{e,\ell}$ use (8.4). On this common form core the two energies differ by $\langle B_\ell f, f \rangle$, so their form norms, after adding $(1 + \|B_\ell\|)\|f\|^2$, are equivalent. Their completions therefore have the same form domain, and polarization extends the bounded difference identity to that domain. The representation theorem for closed semi-bounded forms [Kat95, Theorem VI.2.1] now gives $A_{0,\ell} = \mathcal{C}_{e,\ell} + B_\ell$ with equality of operator domains. This argument does not use finite supports as an operator core. The dilation identity $U^{-1} \mathcal{C}_{e,\ell} U = \tau \mathcal{C}_{e,\ell}$ follows directly from the closed form. Translation of the compact corner now gives (8.20). The resolvent identity proves (8.22). The bounded-perturbation Duhamel formula [EN00, Corollary III.1.7], with both heat semigroups contractive, proves (8.23). ■

In particular the test bands $\ell = 1$ and $\ell = -1$ contract to exactly (8.5), without a change of Casimir normalization. The bounds are not uniform over all angular momenta. That distinction is necessary, as Proposition 8.7 below shows.

Theorem 8.5 (Full Hilbert-space and form contraction) *The operators A_N converge to $\mathcal{E}_e$ in the strong resolvent sense. Their closed forms*

$$(8.24) \quad q_N(\xi) = \tau^N \mathcal{E}_c(\mathcal{J}_N^* \xi)$$

(with zero energy on $R_N^\perp \mathcal{H}_e$) converge to $\mathcal{E}_e$ in the sense of Mosco [Mos94]. The same is true for the lower semicontinuous extended-valued forms $\hat{q}_N$ defined by $\hat{q}_N(\xi) = q_N(\xi)$ when $\xi \in R_N \mathcal{H}_e$ and $\hat{q}_N(\xi) = +\infty$ otherwise. Their effective domains need not be dense in the ambient Hilbert space. Both versions also converge after tensoring with any finite-dimensional matrix Hilbert-Schmidt space. Finally

$$(8.25) \quad e^{-tA_N} \xi \rightarrow e^{-t\mathcal{E}_e} \xi$$

uniformly for t in every compact subset of $[0, \infty)$, for each $\xi \in \mathcal{H}_e$.

Proof : The resolvent and heat claims follow immediately from Theorem 8.4 on any finite collection of bands. Orthogonal band truncations and the uniform bounds λ^{-1} and 1 for the relevant resolvents and semigroups then prove the full assertions, including uniformity in compact time intervals.

Here is also a direct proof of Mosco convergence, which verifies the boundary realization independently. In band ℓ , set $a_N = k_\ell - N$ and, for $k \geq a_N$,

$$c_{\ell,k}^{(N)} = \sqrt{(1 - \tau^{N+k})(1 - \tau^{N+k+\ell})}, \quad v_{\ell,k}^{(N)} = 1 + \tau^{-\ell} - (1 + \tau)\tau^{N+k} - r^{-\ell}(c_{\ell,k+1}^{(N)} + c_{\ell,k}^{(N)}).$$

Substituting (8.9) into (8.12) gives

$$(8.26) \quad q_{N,\ell}(f) = (1 - \tau) \left(r^{-\ell} \sum_{k \geq a_N} c_{\ell,k+1}^{(N)} |\Delta f(k)|^2 + \sum_{k \geq a_N} v_{\ell,k}^{(N)} |f(k)|^2 \right).$$

The coordinates below a_N make no contribution. All summands are nonnegative, and for fixed ℓ, k their coefficients tend respectively to $r^{-\ell}$ and $(1 - r^{-\ell})^2$. If $\xi_N \rightarrow \xi$ in $\mathcal{H}_e$, every fixed matrix entry converges. Retain an arbitrary finite collection of summands in (8.26), pass to the limit and then take the supremum over finite collections. This gives

$$\mathcal{E}_e(\xi) \leq \liminf_N q_N(\xi_N).$$

For the upper bound, finite matrices are a form core for $\mathcal{E}_e$. For any fixed such matrix F , one has $R_N F = F$ eventually and $q_N(F) \rightarrow \mathcal{E}_e(F)$. Approximate a finite-energy ξ in form norm by finite matrices F_m , and choose an increasing sequence of cutoffs in N so that the error $|q_N(F_m) - \mathcal{E}_e(F_m)|$ is small and $R_N F_m = F_m$ before advancing to F_{m+1} . This diagonal choice is a strong recovery sequence. For infinite-energy ξ the recovery upper bound is automatic. The argument applies unchanged to the $+\infty$ version; its recovery sequences lie in the appropriate corners. It also applies entrywise to any fixed finite matrix amplification. Recall that Mosco convergence is equivalent to strong resolvent convergence [Mos94, Theorem 2.4.1 and Corollary 2.6.1], see also [KS03]. The argument above proves the two Mosco conditions directly, for both choices of the transported extended-valued form. $\blacksquare$

Remark 8.6 (Zero extension is not Markovian) The matrix-level statement refers to the genuine completely Dirichlet forms on the *varying corners*, or to the preceding quadratic-form

convergence after matrix amplification. It does not say that q_N with zero energy on the orthogonal complement is a Dirichlet form for the fixed full algebra. In fact it generally is not. At the algebraic level its heat implementation would act as $S_{\tau_N t}^c$ on $Q_N x Q_N$ and as the identity on the other three matrix blocks. Choose $i \geq -N$ and $j < -N$, and apply it to $(e_i + e_j)(e_i + e_j)^*$. The principal two-by-two matrix at i, j is

$$\begin{pmatrix} P_{\tau_N t}^c(i + N, i + N) & 1 \\ 1 & 1 \end{pmatrix}.$$

For $t > 0$ the return probability in the upper left entry is strictly less than one, so the determinant is negative. Thus the zero-extended Hilbert heat operator is not a positive algebra map. The completely positive comparison maps will instead be the compressed maps of Section 8.5.

Proposition 8.7 (Failure of global norm convergence) *For every $N \geq 0$, $t > 0$ and $\lambda > 0$,*

$$(8.27) \quad \|e^{-tA_N} - e^{-t\mathcal{C}_e}\| = 1,$$

$$(8.28) \quad \|(A_N + \lambda)^{-1} - (\mathcal{C}_e + \lambda)^{-1}\| = \lambda^{-1}.$$

The same equalities hold with the first operators replaced, respectively, by the compressed operators $\mathcal{J}_N e^{-t\tau^N \mathcal{C}_c} \mathcal{J}_N^$ and $\mathcal{J}_N(\lambda + \tau^N \mathcal{C}_c)^{-1} \mathcal{J}_N^*$. On any fixed finite collection of bands, however, the zero-extended operators satisfy the norm bounds (8.22)–(8.23).*

Proof : Fix $j = -N - 1$ and let $i \rightarrow +\infty$. The normalized matrix unit $\xi_i = w_j^{-\frac{1}{2}} e_{ij}$ belongs to $R_N^\perp \mathcal{H}_e$ and to band $\ell = i - j$. From (8.4), in form order,

$$\mathcal{C}_{e,\ell} \geq (1 - r^{-\ell})^2 M_{\tau^{-k}}.$$

The order-reversing property of the resolvent, equivalently its variational characterization, gives

$$0 \leq \langle (\lambda + \mathcal{C}_e)^{-1} \xi_i, \xi_i \rangle \leq \frac{1}{\lambda + (1 - r^{-\ell})^2 \tau^{-j}} \rightarrow 0.$$

Since the resolvent is bounded by λ^{-1} , its value at ξ_i tends to zero in norm. Also $e^{-2tu} \leq C_{t,\lambda}(\lambda + u)^{-1}$ for $u \geq 0$, so $\|e^{-t\mathcal{C}_e} \xi_i\| \rightarrow 0$. On these vectors e^{-tA_N} is the identity and $(\lambda + A_N)^{-1}$ is λ^{-1} times the identity. This proves the lower bounds in (8.27)–(8.28). The upper bounds follow because the differences are differences of positive operators in $[0, I]$ or $[0, \lambda^{-1}I]$.

For the compressed operators, the compact spectrum (8.3), with finite multiplicities $2l + 1$ and $\Lambda_l \rightarrow \infty$, makes both operators compact. By contrast $\mathcal{C}_e$ has eigenvalues τ^p of infinite multiplicity, with $\tau^p \rightarrow 0$ as $p \rightarrow +\infty$. The essential norms of its heat operator and resolvent are therefore 1 and λ^{-1} . Distance to a compact operator is at least the essential norm. The same upper bounds prove equality. $\blacksquare$

8.5 Contraction of spherical eigenoperators and completely positive maps

We now pass from Hilbert convergence to the norm topology of the quantum homogeneous spaces. This passage is not a consequence of strong resolvent convergence alone.

For $m, s \geq 0$ write the little Jacobi polynomial explicitly as

$$(8.29) \quad p_m(x; \tau^s, \tau^s; \tau) = \sum_{a=0}^m \frac{(\tau^{-m}; \tau)_a (\tau^{m+2s+1}; \tau)_a}{(\tau^{s+1}; \tau)_a (\tau; \tau)_a} (\tau x)^a.$$

It is the degree- m eigenpolynomial in (8.16), normalized to be one at zero. One can verify this directly by equating coefficients in that equation; orthogonality and uniqueness follow from its detailed balance and triangularity.

Theorem 8.8 (Norm convergence of nonradial spherical coefficients) *Fix $p \in \mathbb{Z}$ and $s \geq 0$. For $N \geq \max(0, p + s)$ set*

$$(8.30) \quad F_{p,s}^{(N)}(k) = \begin{cases} r^{sk} \sqrt{(\tau^{N+k+1}; \tau)_s} p_{N-p-s}(\tau^{N+k}; \tau^s, \tau^s; \tau), & k \geq -N, \\ 0, & k < -N. \end{cases}$$

This is the band coefficient of an operator $X_{p,s}^{(N)} \in \iota_N(\mathcal{P}_c)$ belonging to the compact spin $N - p$. It satisfies

$$(8.31) \quad A_{N,s} F_{p,s}^{(N)} = \lambda_{N,p} F_{p,s}^{(N)}, \quad \lambda_{N,p} = \tau^p (1 - \tau^{N-p})(1 - \tau^{N-p+1}).$$

Then $F_{p,s}^{(N)} \rightarrow f_{p,s}$ both in H_w and uniformly on $\mathbb{Z}$. In particular $X_{p,s}^{(N)} \rightarrow X_{p,s}$ in operator norm. The analogous assertions for negative bands follow by adjoints. The normalized rank-one spectral projections converge in norm in each fixed band.

Proof : The formula in Theorem 8.2 identifies the compact eigenfunction as $g_s p_m$. Translating and multiplying by an overall nonzero scalar gives (8.30); then (8.31) follows from (8.3).

Set $m = N - p - s$. In each term of (8.29) use

$$(\tau^{-m}; \tau)_a = (-1)^a \tau^{-ma + a(a-1)/2} \prod_{b=0}^{a-1} (1 - \tau^{m-b}).$$

For fixed k , the absolute value of the a -th summand in the translated polynomial is bounded, uniformly in N , by

$$(8.32) \quad \frac{\tau^{a(p+s+k+1) + a(a-1)/2}}{(\tau^{s+1}; \tau)_\infty (\tau; \tau)_\infty}.$$

The majorant is summable. Each fixed coefficient converges to its ${}_1\phi_1$ counterpart, and the square-root product in (8.30) converges to one. Dominated convergence proves pointwise convergence to (8.6).

To upgrade it to Hilbert convergence, choose an interval enclosing τ^p and no other point of $\tau^\mathbb{Z} \cup \{0\}$. For sufficiently large N the only eigenvalue of $A_{N,s}$ in this interval is $\lambda_{N,p}$, of multiplicity one. This follows directly from monotonicity of Λ_l and (8.31). The norm-resolvent convergence of Theorem 8.4 gives norm convergence of the corresponding projections to the projection onto $\mathbb{C}f_{p,s}$. Apply the projections to $f_{p,s}$ and normalize to get unit vectors $g_N \rightarrow f_{p,s}/\|f_{p,s}\|$. Each $F_{p,s}^{(N)}$ is a scalar multiple of g_N . Evaluation at a fixed coordinate where $f_{p,s}$ is nonzero, together with pointwise convergence, proves convergence of this scalar to $\|f_{p,s}\|$. Thus the claimed H_w convergence, and in particular uniform boundedness of the H_w norms, follows.

For $k \rightarrow -\infty$ the bound $|F_{p,s}^{(N)}(k)| \leq w_k^{-1/2} \|F_{p,s}^{(N)}\|_{H_w}$ makes the negative tails uniformly small. For $k \rightarrow +\infty$, (8.32) gives $|F_{p,s}^{(N)}(k)| \leq C_{p,s,r} r^{sk}$ when $s > 0$. When $s = 0$, its zeroth term is one and its remaining terms give $|F_{p,0}^{(N)}(k) - 1| \leq C_{p,r} \tau^k$ for all sufficiently large k , uniformly in N . The same estimates hold for the limit. Pointwise convergence on the finite remaining interval proves uniform convergence.

For $s > 0$ the adjoint of a band- s matrix has band- $(-s)$ coefficients shifted by s : specifically $f_{p,s}(k - s) = r^{-s^2} f_{p,-s}(k)$. Thus $r^{s^2} (X_{p,s}^{(N)})^*$ converges to $X_{p,-s}$ in both relevant norms, with the same eigenvalue. This also fixes the sign convention for the negative bands. ■

Theorem 8.9 (Point-norm quantum Markov contraction) For $x \in \mathcal{P}_e$ and $t \geq 0$ define

$$(8.33) \quad \tilde{S}_t^{(N)}(x) = \iota_N(S_{\tau_N t}^c(\rho_N(x))).$$

These are completely positive contractions; their value at $t = 0$ is $x \mapsto Q_N x Q_N$. For every $x \in \mathcal{P}_e$ and $T < \infty$,

$$(8.34) \quad \sup_{0 \leq t \leq T} \|\tilde{S}_t^{(N)}(x) - S_t^e(x)\| \rightarrow 0.$$

The same assertion holds after every fixed finite matrix amplification. The normal extensions converge point- σ -weakly on $B(H)$, also locally uniformly in time. On their corners the semigroups preserve ψ , are symmetric for its GNS and KMS realizations, and commute with its modular group.

Proof : Compression, S_t^c and ι_N are completely positive contractions. The identity $\rho_N \iota_N = \text{id}$ gives the composition law for (8.33), including its compression value at zero. Formula (8.8) proves invariant-weight preservation on the corner. The GNS implementation of (8.33) is exactly $\mathcal{J}_N e^{-t\tau^N \mathcal{L}_c} \mathcal{J}_N^*$: the two factors $\tau^{\frac{N}{2}}$ and $\tau^{-\frac{N}{2}}$ cancel. The modular group is diagonal on matrix bands, so the same bandwise implementation holds for the symmetric embedding and commutes with the modular group.

For a limiting eigenoperator $X = X_{p,\ell}$ choose X_N from Theorem 8.8. Both the compact and plane semigroups act on these operators with the respective eigenvalues $\lambda_{N,p}$ and τ^p . Contractivity gives

$$\sup_{t \leq T} \|\tilde{S}_t^{(N)} X - S_t^e X\| \leq 2\|X_N - X\| + T|\lambda_{N,p} - \tau^p| \|X\| \rightarrow 0.$$

The norm density of their span in $\mathcal{P}_e$ proves (8.34). Finite matrices over $\mathcal{P}_e$ are handled entrywise; complete contractivity supplies the needed uniform operator bounds.

For the normal statement, let F_m increase through finite coordinate projections to 1. Fix two vector test states. By Stinespring and Cauchy–Schwarz, the matrix entries of $\tilde{S}_t^{(N)}(x - F_m x F_m)$ are bounded by $2\|x\|$ times suitable square roots of entries of $\tilde{S}_t^{(N)}(1 - F_m)$. Here $\tilde{S}_t^{(N)}(1) = Q_N$. For fixed m , (8.34) applies to F_m and $F_m x F_m$. The limiting semigroup is normal and unital, so $S_t^e(F_m) \uparrow 1$ strongly. The relevant scalar functions are continuous in t ; monotonicity in m and compactness of $[0, T]$ make this convergence uniform in t . First choose m and then let $N \rightarrow \infty$. The matrix entries converge uniformly in time. Uniform boundedness and finite-rank approximation of trace-class test operators give point- σ -weak convergence. ■

Theorem 8.9 concerns the full noncommutative quotient, not merely radial functions. In particular it applies to P_0 as well as to all off-diagonal matrix units. It does not assert convergence in completely bounded norm, or uniform convergence over the unit ball of $\mathcal{P}_e$. In fact convergence over that unit ball is impossible for these compressed maps. For $j < -N$, $\tilde{S}_t^{(N)}(e_{jj}) = 0$, whereas the probability of no radial jump gives $(S_t^e(e_{jj}))_{jj} \geq e^{-2t\tau^{-j}}$. Letting $j \rightarrow -\infty$ yields $\|\tilde{S}_t^{(N)} - S_t^e\|_{B(\mathcal{P}_e)} \geq 1$, and hence also excludes completely bounded norm convergence.

8.6 A quantitative contraction of entrance laws

Let μ_t^c be the entrance law at the accumulation point for the compact radial heat process, and μ_t^e the corresponding plane law. For $t > 0$, both have no atom at that point. Define a probability vector on $\mathbb{Z}$ by

$$(8.35) \quad \mu_{N,t}(k) = \begin{cases} \mu_{\tau_N t}^c(N+k), & k \geq -N, \\ 0, & k < -N. \end{cases}$$

Theorem 8.10 (Total-variation estimate) *For every $N \geq 0$ and $t > 0$,*

$$(8.36) \quad \|\mu_{N,t} - \mu_t^e\|_{\ell^1_{\mathbb{Z}}} \leq 2 \left(1 - e^{-(1+\tau)\tau^N t}\right) \leq 2(1+\tau)t\tau^N.$$

Thus convergence is uniform on $0 < t \leq T$ for each finite T . At $t = 0$ both measures are interpreted as the Dirac mass at the common accumulation point, rather than as probability vectors on $\mathbb{Z}$.

Proof : The plane chain has rates $q(k, k \pm 1) = \tau^{-k}$. After translating the compact chain and changing time, its rates for $k \geq -N$ are

$$(8.37) \quad q_N(k, k+1) = \tau^{-k} - \tau^{N+1}, \quad q_N(k, k-1) = \tau^{-k} - \tau^N.$$

The second one is zero at $k = -N$. Complete this process to all of $\mathbb{Z}$ by making the points below $-N$ absorbing.

Both minimal chains are conservative. For the plane this follows from recurrence of the symmetric embedded walk and the holding times at a fixed site. For the compact embedded walk, its scale ratios telescope:

$$\prod_{k=1}^m \frac{q_c(k, k-1)}{q_c(k, k+1)} = \prod_{k=1}^m \frac{1 - \tau^k}{1 - \tau^{k+1}} = \frac{1 - \tau}{1 - \tau^{m+1}}.$$

The associated scale sum diverges, so the walk hits zero almost surely and returns there indefinitely. Its independent positive holding times there have divergent sum. This proves nonexplosion. The form-core statements ensure that the minimal chains are the specified self-adjoint realizations, not additional boundary extensions.

All q_N are dominated by q and

$$(8.38) \quad \sup_{k \in \mathbb{Z}} \sum_{\varepsilon \in \{-1, 1\}} (q(k, k + \varepsilon) - q_N(k, k + \varepsilon)) \leq c_N \stackrel{\text{def}}{=} (1 + \tau)\tau^N.$$

Above $-N$ the sum is exactly c_N . Below it, the sum is $2\tau^{-k} \leq 2\tau^{N+1} \leq c_N$. Couple the two chains from the same site, see [Lin92] for the coupling method. While they agree, use common jumps with rates q_N and extra jumps of the plane chain with the nonnegative rate differences. The separation hazard is at most c_N . Nonexplosion justifies the construction by localization and passage to the limit. Therefore separation before t has probability at most $1 - e^{-c_N t}$, and the ℓ^1 distance of the transition rows is at most twice this probability.

Now let the common starting site tend to $+\infty$. The compact and plane transition rows converge in ℓ^1 to their respective entrance laws, by [Arh26, Proposition 3.3] and Proposition 7.12. Translation of the compact law is exactly (8.35). Passing to the limit proves (8.36). ■

The same coupling also bounds the operator norm of the difference between the plane radial semigroup and the completed, absorbing-below- $-N$ radial semigroup by $2(1 - e^{-c_N t})$. This completed radial semigroup differs from the compressed comparison at its outside points. Neither statement contradicts Proposition 8.7 about full angular Hilbert operators.

In the full plane-motion representation $M_e = B(H) \bar{\otimes} L^\infty(\mathbb{T})$ define, for $t > 0$,

$$(8.39) \quad \widehat{\omega}_{N,t}(a) = \sum_k \mu_{N,t}(k) \int_{\mathbb{T}} a_{kk}(z) dm(z), \quad \omega_t^e(a) = \sum_k \mu_t^e(k) \int_{\mathbb{T}} a_{kk}(z) dm(z).$$

Corollary 8.11 *These are normal states and*

$$(8.40) \quad \|\widehat{\omega}_{N,t} - \omega_t^e\| = \|\mu_{N,t} - \mu_t^e\|_1 \leq 2(1 - e^{-c_N t}).$$

The equality of norms also holds on $C_0(E_r(2))$. At zero the common functional on that C^ -algebra is the Haar idempotent of the torus, not a normal state of M_e and not the counit.*

Proof : The upper bound follows directly from the diagonal formula. Finite diagonal sign tests lie in $C_0(E_r(2))$ and recover arbitrarily large finite subsums of the ℓ^1 norm, proving the reverse inequality. The description at zero follows from torus averaging and evaluation at the accumulation point. It annihilates each finite diagonal spectral projection, so cannot be normal on M_e . ■

The transported states $\widehat{\omega}_{N,t}$ are not asserted to form convolution semigroups for the *fixed* plane coproduct. Their semigroup law belongs to the transported compact structure. The limiting convolution semigroup starts at the torus idempotent; its action on the quotient starts at the identity.

8.7 Compatibility with the quantum-group contraction

This section uses a published theorem, rather than deriving coproduct convergence from the preceding Hilbert convergence. Write $\mathcal{A}_c = C(\mathrm{SU}_r(2))$, $\mathcal{A}_e = C_0(E_r(2))$. In the supplied compact convention

$$\Delta_c(b) = b \otimes b - r\gamma^* \otimes \gamma, \quad \Delta_c(\gamma) = \gamma \otimes b + b^* \otimes \gamma.$$

In the Euclidean convention

$$\Delta_e(v) = v \otimes v, \quad \Delta_e(n) = v \otimes n + n \otimes v^*.$$

Their faithful concrete representations differ in the circular orientation needed for the contraction. Put

$$(8.41) \quad \kappa_0(b) \stackrel{\text{def}}{=} v\sqrt{1-AP_0}, \quad \kappa_0(\gamma) \stackrel{\text{def}}{=} n^*P_0, \quad \kappa_N \stackrel{\text{def}}{=} \mathrm{Ad}(v^N)\kappa_0.$$

Here the first expression is functional calculus only on P_0H . On matrix entries κ_0 is the corner embedding followed by $z \mapsto \bar{z}$. It is a $*$ -isomorphism onto $P_0\mathcal{A}_eP_0$, and κ_N is one onto $Q_N\mathcal{A}_eQ_N$. For the sphere and plane twists of the manuscripts it satisfies

$$(8.42) \quad \kappa_N\theta_c(x) = \Theta_e(\iota_N(x)).$$

In particular it identifies the *right* fixed-point corners, without silently replacing them by left quotients. This is why the adjoint on n in (8.41) matters.

Proposition 8.12 (Published C^* -contraction input) *With the above conventions, define completely positive contractions*

$$(8.43) \quad \Gamma_N(a) \stackrel{\text{def}}{=} (\kappa_N \otimes \kappa_N)\Delta_c(\kappa_N^{-1}(Q_N a Q_N)), \quad a \in \mathcal{A}_e.$$

Then $\|\Gamma_N(a) - \Delta_e(a)\| \rightarrow 0$ for every $a \in \mathcal{A}_e$.

Source and convention check. This is the real positive parameter instance of Woronowicz's C^* -contraction [Wor92], explicitly stated as [RR21, Theorem 6.6, equation (6.7)]. In that source the braiding is trivial for $q = r > 0$ and the tensor products are ordinary spatial tensor products. Its concrete compact generator in (6.1) has the reverse circular shift, giving exactly $\gamma \mapsto n^*P_0$ in our convention. Its inner automorphisms are $\text{Ad}(v^N)$. The norm limit on the increasing corner algebra extends to every a by contractivity. We use that published theorem with these substitutions; no claim of a new coproduct-contraction theorem is made here. In particular this input includes the identification of the closed affiliated sums, something that a formal limit on generators alone would not establish.

There is also a useful direct check on the norm topology of this input. Inside the initial corner write $\alpha = \kappa_0(b)$, $\gamma_0 = \kappa_0(\gamma)$ and $t_N = (\tau A; \tau)_N P_0$. Then $\alpha^N v^{-N} = t_N^{\frac{1}{2}}$ converges in norm to an element invertible in the corner. Put $a_1 \stackrel{\text{def}}{=} \alpha \otimes \alpha$ and $b_1 \stackrel{\text{def}}{=} -r\gamma_0^* \otimes \gamma_0$, so $a_1 b_1 = \tau b_1 a_1$. The finite q -binomial expansion [Kas95, Proposition IV.2.2], applied to the pair (b_1, a_1) since $a_1 b_1 = \tau b_1 a_1$, of

$$U_N = \Delta_c(\alpha^N)(v^{-N} \otimes v^{-N}) = \sum_{j=0}^N \begin{bmatrix} N \\ j \end{bmatrix}_{\tau} b_1^j (t_{N-j}^{\frac{1}{2}} \otimes t_{N-j}^{\frac{1}{2}}) (v^{-j} \otimes v^{-j})$$

converges in norm: the j -th term has norm at most $r^j/(\tau; \tau)_{\infty}$, and each fixed term converges. Here the compact coproduct is transported to the initial corner. For a in that corner,

$$\Gamma_N(a) = U_N^* \Delta_c(t_N^{-\frac{1}{2}} a t_N^{-\frac{1}{2}}) U_N.$$

Thus the norm limit exists on the initial corner. The identity $\Gamma_N(\text{Ad}(v^m)a) = \text{Ad}(v^m \otimes v^m) \Gamma_{N-m}(a)$, valid for $N \geq m$, gives convergence on every translated corner, and contractivity gives convergence on the whole algebra. The identification of this norm limit with the prescribed Δ_e , including its affiliated generators, is the published part of the input. In these corner formulas $(a; \tau)_0$ means the corner unit when it acts as an operator.

Theorem 8.13 (Coactions and covariance) *Let $\delta_N = \Gamma_N|_{\mathcal{P}_e}$, with the identifications (8.42). Then $\delta_N(x) \rightarrow \delta_e(x)$ in norm for every $x \in \mathcal{P}_e$. In particular $\delta_e(\mathcal{P}_e) \subset \mathcal{A}_e \otimes \mathcal{P}_e$ in this model. The comparison dynamics satisfy the exact corner covariance identity*

$$(8.44) \quad \delta_N \tilde{S}_t^{(N)} = (\text{Id} \otimes \tilde{S}_t^{(N)}) \delta_N.$$

Both sides converge in norm to $\delta_e S_t^e = (\text{Id} \otimes S_t^e) \delta_e$, locally uniformly in t . Moreover, for $x \in \mathcal{P}_e$,

$$(8.45) \quad \sup_{0 \leq t \leq T} \|\tilde{S}_t^{(N)} x - S_t^e x\| \leq \|\Gamma_N x - \Delta_e x\| + 2(1 - e^{-c_N T}) \|x\|.$$

Proof : The dictionary (8.42) follows by applying κ_N to $\theta_c(e_{ij}) = e_{ij} \otimes z^{i-j}$. Its value is $e_{i-N, j-N} \otimes z^{j-i} = \Theta_e(e_{i-N, j-N})$. It follows that every $\delta_N(x)$ belongs to $\mathcal{A}_e \otimes \mathcal{P}_e$. Proposition 8.12 and closedness of that tensor product prove the first assertion and its stated inclusion. The inclusion is a feature of this contraction, not a general property of noncompact homogeneous spaces.

For x in the compressed corner, compact covariance transported by κ_N gives (8.44). For a general x , both sides first compress by Q_N , so the identity still holds. Since the maps are completely contractive, point-norm convergence extends to their tensor product with the identity on $\mathcal{A}_e$, by density of finite sums of elementary tensors. For the left-hand side, the family $\{S_t^e(x) : 0 \leq t \leq T\}$ is norm compact, and the uniformly bounded maps $\delta_N - \delta_e$, which

converge pointwise in norm, converge uniformly on this compact set. For the right-hand side, approximate $\delta_e(x)$ in $\mathcal{A}_e \otimes \mathcal{P}_e$ by finite sums of elementary tensors and use Theorem 8.9 uniformly on $[0, T]$. These two observations, together with complete contractivity and $\delta_N(x) \rightarrow \delta_e(x)$, prove convergence of the covariance identity locally uniformly in time.

The normal state in (8.39) restricts to the compact entrance state on $\kappa_N(\mathcal{A}_c)$: its diagonal integral there is exactly the sum with weights $\mu_{\tau_N t}^c(k)$. Hence

$$\tilde{S}_t^{(N)}x = (\text{Id} \otimes \hat{\omega}_{N,t})\Gamma_N(x), \quad S_t^e x = (\text{Id} \otimes \omega_t^e)\Delta_e(x).$$

Subtracting and using Corollary 8.11 proves (8.45) for $t > 0$. At zero, both states on $\mathcal{A}_e$ are the torus idempotent and the same calculation applies. This supplies a second proof of point-norm dynamical convergence, now using the quantum-group contraction input rather than the norm convergence of eigenoperators. ■

Remark 8.14 The deformation parameter r is fixed throughout; $N \rightarrow \infty$ enlarges the coordinate corner and changes the clock. There is no assertion that $\text{SU}_r(2)$ maps to $\text{E}_r(2)$ by a quantum-group homomorphism. The maps Γ_N are cornerwise transported coproducts, and the maps $\tilde{S}_t^{(N)}$ start at compression on the ambient algebra. Likewise the convolution states start at the torus Haar idempotent, not at the counit of the entire group. These distinctions are compatible with the no-Feller-lifting theorems of both manuscripts. The linking-groupoid and cocycle-deformation constructions of [DeC10, DK15] give related, but different, relations between the two quantum groups; they are not used as a substitute for the norm-contraction input.

8.8 Conclusions, dependencies and limits of the result

The central new analytic calculation is (8.11), with the positive potential (8.13). It yields the bounded band defect (8.20) and the full Mosco limit, selecting exactly the self-adjoint realization of Theorem 7.3. The nonradial spherical coefficients converge in both GNS and algebra norm. Consequently the contraction extends to completely positive dynamics on the entire quotient algebra. The elementary rate comparison (8.38) gives quantitative convergence of the entrance laws and of the associated normal states. Adding the published quantum-group contraction makes the dynamics compatible with the limiting coaction.

The convergence is stronger than a radial special-function limit, but weaker than global Hilbert operator-norm convergence, which is explicitly false for the comparison conventions of the question. Uniform norm estimates are available after restricting the angular momentum to any fixed finite set. The zero-extension used to formulate a self-adjoint operator on a common Hilbert space must not be confused with a completely positive extension to the whole von Neumann algebra.

This section does not establish a uniform completely bounded norm rate, a classification of covariant generators, a higher-rank contraction, or an independent reconstruction of Woronowicz's quantum-group contraction. The existence and integrated covariance of the limiting plane semigroup were established in Section 7; the present arguments prove their compatibility with the compact limit. The C^* -contraction itself and the underlying special-function transforms are established inputs. The additional conclusions here concern the chosen closed realizations, their nonradial form contraction and the completely positive heat maps.

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Data availability No data sets were generated during this study.

Bibliography

- [AV91] I. Ya. Aref'eva and I. V. Volovich. Quantum group particles and non-archimedean geometry. *Phys. Lett. B* 268 (1991), no. 2, 179–187. doi:10.1016/0370-2693(91)90801-V. Section numbering refers to CERN-TH.6137/91 (June 1991). 2
- [Arh26] C. Arhancet. A bridge between analysis on local fields and quantum groups: the compact case. Preprint (2026). Numbering refers to the companion submission version. 2, 3, 4, 5, 6, 7, 8, 13, 17, 19, 20, 24, 26, 38, 40, 47
- [AK26] C. Arhancet and C. Kriegler. Functional calculus and semilinear evolution equations for the Taibleson operator on non-Archimedean local fields. *J. Evol. Equ.* 26 (2026), no. 2, Paper No. 63. doi:10.1007/s00028-026-01216-8. Equation numbering refers to arXiv:2407.10508v5. 6
- [Baa95] S. Baa. Représentation régulière du groupe quantique des déplacements de Woronowicz. Recent advances in operator algebras (Orléans, 1992), *Astérisque* 232 (1995), 11–48. 19
- [BS93] S. Baa and G. Skandalis. Unitaires multiplicatifs et dualité pour les produits croisés de C^* -algèbres. *Ann. Sci. École Norm. Sup. (4)* 26 (1993), no. 4, 425–488. 28
- [BGP+14] A. Bendikov, A. Grigor'yan, C. Pittet and W. Woess. Isotropic Markov semigroups on ultra-metric spaces. *Russian Math. Surveys* 69 (2014), no. 4, 589–680. doi:10.1070/RM2014v069n04ABEH004907. 7
- [Bha97] R. Bhatia. Matrix analysis. Graduate Texts in Mathematics 169, Springer, New York, 1997. 37
- [Bia06] P. Biane. Le théorème de Pitman, le groupe quantique $SU_q(2)$, et une question de P. A. Meyer. In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX, *Lecture Notes in Math.* 1874, Springer, Berlin, 2006, 61–75. 4, 24
- [Cho74] M.-D. Choi. A Schwarz inequality for positive linear maps on C^* -algebras. *Illinois J. Math.* 18 (1974), 565–574. 23
- [Cip97] F. Cipriani. Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras. *J. Funct. Anal.* 147 (1997), no. 2, 259–300. 23
- [DeC10] K. De Commer. On a Morita equivalence between the duals of quantum $SU(2)$ and quantum $\tilde{E}(2)$. Preprint, arXiv:0912.4350 (2010). 50
- [DLMF] NIST Digital Library of Mathematical Functions. https://dlmf.nist.gov/, Release 1.2.8 of 2026-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl and M. A. McClain, eds. 8, 26, 32
- [EN00] K.-J. Engel and R. Nagel. One-parameter semigroups for linear evolution equations. Graduate Texts in Mathematics 194, Springer, New York, 2000. 42
- [FOT11] M. Fukushima, Y. Oshima and M. Takeda. Dirichlet forms and symmetric Markov processes. Second revised and extended edition. *De Gruyter Studies in Mathematics* 19, Walter de Gruyter, Berlin, 2011. 33
- [GR04] G. Gasper and M. Rahman. Basic hypergeometric series. Second edition, *Encyclopedia of Mathematics and its Applications* 96, Cambridge University Press, Cambridge, 2004. doi:10.1017/CBO9780511526251. 8, 26, 32
- [Izu02] M. Izumi. Non-commutative Poisson boundaries and compact quantum group actions. *Adv. Math.* 169 (2002), no. 1, 1–57. doi:10.1006/aima.2001.2053.
- [Kas95] C. Kassel. Quantum groups. Graduate Texts in Mathematics 155, Springer, New York, 1995. 49

- [Kat95] T. Kato. Perturbation theory for linear operators. Reprint of the 1980 edition, Classics in Mathematics, Springer, Berlin, 1995. [37](#), [42](#)
- [KNR] M. Kalantar, M. Neufang and Z.-J. Ruan. Poisson boundaries over locally compact quantum groups. Preprint, [arXiv:1111.5828](#).
- [Koc01] A. N. Kochubei. Pseudo-differential equations and stochastics over non-Archimedean fields. Monographs and Textbooks in Pure and Applied Mathematics 244, Marcel Dekker, New York, 2001. [6](#)
- [Koe94] H. T. Koelink. The quantum group of plane motions and the Hahn–Exton q -Bessel function. Duke Math. J. 76 (1994), 483–508. [3](#), [21](#)
- [KS92] T. H. Koornwinder and R. F. Swarttouw. On q -analogues of the Fourier and Hankel transforms. Trans. Amer. Math. Soc. 333 (1992), no. 1, 445–461. Numbering refers to [arXiv:1208.2521](#). [3](#), [4](#), [8](#), [12](#), [15](#), [21](#), [32](#), [39](#)
- [Koz02] S. V. Kozyrev. Wavelet theory as p -adic spectral analysis. Izv. Math. 66 (2002), no. 2, 367–376. [7](#)
- [KV00] J. Kustermans and S. Vaes. Locally compact quantum groups. Ann. Sci. École Norm. Sup. (4) 33 (2000), no. 6, 837–934. [20](#)
- [KS03] K. Kuwae and T. Shioya. Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry. Comm. Anal. Geom. 11 (2003), no. 4, 599–673. [43](#)
- [Lan95] E. C. Lance. Hilbert C^* -modules. A toolkit for operator algebraists. London Mathematical Society Lecture Note Series 210, Cambridge University Press, Cambridge, 1995. [35](#)
- [Lin76] G. Lindblad. On the generators of quantum dynamical semigroups. Comm. Math. Phys. 48 (1976), no. 2, 119–130. [35](#)
- [Lin92] T. Lindvall. Lectures on the coupling method. Wiley, New York, 1992. [47](#)
- [Mos94] U. Mosco. Composite media and asymptotic Dirichlet forms. J. Funct. Anal. 123 (1994), no. 2, 368–421. [43](#)
- [Nor97] J. R. Norris. Markov chains. Cambridge Series in Statistical and Probabilistic Mathematics 2, Cambridge University Press, Cambridge, 1997. [33](#)
- [Pal96] A. Pal. Haar measure on $E_q(2)$. Pacific J. Math. 176 (1996), no. 1, 217–233. [doi:10.2140/pjm.1996.176.217](#). [2](#), [19](#), [20](#), [27](#), [31](#), [39](#)
- [Pau02] V. Paulsen. Completely bounded maps and operator algebras. Cambridge Studies in Advanced Mathematics 78, Cambridge University Press, Cambridge, 2002. [36](#)
- [RS80] M. Reed and B. Simon. Methods of modern mathematical physics. I. Functional analysis. Second edition, Academic Press, New York, 1980. [14](#)
- [SSV10] R. L. Schilling, R. Song and Z. Vondraček. Bernstein functions. Theory and applications. De Gruyter Studies in Mathematics 37, Walter de Gruyter, Berlin, 2010. [37](#)
- [SV19] A. Skalski and A. Viselter. Convolution semigroups on locally compact quantum groups and noncommutative Dirichlet forms. J. Math. Pures Appl. 124 (2019), 59–105. [doi:10.1016/j.matpur.2018.04.007](#). [34](#), [36](#)
- [Tai75] M. H. Taibleson. Fourier analysis on local fields. Mathematical Notes 15, Princeton University Press, Princeton, NJ, 1975. [5](#)
- [Tak03] M. Takesaki. Theory of operator algebras. II. Encyclopaedia of Mathematical Sciences 125, Springer, Berlin, 2003. [22](#)
- [Tes00] G. Teschl. Jacobi operators and completely integrable nonlinear lattices. Mathematical Surveys and Monographs 72, American Mathematical Society, Providence, RI, 2000. [11](#), [12](#)
- [Vae05] S. Vaes. A new approach to induction and imprimitivity results. J. Funct. Anal. 229 (2005), no. 2, 317–374. [28](#)
- [VK89] L. L. Vaksman and L. I. Korogodskii. An algebra of bounded functions on the quantum group of motions of the plane, and q -analogues of Bessel functions. Dokl. Akad. Nauk SSSR 304 (1989), no. 5, 1036–1040; English translation in Soviet Math. Dokl. 39 (1989), no. 1, 173–177. [3](#), [21](#)
- [Wor91] S. L. Woronowicz. Quantum $E(2)$ group and its Pontryagin dual. Lett. Math. Phys. 23 (1991), no. 4, 251–263. [2](#), [19](#), [27](#)
- [Wor91b] S. L. Woronowicz. Unbounded elements affiliated with C^* -algebras and non-compact quantum groups. Comm. Math. Phys. 136 (1991), no. 2, 399–432. [27](#)

- [Wor92] S. L. Woronowicz. Quantum $SU(2)$ and $E(2)$ groups. Contraction procedure. Comm. Math. Phys. 149 (1992), no. 3, 637–652. [2](#), [19](#), [20](#), [27](#), [49](#)
- [Zun22] W. A. Zúñiga-Galindo. Non-Archimedean quantum mechanics via quantum groups. Nuclear Phys. B (2022), Paper No. 116021. [doi:10.1016/j.nuclphysb.2022.116021](#), [arXiv:2210.03071](#). [2](#)
- [AD99] H. Ahmedov and I. H. Duru. Green function on the quantum plane. J. Phys. A: Math. Gen. 32 (1999), 6255. [doi:10.1088/0305-4470/32/35/304](#), [arXiv:math/9812108](#). [27](#)
- [BCGST96] F. Bonechi, N. Ciccoli, R. Giachetti, E. Sorace and M. Tarlini. Free q -Schrödinger equation from homogeneous spaces of the 2-dim Euclidean quantum group. Comm. Math. Phys. 175 (1996), 161–176. [doi:10.1007/BF02101628](#). Numbering refers to [arXiv:hep-th/9409048v3](#), in particular Proposition 3.7 and (4.2). [27](#), [29](#)
- [DK15] K. De Commer and E. Koelink. A q -Hankel transform associated to the quantum linking groupoid for the quantum $SU(2)$ and $E(2)$ groups. Proc. Amer. Math. Soc. 143 (2015), 2515–2526. [arXiv:1308.2859](#). [50](#)
- [RR21] A. Rahaman and S. Roy. Quantum $E(2)$ groups for complex deformation parameters. Rev. Math. Phys. 33 (2021), no. 6, 2150021. [doi:10.1142/S0129055X21500215](#). Numbering refers to [arXiv:2004.10005v4](#), Theorem 6.6 and (6.7). [39](#), [49](#)

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