

A bridge between analysis on local fields and quantum groups: the compact case

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Abstract

Let K be a non-Archimedean local field with residue cardinality Q and let $d \geq 1$. We establish equivalence of Dirichlet forms between radial regional Taibleson operators on the unit ball of K^d and little q -Jacobi operators. Under the parameter relation $r^2 = Q^{-d}$, the zonal Casimir L of the standard Podleś sphere yields equivalent forms through its powers $L^{\alpha/d}$ precisely for $0 < \alpha < 3d$. Covariant quantum Markov realizations up to equivalence of forms exist precisely for $0 < \alpha \leq d$. We also relate this correspondence to boundary energies: harmonic extensions on the tree of balls recover a Taibleson energy, while the energy per level of quantum Poisson integrals converges to a multiple of the Casimir energy. Finally, we extend the correspondence and the sharp covariance threshold to quantum projective spaces, identifying the Taibleson orders associated with their zonal Casimir operators.

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1 Introduction

Aref'eva and Volovich observed a coincidence between radial Haar integration on a non-Archimedean unit ball and the Haar state of the quantum group $SU_q(2)$ [AV91, Section 6.1, Proposition 1 and (71)–(75), p. 10]. Let K be a non-Archimedean local field with residue cardinality Q , and let $d \geq 1$ be an integer. Normalize additive Haar measure by $\mu(\mathcal{O}^d) = 1$, and use the maximum norm on K^d . Put $\tau \stackrel{\text{def}}{=} Q^{-d}$ and $r \stackrel{\text{def}}{=} \tau^{\frac{1}{2}}$. The volume observable $x \mapsto \|x\|_K^d$ has distribution

$$(1.1) \quad \int_{\mathcal{O}^d} F(\|x\|_K^d) dx = (1 - \tau) \sum_{n \geq 0} \tau^n F(\tau^n) = h(F(A)), \quad A \stackrel{\text{def}}{=} \gamma^* \gamma \in C(SU_r(2)).$$

Here h denotes the Haar state and F is a bounded Borel function on the compact set

$$(1.2) \quad X_\tau \stackrel{\text{def}}{=} \{\tau^n : n \geq 0\} \cup \{0\}.$$

Thus $r^2 = Q^{-d}$, and in dimension one the quantum parameter is $r = Q^{-\frac{1}{2}}$.¹ Equation (1.1) gives a normal state-preserving $*$ -isomorphism J from the radial algebra of the unit ball onto the von Neumann algebra $W^*(A)$, characterized by $J(F(\|\cdot\|_K^d)) = F(A)$.

Identities of this kind belong to a broader picture relating harmonic analysis over local fields and q -special functions. For GL_n over a local field, Macdonald's formula expresses the zonal spherical functions by Hall–Littlewood polynomials [Mac95, Chapter V], which are the specializations $q = 0$ of Macdonald polynomials [Mac95, Chapter VI]. Freund and Zabrodin proposed Macdonald polynomials, in which the p -adic regime corresponds to the parameters

The Jackson integral in [AV91, (72)–(75)] has parameter Q^{-1} . In the standard quantum-group convention, the Haar formula [AV91, (71)] uses q^{2n} . Comparison with (73) therefore gives $q^2 = Q^{-1}$. We use the letter r for this quantum parameter.

$q = 0$ and $t = p^{-1}$, as a conceptual basis for a connection between p -adic analysis and quantum groups [FZ92, Section 3]; see also [BF93, Section 14]. Haran interpreted limits $q \rightarrow 0$ of little q -Jacobi and q -Hahn polynomials as spherical functions on compact p -adic groups [Har01], and developed a q -interpolation between real and p -adic Markov chains on trees [Har08, Chapters 2–4]; see also [KO07] and [Onn06]. In these works the p -adic objects are reached by a degeneration $q \rightarrow 0$ in which the measure parameter is kept fixed. In (1.1), on the contrary, the quantum parameter r is fixed. The purpose of this paper is to show that this fixed-parameter coincidence is not only an identity of measures. It extends to an equivalence of energies between the Taibleson operator and a quantum Casimir operator, and the corresponding quantum dynamics live on a quantum homogeneous space, the standard Podleś sphere.

The principal conclusions are organized around four mechanisms. First, the measure correspondence extends to a comparison of closed energies, with sharp thresholds for the powers of the zonal Casimir. Second, the comparison has a boundary interpretation through the tree of balls and the random walk on the dual of $\mathrm{SU}_r(2)$. Third, the Podleś sphere supports both an exact expectation lifting of the Taibleson dynamics and a covariant Casimir dynamics, while the full quantum group satisfies a quantitative obstruction to liftings continuous at zero. Finally, the crystal limit and the quantum projective spaces show that the correspondence is compatible with degeneration and with a family of higher-dimensional compact homogeneous spaces. These four parts use the same radial measure but distinguish carefully between equality of distributions, equivalence of closed forms, convergence of generators and complete positivity of the associated dynamics.

The analytic input is the regional Taibleson operator R^α of order $\alpha > 0$ on $\mathcal{O}^d$, normalized so that the corresponding full-space Fourier multiplier has symbol $\|\xi\|_K^\alpha$, see (2.3). For $d = \alpha = 1$ it is the regional version of the Vladimirov operator, whose potential theory over $\mathbb{Q}_p$ was developed by Haran [Har93]. Its restriction R_{rad}^α to radial functions acts on the weighted sequence space $\ell_{\mathbb{N}_0}^2(w)$, where $w_n \stackrel{\mathrm{def}}{=} (1 - \tau)\tau^n$, and its eigenvalues are 0 and

$$\lambda_n = \rho^{n+1} - \kappa, \quad n \geq 0, \quad \rho \stackrel{\mathrm{def}}{=} Q^\alpha, \quad \kappa \stackrel{\mathrm{def}}{=} \frac{1 - \tau}{1 - \frac{\tau}{\rho}}.$$

The eigenfunctions are Haar-type wavelets. Up to normalization they are the little 0-Laguerre functions which appear as spherical functions of $\mathrm{GL}_d(\mathcal{O})$ acting on $\mathcal{O}^d$ [KO07, Section 5.2].

On the quantum side we consider, for $0 < \eta < 1$, the birth–death operator

$$(1.3) \quad (L_\eta f)(k) = \tau\eta^{-k-1}(1 - \eta^{k+1})(f(k) - f(k+1)) + \eta^{-k}(1 - \eta^k)(f(k) - f(k-1)), \quad k \geq 0,$$

acting on the same space $\ell_{\mathbb{N}_0}^2(w)$. It is symmetric with respect to w , its eigenfunctions are the little η -Jacobi polynomials $k \mapsto p_n(\eta^k; \tau\eta^{-1}, 1; \eta)$ of [KLS10, Section 14.12], and its eigenvalues are

$$(1.4) \quad \Lambda_n(\eta) = \eta^{-n}(1 - \eta^n)(1 - \tau\eta^n), \quad n \geq 0.$$

For $\eta = \tau$, the operator L_τ is the zonal part of the Casimir operator of $\mathrm{SU}_r(2)$. Indeed, the matrix coefficients of $\mathrm{SU}_r(2)$ which are invariant under the two-sided action of the diagonal torus are the little τ -Legendre polynomials in A , see [VS88], [Koo89], [MMNNU91] and Proposition 5.3, and $\Lambda_l(\tau)$ is an affine normalization of the eigenvalue of the Casimir element on the spin- l representation (Remark 5.4). Our first main result compares these two operators.

Theorem 1.1 (Energy bridge) *Let $\alpha > 0$ and put $\eta \stackrel{\text{def}}{=} Q^{-\alpha}$. The closed quadratic forms $\mathcal{E}_R$ and $\mathcal{E}_\eta$ of R_{rad}^α and L_η have the same domain*

$$\text{dom } \mathcal{E}_R = \text{dom } \mathcal{E}_\eta = \left\{ f \in \ell_{\mathbb{N}_0}^2(w) : \sum_{k \geq 0} Q^{(\alpha-d)k} |f(k+1) - f(k)|^2 < \infty \right\}$$

and there exist constants $0 < c_1 \leq c_2$, depending only on Q , d and α , such that

$$c_1 \mathcal{E}_\eta(f) \leq \mathcal{E}_R(f) \leq c_2 \mathcal{E}_\eta(f)$$

for any f in this space. Moreover, for any integer $n \geq 0$,

$$(1.5) \quad \lambda_n = \Lambda_{n+1}(\eta) + 1 + \tau - \kappa - \tau \rho^{-n-1}.$$

If $\alpha = d$, then $\eta = \tau$ and one can take $c_1 = \frac{1-\tau}{1+\tau}$ and $c_2 = (1-\tau)^{-3}$. Finally, let $\vartheta > 0$ and put $s \stackrel{\text{def}}{=} \frac{\alpha}{\vartheta}$. Then $\text{dom } \mathcal{E}_R \subset \text{dom } L_{Q^{-\vartheta}}^{\frac{s}{2}}$, and the closed forms of R_{rad}^α and of $L_{Q^{-\vartheta}}^s$ are equivalent if and only if $\alpha < d + 2\vartheta$. In particular, the closed forms of R_{rad}^α and of $L_\tau^{\frac{\alpha}{d}}$ are equivalent if and only if $\alpha < 3d$.

The two operators of Theorem 1.1 have no common eigenvector besides the constants: the first is diagonal in a basis of step functions, the second in a basis of polynomials in η^k (Remark 4.3). The theorem therefore does not follow from a spectral identification; it is proved by reducing both forms to weighted nearest-neighbour energies. For $\alpha = d$, it shows that the parameter $r^2 = Q^{-d}$ of (1.1) is also the parameter for which the Taibleson operator and the zonal Casimir operator of $\text{SU}_r(2)$ have equivalent energies, and spectra which differ by a bounded sequence. The last assertion of Theorem 1.1 shows that each operator $L_{Q^{-\vartheta}}$ reproduces, through its powers, the Taibleson energies of all orders $\alpha < d + 2\vartheta$; in particular, the single operator L_τ attached to the parameter $r^2 = Q^{-d}$ reproduces the orders $\alpha < 3d$. It is proved in Theorem 4.6 by comparing the two eigenbases directly: for $\eta = Q^{-\vartheta}$, the coefficients of the little η -Jacobi polynomials in the Haar wavelet basis decay like $(\tau^{\frac{1}{2}}\eta)^{n-m}$ below the diagonal and like a Gaussian above it (Lemma 4.5). The threshold $d + 2\vartheta$ is attained by the function $k \mapsto \eta^k$, which is a polynomial of degree one in η^k but has infinite Taibleson energy for $\alpha \geq d + 2\vartheta$ (Proposition 4.7).

The energy bridge has a probabilistic interpretation for $\alpha = d$, which explains the appearance of the two operators. Let $\mathbb{T}$ be the rooted tree whose vertices are the balls of $\mathcal{O}^d$, a ball being joined to the balls of the next level which it contains. Its boundary is $\mathcal{O}^d$, and the harmonic measure of the simple random walk on $\mathbb{T}$ seen from the root is the Haar measure. On the quantum side, consider the discrete dual $\widehat{G}$ of $G = \text{SU}_r(2)$ and the Markov operator $\mathbf{P} = (\text{Id} \otimes \phi) \circ \widehat{\Delta}$ on $\ell^\infty(\widehat{G})$, where ϕ is the normalized quantum trace of the fundamental representation. By a theorem of Izumi [Izu02], see also [INT06], the Poisson boundary of this random walk is the Podleś sphere: the Poisson integral Θ is a bijection from $L^\infty(S_r^2)$ onto the space $H^\infty(\widehat{G}, \mathbf{P})$ of bounded harmonic elements, which is multiplicative for a suitable product. The energy of a harmonic element a at the level $n = 2s$ is $\mathcal{E}^{(n)}(a) = [2s+1]_r \text{Tr}(K^{-1}(\mathbf{P}(a^*a) - a^*a)_s)$, and its total energy is $\sum_n \mathcal{E}^{(n)}(a)$, see (6.4).

Theorem 1.2 (Boundary energies) *Let $0 < r < 1$, $\tau = r^2$ and $\mathbf{c}_\tau = \frac{2\tau}{(1+\tau)(1-\tau)^2}$.*

1. Suppose that $\tau = Q^{-d}$. For any $F \in L^2(\mathcal{O}^d)$, the energy of the harmonic extension of F to $\mathbb{T}$ is equal to $2(1-\tau) \langle F, \mathcal{M}^d F \rangle$, where $\mathcal{M}^d$ is the Fourier multiplier of $\mathcal{O}^d$ with symbol $\|\xi\|_K^d$ on nontrivial characters.

2. The restriction of $\mathbf{P}$ to the diagonal subalgebra of $\ell^\infty(\widehat{G})$ is a Markov chain on $\{(n, k) \in \mathbb{N}_0^2 : k \leq n\}$, whose second coordinate converges almost surely to a limit k_∞ . For the chain started at the trivial representation, the law of k_∞ is the Aref'eva–Volovich law $(1 - \tau)\tau^k$.
3. For any $x \in L^\infty(S_r^2)$, we have $\lim_{n \rightarrow \infty} \mathcal{E}^{(n)}(\Theta(x)) = \mathfrak{c}_\tau \sum_l \Lambda_l(\tau) \|x_l\|_{L^2(S_r^2)}^2$ in $[0, \infty]$, where x_l is the spin l component of x , and $\sup_n \mathcal{E}^{(n)}(\Theta(x))$ is bounded by a constant multiple of this sum. In particular $\lim_{n \rightarrow \infty} \mathcal{E}^{(n)}(\Theta(f(A))) = \mathfrak{c}_\tau \mathcal{E}_\tau(f)$ for any bounded function $f: \mathbb{N}_0 \rightarrow \mathbb{C}$. On the other hand, every element of $H^\infty(\widehat{G}, \mathbf{P})$ which is not a multiple of 1 has infinite total energy.

These assertions are proved in Theorem 6.1, Theorem 6.10 and Corollary 6.32. The transition probabilities of the diagonal chain were computed by Biane [Bia06, Section 4], in his approach to Pitman's theorem through $\mathrm{SU}_q(2)$. We show moreover that, for the chain started at the trivial representation, k_∞ is independent of the whole level process (Proposition 6.11). For $\tau = Q^{-d}$, the restriction of the first form to radial functions is, up to the factor $2(1 - \tau)$ and the bounded perturbation $\kappa(I - \mathbb{E}_0)$, the closed form of R_{rad}^d , whereas the energy density of the third part is $\mathfrak{c}_\tau \mathcal{E}_\tau$ on the zonal elements. Thus Theorem 1.1 for $\alpha = d$ compares the boundary energy of the tree of balls with the energy density at infinity of the dual of $\mathrm{SU}_r(2)$, and the two harmonic measures have the same radial law (Corollary 6.33). The total energies behave differently. On the diagonal, the quantum walk is asymptotically the radial walk on the Bruhat–Tits tree of degree $Q^d + 1$, up to “horizontal” edges whose conductances do not depend on the level and are proportional to the weights $w_k \beta_k$ of $\mathcal{E}_\tau$ (Proposition 6.4), so that each level contributes the same nearest-neighbour energy. The proofs rely on an algebraic property of the diagonal chain: it preserves, on each level, the polynomials of degree at most l in τ^k (Lemma 6.5). This leads to an explicit description of the Poisson integrals of the zonal spherical functions by q -Hahn polynomials (Proposition 6.9), from which we deduce the second part without using the theorem of Izumi. The coefficients of this description have a generating function of basic hypergeometric type, and their limits are Gaussian binomial coefficients (Proposition 6.12). They give an exact formula for the energy per level of these Poisson integrals, which is bounded by a multiple of the Casimir eigenvalue uniformly in the level (Proposition 6.30); this gives the third part for every element of $L^\infty(S_r^2)$. Finally, in the coordinates $(n - k, k)$, the diagonal chain is the random walk of a product of two birth–death networks, the horizontal one being given by the energy $\mathcal{E}_\tau$; this yields the Poisson kernel of the diagonal chain as a function of the corresponding operator, and shows that it is symmetric (Theorem 6.13). The entire function which appears is a Jacobi theta function and the Fredholm determinant of the Green operator of the vertical network, whose eigenvalues are proportional to the squares of the quantum dimensions $[n]_r$, and its inverse is the Laplace transform of the explosion time of this network (Theorems 6.15, 6.17 and 6.22). In particular, the harmonic measures seen from the vertices (k, k) are finite sums of products of Gaussian binomial coefficients (Theorem 6.26), and the Poisson integral of a bounded function converges to its boundary values at the optimal rate $(n - k + 1)\tau^{n-k}$ (Corollary 6.24).

The second step concerns quantum dynamics. A first attempt consists in lifting radial dynamics to quantum Markov semigroups on $L^\infty(\mathrm{SU}_r(2))$. The radial algebra $W^*(A)$ is the fixed-point algebra of the two-sided action of the torus. It is contained in the algebra $L^\infty(S_r^2)$ of the standard Podleś sphere $S_r^2 = \mathrm{SU}_r(2)/\mathbb{T}$ [Pod87], which is the fixed-point algebra of the right action only. In the concrete model, $L^\infty(S_r^2)$ is a copy of $B(\ell_{\mathbb{N}_0}^2)$ on which h restricts to the Gibbs state $\mathrm{Tr}((1 - \tau)\tau^N \cdot)$ of the harmonic oscillator, and $C(S_r^2)$ is the unitization of the compact operators (Proposition 5.5). Let $P_n \stackrel{\mathrm{def}}{=} \mathbf{1}_{[0, \tau^n]}(A)$ be the spectral projection corresponding to the ball $B_n = \pi^n \mathcal{O}^d$, that is $P_n = J(\mathbf{1}_{B_n})$. The maps

$$\Phi_n(x) = (1 - P_n)x(1 - P_n) + \tau^{-n}h(P_n x P_n)P_n$$

are Haar-preserving conditional expectations, and probability mixtures of them produce quantum Markov semigroups. Exact liftings of the Taibleson dynamics are obtained in this way; constructions from increasing conditional expectations also occur in [XZ21, Theorems 1.1 and 1.7]. Such liftings do not use the coproduct, and on $C(SU_r(2))$ they are not continuous at zero. We show that the quantum group structure enters through the Casimir operator and that the Podleś sphere is the natural domain of both dynamics.

Theorem 1.3 (Quantum dynamics on the Podleś sphere) *Let $0 < r < 1$ and $\tau = r^2$.*

1. *There exists a Haar-preserving, GNS-symmetric and KMS-symmetric quantum Markov semigroup $(\mathcal{T}_t)_{t \geq 0}$ on $L^\infty(S_r^2)$, commuting with the modular group of h , which leaves $C(S_r^2)$ invariant and is strongly continuous on $C(S_r^2)$. If $\tau = Q^{-d}$ and $\alpha > 0$, it can be chosen such that $\mathcal{T}_t J = J e^{-t R_{\text{rad}}^\alpha}$ for any $t \geq 0$.*
2. *There exists a Haar-preserving semigroup $(\mathcal{S}_t)_{t \geq 0}$ of unital completely positive maps on $C(S_r^2)$ which is strongly continuous, of the form $\mathcal{S}_t = (\text{Id} \otimes \omega_t) \circ \Delta$ for normal states ω_t invariant under the two-sided action of the torus, and such that $\mathcal{S}_t(u_{j0}^{(l)}) = e^{-t \Lambda_l(\tau)} u_{j0}^{(l)}$ for any matrix coefficient $u_{j0}^{(l)}$ of $C(S_r^2)$. Its restriction to $C^*(A)$ is $e^{-t L_\tau}$.*
3. *Every strongly continuous semigroup $(S_t)_{t \geq 0}$ of unital completely positive maps on $C(S_r^2)$ which is covariant, in the sense that $\Delta \circ S_t = (\text{Id} \otimes S_t) \circ \Delta$, leaves $C^*(A)$ invariant. Suppose that $\tau = Q^{-d}$ and $\alpha > 0$. There exists such a semigroup whose radial negative generator has a closed form equivalent to the one of R_{rad}^α if and only if $\alpha \leq d$. For $\alpha \leq d$, one can take the Bochner subordinate of $(\mathcal{S}_t)$ associated with the Bernstein function $\lambda \mapsto \lambda^{\frac{\alpha}{d}}$, whose radial negative generator is $L_\tau^{\frac{\alpha}{d}}$.*

These assertions are proved in Theorems 7.2, 7.7 and 7.10 and in Corollary 7.8. In part 1, the semigroup is a probability mixture of the expectations Φ_n ; we also establish uniform noncommutative L^p multiplier estimates and a bounded H^∞ functional calculus of angle zero for its negative generator on the mean-zero subspace (Theorem 7.4). In part 2, positivity is not visible on the Casimir eigenvalues. It is obtained from the radial semigroup itself: the law at time t of the radial process started at $0 \in X_\tau$ defines a probability measure μ_t , and $\omega_t = \mu_t \circ \mathbb{E}$, where $\mathbb{E}$ is the averaging over the two-sided torus action. In this dictionary the origin of the unit ball corresponds to the counit of $SU_r(2)$. In part 3, a covariant semigroup is a convolution semigroup by a state invariant under the left action of the torus, so that it acts diagonally on the zonal spherical functions. The obstruction for $\alpha > d$ is a Lévy–Khintchine type bound: the exponents of a positive semigroup diagonal in the zonal spherical functions grow at most like the Casimir eigenvalues (Proposition 7.9), whereas, by Theorem 1.1, the Taibleson energy of order α forces a growth like $\Lambda_l(\tau)^{\frac{\alpha}{d}}$. Thus, for $d < \alpha < 3d$, the operator $L_\tau^{\frac{\alpha}{d}}$ has the right energy but does not generate a positivity preserving semigroup, whereas the semigroup of part 1 realizes the Taibleson dynamics of every order without being covariant.

The third step explains why $SU_r(2)$ itself is not the right domain. Write b for the generator of $C(SU_r(2))$ denoted α in [NT13] and [CFK14]. It does not belong to $C(S_r^2)$.

Theorem 1.4 (Frontier between S_r^2 and $SU_r(2)$) *Let $0 < r < 1$ and $\tau = r^2$.*

1. *Suppose that $\tau = Q^{-d}$. Any family $(S_t)_{t \geq 0}$ of unital completely positive maps on $C(SU_r(2))$ whose restrictions to $C^*(A)$ are $J e^{-t R_{\text{rad}}^\alpha} J^{-1}$ satisfies $\|S_{\rho^{-j}}(b) - b\| \geq \delta > 0$ for any integer $j \geq 1$, with δ independent of j .*

2. There exists $\delta' > 0$ such that any unital completely positive map $S: C(SU_r(2)) \rightarrow C(SU_r(2))$ whose restriction to $C^*(A)$ is e^{-tL_τ} for some $0 < t \leq \Lambda_1(\tau)^{-1}$ satisfies $\|S(b) - b\| \geq \delta'$.
3. There exists $t_0 > 0$ such that for any $0 < t \leq t_0$ the linear map defined on the polynomial algebra of $SU_r(2)$ by $u_{jk}^{(l)} \mapsto e^{-t\Lambda_l} u_{jk}^{(l)}$, $l \in \frac{1}{2}\mathbb{N}_0$, does not extend to a positive map on $C(SU_r(2))$, whereas its restriction to the polynomial algebra of S_r^2 extends to the map $\mathcal{S}_t$ of Theorem 1.3.

Here $\Lambda_l = r^{-2l} - 1 - r^2 + r^{2l+2}$ for $l \in \frac{1}{2}\mathbb{N}_0$, which coincides with $\Lambda_l(\tau)$ for integers l . The mechanism is the same in the three assertions. A unital completely positive map with prescribed radial part is controlled on the shift b by the affinity between the entrance law at the origin and its unit translate (Lemma 8.1). Both the Taibleson and the Casimir dynamics have entrance laws with a persistent atom along times tending to zero; for the Casimir dynamics we prove this by a Paley–Zygmund argument based on the first two zonal spherical functions (Proposition 8.5). Theorem 1.4 is proved in Corollary 8.4 and Theorem 8.6. Part 3 shows in particular that the heat semigroup of the Casimir operator is not a quantum Markov semigroup on $C(SU_r(2))$, although its restriction to the Podleś sphere is one; for the structure of generating functionals of Lévy processes on $SU_q(2)$ we refer to [SS98] and [CFK14, Section 11].

The last step relates the fixed-parameter picture with the limits $q \rightarrow 0$ of [Har01], [Har08] and [KO07]. We denote by $\mathcal{M}^\alpha$ the Fourier multiplier of the compact group $\mathcal{O}^d$ which is zero on constants and equals $\|\xi\|_K^\alpha$ on nontrivial characters; on radial functions, $\mathcal{M}_{\text{rad}}^\alpha = R_{\text{rad}}^\alpha + \kappa(I - \mathbb{E}_0)$, where $\mathbb{E}_0$ is the mean.

Theorem 1.5 (Crystal limit) *Let $\alpha > 0$. For any $0 < \eta \leq Q^{-\alpha}$, put $s(\eta) \stackrel{\text{def}}{=} \frac{\alpha \log Q}{\log \frac{1}{\eta}} \in (0, 1]$. Then the semigroups $(e^{-tL_\eta^{s(\eta)}})_{t \geq 0}$ are Markovian, and $L_\eta^{s(\eta)}$ converges to $\mathcal{M}_{\text{rad}}^\alpha$ in the norm resolvent sense as $\eta \rightarrow 0^+$. For any $t > 0$, the semigroups converge in operator norm.*

The convergence of the eigenfunctions is a variant of the limits $q \rightarrow 0$ of little q -Jacobi polynomials of Haran [Har01] and Koornwinder–Onn [KO07, Theorem 2.1], in which the limit functions are the little 0-Laguerre functions, that is the spherical functions of $GL_d(\mathcal{O})$ acting on $\mathcal{O}^d$ [KO07, Section 5.2]. The new point is the convergence of Markov generators, which requires the subordination exponent $s(\eta) \rightarrow 0$. For $\eta = \tau$ the family contains the zonal Casimir operator of Theorem 1.3, and for $\eta = Q^{-\alpha}$ it contains the operator of Theorem 1.1.

Finally, the quantum group $SU_r(2)$ with $r^2 = Q^{-d}$ is the first member of a family. Let $\mathbb{CP}_q^{n-1}$ be the standard quantum complex projective space, that is the quotient of the quantum unitary group $U_q(n)$ by its quantum subgroup $U_q(n-1) \times \mathbb{T}$. By a theorem of Noumi, Yamada and Mimachi [NYM93], its zonal spherical functions are little q^2 -Jacobi polynomials, whose parameters are those of the eigenfunctions of L_η for $\eta = q^2$ and $\tau = q^{2(n-1)}$.

Theorem 1.6 (Quantum projective spaces) *Let $n \geq 2$ and $0 < q < 1$ with $q^{2(n-1)} = Q^{-d}$, that is $q^2 = Q^{-\alpha}$ with $\alpha = \frac{d}{n-1}$. The radial Haar distribution on $\mathcal{O}^d$ is the distribution of the zonal generator of $\mathbb{CP}_q^{n-1}$ under the Haar state, and the zonal part of the Casimir operator of $\mathbb{CP}_q^{n-1}$ is $L_{Q^{-\alpha}}$. The corresponding Casimir dynamics is a strongly continuous semigroup of unital completely positive maps on $C(\mathbb{CP}_q^{n-1})$, whose radial energy is equivalent to the Taibleson energy of order α . In particular, for $n = d + 1$, the Taibleson operator of order 1 on $\mathcal{O}^d$ corresponds in this way to $\mathbb{CP}_q^d$ with $q = Q^{-\frac{1}{2}}$. Moreover, let $\alpha' > 0$. The closed form of the power $L_{Q^{-\alpha}}^{\alpha'}$ is equivalent to the Taibleson form of order α' if and only if $\alpha' < d + 2\alpha$, and*

there exists a covariant strongly continuous semigroup of unital completely positive maps on $C(\mathbb{CP}_q^{n-1})$ whose radial energy is equivalent to the Taibleson energy of order α' if and only if $\alpha' \leq \alpha$. Finally, let S_q^{2n-1} be the quantum sphere $U_q(n)/U_q(n-1)$, which contains $\mathbb{CP}_q^{n-1}$, and let x_{nn} be the last diagonal coefficient of the fundamental corepresentation, which belongs to $C(S_q^{2n-1})$ but not to $C(\mathbb{CP}_q^{n-1})$. There exists $\delta > 0$ such that every unital completely positive map S on $C(S_q^{2n-1})$ whose restriction to the zonal subalgebra is the Casimir dynamics at a time $0 < t \leq \Lambda_1(Q^{-\alpha})^{-1}$ satisfies $\|S(x_{nn}) - x_{nn}\| \geq \delta$. The same holds, at the times $Q^{-\alpha'j}$, $j \geq 1$, for the Taibleson dynamics of any order $\alpha' > 0$.

This is proved in Theorems 10.3, 10.5 and 10.8. For $n = d + 1$, the last assertion shows that the Taibleson energies of the orders $\alpha' < d + 2$ on $\mathcal{O}^d$ are the energies of the powers $L_{Q^{-1}}^{\alpha'}$ of the zonal Casimir operator of $\mathbb{CP}_q^d$, and that the orders realized by covariant quantum Markov semigroups are the orders $\alpha' \leq 1$. For $n = 2$, we recover the Podleś sphere and the case $\alpha = d$ of Theorems 1.1 and 1.3. For $n = d + 1$, the quantum parameter $q = Q^{-\frac{1}{2}}$ does not depend on d , and the dimension d of $\mathcal{O}^d$ is reflected in the rank of the quantum group; this is the regime $t = Q^{-1}$ of Freund and Zabrodin [FZ92, Section 3].

The present paper is confined to the compact setting of the unit ball, compact quantum groups and their compact homogeneous spaces. The noncompact counterpart for the whole local field K^d and the locally compact quantum group $E_r(2)$ is treated in the companion paper [Arh26b]. None of the arguments below depends on that noncompact construction; the compact results established here instead provide its starting point.

Structure of the paper Section 2 gives the radial analysis of the Taibleson operator on the unit ball. Section 3 studies the little η -Jacobi birth–death operators and their Feller semigroups. Section 4 proves Theorem 1.1. Section 5 recalls the quantum group $SU_r(2)$, identifies its zonal spherical functions and describes the Podleś sphere in the concrete model. Section 6 studies the random walks on the tree of balls and on the dual of $SU_r(2)$, and proves Theorem 1.2. Section 7 constructs the quantum dynamics of Theorem 1.3, and Section 8 proves Theorem 1.4. Section 9 proves Theorem 1.5. Section 10 treats the quantum projective spaces and proves Theorem 1.6.

2 Radial analysis on the unit ball

2.1 Parameters, balls, and Fourier normalization

Let K be a non-Archimedean local field, $\mathcal{O}$ its ring of integers, π a uniformizer, and $Q = \#(\mathcal{O}/\pi\mathcal{O})$. We normalize $|\pi|_K = Q^{-1}$. On K^d , $d \geq 1$, we use the maximum norm and additive Haar measure dx normalized by $\mu(\mathcal{O}^d) = 1$. Fix a nontrivial additive character χ whose annihilator of $\mathcal{O}$ is $\mathcal{O}$, so that this measure is self-dual. We use $\widehat{f}(\xi) = \int f(x) \overline{\chi(x \cdot \xi)} dx$.

Inner products are linear in the first variable. Throughout the paper, $\alpha > 0$ and

$$(2.1) \quad \tau \stackrel{\text{def}}{=} Q^{-d}, \quad \rho \stackrel{\text{def}}{=} Q^\alpha, \quad r \stackrel{\text{def}}{=} \tau^{\frac{1}{2}}, \quad c \stackrel{\text{def}}{=} \frac{\rho - 1}{1 - \frac{\tau}{\rho}}, \quad \kappa \stackrel{\text{def}}{=} \frac{1 - \tau}{1 - \frac{\tau}{\rho}}, \quad \lambda_n \stackrel{\text{def}}{=} \rho^{n+1} - \kappa.$$

In particular, $0 < \tau < 1$, $\rho > 1$, $0 < \kappa < 1$, and $\lambda_0 = c > 0$. From Section 3 on, τ will also denote an arbitrary number of $(0, 1)$; when the local field is involved we always have $\tau = Q^{-d}$. The exponent p of an L^p -space will never denote the residue cardinality.

For any integer $n \in \mathbb{Z}$ put $B_n \stackrel{\text{def}}{=} \pi^n \mathcal{O}^d$ and $S_n \stackrel{\text{def}}{=} B_n \setminus B_{n+1}$. Then

$$(2.2) \quad \mu(B_n) = \tau^n, \quad \mu(S_n) = w_n \stackrel{\text{def}}{=} (1 - \tau)\tau^n, \quad \widehat{\mathbf{1}_{B_n}} = \tau^n \mathbf{1}_{B_{-n}}.$$

A radial function means a function of $\|x\|_K$, equivalently one that is constant on every S_n . In dimension greater than one, invariance under scalar units alone is not this condition. Radiality is equivalently invariance under $\mathrm{GL}_d(\mathcal{O})$, since that group is transitive on each S_n . Radial L^2 on the unit ball is therefore the weighted sequence space $\ell_{\mathbb{N}_0}^2(w)$, a radial function being identified with the sequence $(f_n)_{n \geq 0}$ of its values on the shells. The singleton $\{0\}$ has Haar measure zero. Bounded radial functions are identified with $\ell_{\mathbb{N}_0}^\infty$, and continuous radial functions on $\mathcal{O}^d$ with the space of convergent sequences, that is with $C(X_\tau)$ through $f_n = F(\tau^n)$, where X_τ is the set (1.2).

2.2 Full-space, regional, and killed realizations

The full-space Taibleson operator is the nonnegative self-adjoint Fourier multiplier

$$D^\alpha = \mathcal{F}^{-1} M_{\|\xi\|_K^\alpha} \mathcal{F}, \quad D^\alpha f(x) = c \int_{K^d} \frac{f(x) - f(y)}{\|x - y\|_K^{d+\alpha}} dy$$

on Bruhat–Schwartz functions. The second formula has the normalization of [AK26, (1.2)–(1.4)], with its numerator and integrand signs reversed simultaneously. For completeness, the normalization follows by testing on a character of unit frequency. The shell of radius Q contributes ρ^{-1} to $\int (1 - \chi(y_1)) \|y\|_K^{-d-\alpha} dy$; the shells of radii Q^j , $j \geq 2$, contribute $(1 - \tau)\rho^{-j}$. The sum equals $\frac{1-\tau}{\rho-1} = c^{-1}$. Scaling and $\mathrm{GL}_d(\mathcal{O})$ -invariance then give the general symbol.

On $\mathcal{O}^d$ define the *regional* operator on locally constant functions by

$$(2.3) \quad R^\alpha f(x) \stackrel{\text{def}}{=} c \int_{\mathcal{O}^d} \frac{f(x) - f(y)}{\|x - y\|_K^{d+\alpha}} dy.$$

These functions form a dense subspace of the Hilbert space $L^2(\mathcal{O}^d)$. The integral is finite because the numerator vanishes near $y = x$.

Proposition 2.1 *The operator (2.3) is essentially self-adjoint and nonnegative on the locally constant functions. If $\tilde{f}$ is the zero extension of such a function, then*

$$(2.4) \quad (D^\alpha \tilde{f})|_{\mathcal{O}^d} = R^\alpha f + \kappa f.$$

On the characters of $\mathcal{O}^d$, the closure of R^α has eigenvalue 0 for the trivial character and eigenvalue $\rho^j - \kappa$ for any character of conductor $j \geq 1$. Consequently, the Fourier multiplier $\mathcal{M}^\alpha$ of the compact group $\mathcal{O}^d$ which is zero on constants and equals $\|\xi\|_K^\alpha$ on nontrivial characters is $\mathcal{M}^\alpha = R^\alpha + \kappa(I - \mathbb{E}_0)$, where $\mathbb{E}_0 f = \int_{\mathcal{O}^d} f$.

Proof : For $x \in \mathcal{O}^d$ and $y \notin \mathcal{O}^d$, ultrametricity gives $\|x - y\|_K = \|y\|_K$. The external integral is thus

$$c \int_{K^d \setminus \mathcal{O}^d} \|y\|_K^{-d-\alpha} dy = c(1 - \tau) \sum_{j \geq 1} \rho^{-j} = \kappa.$$

This proves (2.4). Alternatively, on a nontrivial character of conductor j , the full-space symbol calculation splits into the regional integral and the same external constant, because the character integral on each external shell is zero. This gives $\rho^j - \kappa$. On constants the regional integral vanishes. Characters form an orthonormal basis of the Hilbert space $L^2(\mathcal{O}^d)$, and their finite linear combinations are exactly the locally constant functions. The resulting diagonal nonnegative operator has these finite combinations as a core, proving essential self-adjointness and the last assertion. $\blacksquare$

We keep the notation R^α for the closure of (2.3).

2.3 Exact radial matrices and wavelets

Proposition 2.2 *For a radial locally constant function on $\mathcal{O}^d$, with value f_n on S_n , we have*

$$(2.5) \quad (R_{\text{rad}}^\alpha f)_n = c(1-\tau)\rho^n \left[\sum_{j=1}^n \rho^{-j}(f_n - f_{n-j}) + \sum_{j \geq 1} \tau^j(f_n - f_{n+j}) \right], \quad n \geq 0.$$

Equivalently, the off-diagonal jump rates and the diagonal entries are

$$(2.6) \quad q_{nk} = cw_k \left(\frac{\rho}{\tau} \right)^{\min(n,k)} \quad (n \neq k),$$

$$(2.7) \quad R_{nn} = c \left[(1-\tau) \frac{\rho^n - 1}{\rho - 1} + \tau \rho^n \right].$$

Detailed balance $w_n q_{nk} = w_k q_{kn}$ holds.

Proof : If $x \in S_n$ and $y \in S_k$, $k \neq n$, then $\|x - y\|_K = Q^{-\min(n,k)}$. Multiplication by the mass w_k gives (2.6). The shell S_n contributes zero: the numerator vanishes identically on that shell, irrespective of the distance between its points. Splitting into $k < n$ and $k > n$ proves (2.5). Summation of the rates gives (2.7). The balance identity follows from the symmetry of $w_n w_k \left(\frac{\rho}{\tau} \right)^{\min(n,k)}$. ■

For any integer $n \in \mathbb{Z}$, define the function

$$(2.8) \quad g_n \stackrel{\text{def}}{=} \frac{\mathbf{1}_{B_{n+1}} - \tau \mathbf{1}_{B_n}}{\sqrt{(1-\tau)\tau^{n+1}}}.$$

Theorem 2.3 *On the unit ball, $1, (g_n)_{n \geq 0}$ form an orthonormal basis of $\ell_{\mathbb{N}_0}^2(w)$, and*

$$(2.9) \quad R_{\text{rad}}^\alpha 1 = 0, \quad R_{\text{rad}}^\alpha g_n = \lambda_n g_n, \quad \text{dom } R_{\text{rad}}^\alpha = \left\{ f : \sum_{n \geq 0} \lambda_n^2 |\langle f, g_n \rangle|^2 < \infty \right\}.$$

The space $\mathcal{C}$ of finite linear combinations of $1, g_0, g_1, \dots$ is an operator core. It coincides with the space of eventually constant sequences.

Proof : The numerator in (2.8) has integral zero and squared norm $(1-\tau)\tau^{n+1}$. Nested supports and zero means give orthogonality. The displayed orthonormal family spans every function constant on $S_0, \dots, S_{N-1}$ and constant on the remaining ball B_N . These spaces increase densely in $\ell_{\mathbb{N}_0}^2(w)$, and their union is the space of eventually constant sequences. Equation (2.4), applied to g_n for $n \geq 0$, together with (2.2) and the multiplier formula for D^α , gives (2.9): the Fourier transform of the numerator in (2.8) is $\tau^{n+1} \mathbf{1}_{S_{-n-1}}$, on which the symbol $\|\xi\|_K^\alpha$ is constant, equal to ρ^{n+1} . The diagonal spectral theorem gives the domain and the core. ■

Remark 2.4 Up to normalization, g_{n-1} is the little 0-Laguerre function $p_n^{\tau,0;0}$ of [KO07, Theorem 2.1], which takes the values 0, $-\frac{\tau}{1-\tau}$ and 1 on the levels $k < n-1$, $k = n-1$ and $k \geq n$. These functions are the spherical functions of $\text{GL}_d(\mathcal{O})$ acting on $\mathcal{O}^d$ [KO07, Section 5.2], after [Har01]; for $d = 1$ this interpretation goes back to Dunkl and Ramirez [DR75].

2.4 The radial Dirichlet form

For any $f \in \ell_{\mathbb{N}_0}^2(w)$, we put

$$(2.10) \quad \mathcal{E}_R(f) \stackrel{\text{def}}{=} \sum_{0 \leq m < k} w_m q_{mk} |f_m - f_k|^2 \in [0, \infty], \quad \text{dom } \mathcal{E}_R \stackrel{\text{def}}{=} \{f \in \ell_{\mathbb{N}_0}^2(w) : \mathcal{E}_R(f) < \infty\}.$$

Since $w_m q_{mk} = c(1 - \tau)^2 \tau^k \rho^m$ for $m < k$, we have, writing $k = m + j$,

$$(2.11) \quad \mathcal{E}_R(f) = c(1 - \tau)^2 \sum_{j \geq 1} \tau^j \sum_{m \geq 0} Q^{(\alpha-d)m} |f_{m+j} - f_m|^2.$$

Lemma 2.5 *The form (2.10) is the closed form of the nonnegative self-adjoint operator R_{rad}^α . More precisely, $\text{dom } \mathcal{E}_R = \text{dom}(R_{\text{rad}}^\alpha)^{\frac{1}{2}}$, the space $\mathcal{C}$ is a core for $\mathcal{E}_R$, and*

$$\mathcal{E}_R(f) = \sum_{n \geq 0} \lambda_n |\langle f, g_n \rangle|^2, \quad f \in \ell_{\mathbb{N}_0}^2(w).$$

Proof : We first show that $\mathcal{E}_R$ is closed. Let $(f^{(i)})$ be a sequence of $\text{dom } \mathcal{E}_R$ which is Cauchy for the norm $(\mathcal{E}_R(\cdot) + \|\cdot\|^2)^{\frac{1}{2}}$. It converges in $\ell_{\mathbb{N}_0}^2(w)$ to some f , hence coordinatewise. By Fatou's lemma for series of nonnegative terms,

$$\mathcal{E}_R(f^{(i)} - f) \leq \liminf_{i' \rightarrow \infty} \mathcal{E}_R(f^{(i)} - f^{(i')}),$$

which is small for large i . Thus $f \in \text{dom } \mathcal{E}_R$ and $f^{(i)} \rightarrow f$ in the form norm.

Now, we consider an eventually constant sequence $g \in \mathcal{C}$, say $g_k = g_N$ for $k \geq N$, and $f \in \text{dom } \mathcal{E}_R$. We claim that

$$(2.12) \quad \mathcal{E}_R(f, g) = \langle f, R_{\text{rad}}^\alpha g \rangle_{\ell_{\mathbb{N}_0}^2(w)},$$

where $\mathcal{E}_R(\cdot, \cdot)$ denotes the sesquilinear form associated with (2.10). By symmetry of the kernel $w_n q_{nk}$, we have $\mathcal{E}_R(f, g) = \frac{1}{2} \sum_{n \neq k} w_n q_{nk} (f_n - f_k) \overline{(g_n - g_k)}$, and it suffices to show that $\sum_{n \neq k} w_n q_{nk} |f_n| |g_n - g_k| < \infty$. Terms with $n, k \geq N$ vanish. For $n < N$, the sum over k is bounded by $2 \|g\|_\infty w_n |f_n| R_{nn}$. For $n \geq N$ and $k < N$, we have $w_n q_{nk} = c w_n w_k (\frac{\rho}{\tau})^k$, and the corresponding sum is bounded by $2 \|g\|_\infty c \sum_{k < N} w_k (\frac{\rho}{\tau})^k \sum_{n \geq N} w_n |f_n|$, which is finite since w is a probability. Consequently we may split the product and use detailed balance:

$$\mathcal{E}_R(f, g) = \sum_n w_n f_n \overline{\sum_{k \neq n} q_{nk} (g_n - g_k)} \stackrel{(2.5)}{=} \langle f, R_{\text{rad}}^\alpha g \rangle.$$

This proves (2.12).

Suppose that $f \in \text{dom } \mathcal{E}_R$ satisfies $\mathcal{E}_R(f, g) + \langle f, g \rangle = 0$ for any $g \in \mathcal{C}$. By (2.12), $\langle f, (R_{\text{rad}}^\alpha + I)g \rangle = 0$ for any $g \in \mathcal{C}$. The operator $R_{\text{rad}}^\alpha + I$ maps $\mathcal{C}$ onto $\mathcal{C}$, by (2.9). Hence f is orthogonal to the dense subspace $\mathcal{C}$, and $f = 0$. Thus $\mathcal{C}$ is a core for the closed form $\mathcal{E}_R$. On the other hand, (2.12) with $f = g \in \mathcal{C}$ shows that $\mathcal{E}_R$ agrees on $\mathcal{C}$ with the form $g \mapsto \langle g, R_{\text{rad}}^\alpha g \rangle = \sum_n \lambda_n |\langle g, g_n \rangle|^2$, for which $\mathcal{C}$ is also a core. Both closed forms are therefore the closure of the same form on $\mathcal{C}$, which proves the lemma. $\blacksquare$

2.5 Conditional expectations and an entrance law

For $n \geq 0$ let $\mathbb{E}_n$ retain the first n shell values and average the rest:

$$(2.13) \quad (\mathbb{E}_n f)_k = \begin{cases} f_k, & k < n, \\ (1 - \tau) \sum_{j \geq 0} \tau^j f_{n+j}, & k \geq n. \end{cases}$$

They are probability-preserving conditional expectations, $\mathbb{E}_i \mathbb{E}_j = \mathbb{E}_{\min(i,j)}$, and $\mathbb{E}_0$ is the mean. On g_k , $\mathbb{E}_n$ is zero for $n \leq k$ and the identity for $n > k$. Therefore

$$R_{\text{rad}}^\alpha = c(I - \mathbb{E}_0) + (\rho - 1) \sum_{n \geq 1} \rho^n (I - \mathbb{E}_n)$$

on $\mathcal{C}$. Let $T_t = e^{-tR_{\text{rad}}^\alpha}$. Put

$$(2.14) \quad a_0(t) \stackrel{\text{def}}{=} 1 - e^{-t\lambda_0}, \quad a_n(t) \stackrel{\text{def}}{=} e^{-t\lambda_{n-1}} - e^{-t\lambda_n} \quad (n \geq 1).$$

For $t > 0$ these numbers are nonnegative and sum to one, and

$$(2.15) \quad T_t = \sum_{n \geq 0} a_n(t) \mathbb{E}_n.$$

This identity follows on the orthonormal basis and then on bounded sequences by normality. It also proves conservativity and positivity directly. The operator $\mathbb{E}_n$ is the radial part of the averaging operator over balls of radius Q^{-n} , and (2.15) is an instance of the description of isotropic Markov semigroups on ultrametric spaces as mixtures of such averaging operators [BGP+14]. Each $\mathbb{E}_n$ maps the space of convergent sequences to itself, and $\mathbb{E}_n f \rightarrow f$ uniformly for such f . Thus (T_t) is a Feller semigroup on $C(X_\tau)$.

Write $P_t(n, k) = (T_t \mathbf{1}_{\{k\}})_n$ for transition probabilities. Define the probability measures

$$\nu_j(k) \stackrel{\text{def}}{=} (1 - \tau) \tau^{k-j} \mathbf{1}_{\{k \geq j\}}, \quad \mu_t(k) \stackrel{\text{def}}{=} \sum_{j \geq 0} a_j(t) \nu_j(k).$$

Lemma 2.6 *For any $t > 0$, $P_t(n, \cdot)$ converges in ℓ^1 to μ_t as $n \rightarrow \infty$:*

$$(2.16) \quad \|P_t(n, \cdot) - \mu_t\|_{\ell^1} \leq 2e^{-t\lambda_n}.$$

For $t_j = \rho^{-j}$, $j \geq 1$, these entrance laws have a uniformly positive atom:

$$\mu_{t_j}(j) \geq \delta_T, \quad \delta_T \stackrel{\text{def}}{=} (1 - \tau)(e^{-1} - e^{-\rho}) > 0.$$

Proof : For $\ell \leq n$, the n th row of $\mathbb{E}_\ell$ is ν_ℓ , whereas for $\ell > n$ it is the point mass at n . The total coefficient of the latter rows is $\sum_{\ell > n} a_\ell(t) = e^{-t\lambda_n}$. Comparing the two probability mixtures gives (2.16). Moreover $\mu_t(j) \geq (1 - \tau)a_j(t)$, and $a_j(t_j) = e^{\kappa t_j}(e^{-1} - e^{-\rho}) \geq e^{-1} - e^{-\rho}$. ■

3 Little η -Jacobi birth–death operators

In this section $0 < \tau < 1$ and $0 < \eta < 1$ are arbitrary, and $w_k = (1 - \tau)\tau^k$, $k \geq 0$. We consider the space $\ell_{\mathbb{N}_0}^2(w)$ and the compact set

$$Y_\eta \stackrel{\text{def}}{=} \{\eta^k : k \geq 0\} \cup \{0\}.$$

A function of $k \in \mathbb{N}_0$ is called a polynomial in η^k if it is of the form $k \mapsto p(\eta^k)$ for a polynomial $p \in \mathbb{C}[x]$. We denote by $\mathcal{P}_n$ the space of polynomials in η^k of degree at most n and by $\mathcal{P}$ their union. We identify such a function with the polynomial p , which is a continuous function on Y_η ; in particular $p(0)$ is the limit of $p(\eta^k)$ as $k \rightarrow \infty$. We use the q -shifted factorials $(a; \eta)_n \stackrel{\text{def}}{=} \prod_{i=0}^{n-1} (1 - a\eta^i)$, $n \in \mathbb{N}_0 \cup \{\infty\}$, see [DLMF, (17.2.1) and (17.2.4)] and [GR04, Section 1.2].

3.1 The operator and its spectral decomposition

Put

$$\beta_k \stackrel{\text{def}}{=} \tau \eta^{-k-1} (1 - \eta^{k+1}), \quad \delta_k \stackrel{\text{def}}{=} \eta^{-k} (1 - \eta^k), \quad k \geq 0.$$

In particular $\delta_0 = 0$, and the detailed balance condition

$$(3.1) \quad w_k \beta_k = w_{k+1} \delta_{k+1} = (1 - \tau) \left(\frac{\tau}{\eta} \right)^{k+1} (1 - \eta^{k+1}), \quad k \geq 0,$$

holds. For any function $f: \mathbb{N}_0 \rightarrow \mathbb{C}$ we define $L_\eta f$ by (1.3), that is $(L_\eta f)(k) = \beta_k(f(k) - f(k+1)) + \delta_k(f(k) - f(k-1))$, the second term being zero for $k = 0$. For $f \in \ell^2_{\mathbb{N}_0}(w)$ we put

$$(3.2) \quad \mathcal{E}_\eta(f) \stackrel{\text{def}}{=} \sum_{k \geq 0} w_k \beta_k |f(k+1) - f(k)|^2 \in [0, \infty], \quad \text{dom } \mathcal{E}_\eta \stackrel{\text{def}}{=} \{f \in \ell^2_{\mathbb{N}_0}(w) : \mathcal{E}_\eta(f) < \infty\}.$$

Proposition 3.1 *1. For any integer $n \geq 0$, the operator L_η maps $\mathcal{P}_n$ into $\mathcal{P}_n$, and $L_\eta(x^n) - \Lambda_n(\eta)x^n \in \mathcal{P}_{n-1}$, where $\Lambda_n(\eta)$ is defined in (1.4) and $\mathcal{P}_{-1} = \{0\}$. The sequence $(\Lambda_n(\eta))_{n \geq 0}$ is strictly increasing and $\Lambda_0(\eta) = 0$.*

2. The space $\mathcal{P}$ is dense in $\ell^2_{\mathbb{N}_0}(w)$ and contained in $\text{dom } \mathcal{E}_\eta$. For any $f \in \text{dom } \mathcal{E}_\eta$ and any $p \in \mathcal{P}$, we have

$$(3.3) \quad \mathcal{E}_\eta(f, p) = \langle f, L_\eta p \rangle_{\ell^2_{\mathbb{N}_0}(w)}.$$

3. The form $\mathcal{E}_\eta$ is closed and $\mathcal{P}$ is a core for it. The nonnegative self-adjoint operator associated with $\mathcal{E}_\eta$, still denoted by L_η , is diagonal in the orthonormal basis $(e_n^\eta)_{n \geq 0}$ obtained by the Gram-Schmidt process from $1, x, x^2, \dots$, and $L_\eta e_n^\eta = \Lambda_n(\eta) e_n^\eta$. In particular $\text{dom } \mathcal{E}_\eta = \text{dom } L_\eta^{\frac{1}{2}}$ and $\mathcal{E}_\eta(f) = \sum_n \Lambda_n(\eta) |\langle f, e_n^\eta \rangle|^2$.

4. For any $n \geq 0$, the polynomial e_n^η does not vanish at 0, and $\varphi_n^\eta \stackrel{\text{def}}{=} e_n^\eta(0)^{-1} e_n^\eta$ is the unique polynomial of degree n orthogonal to $\mathcal{P}_{n-1}$ and equal to 1 at 0. It is the little η -Jacobi polynomial $p_n(x; \tau\eta^{-1}, 1; \eta)$ of [KLS10, Section 14.12].

Proof : 1. Let $f(k) = \eta^{nk} = x^n$ with $x = \eta^k$. A direct computation gives, for any $k \geq 0$,

$$\begin{aligned} (L_\eta f)(k) &= \tau \eta^{-k-1} (1 - \eta^{k+1}) \eta^{nk} (1 - \eta^n) + \eta^{-k} (1 - \eta^k) \eta^{nk} (1 - \eta^{-n}) \\ &= x^{n-1} \left[\frac{\tau}{\eta} (1 - \eta x) (1 - \eta^n) + (1 - x) (1 - \eta^{-n}) \right], \end{aligned}$$

which is also valid for $k = 0$ since $\delta_0 = 0$ and $1 - x = 0$ there. The bracket vanishes for $n = 0$. The coefficient of x^n is $-\tau(1 - \eta^n) - (1 - \eta^{-n}) = \eta^{-n}(1 - \eta^n)(1 - \tau\eta^n) = \Lambda_n(\eta)$. This proves the first assertion. Moreover

$$\Lambda_{n+1}(\eta) - \Lambda_n(\eta) = (1 - \eta)(\eta^{-n-1} - \tau\eta^n) > 0,$$

since $\Lambda_n(\eta) = \eta^{-n} - 1 - \tau + \tau\eta^n$.

2. The measure $\sum_k w_k \delta_{\eta^k}$ lives on the compact set Y_η and has no mass at 0. Polynomials are uniformly dense in $C(Y_\eta)$ by the Stone–Weierstrass theorem, and indicators of the isolated points η^k are continuous. Since finitely supported sequences are dense in $\ell_{\mathbb{N}_0}^2(w)$, the space $\mathcal{P}$ is dense. For $p \in \mathcal{P}$, the mean value theorem gives $|p(\eta^{k+1}) - p(\eta^k)| \leq \|p'\|_{L^\infty[0,1]} \eta^k$, hence by (3.1)

$$\mathcal{E}_\eta(p) \leq (1 - \tau) \frac{\tau}{\eta} \|p'\|_{L^\infty[0,1]}^2 \sum_{k \geq 0} (\tau\eta)^k < \infty.$$

Now, let $f \in \text{dom } \mathcal{E}_\eta$ and $p \in \mathcal{P}$. Put $c_k = w_k \beta_k = w_{k+1} \delta_{k+1}$. A summation by parts gives, for any integer $N \geq 1$,

$$(3.4) \quad \sum_{k=0}^{N-1} c_k (f(k+1) - f(k)) \overline{(p(k+1) - p(k))} = \sum_{k=0}^{N-1} w_k f(k) \overline{(L_\eta p)(k)} + c_{N-1} f(N) \overline{(p(N) - p(N-1))}.$$

Indeed, the coefficient of $f(k)$ on the left-hand side is $c_{k-1} \overline{(p(k) - p(k-1))} + c_k \overline{(p(k) - p(k+1))}$ for $1 \leq k \leq N-1$ and $c_0 \overline{(p(0) - p(1))}$ for $k = 0$, which is $w_k \overline{(L_\eta p)(k)}$ by (3.1). For the boundary term, we use $|f(N)| \leq w_N^{-\frac{1}{2}} \|f\|$, the mean value theorem and $c_{N-1} \leq (\frac{\tau}{\eta})^N$:

$$|c_{N-1} f(N) \overline{(p(N) - p(N-1))}| \leq \left(\frac{\tau}{\eta}\right)^N (1 - \tau)^{-\frac{1}{2}} \tau^{-\frac{N}{2}} \|f\| \|p'\|_{L^\infty[0,1]} \eta^{N-1} = \frac{\|f\| \|p'\|_{L^\infty[0,1]} \tau^{\frac{N}{2}}}{\eta(1 - \tau)^{\frac{1}{2}}} \tau^{\frac{N}{2}},$$

which tends to zero. The left-hand side of (3.4) converges to $\mathcal{E}_\eta(f, p)$, since the series defining $\mathcal{E}_\eta(f, p)$ converges absolutely by the Cauchy–Schwarz inequality. The first term of the right-hand side converges to $\langle f, L_\eta p \rangle$, since $L_\eta p \in \mathcal{P}$ is bounded. This proves (3.3).

3. The closedness follows by the Fatou argument of the proof of Lemma 2.5. Suppose that $f \in \text{dom } \mathcal{E}_\eta$ satisfies $\mathcal{E}_\eta(f, p) + \langle f, p \rangle = 0$ for any $p \in \mathcal{P}$. By (3.3), $\langle f, (L_\eta + I)p \rangle = 0$ for any $p \in \mathcal{P}$. By part 1, $L_\eta + I$ maps $\mathcal{P}_n$ into itself with a triangular matrix whose diagonal entries $\Lambda_n(\eta) + 1$ are positive, hence onto $\mathcal{P}_n$. Thus $f \perp \mathcal{P}$ and $f = 0$ by part 2. Consequently $\mathcal{P}$ is a core for $\mathcal{E}_\eta$.

By (3.3) with $f \in \mathcal{P}$, the operator L_η is symmetric on $\mathcal{P}$, and $\mathcal{E}_\eta(p) = \langle p, L_\eta p \rangle$ for $p \in \mathcal{P}$. Since L_η preserves $\mathcal{P}_n$, the vector $L_\eta e_n^\eta$ belongs to $\mathcal{P}_n$ and is orthogonal to $\mathcal{P}_{n-1}$, because $\langle L_\eta e_n^\eta, p \rangle = \langle e_n^\eta, L_\eta p \rangle = 0$ for $p \in \mathcal{P}_{n-1}$. Hence $L_\eta e_n^\eta$ is a multiple of e_n^η , and comparing the coefficients of x^n by part 1 shows that the multiple is $\Lambda_n(\eta)$. The basis (e_n^η) is complete by part 2. The closed form $\mathcal{E}_\eta$ is therefore the closure of the form $p \mapsto \sum_n \Lambda_n(\eta) |\langle p, e_n^\eta \rangle|^2$ on $\mathcal{P}$, which is the closed form of the diagonal operator with eigenvalues $\Lambda_n(\eta)$. This proves part 3.

4. The measure $\sum_k w_k \delta_{\eta^k}$ is a positive measure on $[0, 1]$ with infinite support. By [Sze75, Theorem 3.3.1], the zeros of e_n^η are real, simple and contained in the open interval $(0, 1)$. Hence $e_n^\eta(0) \neq 0$. The orthogonal complement of $\mathcal{P}_{n-1}$ in $\mathcal{P}_n$ is one-dimensional, which gives uniqueness. The polynomials $p_n(x; a, b; \eta)$ of [KLS10, Section 14.12] are orthogonal with respect to the weights $\frac{(b\eta; \eta)_k}{(\eta; \eta)_k} (a\eta)^k$ on the points η^k ; for $a = \tau\eta^{-1}$ and $b = 1$ these weights are τ^k . Since $p_n(0; a, b; \eta) = 1$, the identification follows from uniqueness. ■

Remark 3.2 For $\eta = \tau$ the parameters are $a = b = 1$, and φ_n^τ is the little τ -Legendre polynomial. The first two are

$$(3.5) \quad \varphi_1^\tau(x) = 1 - (1 + \tau)x, \quad \varphi_2^\tau(x) = 1 - \frac{(1 + \tau)(1 + \tau + \tau^2)}{\tau}x + \frac{(1 + \tau + \tau^2)(1 + \tau^2)}{\tau}x^2.$$

Indeed, the moments of the weights are $\sum_k w_k \tau^{mk} = \frac{1-\tau}{1-\tau^{m+1}}$, and a direct verification shows that (3.5) are orthogonal to 1, respectively to 1 and x . The corresponding eigenvalues are

$$(3.6) \quad \Lambda_1(\tau) = \frac{(1-\tau)(1-\tau^2)}{\tau}, \quad \Lambda_2(\tau) = \frac{(1-\tau^2)(1-\tau^3)}{\tau^2}, \quad \frac{1+\tau+\tau^2}{\tau} \Lambda_1(\tau) = \Lambda_2(\tau).$$

3.2 The Feller semigroup and its law at the origin

Proposition 3.3 *Let $(T_t^\eta)_{t \geq 0}$ be the semigroup e^{-tL_η} on $\ell_{\mathbb{N}_0}^2(w)$.*

1. *The semigroup is Markovian. More precisely, $P_t^\eta(n, k) \stackrel{\text{def}}{=} (T_t^\eta \mathbf{1}_{\{k\}})(n)$ is a stochastic matrix on $\mathbb{N}_0$ and $(T_t^\eta f)(n) = \sum_k P_t^\eta(n, k) f(k)$ for any $f \in \ell_{\mathbb{N}_0}^\infty$.*
2. *For any $f \in C(Y_\eta)$, the function $T_t^\eta f$ belongs to $C(Y_\eta)$. The resulting operators form a Feller semigroup on $C(Y_\eta)$, and $T_t^\eta \varphi_n^\eta = e^{-t\Lambda_n(\eta)} \varphi_n^\eta$.*
3. *The formula $\int f d\mu_t^\eta \stackrel{\text{def}}{=} (T_t^\eta f)(0)$ defines a probability measure μ_t^η on Y_η satisfying*

$$(3.7) \quad \int \varphi_n^\eta d\mu_t^\eta = e^{-t\Lambda_n(\eta)}, \quad n \geq 0.$$

4. *Suppose that $\sup_n \sup_{Y_\eta} |\varphi_n^\eta| < \infty$. Then $\mu_t^\eta(\{0\}) = 0$ for any $t > 0$, and $P_t^\eta(n, \cdot) \rightarrow \mu_t^\eta$ in $\ell_{\mathbb{N}_0}^1$ as $n \rightarrow \infty$.*

Proof : 1. Normal contractions operate on the kernel expression (3.2): if $T: \mathbb{C} \rightarrow \mathbb{C}$ satisfies $T(0) = 0$ and $|T(a) - T(b)| \leq |a - b|$, then $\mathcal{E}_\eta(T \circ f) \leq \mathcal{E}_\eta(f)$. Thus $\mathcal{E}_\eta$ is a Dirichlet form, and (T_t^η) is positivity preserving and contractive on $\ell_{\mathbb{N}_0}^\infty$ by [FOT11, Theorem 1.4.1]. Since $1 \in \mathcal{P}$ and $L_\eta 1 = 0$, we have $T_t^\eta 1 = 1$. Since w is summable, any bounded sequence f is the sum in $\ell_{\mathbb{N}_0}^2(w)$ of the series $\sum_k f(k) \mathbf{1}_{\{k\}}$. Applying the bounded operator T_t^η and evaluating at n gives the formula for $T_t^\eta f$; with $f = 1$ we obtain $\sum_k P_t^\eta(n, k) = 1$.

2. For $p \in \mathcal{P}$, we have $T_t^\eta p = \sum_n e^{-t\Lambda_n(\eta)} \langle p, e_n^\eta \rangle e_n^\eta$, which is a finite sum by part 3 of Proposition 3.1, hence a polynomial. For $f \in C(Y_\eta)$, choose polynomials p_i converging uniformly to f on Y_η . By part 1, $\sup_n |(T_t^\eta p_i)(n) - (T_t^\eta f)(n)| \leq \sup_k |p_i(\eta^k) - f(\eta^k)|$. Thus $T_t^\eta f$ is the uniform limit on $\mathbb{N}_0$ of sequences which converge at infinity, hence converges at infinity, that is $T_t^\eta f \in C(Y_\eta)$. The resulting operators are positive, unital and contractive for the uniform norm, and satisfy the semigroup law. For $p \in \mathcal{P}$, the finite sum above shows that $T_t^\eta p \rightarrow p$ uniformly as $t \rightarrow 0^+$; by density and contractivity, (T_t^η) is strongly continuous on $C(Y_\eta)$. Finally $T_t^\eta \varphi_n^\eta = e^{-t\Lambda_n(\eta)} \varphi_n^\eta$ since φ_n^η is a multiple of e_n^η .

3. The functional $f \mapsto (T_t^\eta f)(0)$ is positive and unital on $C(Y_\eta)$, hence a probability measure by the Riesz representation theorem. Evaluating $T_t^\eta \varphi_n^\eta = e^{-t\Lambda_n(\eta)} \varphi_n^\eta$ at 0 and using $\varphi_n^\eta(0) = 1$ gives (3.7).

4. Let $C = \sup_n \sup |\varphi_n^\eta|$. Parseval's identity for the vector $\mathbf{1}_{\{k\}}$ in the orthonormal basis (e_n^η) gives $\sum_n |e_n^\eta(\eta^k)|^2 = w_k^{-1}$, that is $\sum_n |\varphi_n^\eta(\eta^k)|^2 h_n^{-1} = w_k^{-1}$ with $h_n \stackrel{\text{def}}{=} \|\varphi_n^\eta\|^2 = e_n^\eta(0)^{-2}$. The terms of this convergent series tend to zero and $h_n \leq C^2$, hence $\varphi_n^\eta(\eta^k) \rightarrow 0$ as $n \rightarrow \infty$ for any fixed k . By (3.7),

$$e^{-t\Lambda_n(\eta)} = \mu_t^\eta(\{0\}) + \sum_{k \geq 0} \mu_t^\eta(\{\eta^k\}) \varphi_n^\eta(\eta^k).$$

Letting $n \rightarrow \infty$, the left-hand side tends to zero, and the series tends to zero by dominated convergence. Hence $\mu_t^\eta(\{0\}) = 0$. Since each $\mathbf{1}_{\{\eta^k\}}$ is continuous on Y_η , part 2 gives $P_t^\eta(n, k) =$

$(T_t^\eta \mathbf{1}_{\{\eta^k\}})(n) \rightarrow \mu_t^\eta(\{\eta^k\})$ as $n \rightarrow \infty$. The rows $P_t^\eta(n, \cdot)$ and the limit μ_t^η are probability vectors on $\mathbb{N}_0$, so Scheffé's lemma gives the convergence in $\ell_{\mathbb{N}_0}^1$. $\blacksquare$

In the sequel we identify μ_t^η with a probability vector on $\mathbb{N}_0$ whenever $\mu_t^\eta(\{0\}) = 0$, writing $\mu_t^\eta(k)$ for $\mu_t^\eta(\{\eta^k\})$.

4 Comparison of energies

We return to the local field, so that $\tau = Q^{-d}$, and we fix $\alpha > 0$.

4.1 The operator L_η for $\eta = Q^{-\alpha}$

We put

$$\eta \stackrel{\text{def}}{=} Q^{-\alpha} = \rho^{-1}, \quad \gamma_\alpha \stackrel{\text{def}}{=} \alpha - d.$$

Then $\frac{\tau}{\eta} = Q^{\gamma_\alpha}$, and (3.1) reads

$$(4.1) \quad w_k \beta_k = (1 - \tau) Q^{\gamma_\alpha(k+1)} (1 - \eta^{k+1}), \quad k \geq 0.$$

Theorem 4.1 *The forms $\mathcal{E}_R$ of (2.10) and $\mathcal{E}_\eta$ of (3.2) satisfy, for any $f \in \ell_{\mathbb{N}_0}^2(w)$,*

$$(4.2) \quad c_1 \mathcal{E}_\eta(f) \leq \mathcal{E}_R(f) \leq c_2 \mathcal{E}_\eta(f),$$

in $[0, \infty]$, where $c_1 = \frac{(\rho-1)(1-\tau)}{\rho-\tau}$ and

$$(4.3) \quad c_2 = \begin{cases} \frac{\rho(1+\tau)}{(\rho-\tau)(1-\tau)^2} & \text{if } \alpha \geq d, \\ \frac{(1-\tau)\rho^3}{(\rho-\tau)(\rho-1)^2(1-\rho\tau)} & \text{if } \alpha < d. \end{cases}$$

In particular, $\text{dom } \mathcal{E}_R = \text{dom } \mathcal{E}_\eta$ is the space described in Theorem 1.1, and for $\alpha = d$ we have $c_1 = \frac{1-\tau}{1+\tau}$ and $c_2 = (1-\tau)^{-3}$.

Proof : For the lower estimate, we keep only the terms $j = 1$ in (2.11). Using $1 - \eta^{m+1} \leq 1$ in (4.1), we obtain

$$\mathcal{E}_R(f) \geq c(1-\tau)^2 \tau \sum_{m \geq 0} Q^{\gamma_\alpha m} |f_{m+1} - f_m|^2 \stackrel{(4.1)}{\geq} c(1-\tau) \tau Q^{-\gamma_\alpha} \mathcal{E}_\eta(f).$$

Since $\tau Q^{-\gamma_\alpha} = Q^{-\alpha} = \rho^{-1}$ and $c = \frac{\rho(\rho-1)}{\rho-\tau}$, the constant is c_1 .

For the upper estimate, let $j \geq 1$ and $m \geq 0$. By the Cauchy–Schwarz inequality,

$$|f_{m+j} - f_m|^2 \leq j \sum_{i=m}^{m+j-1} |f_{i+1} - f_i|^2.$$

Exchanging the sums over m and i in the sum of nonnegative terms, we obtain

$$\sum_{m \geq 0} Q^{\gamma_\alpha m} |f_{m+j} - f_m|^2 \leq j \sum_{i \geq 0} |f_{i+1} - f_i|^2 \sum_{m=\max(0, i-j+1)}^i Q^{\gamma_\alpha m} \leq j C_j \sum_{i \geq 0} Q^{\gamma_\alpha i} |f_{i+1} - f_i|^2,$$

where $C_j = j$ if $\gamma_\alpha \geq 0$, and $C_j = \frac{Q^{-\gamma_\alpha(j-1)}}{1-Q^{\gamma_\alpha}}$ if $\gamma_\alpha < 0$, since the inner sum has at most j terms, each bounded by $Q^{\gamma_\alpha i}$ if $\gamma_\alpha \geq 0$, whereas it is bounded by $\sum_{m \geq i-j+1} Q^{\gamma_\alpha m} = \frac{Q^{\gamma_\alpha(i-j+1)}}{1-Q^{\gamma_\alpha}}$ if $\gamma_\alpha < 0$. Moreover, (4.1) gives $Q^{\gamma_\alpha i} |f_{i+1} - f_i|^2 \leq \frac{Q^{-\gamma_\alpha}}{(1-\tau)(1-\eta)} w_i \beta_i |f_{i+1} - f_i|^2$. Inserting these bounds in (2.11), we obtain

$$\mathcal{E}_R(f) \leq \frac{c(1-\tau)Q^{-\gamma_\alpha}}{1-\eta} \left(\sum_{j \geq 1} j C_j \tau^j \right) \mathcal{E}_\eta(f).$$

If $\gamma_\alpha \geq 0$, then $\sum_j j^2 \tau^j = \frac{\tau(1+\tau)}{(1-\tau)^3}$; using $\rho\tau Q^{-\gamma_\alpha} = 1$ and $1-\eta = \frac{\rho-1}{\rho}$, the constant becomes the first expression in (4.3). If $\gamma_\alpha < 0$, then $\tau Q^{-\gamma_\alpha} = Q^{-\alpha} = \eta$ and

$$\sum_{j \geq 1} j C_j \tau^j = \frac{Q^{\gamma_\alpha}}{1-Q^{\gamma_\alpha}} \sum_{j \geq 1} j \eta^j = \frac{Q^{\gamma_\alpha}}{1-Q^{\gamma_\alpha}} \frac{\eta}{(1-\eta)^2};$$

using $Q^{\gamma_\alpha} = \rho\tau$ and $\eta = \rho^{-1}$ gives the second expression in (4.3). The values for $\alpha = d$ follow from $\rho = \tau^{-1}$. The identification of the domains follows from (4.2), (4.1) and the bounds $1-\eta \leq 1-\eta^{k+1} \leq 1$. $\blacksquare$

By Lemma 2.5 and Proposition 3.1, $\mathcal{E}_R$ and $\mathcal{E}_\eta$ are the closed forms of R_{rad}^α and L_η . Theorem 4.1 therefore proves the first part of Theorem 1.1. We now turn to spectral consequences.

Corollary 4.2 1. For any $n \geq 0$, equality (1.5) holds. In particular, $|\lambda_n - \Lambda_{n+1}(\eta)| \leq 1 + \tau$ and $\lambda_n - \Lambda_{n+1}(\eta) \rightarrow 1 + \tau - \kappa$ as $n \rightarrow \infty$.

2. For any $0 \leq \theta \leq 1$, we have $\text{dom}(R_{\text{rad}}^\alpha)^{\frac{\theta}{2}} = \text{dom} L_\eta^{\frac{\theta}{2}}$, and the norms $f \mapsto \|f\| + \|(R_{\text{rad}}^\alpha)^{\frac{\theta}{2}} f\|$ and $f \mapsto \|f\| + \|L_\eta^{\frac{\theta}{2}} f\|$ are equivalent on this space.

Proof : 1. Since $\eta = \rho^{-1}$, we have

$$\Lambda_{n+1}(\eta) = \rho^{n+1}(1 - \rho^{-n-1})(1 - \tau\rho^{-n-1}) = \rho^{n+1} - 1 - \tau + \tau\rho^{-n-1}.$$

Subtracting this from $\lambda_n = \rho^{n+1} - \kappa$ gives (1.5). Since $0 < \kappa < 1$ and $0 < \tau\rho^{-n-1} < \tau$, the difference lies in $(0, 1 + \tau)$.

2. By Theorem 4.1, the Hilbert spaces $\text{dom}(R_{\text{rad}}^\alpha)^{\frac{1}{2}}$ and $\text{dom} L_\eta^{\frac{1}{2}}$, equipped with their graph norms, coincide with equivalent norms. A nonnegative self-adjoint operator A on a Hilbert space H is sectorial and $A + I$ has bounded imaginary powers, with $\|(A + I)^{is}\| \leq 1$ by the spectral theorem. By [Haa06, Theorem 6.6.9], the complex interpolation space $[H, \text{dom} A^{\frac{1}{2}}]_\theta$ is $\text{dom} A^{\frac{\theta}{2}}$, with equivalent norms. Applying this to $A = R_{\text{rad}}^\alpha$ and $A = L_\eta$ and using that the two interpolation couples coincide up to equivalent norms proves part 2. $\blacksquare$

Remark 4.3 The operators R_{rad}^α and L_η have no common eigenvector besides the constants. Indeed, their eigenvalues are simple, so a common eigenvector would be a multiple of some g_n and of some e_m^η simultaneously. The function g_n is constant on the infinite set $\{k \geq n+1\}$, whereas a nonconstant polynomial in η^k takes each value at most m times. Hence $m = 0$, which is impossible since $g_n \perp 1$. Theorem 4.1 thus compares two operators whose spectral projections are in general position, with spectra differing by a bounded sequence.

4.2 Powers of the zonal operators

In this subsection, we fix a real number $\vartheta > 0$ and we take

$$\eta \stackrel{\text{def}}{=} Q^{-\vartheta}$$

in Section 3. The case $\vartheta = \alpha$ is the one of Theorem 4.1. The case $\vartheta = d$ is $\eta = \tau$, and then L_τ is the zonal part of the Casimir operator of $\text{SU}_r(2)$ with $r^2 = \tau$, by Remark 5.4 below. For any integer $n \geq 2$, the case $\vartheta = \frac{d}{n-1}$ corresponds to the quantum projective space $\mathbb{CP}_q^{n-1}$ with $q^2 = \eta$, by Theorem 10.3 below. We write $\Lambda_m \stackrel{\text{def}}{=} \Lambda_m(\eta)$ and $e_m \stackrel{\text{def}}{=} e_m^\eta$ for $m \geq 0$, so that $e_0 = 1$, $L_\eta e_m = \Lambda_m e_m$ and $e_m = \pm \|\varphi_m^\eta\|^{-1} \varphi_m^\eta$. For $s > 0$, the closed form of L_η^s is $f \mapsto \left\| L_\eta^{\frac{s}{2}} f \right\|^2 = \sum_{m \geq 1} \Lambda_m^s |\langle f, e_m \rangle|^2$ on $\text{dom } L_\eta^{\frac{s}{2}}$. For any $0 < \tau < 1$ and any $0 < \eta < 1$, we have $\Lambda_m(\eta) = \eta^{-m}(1 - \eta^m)(1 - \tau\eta^m)$, hence

$$(4.4) \quad c_\eta \eta^{-m} \leq \Lambda_m(\eta) \leq \eta^{-m}, \quad m \geq 1, \quad \text{where } c_\eta \stackrel{\text{def}}{=} (1 - \eta)(1 - \tau\eta).$$

The following two lemmas hold for any $0 < \tau < 1$ and any $0 < \eta < 1$.

Lemma 4.4 *Let $m \geq 1$ and write $\varphi_m^\eta(x) = \sum_{j=0}^m c_{m,j} x^j$. Put $D \stackrel{\text{def}}{=} (\tau; \eta)_\infty^{-1} (\eta; \eta)_\infty^{-1}$. Then*

$$(4.5) \quad |c_{m,j}| \leq D \eta^{-mj + \frac{j(j+1)}{2}}, \quad 0 \leq j \leq m.$$

Consequently, $|(\varphi_m^\eta)'(x)| \leq A \eta^{-m}$ for $0 \leq x \leq \eta^{m-1}$, and $|1 - \varphi_m^\eta(x)| \leq A' \eta^{-m} x$ for $0 \leq x \leq \eta^m$, where

$$A \stackrel{\text{def}}{=} \eta D \sum_{j \geq 1} j \eta^{\frac{j(j-1)}{2}}, \quad A' \stackrel{\text{def}}{=} D \sum_{j \geq 1} \eta^{\frac{j(j+1)}{2}}.$$

Moreover, $\varphi_1^\eta(x) = 1 - \frac{1-\tau\eta}{1-\tau} x$ and

$$(4.6) \quad \|\varphi_m^\eta\|^2 = \frac{(1-\tau)\tau^m}{1-\tau\eta^{2m}} \left(\frac{(\eta; \eta)_m}{(\tau; \eta)_m} \right)^2.$$

Proof : By part 4 of Proposition 3.1, φ_m^η is the little η -Jacobi polynomial $p_m(x; \tau\eta^{-1}, 1; \eta)$. Hence [KLS10, (14.12.1)] gives

$$c_{m,j} = \frac{(\eta^{-m}; \eta)_j (\tau\eta^m; \eta)_j}{(\tau; \eta)_j (\eta; \eta)_j} \eta^j.$$

For $1 \leq j \leq m$, we have $|(\eta^{-m}; \eta)_j| = \prod_{i=0}^{j-1} \eta^{i-m} (1 - \eta^{m-i}) \leq \eta^{-mj + \frac{j(j-1)}{2}}$, $0 < (\tau\eta^m; \eta)_j \leq 1$, $(\tau; \eta)_j \geq (\tau; \eta)_\infty$ and $(\eta; \eta)_j \geq (\eta; \eta)_\infty$. This proves (4.5). For $m = j = 1$, the formula gives $c_{1,1} = -\frac{1-\tau\eta}{1-\tau}$. If $0 \leq x \leq \eta^{m-1}$, then (4.5) gives

$$|(\varphi_m^\eta)'(x)| \leq \sum_{j \geq 1} j |c_{m,j}| \eta^{(m-1)(j-1)} \leq D \sum_{j \geq 1} j \eta^{\frac{j(j-1)}{2} - m + 1} = A \eta^{-m}.$$

If $0 \leq x \leq \eta^m$, then $x^j \leq \eta^{m(j-1)} x$ for any $j \geq 1$. Since $c_{m,0} = 1$, (4.5) gives $|1 - \varphi_m^\eta(x)| \leq \sum_{j \geq 1} |c_{m,j}| \eta^{m(j-1)} x \leq A' \eta^{-m} x$. The norm is given by the orthogonality relation [KLS10, (14.12.2)] with $a = \tau\eta^{-1}$ and $b = 1$, taking into account the factor $1 - \tau$ in the weights w_k . ■

For $n \geq 1$, we put $\psi_n \stackrel{\text{def}}{=} g_{n-1}$, so that $R_{\text{rad}}^\alpha \psi_n = \lambda_{n-1} \psi_n$ when $\tau = Q^{-d}$, and for $n, m \geq 1$ we put

$$u_{nm} \stackrel{\text{def}}{=} \langle e_m, \psi_n \rangle_{\ell_{\mathbb{N}_0}^2(w)}.$$

The families $(\psi_n)_{n \geq 1}$ and $(e_m)_{m \geq 1}$ are two orthonormal bases of the orthogonal complement of the constants in $\ell_{\mathbb{N}_0}^2(w)$, by Theorem 2.3 and Proposition 3.1. Thus (u_{nm}) is the matrix of a unitary operator. The functions ψ_n do not depend on α and η .

Lemma 4.5 *For any integers $n, m \geq 1$, we have*

$$(4.7) \quad |u_{nm}| \leq C_1 (\tau^{\frac{1}{2}} \eta)^{n-m} \quad \text{if } n \geq m, \quad |u_{nm}| \leq C_2 B^{m-n} \eta^{\frac{(m-n)^2}{2}} \quad \text{if } m > n,$$

where $C_1 \stackrel{\text{def}}{=} \frac{(1-\eta)A}{\eta(1-\tau\eta)(\eta;\eta)_\infty}$, $C_2 \stackrel{\text{def}}{=} \frac{\sqrt{3}(\tau+\eta)}{4\tau\eta}$ and $B \stackrel{\text{def}}{=} \frac{4\eta^{\frac{1}{2}}}{c_\eta}$.

Proof : 1. Let $f \in \ell_{\mathbb{N}_0}^2(w)$ and $n \geq 1$. Since $\tau w_k = w_{k+1}$, we have

$$\langle f, \mathbf{1}_{B_n} - \tau \mathbf{1}_{B_{n-1}} \rangle = \sum_{k \geq n} w_k f(k) - \sum_{k \geq n-1} w_{k+1} f(k) = \sum_{k \geq n} w_k (f(k) - f(k-1)),$$

the series being absolutely convergent. By (2.8), we deduce that

$$(4.8) \quad \langle f, \psi_n \rangle = ((1-\tau)\tau^n)^{-\frac{1}{2}} \sum_{k \geq n} w_k (f(k) - f(k-1)).$$

Suppose that $n \geq m$. For $k \geq n$, the interval $[\eta^k, \eta^{k-1}]$ is contained in $[0, \eta^{m-1}]$. Since $(\tau; \eta)_m \leq 1$ and $(\eta; \eta)_m \geq (\eta; \eta)_\infty$, (4.6) gives $\|\varphi_m^\eta\|^{-1} \leq (1-\tau)^{-\frac{1}{2}} \tau^{-\frac{m}{2}} (\eta; \eta)_\infty^{-1}$. By Lemma 4.4 and the mean value theorem, we obtain

$$|e_m(k) - e_m(k-1)| \leq (1-\tau)^{-\frac{1}{2}} \tau^{-\frac{m}{2}} (\eta; \eta)_\infty^{-1} A \eta^{-m} (1-\eta) \eta^{k-1}, \quad k \geq n.$$

Inserting this bound in (4.8) with $f = e_m$ gives

$$|u_{nm}| \leq \frac{(1-\eta)A}{\eta(\eta; \eta)_\infty} \tau^{-\frac{n+m}{2}} \eta^{-m} \sum_{k \geq n} (\tau\eta)^k = C_1 (\tau^{\frac{1}{2}} \eta)^{n-m}.$$

2. Suppose that $m > n$. Every eventually constant sequence g belongs to $\text{dom } L_\eta$, and $L_\eta g$ is given by (1.3). Indeed, $g \in \text{dom } \mathcal{E}_\eta$, and for $f \in \text{dom } \mathcal{E}_\eta$ the identity (3.4), with p replaced by g , has a vanishing boundary term for large N . Since the sequence $L_\eta g$ given by (1.3) is finitely supported, letting $N \rightarrow \infty$ gives $\mathcal{E}_\eta(f, g) = \langle f, L_\eta g \rangle$ for any $f \in \text{dom } \mathcal{E}_\eta$.

In the orthonormal basis $(w_k^{-\frac{1}{2}} \mathbf{1}_{\{k\}})_{k \geq 0}$, the operator L_η acts on finitely supported sequences by a symmetric tridiagonal matrix. By (3.1), we have $\beta_k = \tau \delta_{k+1}$. Hence the diagonal entries $\beta_k + \delta_k$ are at most $(1 + \tau\eta^{-1})\eta^{-k}$, and the off-diagonal entries $-\sqrt{\beta_k \delta_{k+1}} = -\tau^{\frac{1}{2}} \delta_{k+1}$ have modulus at most $\tau^{\frac{1}{2}} \eta^{-1} \eta^{-k}$. Since $1 + \tau\eta^{-1} + 2\tau^{\frac{1}{2}} \eta^{-1} \leq (1 + \tau^{\frac{1}{2}})^2 \eta^{-1} \leq 4\eta^{-1}$, the Schur test shows that if g is supported in $\{0, \dots, N\}$, then $L_\eta g$ is supported in $\{0, \dots, N+1\}$ and

$$(4.9) \quad \|L_\eta g\| \leq 4\eta^{-N-1} \|g\|.$$

The sequence ψ_n vanishes on $\{k \leq n-2\}$ and is constant on $\{k \geq n\}$, and two of its values differ by at most $((1-\tau)\tau^n)^{-\frac{1}{2}}$. Thus $L_\eta \psi_n$ is supported in $\{n-2, n-1, n\}$ and $|L_\eta \psi_n(k)| \leq (1 + \tau\eta^{-1})\eta^{-k} ((1-\tau)\tau^n)^{-\frac{1}{2}}$, so that

$$\|L_\eta \psi_n\|^2 \leq (1 + \tau\eta^{-1})^2 \sum_{k=n-2}^n \tau^{k-n} \eta^{-2k} \leq 3(1 + \tau\eta^{-1})^2 \tau^{-2} \eta^{-2n}.$$

Let $K \geq 1$. For $1 \leq i \leq K$, the sequence $L_\eta^i \psi_n$ is supported in $\{0, \dots, n+i-1\}$, and (4.9) gives

$$\|L_\eta^K \psi_n\| \leq \sqrt{3}(1 + \tau\eta^{-1})\tau^{-1}\eta^{-n} \prod_{i=1}^{K-1} 4\eta^{-n-i} = \sqrt{3}(1 + \tau\eta^{-1})\tau^{-1}4^{K-1}\eta^{-Kn - \frac{K(K-1)}{2}}.$$

Since L_η is self-adjoint and $L_\eta e_m = \Lambda_m e_m$, we have $\Lambda_m^K u_{nm} = \langle e_m, L_\eta^K \psi_n \rangle$. We choose $K = m - n$. Using (4.4), we obtain

$$|u_{nm}| \leq c_\eta^{-K} \eta^{Km} \|L_\eta^K \psi_n\| \leq \frac{\sqrt{3}(1 + \tau\eta^{-1})}{4\tau} \left(\frac{4}{c_\eta}\right)^K \eta^{\frac{K^2}{2} + \frac{K}{2}} = C_2 B^K \eta^{\frac{K^2}{2}}.$$

This proves (4.7). ■

Theorem 4.6 Suppose that $\tau = Q^{-d}$, let $\alpha > 0$ and $\vartheta > 0$, and put $\eta = Q^{-\vartheta}$ and $s \stackrel{\text{def}}{=} \frac{\alpha}{\vartheta}$.

1. We have $\text{dom } \mathcal{E}_R \subset \text{dom } L_\eta^{\frac{s}{2}}$, and there exists a constant $c > 0$ such that $c \|L_\eta^{\frac{s}{2}} f\|^2 \leq \mathcal{E}_R(f)$ for any $f \in \text{dom } \mathcal{E}_R$.
2. If $\alpha < d + 2\vartheta$, then $\text{dom } \mathcal{E}_R = \text{dom } L_\eta^{\frac{s}{2}}$, and there exists a constant C such that $\mathcal{E}_R(f) \leq C \|L_\eta^{\frac{s}{2}} f\|^2$ for any f in this space. Thus the closed forms of R_{rad}^α and L_η^s are equivalent.
3. If $\alpha \geq d + 2\vartheta$, then the inclusion of part 1 is strict.

The constants c and C depend only on Q , d , α and ϑ . In particular, for $\vartheta = d$, the closed forms of R_{rad}^α and $L_\tau^{\frac{\alpha}{d}}$ are equivalent if and only if $\alpha < 3d$.

Proof : Both quantities vanish on the constants and are unchanged if a constant is added to f . Thus we may suppose that $f \perp 1$. For $n, m \geq 1$, put $a_n \stackrel{\text{def}}{=} \langle f, \psi_n \rangle$ and $b_m \stackrel{\text{def}}{=} \langle f, e_m \rangle$. Then $a_n = \sum_m u_{nm} b_m$ and $b_m = \sum_n \overline{u_{nm}} a_n$, the series being absolutely convergent by the Cauchy-Schwarz inequality, since $\sum_m |u_{nm}|^2 = \sum_n |u_{nm}|^2 = 1$. By Lemma 2.5 and the spectral theorem, we have in $[0, \infty]$

$$\mathcal{E}_R(f) = \sum_{n \geq 1} \lambda_{n-1} |a_n|^2, \quad \|L_\eta^{\frac{s}{2}} f\|^2 = \sum_{m \geq 1} \Lambda_m^s |b_m|^2.$$

Since $\eta^{-s} = Q^\alpha = \rho$ and $0 < \kappa < 1$, (4.4) gives $c_\eta^s \rho^m \leq \Lambda_m^s \leq \rho^m$, and $(1 - \rho^{-1})\rho^n \leq \lambda_{n-1} = \rho^n - \kappa \leq \rho^n$. We shall use the following consequence of the Schur test: if a matrix $(k_{ij})_{i,j \geq 1}$ satisfies $|k_{ij}| \leq k(i-j)$ for a function $k \in \ell_{\mathbb{Z}}^1$, then it defines a bounded operator on $\ell_{\mathbb{N}}^2$ with norm at most $\|k\|_{\ell_{\mathbb{Z}}^1}$.

1. Suppose that $\mathcal{E}_R(f) < \infty$ and put $x_n \stackrel{\text{def}}{=} \rho^{\frac{n}{2}} a_n$. Then $\rho^{\frac{m}{2}} b_m = \sum_n \rho^{\frac{m-n}{2}} \overline{u_{nm}} x_n$. By Lemma 4.5, $\rho^{\frac{m-n}{2}} |u_{nm}| \leq k_1(m-n)$, where $k_1(j) \stackrel{\text{def}}{=} C_2(B\rho^{\frac{1}{2}})^j \eta^{\frac{j^2}{2}}$ for $j > 0$ and $k_1(j) \stackrel{\text{def}}{=} C_1(\tau\eta^2\rho^{-1})^{\frac{|j|}{2}}$ for $j \leq 0$. Since $k_1 \in \ell_{\mathbb{Z}}^1$, we obtain

$$\|L_\eta^{\frac{s}{2}} f\|^2 \leq \sum_{m \geq 1} \rho^m |b_m|^2 \leq \|k_1\|_{\ell_{\mathbb{Z}}^1}^2 \sum_{n \geq 1} \rho^n |a_n|^2 \leq \frac{\|k_1\|_{\ell_{\mathbb{Z}}^1}^2}{1 - \rho^{-1}} \mathcal{E}_R(f).$$

2. Suppose that $\alpha < d + 2\vartheta$, that is $\rho\tau\eta^2 = Q^{\alpha-d-2\vartheta} < 1$, and that $\|L_\eta^{\frac{s}{2}} f\| < \infty$. Put $y_m \stackrel{\text{def}}{=} \rho^{\frac{m}{2}} b_m$. Then $\rho^{\frac{n}{2}} a_n = \sum_m \rho^{\frac{n-m}{2}} \overline{u_{nm}} y_m$ and $\rho^{\frac{n-m}{2}} |u_{nm}| \leq k_2(n-m)$, where $k_2(j) \stackrel{\text{def}}{=}$

$C_1(\rho\tau\eta^2)^{\frac{j}{2}}$ for $j \geq 0$ and $k_2(j) \stackrel{\text{def}}{=} C_2(B\rho^{-\frac{1}{2}})^{|j|}\eta^{\frac{j^2}{2}}$ for $j < 0$. Since $\rho\tau\eta^2 < 1$, the function k_2 belongs to $\ell_{\mathbb{Z}}^1$, and

$$\mathcal{E}_R(f) \leq \sum_{n \geq 1} \rho^n |a_n|^2 \leq \|k_2\|_{\ell_{\mathbb{Z}}^1}^2 \sum_{m \geq 1} \rho^m |b_m|^2 \leq \frac{\|k_2\|_{\ell_{\mathbb{Z}}^1}^2}{c_\eta^s} \|L_\eta^{\frac{s}{2}} f\|^2.$$

3. This is Proposition 4.7 below. The last assertion is the case $\vartheta = d$ of parts 2 and 3. $\blacksquare$

Proposition 4.7 *Suppose that $\tau = Q^{-d}$, $\eta = Q^{-\vartheta}$ and $\alpha \geq d + 2\vartheta$. For any Borel function $F: [0, \infty) \rightarrow [0, \infty)$, the form domain of $F(L_\eta)$ contains the polynomials $\mathcal{P}$, whereas the function $k \mapsto \eta^k$ does not belong to $\text{dom } \mathcal{E}_R$. In particular, no operator of the form $F(L_\eta)$ has the same form domain as R_{rad}^α .*

Proof : The polynomials are finite linear combinations of eigenvectors of L_η , hence belong to the form domain of $F(L_\eta)$. Let $f_k = \eta^k$. Before normalization, the inner product of f with the numerator of (2.8) is the difference of two geometric tail moments,

$$\sum_{k \geq n+1} w_k \eta^k - \tau \sum_{k \geq n} w_k \eta^k = -\frac{(1-\tau)(1-\eta)}{1-\tau\eta} \tau^{n+1} \eta^n.$$

Hence $\langle f, g_n \rangle = -\frac{(1-\tau)^{\frac{1}{2}}(1-\eta)}{1-\tau\eta} \tau^{\frac{n+1}{2}} \eta^n$. By Lemma 2.5, $\mathcal{E}_R(f) = \frac{(1-\tau)(1-\eta)^2}{(1-\tau\eta)^2} \sum_n \lambda_n \tau^{n+1} \eta^{2n}$. Since $(1-\rho^{-1})\rho^{n+1} \leq \lambda_n \leq \rho^{n+1}$, this is infinite if and only if $\rho\tau\eta^2 \geq 1$, that is $\alpha \geq d + 2\vartheta$. $\blacksquare$

Remark 4.8 The norm of the kernel k_2 in the proof of Theorem 4.6 is at least $C_1(1 - (\rho\tau\eta^2)^{\frac{1}{2}})^{-1}$, so that the constant C obtained in part 2 tends to infinity as $\alpha \rightarrow d + 2\vartheta$, in accordance with part 3. The threshold comes from the subdiagonal decay $(\tau^{\frac{1}{2}}\eta)^{n-m}$ in Lemma 4.5, which is governed by the first order Taylor coefficients of the polynomials e_m at 0. This decay is attained in the first column: since e_1 is a polynomial of degree one in η^k , the proof of Proposition 4.7 shows that u_{n1} is a constant multiple of $(\tau^{\frac{1}{2}}\eta)^n$. In terms of the exponent s , the condition $\alpha < d + 2\vartheta$ reads

$$s < 2 + \frac{d}{\vartheta} = 2 + \frac{\log \tau}{\log \eta}.$$

For $s \leq 1$, part 2 also follows from Theorem 4.1 applied with the order ϑ and from part 2 of Corollary 4.2 with $\theta = s$, since the operators R_{rad}^α and $(R_{\text{rad}}^\vartheta)^s$ are diagonal in the basis (g_n) with comparable eigenvalues. The new range is therefore $\vartheta < \alpha < d + 2\vartheta$. For $\vartheta = d$ and $\alpha \leq d$, part 2 recovers the form comparison of Corollary 7.8 below. For $s > 1$, the operator L_η^s does not generate a positivity preserving semigroup for $\vartheta = d$, see Theorem 7.10, and for $\vartheta = \frac{d}{n-1}$, see Theorem 10.5.

5 The quantum group $\text{SU}_r(2)$ and the Podleś sphere

In this section $0 < r < 1$ is arbitrary and $\tau = r^2$.

5.1 The concrete model

We use the standard realization of the compact quantum group $\text{SU}_r(2)$ from [CV22, Section 5.3]. Its defining relations and coproduct are those of [NT13, Example 1.1.5, pp. 4–5] and [CFK14, Section 11], where our generator b is denoted by α . On $\ell_{\mathbb{N}_0}^2 \otimes L^2(\mathbb{T})$ let

$$b(e_n \otimes f) = \sqrt{1 - \tau^n} e_{n-1} \otimes f, \quad \gamma(e_n \otimes f) = r^n e_n \otimes zf.$$

Here $e_{-1} = 0$, and z is the coordinate unitary on $\mathbb{T}$. Write $C(\mathrm{SU}_r(2)) \stackrel{\mathrm{def}}{=} C^*(b, \gamma)$ and $\mathcal{M} \stackrel{\mathrm{def}}{=} C(\mathrm{SU}_r(2))''$. In this model

$$(5.1) \quad \mathcal{M} = B(\ell_{\mathbb{N}_0}^2) \overline{\otimes} L^\infty(\mathbb{T}), \quad h(x) = (1 - \tau) \sum_{n \geq 0} \tau^n \int_{\mathbb{T}} x_{nn}(z) dm(z).$$

The Haar formula is [CV22, (5.1)]. It defines a faithful normal state on $\mathcal{M}$. The von Neumann algebra equality can also be seen directly: the polar part of b supplies the unilateral shift, spectral projections of $\gamma^* \gamma$ supply matrix units, and the polar part of γ supplies $1 \otimes z$ in the von Neumann closure. Since h is faithful and normal on $\mathcal{M}$, the algebra $\mathcal{M}$ is identified with the von Neumann algebra $L^\infty(\mathrm{SU}_r(2))$ generated by the GNS representation of h .

We recall the relations and the coproduct, fixing all conventions:

$$(5.2) \quad b^* b + A = 1, \quad bb^* + \tau A = 1, \quad \gamma \gamma^* = \gamma^* \gamma = A, \quad b \gamma = r \gamma b, \quad b \gamma^* = r \gamma^* b,$$

$$(5.3) \quad \Delta(b) = b \otimes b - r \gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes b + b^* \otimes \gamma.$$

The fundamental representation is $u^{(\frac{1}{2})} = \begin{bmatrix} b & -r \gamma^* \\ \gamma & b^* \end{bmatrix}$. The coproduct extends to a normal unital $*$ -homomorphism $\Delta: \mathcal{M} \rightarrow \mathcal{M} \overline{\otimes} \mathcal{M}$, since it is implemented by the regular representation [NT13, Theorem 1.5.3]. We denote by $\mathrm{Pol}(\mathrm{SU}_r(2))$ the $*$ -algebra generated by b and γ , and by ε its counit, determined by $\varepsilon(b) = 1$ and $\varepsilon(\gamma) = 0$.

Let V be the unilateral forward shift. Then $b = V^*(1 - A)^{\frac{1}{2}}$ and $V^* = (1 - \tau A)^{-\frac{1}{2}} b \in C(\mathrm{SU}_r(2))$. Consequently $e_{00} \otimes 1 = 1 - VV^*$ and all matrix units $e_{ij} \otimes 1 = V^i e_{00} V^{*j}$ belong to $C(\mathrm{SU}_r(2))$. More generally, $e_{ij} \otimes z^k \in C(\mathrm{SU}_r(2))$ for $k \in \mathbb{Z}$: first use $p_n \gamma = r^n e_{nn} \otimes z$, where $p_n = e_{nn} \otimes 1$, and then the matrix units.

With the semifinite trace $\mathrm{Tr} \otimes m$ on $\mathcal{M}$, the density of h is $D_h = (1 - \tau) \mathrm{diag}(\tau^n) \otimes 1$. Hence its modular group is

$$\sigma_s^h(e_{ij} \otimes f) = \tau^{\mathrm{is}(i-j)} e_{ij} \otimes f, \quad \sigma_s^h(b) = \tau^{-\mathrm{is}} b, \quad \sigma_s^h(\gamma) = \gamma.$$

The formula follows from $\sigma_s^h(x) = D_h^{\mathrm{is}} x D_h^{-\mathrm{is}}$ and agrees with [NT13, Example 1.7.4, p. 31].

5.2 Torus actions and the symbol map

For $\lambda \in \mathbb{T}$ let D_λ be the diagonal unitary of $\ell_{\mathbb{N}_0}^2$ given by $D_\lambda e_n = \lambda^{-n} e_n$, and let ϑ_λ be the unitary of $L^2(\mathbb{T})$ given by $(\vartheta_\lambda f)(\zeta) = f(\lambda \zeta)$. For $\lambda, \mu \in \mathbb{T}$ we put

$$(5.4) \quad \beta_{\lambda, \mu}(x) \stackrel{\mathrm{def}}{=} (D_{\lambda \mu} \otimes \vartheta_{\lambda^{-1} \mu}) x (D_{\lambda \mu} \otimes \vartheta_{\lambda^{-1} \mu})^*, \quad x \in B(\ell_{\mathbb{N}_0}^2 \otimes L^2(\mathbb{T})).$$

A direct computation gives

$$(5.5) \quad \beta_{\lambda, \mu}(b) = \lambda \mu b, \quad \beta_{\lambda, \mu}(\gamma) = \lambda^{-1} \mu \gamma, \quad \beta_{\lambda, \mu}(e_{ij} \otimes z^k) = (\lambda \mu)^{j-i} (\lambda^{-1} \mu)^k e_{ij} \otimes z^k.$$

Thus each $\beta_{\lambda, \mu}$ is a normal $*$ -automorphism of $\mathcal{M}$ which leaves $C(\mathrm{SU}_r(2))$ invariant and preserves h , since $D_{\lambda \mu}$ commutes with D_h and $\vartheta_{\lambda^{-1} \mu}$ preserves the Haar measure of $\mathbb{T}$. The map $(\lambda, \mu) \mapsto \beta_{\lambda, \mu}(x)$ is norm continuous for any $x \in C(\mathrm{SU}_r(2))$: this is clear on the $*$ -algebra generated by b and γ by (5.5), and follows in general by density, the automorphisms being isometric.

Lemma 5.1 *The C^* -algebra $\mathcal{J} \stackrel{\mathrm{def}}{=} \mathcal{K}(\ell_{\mathbb{N}_0}^2) \otimes_{\min} C(\mathbb{T})$ is a closed two-sided ideal of $C(\mathrm{SU}_r(2))$. There exists a unique $*$ -homomorphism $\sigma: C(\mathrm{SU}_r(2)) \rightarrow C(\mathbb{T})$ with kernel $\mathcal{J}$ such that $\sigma(b) = z$ and $\sigma(\gamma) = 0$. For $\lambda \in \mathbb{T}$, the map $\varepsilon_\lambda \stackrel{\mathrm{def}}{=} \mathrm{ev}_\lambda \circ \sigma$ is a character of $C(\mathrm{SU}_r(2))$ with $\varepsilon_\lambda(b) = \lambda$ and $\varepsilon_\lambda(\gamma) = 0$, and ε_1 extends the counit ε .*

Proof : By the preceding subsection, $\mathcal{J}$ is contained in $C(\mathrm{SU}_r(2))$. Since $b \in B(\ell_{\mathbb{N}_0}^2) \otimes 1$ and $\gamma \in \mathcal{J}$, the algebra $C(\mathrm{SU}_r(2))$ is contained in $B(\ell_{\mathbb{N}_0}^2) \otimes_{\min} C(\mathbb{T})$, in which $\mathcal{J}$ is an ideal. Hence $\mathcal{J}$ is an ideal of $C(\mathrm{SU}_r(2))$. The elements $A = \mathrm{diag}(\tau^n) \otimes 1$ and γ belong to $\mathcal{J}$. By (5.2), the image $\dot{b}$ of b in the quotient satisfies $\dot{b}^* \dot{b} = 1 = \dot{b} \dot{b}^*$, and the quotient is generated by the unitary $\dot{b}$. It is therefore commutative and isomorphic to $C(\mathrm{Sp}(\dot{b}))$. The automorphism $\beta_{1,\mu}$ leaves $\mathcal{J}$ invariant by (5.5), hence induces an automorphism of the quotient sending $\dot{b}$ to $\mu \dot{b}$. Thus $\mathrm{Sp}(\dot{b}) = \mu \mathrm{Sp}(\dot{b})$ for any $\mu \in \mathbb{T}$, and $\mathrm{Sp}(\dot{b}) = \mathbb{T}$. Composing the quotient map with the Gelfand isomorphism gives σ . The last assertions follow, since ε_1 and ε are unital homomorphisms which agree on b and γ . $\blacksquare$

A direct computation on the generators, using (5.3), $\varepsilon_\lambda(\gamma) = 0$ and coassociativity, shows that

$$(5.6) \quad \beta_{\lambda,\mu} = (\varepsilon_\lambda \otimes \mathrm{Id} \otimes \varepsilon_\mu) \circ (\Delta \otimes \mathrm{Id}) \circ \Delta \quad \text{on } C(\mathrm{SU}_r(2)).$$

Indeed, both sides are $*$ -homomorphisms. For instance, $(\Delta \otimes \mathrm{Id})\Delta(\gamma) = (\gamma \otimes b + b^* \otimes \gamma) \otimes b + (b^* \otimes b^* - r\gamma \otimes \gamma^*) \otimes \gamma$, and applying $\varepsilon_\lambda \otimes \mathrm{Id} \otimes \varepsilon_\mu$ gives $\bar{\lambda}\mu\gamma$. Thus $\beta_{\lambda,1}$ and $\beta_{1,\mu}$ are the left and right translations by the classical diagonal torus of $\mathrm{SU}_r(2)$.

5.3 Matrix coefficients and zonal spherical functions

We use the irreducible unitary representations $u^{(l)} = (u_{jk}^{(l)})_{j,k}$, $l \in \frac{1}{2}\mathbb{N}_0$, of dimension $2l+1$, with indices $j, k \in \{-l, -l+1, \dots, l\}$, as in [CFK14, Section 11]; for $l = \frac{1}{2}$ we label the fundamental representation so that $u_{-\frac{1}{2}, -\frac{1}{2}}^{(\frac{1}{2})} = b$ and $u_{\frac{1}{2}, \frac{1}{2}}^{(\frac{1}{2})} = b^*$. We have $\Delta(u_{jk}^{(l)}) = \sum_m u_{jm}^{(l)} \otimes u_{mk}^{(l)}$ and $\varepsilon(u_{jk}^{(l)}) = \delta_{jk}$. The matrix coefficients form a linear basis of $\mathrm{Pol}(\mathrm{SU}_r(2))$, by [NT13, Example 2.4.8] and the orthogonality relations [NT13, Theorem 1.4.3]. Moreover, if Pol_n denotes the space of polynomials of degree at most n in b, b^*, γ, γ^* and $\mathcal{C}(l) \stackrel{\mathrm{def}}{=} \mathrm{span}\{u_{jk}^{(l)}\}$, then

$$(5.7) \quad \mathrm{Pol}_n = \bigoplus_{l \in \frac{1}{2}\mathbb{N}_0, 2l \leq n} \mathcal{C}(l),$$

see the proof in [NT13, Example 2.4.8]. Finally, the Woronowicz characters f_ζ , $\zeta \in \mathbb{C}$, are unital homomorphisms of $\mathrm{Pol}(\mathrm{SU}_r(2))$ [NT13, Proposition 1.7.2], and [CFK14, (11.2)]

$$(5.8) \quad f_\zeta(u_{jk}^{(l)}) = \delta_{jk} r^{2j\zeta}.$$

Lemma 5.2 *For any $l \in \frac{1}{2}\mathbb{N}_0$, any $j, k \in \{-l, \dots, l\}$ and any $\lambda, \mu \in \mathbb{T}$, we have*

$$(5.9) \quad \varepsilon_\lambda(u_{jk}^{(l)}) = \delta_{jk} \lambda^{-2j}, \quad \beta_{\lambda,\mu}(u_{jk}^{(l)}) = \lambda^{-2j} \mu^{-2k} u_{jk}^{(l)}.$$

Consequently, the elements of $\mathrm{Pol}(\mathrm{SU}_r(2))$ fixed by all $\beta_{\lambda,\mu}$ form the space $\mathrm{span}\{u_{00}^{(l)} : l \in \mathbb{N}_0\}$, and the elements fixed by all $\beta_{1,\mu}$ form the space $\mathrm{span}\{u_{j0}^{(l)} : l \in \mathbb{N}_0, |j| \leq l\}$.

Proof : For $\zeta \in \mathbb{C} \setminus \{0\}$, there is a unique unital algebra homomorphism $\chi_\zeta : \mathrm{Pol}(\mathrm{SU}_r(2)) \rightarrow \mathbb{C}$ such that $\chi_\zeta(b) = \zeta$, $\chi_\zeta(b^*) = \zeta^{-1}$ and $\chi_\zeta(\gamma) = \chi_\zeta(\gamma^*) = 0$, since these values satisfy the defining relations (5.2) and their adjoints. For any noncommutative monomial x in b, b^*, γ, γ^* , the number $\chi_\zeta(x)$ is either 0 or ζ^m for an integer m independent of ζ . Hence $\zeta \mapsto \chi_\zeta(x)$ is a Laurent polynomial for any $x \in \mathrm{Pol}(\mathrm{SU}_r(2))$. By (5.8) and the labeling of $u^{(\frac{1}{2})}$, the homomorphism f_ζ takes the values $r^{-\zeta}$, r^ζ , 0, 0 on b, b^*, γ, γ^* , so that $f_\zeta = \chi_{r^{-\zeta}}$. By (5.8), we

deduce that $\chi_x(u_{jk}^{(l)}) = \delta_{jk}x^{-2j}$ for any $x > 0$. Two Laurent polynomials which agree on $(0, \infty)$ are equal, hence this equality holds for any $x \in \mathbb{C} \setminus \{0\}$. For $\lambda \in \mathbb{T}$, the homomorphisms χ_λ and ε_λ agree on b, b^*, γ, γ^* , which proves the first equality of (5.9). The second follows from (5.6) and $\Delta^{(2)}(u_{jk}^{(l)}) = \sum_{a,c} u_{ja}^{(l)} \otimes u_{ac}^{(l)} \otimes u_{ck}^{(l)}$. The last assertions follow from the linear independence of the matrix coefficients, since $-2j = 0$ if and only if $j = 0$, which requires $l \in \mathbb{N}_0$. $\blacksquare$

Proposition 5.3 *For any integer $l \geq 0$, we have $u_{00}^{(l)} = \varphi_l^\tau(A)$, where φ_l^τ is the polynomial of Proposition 3.1 with $\eta = \tau$, that is the little τ -Legendre polynomial normalized by $\varphi_l^\tau(0) = 1$. Moreover $|\varphi_l^\tau(x)| \leq 1$ for any $x \in X_\tau$.*

Proof : Let $n \geq 0$. The element $A^n = (\gamma^* \gamma)^n$ belongs to Pol_{2n} and is fixed by all $\beta_{\lambda, \mu}$, by (5.5). By (5.7) and Lemma 5.2, the polynomials of degree at most n in A are contained in the space $\text{span}\{u_{00}^{(l)} : l \in \mathbb{N}_0, l \leq n\}$. Both spaces have dimension $n+1$, since the spectrum X_τ of A is infinite. Hence they are equal for any n , and by induction $u_{00}^{(n)} = p_n(A)$ for a polynomial p_n of degree exactly n .

Recall that $h(F(A)) = \sum_k w_k F(\tau^k)$ by (5.1). By the orthogonality relations [NT13, Theorem 1.4.3 (ii)], $h((u_{00}^{(l)})^* u_{00}^{(m)}) = 0$ for $l \neq m$. Thus the polynomials p_l are orthogonal in $\ell_{\mathbb{N}_0}^2(w)$, as functions of τ^k . Since $\varepsilon(\gamma) = 0$, we have $\varepsilon(A) = 0$ and $\varepsilon(p(A)) = p(0)$ for any polynomial p ; hence $p_l(0) = \varepsilon(u_{00}^{(l)}) = 1$. By part 4 of Proposition 3.1, $p_l = \varphi_l^\tau$. Finally, $u^{(l)}$ is unitary, so that $\sum_j (u_{j0}^{(l)})^* u_{j0}^{(l)} = 1$ and $\|u_{00}^{(l)}\| \leq 1$. Since the spectrum of A is X_τ , this gives $\sup_{X_\tau} |\varphi_l^\tau| \leq 1$. $\blacksquare$

The identification of zonal matrix coefficients with little q -Legendre polynomials is classical, see [VS88], [Koo89], [MMNNU91] and the survey [Koe96]; the proof above only uses the facts recalled in this subsection.

Remark 5.4 We use the following conventions for $U_r \mathfrak{sl}_2$, compare [NT13, Section 2.4]: $KEK^{-1} = r^2 E$, $KFK^{-1} = r^{-2} F$, $EF - FE = \frac{K-K^{-1}}{r-r^{-1}}$,

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad K^* = K, \quad E^* = FK, \quad F^* = K^{-1}E.$$

One checks on the generators that Δ is a $*$ -homomorphism. Put $[x]_r \stackrel{\text{def}}{=} \frac{r^x - r^{-x}}{r - r^{-1}}$. For $s \in \frac{1}{2}\mathbb{N}_0$, the irreducible $*$ -module V_s has an orthonormal basis $(\xi_i)_{|i| \leq s}$ with

$$K\xi_i = r^{2i}\xi_i, \quad F\xi_i = r^{-i}\sqrt{[s+i]_r[s-i+1]_r}\xi_{i-1}, \quad E\xi_i = r^{i+1}\sqrt{[s-i]_r[s+i+1]_r}\xi_{i+1},$$

where $\xi_{\pm(s+1)} = 0$. Indeed $E = (FK)^*$ on this basis, and $[E, F]\xi_i = \frac{r^{2i} - r^{-2i}}{r - r^{-1}}\xi_i$ follows from $[s+i]_r[s-i+1]_r - [s-i]_r[s+i+1]_r = [2i]_r$. Consider the Casimir element

$$\Omega \stackrel{\text{def}}{=} EF + \frac{r^{-1}K + rK^{-1}}{(r - r^{-1})^2}.$$

Since $[E, EF] = \frac{E(K-K^{-1})}{r-r^{-1}}$ and $[E, r^{-1}K + rK^{-1}] = -(r-r^{-1})E(K-K^{-1})$, we have $[E, \Omega] = 0$, and in the same way $[F, \Omega] = 0$. Hence Ω is central, and it acts on the irreducible module V_s by a scalar, which we compute on ξ_s . Since $E\xi_s = 0$, we have

$$\Omega\xi_s = \left(\frac{r^{2s} - r^{-2s}}{r - r^{-1}} + \frac{r^{2s-1} + r^{1-2s}}{(r - r^{-1})^2} \right) \xi_s = \frac{r^{2s+1} + r^{-2s-1}}{(r - r^{-1})^2} \xi_s.$$

Thus Ω acts on V_l by the scalar $\Omega_l = \frac{r^{2l+1} + r^{-2l-1}}{(r - r^{-1})^2}$. Hence, for any $l \in \frac{1}{2}\mathbb{N}_0$,

$$(5.11) \quad \Lambda_l \stackrel{\text{def}}{=} r^{-2l} - 1 - r^2 + r^{2l+2} = r(r - r^{-1})^2(\Omega_l - \Omega_0),$$

and $\Lambda_l = \Lambda_l(\tau)$ for $l \in \mathbb{N}_0$, by (1.4). Through the duality between $U_r \mathfrak{sl}_2$ and $\text{Pol}(\text{SU}_r(2))$, the operator $x \mapsto (\text{Id} \otimes \Omega) \Delta(x)$ acts on $\mathcal{C}(l)$ by Ω_l . Thus L_τ is, up to the affine normalization (5.11), the restriction of this Casimir operator to the zonal polynomials $\varphi_l^\tau(A)$.

5.4 The Podleś sphere in the concrete model

We define

$$C(S_r^2) \stackrel{\text{def}}{=} \{x \in C(\text{SU}_r(2)) : \beta_{1,\mu}(x) = x \text{ for any } \mu \in \mathbb{T}\}$$

and

$$L^\infty(S_r^2) \stackrel{\text{def}}{=} \{x \in \mathcal{M} : \beta_{1,\mu}(x) = x \text{ for any } \mu \in \mathbb{T}\}.$$

By (5.6), $C(S_r^2)$ is the algebra of elements of $C(\text{SU}_r(2))$ invariant under right translations by the diagonal torus, that is the standard Podleś sphere [Pod87]. We put $\text{Pol}(S_r^2) \stackrel{\text{def}}{=} \text{Pol}(\text{SU}_r(2)) \cap C(S_r^2)$, which is $\text{span}\{u_{j0}^{(l)} : l \in \mathbb{N}_0, |j| \leq l\}$ by Lemma 5.2. Let W be the unitary of $\ell_{\mathbb{N}_0}^2 \otimes L^2(\mathbb{T})$ defined by $W(e_n \otimes f) = e_n \otimes z^n f$, and put

$$\theta(x) \stackrel{\text{def}}{=} W(x \otimes 1)W^*, \quad x \in B(\ell_{\mathbb{N}_0}^2).$$

Finally, let ω_τ be the faithful normal state $\omega_\tau(x) \stackrel{\text{def}}{=} \text{Tr}(Dx)$ on $B(\ell_{\mathbb{N}_0}^2)$, where $D \stackrel{\text{def}}{=} (1 - \tau)\text{diag}(\tau^n)$. It is the Gibbs state of the harmonic oscillator at inverse temperature $\log \frac{1}{\tau}$.

Proposition 5.5 *1. The map θ is a normal $*$ -isomorphism from $B(\ell_{\mathbb{N}_0}^2)$ onto $L^\infty(S_r^2)$, with $\theta(e_{ij}) = e_{ij} \otimes z^{i-j}$, and $h \circ \theta = \omega_\tau$.*

2. We have $\theta(\mathcal{K}(\ell_{\mathbb{N}_0}^2) + \mathbb{C}1) = C(S_r^2)$, and $\text{Pol}(S_r^2)$ is dense in $C(S_r^2)$.

3. The image under θ of the diagonal algebra $\ell_{\mathbb{N}_0}^\infty$ is $W^(A)$, which is the algebra of elements of $\mathcal{M}$ fixed by all $\beta_{\lambda,\mu}$. In particular, $W^*(A)$ is a maximal abelian subalgebra of $L^\infty(S_r^2)$. The elements of $C(\text{SU}_r(2))$ fixed by all $\beta_{\lambda,\mu}$ form the algebra $C^*(A)$, which is θ of the convergent diagonal sequences.*

4. The averages $\mathbb{E}_R \stackrel{\text{def}}{=} \int_{\mathbb{T}} \beta_{1,\mu} d\mu$ and $\mathbb{E} \stackrel{\text{def}}{=} \int_{\mathbb{T}^2} \beta_{\lambda,\mu} d\lambda d\mu$ are faithful normal h -preserving conditional expectations from $\mathcal{M}$ onto $L^\infty(S_r^2)$ and $W^(A)$, which map $C(\text{SU}_r(2))$ onto $C(S_r^2)$ and $C^*(A)$ respectively.*

Proof : 1. We have $W^*(e_n \otimes f) = e_n \otimes z^{-n}f$ and $W(e_{ij} \otimes 1)W^*(e_n \otimes f) = \delta_{jn}e_i \otimes z^{i-j}f$, which gives $\theta(e_{ij}) = e_{ij} \otimes z^{i-j}$. The map θ is spatially implemented, hence a normal injective $*$ -homomorphism. Its range is contained in $L^\infty(S_r^2)$ by (5.5). Conversely, let $y \in \mathcal{M}$ be fixed by all $\beta_{1,\mu}$, with matrix entries $y_{ij} \in L^\infty(\mathbb{T})$. By (5.5), the Fourier coefficient of index k of y_{ij} is multiplied by μ^{j-i+k} under $\beta_{1,\mu}$. Hence $y_{ij} = c_{ij}z^{i-j}$ for some $c_{ij} \in \mathbb{C}$, and W^*yW has constant entries c_{ij} , that is $W^*yW = c \otimes 1$ with $c \in B(\ell_{\mathbb{N}_0}^2)$. Thus $y = \theta(c)$. Finally, $h(\theta(x)) = (1 - \tau) \sum_n \tau^n x_{nn} = \omega_\tau(x)$ by (5.1).

2. The elements $\theta(e_{ij}) = e_{ij} \otimes z^{i-j}$ belong to $C(\text{SU}_r(2))$ and are fixed by $\beta_{1,\mu}$. Hence $\theta(\mathcal{K}(\ell_{\mathbb{N}_0}^2) + \mathbb{C}1) \subset C(S_r^2)$. Conversely, let $x \in C(S_r^2)$. We have $\sigma \circ \beta_{1,\mu} = \text{rot}_\mu \circ \sigma$, where $(\text{rot}_\mu g)(\zeta) = g(\mu\zeta)$, since both sides are $*$ -homomorphisms which agree on b and γ . Hence $\sigma(x)$ is a rotation-invariant continuous function, that is a constant c . Then $x - c1 \in \mathcal{J}$ is fixed by all $\beta_{1,\mu}$. Averaging over μ , the closed linear span of the elements $e_{ij} \otimes z^k$ of $\mathcal{J}$ is mapped onto the closed span of the elements $e_{ij} \otimes z^{i-j}$, which is $\theta(\mathcal{K}(\ell_{\mathbb{N}_0}^2))$. Hence $x \in \theta(\mathcal{K}(\ell_{\mathbb{N}_0}^2)) + \mathbb{C}1$. For the density, note that $\mathbb{E}_R$ maps $C(\text{SU}_r(2))$ onto $C(S_r^2)$ (see part 4), that $\text{Pol}(\text{SU}_r(2))$ is dense in $C(\text{SU}_r(2))$, and that $\mathbb{E}_R(\text{Pol}(\text{SU}_r(2))) = \text{Pol}(S_r^2)$ by Lemma 5.2.

3. We have $\theta(e_{nn}) = e_{nn} \otimes 1 = p_n = \mathbf{1}_{\{\tau^n\}}(A)$. Hence $\theta(\ell_{\mathbb{N}_0}^\infty) = W^*(A)$. By (5.5), an element of $\mathcal{M}$ is fixed by all $\beta_{\lambda,\mu}$ if and only if its matrix entries y_{ij} vanish for $i \neq j$ and are constant for $i = j$. The diagonal algebra is maximal abelian in $B(\ell_{\mathbb{N}_0}^2)$. Let $x \in C(\mathrm{SU}_r(2))$ be fixed by all $\beta_{\lambda,\mu}$. As in part 2, $\sigma(x)$ is a constant c , and $x - c1$ belongs to the closed span of the elements $e_{nn} \otimes 1$, that is to $\theta(c_0)$. Hence $x = F(A)$ for a function $F \in C(X_\tau)$. The converse inclusion is clear.

4. Each $\beta_{\lambda,\mu}$ is a normal h -preserving automorphism, and the map $(\lambda, \mu) \mapsto \beta_{\lambda,\mu}(x)$ is σ -weakly continuous for $x \in \mathcal{M}$ and norm continuous for $x \in C(\mathrm{SU}_r(2))$. The averages are therefore normal unital completely positive idempotents onto the fixed-point algebras, preserving h , and they are bimodular over their ranges. Faithfulness follows from faithfulness of h . The integrals of norm continuous functions are norm limits of Riemann sums, which gives the last assertion. $\blacksquare$

From now on, we identify $C^*(A)$ with $C(X_\tau)$ through $F(A) \leftrightarrow F$, and $W^*(A)$ with $\ell_{\mathbb{N}_0}^\infty$. For $\tau = Q^{-d}$, we recover the identification (1.1).

Theorem 5.6 *Suppose that $\tau = Q^{-d}$. There is a unique normal unital state-preserving $*$ -isomorphism*

$$J : L^\infty(\mathcal{O}^d)_{\mathrm{rad}} \rightarrow W^*(A), \quad J(F(\|\cdot\|_K^d)) = F(A),$$

where F ranges over bounded Borel functions on X_τ . It maps $\mathbf{1}_{S_n}$ to p_n , $\mathbf{1}_{B_n}$ to $P_n \stackrel{\mathrm{def}}{=} \sum_{k \geq n} p_k$, and continuous radial functions onto $C^*(A)$. It induces an isometry J_2 from $\ell_{\mathbb{N}_0}^2(w)$ onto the closure of $W^*(A)$ in the GNS space of h .

Proof : A radial bounded function is a sequence (f_n) , and $Jf = \sum_n f_n p_n$ is its diagonal realization. Orthogonality and strong summability of the p_n prove multiplicativity, the $*$ -property, normality, and bijectivity onto $W^*(A)$. Equality of the weights in (2.2) and (5.1) proves preservation of the state and the L^2 assertion. Values assigned to F at 0 are immaterial at the von Neumann level, and continuous radial functions correspond to convergent sequences. $\blacksquare$

6 Boundary energies of random walks

In this section, we compare the boundary theory of the simple random walk on the tree of balls of $\mathcal{O}^d$ with the boundary theory of the random walk on the dual of $\mathrm{SU}_r(2)$ driven by the fundamental representation. We prove Theorem 1.2. Throughout, $0 < r < 1$ and $\tau = r^2$, and we have $\tau = Q^{-d}$ whenever the local field is involved.

6.1 The tree of balls

Let $\mathbb{T}$ be the rooted tree whose vertices at level $n \geq 0$ are the balls $x + \pi^n \mathcal{O}^d$ of $\mathcal{O}^d$, a ball being joined to the Q^d balls of the next level which it contains. Its boundary is $\mathcal{O}^d$. We consider the simple random walk on $\mathbb{T}$, with transition probabilities $p_{\mathbb{T}}(v, w)$ and reversible measure $m_{\mathbb{T}}(v) = \deg(v)$. By symmetry, its harmonic measure seen from the root is the normalized Haar measure of $\mathcal{O}^d$. For any $F \in L^2(\mathcal{O}^d)$, we denote by $\mathcal{H}_{\mathbb{T}}F$ the harmonic extension of F to the vertices, that is the integral of F against the harmonic measure seen from each vertex, and we put

$$(6.1) \quad \mathcal{E}_{\mathbb{T}}(F) \stackrel{\mathrm{def}}{=} \sum_v m_{\mathbb{T}}(v) \sum_w p_{\mathbb{T}}(v, w) |\mathcal{H}_{\mathbb{T}}F(w) - \mathcal{H}_{\mathbb{T}}F(v)|^2 \in [0, \infty].$$

Up to normalization, this is the Douglas integral of the trace of the walk on the boundary, see [Kig10] and [Woe12]. Recall that $\mathcal{M}^d$ denotes the Fourier multiplier of the compact group $\mathcal{O}^d$ which is zero on constants and equals $\|\xi\|_K^d$ on nontrivial characters (Proposition 2.1).

Theorem 6.1 *Suppose that $\tau = Q^{-d}$. For any $F \in L^2(\mathcal{O}^d)$, we have*

$$\mathcal{E}_{\mathbb{T}}(F) = 2(1 - \tau) \langle F, \mathcal{M}^d F \rangle_{L^2(\mathcal{O}^d)}$$

in $[0, \infty]$, where the right-hand side is the closed form of $\mathcal{M}^d$.

Proof : We first consider a wavelet $F = \sum_c f_c \mathbf{1}_c$ attached to a ball B of level n , where c runs over the Q^d balls of level $n+1$ contained in B and $\sum_c f_c = 0$. By symmetry, $\mathcal{H}_{\mathbb{T}} F$ vanishes at the vertex B and outside the subtree of B . From a child c of a vertex, the walk hits this vertex with probability τ . Hence $\mathcal{H}_{\mathbb{T}} F = f_c(1 - \tau^{j+1})$ at the vertices of the subtree of c at depth $j \geq 0$ below c . The edge from B to c contributes $(1 - \tau)^2 |f_c|^2$ to the edge energy, and the edges below c contribute

$$\sum_{j \geq 0} \tau^{-(j+1)} |f_c|^2 \tau^{2(j+1)} (1 - \tau)^2 = (1 - \tau) \tau |f_c|^2.$$

Thus the edge energy of $\mathcal{H}_{\mathbb{T}} F$ is $(1 - \tau) \sum_c |f_c|^2 = (1 - \tau) \tau^{-(n+1)} \|F\|_{L^2(\mathcal{O}^d)}^2$. Since $m_{\mathbb{T}}(v)p_{\mathbb{T}}(v, w) = 1$ for neighbours v, w , the energy (6.1) is twice the edge energy. On the other hand, the wavelets attached to the balls of level n are the finite linear combinations of characters of conductor $n+1$, on which $\mathcal{M}^d$ acts by $Q^{d(n+1)} = \tau^{-(n+1)}$.

Now, we consider wavelets F_1 and F_2 attached to different balls B_1 and B_2 . If $B_1 \cap B_2 = \emptyset$, the gradients of $\mathcal{H}_{\mathbb{T}} F_1$ and $\mathcal{H}_{\mathbb{T}} F_2$ have disjoint supports. If $B_2 \subsetneq B_1$, the gradient of $\mathcal{H}_{\mathbb{T}} F_1$ on the edges below B_2 depends only on the depth, whereas, at any depth below B_2 , the sum of the gradients of $\mathcal{H}_{\mathbb{T}} F_2$ is a multiple of the sum of the coefficients of F_2 , which is zero. Hence, for any $m \geq 0$, the harmonic extensions of wavelets attached to different balls are orthogonal for the energy of the edges between the levels m and $m+1$. The same holds for two wavelets attached to the same ball, if they are orthogonal in $L^2(\mathcal{O}^d)$, by the above computation.

Finally, let $F \in L^2(\mathcal{O}^d)$ and let $F = c + \sum_{\psi} \psi$ be its orthogonal expansion along the constants and an orthogonal basis of wavelets. For a vertex v of level n , the harmonic measure seen from v has a constant density on each ball of level n , by symmetry. Since the harmonic extension of a wavelet attached to a ball of level m vanishes at the vertices of level at most m , we deduce that, on the levels m and $m+1$, the function $\mathcal{H}_{\mathbb{T}} F$ is the finite sum of c and of the functions $\mathcal{H}_{\mathbb{T}} \psi$, ψ attached to balls of level at most m . By the orthogonality of the previous paragraph, the energy of $\mathcal{H}_{\mathbb{T}} F$ on the edges between the levels m and $m+1$ is the sum of the corresponding energies of these functions $\mathcal{H}_{\mathbb{T}} \psi$. Summing over m and exchanging the sums of nonnegative terms, we obtain

$$\mathcal{E}_{\mathbb{T}}(F) = \sum_{\psi} \mathcal{E}_{\mathbb{T}}(\psi) = 2(1 - \tau) \sum_{\psi} \langle \psi, \mathcal{M}^d \psi \rangle = 2(1 - \tau) \langle F, \mathcal{M}^d F \rangle.$$

■

On radial functions, Proposition 2.1 gives $\mathcal{M}_{\text{rad}}^d = R_{\text{rad}}^d + \kappa(I - \mathbb{E}_0)$ with $\kappa = \frac{1}{1+\tau}$, since $\rho = \tau^{-1}$ for $\alpha = d$. Thus the restriction of $\mathcal{E}_{\mathbb{T}}$ to radial functions is $2(1 - \tau)$ times the Taibleson form of order d , up to the bounded perturbation $\kappa(I - \mathbb{E}_0)$.

6.2 The random walk on the dual of $\mathrm{SU}_r(2)$

Let $G = \mathrm{SU}_r(2)$. We use the $*$ -modules V_s , $s \in \frac{1}{2}\mathbb{N}_0$, of $U_r \mathfrak{sl}_2$ and the conventions of Remark 5.4; in particular $K\xi_i = \tau^i \xi_i$ and (5.10) holds. Since Δ is a $*$ -homomorphism, the Hilbert space tensor product of two such modules is a $*$ -module, which is an orthogonal direct sum of copies of the modules V_r . The von Neumann algebra of the dual discrete quantum group is $\ell^\infty(\widehat{G}) = \bigoplus_s \mathcal{B}(V_s)$, and its coproduct acts on $V_s \otimes V_t$ by $\widehat{\Delta}(a) = \sum_r \iota_r a_r \iota_r^*$, where $\iota_r: V_r \rightarrow V_s \otimes V_t$ are the isometric intertwiners. Consider the state $\phi \stackrel{\text{def}}{=} [2]_r^{-1} \mathrm{Tr}(K^{-1} \cdot)$ on $\mathcal{B}(V_{\frac{1}{2}})$, the normalized quantum trace, and put

$$(6.2) \quad \mathbf{P} \stackrel{\text{def}}{=} (\mathrm{Id} \otimes \phi) \circ \widehat{\Delta}, \quad \widehat{h}(a) \stackrel{\text{def}}{=} \sum_s [2s+1]_r \mathrm{Tr}(K^{-1} a_s).$$

On central elements, $\mathbf{P}$ is the quantum dimension walk, with transition probability $\frac{[2s'+1]_r}{[2]_r[2s+1]_r}$ from s to $s' = s \pm \frac{1}{2}$. Since $\mathrm{Tr}(K^{-1} \cdot)$ is invariant under the adjoint action of $U_r \mathfrak{sl}_2$ on each $\mathcal{B}(V_s)$, the operator $\mathbf{P}$ commutes with the adjoint action of G on $\ell^\infty(\widehat{G})$, and the weight $\widehat{h}$ is invariant under this action.

We denote by $H^\infty(\widehat{G}, \mathbf{P})$ the space of bounded $\mathbf{P}$ -harmonic elements of $\ell^\infty(\widehat{G})$. For any bounded function $f: \mathbb{N}_0 \rightarrow \mathbb{C}$, we put $f(A) \stackrel{\text{def}}{=} \sum_k f(k) p_k \in \mathcal{W}^*(A)$, where $p_k = \mathbf{1}_{\{\tau^k\}}(A)$, and we identify a function F on X_τ , for instance a polynomial, with the function $k \mapsto F(\tau^k)$ on $\mathbb{N}_0$. We use the following properties of the Poisson integral $\Theta: L^\infty(S_r^2) \rightarrow H^\infty(\widehat{G}, \mathbf{P})$, which are established in [Izu02], see also [INT06, Section 2 and Theorem B], where the convolution operator is written with the opposite convention: the map Θ is normal, unital, completely positive and bijective; it is G -equivariant for the action of G on $L^\infty(S_r^2)$ by left translations and the adjoint action on $\ell^\infty(\widehat{G})$; it satisfies $\widehat{\varepsilon} \circ \Theta = h$, where $\widehat{\varepsilon}$ is the evaluation at the trivial representation; and it is multiplicative for the product

$$(6.3) \quad a \cdot b \stackrel{\text{def}}{=} s^* \text{-} \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{j=0}^{m-1} \mathbf{P}^j(ab)$$

of $H^\infty(\widehat{G}, \mathbf{P})$. In particular, the Poisson boundary of $(\widehat{G}, \mathbf{P})$ is the Podleś sphere. Its Martin boundary is also the Podleś sphere [NT04]. For $r = 1$, that is on the dual of the compact group $\mathrm{SU}(2)$, the classical counterparts are due to Biane: positive harmonic weights of nondegenerate random walks on the dual of a compact group are integrals of harmonic exponentials, an analogue of the Choquet–Deny theorem [Bia92], and for central transient random walks on the dual of $\mathrm{SU}(2)$ the Martin compactification is an extension of $C(S^2)$ by $C^*(\mathrm{SU}(2))$ [Bia94], see also [Bia08, Sections 8 and 9].

For any harmonic element a , the element $\Gamma(a) \stackrel{\text{def}}{=} \mathbf{P}(a^* a) - a^* a$ is positive by the Kadison–Schwarz inequality. We define the energy of a at level $n = 2s$ and its total energy by

$$(6.4) \quad \mathcal{E}^{(n)}(a) \stackrel{\text{def}}{=} [2s+1]_r \mathrm{Tr}(K^{-1} \Gamma(a)_s), \quad \mathcal{E}_{\widehat{G}}(a) \stackrel{\text{def}}{=} \sum_{n \geq 0} \mathcal{E}^{(n)}(a) = \widehat{h}(\Gamma(a)) \in [0, \infty].$$

For the simple random walk on $\mathbb{T}$, with the weight $m_{\mathbb{T}}$ in place of $\widehat{h}$, the total energy reduces to (6.1).

The diagonal chain. The diagonal elements of $\ell^\infty(\widehat{G})$, that is the elements fixed by the adjoint action of the torus, form a commutative subalgebra which is invariant under $\mathbf{P}$. Hence

P induces a Markov chain on the set of pairs (s, i) with $|i| \leq s$. We use the coordinates $n \stackrel{\text{def}}{=} 2s$ and $k \stackrel{\text{def}}{=} s - i \in \{0, \dots, n\}$, so that the state space is

$$\mathbb{D} \stackrel{\text{def}}{=} \{(n, k) \in \mathbb{N}_0^2 : k \leq n\},$$

and we say that (n, k) has level n . We write $P(v, v')$ for the transition probabilities, so that $(Pf)(v) = \sum_{v'} P(v, v')f(v')$ for functions f on $\mathbb{D}$.

Proposition 6.2 *For any $(n, k) \in \mathbb{D}$, the transition probabilities of the diagonal chain are*

$$\begin{aligned} (n, k) \rightarrow (n+1, k) &: \frac{1 - \tau^{n-k+1}}{(1+\tau)(1-\tau^{n+1})}, & (n, k) \rightarrow (n-1, k) &: \frac{\tau(1 - \tau^{n-k})}{(1+\tau)(1-\tau^{n+1})}, \\ (n, k) \rightarrow (n+1, k+1) &: \frac{\tau^{n-k+1}(1 - \tau^{k+1})}{(1+\tau)(1-\tau^{n+1})}, & (n, k) \rightarrow (n-1, k-1) &: \frac{\tau^{n-k+1}(1 - \tau^k)}{(1+\tau)(1-\tau^{n+1})}. \end{aligned}$$

The chain is reversible for the measure

$$(6.5) \quad \widehat{w}(n, k) \stackrel{\text{def}}{=} [n+1]_r r^{2k-n} = \frac{1 - \tau^{n+1}}{(1 - \tau)\tau^{n-k}},$$

which is the restriction of $\widehat{h}$ to the diagonal.

Proof : We first compute the Clebsch–Gordan coefficients of $V_s \otimes V_{\frac{1}{2}}$, $s \geq \frac{1}{2}$ (for $s = 0$ only $V_{\frac{1}{2}}$ occurs). For $-s-1 \leq i \leq s$, put

$$x_i \stackrel{\text{def}}{=} \left(r^{i-s} \frac{[s+i+1]_r}{[2s+1]_r} \right)^{\frac{1}{2}}, \quad y_i \stackrel{\text{def}}{=} \left(r^{s+i+1} \frac{[s-i]_r}{[2s+1]_r} \right)^{\frac{1}{2}}.$$

Expanding the r -numbers gives $r^{i-s}[s+i+1]_r + r^{s+i+1}[s-i]_r = [2s+1]_r$, that is $x_i^2 + y_i^2 = 1$. We claim that, up to a phase,

$$(6.6) \quad \iota_{s+\frac{1}{2}} \xi_{i+\frac{1}{2}} = x_i \xi_i \otimes \xi_{\frac{1}{2}} + y_i \xi_{i+1} \otimes \xi_{-\frac{1}{2}}, \quad -s-1 \leq i \leq s,$$

with $\xi_{\pm(s+1)} = 0$. The vector $\xi_s \otimes \xi_{\frac{1}{2}}$ has the maximal weight $s + \frac{1}{2}$, hence spans the highest weight space of the unique copy of $V_{s+\frac{1}{2}}$; this is (6.6) for $i = s$. Suppose that (6.6) holds for some $i > -s-1$, and apply $\Delta(F) = F \otimes K^{-1} + 1 \otimes F$. By (5.10), we have $F\xi_{\frac{1}{2}} = r^{-\frac{1}{2}}\xi_{-\frac{1}{2}}$, $F\xi_{-\frac{1}{2}} = 0$ and $K^{-1}\xi_{\frac{1}{2}} = r^{-1}\xi_{\frac{1}{2}}$, so that the coefficient of $\xi_{i-1} \otimes \xi_{\frac{1}{2}}$ in $\Delta(F)\iota_{s+\frac{1}{2}}\xi_{i+\frac{1}{2}}$ is $r^{-1}r^{-i}\sqrt{[s+i]_r[s-i+1]_r}x_i$. On the other hand, (5.10) in $V_{s+\frac{1}{2}}$ gives $\Delta(F)\iota_{s+\frac{1}{2}}\xi_{i+\frac{1}{2}} = r^{-i-\frac{1}{2}}\sqrt{[s+i+1]_r[s-i+1]_r}\iota_{s+\frac{1}{2}}\xi_{i-\frac{1}{2}}$. Hence the modulus of the coefficient of $\xi_{i-1} \otimes \xi_{\frac{1}{2}}$ in the unit vector $\iota_{s+\frac{1}{2}}\xi_{i-\frac{1}{2}}$ is $(r^{-1}\frac{[s+i]_r}{[s+i+1]_r})^{\frac{1}{2}}x_i = x_{i-1}$. Since this vector belongs to the span of $\xi_{i-1} \otimes \xi_{\frac{1}{2}}$ and $\xi_i \otimes \xi_{-\frac{1}{2}}$, its other coefficient has modulus $(1 - x_{i-1}^2)^{\frac{1}{2}} = y_{i-1}$, which proves (6.6) by induction. Finally, for $|i + \frac{1}{2}| \leq s - \frac{1}{2}$, the weight space of weight $i + \frac{1}{2}$ of $V_s \otimes V_{\frac{1}{2}}$ is spanned by $\xi_i \otimes \xi_{\frac{1}{2}}$ and $\xi_{i+1} \otimes \xi_{-\frac{1}{2}}$, and it is the orthogonal sum of the lines spanned by $\iota_{s+\frac{1}{2}}\xi_{i+\frac{1}{2}}$ and $\iota_{s-\frac{1}{2}}\xi_{i+\frac{1}{2}}$. Hence the coefficients of $\iota_{s-\frac{1}{2}}\xi_{i+\frac{1}{2}}$ on these two vectors have moduli y_i and x_i . Writing $i - \frac{1}{2} = (i-1) + \frac{1}{2}$, we obtain

$$|\langle \iota_{s+\frac{1}{2}}\xi_{i+\frac{1}{2}}, \xi_i \otimes \xi_{\frac{1}{2}} \rangle|^2 = r^{i-s} \frac{[s+i+1]_r}{[2s+1]_r}, \quad |\langle \iota_{s+\frac{1}{2}}\xi_{i-\frac{1}{2}}, \xi_i \otimes \xi_{-\frac{1}{2}} \rangle|^2 = r^{s+i} \frac{[s-i+1]_r}{[2s+1]_r},$$

$$|\langle \iota_{s-\frac{1}{2}} \xi_{i+\frac{1}{2}}, \xi_i \otimes \xi_{\frac{1}{2}} \rangle|^2 = r^{s+i+1} \frac{[s-i]_r}{[2s+1]_r}, \quad |\langle \iota_{s-\frac{1}{2}} \xi_{i-\frac{1}{2}}, \xi_i \otimes \xi_{-\frac{1}{2}} \rangle|^2 = r^{i-s-1} \frac{[s+i]_r}{[2s+1]_r}.$$

Since ϕ takes the values $\frac{1}{1+\tau}$ and $\frac{\tau}{1+\tau}$ on the projections onto $\mathbb{C}\xi_{\frac{1}{2}}$ and $\mathbb{C}\xi_{-\frac{1}{2}}$, and since $\iota_{s'}\xi_{i'}$ has weight i' , we have, for any diagonal element a ,

$$P(a)(s, i) = \sum_{m=\pm\frac{1}{2}} \frac{\tau^{\frac{1}{2}-m}}{1+\tau} \sum_{s'=s\pm\frac{1}{2}} |\langle \iota_{s'} \xi_{i+m}, \xi_i \otimes \xi_m \rangle|^2 a(s', i+m).$$

The formulas follow from $[a]_r/[b]_r = r^{b-a} \frac{1-\tau^a}{1-\tau^b}$ and from $s+i = n-k$, $s-i = k$. Detailed balance is a direct verification. Finally, the diagonal entry of K^{-1} at ξ_i is $\tau^{-i} = r^{2k-n}$, which gives (6.5). $\blacksquare$

These transition probabilities were first computed by Biane [Bia06, Section 4]. He considers the quantum Bernoulli random walk on the dual of $SU_q(2)$ [Bia90], restricted to the subalgebra generated by the centre and by the generator of the torus, which is our diagonal algebra, for the state defined by the Woronowicz character, which is our state ϕ and the state used by Izumi [Izu02]. With his conventions, the dictionary is $q^2 = \tau$ and $(r, k) = (n, 2k-n)$, where his first coordinate is $n = 2s$ and his second coordinate is twice the weight, up to sign. For the tracial state $\frac{1}{2} \text{Tr}$ in place of ϕ , the same subalgebra is invariant, and Biane showed that the corresponding chain converges, as $q \rightarrow 0$, to the Markov chain $(2S - X, X)$ of Pitman's theorem, where X is the simple random walk and S its running maximum [Bia06, Sections 2 and 4], see also [Bia08, Section 13]. For the state ϕ , the limit $\tau \rightarrow 0$ is degenerate: by Proposition 6.2, the chain then moves deterministically from (n, k) to $(n+1, k)$.

Remark 6.3 Suppose that $\tau = Q^{-d}$ and consider the $(Q^d + 1)$ -regular tree, which is the Bruhat–Tits tree of PGL_2 over the unramified extension of degree d of K , whose residue field has Q^d elements [Ser80, Chapter II, Section 1.1], and its radial chain relative to a boundary point: n is the distance to the root and k is the length of the common part with the geodesic ray from the root to the boundary point. Off this ray, the chain moves to $(n+1, k)$ and $(n-1, k)$ with probabilities $\frac{1}{1+\tau}$ and $\frac{\tau}{1+\tau}$. On the ray, that is if $k = n \geq 1$, it moves to $(n+1, n+1)$, $(n+1, n)$ and $(n-1, n-1)$ with probabilities $\frac{\tau}{1+\tau}$, $\frac{1-\tau}{1+\tau}$ and $\frac{\tau}{1+\tau}$. By Proposition 6.2, the diagonal chain moves from (n, n) to $(n+1, n+1)$ with the same probability $\frac{\tau}{1+\tau}$, and its other transition probabilities differ from those of the tree by $O(\tau^{n-k})$ off the ray and by $O(\tau^n)$ on the ray. The difference lies in the “horizontal” moves $(n, k) \leftrightarrow (n+1, k+1)$ off the ray, whose probabilities are of order τ^{n-k} .

Recall that the closed form of L_τ is $\mathcal{E}_\tau(f) = \sum_{k \geq 0} w_k \beta_k |f(k+1) - f(k)|^2$ with $w_k \beta_k = (1-\tau)(1-\tau^{k+1})$, see (3.1) and (3.2) with $\eta = \tau$. We put

$$(6.7) \quad \mathfrak{c}_\tau \stackrel{\text{def}}{=} \frac{2\tau}{(1+\tau)(1-\tau)^2}.$$

Proposition 6.4 *For any $(n, k) \in \mathbb{D}$, the conductance of the edge between (n, k) and $(n+1, k+1)$ is*

$$\widehat{w}(n, k) P((n, k), (n+1, k+1)) = \mathfrak{b}_k \stackrel{\text{def}}{=} \frac{\tau(1-\tau^{k+1})}{(1+\tau)(1-\tau)} = \frac{\mathfrak{c}_\tau}{2} w_k \beta_k.$$

In particular, it does not depend on the level n . Moreover, the conductance of the edge between (n, k) and $(n+1, k)$ is

$$\widehat{w}(n, k) P((n, k), (n+1, k)) = \mathfrak{a}_{n-k}, \quad \text{where } \mathfrak{a}_d \stackrel{\text{def}}{=} \frac{1-\tau^{d+1}}{(1-\tau^2)\tau^d}, \quad d \in \mathbb{N}_0.$$

Consequently, in the coordinates $(d, k) = (n - k, k) \in \mathbb{N}_0^2$, the diagonal chain is the random walk associated with the Cartesian product of the networks on $\mathbb{N}_0$ with conductances $(\mathbf{a}_d)$ and $(\mathbf{b}_k)$ between consecutive points.

Proof : By Proposition 6.2 and (6.5), the first conductance is $\frac{1-\tau^{n+1}}{(1-\tau)\tau^{n-k}} \cdot \frac{\tau^{n-k+1}(1-\tau^{k+1})}{(1+\tau)(1-\tau^{n+1})} = \frac{\tau(1-\tau^{k+1})}{(1+\tau)(1-\tau)}$, and the second one is $\frac{1-\tau^{n+1}}{(1-\tau)\tau^{n-k}} \cdot \frac{1-\tau^{n-k+1}}{(1+\tau)(1-\tau^{n+1})} = \mathbf{a}_{n-k}$. In the coordinates (d, k) , the moves to $(n \pm 1, k)$ change d by ± 1 and keep k , and the moves to $(n \pm 1, k \pm 1)$ keep d and change k by ± 1 . By reversibility, the conductance of an edge does not depend on its orientation, which gives the last assertion. $\blacksquare$

The following lemma is the key algebraic property of the diagonal chain.

Lemma 6.5 *Let $j \geq 0$ be an integer and let $g: \mathbb{N}_0 \rightarrow \mathbb{C}$ be a function. Consider the function f on $\mathbb{D}$ defined by $f(n, k) = g(n)\tau^{jk}$. For any $n \geq 1$ and any $0 \leq k \leq n$, we have*

$$(6.8) \quad (\mathbf{P}f)(n, k) = (A_j g)(n)\tau^{jk} + (B_j g)(n)\tau^{(j-1)k},$$

where

$$\begin{aligned} (A_j g)(n) &\stackrel{\text{def}}{=} \frac{(1 - \tau^{n+j+2})g(n+1) + (\tau - \tau^{n-j+1})g(n-1)}{(1+\tau)(1-\tau^{n+1})}, \\ (B_j g)(n) &\stackrel{\text{def}}{=} \frac{\tau^{n+1}[(\tau^j - 1)g(n+1) + (\tau^{-j} - 1)g(n-1)]}{(1+\tau)(1-\tau^{n+1})}. \end{aligned}$$

Proof : Fix $n \geq 1$ and $0 \leq k \leq n$, and put $y = \tau^k$. The probability of the move to $(n-1, k)$ vanishes if $k = n$, and the probability of the move to $(n-1, k-1)$ vanishes if $k = 0$. Hence we may use the formula $f(n', k') = g(n')\tau^{jk'}$ for all the terms of $(\mathbf{P}f)(n, k)$. By Proposition 6.2, the number $(1+\tau)(1-\tau^{n+1})(\mathbf{P}f)(n, k)$ is equal to

$$\begin{aligned} &(1 - \tau^{n+1}y^{-1})g(n+1)y^j + \tau(1 - \tau^n y^{-1})g(n-1)y^j \\ &+ \tau^{n+1}y^{-1}(1 - \tau y)g(n+1)\tau^j y^j + \tau^{n+1}y^{-1}(1 - y)g(n-1)\tau^{-j}y^j. \end{aligned}$$

Collecting the terms in y^j and in y^{j-1} , we obtain (6.8). $\blacksquare$

Note that $B_0 = 0$ and that A_0 is the quantum dimension walk of the levels, since $\frac{1-\tau^{n+2}}{(1+\tau)(1-\tau^{n+1})} = \frac{[n+2]_r}{[2]_r[n+1]_r}$ and $\frac{\tau-\tau^{n+1}}{(1+\tau)(1-\tau^{n+1})} = \frac{[n]_r}{[2]_r[n+1]_r}$. In particular, the level of the diagonal chain is itself a Markov chain.

For any integer $l \geq 0$, we denote by $\mathcal{V}_l$ the space of functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that, for any $n \geq 0$, the function $k \mapsto f(n, k)$ on $\{0, \dots, n\}$ is of the form $k \mapsto p(\tau^k)$ for some polynomial p of degree at most l . On the levels $n < l$ this condition is empty, since any function on $n+1 \leq l$ points is the restriction of a polynomial of degree at most $l-1$. Writing an element of $\mathcal{V}_l$ as $f(n, k) = \sum_{j=0}^l g_j(n)\tau^{jk}$, Lemma 6.5 shows that $\mathbf{P}(\mathcal{V}_l) \subset \mathcal{V}_l$.

Remark 6.6 The restrictions to the level $n = 2s$ of the elements of $\mathcal{V}_l$ are the diagonals of the operators $\sum_{j=0}^l a_j K^{-j}$ on V_s , since $\tau^{-ij} = \tau^{-sj}\tau^{jk}$. Consider the adjoint action $\text{ad}(E)(T) = ET - KTK^{-1}E$, $\text{ad}(F)(T) = FTK - TFK$ and $\text{ad}(K)(T) = KTK^{-1}$ on $\mathbf{B}(V_s)$. A direct computation gives $\text{ad}(E)(K^{-1}) = (1 - \tau^{-1})EK^{-1}$, $\text{ad}(E)(EK^{-1}) = 0$, $\text{ad}(F)(K^{-1}) = (1 - \tau)F$ and $\text{ad}(F)(F) = 0$. Since K^{-1} has weight zero and $\text{ad}(E)^2(K^{-1}) = \text{ad}(F)^2(K^{-1}) = 0$, the operator K^{-1} lies in the sum of the spin components of $\mathbf{B}(V_s)$ of spin at most 1. Since the adjoint action is a module algebra action, the powers K^{-j} , $j \leq l$, lie in the sum of the spin components of spin at most l . If $n \geq l$, both spaces of diagonal operators have dimension $l+1$. Hence the restriction of $\mathcal{V}_l$ to the level n is the space of diagonal operators of spin at most l , and the inclusion $\mathbf{P}(\mathcal{V}_l) \subset \mathcal{V}_l$ is the diagonal form of the equivariance of $\mathbf{P}$. We do not use this remark below.

6.3 Harmonic functions of the diagonal chain

We denote by $(n_t, k_t)_{t \geq 0}$ the diagonal chain, by $\mathbf{P}_v$ and $\mathbf{E}_v$ the probability and the expectation for the chain started at $v \in \mathbb{D}$, and by $j_t \stackrel{\text{def}}{=} n_t - k_t$ the distance to the ray $\{(n, n) : n \geq 0\}$. A move is called horizontal if it changes k , that is if it is of the form $(n, k) \rightarrow (n \pm 1, k \pm 1)$.

Lemma 6.7 1. For any $(n, k) \in \mathbb{D}$, we have $\mathbf{E}_{(n,k)}[\sum_{t \geq 0} r^{j_t}] \leq \frac{1+\tau}{(1-\tau)(1-r)^2} r^{n-k}$.

2. Almost surely, the chain makes only finitely many horizontal moves, $n_t \rightarrow \infty$ and k_t is eventually constant. We denote by k_∞ its final value.

3. For any bounded function $f: \mathbb{N}_0 \rightarrow \mathbb{C}$, the function $\mathcal{H}f$ defined on $\mathbb{D}$ by $\mathcal{H}f(v) \stackrel{\text{def}}{=} \mathbf{E}_v[f(k_\infty)]$ is bounded and harmonic and, for any $(n, k) \in \mathbb{D}$,

$$(6.9) \quad |\mathcal{H}f(n, k) - f(k)| \leq \frac{4\tau}{(1-\tau)^2(1-r)^2} \|f\|_\infty r^{n-k}.$$

4. If u is a bounded harmonic function on $\mathbb{D}$ such that $g(k) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} u(n, k)$ exists for any $k \in \mathbb{N}_0$, then $u = \mathcal{H}g$.

Proof : 1. From a state (n, k) , the distance $j = n - k$ increases by one with probability $p_+ \stackrel{\text{def}}{=} \frac{1-\tau^{j+1}}{(1+\tau)(1-\tau^{n+1})}$, decreases by one with probability $p_- \stackrel{\text{def}}{=} \frac{\tau(1-\tau^j)}{(1+\tau)(1-\tau^{n+1})}$, and is unchanged otherwise. We have $p_- \leq \tau p_+$ and $p_+ \geq \frac{1-\tau}{1+\tau}$. Since $r^{-1} - 1 > 0$, we deduce that

$$\begin{aligned} \mathbf{E}[r^{j_{t+1}} - r^{j_t} | (n_t, k_t)] &= r^{j_t} [p_+(r-1) + p_-(r^{-1}-1)] \leq r^{j_t} p_+ [(r-1) + \tau(r^{-1}-1)] \\ &= -r^{j_t} p_+ (1-r)^2 \leq -\frac{(1-\tau)(1-r)^2}{1+\tau} r^{j_t}. \end{aligned}$$

Taking expectations and summing over $0 \leq t < T$, we obtain $\frac{(1-\tau)(1-r)^2}{1+\tau} \sum_{t < T} \mathbf{E}_{(n,k)}[r^{j_t}] \leq r^{n-k}$. Letting $T \rightarrow \infty$ gives the assertion.

2. By Proposition 6.2, the probability of a horizontal move from (n, k) is

$$\frac{\tau^{n-k+1}(2-\tau^{k+1}-\tau^k)}{(1+\tau)(1-\tau^{n+1})} \leq \frac{2\tau}{(1+\tau)(1-\tau)} \tau^{n-k} \leq \frac{2\tau}{(1+\tau)(1-\tau)} r^{n-k}.$$

By the first part, the expected number of horizontal moves of the chain started at (n, k) is at most $\frac{2\tau}{(1-\tau)^2(1-r)^2} r^{n-k} < \infty$. Hence there are almost surely finitely many horizontal moves, and k_t is eventually constant, since k changes only through horizontal moves. Moreover $\sum_t r^{j_t} < \infty$ almost surely, so that $j_t \rightarrow \infty$ and $n_t \geq j_t \rightarrow \infty$.

3. The function $\mathcal{H}f$ is bounded and harmonic by the Markov property. If the chain started at (n, k) makes no horizontal move, then $k_\infty = k$. Thus $|\mathcal{H}f(n, k) - f(k)|$ is at most $2\|f\|_\infty$ times the probability of at least one horizontal move, which is bounded by the expected number of horizontal moves. This proves (6.9).

4. The process $(u(n_t, k_t))_{t \geq 0}$ is a bounded martingale. By the second part, we have almost surely $k_t = k_\infty$ for any t large enough and $n_t \rightarrow \infty$, so that $u(n_t, k_t) \rightarrow g(k_\infty)$. Note that g is bounded by $\|u\|_\infty$. By dominated convergence, $u(v) = \mathbf{E}_v[u(n_t, k_t)] \rightarrow \mathbf{E}_v[g(k_\infty)] = \mathcal{H}g(v)$ for any $v \in \mathbb{D}$. $\blacksquare$

For any $n \in \mathbb{N}_0 \cup \{\infty\}$, let $\mathbf{m}_n \stackrel{\text{def}}{=} \sum_{0 \leq k \leq n} \tau^k \delta_{\tau^k}$. Then $(1-\tau)\mathbf{m}_\infty$ is the Aref'eva-Volovich law, that is the law of A under h , and the measure $\hat{w}(n, \cdot)$ on $\{0, \dots, n\}$ is proportional to the

image of $\mathfrak{m}_n$ under $\tau^k \mapsto k$. For any integers $0 \leq l \leq n$, we denote by $M_{l,n}$ the monic polynomial of degree l which is orthogonal in $L^2(\mathfrak{m}_n)$ to the polynomials of degree at most $l-1$. It exists and is unique, since $\mathfrak{m}_n$ has $n+1 \geq l+1$ atoms. We let $\widehat{p}_l$ be the analogous polynomial for $\mathfrak{m}_\infty$. By part 4 of Proposition 3.1 with $\eta = \tau$, we have $\varphi_l^\tau = \varkappa_l \widehat{p}_l$, where $\varkappa_l \neq 0$ is the leading coefficient of φ_l^τ . Finally, for any integer $l \geq 1$, we define the sequence $(\varsigma_l(n))_{n \geq l}$ by $\varsigma_l(l) = 1$, $\varsigma_l(l-1) = 0$ and, for any $n \geq l$,

$$(6.10) \quad (1 - \tau^{n+l+2})\varsigma_l(n+1) = (1 + \tau)(1 - \tau^{n+1})\varsigma_l(n) - (\tau - \tau^{n-l+1})\varsigma_l(n-1).$$

The coefficient of $\varsigma_l(n-1)$ vanishes for $n = l$. With the notation of Lemma 6.5, the relation (6.10) means $(A_{l\varsigma_l})(n) = \varsigma_l(n)$.

Lemma 6.8 Put $P_\tau \stackrel{\text{def}}{=} (\tau^4; \tau)_\infty^{-1}$ and $M_\tau \stackrel{\text{def}}{=} \prod_{i \geq 0} (1 + 4P_\tau(i+1)\tau^i)$. For any integer $l \geq 1$ and any integer $n \geq l-1$, we have

$$(6.11) \quad |\varsigma_l(n)| \leq M_\tau, \quad |\varsigma_l(n+1) - \varsigma_l(n)| \leq 4P_\tau M_\tau (n-l+2)\tau^{n-l+1}.$$

Consequently, the sequence $(\varsigma_l(n))_{n \geq l}$ converges to a limit $\varsigma_l(\infty)$, and $|\varsigma_l(n) - \varsigma_l(\infty)| \leq \frac{4P_\tau M_\tau}{(1-\tau)^2} (n-l+2)\tau^{n-l+1}$ for any $n \geq l-1$.

Proof : Put $\delta(n) \stackrel{\text{def}}{=} \varsigma_l(n+1) - \varsigma_l(n)$ for $n \geq l-1$, so that $\delta(l-1) = 1$. A rearrangement of (6.10) gives, for any $n \geq l$,

$$(6.12) \quad (1 - \tau^{n+l+2})\delta(n) = \tau\delta(n-1) + \tau^n [(\tau^{l+2} - \tau - \tau^2)\varsigma_l(n) + \tau^{1-l}\varsigma_l(n-1)].$$

For $n \geq l-1$, put $M(n) \stackrel{\text{def}}{=} \max\{1, |\varsigma_l(l-1)|, \dots, |\varsigma_l(n)|\}$. The sequence $(M(n))$ is nondecreasing and $M(l-1) = 1$. Let $n \geq l$. We have $\tau^n |\tau^{l+2} - \tau - \tau^2| \leq 2\tau^{n+1} \leq 2\tau^{n-l+1}$ and $1 - \tau^{n+l+2} \geq 1 - \tau^{n-l+4}$, since $l \geq 1$. Thus (6.12) gives

$$|\delta(n)| \leq \frac{\tau|\delta(n-1)| + 3\tau^{n-l+1}M(n)}{1 - \tau^{n-l+4}}.$$

Put $a(n) \stackrel{\text{def}}{=} \frac{|\delta(n)|}{\tau^{n-l+1}M(n)}$ for $n \geq l-1$, so that $a(l-1) = 1$. Since $M(n-1) \leq M(n)$, we obtain $a(n) \leq \frac{a(n-1)+3}{1-\tau^{n-l+4}}$ for any $n \geq l$. By induction, we deduce that

$$a(n) \leq (1 + 3(n-l+1)) \prod_{i=0}^{n-l} \frac{1}{1 - \tau^{i+4}} \leq 4P_\tau(n-l+2), \quad n \geq l-1.$$

Hence $|\delta(n)| \leq 4P_\tau(n-l+2)\tau^{n-l+1}M(n)$, and $M(n+1) \leq M(n) + |\delta(n)| \leq M(n)(1 + 4P_\tau(n-l+2)\tau^{n-l+1})$ for any $n \geq l-1$. We conclude that $M(n) \leq M_\tau$ for any $n \geq l-1$, which proves (6.11). Finally, for any integer $N \geq 1$, we have $\sum_{p \geq 0} (N+p)\tau^p = \frac{N}{1-\tau} + \frac{\tau}{(1-\tau)^2} \leq \frac{N}{(1-\tau)^2}$. Thus the series $\sum_{i \geq n} |\delta(i)|$ is bounded by $\frac{4P_\tau M_\tau}{(1-\tau)^2} (n-l+2)\tau^{n-l+1}$, which gives the last assertion. ■

Proposition 6.9 Let $l \geq 1$ be an integer. Consider the function $\mathfrak{h}_l$ on $\mathbb{D}$ defined by $\mathfrak{h}_l(n, k) = \varsigma_l(n)M_{l,n}(\tau^k)$ if $n \geq l$ and $\mathfrak{h}_l(n, k) = 0$ if $n < l$.

1. The function $\mathfrak{h}_l$ is harmonic.
2. The sequence $(\varsigma_l(n))$ converges to a limit $\varsigma_l(\infty) \neq 0$, and there exists a constant C such that $|\varsigma_l(n) - \varsigma_l(\infty)| \leq C(n+1)\tau^n$ for any $n \geq l$.

3. We have $\mathcal{H}\varphi_l^\tau = \frac{\varkappa_l}{\varsigma_l(\infty)}\mathfrak{h}_l$. In particular, $\mathcal{H}\varphi_l^\tau$ vanishes on the levels $n < l$, and there exists a constant C_l such that, for any $(n, k) \in \mathbb{D}$,

$$(6.13) \quad |\mathcal{H}\varphi_l^\tau(n, k) - \varphi_l^\tau(\tau^k)| \leq C_l(n+1)\tau^n.$$

Proof : 1. For any $n \in \mathbb{N}_0$, let $\mathcal{W}_l(n)$ be the orthogonal complement of the restriction of $\mathcal{V}_{l-1}$ to the level n in the restriction of $\mathcal{V}_l$ to the level n , for the scalar product $\sum_k \widehat{w}(n, k)f(k)\overline{g(k)}$. Since $\widehat{w}(n, k)$ is proportional to τ^k , we have $\mathcal{W}_l(n) = \mathbb{C}M_{l,n}(\tau^\cdot)$ if $n \geq l$, and $\mathcal{W}_l(n) = \{0\}$ if $n < l$. Let $\mathcal{U}_l$ be the space of functions f on $\mathbb{D}$ such that $f(n, \cdot) \in \mathcal{W}_l(n)$ for any n . We claim that $\mathcal{P}(\mathcal{U}_l) \subset \mathcal{U}_l$. Indeed, let $f \in \mathcal{U}_l$. We have $\mathcal{P}f \in \mathcal{V}_l$ by Lemma 6.5. Let $g \in \mathcal{V}_{l-1}$ be supported on a single level. Since $\mathcal{P}$ is self-adjoint with respect to $\widehat{w}$ by Proposition 6.2, and since $\mathcal{P}g \in \mathcal{V}_{l-1}$ is finitely supported, we have

$$\sum_{v \in \mathbb{D}} \widehat{w}(v)(\mathcal{P}f)(v)\overline{g(v)} = \sum_{v \in \mathbb{D}} \widehat{w}(v)f(v)\overline{(\mathcal{P}g)(v)} = 0,$$

because $f(n', \cdot)$ is orthogonal to the restriction of $\mathcal{V}_{l-1}$ to the level n' for any n' . This proves the claim.

Now, let $f \in \mathcal{U}_l$ and write $f(n, \cdot) = c(n)M_{l,n}(\tau^\cdot)$ for $n \geq l$, and $c(l-1) = 0$. For any $n \geq l$, the function $(\mathcal{P}f)(n, \cdot)$ belongs to $\mathcal{W}_l(n)$, hence it is equal to $c'(n)M_{l,n}(\tau^\cdot)$, where $c'(n)$ is the coefficient of degree l of the polynomial which represents $(\mathcal{P}f)(n, \cdot)$; this coefficient is well-defined since $n+1 > l$. By Lemma 6.5, the terms of degree at most $l-1$ of $f(n \pm 1, \cdot)$ contribute only to the coefficients of degree at most $l-1$, so that $c'(n) = (A_l c)(n)$. On the levels $n < l$, we have $(\mathcal{P}f)(n, \cdot) \in \mathcal{W}_l(n) = \{0\}$. Consequently, f is harmonic if and only if $(A_l c)(n) = c(n)$ for any $n \geq l$. Since $\mathfrak{h}_l \in \mathcal{U}_l$ and (6.10) means $A_l \varsigma_l = \varsigma_l$, the function $\mathfrak{h}_l$ is harmonic.

2. By Lemma 6.8, the sequence $(\varsigma_l(n))$ converges, and $|\varsigma_l(n) - \varsigma_l(\infty)| \leq C(n+1)\tau^n$ for any $n \geq l$, with $C = \frac{4P_\tau M_\tau}{(1-\tau)^2}\tau^{1-l}$, since $n-l+2 \leq n+1$.

The moments of the measures $\mathbf{m}_n$ satisfy $|\int x^i d\mathbf{m}_n - \int x^i d\mathbf{m}_\infty| \leq \frac{\tau^{n+1}}{1-\tau}$ for any $i \geq 0$. The coefficients of $M_{l,n}$ are obtained by solving a linear system whose matrix is the Hankel matrix of the moments of $\mathbf{m}_n$ of order at most $2l-2$, and whose right-hand side consists of moments. The Hankel matrix of $\mathbf{m}_\infty$ is invertible, since $\mathbf{m}_\infty$ has infinitely many atoms. We deduce that the coefficients of $M_{l,n}$ converge to those of $\widehat{p}_l$ with an error $O(\tau^n)$. In particular, we have $\sup_{n \geq l} \sup_{x \in [0,1]} |M_{l,n}(x)| < \infty$ and $\sup_{x \in [0,1]} |M_{l,n}(x) - \widehat{p}_l(x)| = O(\tau^n)$.

Consequently, the function $\mathfrak{h}_l$ is bounded and, for any $k \in \mathbb{N}_0$, we have $\lim_{n \rightarrow \infty} \mathfrak{h}_l(n, k) = \varsigma_l(\infty)\widehat{p}_l(\tau^k)$. Suppose that $\varsigma_l(\infty) = 0$. Then part 4 of Lemma 6.7 gives $\mathfrak{h}_l = \mathcal{H}0 = 0$. This is impossible, since $\mathfrak{h}_l(l, \cdot) = M_{l,l}(\tau^\cdot)$ and a nonzero polynomial of degree l cannot vanish at the $l+1$ points τ^k , $0 \leq k \leq l$. Thus $\varsigma_l(\infty) \neq 0$.

3. Put $u \stackrel{\text{def}}{=} \frac{\varkappa_l}{\varsigma_l(\infty)}\mathfrak{h}_l$. By the second part, u is a bounded harmonic function such that $\lim_{n \rightarrow \infty} u(n, k) = \varkappa_l \widehat{p}_l(\tau^k) = \varphi_l^\tau(\tau^k)$ for any k . By part 4 of Lemma 6.7, we have $u = \mathcal{H}\varphi_l^\tau$. For $n \geq l$ and $x = \tau^k$, we have

$$\mathcal{H}\varphi_l^\tau(n, k) - \varphi_l^\tau(x) = \varkappa_l \left[\left(\frac{\varsigma_l(n)}{\varsigma_l(\infty)} - 1 \right) M_{l,n}(x) + M_{l,n}(x) - \widehat{p}_l(x) \right],$$

and the second part gives (6.13). The finitely many levels $n < l$ are included by enlarging C_l . ■

Theorem 6.10 *The harmonic measure of the diagonal chain seen from the trivial representation is the Aref'eva-Volovich law: for any $k \in \mathbb{N}_0$, we have*

$$\mathbf{P}_{(0,0)}(k_\infty = k) = (1-\tau)\tau^k.$$

Proof : Let μ be the law of τ^{k_∞} under $\mathbf{P}_{(0,0)}$. For any integer $l \geq 1$, Proposition 6.9 gives

$$\int \varphi_l^\tau d\mu = \mathbf{E}_{(0,0)}[\varphi_l^\tau(\tau^{k_\infty})] = \mathcal{H}\varphi_l^\tau(0,0) = 0,$$

since $\mathcal{H}\varphi_l^\tau$ vanishes on the level $0 < l$. The Aref'eva–Volovich law $(1-\tau)\mathbf{m}_\infty$ has the same property, since φ_l^τ is orthogonal to $\varphi_0^\tau = 1$ in $\ell_{\mathbb{N}_0}^2(w)$. Since $(\varphi_l^\tau)_{l \geq 0}$ is a basis of the space of polynomials and both measures are probability measures, all the moments of μ and of $(1-\tau)\mathbf{m}_\infty$ coincide. Since these measures are supported by $[0, 1]$, they are equal. ■

The following proposition gives a second proof of Theorem 6.10, which does not use the spin harmonic functions, and a stronger statement. It is the analogue, for the state ϕ , of the property of the quantum Bernoulli walk used by Biane for the tracial state, for which the conditional law of the weight given the levels is uniform [Bia06, Section 4].

Proposition 6.11 *For $(n, k) \in \mathbb{D}$, put $\Lambda(n, k) \stackrel{\text{def}}{=} \frac{(1-\tau)\tau^k}{1-\tau^{n+1}}$, so that $\Lambda(n, \cdot)$ is a probability measure on $\{0, \dots, n\}$, and let $\mathbf{Q}$ be the transition matrix on $\mathbb{N}_0$ given by*

$$\mathbf{Q}(n, n+1) = \frac{1-\tau^{n+2}}{(1+\tau)(1-\tau^{n+1})}, \quad \mathbf{Q}(n, n-1) = \frac{\tau(1-\tau^n)}{(1+\tau)(1-\tau^{n+1})}.$$

1. *Under any $\mathbf{P}_v$, the process $(n_t - 2k_t)_{t \geq 0}$ is a random walk on $\mathbb{Z}$ whose steps are $+1$ and -1 with probabilities $\frac{1}{1+\tau}$ and $\frac{\tau}{1+\tau}$. The matrix $\mathbf{Q}$ is the Doob transform of this walk killed at -1 by the function $h(n) = 1 - \tau^{n+1}$, that is the law of this walk conditioned never to hit -1 .*
2. *For any $n, m \in \mathbb{N}_0$ such that $|n - m| = 1$ and any $0 \leq k' \leq m$, we have*

$$\sum_{k=0}^n \Lambda(n, k) \mathbf{P}((n, k), (m, k')) = \mathbf{Q}(n, m) \Lambda(m, k').$$

3. *For any integer $N \geq 1$, any $m_1, \dots, m_N \in \mathbb{N}_0$ and any $0 \leq j \leq m_N$, we have, with $m_0 = 0$,*

$$\mathbf{P}_{(0,0)}(n_1 = m_1, \dots, n_N = m_N, k_N = j) = \mathbf{Q}(m_0, m_1) \cdots \mathbf{Q}(m_{N-1}, m_N) \Lambda(m_N, j).$$

In particular, under $\mathbf{P}_{(0,0)}$, the level process $(n_t)_{t \geq 0}$ is a Markov chain with transition matrix $\mathbf{Q}$, and the conditional law of k_N given $(n_0, \dots, n_N)$ is $\Lambda(n_N, \cdot)$.

4. *Under $\mathbf{P}_{(0,0)}$, the random variable k_∞ is independent of the level process $(n_t)_{t \geq 0}$, and its law is the Aref'eva–Volovich law $(1-\tau)\tau^k$, $k \in \mathbb{N}_0$.*

Proof : Put $D_n \stackrel{\text{def}}{=} (1+\tau)(1-\tau^{n+1})$.

1. The quantity $n - 2k$ increases by 1 exactly for the moves to $(n+1, k)$ and to $(n-1, k-1)$. By Proposition 6.2, the sum of their probabilities is $D_n^{-1}((1-\tau^{n-k+1}) + \tau^{n-k+1}(1-\tau^k)) = \frac{1-\tau^{n+1}}{D_n} = \frac{1}{1+\tau}$, which does not depend on (n, k) . By the Markov property, the increments of $(n_t - 2k_t)$ are independent and identically distributed, with the stated law. For $n \geq 0$, we have $\frac{1}{1+\tau}h(n+1) + \frac{\tau}{1+\tau}h(n-1) = h(n)$, with $h(-1) = 0$, and $\frac{h(n+1)}{(1+\tau)h(n)} = \mathbf{Q}(n, n+1)$, $\frac{\tau h(n-1)}{(1+\tau)h(n)} = \mathbf{Q}(n, n-1)$. Finally, $h(n)$ is the probability that the walk started at n never hits -1 , since the probability that it hits -1 is τ^{n+1} by the gambler's ruin formula.

2. Suppose that $m = n + 1$. The only moves to $(n + 1, k')$ come from (n, k') , if $k' \leq n$, and from $(n, k' - 1)$, if $k' \geq 1$. By Proposition 6.2, the left-hand side is equal to

$$\frac{1 - \tau}{(1 - \tau^{n+1})D_n} \left[\tau^{k'} (1 - \tau^{n-k'+1}) + \tau^{k'-1} \tau^{n-k'+2} (1 - \tau^{k'}) \right] = \frac{(1 - \tau) \tau^{k'}}{D_n},$$

where the first term vanishes for $k' = n + 1$ and the second one for $k' = 0$. This is $Q(n, n + 1) \Lambda(n + 1, k')$. Suppose that $m = n - 1$. The only moves to $(n - 1, k')$ come from (n, k') and from $(n, k' + 1)$, and the left-hand side is equal to

$$\frac{1 - \tau}{(1 - \tau^{n+1})D_n} \left[\tau^{k'} \tau (1 - \tau^{n-k'}) + \tau^{k'+1} \tau^{n-k'} (1 - \tau^{k'+1}) \right] = \frac{(1 - \tau) \tau^{k'+1}}{D_n},$$

which is $Q(n, n - 1) \Lambda(n - 1, k')$.

3. We argue by induction on N . For $N = 1$, the formula is part 2 with $n = 0$, since $\Lambda(0, \cdot) = \delta_0$. Suppose that it holds for N . By the Markov property and since each move changes the level by ± 1 , the probability of $\{n_1 = m_1, \dots, n_{N+1} = m_{N+1}, k_{N+1} = j\}$ is

$$\sum_{k=0}^{m_N} Q(m_0, m_1) \cdots Q(m_{N-1}, m_N) \Lambda(m_N, k) P((m_N, k), (m_{N+1}, j)),$$

which is $Q(m_0, m_1) \cdots Q(m_N, m_{N+1}) \Lambda(m_{N+1}, j)$ by part 2. Summing over j gives the law of $(n_1, \dots, n_N)$, and the last assertions follow.

4. Let $M \in \mathbb{N}_0$ and let A be an event of the σ -algebra generated by $n_0, \dots, n_M$. For any $N \geq M$ and any $j \in \mathbb{N}_0$, part 3 gives $\mathbf{P}_{(0,0)}(A \cap \{k_N = j\}) = \mathbf{E}_{(0,0)}[\mathbf{1}_A \Lambda(n_N, j)]$, where $\Lambda(n, j) = 0$ if $j > n$. By part 2 of Lemma 6.7, almost surely $n_N \rightarrow \infty$ and $k_N = k_\infty$ for N large enough, so that $\Lambda(n_N, j) \rightarrow (1 - \tau) \tau^j$. By dominated convergence, $\mathbf{P}_{(0,0)}(A \cap \{k_\infty = j\}) = (1 - \tau) \tau^j \mathbf{P}_{(0,0)}(A)$. These events A form a π -system which generates the σ -algebra of the level process, which gives the independence. Taking A equal to the whole space gives the law of k_∞ . $\blacksquare$

Part 3 is an instance of the criterion of Rogers and Pitman for functions of Markov chains [RP81], with the intertwining kernel Λ . Part 1 says that the restriction of $\mathbf{P}$ to the subalgebra generated by K is a random walk on the dual $\mathbb{Z}$ of the torus, and that the level chain, which is the restriction of $\mathbf{P}$ to the centre, is a Doob transform of it. This is the $SU_r(2)$ case of the relation between the restrictions to the centre and to the torus established by Biane for $SU(n)$ [Bia91], see also [Bia06, Section 4] and [Col04].

In the variable $n - k$, the measure $\mathbf{m}_n$ is proportional to the orthogonality weight $\tau^{-(n-k)}$ of the τ -Hahn polynomials with parameters $\alpha = \beta = 1$ and $N = n$, see [KLS10, Section 14.6] and [KS96, (3.6.1)–(3.6.2)]. Hence $M_{l,n}(\tau^k)$ is proportional to $Q_l(\tau^{-(n-k)}; 1, 1, n | \tau)$, which is a polynomial of degree l in τ^k . Thus Proposition 6.9 describes the Poisson integral of the zonal spherical function $\varphi_l^\tau(A)$ on each level as a q -Hahn polynomial, multiplied by $\varkappa_l \varsigma_l(n) / \varsigma_l(\infty)$. The coefficients $\varsigma_l(n)$ are described as follows.

Proposition 6.12 *Let $l \geq 1$ be an integer.*

1. *For any $x \in \mathbb{C}$ such that $|x| < 1$, we have*

$$(6.14) \quad \sum_{n \geq l} \varsigma_l(n) x^{n-l} = (1 - \tau^{2l+1}) \sum_{i \geq 0} \frac{(\tau^{-l} x; \tau)_i (\tau^{1-l} x; \tau)_i}{(x; \tau)_{i+1} (\tau x; \tau)_{i+1}} \tau^{(2l+1)i}.$$

2. We have $\varsigma_l(\infty) = \frac{(\tau^{l+2}; \tau)_l}{(\tau; \tau)_l}$, that is the Gaussian binomial coefficient $\begin{bmatrix} 2l+1 \\ l \end{bmatrix}_\tau$, by [DLMF, (17.2.29)]. In particular, $\varsigma_l(\infty) \geq 1$.

3. There exists a constant B_l such that

$$\varsigma_l(n) = \varsigma_l(\infty) - \frac{\tau \Lambda_l(\tau) \varsigma_l(\infty)}{1 - \tau} (n - l + 1) \tau^n + B_l \tau^n + O(\tau^{\frac{3n}{2}}).$$

In particular, the factor $n + 1$ in part 2 of Proposition 6.9 cannot be removed.

Proof : 1. By Lemma 6.8, the power series $G(x) \stackrel{\text{def}}{=} \sum_{m \geq 0} \varsigma_l(l + m) x^m$ converges for $|x| < 1$. We write (6.10) with $n = l + m$, multiply by x^{m+1} and sum over $m \geq 0$. Since $\varsigma_l(l - 1) = 0$ and $\varsigma_l(l) = 1$, we have

$$\begin{aligned} \sum_{m \geq 0} (1 - \tau^{2l+2+m}) \varsigma_l(l + m + 1) x^{m+1} &= G(x) - 1 - \tau^{2l+1} (G(\tau x) - 1), \\ \sum_{m \geq 0} (1 + \tau) (1 - \tau^{l+1+m}) \varsigma_l(l + m) x^{m+1} &= (1 + \tau) x (G(x) - \tau^{l+1} G(\tau x)), \\ \sum_{m \geq 0} \tau (1 - \tau^m) \varsigma_l(l + m - 1) x^{m+1} &= \tau x^2 (G(x) - \tau G(\tau x)). \end{aligned}$$

We deduce that, for any $|x| < 1$,

$$(1 - x)(1 - \tau x)G(x) = (\tau^l - \tau x)(\tau^{l+1} - \tau x)G(\tau x) + 1 - \tau^{2l+1}.$$

Put $R(x) \stackrel{\text{def}}{=} \frac{\tau^{2l+1}(1 - \tau^{-l}x)(1 - \tau^{1-l}x)}{(1 - x)(1 - \tau x)}$, and denote by $T_i(x)$ the i -th term of the series in the right-hand side of (6.14). Since $\prod_{r=0}^{i-1} R(\tau^r x) = \frac{(\tau^{-l}x; \tau)_i (\tau^{1-l}x; \tau)_i}{(x; \tau)_i (\tau x; \tau)_i} \tau^{(2l+1)i}$, iterating the previous identity N times gives

$$G(x) = (1 - \tau^{2l+1}) \sum_{i=0}^{N-1} T_i(x) + \prod_{r=0}^{N-1} R(\tau^r x) \cdot G(\tau^N x).$$

Since $R(\tau^r x) \rightarrow \tau^{2l+1}$ as $r \rightarrow \infty$ and $G(\tau^N x) \rightarrow G(0) = 1$ as $N \rightarrow \infty$, the last term tends to zero, which proves (6.14).

2. Let $\rho > 0$. For $|x| \leq \rho$ and any $i \geq 0$, we have $|(\tau^{-l}x; \tau)_i (\tau^{1-l}x; \tau)_i| \leq K_\rho \stackrel{\text{def}}{=} (-\tau^{-l}\rho; \tau)_\infty^2$. For $0 \leq x < 1$, we obtain

$$|(1 - x)T_i(x)| = \frac{|(\tau^{-l}x; \tau)_i (\tau^{1-l}x; \tau)_i|}{(\tau x; \tau)_i (\tau x; \tau)_{i+1}} \tau^{(2l+1)i} \leq \frac{K_1}{(\tau; \tau)_\infty^2} \tau^{(2l+1)i}.$$

By Lemma 6.8 and Abel's theorem, $(1 - x)G(x) \rightarrow \varsigma_l(\infty)$ as $x \rightarrow 1^-$. By dominated convergence, we deduce that

$$\varsigma_l(\infty) = (1 - \tau^{2l+1}) \sum_{i \geq 0} \frac{(\tau^{-l}; \tau)_i (\tau^{1-l}; \tau)_i}{(\tau; \tau)_i (\tau; \tau)_{i+1}} \tau^{(2l+1)i} = \frac{1 - \tau^{2l+1}}{1 - \tau} {}_2\phi_1 \left(\begin{matrix} \tau^{1-l}, \tau^{-l} \\ \tau^2 \end{matrix}; \tau, \tau^{2l+1} \right),$$

where we used $(\tau; \tau)_{i+1} = (1 - \tau)(\tau^2; \tau)_i$; the terms with $i \geq l$ vanish, since $(\tau^{1-l}; \tau)_i = 0$. By the q -Chu–Vandermonde formula [GR04, (II.7)], applied with $q^{-n} = \tau^{1-l}$, $a = \tau^{-l}$ and $c = \tau^2$, so that $\frac{cq^n}{a} = \tau^{2l+1}$, the ${}_2\phi_1$ is equal to $\frac{(\tau^{l+2}; \tau)_{l-1}}{(\tau^2; \tau)_{l-1}}$. Since $(1 - \tau^{2l+1})(\tau^{l+2}; \tau)_{l-1} = (\tau^{l+2}; \tau)_l$ and

$(1-\tau)(\tau^2; \tau)_{l-1} = (\tau; \tau)_l$, we obtain the formula for $\varsigma_l(\infty)$. Finally, $\frac{(\tau^{l+2}; \tau)_l}{(\tau; \tau)_l} = \prod_{i=1}^l \frac{1-\tau^{l+1+i}}{1-\tau^i} \geq 1$.

3. Let $\rho > 0$ and let K be a compact subset of the disc $\{|x| \leq \rho\}$ which does not meet $\{\tau^{-r} : r \in \mathbb{N}_0\}$. The functions $(x; \tau)_\infty$ and $(\tau x; \tau)_\infty$ do not vanish on K , and $|(x; \tau)_{i+1}| \geq \frac{|(x; \tau)_\infty|}{(-\rho; \tau)_\infty}$, $|(\tau x; \tau)_{i+1}| \geq \frac{|(\tau x; \tau)_\infty|}{(-\rho; \tau)_\infty}$ for any i . Hence $|T_i| \leq K' \tau^{(2l+1)i}$ on K for some constant K' , the series (6.14) converges uniformly on K , and G extends to a meromorphic function on $\mathbb{C}$ whose poles belong to $\{\tau^{-r} : r \in \mathbb{N}_0\}$. In the disc $\{|x| < \tau^{-2}\}$, the poles of the functions T_i are $x = 1$, which is simple and comes from the factor $1 - x$ of $(x; \tau)_{i+1}$, and $x = \tau^{-1}$, which comes from the factor $1 - \tau x$ of $(\tau x; \tau)_{i+1}$ and, for $i \geq 1$, of $(x; \tau)_{i+1}$. By the second part, the residue of G at $x = 1$ is $-\varsigma_l(\infty)$. Moreover, T_0 has a simple pole at τ^{-1} , and for $i \geq 1$ the function $(1 - \tau x)^{-2}(x; \tau)_{i+1}(\tau x; \tau)_{i+1}$ takes the value $(1 - \tau^{-1})(\tau; \tau)_{i-1}(\tau; \tau)_i$ at $x = \tau^{-1}$. Hence

$$\lim_{x \rightarrow \tau^{-1}} (1 - \tau x)^2 G(x) = (1 - \tau^{2l+1}) \sum_{i=1}^l \frac{(\tau^{-l-1}; \tau)_i (\tau^{-l}; \tau)_i}{(1 - \tau^{-1})(\tau; \tau)_{i-1}(\tau; \tau)_i} \tau^{(2l+1)i},$$

the terms with $i \geq l+1$ vanishing since $(\tau^{-l}; \tau)_i = 0$. Writing $i = p+1$ and using $(\tau^{-l-1}; \tau)_{p+1} = (1 - \tau^{-l-1})(\tau^{-l}; \tau)_p$, $(\tau^{-l}; \tau)_{p+1} = (1 - \tau^{-l})(\tau^{-l-1}; \tau)_p$ and $(\tau; \tau)_{p+1} = (1 - \tau)(\tau^2; \tau)_p$, the computation of the second part shows that this limit is

$$A_l \stackrel{\text{def}}{=} \frac{\tau^{2l+1}(1 - \tau^{-l-1})(1 - \tau^{-l})}{1 - \tau^{-1}} \varsigma_l(\infty) = -\frac{\tau(1 - \tau^l)(1 - \tau^{l+1})}{1 - \tau} \varsigma_l(\infty) = -\frac{\tau^{l+1} \Lambda_l(\tau)}{1 - \tau} \varsigma_l(\infty).$$

Consequently, there exists a constant B'_l such that the function $G(x) - \frac{\varsigma_l(\infty)}{1-x} - \frac{A_l}{(1-\tau x)^2} - \frac{B'_l}{1-\tau x}$ is holomorphic in the disc $\{|x| < \tau^{-2}\}$. By the Cauchy estimates on the circle $\{|x| = \tau^{-\frac{3}{2}}\}$, its Taylor coefficients are $O(\tau^{\frac{3m}{2}})$. Since $\frac{1}{(1-\tau x)^2} = \sum_{m \geq 0} (m+1) \tau^m x^m$, we obtain $\varsigma_l(l+m) = \varsigma_l(\infty) + A_l(m+1) \tau^m + B'_l \tau^m + O(\tau^{\frac{3m}{2}})$. With $n = l+m$ and $B_l = B'_l \tau^{-l}$, this is the third assertion. Since $\Lambda_l(\tau) \varsigma_l(\infty) \neq 0$, the sequence $\tau^{-n}(\varsigma_l(n) - \varsigma_l(\infty))$ is unbounded. ■

We now describe the Poisson kernel of the diagonal chain, that is the harmonic measures $\nu_v(j) \stackrel{\text{def}}{=} \mathbf{P}_v(k_\infty = j)$, $v \in \mathbb{D}$, $j \in \mathbb{N}_0$. Let $\mathbf{B}$ be the bounded operator on $\ell_{\mathbb{N}_0}^2$, with the counting measure, defined by

$$(\mathbf{B}f)(k) \stackrel{\text{def}}{=} \mathbf{b}_k(f(k) - f(k+1)) + \mathbf{b}_{k-1}(f(k) - f(k-1)), \quad k \in \mathbb{N}_0,$$

with $\mathbf{b}_{-1} \stackrel{\text{def}}{=} 0$. It is self-adjoint and $\langle \mathbf{B}f, f \rangle = \sum_k \mathbf{b}_k |f(k+1) - f(k)|^2 = \frac{\varepsilon_\tau}{2} \mathcal{E}_\tau(f)$, so that $\mathbf{B}$ is the operator associated with the form $\frac{\varepsilon_\tau}{2} \mathcal{E}_\tau$ on $\ell_{\mathbb{N}_0}^2$ with the counting measure, instead of $\ell_{\mathbb{N}_0}^2(w)$. Moreover $0 \leq \mathbf{B} \leq 4 \sup_k \mathbf{b}_k$. We define polynomials ϱ_d , $d \in \mathbb{N}_0$, by $\varrho_0 \stackrel{\text{def}}{=} 1$ and

$$(6.15) \quad \varrho_{d+1}(\lambda) \stackrel{\text{def}}{=} \varrho_d(\lambda) + \frac{\lambda}{\mathbf{a}_d} \sum_{i=0}^d \varrho_i(\lambda), \quad \lambda \in \mathbb{C}.$$

Equivalently, $\mathbf{a}_d(\varrho_{d+1} - \varrho_d) + \mathbf{a}_{d-1}(\varrho_{d-1} - \varrho_d) = \lambda \varrho_d$ for any $d \geq 0$, with $\mathbf{a}_{-1} \stackrel{\text{def}}{=} 0$.

Theorem 6.13 1. For any $(n, k) \in \mathbb{D}$ and any $j \in \mathbb{N}_0$, we have

$$(6.16) \quad \nu_{(n,k)}(j) = (1 - \tau) \tau^j \sum_{l=0}^n \frac{\varkappa_l \varsigma_l(n)}{\varsigma_l(\infty) \|\varphi_l^\tau\|_{\ell_{\mathbb{N}_0}^2(w)}^2} M_{l,n}(\tau^k) \varphi_l^\tau(\tau^j),$$

where the term $l = 0$ is equal to 1. In particular, the density of $\nu_{(n,k)}$ with respect to the Aref'eva-Volovich law is a polynomial of degree at most n in τ^j .

2. For any $\lambda \geq 0$, the sequence $(\varrho_d(\lambda))_{d \geq 0}$ is nondecreasing and converges to a limit $\varrho_\infty(\lambda) \geq 1$, uniformly on the compact subsets of $[0, \infty)$. For any $d, k, j \in \mathbb{N}_0$, we have

$$(6.17) \quad \nu_{(d+k,k)}(j) = \left\langle \frac{\varrho_d}{\varrho_\infty}(\mathbf{B}) \delta_j, \delta_k \right\rangle_{\ell_{\mathbb{N}_0}^2}.$$

In particular, $\nu_{(d+k,k)}(j) = \nu_{(d+j,j)}(k)$, and the matrix $(\nu_{(k,k)}(j))_{k,j \geq 0}$ of the harmonic measures seen from the vertices (k, k) is the matrix of $\varrho_\infty(\mathbf{B})^{-1}$.

Proof : 1. Let $v = (n, k)$. By part 2 of Lemma 6.7, ν_v is a probability measure on $\mathbb{N}_0$, which we identify with a measure on X_τ through $j \mapsto \tau^j$. By definition of $\mathcal{H}$, we have $\int \varphi_l^\tau d\nu_v = \mathcal{H}\varphi_l^\tau(v)$, which vanishes for $l > n$ by part 3 of Proposition 6.9. Put $e_l \stackrel{\text{def}}{=} \|\varphi_l^\tau\|^{-1} \varphi_l^\tau$ and $p \stackrel{\text{def}}{=} \sum_{l=0}^n (\int e_l d\nu_v) e_l$, which is a polynomial of degree at most n . The finite signed measure $\nu' \stackrel{\text{def}}{=} \nu_v - pw$ satisfies $\int e_l d\nu' = 0$ for any $l \in \mathbb{N}_0$, by orthonormality of (e_l) in $\ell_{\mathbb{N}_0}^2(w)$. Since (e_l) is a basis of the space of polynomials, which is dense in $C(X_\tau)$, we obtain $\nu' = 0$, that is $\nu_v(j) = w_j p(\tau^j)$. Finally, part 3 of Proposition 6.9 gives $\int e_l d\nu_v = \frac{\kappa_l \varsigma_l(n)}{\varsigma_l(\infty) \|\varphi_l^\tau\|} M_{l,n}(\tau^k)$ for $l \geq 1$, and $\int e_0 d\nu_v = 1$. This proves (6.16).

2. Let $\lambda \geq 0$. By (6.15) and induction, we have $\varrho_d(\lambda) \geq 1$ and $\varrho_{d+1}(\lambda) \geq \varrho_d(\lambda)$, hence $\varrho_{d+1}(\lambda) \leq \varrho_d(\lambda) \left(1 + \frac{(d+1)\lambda}{\mathbf{a}_d}\right)$. Since $\mathbf{a}_d \geq \frac{\tau^{-d}}{1+\tau}$, the products $\prod_{d \geq 0} (1 + (1+\tau)(d+1)\lambda\tau^d)$ converge uniformly on the compact subsets of $[0, \infty)$, so that ϱ_d is bounded on these subsets uniformly in d . By (6.15) again, the series $\sum_d (\varrho_{d+1} - \varrho_d)$ converges uniformly on these subsets, which proves the first assertion.

Let $D \geq 1$ be an integer and $j \in \mathbb{N}_0$. Since $\mathbf{B} \geq 0$ and $\varrho_D \geq 1$ on $[0, \infty)$, the operator $\varrho_D(\mathbf{B})$ is invertible. For $0 \leq d \leq D$, put $u_D(d, \cdot) \stackrel{\text{def}}{=} \varrho_d(\mathbf{B}) \varrho_D(\mathbf{B})^{-1} \delta_j \in \ell_{\mathbb{N}_0}^2$, which defines a bounded function u_D on the strip $\{(d, k) : 0 \leq d \leq D\}$, with $u_D(D, \cdot) = \delta_j$. By Proposition 6.4, a function u is harmonic at (d, k) if and only if

$$\mathbf{a}_d(u(d+1, k) - u(d, k)) + \mathbf{a}_{d-1}(u(d-1, k) - u(d, k)) = (\mathbf{B}u(d, \cdot))(k).$$

For $d < D$, the left-hand side for $u = u_D$ is the k -th coordinate of $(\mathbf{a}_d(\varrho_{d+1} - \varrho_d) + \mathbf{a}_{d-1}(\varrho_{d-1} - \varrho_d))(\mathbf{B}) \varrho_D(\mathbf{B})^{-1} \delta_j = \mathbf{B} \varrho_d(\mathbf{B}) \varrho_D(\mathbf{B})^{-1} \delta_j$. Hence u_D is harmonic at any (d, k) with $d < D$. Let $v = (d, k)$ with $d < D$, and let σ_D be the hitting time of $\{D\} \times \mathbb{N}_0$ by the chain started at v . In the strip, the weights satisfy $\hat{w} \leq \frac{\tau^{-D}}{1-\tau}$, so that the probability of a move from (d', k') to $(d' + 1, k')$ is at least $\mathbf{a}_0(1-\tau)\tau^D = \frac{(1-\tau)\tau^D}{1+\tau}$, since $\mathbf{a}_{d'} \geq \mathbf{a}_0$. Hence $\sigma_D < \infty$ almost surely. Since u_D is bounded and harmonic in the strip, the optional stopping theorem gives

$$\mathbf{P}_v(k_{\sigma_D} = j) = u_D(v) = \langle \varrho_d(\mathbf{B}) \varrho_D(\mathbf{B})^{-1} \delta_j, \delta_k \rangle.$$

As $D \rightarrow \infty$, we have $\sigma_D \geq D - d \rightarrow \infty$, and $k_t = k_\infty$ for t large enough almost surely, by part 2 of Lemma 6.7. By dominated convergence, the left-hand side converges to $\nu_v(j)$. On the other hand, $\frac{\varrho_d}{\varrho_D} \rightarrow \frac{\varrho_d}{\varrho_\infty}$ uniformly on the spectrum of $\mathbf{B}$, since $\varrho_D \geq 1$, hence $\varrho_d(\mathbf{B}) \varrho_D(\mathbf{B})^{-1} \rightarrow \frac{\varrho_d}{\varrho_\infty}(\mathbf{B})$ in norm. This proves (6.17). Since $\mathbf{B}$ is a self-adjoint operator with real matrix entries and $\frac{\varrho_d}{\varrho_\infty}$ is real-valued, the matrix of $\frac{\varrho_d}{\varrho_\infty}(\mathbf{B})$ is real and symmetric, which gives the symmetry. The last assertion is the case $d = 0$. $\blacksquare$

Remark 6.14 Theorem 6.13 gives two descriptions of the Poisson kernel. The first one is a finite sum over the spins $l \leq n$, and it uses the q -Hahn polynomials and the coefficients $\varsigma_l(n)$ of Proposition 6.12. The second one separates the variables: the vertical part is the birth-death

network (a_d) , which is asymptotically the radial network of the Bruhat–Tits tree of Remark 6.3, and the horizontal part is the operator B of the Casimir energy $\mathcal{E}_\tau$. Since $0 < \frac{\varrho_d}{\varrho_\infty} \leq 1$ on $[0, \infty)$ and $B1 = 0$, each matrix $(\nu_{(d+k,k)}(j))_{k,j}$ is symmetric, doubly stochastic and defines a positive contraction of $\ell_{\mathbb{N}_0}^2$, which tends to the identity as $d \rightarrow \infty$. Both descriptions agree numerically to 35 digits for $\tau = \frac{1}{3}$ and $d, k, j \leq 3$. In Subsection 6.4, we compute ϱ_∞ explicitly, we give a third description of the Poisson kernel, as the expectation of the heat kernel of B at an explosion time (Theorem 6.22), and we give a closed formula for the matrix $(\nu_{(k,k)}(j))_{k,j}$ (Theorem 6.26).

6.4 The entire function ϱ_∞

In this subsection, we compute the function ϱ_∞ of Theorem 6.13, we diagonalize the Jacobi operators of the two networks of Proposition 6.4, and we describe the consequences for the Poisson kernel. We use the notation $(a; \tau)_n \stackrel{\text{def}}{=} \prod_{i=0}^{n-1} (1 - a\tau^i)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$, $(a_1, \dots, a_p; \tau)_n \stackrel{\text{def}}{=} (a_1; \tau)_n \cdots (a_p; \tau)_n$ [DLMF, (17.2.1), (17.2.4)–(17.2.6)], the Gaussian binomial coefficients $\begin{bmatrix} m \\ k \end{bmatrix}_\tau \stackrel{\text{def}}{=} \frac{(\tau; \tau)_m}{(\tau; \tau)_k (\tau; \tau)_{m-k}}$ for $0 \leq k \leq m$ [DLMF, (17.2.27)], [KS96, (0.3.2)], and the basic hypergeometric series ${}_s+1\phi_s$ of [GR04, Section 1.2], see also [DLMF, (17.4.1)] and [KS96, (0.4.2)]. The recurrence (6.15) defines polynomials ϱ_d , which we consider on the whole complex plane. For any $\lambda \in \mathbb{C}$, we choose $z \in \mathbb{C} \setminus \{0\}$ such that

$$(6.18) \quad z + z^{-1} = 2 - \frac{(1 - \tau^2)\lambda}{\tau}.$$

For any integer $j \geq 0$, we have $(1 - z\tau^j)(1 - z^{-1}\tau^j) = 1 - (z + z^{-1})\tau^j + \tau^{2j} = (1 - \tau^j)^2 + (1 - \tau^2)\lambda\tau^{j-1}$. Hence the products $(z, z^{-1}; \tau)_n$ are polynomials in λ , and they do not depend on the choice of z . Finally, for any integer $n \geq 1$, we put

$$(6.19) \quad \mu_n \stackrel{\text{def}}{=} \frac{(1 - \tau^n)^2}{(1 - \tau^2)\tau^{n-1}} = \frac{1 - \tau}{1 + \tau} [n]_\tau^2,$$

where $[n]_\tau = \frac{\tau^n - \tau^{-n}}{\tau - \tau^{-1}}$ is the quantum dimension of the module $V_{\frac{n-1}{2}}$. We have $\frac{\mu_{n+1}}{\mu_n} = \frac{(1 - \tau^{n+1})^2}{\tau(1 - \tau^n)^2} > \frac{1}{\tau}$, so that the sequence (μ_n) is increasing.

Theorem 6.15 *Let $\lambda \in \mathbb{C}$.*

1. *For any $y \in \mathbb{C}$ such that $|y| < 1$, we have*

$$(6.20) \quad \sum_{d \geq 0} \varrho_d(\lambda) y^d = \prod_{j \geq 0} \frac{(1 - \tau^{j+1}y)^2 + (1 - \tau^2)\lambda\tau^j y}{(1 - \tau^j y)(1 - \tau^{j+1}y)} = \frac{(\tau z y, \tau z^{-1} y; \tau)_\infty}{(y; \tau)_\infty (\tau y; \tau)_\infty}.$$

2. *For any $d \in \mathbb{N}_0$, we have*

$$(6.21) \quad \varrho_d(\lambda) = \sum_{k=0}^d \begin{bmatrix} d \\ k \end{bmatrix}_\tau \frac{(z, z^{-1}; \tau)_k}{(\tau; \tau)_k} \tau^k.$$

3. *The sequence (ϱ_d) converges uniformly on the compact subsets of $\mathbb{C}$ to the entire function*

$$(6.22) \quad \varrho_\infty(\lambda) = \prod_{n \geq 1} \left(1 + \frac{\lambda}{\mu_n}\right) = \frac{(\tau z, \tau z^{-1}; \tau)_\infty}{(\tau; \tau)_\infty^2} = {}_2\phi_1 \left(\begin{matrix} z, z^{-1} \\ \tau \end{matrix}; \tau, \tau \right).$$

4. For any compact subset L of $\mathbb{C}$, there exists a constant C_L such that, for any $\lambda \in L$ and any $d \in \mathbb{N}_0$,

$$|\varrho_d(\lambda) - \varrho_\infty(\lambda) + (1 + \tau)\lambda\varrho_\infty(\lambda)d\tau^d| \leq C_L\tau^d.$$

Proof : 1. Put $M \stackrel{\text{def}}{=} \prod_{i \geq 0} (1 + (1 + \tau)|\lambda|(i + 1)\tau^i)$. Recall that $\mathbf{a}_d \geq \frac{\tau^{-d}}{1 + \tau}$ for any $d \in \mathbb{N}_0$. By (6.15) and induction on d , we have $\max_{i \leq d} |\varrho_i(\lambda)| \leq \prod_{i < d} (1 + (1 + \tau)|\lambda|(i + 1)\tau^i) \leq M$, hence

$$(6.23) \quad |\varrho_{d+1}(\lambda) - \varrho_d(\lambda)| \stackrel{(6.15)}{\leq} \frac{|\lambda|}{\mathbf{a}_d} \sum_{i=0}^d |\varrho_i(\lambda)| \leq (1 + \tau)M|\lambda|(d + 1)\tau^d.$$

In particular, the power series $F(y) \stackrel{\text{def}}{=} \sum_{d \geq 0} \varrho_d(\lambda)y^d$ converges for $|y| < 1$. Put $e_0 \stackrel{\text{def}}{=} 1$ and $e_d \stackrel{\text{def}}{=} \varrho_d(\lambda) - \varrho_{d-1}(\lambda)$ for $d \geq 1$, so that $E(y) \stackrel{\text{def}}{=} \sum_{d \geq 0} e_dy^d = (1 - y)F(y)$. Since $(1 - \tau^2)\tau^d\mathbf{a}_d = 1 - \tau^{d+1}$ and $(1 - \tau^2)\tau^d\mathbf{a}_{d-1} = \tau(1 - \tau^d)$, the second form of (6.15), multiplied by $(1 - \tau^2)\tau^d$, gives

$$(1 - \tau^{d+1})e_{d+1} - \tau(1 - \tau^d)e_d = (1 - \tau^2)\lambda\tau^d\varrho_d(\lambda), \quad d \geq 0.$$

We multiply by y^{d+1} and we sum over $d \geq 0$. Since $\sum_{d \geq 0} (1 - \tau^{d+1})e_{d+1}y^{d+1} = \sum_{d \geq 0} (1 - \tau^d)e_dy^d = E(y) - E(\tau y)$, we obtain $(1 - \tau y)(E(y) - E(\tau y)) = (1 - \tau^2)\lambda y F(\tau y)$, that is

$$(6.24) \quad (1 - y)(1 - \tau y)F(y) = [(1 - \tau y)^2 + (1 - \tau^2)\lambda y]F(\tau y), \quad |y| < 1.$$

Let $|y| < 1$. For any $j \geq 0$, put $R_j(y) \stackrel{\text{def}}{=} \frac{(1 - \tau^{j+1}y)^2 + (1 - \tau^2)\lambda\tau^j y}{(1 - \tau^j y)(1 - \tau^{j+1}y)}$, so that $R_j(y) = 1 + O(\tau^j)$. Applying (6.24) at the points $y, \tau y, \dots, \tau^{N-1}y$, we obtain $F(y) = \prod_{j=0}^{N-1} R_j(y) \cdot F(\tau^N y)$ for any integer $N \geq 1$. Since $F(\tau^N y) \rightarrow F(0) = 1$ as $N \rightarrow \infty$, this proves the first equality of (6.20). By (6.18), we have $(1 - \tau^{j+1}y)^2 + (1 - \tau^2)\lambda\tau^j y = (1 - \tau^{j+1}zy)(1 - \tau^{j+1}z^{-1}y)$, which gives the second equality.

2. Let $|y| < 1$. Since $|\tau y| < 1 = |z \cdot z^{-1}|$, the q -Gauss sum [DLMF, (17.6.1)], applied with the parameters $(z, z^{-1}, \tau y)$, gives

$$\sum_{k \geq 0} \frac{(z, z^{-1}; \tau)_k}{(\tau; \tau)_k (\tau y; \tau)_k} (\tau y)^k = \frac{(\tau z y, \tau z^{-1} y; \tau)_\infty}{(\tau y; \tau)_\infty^2}.$$

We divide by $1 - y$. Since $(1 - y)(\tau y; \tau)_k = (y; \tau)_{k+1}$ and $(1 - y)(\tau y; \tau)_\infty = (y; \tau)_\infty$, the first part gives

$$F(y) = \sum_{k \geq 0} \frac{(z, z^{-1}; \tau)_k}{(\tau; \tau)_k} \tau^k \frac{y^k}{(y; \tau)_{k+1}}.$$

By the q -binomial theorem [KS96, (0.5.2)], applied with the parameter τ^{k+1} (see also [DLMF, (17.2.38)]), we have $\frac{y^k}{(y; \tau)_{k+1}} = \sum_{d \geq k} \begin{bmatrix} d \\ k \end{bmatrix}_\tau y^d$. The double series converges absolutely, since $|(z, z^{-1}; \tau)_k| \leq \prod_{j \geq 0} (1 + |z + z^{-1}|\tau^j + \tau^{2j})$ and $\sum_{d \geq k} \begin{bmatrix} d \\ k \end{bmatrix}_\tau |y|^d = \frac{|y|^k}{(|y|; \tau)_{k+1}} \leq \frac{|y|^k}{(|y|; \tau)_\infty}$. Comparing the coefficients of y^d , we obtain (6.21).

3. The uniform convergence follows from (6.23), and the limit ϱ_∞ is entire. By the first part, the function $E(y) = \sum_{d \geq 0} e_dy^d$ is equal to $\frac{(\tau z y, \tau z^{-1} y; \tau)_\infty}{(\tau y; \tau)_\infty^2}$ on the unit disc, and the right-hand side is holomorphic on the disc $\{|y| < \tau^{-1}\}$. Hence the Taylor series of E converges at $y = 1$, and we deduce that $\varrho_\infty(\lambda) = \sum_{d \geq 0} e_d = E(1) = \frac{(\tau z, \tau z^{-1}; \tau)_\infty}{(\tau; \tau)_\infty^2}$. Since $(1 - \tau^n z)(1 - \tau^n z^{-1}) =$

$(1 - \tau^n)^2 + (1 - \tau^2)\lambda\tau^{n-1}$, we have $\frac{(1 - \tau^n z)(1 - \tau^n z^{-1})}{(1 - \tau^n)^2} \stackrel{(6.19)}{=} 1 + \frac{\lambda}{\mu_n}$, which gives the product over n . The last equality of (6.22) is the q -Gauss sum [DLMF, (17.6.1)] with the parameters (z, z^{-1}, τ) .

4. For $|u| < \tau^{-1}$, put $H(u) \stackrel{\text{def}}{=} \prod_{j \geq 0} (1 - (z + z^{-1})\tau^j u + \tau^{2j} u^2)(\tau u; \tau)_\infty^{-2}$, which is a holomorphic function of $(\lambda, u) \in \mathbb{C} \times \{|u| < \tau^{-1}\}$ by (6.18). By the third part, $E(y) = \frac{H(\tau y)}{(1 - \tau y)^2}$. Write $H(u) = H(1) + (u - 1)H_1(u)$, where H_1 is holomorphic in the same domain. The function

$$E(y) - \frac{H(1)}{(1 - \tau y)^2} + \frac{H_1(1)}{1 - \tau y} = -\frac{H_1(\tau y) - H_1(1)}{1 - \tau y}$$

is holomorphic in $(\lambda, y) \in \mathbb{C} \times \{|y| < \tau^{-2}\}$. By the Cauchy estimates on the circle $\{|y| = \tau^{-\frac{3}{2}}\}$, its Taylor coefficients are bounded by $C'_L \tau^{\frac{3d}{2}}$ for any $\lambda \in L$. Since $\frac{1}{(1 - \tau y)^2} = \sum_{d \geq 0} (d + 1)\tau^d y^d$, we obtain $e_d = H(1)(d + 1)\tau^d - H_1(1)\tau^d + O(\tau^{\frac{3d}{2}})$ uniformly on L . Summing over the indices larger than d , we deduce that

$$\varrho_\infty(\lambda) - \varrho_d(\lambda) = \frac{H(1)}{1 - \tau} d\tau^{d+1} + O(\tau^d)$$

uniformly on L . Finally, $H(1) = \frac{(z, z^{-1}; \tau)_\infty}{(\tau; \tau)_\infty^2} = (1 - z)(1 - z^{-1})\varrho_\infty(\lambda) \stackrel{(6.18)}{=} \frac{(1 - \tau^2)\lambda}{\tau} \varrho_\infty(\lambda)$, which gives the fourth part. $\blacksquare$

Remark 6.16 Put $Q = \tau^{-1}$ and $x = \frac{z + z^{-1}}{2} = 1 - \frac{(1 - \tau^2)\lambda}{2\tau}$. A direct computation gives $\frac{(Q^{-d}, z, z^{-1}; Q)_k}{(Q; Q)_k^2} Q^k = \begin{bmatrix} d \\ k \end{bmatrix}_\tau \frac{(z, z^{-1}; \tau)_k}{(\tau; \tau)_k} \tau^k$. Hence (6.21) reads

$$\varrho_d(\lambda) = {}_3\phi_2 \left(\begin{matrix} Q^{-d}, z, z^{-1} \\ Q, 0 \end{matrix}; Q, Q \right) = \frac{Q_d(x; 1, Q|Q)}{(Q; Q)_d},$$

where $Q_d(\cdot; a, b|q)$ denotes the Al-Salam–Chihara polynomials [KS96, (3.8.1)], here with the base $Q > 1$. Thus the polynomials ϱ_d are q^{-1} -Al-Salam–Chihara polynomials, see [AI84], [DLMF, Section 18.28(iv)] and [CCSW24]. In this language, the product formula (6.22) is the case $q = \tau$, $(a, b) = (1, \tau^{-1})$ of the large degree asymptotics of these polynomials, which is obtained by Darboux’s method in [CCSW24, Lemma 4.17]. The above proof is self-contained, and it gives in addition the generating function (6.20) and the rate of convergence of the fourth part. Theorem 6.26 shows that the polynomials of the horizontal network are the Al-Salam–Chihara polynomials with the base τ and the parameters $(1, \tau)$. This duality between the bases τ and τ^{-1} reflects the relation $\mathbf{a}_d(\tau) = \mathbf{b}_d(\tau^{-1})$ between the conductances of Proposition 6.4. Finally, by (6.19), the product formula reads

$$\varrho_\infty(\lambda) = \prod_{s \in \frac{1}{2}\mathbb{N}_0} \left(1 + \frac{1 + \tau}{1 - \tau} \cdot \frac{\lambda}{(\dim_r V_s)^2} \right),$$

where $\dim_r V_s = [2s + 1]_\tau$ is the quantum dimension of V_s . We do not know a representation theoretic explanation of this formula.

The function ϱ_∞ is also a Fredholm determinant. For any $m \in \mathbb{N}_0$, put $u_m \stackrel{\text{def}}{=} \mathbf{1}_{\{0, \dots, m\}} \in \ell_{\mathbb{N}_0}^2$. We consider the operator K on $\ell_{\mathbb{N}_0}^2$, with the counting measure, whose matrix is

$$K(i, j) \stackrel{\text{def}}{=} \sum_{m \geq \max(i, j)} \frac{1}{\mathbf{a}_m}, \quad i, j \in \mathbb{N}_0.$$

For any $N \in \mathbb{N}_0$, let J_N be the matrix of the network $(\mathbf{a}_d)$ on $\{0, \dots, N\}$ with the Dirichlet condition at $N+1$, that is

$$(J_N f)(d) \stackrel{\text{def}}{=} \mathbf{a}_d(f(d) - f(d+1)) + \mathbf{a}_{d-1}(f(d) - f(d-1)), \quad 0 \leq d \leq N,$$

with $f(N+1) = 0$ and $\mathbf{a}_{-1} = 0$, and let K_N be the matrix $K_N(i, j) \stackrel{\text{def}}{=} \sum_{m=\max(i, j)}^N \frac{1}{\mathbf{a}_m}$, $0 \leq i, j \leq N$. For a positive trace class operator T with eigenvalues $\kappa_1 \geq \kappa_2 \geq \dots$, counted with multiplicity, the Fredholm determinant is $\det(\text{Id} + \lambda T) = \prod_i (1 + \lambda \kappa_i)$, see [Sim05].

Theorem 6.17 1. The series $\sum_{m \geq 0} \frac{1}{\mathbf{a}_m} \langle \cdot, u_m \rangle u_m$ converges in trace norm to K , and K is positive, injective and trace class, with

$$\text{Tr } K = \sum_{m \geq 0} \frac{m+1}{\mathbf{a}_m} = (1 - \tau^2) \sum_{m \geq 1} \frac{m\tau^{m-1}}{1 - \tau^m} = \sum_{n \geq 1} \frac{1}{\mu_n} = \varrho'_\infty(0).$$

2. For any $N \in \mathbb{N}_0$, we have $\det J_N = \prod_{m=0}^N \mathbf{a}_m$, $J_N^{-1} = K_N$ and, for any $\lambda \in \mathbb{C}$, $\varrho_{N+1}(\lambda) = \det(\text{Id} + \lambda K_N)$.

3. For any $\lambda \in \mathbb{C}$, we have $\det(\text{Id} + \lambda K) = \varrho_\infty(\lambda)$.

4. The eigenvalues of K are the numbers $\frac{1}{\mu_n}$, $n \geq 1$, and they are simple. For any $n \geq 1$, the sequence $v_n \stackrel{\text{def}}{=} (\varrho_m(-\mu_n))_{m \geq 0}$ is an eigenvector for $\frac{1}{\mu_n}$, and

$$\sum_{m \geq 0} \varrho_m(-\mu_n) y^m = \frac{(\tau^{1-n} y; \tau)_{n-1}}{(\tau y; \tau)_n}, \quad \|v_n\|_{\ell_{\mathbb{N}_0}^2}^2 = \frac{1}{\varpi_n}, \quad \text{where } \varpi_n \stackrel{\text{def}}{=} \tau^{n(n-1)}(1 - \tau^{2n}).$$

Consequently, $(\sqrt{\varpi_n} v_n)_{n \geq 1}$ is an orthonormal basis of $\ell_{\mathbb{N}_0}^2$ and, for any $i, j \in \mathbb{N}_0$,

$$(6.25) \quad \sum_{n \geq 1} \varpi_n \varrho_i(-\mu_n) \varrho_j(-\mu_n) = \delta_{ij}.$$

Proof : 1. The operator $\langle \cdot, u_m \rangle u_m$ is positive of rank one, with trace $\|u_m\|^2 = m+1$, and $\frac{m+1}{\mathbf{a}_m} \leq (1 + \tau)(m+1)\tau^m$. Hence the series converges in trace norm to a positive trace class operator, whose matrix is $\sum_m \frac{1}{\mathbf{a}_m} u_m(i) u_m(j) = K(i, j)$. If $\langle Kf, f \rangle = \sum_m \frac{1}{\mathbf{a}_m} |\langle f, u_m \rangle|^2 = 0$, then $\langle f, u_m \rangle = 0$ for any m , hence $f = 0$, since $\delta_m = u_m - u_{m-1}$. We have $\text{Tr } K = \sum_m \frac{m+1}{\mathbf{a}_m} = (1 - \tau^2) \sum_{m \geq 1} \frac{m\tau^{m-1}}{1 - \tau^m}$. On the other hand, $\varrho'_\infty(0) \stackrel{(6.22)}{=} \sum_n \frac{1}{\mu_n} = (1 - \tau^2) \sum_{n \geq 1} \frac{\tau^{n-1}}{(1 - \tau^n)^2}$, and the two Lambert series are equal, since $\sum_{m \geq 1} \frac{m\tau^{m-1}}{1 - \tau^m} = \sum_{m, i \geq 1} m\tau^{mi} = \sum_{i \geq 1} \frac{\tau^i}{(1 - \tau^i)^2}$.

2. For $0 \leq n \leq N$, the leading principal submatrix of size $n+1$ of $J_N + \lambda$ is $J_n + \lambda$, since the diagonal entry of J_N at n is $\mathbf{a}_n + \mathbf{a}_{n-1}$. Expanding along the last row, the determinants $\Delta_n \stackrel{\text{def}}{=} \det(J_n + \lambda)$ satisfy $\Delta_n = (\mathbf{a}_n + \mathbf{a}_{n-1} + \lambda)\Delta_{n-1} - \mathbf{a}_{n-1}^2 \Delta_{n-2}$, with $\Delta_{-1} = 1$ and $\Delta_{-2} = 0$. By the second form of (6.15), we have $\mathbf{a}_n \varrho_{n+1} = (\mathbf{a}_n + \mathbf{a}_{n-1} + \lambda)\varrho_n - \mathbf{a}_{n-1} \varrho_{n-1}$, so that the numbers $P_n \stackrel{\text{def}}{=} (\prod_{m=0}^n \mathbf{a}_m) \varrho_{n+1}(\lambda)$ satisfy the same recurrence, with $P_{-1} = \varrho_0 = 1$ and $P_0 = \mathbf{a}_0 + \lambda = \Delta_0$. Hence $\det(J_N + \lambda) = (\prod_{m=0}^N \mathbf{a}_m) \varrho_{N+1}(\lambda)$. For $\lambda = 0$, we obtain $\det J_N = \prod_{m=0}^N \mathbf{a}_m$, since $\varrho_{N+1}(0) = 1$. Now fix j and put $g(i) \stackrel{\text{def}}{=} K_N(i, j)$ for $0 \leq i \leq N$ and $g(N+1) = 0$. Then $g(i) - g(i+1) = \frac{1}{\mathbf{a}_i} \mathbf{1}_{i \geq j}$, so that $(J_N g)(i) = \mathbf{1}_{i \geq j} - \mathbf{1}_{i-1 \geq j} = \delta_{ij}$. Thus $J_N^{-1} = K_N$ and $\varrho_{N+1}(\lambda) = \frac{\det(J_N + \lambda)}{\det J_N} = \det(\text{Id} + \lambda K_N)$.

3. We identify K_N with the operator $\sum_{m=0}^N \frac{1}{a_m} \langle \cdot, u_m \rangle u_m$ on $\ell_{\mathbb{N}_0}^2$, which has the same nonzero eigenvalues. We have $0 \leq K_N \leq K_{N+1} \leq K$ and $\|K - K_N\| \leq \text{Tr}(K - K_N) = \sum_{m>N} \frac{m+1}{a_m} \rightarrow 0$. By the min-max principle [RS78, Theorem XIII.1], applied to $-K_N$ and $-K$, the i -th eigenvalue $\kappa_i(K_N)$ satisfies $\kappa_i(K_N) \leq \kappa_i(K)$ and $|\kappa_i(K_N) - \kappa_i(K)| \leq \|K - K_N\|$. Let $\lambda \geq 0$. We have $0 \leq \log(1 + \lambda \kappa_i(K_N)) \leq \lambda \kappa_i(K)$ and $\sum_i \kappa_i(K) = \text{Tr } K < \infty$. By dominated convergence and the second part, we deduce that

$$\log \det(\text{Id} + \lambda K) = \lim_{N \rightarrow \infty} \sum_i \log(1 + \lambda \kappa_i(K_N)) = \lim_{N \rightarrow \infty} \log \varrho_{N+1}(\lambda) = \log \varrho_{\infty}(\lambda).$$

Both $\lambda \mapsto \prod_i (1 + \lambda \kappa_i(K))$ and ϱ_{∞} are entire functions, hence they are equal on $\mathbb{C}$.

4. By the third part and (6.22), we have $\prod_i (1 + \lambda \kappa_i(K)) = \prod_n (1 + \frac{\lambda}{\mu_n})$ for any $\lambda \in \mathbb{C}$. Comparing the zeros with their multiplicities, and since the μ_n are distinct, we deduce that the nonzero eigenvalues of K are the $\frac{1}{\mu_n}$, each of them being simple. Let $f \in \ell_{\mathbb{N}_0}^2$. For any $m \in \mathbb{N}_0$, we have $(Kf)(m) = \sum_{l \geq m} \frac{1}{a_l} \sum_{i \leq l} f(i)$, the double series being absolutely convergent, so that $(Kf)(m+1) - (Kf)(m) = -\frac{1}{a_m} \sum_{i \leq m} f(i)$. Hence, if $Kf = \frac{1}{\mu_n} f$ with $f \neq 0$, then $f(m+1) - f(m) = -\frac{\mu_n}{a_m} \sum_{i \leq m} f(i)$ for any m , which is (6.15) with $\lambda = -\mu_n$. Thus $f = f(0)v_n$, and v_n is an eigenvector.

For $\lambda = -\mu_n$, we can choose $z = \tau^{-n}$ in (6.18), since $\tau^{-n} + \tau^n = 2 + \frac{(1-\tau^n)^2}{\tau^n} = 2 + \frac{(1-\tau^2)\mu_n}{\tau}$. Then $(\tau z y; \tau)_{\infty} = (\tau^{1-n} y; \tau)_{n-1} (y; \tau)_{\infty}$ and $(\tau z^{-1} y; \tau)_{\infty} = (\tau^{n+1} y; \tau)_{\infty}$, and (6.20) gives the generating function of v_n . It is a rational function whose poles are the simple poles $\tau^{-1}, \dots, \tau^{-n}$. The pole τ^{-1} gives the leading term $\varrho_m(-\mu_n) = c_n \tau^m + O(\tau^{2m})$, where

$$c_n = \frac{(\tau^{-n}; \tau)_{n-1}}{(\tau; \tau)_{n-1}} = (-1)^{n-1} \tau^{1 - \frac{n(n+1)}{2}} \frac{1 - \tau^n}{1 - \tau}.$$

For $\lambda, \lambda' \in \mathbb{C}$, put $T_m \stackrel{\text{def}}{=} a_m (\varrho_{m+1}(\lambda) \varrho_m(\lambda') - \varrho_m(\lambda) \varrho_{m+1}(\lambda'))$ and $T_{-1} = 0$. The second form of (6.15) gives $T_m - T_{m-1} = (\lambda - \lambda') \varrho_m(\lambda) \varrho_m(\lambda')$. Summing over $0 \leq m \leq N$, dividing by $\lambda - \lambda'$ and letting $\lambda' \rightarrow \lambda$, we obtain the confluent form of the Christoffel–Darboux formula, compare [Sze75, Theorem 3.2.2]

$$\sum_{m=0}^N \varrho_m(\lambda)^2 = a_N (\varrho'_{N+1}(\lambda) \varrho_N(\lambda) - \varrho'_N(\lambda) \varrho_{N+1}(\lambda)).$$

Let $\lambda = -\mu_n$ and $N \rightarrow \infty$. By Theorem 6.15, $\varrho'_N \rightarrow \varrho'_{\infty}$ locally uniformly, and $a_N \varrho_N(-\mu_n) \rightarrow \frac{c_n}{1-\tau^2}$, $a_N \varrho_{N+1}(-\mu_n) \rightarrow \frac{\tau c_n}{1-\tau^2}$. Hence $\|v_n\|^2 = \frac{c_n}{1+\tau} \varrho'_{\infty}(-\mu_n)$. In (6.22), the factor $1 - \tau^n z$ of $(\tau z; \tau)_{\infty}$ vanishes at $z = \tau^{-n}$. Since $\frac{dz}{d\lambda} = -\frac{1-\tau^2}{\tau(1-z^{-2})}$, its derivative with respect to λ at $-\mu_n$ is $\frac{\tau^{n-1}(1-\tau^2)}{1-\tau^{2n}}$. The other factors take the value $\frac{(-1)^{n-1} \tau^{-\frac{n(n-1)}{2}} (\tau; \tau)_{n-1} (\tau; \tau)_{\infty} (\tau^{n+1}; \tau)_{\infty}}{(\tau; \tau)_{\infty}^2} = \frac{(-1)^{n-1} \tau^{-\frac{n(n-1)}{2}}}{1-\tau^n}$. We deduce that

$$(6.26) \quad \varrho'_{\infty}(-\mu_n) = (-1)^{n-1} \tau^{-\frac{(n-1)(n-2)}{2}} \frac{1 - \tau^2}{(1 - \tau^n)(1 - \tau^{2n})},$$

and $\|v_n\|^2 = \frac{\tau^{-n(n-1)}}{1-\tau^{2n}} = \frac{1}{\varpi_n}$. Finally, K is a compact, self-adjoint and injective operator, whose eigenspaces are the lines $\mathbb{C}v_n$. Hence $(\sqrt{\varpi_n} v_n)$ is an orthonormal basis, and (6.25) is the Parseval identity $\langle \delta_i, \delta_j \rangle = \sum_n \varpi_n \langle \delta_i, v_n \rangle \langle v_n, \delta_j \rangle$. $\blacksquare$

Remark 6.18 The Jacobi operator $J \stackrel{\text{def}}{=} K^{-1}$ of the vertical network has the spectrum $\{\mu_n : n \geq 1\}$, and by (6.25) its spectral measure at δ_0 is $\sum_n \varpi_n \delta_{\mu_n}$. The total mass is $\sum_n (\tau^{n(n-1)} - \tau^{n(n+1)}) = 1$. In the variable x of Remark 6.16, the masses lie at the points $x_n = \frac{\tau^n + \tau^{-n}}{2}$, and this measure is the orthogonality measure of the q^{-1} -Al-Salam–Chihara polynomials $Q_d(x; 1, Q|Q)$, a special case of the discrete orthogonality of [AI84], see [CCSW24, Theorem 4.16]. The eigenvectors are little τ -Jacobi polynomials: for any $m \in \mathbb{N}_0$ and $n \geq 1$,

$$\varrho_m(-\mu_n) = (-1)^{n-1} \tau^{m - \frac{(n-1)(n+2)}{2}} \frac{1 - \tau^n}{1 - \tau} p_{n-1}(\tau^m; \tau, 1|\tau),$$

where $p_k(x; a, b|q) = {}_2\phi_1(q^{-k}, abq^{k+1}; aq; q, qx)$ [KS96, (3.12.1)]. Indeed, by the generating function of Theorem 6.17, $m \mapsto \tau^{-m} \varrho_m(-\mu_n)$ is a polynomial of degree $n-1$ in τ^m , and these polynomials are orthogonal for the measure $\sum_m \tau^{2m} \delta_{\tau^m}$, which is the orthogonality measure of the little τ -Jacobi polynomials with $(a, b) = (\tau, 1)$ [KS96, (3.12.2)]; the constant follows from $\varrho_0 = 1$ and from the q -Chu–Vandermonde sum [DLMF, (17.6.3)]. This is an instance of the duality between q^{-1} -Al-Salam–Chihara and little q -Jacobi polynomials [AK04].

Corollary 6.19 1. The zeros of ϱ_∞ are the numbers $-\mu_n$, $n \geq 1$, and they are simple. The entire function ϱ_∞ has order 0, $\max_{|\lambda|=R} |\varrho_\infty(\lambda)| = \varrho_\infty(R)$, and $\log \varrho_\infty(R) = \frac{(\log R)^2}{2 \log \frac{1}{\tau}} + O(\log R)$ as $R \rightarrow \infty$.

2. Let W_n be the Chebyshev polynomials of the fourth kind, $W_n(\cos \theta) = \frac{\sin((n+\frac{1}{2})\theta)}{\sin \frac{\theta}{2}}$ [DLMF, (18.5.4)]. For any $\lambda \in \mathbb{C}$, with $x = 1 - \frac{(1-\tau^2)\lambda}{2\tau}$, we have

$$\varrho_\infty(\lambda) = \frac{1}{(\tau; \tau)_\infty^3} \sum_{n \geq 0} (-1)^n \tau^{\frac{n(n+1)}{2}} W_n(x) = \frac{\theta_1(u, \tau^{\frac{1}{2}})}{\theta'_1(0, \tau^{\frac{1}{2}}) \sin u},$$

where $u \in \mathbb{C}$ is such that $\sin^2 u = \frac{(1-\tau^2)\lambda}{4\tau}$ and θ_1 is the Jacobi theta function [DLMF, (20.2.1)], the last expression being 1 for $u = 0$.

3. The function $f(z) = \frac{(\tau z, \tau z^{-1}; \tau)_\infty}{(\tau; \tau)_\infty^2}$ satisfies $(1-\tau z)f(\tau z) = (1-z^{-1})f(z)$ for any $z \in \mathbb{C} \setminus \{0\}$, and any holomorphic function on $\mathbb{C} \setminus \{0\}$ which satisfies this functional equation is a constant multiple of f .

4. Write $\varrho_\infty(\lambda) = \sum_{n \geq 0} c_n \lambda^n$. For any $n \geq 1$, we have $c_n > 0$ and

$$\begin{aligned} c_n &= (1 - \tau^2)^n \sum_{1 \leq i_1 < \dots < i_n} \prod_{l=1}^n \frac{\tau^{i_l-1}}{(1 - \tau^{i_l})^2} \\ &= \frac{1}{(\tau; \tau)_\infty^3} \left(\frac{\tau^2 - 1}{\tau} \right)^n \sum_{m \geq n} (-1)^m \frac{2m+1}{2n+1} \binom{m+n}{2n} \tau^{\frac{m(m+1)}{2}}. \end{aligned}$$

5. For any integer $k \geq 1$, let $E_{2k}(\tau) \stackrel{\text{def}}{=} 1 - \frac{4k}{B_{2k}} \sum_{N \geq 1} \sigma_{2k-1}(N) \tau^N$ be the Eisenstein series of weight $2k$ [Zag08], written as a power series in τ , where B_{2k} is the Bernoulli number [DLMF, Section 24.2] and $\sigma_{2k-1}(N) = \sum_{e|N} e^{2k-1}$. For any $u \in \mathbb{C}$ such that $|u| < \min(\pi, \frac{1}{2} \log \frac{1}{\tau})$, we have

$$(6.27) \quad \frac{\sin u}{u} \varrho_\infty\left(\frac{4\tau \sin^2 u}{1 - \tau^2}\right) = \exp\left(\sum_{k \geq 1} \frac{(-1)^k 2^{2k-1} B_{2k}}{k(2k)!} E_{2k}(\tau) u^{2k}\right).$$

Consequently, for any $n \geq 1$, the number $\left(\frac{4\tau}{1-\tau^2}\right)^n c_n$ is a polynomial with rational coefficients in $E_2(\tau)$, $E_4(\tau)$ and $E_6(\tau)$, whose homogeneous components have weights $0, 2, \dots, 2n$, the weight of E_{2k} being $2k$. For instance,

$$c_1 = \frac{1-\tau^2}{24\tau}(1-E_2), \quad c_2 = \left(\frac{1-\tau^2}{4\tau}\right)^2 \left(\frac{E_2^2}{72} - \frac{E_2}{12} - \frac{E_4}{180} + \frac{3}{40}\right).$$

Proof : 1. The first assertion follows from (6.22), since the μ_n are distinct. By the fourth part below, $c_n > 0$ for any n , hence $\max_{|\lambda|=R} |\varrho_\infty(\lambda)| = \varrho_\infty(R)$. Put $\ell = \log \frac{1}{\tau}$. By (6.19), there exists $C \geq 1$ such that $C^{-1}\tau^{-n} \leq \mu_n \leq C\tau^{-n}$ for any $n \geq 1$. Let $R \geq 1$ and let N be the integer part of $\frac{\log R}{\ell}$. For $n \leq N$, we have $\log(1 + \frac{R}{\mu_n}) = \log R - n\ell + O(1)$, and for $n > N$, we have $\log(1 + \frac{R}{\mu_n}) \leq \frac{R}{\mu_n} \leq CR\tau^n$, whose sum over $n > N$ is $O(1)$. Hence

$$\log \varrho_\infty(R) = N \log R - \frac{N(N+1)}{2} \ell + O(N) = \frac{(\log R)^2}{2\ell} + O(\log R).$$

In particular, ϱ_∞ has order 0.

2. For $n \in \mathbb{N}_0$ and $w \in \mathbb{C} \setminus \{0\}$, put $D_n(w) \stackrel{\text{def}}{=} \sum_{|i| \leq n} w^i$, so that $D_n(e^{i\theta}) = W_n(\cos \theta)$ and $D_n(w) = W_n(\frac{w+w^{-1}}{2})$, both sides being Laurent polynomials. Let $z \neq 1$. By Jacobi's triple product identity [DLMF, (17.8.1)], $\sum_{m \in \mathbb{Z}} (-1)^m \tau^{\frac{m(m-1)}{2}} z^m = (\tau, z, \tau z^{-1}; \tau)_\infty$. Since $(z; \tau)_\infty = (1-z)(\tau z; \tau)_\infty$, (6.22) gives

$$\varrho_\infty(\lambda) = \frac{1}{(1-z)(\tau; \tau)_\infty^3} \sum_{m \in \mathbb{Z}} (-1)^m \tau^{\frac{m(m-1)}{2}} z^m.$$

We group the terms m and $1-m$, $m \geq 1$, which have the same coefficient up to sign. Since $\frac{z^m - z^{1-m}}{1-z} = -D_{m-1}(z)$, we obtain the first formula with $n = m-1$, and it extends to $z = 1$ by continuity. Now write $z = e^{2iu}$. Then $D_n(z) = \frac{\sin((2n+1)u)}{\sin u}$ and $\sin^2 u = \frac{2-z-z^{-1}}{4} \stackrel{(6.18)}{=} \frac{(1-\tau^2)\lambda}{4\tau}$. By [DLMF, (20.2.1)], we have $\theta_1(u, \tau^{\frac{1}{2}}) = 2\tau^{\frac{1}{8}} \sum_{n \geq 0} (-1)^n \tau^{\frac{n(n+1)}{2}} \sin((2n+1)u)$. Hence $\varrho_\infty(\lambda) = \frac{\theta_1(u, \tau^{\frac{1}{2}})}{2\tau^{\frac{1}{8}}(\tau; \tau)_\infty^3 \sin u}$. Letting $u \rightarrow 0$ and using $\varrho_\infty(0) = 1$, we obtain $\theta'_1(0, \tau^{\frac{1}{2}}) = 2\tau^{\frac{1}{8}}(\tau; \tau)_\infty^3$, in accordance with [DLMF, (20.4.2)], which gives the second formula.

3. We have $f(\tau z) = \frac{(\tau^2 z, z^{-1}; \tau)_\infty}{(\tau; \tau)_\infty^2}$, $(\tau z; \tau)_\infty = (1-\tau z)(\tau^2 z; \tau)_\infty$ and $(z^{-1}; \tau)_\infty = (1-z^{-1})(\tau z^{-1}; \tau)_\infty$, which gives the functional equation. Let g be a holomorphic function on $\mathbb{C} \setminus \{0\}$ which satisfies it. The equation at $z = \tau^{-1}$ gives $g(\tau^{-1}) = 0$, and the equation at $z = \tau^{-k}$, $k \geq 2$, gives $(1-\tau^{1-k})g(\tau^{1-k}) = (1-\tau^k)g(\tau^{-k})$, so that $g(\tau^{-k}) = 0$ for any $k \geq 1$ by induction. Similarly, the equation at $z = \tau^k$, $k \geq 1$, gives $(1-\tau^{k+1})g(\tau^{k+1}) = (1-\tau^{-k})g(\tau^k)$, and the equation at $z = 1$ gives $g(\tau) = 0$; hence $g(\tau^k) = 0$ for any $k \geq 1$. The zeros of f are the simple zeros $\tau^{\pm k}$, $k \geq 1$. Hence $h = \frac{g}{f}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and $h(\tau z) = h(z)$. Thus h defines a holomorphic function on the compact Riemann surface $(\mathbb{C} \setminus \{0\})/\tau^\mathbb{Z}$, and h is constant.

4. The first formula follows from the expansion of the absolutely convergent product (6.22), and it shows that $c_n > 0$. For the second one, we use the second part and the expansion

$$W_m(x) = \sum_{j=0}^m \frac{2m+1}{2j+1} \binom{m+j}{2j} (2x-2)^j, \quad \text{where } 2x-2 = -\frac{(1-\tau^2)\lambda}{\tau}.$$

It holds for $m = 0$ and $m = 1$, since $W_1(x) = 2x+1$, and it follows by induction from the relation $D_{m+1}(w) = (w+w^{-1})D_m(w) - D_{m-1}(w)$, since a direct computation shows that $(2m+3)(m+$

$j+1)(m+j) = 2(2m+1)(m+j)(m+1-j) + 2j(2m+1)(2j+1) - (2m-1)(m+1-j)(m-j)$. The exchange of the sums over m and j is legitimate, since $\frac{2m+1}{2j+1} \binom{m+j}{2j} \leq (2m+1)4^m$ and the coefficients $\tau^{\frac{m(m+1)}{2}}$ decrease faster than any geometric sequence.

5. Let $|u| < \min(\pi, \frac{1}{2} \log \frac{1}{\tau})$ and $z = e^{2iu}$, so that $|\tau^n z^{\pm 1}| < 1$ for any $n \geq 1$. Taking logarithms in (6.22), we obtain

$$\log \varrho_\infty(\lambda) = - \sum_{n \geq 1} \sum_{m \geq 1} \frac{\tau^{nm}}{m} (z^m + z^{-m} - 2) = \sum_{m \geq 1} \frac{4 \sin^2(mu)}{m} \cdot \frac{\tau^m}{1 - \tau^m},$$

the double series being absolutely convergent. We expand $4 \sin^2(mu) = -2 \sum_{k \geq 1} \frac{(-1)^k (2mu)^{2k}}{(2k)!}$. Since $e^{2|u|}\tau < 1$, the series $\sum_m \cosh(2m|u|) \frac{\tau^m}{m(1-\tau^m)}$ converges, and we may exchange the sums. For any $k \geq 1$, put

$$S_k \stackrel{\text{def}}{=} \sum_{m \geq 1} \frac{m^{2k-1} \tau^m}{1 - \tau^m} = \sum_{N \geq 1} \sigma_{2k-1}(N) \tau^N = \frac{B_{2k}}{4k} (1 - E_{2k}(\tau)).$$

We obtain $\log \varrho_\infty(\lambda) = - \sum_{k \geq 1} \frac{(-1)^k 2^{2k-1} B_{2k}}{k(2k)!} (1 - E_{2k}(\tau)) u^{2k}$. On the other hand, the product formula of the sine [DLMF, (4.22.1)] and the values $\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!}$ [DLMF, (25.6.2)] give $\log \frac{\sin u}{u} = - \sum_{k \geq 1} \frac{\zeta(2k)}{k \pi^{2k}} u^{2k} = \sum_{k \geq 1} \frac{(-1)^k 2^{2k-1} B_{2k}}{k(2k)!} u^{2k}$ for $|u| < \pi$, which is also [DLMF, (4.19.7)]. Adding the two expansions gives (6.27). Since $u^2 = \arcsin^2 \sqrt{s}$ is a power series in $s = \frac{(1-\tau^2)\lambda}{4\tau}$ with rational coefficients and without constant term, the coefficient of s^n in (6.27) is a polynomial with rational coefficients in the E_{2k} , $k \leq n$, whose monomials have weights at most $2n$. For $k \geq 2$, E_{2k} is a polynomial with rational coefficients in E_4 and E_6 [Zag08]. The values of c_1 and c_2 follow from the expansion up to the order 2. ■

The second-order recurrence of the vertical network is also a q -difference equation in the variable $x = \tau^d$, which we now compare with Heine's equation.

Proposition 6.20 *Let $\lambda \in \mathbb{C}$, and consider the q -difference equation*

$$(6.28) \quad (1 - \tau x) f(\tau x) - [1 + \tau - \tau(z + z^{-1})x] f(x) + \tau(1 - x) f(\tau^{-1}x) = 0.$$

1. The function $\tau^d \mapsto \varrho_d(\lambda)$ satisfies (6.28) at the points $x = \tau^d$, $d \in \mathbb{N}_0$, the last term being zero for $d = 0$.
2. The function $\psi(x) \stackrel{\text{def}}{=} x_2 \phi_1(\tau z, \tau z^{-1}; \tau^2; \tau, \tau x)$ is holomorphic on the disc $\{|x| < \tau^{-1}\}$, and it satisfies (6.28) for any $|x| < 1$. Moreover, the characteristic exponents of (6.28) at $x = 0$ are 0 and 1, and if $\lambda \neq 0$, no function f which is holomorphic on a disc centred at 0 with $f(0) \neq 0$ satisfies (6.28) on a smaller disc.
3. For any $d \in \mathbb{N}_0$, put $\psi_d \stackrel{\text{def}}{=} \psi(\tau^d)$. Then $\mathbf{a}_d(\varrho_{d+1}(\lambda) \psi_d - \varrho_d(\lambda) \psi_{d+1}) = \frac{\varrho_\infty(\lambda)}{1+\tau}$. In particular,

$$\varrho_\infty(\lambda) = (1 + (1 + \tau)\lambda) {}_2\phi_1\left(\begin{matrix} \tau z, \tau z^{-1} \\ \tau^2 \end{matrix}; \tau, \tau\right) - \tau {}_2\phi_1\left(\begin{matrix} \tau z, \tau z^{-1} \\ \tau^2 \end{matrix}; \tau, \tau^2\right).$$

Proof : 1. This is the relation $(1 - \tau^{d+1})e_{d+1} - \tau(1 - \tau^d)e_d = (1 - \tau^2)\lambda \tau^d \varrho_d$ of the proof of Theorem 6.15, rewritten with $x = \tau^d$ and (6.18).

2. Let $a, b, c \in \mathbb{C}$ with $c \notin \tau^{-\mathbb{N}_0}$, and let $v(x) = \sum_{n \geq 0} v_n x^n = {}_2\phi_1(a, b; c; \tau, x)$, so that $(1 - \tau^{n+1})(1 - c\tau^n)v_{n+1} = (1 - a\tau^n)(1 - b\tau^n)v_n$. Comparing the coefficients, we obtain Heine's q -difference equation, compare [DLMF, (17.6.27)], where it is written with the q -derivative

$$(c - ab\tau x)v(\tau^2 x) - [c + \tau - (a + b)\tau x]v(\tau x) + \tau(1 - x)v(x) = 0, \quad |x| < 1.$$

For $(a, b, c) = (\tau z, \tau z^{-1}, \tau^2)$, we have $ab = \tau^2$ and $a + b = \tau(z + z^{-1})$, and a direct computation gives that the left-hand side of (6.28) for $f(x) = xv(\tau x)$ is equal to $\frac{x}{\tau}$ times the left-hand side of Heine's equation at the point x . This proves the first assertion. If $f(x) = x^s(1 + O(x))$, the coefficient of x^s in (6.28) is $\tau^s - (1 + \tau) + \tau^{1-s} = \tau^{-s}(\tau^s - 1)(\tau^s - \tau)$, so that the characteristic exponents are 0 and 1. Let $f(x) = \sum_{n \geq 0} f_n x^n$ be a solution near 0. The coefficient of x in (6.28) is $f_1(\tau - 1 - \tau + 1) + f_0(\tau(z + z^{-1}) - 2\tau) = -(1 - \tau^2)\lambda f_0$, so that $f_0 = 0$ if $\lambda \neq 0$.

3. Put $W_d \stackrel{\text{def}}{=} \mathbf{a}_d(\varrho_{d+1}(\lambda)\psi_d - \varrho_d(\lambda)\psi_{d+1})$. For any $d \geq 1$, both sequences satisfy $\mathbf{a}_d g_{d+1} = (\mathbf{a}_d + \mathbf{a}_{d-1} + \lambda)g_d - \mathbf{a}_{d-1}g_{d-1}$, by the first two parts, hence $W_d = W_{d-1}$. Moreover $\psi_d = \tau^d(1 + O(\tau^d))$, so that $\mathbf{a}_d(\psi_d - \psi_{d+1}) \rightarrow \frac{1-\tau}{1-\tau^2} = \frac{1}{1+\tau}$, while $\mathbf{a}_d(\varrho_{d+1} - \varrho_d)\psi_d = \lambda\psi_d \sum_{i \leq d} \varrho_i \rightarrow 0$ by (6.15) and the boundedness of $(\varrho_i(\lambda))$. Writing $W_d = \mathbf{a}_d(\varrho_{d+1} - \varrho_d)\psi_d + \varrho_d \mathbf{a}_d(\psi_d - \psi_{d+1})$, we deduce that $W_d = \lim_{d' \rightarrow \infty} W_{d'} = \frac{\varrho_\infty(\lambda)}{1+\tau}$. The last formula is $W_0 = \frac{\varrho_\infty(\lambda)}{1+\tau}$, since $\mathbf{a}_0 = \frac{1}{1+\tau}$ and $\varrho_1(\lambda) = 1 + (1 + \tau)\lambda$. ■

Remark 6.21 Proposition 6.20 describes a resonant case. The parameter $c = \tau^2$ of Heine's equation is the square of the base, which is the q -analogue of an integer value of the parameter c of Gauss's equation. Accordingly, (6.28) has a solution which is holomorphic at 0, namely ψ , but no second one, and the solution ϱ_d contains the term $-(1 + \tau)\lambda\varrho_\infty(\lambda)d\tau^d$ of Theorem 6.15, where $d = \frac{\log x}{\log \tau}$ plays the role of a logarithm. In particular, a naive specialization of a ${}_2\phi_1$ in the variable $x = \tau^d$ cannot give ϱ_d ; the closed form (6.21) is a terminating series in the base τ^{-1} (Remark 6.16), which diverges for generic x . The third part says that $(1 + \tau)^{-1}\varrho_\infty$ is the Wronskian of the solution (ϱ_d) , which is regular at $d = 0$, and of the solution (ψ_d) , which is minimal at infinity: ϱ_∞ is the Jost function of the vertical network, and its zeros are the values of λ for which the minimal solution satisfies the boundary condition at 0. This gives another proof of the fact that v_n is square summable in Theorem 6.17. The same computation as in the proof of the third part gives the Weyl function $\langle (J + \lambda)^{-1}\delta_0, \delta_0 \rangle = \frac{(1+\tau)\psi_0}{\varrho_\infty(\lambda)}$ for $\lambda \geq 0$, whose residues give again the weights ϖ_n .

We now give a probabilistic meaning to ϱ_∞ . Let $X = (X_t)_{0 \leq t < \zeta}$ be the minimal continuous-time Markov chain on $\mathbb{N}_0$ which jumps from d to $d + 1$ with the rate $\mathbf{a}_d$ and from d to $d - 1$ with the rate $\mathbf{a}_{d-1}$, where ζ is its explosion time, that is the limit of its jump times [Nor97, Chapter 2]. We denote by $\mathbb{P}_d$ and $\mathbb{E}_d$ the probability and the expectation for the chain started at d , and by $T_D \stackrel{\text{def}}{=} \inf\{t : X_t = D\}$ the hitting time of $D \in \mathbb{N}_0$. Let $Y = (Y_t)_{t \geq 0}$ be a continuous-time Markov chain on $\mathbb{N}_0$, independent of X , which jumps from k to $k + 1$ with the rate $\mathbf{b}_k$ and from k to $k - 1$ with the rate $\mathbf{b}_{k-1}$. Since these rates are bounded, Y does not explode and its generator is the bounded operator $-\mathbf{B}$, so that $\mathbf{P}(Y_t = j | Y_0 = k) = \langle e^{-t\mathbf{B}}\delta_j, \delta_k \rangle$ for any $t \geq 0$ and any $j, k \in \mathbb{N}_0$.

Theorem 6.22 1. For any $d \in \mathbb{N}_0$ and any $\lambda \geq 0$, we have

$$(6.29) \quad \mathbb{E}_d[e^{-\lambda\zeta}] = \frac{\varrho_d(\lambda)}{\varrho_\infty(\lambda)}.$$

In particular, $\mathbb{P}_d(\zeta < \infty) = 1$ and $\mathbb{E}_d[\zeta] = \sum_{m \geq d} \frac{m+1}{\mathbf{a}_m}$. Under $\mathbb{P}_0$, the explosion time ζ has the law of $\sum_{n \geq 1} \frac{\epsilon_n}{\mu_n}$, where $(\epsilon_n)_{n \geq 1}$ are independent exponential ran-

dom variables with parameter 1, and its density is $\sum_{n \geq 1} R_n e^{-\mu_n t}$, $t > 0$, where $R_n = (-1)^{n-1} \tau^{\frac{(n-1)(n-2)}{2}} \frac{(1-\tau^n)(1-\tau^{2n})}{1-\tau^2}$.

2. Suppose that $(X_0, Y_0) = (d, k)$, and let $J_0 = 0 < J_1 < J_2 < \dots$ be the jump times of the pair (X, Y) on $[0, \zeta)$. Then $J_l \rightarrow \zeta$ almost surely, the sequence $(X_{J_l} + Y_{J_l}, Y_{J_l})_{l \geq 0}$ is the diagonal chain started at $(d+k, k)$, and its final value k_∞ is equal to Y_ζ almost surely. Consequently, for any $j \in \mathbb{N}_0$,

$$\nu_{(d+k, k)}(j) = \mathbb{E}_d[\langle e^{-\zeta \mathbf{B}} \delta_j, \delta_k \rangle] = \left\langle \frac{\varrho_d}{\varrho_\infty}(\mathbf{B}) \delta_j, \delta_k \right\rangle,$$

and $\varrho_\infty(\mathbf{B})^{-1} = \prod_{n \geq 1} \mu_n(\mu_n + \mathbf{B})^{-1}$, the product converging in norm.

3. For any $d, k \in \mathbb{N}_0$, we have

$$\sum_{j \geq 0} |\nu_{(d+k, k)}(j) - \delta_{jk}| \leq 2(\mathbf{b}_k + \mathbf{b}_{k-1}) \mathbb{E}_d[\zeta] \leq \frac{4\tau}{(1-\tau)^3} (d+1) \tau^d,$$

$$\text{and } \lim_{d \rightarrow \infty} \frac{1}{d\tau^d} \sum_{j \geq 0} |\nu_{(d+k, k)}(j) - \delta_{jk}| = 2(1+\tau)(\mathbf{b}_k + \mathbf{b}_{k-1}).$$

Proof : 1. Let $\lambda \geq 0$ and $D > 0$. For $0 \leq d \leq D$, put $g(d) \stackrel{\text{def}}{=} \mathbb{E}_d[e^{-\lambda T_D}]$, so that $g(D) = 1$. Let $0 \leq d < D$. Under $\mathbb{P}_d$, the first jump time is exponential with parameter $\mathbf{a}_d + \mathbf{a}_{d-1}$ and independent of the first jump, which goes to $d+1$ with probability $\frac{\mathbf{a}_d}{\mathbf{a}_d + \mathbf{a}_{d-1}}$. By the strong Markov property at the first jump time, we deduce that $(\mathbf{a}_d + \mathbf{a}_{d-1} + \lambda)g(d) = \mathbf{a}_d g(d+1) + \mathbf{a}_{d-1} g(d-1)$, that is $\mathbf{a}_d(g(d+1) - g(d)) + \mathbf{a}_{d-1}(g(d-1) - g(d)) = \lambda g(d)$. By induction on d and the second form of (6.15), we obtain $g(d) = g(0)\varrho_d(\lambda)$ for $0 \leq d \leq D$, hence $g(d) = \frac{\varrho_d(\lambda)}{\varrho_D(\lambda)}$, since $\varrho_D(\lambda) \geq 1$. Here $T_D = \infty$ if X does not visit D , and $e^{-\lambda \infty} = 0$ if $\lambda > 0$. Under $\mathbb{P}_d$, the hitting times T_D , $D \geq d$, are nondecreasing in D , and $\zeta = \lim_{D \rightarrow \infty} T_D$ almost surely. Indeed, if D_l is larger than the first $l+1$ states visited by X , then the l -th jump time is smaller than T_{D_l} ; conversely, since the jump rates are bounded on $\{0, \dots, D\}$, the chain cannot make infinitely many jumps in $\{0, \dots, D\}$ in a finite time, so that $T_D \leq \zeta$ almost surely on $\{\zeta < \infty\}$, see also [GMZ12]. By monotone convergence, we obtain (6.29) for $\lambda > 0$. Letting $\lambda \rightarrow 0^+$, we obtain $\mathbb{P}_d(\zeta < \infty) = 1$. The right-hand side of (6.29) is holomorphic on a neighbourhood of 0, so that ζ has moments of any order, and $\mathbb{E}_d[\zeta] = \varrho'_\infty(0) - \varrho'_d(0)$. By (6.15), $\varrho'_m(0) = \sum_{i < m} \frac{i+1}{\mathbf{a}_i}$ for any $m \in \mathbb{N}_0 \cup \{\infty\}$, which gives $\mathbb{E}_d[\zeta]$.

By (6.22), $\mathbb{E}_0[e^{-\lambda \zeta}] = \prod_n \frac{\mu_n}{\mu_n + \lambda} = \mathbf{E}[e^{-\lambda \sum_n \epsilon_n / \mu_n}]$, the series $\sum_n \frac{\epsilon_n}{\mu_n}$ being almost surely convergent since $\sum_n \frac{1}{\mu_n} < \infty$. Laplace transforms determine the laws of nonnegative random variables, which gives the law of ζ . By (6.26), $R_n = \frac{1}{\varrho'_\infty(-\mu_n)}$, and $\sum_n |R_n| < \infty$. Let $\rho_N \stackrel{\text{def}}{=} \sqrt{\mu_N \mu_{N+1}}$ and $|\lambda| = \rho_N$. Since $\mu_{n+1} \geq \tau^{-1} \mu_n$ for any n , we have $|1 + \frac{\lambda}{\mu_n}| \geq \frac{\rho_N}{\mu_n} - 1 \geq \tau^{-(N-n)-\frac{1}{2}} - 1$ for $n \leq N$, and $|1 + \frac{\lambda}{\mu_n}| \geq 1 - \frac{\rho_N}{\mu_n} \geq 1 - \tau^{(n-N-1)+\frac{1}{2}}$ for $n > N$. Hence $\min_{|\lambda|=\rho_N} |\varrho_\infty(\lambda)| \geq \prod_{i=0}^{N-1} (\tau^{-i-\frac{1}{2}} - 1) \cdot (\tau^{\frac{1}{2}}; \tau)_\infty$, which tends to ∞ as $N \rightarrow \infty$. The function $\frac{1}{\varrho_\infty(\lambda)} - \sum_n \frac{R_n}{\lambda + \mu_n}$ is entire, since the poles of the two terms are simple and have the same residues. It tends to 0 uniformly on the circles $\{|\lambda| = \rho_N\}$ as $N \rightarrow \infty$, since $|\lambda + \mu_n| \geq (1 - \tau^{\frac{1}{2}})\rho_N$ for $|\lambda| = \rho_N$ and any n . By the maximum principle, it vanishes. Hence, for $\lambda \geq 0$,

$$\frac{1}{\varrho_\infty(\lambda)} = \sum_{n \geq 1} \frac{R_n}{\lambda + \mu_n} = \int_0^\infty e^{-\lambda t} \sum_{n \geq 1} R_n e^{-\mu_n t} dt,$$

by Fubini's theorem, since $\sum_n \frac{|R_n|}{\mu_n} < \infty$. This gives the density.

2. By the memoryless property of the exponential holding times and the independence of X and Y , the pair (X, Y) is, up to time ζ , a continuous-time Markov chain on $\mathbb{N}_0^2$ which jumps from (d', k') to $(d' \pm 1, k')$ and to $(d', k' \pm 1)$ with the rates $\mathbf{a}_{d'}$, $\mathbf{a}_{d'-1}$, $\mathbf{b}_{k'}$ and $\mathbf{b}_{k'-1}$. Its jump chain has the transition probabilities $\frac{\text{cond}(v, v')}{\sum_{v''} \text{cond}(v, v'')}$, where $\text{cond}(v, v') = \hat{w}(v)P(v, v')$ are the conductances of Proposition 6.4, written in the coordinates $(n - k, k)$, see [Nor97, Section 2.6]. Since $\sum_{v''} \text{cond}(v, v'') = \hat{w}(v) \sum_{v''} P(v, v'') = \hat{w}(v)$, this is the diagonal chain in the coordinates $(n - k, k)$. By part 2 of Lemma 6.7, $X_{J_l} \rightarrow \infty$ as $l \rightarrow \infty$. Since X is constant between the times J_l , each T_D is one of the J_l , and each J_l is smaller than T_D for D large enough. Hence $\sup_l J_l = \lim_D T_D = \zeta$. The process Y has finitely many jumps on $[0, \zeta]$, since $\zeta < \infty$, and almost surely no jump at the time ζ , since ζ is independent of Y and has no atom, being the sum of the first holding time of X , which has a density, and of an independent variable. Hence $k_\infty = \lim_l Y_{J_l} = Y_\zeta$ almost surely. By independence, we deduce that $\nu_{(d+k, k)}(j) = \mathbb{E}_d[p_\zeta(k, j)]$, where $p_t(k, j) = \langle e^{-t\mathbf{B}} \delta_j, \delta_k \rangle$. Let $\mathbf{s}_{j, k}$ be the complex spectral measure of $\mathbf{B}$ associated with δ_j and δ_k , which is supported by the spectrum $\sigma(\mathbf{B}) \subset [0, \infty)$. By Fubini's theorem and (6.29), $\mathbb{E}_d[p_\zeta(k, j)] = \int \mathbb{E}_d[e^{-\lambda\zeta}] d\mathbf{s}_{j, k}(\lambda) = \langle \frac{\varrho_d}{\varrho_\infty}(\mathbf{B}) \delta_j, \delta_k \rangle$. Finally, $\|\text{Id} - \mu_n(\mu_n + \mathbf{B})^{-1}\| \leq \frac{\|\mathbf{B}\|}{\mu_n}$, which is summable, so that the product converges in norm, and its limit is $\varrho_\infty(\mathbf{B})^{-1}$ by the spectral theorem and (6.22).

3. Put $q_k \stackrel{\text{def}}{=} \mathbf{b}_k + \mathbf{b}_{k-1}$, which is the jump rate of Y at k , and $\bar{q} \stackrel{\text{def}}{=} \sup_i q_i \leq \frac{2\tau}{1-\tau^2}$. We have $\sum_j |\nu_{(d+k, k)}(j) - \delta_{jk}| = 2\mathbf{P}(Y_\zeta \neq k)$. If $Y_\zeta \neq k$, then Y jumps before ζ , and the probability of this event is $\mathbb{E}_d[1 - e^{-q_k\zeta}] \leq q_k \mathbb{E}_d[\zeta]$. Moreover, $\frac{m+1}{\mathbf{a}_m} \leq (1+\tau)(m+1)\tau^m$ and $\sum_{m \geq d} (m+1)\tau^m \leq \frac{(d+1)\tau^d}{(1-\tau)^2}$, which gives the inequalities. Conversely, if Y jumps once on $[0, \zeta)$, then $Y_\zeta \neq k$, and, for any $t \geq 0$, the probability that Y started at k jumps once on $[0, t]$ is at least $q_k t e^{-\bar{q}t} \geq q_k(t - \bar{q}t^2)$. Hence $\mathbf{P}(Y_\zeta \neq k) \geq q_k(\mathbb{E}_d[\zeta] - \bar{q}\mathbb{E}_d[\zeta^2])$. We have $\mathbb{E}_d[\zeta] = (1-\tau^2) \sum_{m \geq d} \frac{(m+1)\tau^m}{1-\tau^{m+1}} = (1+\tau)d\tau^d + O(\tau^d)$. On the other hand, (6.15) gives $\varrho_\infty - \varrho_d = \lambda G_d$, where $G_d(\lambda) = \sum_{m \geq d} \frac{1}{\mathbf{a}_m} \sum_{i \leq m} \varrho_i(\lambda)$, so that $\mathbb{E}_d[e^{-\lambda\zeta}] = 1 - \frac{\lambda G_d(\lambda)}{\varrho_\infty(\lambda)}$ and

$$\begin{aligned} \mathbb{E}_d[\zeta^2] &= -2 \left(\frac{G_d}{\varrho_\infty} \right)'(0) = 2 \sum_{m \geq d} \frac{1}{\mathbf{a}_m} \sum_{i \leq m} (\varrho'_\infty(0) - \varrho'_i(0)) \\ &= 2 \sum_{m \geq d} \frac{1}{\mathbf{a}_m} \sum_{i \leq m} \mathbb{E}_i[\zeta] \leq 2 \left(\sum_{i \geq 0} \mathbb{E}_i[\zeta] \right) \sum_{m \geq d} \frac{1}{\mathbf{a}_m}, \end{aligned}$$

which is $O(\tau^d)$, since $\sum_i \mathbb{E}_i[\zeta] = \sum_m \frac{(m+1)^2}{\mathbf{a}_m} < \infty$. We deduce that $\mathbf{P}(Y_\zeta \neq k) = q_k(1+\tau)d\tau^d + O(\tau^d)$, which gives the limit. $\blacksquare$

Remark 6.23 Theorem 6.22 is a q -deformation of a classical object studied by Biane, Pitman and Yor [BPY01]. By (6.19), the normalized explosion time $\zeta_\tau \stackrel{\text{def}}{=} \frac{1-\tau}{1+\tau} \zeta$ satisfies, under $\mathbb{P}_0$,

$$\mathbb{E}_0[e^{-\lambda\zeta_\tau}] = \prod_{n \geq 1} \left(1 + \frac{\lambda}{[n]_\tau^2} \right)^{-1}, \quad \lambda \geq 0.$$

Since $[n]_\tau = \sum_{j=0}^{n-1} \tau^j \geq n$ and $[n]_\tau \rightarrow n$ as $\tau \rightarrow 1^-$, dominated convergence and Euler's product $\sinh z = z \prod_{n \geq 1} (1 + \frac{z^2}{n^2\pi^2})$ [DLMF, (4.36.1)] show that this Laplace transform tends to $\frac{\pi\sqrt{\lambda}}{\sinh(\pi\sqrt{\lambda})}$ as $\tau \rightarrow 1^-$. Thus ζ_τ converges in law to $Y \stackrel{\text{def}}{=} \sum_{n \geq 1} \frac{\varepsilon_n}{n^2}$, which has the law

of $\frac{\pi^2}{2}T$, where T is the hitting time of 1 by a three-dimensional Bessel process started at 0, since $\mathbf{E}[e^{-\lambda T}] = \frac{\sqrt{2\lambda}}{\sinh \sqrt{2\lambda}}$. The density of Y is $\sum_{n \geq 1} 2(-1)^{n-1} n^2 e^{-n^2 t}$, a derivative of a Jacobi theta function, and this law and its relations with the Riemann zeta function are studied in [BPY01] and [Bia03]. Accordingly, $\frac{1+\tau}{1-\tau} R_n = (-1)^{n-1} \tau^{\frac{(n-1)(n-2)}{2}} \frac{(1-\tau^n)(1-\tau^{2n})}{(1-\tau)^2} \rightarrow 2(-1)^{n-1} n^2$, and $\varrho_\infty\left(\frac{1+\tau}{1-\tau}\lambda\right) = \prod_{n \geq 1} \left(1 + \frac{\lambda}{[n]_\tau^2}\right)$ tends to $\frac{\sinh(\pi\sqrt{\lambda})}{\pi\sqrt{\lambda}}$ uniformly on the compact subsets of $\mathbb{C}$, so that the theta quotient of Corollary 6.19 is a q -analogue of this function. Finally, let $s \in \mathbb{C}$ with $\operatorname{Re} s > -1$. Since $\sum_n |R_n| \mu_n^{-\operatorname{Re} s - 1} < \infty$, Fubini's theorem and the density of Theorem 6.22 give

$$\mathbb{E}_0[\zeta^s] = \Gamma(s+1) \sum_{n \geq 1} R_n \mu_n^{-s-1},$$

a Dirichlet series in the squares of the quantum dimensions $[n]_\tau$. Formally, as $\tau \rightarrow 1^-$, the corresponding series for ζ_τ becomes $2\Gamma(s+1)(1-2^{1-2s})\zeta(2s) = \mathbf{E}[Y^s]$, see [Bia03]. We do not know whether this q -analogue satisfies a functional equation.

Corollary 6.24 *For any bounded function $f: \mathbb{N}_0 \rightarrow \mathbb{C}$ and any $(n, k) \in \mathbb{D}$, we have*

$$|\mathcal{H}f(n, k) - f(k)| \leq \frac{4\tau}{(1-\tau)^3} \|f\|_\infty (n-k+1)\tau^{n-k}.$$

For $f = \mathbf{1}_{\{k\}}$, the left-hand side is equivalent to $(1+\tau)(\mathfrak{b}_k + \mathfrak{b}_{k-1})(n-k)\tau^{n-k}$ as $n \rightarrow \infty$.

Proof : We have $\mathcal{H}f(n, k) - f(k) = \sum_j \nu_{(n,k)}(j)(f(j) - f(k))$, which is bounded by $\|f\|_\infty \sum_j |\nu_{(n,k)}(j) - \delta_{jk}|$, and part 3 of Theorem 6.22 with $d = n-k$ concludes. For $f = \mathbf{1}_{\{k\}}$, the left-hand side is $1 - \nu_{(n,k)}(k) = \frac{1}{2} \sum_j |\nu_{(n,k)}(j) - \delta_{jk}|$. ■

Thus the rate $\tau^{n-k} = \tau^{\frac{n-k}{2}}$ of (6.9) can be replaced by $(n-k+1)\tau^{n-k}$, and this rate is optimal. Theorem 6.22 also explains the symmetry of Theorem 6.13: the discrete chain is reversible for $\widehat{w}$, whereas, after the time change by the holding times, the horizontal chain Y is reversible for the counting measure and is observed at a random time ζ which does not depend on it. Finally, we describe the horizontal side explicitly.

Lemma 6.25 *For any $k, n \in \mathbb{N}_0$, put*

$$\widetilde{\varrho}_k \stackrel{\text{def}}{=} {}_3\phi_2\left(\begin{matrix} \tau^{-k}, z, z^{-1} \\ \tau, 0 \end{matrix}; \tau, \tau\right), \quad \mathbf{H}_n \stackrel{\text{def}}{=} {}_3\phi_2\left(\begin{matrix} \tau^{-n}, z, z^{-1} \\ \tau, \tau \end{matrix}; \tau, \tau\right),$$

which are polynomials in λ . Then, for any $k \in \mathbb{N}_0$,

$$\widetilde{\varrho}_k = \sum_{n=0}^k \frac{(\tau; \tau)_k}{(\tau; \tau)_{k-n}} \tau^{k-n} \mathbf{H}_n.$$

Proof : Put $\varphi_i \stackrel{\text{def}}{=} (z, z^{-1}; \tau)_i$. We have $\widetilde{\varrho}_k = \sum_i A_{k,i} \varphi_i$ and $\mathbf{H}_n = \sum_i A_{n,i} \frac{\varphi_i}{(\tau; \tau)_i}$, where $A_{k,i} = \frac{(\tau^{-k}; \tau)_i}{(\tau; \tau)_i^2} \tau^i$. Since (φ_i) is a basis of the polynomials in λ , we must show that $A_{k,i} = \sum_{n=i}^k \frac{(\tau; \tau)_k}{(\tau; \tau)_{k-n}} \tau^{k-n} \frac{A_{n,i}}{(\tau; \tau)_i}$ for any $0 \leq i \leq k$. Using $(\tau^{-n}; \tau)_i = (-1)^i \tau^{\frac{i(i-1)}{2} - ni} \frac{(\tau; \tau)_n}{(\tau; \tau)_{n-i}}$, writing $n = i + l$ and $N = k - i$, and simplifying, this identity becomes

$$\sum_{l=0}^N \left[\begin{matrix} N \\ l \end{matrix} \right]_\tau \prod_{p=0}^{l-1} (\xi - \tau^p) = \xi^N, \quad \text{where } \xi = \tau^{-(i+1)}.$$

This holds for any $\xi \in \mathbb{C}$, by induction on N , using the relation $\begin{bmatrix} N+1 \\ l \end{bmatrix}_\tau = \tau^l \begin{bmatrix} N \\ l \end{bmatrix}_\tau + \begin{bmatrix} N \\ l-1 \end{bmatrix}_\tau$, since $\tau^l + (\xi - \tau^l) = \xi$. $\blacksquare$

Theorem 6.26 *Let $\tilde{\varrho}_k$ be the polynomials of Lemma 6.25, that is $\tilde{\varrho}_k(\lambda) = \frac{Q_k(x; 1, \tau|\tau)}{(\tau; \tau)_k}$ with $x = 1 - \frac{(1-\tau^2)\lambda}{2\tau}$, and for $\theta \in [0, \pi]$, put $\lambda_\theta \stackrel{\text{def}}{=} \frac{2\tau(1-\cos \theta)}{1-\tau^2}$.*

1. *The sequence $(\tilde{\varrho}_k(\lambda))_{k \geq 0}$ satisfies $B\tilde{\varrho}(\lambda) = \lambda\tilde{\varrho}(\lambda)$, the spectrum of B is $[0, \frac{4\tau}{1-\tau^2}]$, and the spectral measure $\mathbf{s}$ of B at δ_0 is the image under $\theta \mapsto \lambda_\theta$ of the measure*

$$(\tau; \tau)_\infty^2 \frac{|(e^{2i\theta}; \tau)_\infty|^2}{|(e^{i\theta}, \tau e^{i\theta}; \tau)_\infty|^2} \frac{d\theta}{2\pi}, \quad \theta \in [0, \pi].$$

Moreover, $\langle g(B)\delta_j, \delta_k \rangle = \int g\tilde{\varrho}_j\tilde{\varrho}_k d\mathbf{s}$ for any bounded Borel function g on $[0, \frac{4\tau}{1-\tau^2}]$ and any $j, k \in \mathbb{N}_0$.

2. *For any $d, k, j \in \mathbb{N}_0$,*

$$\nu_{(d+k, k)}(j) = \frac{(\tau; \tau)_\infty^4}{2\pi} \int_0^\pi \varrho_d(\lambda_\theta) \tilde{\varrho}_j(\lambda_\theta) \tilde{\varrho}_k(\lambda_\theta) \frac{|(e^{2i\theta}; \tau)_\infty|^2}{|(e^{i\theta}; \tau)_\infty|^2 |(\tau e^{i\theta}; \tau)_\infty|^4} d\theta.$$

3. *For any $j, k \in \mathbb{N}_0$,*

$$(6.30) \quad \nu_{(k, k)}(j) = \sum_{n=0}^{\min(j, k)} \begin{bmatrix} j \\ n \end{bmatrix}_\tau \begin{bmatrix} k \\ n \end{bmatrix}_\tau (\tau; \tau)_n (\tau; \tau)_{n+1} \tau^{j+k-2n},$$

and, for any $d \in \mathbb{N}_0$, $\nu_{(d+k, k)}(j) = \sum_{i \geq 0} \langle \varrho_d(B)\delta_i, \delta_k \rangle \nu_{(i, i)}(j)$, where $\langle \varrho_d(B)\delta_i, \delta_k \rangle = 0$ if $|i - k| > d$.

Proof : 1. By [KS96, (3.8.1)] with $(a, b) = (1, \tau)$, we have $\tilde{\varrho}_k = \frac{Q_k(x; 1, \tau|\tau)}{(\tau; \tau)_k}$, and the recurrence relation [KS96, (3.8.4)] reads $(1 - \tau^{k+1})\tilde{\varrho}_{k+1} + (1 - \tau^k)\tilde{\varrho}_{k-1} = (2x - \tau^k - \tau^{k+1})\tilde{\varrho}_k$. We multiply by $\frac{\tau}{1-\tau^2}$. Since $\frac{\tau(1-\tau^{k+1})}{1-\tau^2} = \mathbf{b}_k$ and $\frac{\tau(2x - \tau^k - \tau^{k+1})}{1-\tau^2} = \mathbf{b}_k + \mathbf{b}_{k-1} - \lambda$, we obtain $\mathbf{b}_k(\tilde{\varrho}_k - \tilde{\varrho}_{k+1}) + \mathbf{b}_{k-1}(\tilde{\varrho}_k - \tilde{\varrho}_{k-1}) = \lambda\tilde{\varrho}_k$, that is $B\tilde{\varrho} = \lambda\tilde{\varrho}$. Since B is a bounded Jacobi operator with nonzero off-diagonal entries and $\tilde{\varrho}_0 = 1$, the map $\delta_k \mapsto \tilde{\varrho}_k$ extends to a unitary operator from $\ell_{\mathbb{N}_0}^2$ onto $L^2(\mathbf{s})$ which intertwines B with the multiplication by λ , see [Tes00]. This gives the last assertion. To identify $\mathbf{s}$, let $\frac{1}{2} \leq a < 1$ and $w_a(\theta) \stackrel{\text{def}}{=} \frac{|(e^{2i\theta}; \tau)_\infty|^2}{|(ae^{i\theta}, \tau e^{i\theta}; \tau)_\infty|^2}$. By [KS96, (3.8.2)], the polynomials $\frac{Q_k(x; a, \tau|\tau)}{((\tau; \tau)_k(a\tau; \tau)_k)^{1/2}}$ are orthonormal for the probability measure $(\tau, a\tau; \tau)_\infty w_a(\theta) \frac{d\theta}{2\pi}$ on $[0, \pi]$, with $x = \cos \theta$. Their recurrence coefficients, given by [KS96, (3.8.4)], converge to those of $(\tilde{\varrho}_k)$ as $a \rightarrow 1$, so that the moments of these measures, in the variable λ_θ , converge to the moments $\langle B^p \delta_0, \delta_0 \rangle$ of $\mathbf{s}$. On the other hand, $|1 - e^{i\theta}|^2 \leq 2|1 - ae^{i\theta}|^2$ for $a \geq \frac{1}{2}$, so that w_a is bounded uniformly in a , and $w_a \rightarrow w_1$ pointwise, where w_1 is the density of the statement, which is continuous on $[0, \pi]$. Indeed, $w_a = w_1 \frac{|(e^{i\theta}; \tau)_\infty|^2}{|(ae^{i\theta}; \tau)_\infty|^2}$ and the last fraction is bounded by $2 \prod_{j \geq 1} \left(\frac{1+\tau^j}{1-\tau^j}\right)^2$. By dominated convergence, the moments of $\mathbf{s}$ are those of the measure of the statement. Since both measures have compact supports, they are equal. Its support is $\{\lambda_\theta : \theta \in [0, \pi]\} = [0, \frac{4\tau}{1-\tau^2}]$, which is the spectrum of B .

2. For $\theta \in [0, \pi]$, the number $z = e^{i\theta}$ satisfies (6.18) with $\lambda = \lambda_\theta$, so that $\varrho_\infty(\lambda_\theta) \stackrel{(6.22)}{=} \frac{|(\tau e^{i\theta}; \tau)_\infty|^2}{(\tau; \tau)_\infty^2}$. The second part follows from (6.17) and from the first part with $g = \frac{\varrho_d}{\varrho_\infty}$.

3. Let $d = 0$. By the second part, $\nu_{(k,k)}(j) = (\tau; \tau)_\infty^4 \int_0^\pi \tilde{\varrho}_j \tilde{\varrho}_k w_{1,\tau,\tau} \frac{d\theta}{2\pi}$, where $w_{a,b,c}(\theta) = \frac{|(e^{2i\theta}; \tau)_\infty|^2}{|(ae^{i\theta}, be^{i\theta}, ce^{i\theta}; \tau)_\infty|^2}$ is the weight of the continuous dual τ -Hahn polynomials. By [KS96, (3.3.1) and (3.3.2)] with $(a, b, c) = (a, \tau, \tau)$, $0 < a < 1$, and by the same limit $a \rightarrow 1$ as in the first part, the polynomials H_n of Lemma 6.25 satisfy

$$\int_0^\pi H_m H_n w_{1,\tau,\tau} \frac{d\theta}{2\pi} = \frac{\delta_{mn}}{(\tau; \tau)_n^4 (\tau^{n+1}; \tau)_\infty^3 (\tau^{n+2}; \tau)_\infty} = \frac{1 - \tau^{n+1}}{(\tau; \tau)_\infty^4} \delta_{mn}.$$

By Lemma 6.25, we deduce that

$$\nu_{(k,k)}(j) = \sum_{n \geq 0} \frac{(\tau; \tau)_j (\tau; \tau)_k}{(\tau; \tau)_{j-n} (\tau; \tau)_{k-n}} \tau^{j+k-2n} (1 - \tau^{n+1}),$$

the sum being over $n \leq \min(j, k)$, which is (6.30), since $\frac{(\tau; \tau)_j}{(\tau; \tau)_{j-n}} = [j]_\tau (\tau; \tau)_n$. Finally, $\frac{\varrho_d}{\varrho_\infty}(\mathbf{B}) = \varrho_d(\mathbf{B}) \varrho_\infty(\mathbf{B})^{-1}$, the matrix of $\varrho_\infty(\mathbf{B})^{-1}$ is $(\nu_{(i,i)}(j))_{i,j}$ by Theorem 6.13, and $\varrho_d(\mathbf{B})$ is a polynomial of degree d in the tridiagonal operator $\mathbf{B}$. ■

Remark 6.27 For instance, (6.30) gives $\nu_{(1,1)}(1) = (1 - \tau)(1 - \tau + \tau^3)$, and the last part of Theorem 6.26 gives $\nu_{(1,0)}(0) = (1 + \tau)\nu_{(0,0)}(0) - \tau\nu_{(1,1)}(0) = (1 - \tau)(1 + \tau - \tau^2)$, since $\varrho_1(\mathbf{B}) = \text{Id} + (1 + \tau)\mathbf{B}$. The terms of (6.30) are nonnegative, which makes the positivity of the Poisson kernel visible, and the matrix $(\nu_{(k,k)}(j))$ factorizes as $\mathbf{C}\mathbf{A}\mathbf{C}^T$, where $\mathbf{C}$ is the lower triangular matrix $\mathbf{C}(k, n) = \frac{(\tau; \tau)_k}{(\tau; \tau)_{k-n}} \tau^{k-n}$ and $\mathbf{A} = \text{diag}(1 - \tau^{n+1})$.

Concerning the literature: the evaluation (6.22) is a special case of [CCSW24, Lemma 4.17], the discrete orthogonality of Theorem 6.17 is a special case of [AI84], see [CCSW24, Theorem 4.16], the eigenvectors of Remark 6.18 are given by the duality of [AK04], and the law of the explosion time of a birth-death process with an exit boundary as a sum of independent exponential variables is due to [GMZ12], which extends a theorem of Keilson for finite chains, see [Fil09]. The transition probabilities of the diagonal chain are due to Biane [Bia06, Section 4], see the paragraph after Proposition 6.2, and its restrictions to the centre and to the torus are those considered in [Bia91] for $\text{SU}(n)$ and in [Col04]. The identification of the objects above with the Poisson kernel of the diagonal chain, Theorem 6.22, Corollary 6.24, (6.30) and parts 3 and 4 of Proposition 6.11 are new, as far as we know. The Martin boundary theory of quantum random walks and their restrictions to commutative subalgebras were studied in [Bia94], [NT04], [Col04] and [INT06]. The formulas of this subsection were also checked numerically, with 20 to 40 digits, for $\tau \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$, and (6.29) and (6.30) were compared with Monte Carlo simulations of the chain X and of the chain of Proposition 6.2.

6.5 Energy of Poisson integrals

Let $\text{cond}(v, v') \stackrel{\text{def}}{=} \widehat{w}(v)P(v, v') = \text{cond}(v', v)$ be the conductances of the diagonal chain. For a bounded harmonic function u on $\mathbb{D}$, we have $\Gamma(u)(v) = \sum_{v'} P(v, v') |u(v')|^2 - |u(v)|^2 = \sum_{v'} P(v, v') |u(v') - u(v)|^2$, hence

$$(6.31) \quad \mathcal{E}^{(n)}(u) = \sum_{k=0}^n \sum_{v'} \text{cond}((n, k), v') |u(v') - u(n, k)|^2,$$

where v' runs over the neighbours of (n, k) . Thus $\mathcal{E}^{(n)}(u)$ is the sum of the energies of the edges between the levels n and $n + 1$ and of the edges between the levels $n - 1$ and n . By

Proposition 6.4 and reversibility, the conductance of the horizontal edge between (n, k) and $(n+1, k+1)$ is $\frac{\mathfrak{c}_\tau}{2} w_k \beta_k$ for any $n \geq k$. By Proposition 6.2, the conductances of the vertical edges satisfy

$$(6.32) \quad \text{cond}((n, k), (n \pm 1, k)) \leq \frac{1}{(1 - \tau)\tau^{n-k}}.$$

Proposition 6.28 *For any bounded function $f: \mathbb{N}_0 \rightarrow \mathbb{C}$, we have*

$$\liminf_{n \rightarrow \infty} \mathcal{E}^{(n)}(\mathcal{H}f) \geq \mathfrak{c}_\tau \mathcal{E}_\tau(f) \in [0, \infty].$$

In particular, if f is not constant, then $\mathcal{E}_{\widehat{G}}(\mathcal{H}f) = \infty$.

Proof : Keeping only the horizontal edges in (6.31), we obtain for any integers $0 \leq J < n$

$$\mathcal{E}^{(n)}(\mathcal{H}f) \geq \frac{\mathfrak{c}_\tau}{2} \sum_{k=0}^J w_k \beta_k \left(|\mathcal{H}f(n+1, k+1) - \mathcal{H}f(n, k)|^2 + |\mathcal{H}f(n, k+1) - \mathcal{H}f(n-1, k)|^2 \right).$$

By (6.9), each term of the sum converges to $2w_k \beta_k |f(k+1) - f(k)|^2$ as $n \rightarrow \infty$. Hence $\liminf_n \mathcal{E}^{(n)}(\mathcal{H}f) \geq \mathfrak{c}_\tau \sum_{k=0}^J w_k \beta_k |f(k+1) - f(k)|^2$, and we let $J \rightarrow \infty$. If f is not constant, then $\mathcal{E}_\tau(f) > 0$, since the weights $w_k \beta_k$ are positive, and the series $\sum_n \mathcal{E}^{(n)}(\mathcal{H}f)$ diverges. ■

Theorem 6.29 *Let F be a polynomial, identified with the function $k \mapsto F(\tau^k)$. Then*

$$\mathcal{E}^{(n)}(\mathcal{H}F) = \mathfrak{c}_\tau \mathcal{E}_\tau(F) + O((n+1)^2 \tau^n).$$

In particular, $\lim_{n \rightarrow \infty} \mathcal{E}^{(n)}(\mathcal{H}F) = \mathfrak{c}_\tau \mathcal{E}_\tau(F) = \mathfrak{c}_\tau \langle F, L_\tau F \rangle_{\ell_{\mathbb{N}_0}^2(w)}$.

Proof : Write $F = \sum_{l=0}^L a_l \varphi_l^\tau$. By linearity, (6.13) and $\mathcal{H}\varphi_0^\tau = 1$, there exists a constant C_F such that $|\mathcal{H}F(n, k) - F(\tau^k)| \leq C_F(n+1)\tau^n$ for any $(n, k) \in \mathbb{D}$. Moreover, $|F(\tau^{k+1}) - F(\tau^k)| \leq \|F'\|_{L^\infty([0,1])} \tau^k$.

We first consider the vertical edges at the level $n \geq 1$. For any k , we have $|\mathcal{H}F(n \pm 1, k) - \mathcal{H}F(n, k)| \leq 2C_F(n+2)\tau^{n-1}$ whenever $(n \pm 1, k) \in \mathbb{D}$. Since $\sum_{k=0}^n \tau^{-(n-k)} \leq \frac{\tau^{-n}}{1-\tau}$, the bound (6.32) shows that the contribution of the vertical edges to (6.31) is at most

$$2 \sum_{k=0}^n \frac{\tau^{-(n-k)}}{1-\tau} \cdot 4C_F^2(n+2)^2 \tau^{2n-2} \leq \frac{8C_F^2}{\tau^2(1-\tau)^2} (n+2)^2 \tau^n.$$

Now, we turn to the horizontal edges between the levels n and $n+1$. We write $\mathcal{H}F(n+1, k+1) - \mathcal{H}F(n, k) = F(\tau^{k+1}) - F(\tau^k) + \varepsilon_n(k)$, with $|\varepsilon_n(k)| \leq 2C_F(n+2)\tau^n$. Since $\frac{\mathfrak{c}_\tau}{2} w_k \beta_k \leq \frac{\mathfrak{c}_\tau(1-\tau)}{2}$, we obtain

$$\begin{aligned} & \left| \frac{\mathfrak{c}_\tau}{2} \sum_{k=0}^n w_k \beta_k \left(|\mathcal{H}F(n+1, k+1) - \mathcal{H}F(n, k)|^2 - |F(\tau^{k+1}) - F(\tau^k)|^2 \right) \right| \\ & \leq C \sum_{k=0}^n (\tau^k(n+2)\tau^n + (n+2)^2 \tau^{2n}) \end{aligned}$$

for some constant C , and the right-hand side is $O((n+1)\tau^n)$. Furthermore, we have $\sum_{k>n} w_k \beta_k |F(\tau^{k+1}) - F(\tau^k)|^2 = O(\tau^{2n})$. Hence the horizontal edges between the levels n and

$n + 1$ contribute $\frac{\varepsilon}{2} \mathcal{E}_\tau(F) + O((n + 1)\tau^n)$. In the same way, the horizontal edges between the levels $n - 1$ and n contribute $\frac{\varepsilon}{2} \mathcal{E}_\tau(F) + O((n + 1)\tau^n)$. Adding the three contributions, we obtain the first assertion. The last equality is (3.3). $\blacksquare$

For integers $0 \leq l \leq m$, we put $h_{l,m} \stackrel{\text{def}}{=} \sum_{k=0}^m \tau^k M_{l,m}(\tau^k)^2$, and $h_{l,\infty} \stackrel{\text{def}}{=} \sum_{k \geq 0} \tau^k \widehat{p}_l(\tau^k)^2$. For any function u on $\mathbb{D}$ and any integer $m \geq 0$, we denote by

$$\varepsilon_m(u) \stackrel{\text{def}}{=} \sum_{k=0}^m \sum_{k'=0}^{m+1} \text{cond}((m, k), (m + 1, k')) |u(m + 1, k') - u(m, k)|^2$$

the energy of the edges between the levels m and $m + 1$, and we put $\varepsilon_{-1}(u) \stackrel{\text{def}}{=} 0$. By (6.31), we have $\mathcal{E}^{(n)}(u) = \varepsilon_{n-1}(u) + \varepsilon_n(u)$ for any $n \geq 0$.

Proposition 6.30 *Let $l \geq 1$ be an integer and put $u_l \stackrel{\text{def}}{=} \mathcal{H}\varphi_l^\tau$.*

1. *For any integer $m \geq l$, we have $(1 - \tau^{m+1-l})h_{l,m+1} = (1 - \tau^{m+l+2})h_{l,m}$. Consequently,*

$$h_{l,m} = \frac{(\tau^{2l+2}; \tau)_{m-l}(\tau; \tau)_\infty}{(\tau; \tau)_{m-l}(\tau^{2l+2}; \tau)_\infty} h_{l,\infty} \leq \frac{h_{l,\infty}}{(\tau; \tau)_\infty}, \quad m \geq l.$$

2. *We have $\varepsilon_m(u_l) = 0$ for $m \leq l - 2$, $\varepsilon_{l-1}(u_l) = \frac{(1-\tau^l)\mathcal{K}_l^2 h_{l,l}}{(1-\tau^2)\tau^{l-1}\zeta_l(\infty)^2}$ and, for any integer $m \geq l$,*

$$(6.33) \quad \varepsilon_m(u_l) = \frac{\mathcal{K}_l^2 h_{l,m}}{(1-\tau^2)\tau^m \zeta_l(\infty)^2} \left[(1 - \tau^{m+2})_{\zeta_l(m)}^2 - 2(1 - \tau^{m+l+2})_{\zeta_l(m)} \zeta_l(m+1) + \frac{(1 - \tau^{m+1})(1 - \tau^{m+l+2})}{1 - \tau^{m+1-l}} \zeta_l(m+1)^2 \right].$$

3. *There exists a constant C_τ , depending only on τ , such that $\mathcal{E}^{(n)}(u_l) \leq C_\tau \Lambda_l(\tau) \|\varphi_l^\tau\|_{\ell_{\mathbb{N}_0}^2(w)}^2$ for any integers $l \geq 1$ and $n \geq 0$.*

Proof : By Proposition 6.2, from any vertex (m, k) , the probabilities of a move to the level $m + 1$ and of a move to the level $m - 1$ are

$$p_\uparrow(m) = \frac{1 - \tau^{m+2}}{(1 + \tau)(1 - \tau^{m+1})}, \quad p_\downarrow(m) = \frac{\tau(1 - \tau^m)}{(1 + \tau)(1 - \tau^{m+1})},$$

which do not depend on k . By (6.5), we have $\widehat{w}(m, k) = \frac{1 - \tau^{m+1}}{(1 - \tau)\tau^m} \tau^k$.

1. Let $m \geq l$, so that $m \geq 1$. Let $j \geq 0$ be an integer. Lemma 6.5, applied at the level m to the function $(n', k') \mapsto \mathbf{1}_{\{m+1\}}(n') \tau^{jk'}$, shows that $k \mapsto \sum_{k'} P((m, k), (m + 1, k')) \tau^{jk'}$ is the sum of $\frac{1 - \tau^{m+j+2}}{(1 + \tau)(1 - \tau^{m+1})} \tau^{jk}$ and of a multiple of $\tau^{(j-1)k}$. Similarly, applied at the level $m + 1$ to $(n', k') \mapsto \mathbf{1}_{\{m\}}(n') \tau^{jk'}$, it shows that $k' \mapsto \sum_k P((m + 1, k'), (m, k)) \tau^{jk}$ is the sum of $\frac{\tau - \tau^{m+2-j}}{(1 + \tau)(1 - \tau^{m+2})} \tau^{jk'}$ and of a multiple of $\tau^{(j-1)k'}$. Consider

$$b_m \stackrel{\text{def}}{=} \sum_{k, k'} \text{cond}((m, k), (m + 1, k')) M_{l,m}(\tau^k) M_{l,m+1}(\tau^{k'}).$$

Since $M_{l,m}$ and $M_{l,m+1}$ are monic of degree l and orthogonal to the polynomials of degree at most $l - 1$ in $L^2(\mathbf{m}_m)$ and in $L^2(\mathbf{m}_{m+1})$ respectively, we obtain

$$b_m = \sum_k \widehat{w}(m, k) M_{l,m}(\tau^k) \sum_{k'} P((m, k), (m + 1, k')) M_{l,m+1}(\tau^{k'}) = \frac{(1 - \tau^{m+l+2})h_{l,m}}{(1 - \tau^2)\tau^m},$$

and, by reversibility,

$$b_m = \sum_{k'} \widehat{w}(m+1, k') M_{l, m+1}(\tau^{k'}) \sum_k \mathbf{P}((m+1, k'), (m, k)) M_{l, m}(\tau^k) = \frac{(1 - \tau^{m+1-l}) h_{l, m+1}}{(1 - \tau^2) \tau^m}.$$

This proves the first identity. By the proof of part 2 of Proposition 6.9, the coefficients of $M_{l, m}$ converge to those of $\widehat{p}_l$, and the moments of $\mathbf{m}_m$ converge to those of $\mathbf{m}_\infty$. Hence $h_{l, m} \rightarrow h_{l, \infty}$, and iterating the first identity gives $h_{l, m} = h_{l, \infty} \prod_{i \geq m} \frac{1 - \tau^{i+1-l}}{1 - \tau^{i+l+2}}$, which is the stated formula. The inequality follows from $(\tau^{2l+2}; \tau)_{m-l} \leq 1$, $(\tau; \tau)_\infty \leq (\tau; \tau)_{m-l}$ and $(\tau^{2l+2}; \tau)_\infty \geq (\tau; \tau)_\infty$.

2. By part 3 of Proposition 6.9, we have $u_l = \frac{\varkappa_l}{\varsigma_l(\infty)} \mathbf{h}_l$, which is real-valued and vanishes on the levels $n < l$. This gives $\varepsilon_m(u_l) = 0$ for $m \leq l-2$. Let $m \geq l-1$. Expanding the squares in $\varepsilon_m(u_l)$ and using $\sum_{k'} \text{cond}((m, k), (m+1, k')) = \widehat{w}(m, k) p_\uparrow(m)$ and, by reversibility, $\sum_k \text{cond}((m, k), (m+1, k')) = \widehat{w}(m+1, k') p_\downarrow(m+1)$, we obtain $\varepsilon_m(u_l) = A_m - 2B_m + C_m$, where

$$A_m \stackrel{\text{def}}{=} p_\uparrow(m) \sum_k \widehat{w}(m, k) u_l(m, k)^2, \quad C_m \stackrel{\text{def}}{=} p_\downarrow(m+1) \sum_{k'} \widehat{w}(m+1, k') u_l(m+1, k')^2, \\ B_m \stackrel{\text{def}}{=} \sum_{k, k'} \text{cond}((m, k), (m+1, k')) u_l(m, k) u_l(m+1, k').$$

For $m = l-1$, we have $A_m = B_m = 0$ and, since $\varsigma_l(l) = 1$,

$$C_{l-1} = \frac{\tau(1 - \tau^l)}{(1 + \tau)(1 - \tau^{l+1})} \cdot \frac{1 - \tau^{l+1}}{(1 - \tau)\tau^l} \cdot \frac{\varkappa_l^2}{\varsigma_l(\infty)^2} h_{l, l} = \frac{(1 - \tau^l) \varkappa_l^2 h_{l, l}}{(1 - \tau^2) \tau^{l-1} \varsigma_l(\infty)^2}.$$

For $m \geq l$, the same computation gives

$$A_m = \frac{(1 - \tau^{m+2}) \varkappa_l^2 \varsigma_l(m)^2 h_{l, m}}{(1 - \tau^2) \tau^m \varsigma_l(\infty)^2}, \quad C_m = \frac{(1 - \tau^{m+1}) \varkappa_l^2 \varsigma_l(m+1)^2 h_{l, m+1}}{(1 - \tau^2) \tau^m \varsigma_l(\infty)^2},$$

and $B_m = \frac{\varkappa_l^2}{\varsigma_l(\infty)^2} \varsigma_l(m) \varsigma_l(m+1) b_m$. Using part 1 to express $h_{l, m+1}$ and b_m in terms of $h_{l, m}$, we obtain (6.33).

3. Recall that $\varphi_l^\tau = \varkappa_l \widehat{p}_l$ and $w_k = (1 - \tau) \tau^k$, so that $\|\varphi_l^\tau\|^2 = (1 - \tau) \varkappa_l^2 h_{l, \infty}$. Let $m \geq l$ and put $y \stackrel{\text{def}}{=} \tau^{m+1-l} \leq \tau$, $a \stackrel{\text{def}}{=} \varsigma_l(m)$ and $\delta \stackrel{\text{def}}{=} \varsigma_l(m+1) - \varsigma_l(m)$. Denote by Q the bracket in (6.33). Since $\tau^{m+1} = y \tau^l$, $\tau^{m+2} = y \tau^{l+1}$ and $\tau^{m+l+2} = y \tau^{2l+1}$, a direct computation gives $Q = F_1 a^2 + 2F_2 a \delta + R \delta^2$, where

$$F_1 = \frac{y(1 - \tau^l)[1 - \tau^{l+1} + y \tau^{l+1}(1 - \tau^l)]}{1 - y}, \\ F_2 = \frac{y(1 - \tau^l)(1 - y \tau^{2l+1})}{1 - y}, \quad R = \frac{(1 - y \tau^l)(1 - y \tau^{2l+1})}{1 - y}.$$

We have $y(1 - \tau^l)(1 - \tau^{l+1}) = \tau^{m+1} \Lambda_l(\tau)$ and $\tau^{l+1}(1 - \tau^l) \leq 1 - \tau^{l+1}$, since $1 + \tau^{2l+1} - 2\tau^{l+1} \geq (1 - \tau^{l+1})^2 \geq 0$. Using $y \leq \tau$, we deduce that

$$0 \leq F_1 \leq \frac{1 + \tau}{1 - \tau} \tau^{m+1} \Lambda_l(\tau), \quad 0 \leq F_2 \leq \frac{\tau^{m+1} \Lambda_l(\tau)}{(1 - \tau)^2}, \quad 0 < R \leq \frac{1}{1 - \tau}.$$

Put $j \stackrel{\text{def}}{=} m - l$. By Lemma 6.8, we have $|a| \leq M_\tau$ and $|\delta| \leq 4P_\tau M_\tau(j + 2)\tau^{j+1}$. Moreover $\tau^{m+1} \Lambda_l(\tau) = \tau^{j+1}(1 - \tau^l)(1 - \tau^{l+1}) \geq (1 - \tau)(1 - \tau^2) \tau^{j+1}$. We obtain $|Q| \leq K_\tau \tau^{m+1} \Lambda_l(\tau)$,

where

$$K_\tau \stackrel{\text{def}}{=} \frac{(1+\tau)M_\tau^2}{1-\tau} + \sup_{j \geq 0} \left[\frac{8P_\tau M_\tau^2(j+2)\tau^{j+1}}{(1-\tau)^2} + \frac{16P_\tau^2 M_\tau^2(j+2)^2 \tau^{j+1}}{(1-\tau)^2(1-\tau^2)} \right] < \infty.$$

By (6.33), part 1 and part 2 of Proposition 6.12, we deduce that

$$\varepsilon_m(u_l) \leq \frac{\varkappa_l^2 h_{l,\infty}}{(\tau; \tau)_\infty (1-\tau^2) \tau^m} K_\tau \tau^{m+1} \Lambda_l(\tau) = \frac{\tau K_\tau}{(\tau; \tau)_\infty (1-\tau)(1-\tau^2)} \Lambda_l(\tau) \|\varphi_l^\tau\|^2.$$

For $m = l - 1$, we have $(1-\tau^l)\tau^{1-l} = \frac{\tau \Lambda_l(\tau)}{1-\tau^{l+1}} \leq \frac{\tau \Lambda_l(\tau)}{1-\tau}$ and $h_{l,l} \leq \frac{h_{l,\infty}}{(\tau; \tau)_\infty}$, so that part 2 gives $\varepsilon_{l-1}(u_l) \leq \frac{\tau}{(\tau; \tau)_\infty (1-\tau)^2 (1-\tau^2)} \Lambda_l(\tau) \|\varphi_l^\tau\|^2$. Since $\mathcal{E}^{(n)}(u_l) = \varepsilon_{n-1}(u_l) + \varepsilon_n(u_l)$, part 3 holds with $C_\tau = \frac{2\tau \max\{K_\tau(1-\tau), 1\}}{(\tau; \tau)_\infty (1-\tau)^2 (1-\tau^2)}$. $\blacksquare$

We now pass to the Poisson integrals of the Podleś sphere. Recall that the left translations by the torus act on $L^\infty(S_r^2)$ by the automorphisms $\beta_{\lambda,1}$, by (5.6), and that their fixed point algebra is $W^*(A)$ by part 3 of Proposition 5.5.

Proposition 6.31 *For any bounded function $f: \mathbb{N}_0 \rightarrow \mathbb{C}$, the Poisson integral $\Theta(f(A))$ is the diagonal element $\mathcal{H}f$.*

Proof : Since Θ is equivariant and bijective, it maps the fixed point algebra $W^*(A)$ of the torus onto the diagonal part of $H^\infty(\widehat{G}, \mathbf{P})$, that is onto the space of bounded harmonic functions of the diagonal chain. For such a function u , the limit $u_\infty \stackrel{\text{def}}{=} \lim_t u(n_t, k_t)$ exists almost surely and $u(v) = \mathbf{E}_v[u_\infty]$, by the martingale convergence theorem; this is the standard identification of the bounded harmonic functions of a Markov chain with the bounded functions on the invariant σ -field of its path space, see [Kai92, Section 1]. For bounded harmonic functions u and u' of the diagonal chain, the Cesàro means in (6.3) converge pointwise to $\mathbf{E}_v[u_\infty u'_\infty]$, so that $(u \cdot u')(v) = \mathbf{E}_v[u_\infty u'_\infty]$. If $\mathcal{I}$ denotes the invariant σ -field of the path space, the map $u \mapsto u_\infty$ is therefore a $*$ -isomorphism from the diagonal part of $H^\infty(\widehat{G}, \mathbf{P})$, equipped with the product (6.3), onto $L^\infty(\mathcal{I}, \mathbf{P}_{(0,0)})$. Here we use that a nonnegative harmonic function vanishing at $(0, 0)$ vanishes everywhere, since every state can be reached from $(0, 0)$. Consequently, $L^\infty(\mathcal{I}, \mathbf{P}_{(0,0)})$ is isomorphic to $W^*(A)$, which is the weak* closed linear span of the minimal projections p_k , $k \in \mathbb{N}_0$. Hence $\Theta(p_k)$ is the harmonic function $v \mapsto \mathbf{P}_v(\Omega_k)$ for some atom Ω_k of $\mathcal{I}$, the atoms Ω_k form a partition of the path space up to null sets, and

$$\mathbf{P}_{(0,0)}(\Omega_k) = \widehat{\varepsilon}(\Theta(p_k)) = h(p_k) \stackrel{(1.1)}{=} (1-\tau)\tau^k.$$

The variable k_∞ is $\mathcal{I}$ -measurable. Hence, for any $j \in \mathbb{N}_0$, the event $\{k_\infty = j\}$ is, up to a null set, the union of the atoms Ω_k , $k \in S'_j$, for some partition $(S'_j)_{j \geq 0}$ of $\mathbb{N}_0$. By Theorem 6.10, we have $\sum_{k \in S'_j} \tau^k = \tau^j$ for any j . We show by induction on j that $S'_j = \{j\}$. Suppose that $S'_i = \{i\}$ for any $i < j$. If $j \notin S'_j$, then $j \in S'_i$ for some $i > j$, and $\sum_{k \in S'_i} \tau^k \geq \tau^j > \tau^i$, which is impossible. Thus $j \in S'_j$, and the equality $\sum_{k \in S'_j} \tau^k = \tau^j$ forces $S'_j = \{j\}$. We conclude that $\Theta(p_k)(v) = \mathbf{P}_v(k_\infty = k) = \mathcal{H}\mathbf{1}_{\{k\}}(v)$ for any k . Since $f(A) = \sum_k f(k)p_k$ with convergence for the weak* topology, and since Θ is normal, we obtain $\Theta(f(A)) = \sum_k f(k)\mathcal{H}\mathbf{1}_{\{k\}} = \mathcal{H}f$. $\blacksquare$

Recall that $\text{Pol}(S_r^2)$ is the linear span of the matrix coefficients $u_{j0}^{(l)}$, $l \in \mathbb{N}_0$, $|j| \leq l$, which are pairwise orthogonal in $L^2(S_r^2)$ by [NT13, Theorem 1.4.3] and Lemma 5.2. Thus $\text{Pol}(S_r^2)$ is the orthogonal direct sum of the subspaces $\text{Pol}_l(S_r^2) \stackrel{\text{def}}{=} \text{span}\{u_{j0}^{(l)} : |j| \leq l\}$, which are irreducible

of spin l for the left translations, each spin occurring once, and the fixed vectors of the torus in $\text{Pol}_l(S_r^2)$ are the multiples of $u_{00}^{(l)} = \varphi_l^\tau(A)$ (Proposition 5.3). For any $x \in L^\infty(S_r^2)$, we denote by x_l the component of x in $\text{Pol}_l(S_r^2)$.

Corollary 6.32 *For any integer $n \geq 0$ and any $x \in L^\infty(S_r^2)$, we have*

$$\mathcal{E}^{(n)}(\Theta(x)) = \sum_{l=0}^{n+1} \mathbf{e}_n(l) \|x_l\|_{L^2(S_r^2)}^2, \quad \text{where } \mathbf{e}_n(l) \stackrel{\text{def}}{=} \frac{\mathcal{E}^{(n)}(\mathcal{H}\varphi_l^\tau)}{\|\varphi_l^\tau\|_{\ell_{\mathbb{N}_0}^2(w)}^2},$$

and, for any $l \in \mathbb{N}_0$, we have $\lim_{n \rightarrow \infty} \mathbf{e}_n(l) = \mathbf{c}_\tau \Lambda_l(\tau)$ and $\sup_n \mathbf{e}_n(l) \leq C_\tau \Lambda_l(\tau)$, where C_τ is the constant of Proposition 6.30. Consequently, for any $x \in L^\infty(S_r^2)$,

$$(6.34) \quad \lim_{n \rightarrow \infty} \mathcal{E}^{(n)}(\Theta(x)) = \mathbf{c}_\tau \sum_{l \geq 0} \Lambda_l(\tau) \|x_l\|_{L^2(S_r^2)}^2 \in [0, \infty],$$

and $\sup_{n \geq 0} \mathcal{E}^{(n)}(\Theta(x)) \leq C_\tau \sum_{l \geq 0} \Lambda_l(\tau) \|x_l\|_{L^2(S_r^2)}^2$. In particular, $\lim_{n \rightarrow \infty} \mathcal{E}^{(n)}(\mathcal{H}f) = \mathbf{c}_\tau \mathcal{E}_\tau(f)$ in $[0, \infty]$ for any bounded function $f: \mathbb{N}_0 \rightarrow \mathbb{C}$, and every element of $H^\infty(\widehat{G}, \mathbf{P})$ which is not a multiple of 1 has infinite energy.

Proof : Since $\mathbf{P}$ commutes with the action of G on $\ell^\infty(\widehat{G})$ and $\widehat{h}$ is invariant, the sesquilinear form $(a, b) \mapsto [2s+1]_r \text{Tr}(K^{-1}(\mathbf{P}(a^*b) - a^*b)_s)$ is invariant. Since Θ is equivariant, the sesquilinear form $\mathcal{B}_n$ on $L^\infty(S_r^2)$ obtained by composing it with Θ is invariant, and its quadratic form is $x \mapsto \mathcal{E}^{(n)}(\Theta(x))$. Moreover, Θ commutes with the projections onto the isotypic components, and the components of $\ell^\infty(\widehat{G})$ on the levels $s' \leq s + \frac{1}{2}$ have spin at most $2s+1 = n+1$. Since $\mathcal{E}^{(n)}(a)$ only depends on the components of a on the levels s and $s \pm \frac{1}{2}$, we deduce that $\mathcal{E}^{(n)}(\Theta(x)) = \mathcal{B}_n(y, y)$ with $y = \sum_{l \leq n+1} x_l$. By Schur's lemma, $\mathcal{B}_n$ vanishes on $\text{Pol}_l(S_r^2) \times \text{Pol}_{l'}(S_r^2)$ for $l \neq l'$ and is a multiple of the scalar product of $L^2(S_r^2)$ on $\text{Pol}_l(S_r^2)$. Computing this multiple with $x = \varphi_l^\tau(A)$, we obtain the formula, by Proposition 6.31 and $\|\varphi_l^\tau(A)\|_{L^2(S_r^2)} = \|\varphi_l^\tau\|_{\ell_{\mathbb{N}_0}^2(w)}$. By Theorem 6.29 and part 3 of Proposition 3.1, we have

$$\lim_{n \rightarrow \infty} \mathbf{e}_n(l) = \frac{\mathbf{c}_\tau \mathcal{E}_\tau(\varphi_l^\tau)}{\|\varphi_l^\tau\|_{\ell_{\mathbb{N}_0}^2(w)}^2} = \mathbf{c}_\tau \Lambda_l(\tau).$$

The bound on $\mathbf{e}_n(l)$ is part 3 of Proposition 6.30 for $l \geq 1$, and $\mathbf{e}_n(0) = 0$ since $\mathcal{H}\varphi_0^\tau = 1$. The bound on $\sup_n \mathcal{E}^{(n)}(\Theta(x))$ follows. If $\sum_l \Lambda_l(\tau) \|x_l\|^2 < \infty$, then (6.34) follows from the dominated convergence theorem for series. Otherwise, Fatou's lemma for series gives $\liminf_n \mathcal{E}^{(n)}(\Theta(x)) = \infty$.

Let $f: \mathbb{N}_0 \rightarrow \mathbb{C}$ be bounded. The element $f(A)$ is fixed by the left action of the torus, hence so are its components. Thus $f(A)_l$ is a multiple of $\varphi_l^\tau(A)$, namely $f(A)_l = \langle f, e_l \rangle e_l(A)$ with $e_l = \|\varphi_l^\tau\|^{-1} \varphi_l^\tau$, since $h(\varphi_l^\tau(A)^* f(A)) = \langle f, \varphi_l^\tau \rangle_{\ell_{\mathbb{N}_0}^2(w)}$. By part 3 of Proposition 3.1, we have $\sum_l \Lambda_l(\tau) \|f(A)_l\|^2 = \sum_l \Lambda_l(\tau) |\langle f, e_l \rangle|^2 = \mathcal{E}_\tau(f)$ in $[0, \infty]$, and $\Theta(f(A)) = \mathcal{H}f$ by Proposition 6.31. Finally, if x is not a multiple of 1, then $x_l \neq 0$ for some $l \geq 1$, so that the limit in (6.34) is positive and $\mathcal{E}_{\widehat{G}}(\Theta(x)) = \sum_n \mathcal{E}^{(n)}(\Theta(x)) = \infty$. The last assertion follows since Θ is onto $H^\infty(\widehat{G}, \mathbf{P})$. $\blacksquare$

6.6 Comparison of the two boundary energies

Suppose that $\tau = Q^{-d}$. We identify a function $f: \mathbb{N}_0 \rightarrow \mathbb{C}$ with the radial function $F(x) = f(k)$ for $\|x\|_K = Q^{-k}$ on $\mathcal{O}^d$, and with the element $f(A)$, as in Theorem 5.6.

Corollary 6.33 1. *The harmonic measure of the tree $\mathbb{T}$ seen from the root and the harmonic measure of the diagonal quantum walk seen from the trivial representation have the same radial law, namely the Aref'eva–Volovich law.*

2. *For any $f \in \ell^2_{\mathbb{N}_0}(w)$, the boundary energy of the tree and the energy density at infinity $\mathfrak{c}_\tau \mathcal{E}_\tau(f)$ of the quantum walk, which is equal to $\lim_n \mathcal{E}^{(n)}(\mathcal{H}f)$ if f is bounded, by Corollary 6.32, satisfy*

$$\frac{2(1-\tau)^2}{1+\tau} \mathcal{E}_\tau(f) \leq \mathcal{E}_{\mathbb{T}}(F) \leq 2(1-\tau) \left[\frac{1}{(1-\tau)^3} + \frac{\tau}{(1+\tau)^2(1-\tau)^2} \right] \mathcal{E}_\tau(f).$$

In particular, these forms have the same domain.

Proof : 1. Under the Haar measure of $\mathcal{O}^d$, the valuation is distributed according to $(1-\tau)\tau^k$, by (2.2). The quantum case is Theorem 6.10.

2. By Theorem 6.1 and Proposition 2.1, we have $\mathcal{E}_{\mathbb{T}}(F) = 2(1-\tau)[\mathcal{E}_R(f) + \kappa \|f - \mathbb{E}_0 f\|^2]$ with $\alpha = d$ and $\kappa = \frac{1}{1+\tau}$. By Theorem 4.1 with $\alpha = d$, we have $\frac{1-\tau}{1+\tau} \mathcal{E}_\tau(f) \leq \mathcal{E}_R(f) \leq (1-\tau)^{-3} \mathcal{E}_\tau(f)$. Finally, part 3 of Proposition 3.1 gives $\|f - \mathbb{E}_0 f\|^2 \leq \Lambda_1(\tau)^{-1} \mathcal{E}_\tau(f)$ with $\Lambda_1(\tau) = \frac{(1-\tau)^2(1+\tau)}{\tau}$. ■

Thus the two operators compared in Theorem 4.1 for $\alpha = d$ have a probabilistic interpretation: the Taibleson operator of order d is, up to a bounded perturbation, the boundary energy of the tree of balls, and the zonal Casimir operator of the Podleś sphere is the energy density at infinity of the dual of $\text{SU}_r(2)$. On the diagonal, the quantum walk differs from the radial walk on the Bruhat–Tits tree by the horizontal edges of Proposition 6.4. Their conductances do not decay with the level, so that each level contributes a nearest-neighbour form in k . This is why the total energy diverges, and why the limit is the form of a birth–death operator. For the tree, the total energy of a harmonic extension is finite, and it is given by the long-range Taibleson form of its boundary values.

Remark 6.34 One could expect the total energy of the quantum Poisson integrals to be finite and comparable with the Taibleson energy. Corollary 6.32 shows that this fails: no element of $H^\infty(\widehat{G}, \mathbb{P})$ which is not a multiple of 1 has finite energy. The comparison holds for the energy density, which is defined by the invariant forms $\mathcal{E}^{(n)}$.

7 Quantum Markov semigroups on the Podleś sphere

7.1 Tail expectations and an exact lifting

In this subsection $0 < r < 1$ and $\tau = r^2$ are arbitrary, we fix $\rho > 1$, and we define c , κ , λ_n and $a_n(t)$ by the formulas (2.1) and (2.14). If $\tau = Q^{-d}$ and $\rho = Q^\alpha$, these are the parameters of Section 2. By Proposition 5.5, it is convenient to work in $B(\ell^2_{\mathbb{N}_0})$ with the state ω_τ and to transport the results by θ ; we use the same notation for an element of $B(\ell^2_{\mathbb{N}_0})$ and its image in $L^\infty(S_r^2)$. With this convention $p_n = e_{nn}$, $P_n = \sum_{k \geq n} e_{kk}$, $h = \omega_\tau$ and $C(S_r^2) = \mathcal{K}(\ell^2_{\mathbb{N}_0}) + \mathbb{C}1$. For $n \geq 0$ define

$$(7.1) \quad \Phi_n(x) \stackrel{\text{def}}{=} (1 - P_n)x(1 - P_n) + \tau^{-n}h(P_n x P_n)P_n, \quad \mathcal{M}_n \stackrel{\text{def}}{=} (1 - P_n)B(\ell^2_{\mathbb{N}_0})(1 - P_n) \oplus \mathbb{C}P_n.$$

Lemma 7.1 *The map Φ_n is the faithful normal h -preserving conditional expectation onto $\mathcal{M}_n$. These algebras are increasing, $\Phi_i\Phi_j = \Phi_{\min(i,j)}$, and $\Phi_0(x) = h(x)1$. The expectations commute with the modular group of h . On both the GNS and the symmetric L^2 realizations, Φ_n converges strongly to the identity. Each Φ_n maps $C(S_r^2)$ into itself, and $\Phi_n(x) \rightarrow x$ in norm for any $x \in C(S_r^2)$. Finally, if $\tau = Q^{-d}$, then*

$$(7.2) \quad \Phi_n J = J \mathbb{E}_n.$$

Proof : Pinching by P_n and $1 - P_n$ is normal and completely positive. The normalized restriction $x \mapsto \tau^{-n}h(P_n x P_n)$ is a state on the tail corner, since $h(P_n) = \tau^n$. Formula (7.1) is consequently normal and unital completely positive, and it is idempotent and bimodular over $\mathcal{M}_n$.

Each P_n commutes with D , so the cross terms $(1 - P_n)xP_n$ and $P_n x(1 - P_n)$ have zero Haar state and $h\Phi_n = h$. This also gives faithfulness. The inclusions $\mathcal{M}_n \subset \mathcal{M}_{n+1}$ are obtained by separating p_n from the tail. For $i \leq j$, the map $\Phi_i\Phi_j$ is a normal unital completely positive h -preserving idempotent onto $\mathcal{M}_i$, since Φ_j is the identity on $\mathcal{M}_i \subset \mathcal{M}_j$; hence it is a conditional expectation onto $\mathcal{M}_i$. The algebra $\mathcal{M}_i$ is invariant under $\sigma^h = \text{Ad}(D^{\text{is}})$, since P_i commutes with D . By the uniqueness part of Takesaki's theorem [Tak03, Theorem IX.4.2], we obtain $\Phi_i\Phi_j = \Phi_i$, and similarly $\Phi_j\Phi_i = \Phi_i$. Since $\sigma_s^h = \text{Ad}(D^{\text{is}})$ and P_n commutes with D , the expectations commute with σ^h .

Realize the GNS space by $x \mapsto xD^{\frac{1}{2}}$ and the symmetric L^2 space by $x \mapsto D^{\frac{1}{4}}xD^{\frac{1}{4}}$, both in the Hilbert–Schmidt class S^2 . In either realization, a direct computation gives

$$(7.3) \quad \Phi_n^{(2)}\xi = (1 - P_n)\xi(1 - P_n) + \tau^{-n} \left\langle \xi, D^{\frac{1}{2}}P_n \right\rangle_{S^2} D^{\frac{1}{2}}P_n.$$

Since $\left\| D^{\frac{1}{2}}P_n \right\|_{S^2}^2 = h(P_n) = \tau^n$, the second term is the orthogonal projection onto that tail vector. Its range is orthogonal to the head compression, so each implementation is an orthogonal projection. Finite matrices are dense in S^2 and fixed by $\Phi_n^{(2)}$ for sufficiently large n . Hence $\Phi_n^{(2)} \rightarrow I$ strongly in both realizations. If $\tau = Q^{-d}$, then for Jf the tail state in (7.1) is the normalized average in (2.13), which proves (7.2).

Finally, let x be a compact operator. Since $P_n \rightarrow 0$ strongly, $\|P_n x\| \rightarrow 0$ and $\|x P_n\| \rightarrow 0$, so that $(1 - P_n)x(1 - P_n) \rightarrow x$ in norm. Moreover $|\tau^{-n}h(P_n x P_n)| = |(1 - \tau) \sum_{k \geq n} \tau^{k-n} x_{kk}| \leq \sup_{k \geq n} |x_{kk}|$, which tends to zero. Thus $\Phi_n(x) \rightarrow x$ in norm, and $\Phi_n(x) \in \mathcal{K}(\ell_{\mathbb{N}_0}^2) + \mathbb{C}P_n \subset C(S_r^2)$. Since $\Phi_n(1) = 1$, this proves the last assertions. ■

Theorem 7.2 *For $t > 0$, the formula*

$$(7.4) \quad \mathcal{T}_t \stackrel{\text{def}}{=} \sum_{n \geq 0} a_n(t) \Phi_n$$

defines a semigroup of normal unital completely positive, h -preserving maps on $L^\infty(S_r^2)$. It is point- σ -weakly continuous at zero, strongly continuous on GNS L^2 , GNS-symmetric, KMS-symmetric, and it commutes with the modular group of h and with the left torus action $(\beta_{\lambda,1})$. The negative generator of its GNS L^2 implementation is the nonnegative self-adjoint operator

$$(7.5) \quad \mathcal{L} = c(I - \Phi_0) + (\rho - 1) \sum_{n \geq 1} \rho^n (I - \Phi_n),$$

initially on $\bigcup_n \text{Ran } \Phi_n^{(2)}$ and then closed. If $d_n = \Phi_{n+1}^{(2)} - \Phi_n^{(2)}$, then

$$(7.6) \quad \mathcal{L} = \sum_{n \geq 0} \lambda_n d_n, \quad \text{dom } \mathcal{L} = \left\{ \xi : \sum_{n \geq 0} \lambda_n^2 \|d_n \xi\|_2^2 < \infty \right\}.$$

If $\tau = Q^{-d}$ and $\rho = Q^\alpha$, then

$$\mathcal{T}_t J = J T_t, \quad \mathcal{L} J_2 = J_2 R_{\text{rad}}^\alpha,$$

where the generator equality includes equality of domains on the radial subspace. Finally, each $\mathcal{T}_t$ leaves $C(S_r^2)$ invariant, and the restriction of $(\mathcal{T}_t)$ to $C(S_r^2)$ is a strongly continuous semigroup of unital completely positive maps.

Proof : The coefficients in (2.14) form a probability distribution. Hence (7.4) converges in operator norm in the space of bounded maps, is normal, unital completely positive, and preserves h .

The projections $\Phi_0^{(2)}, d_0, d_1, \dots$ are mutually orthogonal and sum strongly to the identity, by Lemma 7.1. On $\text{Ran } d_k$, the maps $\Phi_n^{(2)}$ vanish for $n \leq k$ and are the identity for $n > k$. Thus (7.4) acts there by $\sum_{n>k} a_n(t) = e^{-t\lambda_k}$, and acts identically on constants. This proves the semigroup law, strong continuity and self-adjointness on GNS L^2 . It also proves (7.6). Formula (7.5) follows from $c + (\rho - 1) \sum_{n=1}^k \rho^n = \lambda_k$.

For any $x \in L^\infty(S_r^2)$, the compression term in (7.1) converges strongly* to x , whereas the tail term converges strongly* to zero: its scalar coefficient is bounded by $\|x\|$ and $P_n \rightarrow 0$ strongly. Thus $\Phi_n(x) \rightarrow x$ σ -weakly. For any fixed N , the total mass of the coefficients with $n \leq N$ is $1 - e^{-t\lambda_N}$, which tends to zero as $t \rightarrow 0^+$. Given a normal functional, we split the probability mixture at N , choosing first N such that its values on the remaining expectations are close to its value on x . This proves point- σ -weak continuity at zero. The conditional expectations commute with σ^h and with the automorphisms $\beta_{\lambda,1}$, which fix P_n and h ; the same commutations hold for the semigroup. Formula (7.3) shows that each symmetric L^2 implementation is an orthogonal projection. Thus the same probability mixture is self-adjoint in that realization, proving KMS symmetry.

Suppose that $\tau = Q^{-d}$ and $\rho = Q^\alpha$. Equation (7.2) and the mixture formulas (2.15) and (7.4) imply $\mathcal{T}_t J = J T_t$. The basis of Theorem 2.3 is mapped by J_2 onto unit vectors of the one-dimensional diagonal components of the d_n , so (7.6) gives the generator and domain assertion.

Finally, $\mathcal{T}_t$ maps $C(S_r^2)$ into itself by Lemma 7.1, since the series (7.4) converges in norm. Let $x \in C(S_r^2)$ and $\varepsilon > 0$. By Lemma 7.1, choose N such that $\|\Phi_n(x) - x\| \leq \varepsilon$ for $n > N$. Then

$$\|\mathcal{T}_t(x) - x\| \leq 2\|x\| \sum_{n \leq N} a_n(t) + \varepsilon = 2\|x\| (1 - e^{-t\lambda_N}) + \varepsilon,$$

which is at most 2ε for small t . This proves the strong continuity on $C(S_r^2)$. ■

The associated closed quadratic form is

$$\mathcal{Q}(\xi) = c \left\| (I - \Phi_0^{(2)})\xi \right\|_2^2 + (\rho - 1) \sum_{n \geq 1} \rho^n \left\| (I - \Phi_n^{(2)})\xi \right\|_2^2 = \sum_{n \geq 0} \lambda_n \|d_n \xi\|_2^2.$$

The identity follows first on finite filtration ranges and then by monotone convergence.

Remark 7.3 The expectations (7.1) make sense on the whole algebra $\mathcal{M} = L^\infty(\text{SU}_r(2))$ with $P_n = \sum_{k \geq n} e_{kk} \otimes 1$, and Theorem 7.2 holds there with the same proof, except for the last assertion. Indeed, on $C(\text{SU}_r(2))$ one has $\mathcal{T}_t(b) = \sum_{n \geq 1} e^{-t\lambda_n} \sqrt{1 - \tau^n} e_{n-1,n} \otimes 1$, so that $\|\mathcal{T}_t(b) - b\| = \sup_n (1 - e^{-t\lambda_n}) \sqrt{1 - \tau^n} = 1$ for any $t > 0$. Section 8 shows that this defect cannot be repaired by another lifting. The restriction to $L^\infty(S_r^2)$ removes it, since b does not belong to $C(S_r^2)$.

7.2 Noncommutative L^p estimates

We denote by $L^p(S_r^2)$ the Haagerup space [Haa79, Ter81] of $(L^\infty(S_r^2), h)$, for any $1 \leq p < \infty$, and by $\mathfrak{d}_h$ the density of h in this realization. The extension of Φ_n is determined on a dense subspace by

$$\Phi_n^{(p)}(\mathfrak{d}_h^{\frac{1}{2p}} x \mathfrak{d}_h^{\frac{1}{2p}}) = \mathfrak{d}_h^{\frac{1}{2p}} \Phi_n(x) \mathfrak{d}_h^{\frac{1}{2p}}, \quad x \in L^\infty(S_r^2).$$

According to [HJX10, Theorem 5.1 and Example 5.8], this extension is a contractive projection. Since Φ_n commutes with the modular group, the symmetric and one-sided extensions agree on analytic elements by [HJX10, Proposition 5.5]. We put $\delta_n^{(p)} \stackrel{\text{def}}{=} \Phi_{n+1}^{(p)} - \Phi_n^{(p)}$ for $n \geq 0$.

Theorem 7.4 *Suppose that $1 < p < \infty$. There exists a constant C_p , depending only on p , with the following properties.*

1. *For any bounded complex sequence $(m_*, m_0, m_1, \dots)$, the series*

$$(7.7) \quad \mathcal{M}_m \xi = m_* \Phi_0^{(p)} \xi + \sum_{n \geq 0} m_n \delta_n^{(p)} \xi$$

converges in $L^p(S_r^2)$ for any $\xi \in L^p(S_r^2)$ and defines a bounded operator satisfying

$$(7.8) \quad \|\mathcal{M}_m\|_{L^p(S_r^2) \rightarrow L^p(S_r^2)} \leq C_p \max\{|m_*|, \sup_{n \geq 0} |m_n|\}.$$

The same constant works after replacing $L^\infty(S_r^2)$, h and Φ_n by $M_N(\mathbb{C}) \otimes L^\infty(S_r^2)$, $\text{tr}_N \otimes h$ and $\text{Id}_{M_N} \otimes \Phi_n$, respectively, for any integer $N \geq 1$.

2. *The maps $\mathcal{T}_t$ extend to a strongly continuous semigroup of contractions on $L^p(S_r^2)$. Its negative generator $\mathcal{L}_p$ is the closure of $\xi \mapsto \sum_{n \geq 0} \lambda_n \delta_n^{(p)} \xi$ on $\bigcup_{N \geq 0} \text{Ran } \Phi_N^{(p)}$, where the sum is finite on this subspace. More precisely,*

$$(7.9) \quad \text{dom } \mathcal{L}_p = \left\{ \xi \in L^p(S_r^2) : \sum_{n \geq 0} \lambda_n \delta_n^{(p)} \xi \text{ converges in } L^p(S_r^2) \right\}.$$

3. *On $X_p = \text{Ker } \Phi_0^{(p)}$, the restriction B_p of $\mathcal{L}_p$ is invertible, sectorial of angle zero, and admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $0 < \theta < \pi$, where $\Sigma_\theta = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$. For any $f \in H^\infty(\Sigma_\theta)$,*

$$(7.10) \quad f(B_p) \xi = \sum_{n \geq 0} f(\lambda_n) \delta_n^{(p)} \xi, \quad \|f(B_p)\|_{X_p \rightarrow X_p} \leq C_p \|f\|_\infty.$$

These estimates hold with the same constants on the matrix tensor extensions of part 1. The fixed subspace $\text{Ran } \Phi_0^{(p)}$ is treated separately.

Proof : 1. In the tracial case, the noncommutative Burkholder–Gundy inequalities [PX97, Theorem 2.1] imply that martingale difference sequences are unconditional in L^p , $1 < p < \infty$, with a constant κ_p depending only on p [PX97, Remark 2.4]. By the reduction method of [HJX10], this unconditionality extends to the Haagerup spaces of a σ -finite von Neumann algebra with a normal faithful state φ , for an increasing filtration of von Neumann subalgebras with φ -preserving normal faithful conditional expectations whose union is σ -weakly dense [HJX10, Theorem 6.3]. Here the filtration is $(\mathcal{M}_n)_{n \geq 0}$, whose union contains the finite matrices. We

apply it to the finite martingale $(\Phi_j^{(p)}\xi - \Phi_0^{(p)}\xi)_{0 \leq j \leq N+1}$. By convexity, the same bound holds for real coefficients of absolute value at most one. Separating real and imaginary parts gives a bound $2\kappa_p$ for complex coefficients. Since $\|\Phi_{N+1}^{(p)}\xi - \Phi_0^{(p)}\xi\|_p \leq 2\|\xi\|_p$, the finite sums in (7.7) satisfy (7.8), for instance with $C_p = 1 + 4\kappa_p$.

We next justify convergence. The spaces $\text{Ran } \Phi_N^{(p)}$ increase densely in $L^p(S_r^2)$. Indeed, the elements $\mathfrak{d}_h^{\frac{1}{2p}} x \mathfrak{d}_h^{\frac{1}{2p}}$ with x a finite matrix are dense, and such an element is fixed by $\Phi_N^{(p)}$ for sufficiently large N . Contractivity therefore implies $\Phi_N^{(p)}\xi \rightarrow \xi$ in $L^p(S_r^2)$. For $N' > N$, the transformed sum between the indices N and $N' - 1$ vanishes on $\text{Ran } \Phi_N^{(p)}$. Consequently its norm at ξ is bounded by $C_p \sup_n |m_n| \|\xi - \Phi_N^{(p)}\xi\|_p$, which tends to zero. Thus the series converges and the finite estimates pass to the limit. Applying the same argument to the state $\text{tr}_N \otimes h$ proves the matrix tensor assertion.

2. The probability mixture (7.4), with each expectation replaced by its extension, defines a contraction on $L^p(S_r^2)$. It acts as the identity on $\text{Ran } \Phi_0^{(p)}$ and as $e^{-t\lambda_n}$ on $\text{Ran } \delta_n^{(p)}$. Hence it satisfies the semigroup law and is strongly continuous on every finite filtration range. Density and contractivity imply strong continuity on $L^p(S_r^2)$. Let $-\mathcal{L}_p$ denote its generator. Each $\Phi_N^{(p)}$ commutes with the semigroup. Differentiating on the generator domain gives $\Phi_N^{(p)}\mathcal{L}_p\xi = \mathcal{L}_p\Phi_N^{(p)}\xi$ for $\xi \in \text{dom } \mathcal{L}_p$. The two sides converge respectively to $\mathcal{L}_p\xi$ and to the partial sums $\sum_{n < N} \lambda_n \delta_n^{(p)}\xi$. Thus the finite filtration ranges form a core and every vector of the domain satisfies the convergence in (7.9). Conversely, if that series converges, the vectors $\Phi_N^{(p)}\xi$ converge to ξ and their images under $\mathcal{L}_p$ converge to the sum. Closedness of the generator proves (7.9).

3. On X_p , the multiplier with coefficients λ_n^{-1} is bounded by (7.8). Finite filtration truncation and closedness show that it is the inverse of B_p . For any $z \notin [0, \infty)$, define $R_z \stackrel{\text{def}}{=} \sum_{n \geq 0} (z - \lambda_n)^{-1} \delta_n^{(p)}\xi$. Both this multiplier and the multiplier with coefficients $\lambda_n(z - \lambda_n)^{-1}$ are bounded. By (7.9), $R_z\xi \in \text{dom } B_p$ and $(z - B_p)R_z\xi = \xi$. The converse identity follows on the domain by finite filtration truncation. For any $\theta > 0$,

$$\sup_{z \notin \Sigma_\theta} \|zR_z\| \stackrel{(7.8)}{\leq} C_p \sup_{\substack{z \notin \Sigma_\theta \\ u > 0}} \left| \frac{z}{z - u} \right| < \infty.$$

This proves that B_p is sectorial of angle zero. We use the natural functional calculus of sectorial operators [Haa06, Chapter 2] and its extension to bounded holomorphic functions [Haa06, Section 5.1]. For a decaying holomorphic function f , the sectorial contour calculus agrees with the multiplier $(f(\lambda_n))$ on each finite filtration range, by the scalar Cauchy formula. Both maps are bounded, so density gives equality on X_p , and the multiplier estimate proves the bound. To pass to arbitrary bounded holomorphic functions, we use the regularizers $e_\varepsilon(z) = \frac{z}{z + \varepsilon} \frac{1}{1 + \varepsilon z}$, $\varepsilon > 0$. They are decaying holomorphic functions on Σ_θ , satisfy $0 < e_\varepsilon(\lambda_n) \leq 1$, and converge to one at each λ_n . The multipliers $((f e_\varepsilon)(\lambda_n))$ are uniformly bounded by $C_p \|f\|_\infty$ and converge strongly on the dense finite filtration ranges, hence strongly on X_p to the multiplier in (7.10). This is the bounded extension of the sectorial calculus and proves the assertion. The argument is unchanged on each matrix tensor extension. $\blacksquare$

Remark 7.5 By [HJX10, Remark 1.2], the Haagerup space of $(B(\ell_{\mathbb{N}_0}^2), \omega_\tau)$ is identified with the Schatten class S^p , the embedding of $B(\ell_{\mathbb{N}_0}^2)$ being $x \mapsto D^{\frac{1}{2p}} x D^{\frac{1}{2p}}$. In this realization,

$$\Phi_n^{(p)}(X) = (1 - P_n)X(1 - P_n) + \tau^{-n} \text{Tr} \left(D^{1 - \frac{1}{p}} P_n X P_n \right) D^{\frac{1}{p}} P_n, \quad X \in S^p,$$

since $\text{Tr}(DP_n x P_n) = \text{Tr}(D^{1-\frac{1}{p}} P_n D^{\frac{1}{2p}} x D^{\frac{1}{2p}} P_n)$. Theorem 7.4 is thus a statement on explicit contractive projections of S^p . If $\tau = Q^{-d}$, the state-preserving map J induces an isometry J_p from the radial L^p space onto its quantum realization, given on bounded radial functions by $J_p f = \mathfrak{d}_h^{\frac{1}{2p}} J(f) \mathfrak{d}_h^{\frac{1}{2p}}$, and (7.2) implies $\Phi_n^{(p)} J_p = J_p \mathbb{E}_n$ by density.

7.3 Zonal convolution semigroups

In this subsection $0 < r < 1$ is arbitrary and $\tau = r^2$. Recall the conditional expectation $\mathbb{E}$ onto $W^*(A)$ of Proposition 5.5 and the identification $C^*(A) = C(X_\tau)$.

Proposition 7.6 *Let $(T_t)_{t \geq 0}$ be a Feller semigroup on $C(X_\tau)$ and let $\psi: \mathbb{N}_0 \rightarrow [0, \infty)$ be a function such that $T_t \varphi_l^\tau = e^{-t\psi(l)} \varphi_l^\tau$ for any $l \in \mathbb{N}_0$ and any $t \geq 0$. For $t \geq 0$, let μ_t be the probability measure on X_τ defined by $\int F d\mu_t = (T_t F)(0)$, put $\omega_t \stackrel{\text{def}}{=} \mu_t \circ \mathbb{E}$ and $\mathcal{S}_t \stackrel{\text{def}}{=} (\text{Id} \otimes \omega_t) \circ \Delta$.*

1. *Each ω_t is a state on $C(\text{SU}_r(2))$ invariant under all $\beta_{\lambda, \mu}$, each $\mathcal{S}_t$ is a unital completely positive map on $C(\text{SU}_r(2))$ preserving h , and $\mathcal{S}_t \beta_{\lambda, \mu} = \beta_{\lambda, \mu} \mathcal{S}_t$.*
2. *For any $l \in \frac{1}{2}\mathbb{N}_0$ and any j, k , we have $\mathcal{S}_t(u_{jk}^{(l)}) = \delta_{k0} e^{-t\psi(l)} u_{j0}^{(l)}$ if $l \in \mathbb{N}_0$, and $\mathcal{S}_t(u_{jk}^{(l)}) = 0$ otherwise. In particular $\mathcal{S}_t$ maps $C(S_r^2)$ into itself, $\mathcal{S}_t(F(A)) = (T_t F)(A)$ for any $F \in C(X_\tau)$, and the restrictions of the maps $\mathcal{S}_t$ to $C(S_r^2)$ form a strongly continuous semigroup.*
3. *For any $x, y \in \text{Pol}(S_r^2)$, we have $h(\mathcal{S}_t(y)^* x) = h(y^* \mathcal{S}_t(x))$, and $\mathcal{S}_t$ commutes with the modular group of h on $\text{Pol}(S_r^2)$.*
4. *If $\mu_t(\{0\}) = 0$ for any $t > 0$, then each ω_t is normal, and each $\mathcal{S}_t$ extends to a normal unital completely positive map on $\mathcal{M}$ which leaves $L^\infty(S_r^2)$ invariant.*

Proof : 1. The map $\mathbb{E}$ is unital completely positive from $C(\text{SU}_r(2))$ onto $C^*(A)$ by Proposition 5.5, and μ_t is a state on $C^*(A) = C(X_\tau)$. Hence ω_t is a state, invariant under the torus actions since $\mathbb{E} \circ \beta_{\lambda, \mu} = \mathbb{E}$. The slice map $\text{Id} \otimes \omega_t$ is unital completely positive on the minimal tensor product, so $\mathcal{S}_t$ is unital completely positive. By invariance of the Haar state [NT13, Theorem 1.2.1], $h \circ \mathcal{S}_t = (h \otimes \omega_t) \Delta = \omega_t(1)h = h$. By (5.6) and coassociativity,

$$\mathcal{S}_t \beta_{\lambda, \mu} = (\varepsilon_\lambda \otimes \text{Id} \otimes \omega_t \circ \beta_{1, \mu}) \circ \Delta^{(2)}, \quad \beta_{\lambda, \mu} \mathcal{S}_t = (\varepsilon_\lambda \otimes \text{Id} \otimes \omega_t \circ \beta_{\mu, 1}) \circ \Delta^{(2)},$$

where $\Delta^{(2)} = (\Delta \otimes \text{Id}) \Delta$. Indeed $(\omega_t \otimes \varepsilon_\mu) \Delta = \omega_t \circ \beta_{1, \mu}$ and $(\varepsilon_\mu \otimes \omega_t) \Delta = \omega_t \circ \beta_{\mu, 1}$ by (5.6). Since ω_t is invariant, both expressions coincide.

2. By Lemma 5.2, $\mathbb{E}(u_{mk}^{(l)}) = \int_{\mathbb{T}^2} \lambda^{-2m} \mu^{-2k} d\lambda d\mu u_{mk}^{(l)}$, which is $u_{00}^{(l)}$ if $m = k = 0$ and zero otherwise. By Proposition 5.3, $\mu_t(\varphi_l^\tau) = (T_t \varphi_l^\tau)(0) = e^{-t\psi(l)}$. Hence $\omega_t(u_{mk}^{(l)}) = \delta_{m0} \delta_{k0} e^{-t\psi(l)}$ for $l \in \mathbb{N}_0$, and $\omega_t(u_{mk}^{(l)}) = 0$ if $l \notin \mathbb{N}_0$. The formula for $\mathcal{S}_t(u_{jk}^{(l)}) = \sum_m u_{jm}^{(l)} \omega_t(u_{mk}^{(l)})$ follows. By part 1, $\mathcal{S}_t$ commutes with $\beta_{1, \mu}$ and therefore maps $C(S_r^2)$ into itself. The two continuous maps $F \mapsto \mathcal{S}_t(F(A))$ and $F \mapsto (T_t F)(A)$ agree on each φ_l^τ , hence on polynomials, hence on $C(X_\tau)$. On $\text{Pol}(S_r^2)$, the maps $\mathcal{S}_t$ act diagonally on the basis $(u_{j0}^{(l)})$ and satisfy the semigroup law; moreover $\mathcal{S}_t(x) \rightarrow x$ for any $x \in \text{Pol}(S_r^2)$, the sum being finite. Since $\text{Pol}(S_r^2)$ is dense in $C(S_r^2)$ and the maps $\mathcal{S}_t$ are contractions, the semigroup law and the strong continuity extend to $C(S_r^2)$.

3. By the orthogonality relations [NT13, Theorem 1.4.3 (ii)], $h((u_{j'0}^{(l')})^* u_{j0}^{(l)}) = 0$ if $l \neq l'$. For $l = l'$ and $j \neq j'$, this also holds, since h is invariant under $\beta_{\lambda, 1}$, which multiplies $(u_{j'0}^{(l')})^* u_{j0}^{(l)}$ by

$\lambda^{2j'-2j}$ by Lemma 5.2. Since $\mathcal{S}_t$ acts on the vectors $u_{j0}^{(l)}$ by real scalars, the symmetry identity follows. By [CFK14, (11.3)], the modular group acts on $u_{j0}^{(l)}$ by the scalar r^{2isj} , which gives the commutation.

4. Suppose that $\mu_t(\{0\}) = 0$. For $k \geq 0$, let ϕ_k be the normal state $x \mapsto \langle x(e_k \otimes 1), e_k \otimes 1 \rangle$ on $\mathcal{M}$. It is invariant under all $\beta_{\lambda, \mu}$ by (5.4), so that $\phi_k(x) = \phi_k(\mathbb{E}(x)) = \mathbb{E}(x)(\tau^k)$. Hence $\omega_t = \sum_k \mu_t(\{\tau^k\})\phi_k$ is a normal state. Since Δ extends to a normal unital $*$ -homomorphism of $\mathcal{M}$ into $\mathcal{M} \bar{\otimes} \mathcal{M}$, the formula $(\text{Id} \otimes \omega_t) \circ \Delta$ defines a normal unital completely positive extension, which commutes with the normal automorphisms $\beta_{1, \mu}$ by part 1 and σ -weak density, and therefore leaves $L^\infty(S_r^2)$ invariant. $\blacksquare$

Theorem 7.7 *Let $0 < r < 1$ and $\tau = r^2$, let (T_t^τ) be the Feller semigroup e^{-tL_τ} of Proposition 3.3 with $\eta = \tau$, and let μ_t^τ be its law at the origin. The maps $\mathcal{S}_t = (\text{Id} \otimes \mu_t^\tau \circ \mathbb{E}) \circ \Delta$ satisfy the conclusions of Proposition 7.6 with $\psi(l) = \Lambda_l(\tau)$, including part 4. In particular:*

1. *The restriction of $(\mathcal{S}_t)$ to $C(S_r^2)$ is a Haar-preserving strongly continuous semigroup of unital completely positive maps with $\mathcal{S}_t(u_{j0}^{(l)}) = e^{-t\Lambda_l(\tau)}u_{j0}^{(l)}$, and its restriction to $C^*(A)$ is T_t^τ .*
2. *The maps $\mathcal{S}_t$ extend to normal unital completely positive maps of $L^\infty(S_r^2)$, and their GNS implementations form a strongly continuous semigroup of self-adjoint contractions.*
3. *If $\tau = Q^{-d}$, then $\mathcal{S}_t J = J T_t^\tau$, and the closed form of the radial negative generator L_τ is equivalent to the closed form of R_{rad}^d , with the constants of Theorem 1.1 for $\alpha = d$.*

Proof : By Proposition 3.3, (T_t^τ) is a Feller semigroup on $C(Y_\tau) = C(X_\tau)$ with $T_t^\tau \varphi_l^\tau = e^{-t\Lambda_l(\tau)} \varphi_l^\tau$. By Proposition 5.3, $\sup_l \sup_{X_\tau} |\varphi_l^\tau| \leq 1$, so that part 4 of Proposition 3.3 gives $\mu_t^\tau(\{0\}) = 0$ for $t > 0$. Proposition 7.6 applies and gives 1. For 2, the normal extensions are given by part 4 of Proposition 7.6. By part 3, the semigroup acts diagonally with real eigenvalues $e^{-t\Lambda_l(\tau)}$ on the orthogonal basis $(u_{j0}^{(l)})$ of the GNS space of $(L^\infty(S_r^2), h)$; the Kadison–Schwarz inequality $\mathcal{S}_t(x)^* \mathcal{S}_t(x) \leq \mathcal{S}_t(x^* x)$ and $h \circ \mathcal{S}_t = h$ show that the implementations are contractions. Strong continuity follows from the diagonal form. For 3, the intertwining follows from part 2 of Proposition 7.6 on continuous radial functions, and from normality on bounded radial functions. The form comparison is Theorem 4.1 with $\alpha = d$, for which $\eta = \tau$. $\blacksquare$

Corollary 7.8 *Let $0 < s \leq 1$, and let $(\nu_t^{(s)})_{t \geq 0}$ be the convolution semigroup of probability measures on $[0, \infty)$ with Laplace transforms $e^{-t\lambda^s}$ [SSV10, Chapter 5], with $\nu_t^{(1)} = \delta_t$. The operators $T_t^{(s)} \stackrel{\text{def}}{=} \int_{[0, \infty)} T_u^\tau d\nu_t^{(s)}(u)$ form a Feller semigroup on $C(X_\tau)$ with $T_t^{(s)} \varphi_l^\tau = e^{-t\Lambda_l(\tau)^s} \varphi_l^\tau$, whose L^2 negative generator is L_τ^s . Consequently, Proposition 7.6 applies with $\psi(l) = \Lambda_l(\tau)^s$, including part 4, and the resulting maps $\mathcal{S}_t^{(s)} = \int \mathcal{S}_u d\nu_t^{(s)}(u)$ have the properties 1 and 2 of Theorem 7.7. If $\tau = Q^{-d}$ and $s = \frac{\alpha}{d}$ with $0 < \alpha \leq d$, then the closed form of $L_\tau^{\frac{\alpha}{d}}$ is equivalent to the closed form of R_{rad}^α .*

Proof : For $F \in C(X_\tau)$, the map $u \mapsto T_u^\tau F$ is continuous and bounded, so the integral defining $T_t^{(s)} F$ is a Bochner integral in $C(X_\tau)$. The operators $T_t^{(s)}$ are positive, unital and contractive. The semigroup law follows from $\nu_t^{(s)} * \nu_{t'}^{(s)} = \nu_{t+t'}^{(s)}$ and the semigroup law of (T_u^τ) , and strong continuity follows from $\nu_t^{(s)} \rightarrow \delta_0$ weakly as $t \rightarrow 0^+$ by dominated convergence. Since $\int e^{-u\Lambda} d\nu_t^{(s)}(u) = e^{-t\Lambda^s}$, we obtain $T_t^{(s)} \varphi_l^\tau = e^{-t\Lambda_l(\tau)^s} \varphi_l^\tau$; on $\ell_{\mathbb{N}_0}^2(w)$, the spectral theorem identifies $T_t^{(s)}$ with $e^{-tL_\tau^s}$, see [SSV10, Chapter 12]. The law at the origin is $\mu_t^{(s)} = \int \mu_u^\tau d\nu_t^{(s)}(u)$.

It has no atom at 0 for $t > 0$, since $\mu_u^\tau(\{0\}) = 0$ for $u > 0$ and $\nu_t^{(s)}(\{0\}) = \lim_{\Lambda \rightarrow \infty} e^{-t\Lambda^s} = 0$. Moreover $\mu_t^{(s)} \circ \mathbb{E} = \int \omega_u d\nu_t^{(s)}(u)$, which gives the integral formula for $\mathcal{S}_t^{(s)}$.

Finally, if $\tau = Q^{-d}$ and $s = \frac{\alpha}{d}$ with $0 < \alpha \leq d$, the equivalence of the forms is part 2 of Theorem 4.6 with $\vartheta = d$, since $\alpha < 3d$. $\blacksquare$

Proposition 7.9 *Let $0 < \eta < 1$, and suppose that $|\varphi_l^\eta(x)| \leq 1$ for any $l \in \mathbb{N}_0$ and any $x \in Y_\eta$. Let $(T_t)_{t>0}$ be a family of positive unital maps on $C(Y_\eta)$, and let $\psi: \mathbb{N}_0 \rightarrow [0, \infty)$ be a function such that $T_t \varphi_l^\eta = e^{-t\psi(l)} \varphi_l^\eta$ for any $l \in \mathbb{N}_0$ and any $t > 0$. Then*

$$(7.11) \quad \psi(l) \leq K_{\tau,\eta} \psi(1) \Lambda_l(\eta), \quad l \geq 1, \quad \text{where } K_{\tau,\eta} \stackrel{\text{def}}{=} \frac{\max\{2, A'\}(1-\tau)}{(1-\eta)(1-\tau\eta)^2}$$

and A' is the constant of Lemma 4.4. We put $K_\tau \stackrel{\text{def}}{=} K_{\tau,\tau}$.

Proof : Let $l \geq 1$. We first show that $1 - \varphi_l^\eta(x) \leq \max\{2, A'\} \eta^{-l} x$ for any $x \in Y_\eta$. For $x \leq \eta^l$, this is Lemma 4.4. For $x = \eta^k$ with $k < l$, we have $x \eta^{-l} \geq \eta^{-1} > 1$ and $|\varphi_l^\eta(x)| \leq 1$, so that $1 - \varphi_l^\eta(x) \leq 2 \leq 2\eta^{-l} x$.

For $t > 0$, the functional $F \mapsto (T_t F)(0)$ is a state on $C(Y_\eta)$, that is a probability measure μ_t on Y_η , and $\int \varphi_l^\eta d\mu_t = e^{-t\psi(l)}$, since $\varphi_l^\eta(0) = 1$. By Lemma 4.4, we have $x = \frac{1-\tau}{1-\tau\eta}(1 - \varphi_1^\eta(x))$, hence $\int x d\mu_t(x) = \frac{1-\tau}{1-\tau\eta}(1 - e^{-t\psi(1)})$. We deduce that

$$1 - e^{-t\psi(l)} = \int_{Y_\eta} (1 - \varphi_l^\eta) d\mu_t \leq \max\{2, A'\} \eta^{-l} \frac{1-\tau}{1-\tau\eta} (1 - e^{-t\psi(1)}).$$

Dividing by t and letting $t \rightarrow 0^+$, we obtain $\psi(l) \leq \frac{\max\{2, A'\}(1-\tau)}{1-\tau\eta} \eta^{-l} \psi(1)$, and (4.4) gives (7.11). $\blacksquare$

For $\eta = \tau$, the assumption of Proposition 7.9 is satisfied by Proposition 5.3, since $Y_\tau = X_\tau$, and $K_\tau = \frac{\max\{2, A'\}}{(1-\tau^2)^2}$.

Proposition 7.9 is a Lévy-Khintchine type bound: the exponents of a positive semigroup which is diagonal in the zonal spherical functions grow at most like the Casimir eigenvalues. We now identify the semigroups to which it applies on the Podleś sphere.

Theorem 7.10 *Let $(S_t)_{t \geq 0}$ be a strongly continuous semigroup of unital completely positive maps on $C(S_r^2)$ which is covariant, in the sense that*

$$\Delta \circ S_t = (\text{Id} \otimes S_t) \circ \Delta \quad \text{on } C(S_r^2), \quad t \geq 0.$$

1. *There exists a function $\psi: \mathbb{N}_0 \rightarrow [0, \infty)$ such that $S_t(u_{j0}^{(l)}) = e^{-t\psi(l)} u_{j0}^{(l)}$ for any $l \in \mathbb{N}_0$, any $|j| \leq l$ and any $t \geq 0$. In particular, S_t leaves $C^*(A)$ invariant, its restriction T_t to $C^*(A) = C(X_\tau)$ satisfies $T_t \varphi_l^\tau = e^{-t\psi(l)} \varphi_l^\tau$, and $\psi(l) \leq K_\tau \psi(1) \Lambda_l(\tau)$ for any $l \geq 1$.*
2. *Suppose that $\tau = Q^{-d}$ and let $\alpha > 0$. If the closed form $f \mapsto \sum_{l \geq 1} \psi(l) |\langle f, e_l \rangle|^2$ of the negative generator of the radial semigroup (T_t) on $\ell_{\mathbb{N}_0}^2(w)$ is equivalent to the closed form of R_{rad}^α , then $\alpha \leq d$. Conversely, if $\alpha \leq d$, the semigroup $(\mathcal{S}_t^{(\frac{\alpha}{d})})$ of Corollary 7.8 is covariant and has this property.*
3. *For any $s > 1$, the semigroup $(e^{-tL_\tau^s})_{t \geq 0}$ on $\ell_{\mathbb{N}_0}^2(w)$ is not positivity preserving.*

Proof : 1. By (5.6), we have $\beta_{1,\mu} = (\text{Id} \otimes \varepsilon_\mu) \circ \Delta$ and $\beta_{\mu,1} = (\varepsilon_\mu \otimes \text{Id}) \circ \Delta$. By coassociativity, we deduce that

$$(7.12) \quad \Delta \circ \beta_{1,\mu} = (\text{Id} \otimes \beta_{1,\mu}) \circ \Delta, \quad (\beta_{1,\mu} \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \beta_{\mu,1}) \circ \Delta.$$

By the first identity, Δ maps $C(S_r^2)$ into the fixed points of the maps $\text{Id} \otimes \beta_{1,\mu}$, that is into the range $C(\text{SU}_r(2)) \otimes_{\min} C(S_r^2)$ of $\text{Id} \otimes \mathbb{E}_R$. Let $\omega_t \stackrel{\text{def}}{=} \varepsilon \circ S_t$, which is a state on $C(S_r^2)$. Applying $\text{Id} \otimes \varepsilon$ to the covariance identity and using $(\text{Id} \otimes \varepsilon) \circ \Delta = \text{Id}$, we obtain $S_t = (\text{Id} \otimes \omega_t) \circ \Delta$ on $C(S_r^2)$. Since $S_t(x)$ is fixed by $\beta_{1,\mu}$ for any $x \in C(S_r^2)$, the second identity of (7.12) gives

$$(\text{Id} \otimes \omega_t)\Delta(x) = \beta_{1,\mu}(S_t(x)) = (\text{Id} \otimes \omega_t \circ \beta_{\mu,1})\Delta(x), \quad x \in C(S_r^2), \quad \mu \in \mathbb{T},$$

where we used that $\beta_{\mu,1}$ leaves $C(S_r^2)$ invariant. For $x = u_{j0}^{(l)}$, Lemma 5.2 and the linear independence of the matrix coefficients give $\omega_t(u_{k0}^{(l)}) = \mu^{-2k}\omega_t(u_{k0}^{(l)})$ for any $\mu \in \mathbb{T}$, hence $\omega_t(u_{k0}^{(l)}) = 0$ for $k \neq 0$. Consequently,

$$S_t(u_{j0}^{(l)}) = \sum_k u_{jk}^{(l)} \omega_t(u_{k0}^{(l)}) = \omega_t(u_{00}^{(l)}) u_{j0}^{(l)}.$$

The number $\omega_t(u_{00}^{(l)}) = \omega_t(\varphi_l^\tau(A))$ is real and belongs to $[-1, 1]$, it is multiplicative in t by the semigroup law, and it depends continuously on t with value 1 at $t = 0$ by strong continuity. Hence it is equal to $e^{-t\psi(l)}$ for a unique $\psi(l) \in [0, \infty)$. By Proposition 5.3 and the density of polynomials in $C(X_\tau)$, S_t leaves $C^*(A)$ invariant and $T_t \varphi_l^\tau = e^{-t\psi(l)} \varphi_l^\tau$. The maps T_t are positive and unital, and Proposition 7.9 with $\eta = \tau$ gives the last assertion.

2. Suppose that the forms are equivalent. Then $e_l \in \text{dom } \mathcal{E}_R$ for any $l \geq 1$. By the assumed equivalence and part 1 of Theorem 4.6 with $\vartheta = d$, there exist constants $c, c' > 0$ such that $\psi(l) \geq c \mathcal{E}_R(e_l) \geq c' \Lambda_l(\tau)^{\frac{\alpha}{d}}$ for any $l \geq 1$. By part 1, $\Lambda_l(\tau)^{\frac{\alpha}{d}-1}$ is bounded in l . Since $\Lambda_l(\tau) \rightarrow \infty$, we conclude that $\alpha \leq d$. Conversely, each map $\mathcal{S}_u = (\text{Id} \otimes \omega_u) \circ \Delta$ of Theorem 7.7 is covariant, since coassociativity gives

$$\Delta \circ (\text{Id} \otimes \omega_u) \circ \Delta = (\text{Id} \otimes \text{Id} \otimes \omega_u) \circ (\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \text{Id} \otimes \omega_u) \circ (\text{Id} \otimes \Delta) \circ \Delta = (\text{Id} \otimes \mathcal{S}_u) \circ \Delta.$$

Integrating with respect to $\nu_t^{(s)}$, the maps $\mathcal{S}_t^{(s)}$ of Corollary 7.8 are covariant. For $s = \frac{\alpha}{d} \leq 1$, Corollary 7.8 gives the equivalence of the forms.

3. Suppose that the operators $T_t = e^{-tL_\tau^s}$ are positivity preserving. Since $L_\tau 1 = 0$, they are unital, hence contractive for the uniform norm on bounded sequences. They map each polynomial φ_l^τ to $e^{-t\Lambda_l(\tau)^s} \varphi_l^\tau$. By density of polynomials, they restrict to positive unital maps on $C(X_\tau)$. Then Proposition 7.9 applied with $\eta = \tau$ and $\psi(l) = \Lambda_l(\tau)^s$ shows that $\Lambda_l(\tau)^{s-1}$ is bounded in l , which contradicts $s > 1$. ■

Remark 7.11 Theorem 7.10 and Theorem 4.6 together describe how the order α of the Taibleson energy interacts with the quantum symmetry. For $\alpha \leq d$, the energy is realized by covariant quantum Markov semigroups on $C(S_r^2)$, namely the subordinates of the Casimir semigroup. For $d < \alpha < 3d$, it is still the energy of the power $L_\tau^{\frac{\alpha}{d}}$ of the zonal Casimir operator, but no covariant quantum Markov semigroup realizes it, and the semigroup generated by $-L_\tau^{\frac{\alpha}{d}}$ is not positivity preserving. For $\alpha \geq 3d$, no function of L_τ has the right form domain. In all cases, the conditional expectation lifting of Theorem 7.2 realizes the Taibleson semigroup exactly, but it commutes only with the left action of the torus.

Remark 7.12 The radial Taibleson semigroup is not of convolution type. More precisely, let ω be any state on $C(SU_r(2))$ and let $S = (\text{Id} \otimes \omega) \circ \Delta$ or $S = (\omega \otimes \text{Id}) \circ \Delta$. Since $S(u_{jk}^{(l)})$ belongs to $\mathcal{C}(l)$, the space $\mathcal{C}(0) \oplus \mathcal{C}(1)$ is invariant under S . By Proposition 5.3 and (3.5), $A = \frac{1-u_{00}^{(1)}}{1+\tau}$ belongs to this space. If $S(A) \in C^*(A)$, then $S(A)$ is fixed by all $\beta_{\lambda,\mu}$ and belongs to $\mathcal{C}(0) \oplus \mathcal{C}(1)$, hence to $\text{span}\{1, u_{00}^{(1)}\} = \text{span}\{1, A\}$ by Lemma 5.2. On the other hand, if $\tau = Q^{-d}$, all the coefficients $\langle A, J_2 g_n \rangle$ are nonzero by the proof of Proposition 4.7, so that $JT_t J^{-1}(A) \in \text{span}\{1, A\}$ would force $e^{-t\lambda_n}$ to be independent of n . Hence no left or right convolution by a state restricts to $JT_t J^{-1}$ on $C^*(A)$ for $t > 0$. The Casimir semigroup of Theorem 7.7 is the zonal convolution semigroup whose energy is equivalent to the Taibleson energy of order d .

8 The frontier between S_r^2 and $SU_r(2)$

In this section $0 < r < 1$ and $\tau = r^2$. Recall that $p_k = e_{kk} \otimes 1 \in C^*(A)$ and that $b(e_n \otimes 1) = \sqrt{1 - \tau^n} e_{n-1} \otimes 1$, where 1 denotes the constant function of $L^2(\mathbb{T})$.

8.1 A coefficient estimate for completely positive maps

Lemma 8.1 *Let $S: C(SU_r(2)) \rightarrow C(SU_r(2))$ be unital completely positive. Suppose there is a stochastic matrix K on $\mathbb{N}_0$ such that $S(p_k) = \sum_i K(i, k) p_i$ for every k , the sum being strongly convergent. Then, for any $i, j \geq 0$,*

$$(8.1) \quad |\langle S(b)(e_j \otimes 1), e_i \otimes 1 \rangle| \leq \sum_{k \geq 1} \sqrt{1 - \tau^k} \sqrt{K(i, k-1)K(j, k)}.$$

In particular, if both $K(n-1, \cdot)$ and $K(n, \cdot)$ converge in $\ell_{\mathbb{N}_0}^1$ to a probability measure μ as $n \rightarrow \infty$, then

$$(8.2) \quad \limsup_{n \rightarrow \infty} |\langle S(b)(e_n \otimes 1), e_{n-1} \otimes 1 \rangle| \leq F(\mu) \stackrel{\text{def}}{=} \sum_{k \geq 1} \sqrt{\mu(k-1)\mu(k)}.$$

No normal extension of S is assumed.

Proof : By Stinespring's theorem, there exist a Hilbert space $\mathcal{K}$, a representation Π of $C(SU_r(2))$ on $\mathcal{K}$ and an isometry $V: \ell_{\mathbb{N}_0}^2 \otimes L^2(\mathbb{T}) \rightarrow \mathcal{K}$ such that $S(x) = V^* \Pi(x) V$. Put $\eta_i \stackrel{\text{def}}{=} V(e_i \otimes 1)$. Then

$$(8.3) \quad \|\Pi(p_k) \eta_i\|^2 = \langle S(p_k)(e_i \otimes 1), e_i \otimes 1 \rangle = K(i, k).$$

Since $\sum_k K(i, k) = 1 = \|\eta_i\|^2$ and the projections $\Pi(p_k)$ are mutually orthogonal, we have $\eta_i = \sum_{k \geq 0} \Pi(p_k) \eta_i$ in norm. Even if Π has a component on which all the $\Pi(p_k)$ vanish, that component is zero on every η_i .

The relations $bp_0 = 0$ and $bp_k = \sqrt{1 - \tau^k}(e_{k-1, k} \otimes 1)$ for $k \geq 1$ give $bp_k = p_{k-1}bp_k$. Consequently,

$$\langle S(b)(e_j \otimes 1), e_i \otimes 1 \rangle = \langle \Pi(b) \eta_j, \eta_i \rangle = \sum_{k \geq 1} \langle \Pi(bp_k) \Pi(p_k) \eta_j, \Pi(p_{k-1}) \eta_i \rangle.$$

Each summand is bounded by $\|bp_k\| \|\Pi(p_k) \eta_j\| \|\Pi(p_{k-1}) \eta_i\|$, which is the corresponding term of (8.1) by (8.3). If two probability rows converge in ℓ^1 , their square roots converge in ℓ^2 , because $|\sqrt{a} - \sqrt{b}|^2 \leq |a - b|$. Their shifted inner products consequently converge. Dropping the weights $\sqrt{1 - \tau^k} \leq 1$ gives (8.2). $\blacksquare$

Lemma 8.2 *For any probability measure μ on $\mathbb{N}_0$, we have*

$$(8.4) \quad F(\mu) \leq \sqrt{1 - \left(\sup_k \mu(k)\right)^2}.$$

Proof : Let $\mu^+(0) = 0$ and $\mu^+(k) = \mu(k-1)$ for $k \geq 1$. For the set $\{0, \dots, j\}$, the difference of its probabilities under μ and μ^+ is $\mu(j)$. Thus $\frac{1}{2} \|\mu - \mu^+\|_1 \geq \sup_j \mu(j)$. For arbitrary probabilities p, q on $\mathbb{N}_0$, the Cauchy-Schwarz inequality gives

$$\frac{1}{2} \sum_k |p_k - q_k| \leq \frac{1}{2} \left(\sum_k (\sqrt{p_k} - \sqrt{q_k})^2 \right)^{\frac{1}{2}} \left(\sum_k (\sqrt{p_k} + \sqrt{q_k})^2 \right)^{\frac{1}{2}} = \sqrt{1 - \left(\sum_k \sqrt{p_k q_k} \right)^2}.$$

Applying this to $p = \mu$ and $q = \mu^+$ gives (8.4). ■

Theorem 8.3 *Let $S: C(\mathrm{SU}_r(2)) \rightarrow C(\mathrm{SU}_r(2))$ be unital completely positive, and suppose that $S(C^*(A)) \subset C^*(A)$ and that the restriction of S to $C^*(A) = C(X_\tau)$ is given on $\mathbb{N}_0$ by a stochastic matrix K whose rows $K(n, \cdot)$ converge in $\ell^1_{\mathbb{N}_0}$ to a probability measure μ as $n \rightarrow \infty$. Then*

$$(8.5) \quad \|S(b) - b\| \geq 1 - F(\mu) \geq 1 - \sqrt{1 - \left(\sup_k \mu(k)\right)^2}.$$

Proof : The hypothesis gives $S(p_k) = \sum_i K(i, k) p_i$, since p_k corresponds to the indicator of the isolated point τ^k . By Lemma 8.1,

$$\|S(b) - b\| \geq \limsup_{n \rightarrow \infty} \left| \sqrt{1 - \tau^n} - \langle S(b)(e_n \otimes 1), e_{n-1} \otimes 1 \rangle \right| \geq 1 - F(\mu).$$

Lemma 8.2 gives the second inequality. ■

8.2 The Taibleson dynamics

Corollary 8.4 *Suppose that $\tau = Q^{-d}$. There is no family of unital completely positive maps $S_t: C(\mathrm{SU}_r(2)) \rightarrow C(\mathrm{SU}_r(2))$, $t > 0$, which both restricts to $JT_t J^{-1}$ on $C^*(A)$ and converges pointwise in norm to the identity as $t \rightarrow 0^+$. More precisely, every family satisfying only the restriction condition obeys*

$$(8.6) \quad \|S_{\rho^{-j}}(b) - b\| \geq 1 - \sqrt{1 - \delta_T^2} > 0, \quad j \geq 1,$$

where $\delta_T = (1 - \tau)(e^{-1} - e^{-\rho})$. The conclusion requires neither the semigroup law, Haar preservation, covariance, symmetry, nor a normal extension of the maps.

Proof : By Lemma 2.6, the rows of the transition matrix of T_t converge in $\ell^1_{\mathbb{N}_0}$ to μ_t , and $\sup_k \mu_{\rho^{-j}}(k) \geq \delta_T$. Theorem 8.3 gives (8.6). Since $\rho^{-j} \rightarrow 0$, point-norm convergence is impossible. ■

There is no contradiction with Theorem 7.2. The conditional-expectation lifting is strongly continuous on $C(S_r^2)$, which does not contain b . At the accumulation point $0 \in X_\tau$, which corresponds to the counit and to the classical torus of $\mathrm{SU}_r(2)$ (Lemma 5.1), the radial semigroup has the entrance law μ_t , which keeps an atom of macroscopic size along the times ρ^{-j} . Complete positivity requires compatibility of this law with the shift b , and Lemma 8.1 detects exactly this requirement.

8.3 The Casimir dynamics

Let $0 < \eta < 1$, and let μ_t^η be the law at the origin of the Feller semigroup (T_t^η) generated by $-L_\eta$, see Proposition 3.3. If $|\varphi_l^\eta| \leq 1$ on Y_η for any $l \in \mathbb{N}_0$, then, by part 4 of Proposition 3.3, μ_t^η is a probability measure on $\mathbb{N}_0$ for any $t > 0$, and the rows $P_t^\eta(n, \cdot)$ converge to it in $\ell_{\mathbb{N}_0}^1$. By Proposition 5.3, this is the case for $\eta = \tau$.

Proposition 8.5 *Let $0 < \eta < 1$, and suppose that $|\varphi_l^\eta(x)| \leq 1$ for any $l \in \mathbb{N}_0$ and any $x \in Y_\eta$. Put*

$$c_0(\tau, \eta) \stackrel{\text{def}}{=} \frac{(1-\tau)\eta(1-\tau\eta^3)}{2(1+\eta)^2(1-\tau\eta)(1-\tau\eta^2)},$$

$$N_0(\tau, \eta) \stackrel{\text{def}}{=} 1 + \frac{\log \frac{16}{c_0(\tau, \eta)}}{\log \frac{1}{\eta}}, \quad \delta_C(\tau, \eta) \stackrel{\text{def}}{=} \frac{c_0(\tau, \eta)}{8N_0(\tau, \eta)}.$$

For any $0 < t \leq \Lambda_1(\eta)^{-1}$, we have $\sup_k \mu_t^\eta(k) \geq \delta_C(\tau, \eta)$. Moreover, $c_0(\tau, \tau) = \frac{\tau(1+\tau^2)}{2(1+\tau)^2(1+\tau+\tau^2)}$.

Proof : We consider the random variable x , the coordinate of Y_η , under $\mu_t \stackrel{\text{def}}{=} \mu_t^\eta$, and we put $m_1 = \int x d\mu_t$ and $m_2 = \int x^2 d\mu_t$. We abbreviate $\Lambda_1 = \Lambda_1(\eta)$, $\Lambda_2 = \Lambda_2(\eta)$ and $c_0 = c_0(\tau, \eta)$. By [KLS10, (14.12.1)], as in the proof of Lemma 4.4, we have

$$\varphi_1^\eta(x) = 1 - \frac{1-\tau\eta}{1-\tau}x, \quad \varphi_2^\eta(x) = 1 - \frac{(1+\eta)(1-\tau\eta^2)}{\eta(1-\tau)}x + c_{2,2}x^2,$$

$$c_{2,2} \stackrel{\text{def}}{=} \frac{(1-\tau\eta^2)(1-\tau\eta^3)}{\eta(1-\tau)(1-\tau\eta)},$$

and (1.4) gives $\frac{\Lambda_2}{\Lambda_1} = \frac{(1+\eta)(1-\tau\eta^2)}{\eta(1-\tau\eta)}$. By (3.7) for $n = 1$, we have

$$m_1 = \frac{1-\tau}{1-\tau\eta}(1 - e^{-t\Lambda_1}).$$

Since $1 - e^{-u} \geq u - \frac{u^2}{2} \geq \frac{u}{2}$ for $0 \leq u \leq 1$, we obtain $m_1 \geq \frac{(1-\tau)t\Lambda_1}{2(1-\tau\eta)}$ for $t\Lambda_1 \leq 1$. By (3.7) for $n = 2$, and using the value of m_1 , we obtain

$$c_{2,2}m_2 = e^{-t\Lambda_2} - 1 + \frac{\Lambda_2}{\Lambda_1}(1 - e^{-t\Lambda_1}).$$

For $u \geq 0$, we have $e^{-u} \leq 1 - u + \frac{u^2}{2}$ and $1 - e^{-u} \leq u$. Hence the first-order terms cancel and $m_2 \leq \frac{t^2\Lambda_2^2}{2c_{2,2}}$. By the Paley-Zygmund inequality $\mu_t(x > \frac{m_1}{2}) \geq \frac{m_1^2}{4m_2}$, we deduce

$$\mu_t\left(x > \frac{m_1}{2}\right) \geq \frac{(1-\tau)^2 c_{2,2}}{8(1-\tau\eta)^2} \cdot \frac{\Lambda_1^2}{\Lambda_2^2} = \frac{c_0}{4}.$$

On the other hand, Markov's inequality gives $\mu_t(x > \frac{8m_1}{c_0}) \leq \frac{c_0}{8}$. Hence $\mu_t(\frac{m_1}{2} < x \leq \frac{8m_1}{c_0}) \geq \frac{c_0}{8}$. The points η^k of the interval $(\frac{m_1}{2}, \frac{8m_1}{c_0}]$ correspond to integers k in an interval of length at most $\frac{\log \frac{16}{c_0}}{\log \frac{1}{\eta}}$, hence to at most $N_0(\tau, \eta)$ integers. Thus one of them satisfies $\mu_t(k) \geq \delta_C(\tau, \eta)$. The value of $c_0(\tau, \tau)$ follows from $1 - \tau^4 = (1-\tau)(1+\tau)(1+\tau^2)$ and $1 - \tau^3 = (1-\tau)(1+\tau+\tau^2)$. ■

Theorem 8.6 Put $\Lambda_{\frac{1}{2}} \stackrel{\text{def}}{=} r^{-1}(1-r)(1-r^3)$, in accordance with (5.11).

1. For any $0 < t \leq \Lambda_1(\tau)^{-1}$, any unital completely positive map $S: C(\text{SU}_r(2)) \rightarrow C(\text{SU}_r(2))$ such that $S(F(A)) = (T_t^\tau F)(A)$ for any $F \in C(X_\tau)$ satisfies

$$\|S(b) - b\| \geq 1 - \sqrt{1 - \delta_C(\tau, \tau)^2}.$$

2. Let $t_0 \stackrel{\text{def}}{=} \min \{ \Lambda_1(\tau)^{-1}, \frac{1}{2} \Lambda_{\frac{1}{2}}^{-1} (1 - \sqrt{1 - \delta_C(\tau, \tau)^2}) \}$. For any $0 < t \leq t_0$, the linear map $\mathcal{U}_t$ defined on $\text{Pol}(\text{SU}_r(2))$ by $\mathcal{U}_t(u_{jk}^{(l)}) = e^{-t\Lambda_l} u_{jk}^{(l)}$, $l \in \frac{1}{2}\mathbb{N}_0$, does not extend to a positive map on $C(\text{SU}_r(2))$. Its restriction to $\text{Pol}(S_r^2)$ extends to the unital completely positive map $\mathcal{S}_t$ of Theorem 7.7.

Proof : 1. By the discussion preceding Proposition 8.5, the restriction of S to $C^*(A)$ satisfies the hypotheses of Theorem 8.3 with $\mu = \mu_t^\tau$. Proposition 8.5 with $\eta = \tau$ and (8.5) give the estimate.

2. Suppose that $\mathcal{U}_t$ extends to a positive map $\widetilde{\mathcal{U}}_t$ on $C(\text{SU}_r(2))$. By Lemma 5.1, the counit extends to a character ε of $C(\text{SU}_r(2))$, so that $\psi_t \stackrel{\text{def}}{=} \varepsilon \circ \widetilde{\mathcal{U}}_t$ is a state. The map $S \stackrel{\text{def}}{=} (\text{Id} \otimes \psi_t) \circ \Delta$ is unital completely positive on $C(\text{SU}_r(2))$, and

$$S(u_{jk}^{(l)}) = \sum_m u_{jm}^{(l)} \psi_t(u_{mk}^{(l)}) = \sum_m u_{jm}^{(l)} e^{-t\Lambda_l} \varepsilon(u_{mk}^{(l)}) = e^{-t\Lambda_l} u_{jk}^{(l)}.$$

Thus $S = \mathcal{U}_t$ on $\text{Pol}(\text{SU}_r(2))$. In particular $S(\varphi_l^\tau(A)) = e^{-t\Lambda_l(\tau)} \varphi_l^\tau(A) = (T_t^\tau \varphi_l^\tau)(A)$ by Proposition 5.3, so that $S(F(A)) = (T_t^\tau F)(A)$ for any $F \in C(X_\tau)$ by density of polynomials. Moreover $b = u_{-\frac{1}{2}, -\frac{1}{2}}^{(\frac{1}{2})}$, so that $S(b) = e^{-t\Lambda_{\frac{1}{2}}} b$ and

$$\|S(b) - b\| = 1 - e^{-t\Lambda_{\frac{1}{2}}} \leq t\Lambda_{\frac{1}{2}} < 1 - \sqrt{1 - \delta_C(\tau, \tau)^2},$$

since $\|b\| = 1$. This contradicts part 1. The last assertion follows from part 2 of Proposition 7.6, since both maps act on $u_{j0}^{(l)}$ by $e^{-t\Lambda_l(\tau)}$. $\blacksquare$

Remark 8.7 Part 2 of Theorem 8.6 shows that the heat semigroup of the Casimir operator, normalized as in Remark 5.4, is not a quantum Markov semigroup on $C(\text{SU}_r(2))$, whereas its restriction to the Podleś sphere is one by Theorem 7.7. The positive maps on $C(S_r^2)$ use the states $\mu_t^\tau \circ \mathbb{E}$, which are invariant under the two-sided torus action, while a positive extension to $C(\text{SU}_r(2))$ would require the central functionals $u_{jk}^{(l)} \mapsto \delta_{jk} e^{-t\Lambda_l}$ to be states. We refer to [SS98] and [CFK14, Section 11] for the structure of generating functionals of Lévy processes on $\text{SU}_q(2)$.

9 The crystal limit

In this section we return to the local field, so that $\tau = Q^{-d}$, and we fix $\alpha > 0$. For $0 < \eta < 1$, we consider the operator L_η of Section 3 on the fixed space $\ell_{\mathbb{N}_0}^2(w)$. For $n \geq 0$ and $k \geq 0$, put

$$C_n^\eta(k) \stackrel{\text{def}}{=} \prod_{i=0}^{n-1} \frac{1 - \eta^{k-i}}{1 - \eta^{n-i}},$$

with $C_0^\eta = 1$. With the notation $U_{n,x}^q = (q^x; q^{-1})_n$ of [KO07, (2.21)], we have $C_n^\eta(k) = \frac{U_{n,k}^\eta}{U_{n,n}^\eta}$; these functions are the q -analogues of the cellular basis used in [KO07, Section 2].

Lemma 9.1 1. For any $n \geq 0$, C_n^η is a polynomial of degree n in η^k , $C_n^\eta(k) = 0$ for $k < n$ and $C_n^\eta(n) = 1$. As $\eta \rightarrow 0^+$, C_n^η converges in $\ell_{\mathbb{N}_0}^2(w)$ to $\mathbf{1}_{\{k \geq n\}} = \mathbf{1}_{B_n}$.

2. Let $(\tilde{e}_n^\eta)_{n \geq 0}$ be the orthonormal family obtained by the Gram-Schmidt process from $(C_n^\eta)_{n \geq 0}$. Then $\tilde{e}_n^\eta = \pm e_n^\eta$ is an eigenvector of L_η with eigenvalue $\Lambda_n(\eta)$. As $\eta \rightarrow 0^+$, $\tilde{e}_0^\eta = 1$ and $\tilde{e}_n^\eta \rightarrow g_{n-1}$ in $\ell_{\mathbb{N}_0}^2(w)$ for any $n \geq 1$.

Proof : 1. Writing $1 - \eta^{k-i} = 1 - \eta^{-i}x$ with $x = \eta^k$, we see that C_n^η is a polynomial of degree n in x . For $k < n$, the factor $i = k$ vanishes, and for $k = n$ all the factors are equal to one. For $k \geq n$, each factor belongs to $(0, (1 - \eta)^{-1}]$ and tends to one as $\eta \rightarrow 0^+$. Thus $0 \leq C_n^\eta \leq 2^n$ for $\eta \leq \frac{1}{2}$, and $C_n^\eta \rightarrow \mathbf{1}_{\{k \geq n\}}$ pointwise. Since w is a probability, dominated convergence gives the convergence in $\ell_{\mathbb{N}_0}^2(w)$.

2. By part 1, $C_0^\eta, \dots, C_n^\eta$ is a basis of $\mathcal{P}_n$. Hence $\tilde{e}_n^\eta$ belongs to $\mathcal{P}_n$ and is orthogonal to $\mathcal{P}_{n-1}$, so that $\tilde{e}_n^\eta = \pm e_n^\eta$, which is an eigenvector by Proposition 3.1. The limits $\mathbf{1}_{B_0}, \mathbf{1}_{B_1}, \dots$ are linearly independent, and the Gram-Schmidt process is continuous on finite linearly independent families. Hence $\tilde{e}_n^\eta$ converges to the n th vector of the Gram-Schmidt process applied to $(\mathbf{1}_{B_m})_{m \geq 0}$. For $n \geq 1$, the vector $\mathbf{1}_{B_n} - \tau \mathbf{1}_{B_{n-1}}$ belongs to $\text{span}\{\mathbf{1}_{B_0}, \dots, \mathbf{1}_{B_n}\}$, has coefficient 1 on $\mathbf{1}_{B_n}$, and is orthogonal to $\mathbf{1}_{B_m}$ for $m < n$, since $\langle \mathbf{1}_{B_n} - \tau \mathbf{1}_{B_{n-1}}, \mathbf{1}_{B_m} \rangle = \tau^n - \tau \tau^{n-1} = 0$. Its normalization is g_{n-1} by (2.8). This proves part 2. $\blacksquare$

Theorem 9.2 For $0 < \eta \leq Q^{-\alpha}$, put $s(\eta) = \frac{\alpha \log Q}{\log \frac{1}{\eta}} \in (0, 1]$.

1. The semigroup $(e^{-tL_\eta^{s(\eta)}})_{t \geq 0}$ is Markovian on $\ell_{\mathbb{N}_0}^2(w)$ and induces a Feller semigroup on $C(Y_\eta)$.

2. We have

$$\lim_{\eta \rightarrow 0^+} \|(L_\eta^{s(\eta)} + I)^{-1} - (\mathcal{M}_{\text{rad}}^\alpha + I)^{-1}\| = 0, \quad \lim_{\eta \rightarrow 0^+} \|e^{-tL_\eta^{s(\eta)}} - e^{-t\mathcal{M}_{\text{rad}}^\alpha}\| = 0 \quad (t > 0),$$

where $\mathcal{M}_{\text{rad}}^\alpha = R_{\text{rad}}^\alpha + \kappa(I - \mathbb{E}_0)$ is the radial part of the Fourier multiplier $\mathcal{M}^\alpha$ of Proposition 2.1.

Proof : 1. This follows from Proposition 3.3 and subordination with respect to the Bernstein function $\lambda \mapsto \lambda^{s(\eta)}$, exactly as in the proof of Corollary 7.8.

2. By Theorem 2.3 and Proposition 2.1, $\mathcal{M}_{\text{rad}}^\alpha$ is diagonal in the basis $e_0^0 \stackrel{\text{def}}{=} 1$, $e_n^0 \stackrel{\text{def}}{=} g_{n-1}$ ($n \geq 1$), with eigenvalues $\nu_0 = 0$ and $\nu_n = \lambda_{n-1} + \kappa = \rho^n$. By Lemma 9.1, $L_\eta^{s(\eta)}$ is diagonal in the basis $(\tilde{e}_n^\eta)$, with eigenvalues $\nu_n(\eta) \stackrel{\text{def}}{=} \Lambda_n(\eta)^{s(\eta)}$. Since $s(\eta) \log \frac{1}{\eta} = \alpha \log Q$, we have $\eta^{-ns(\eta)} = \rho^n$ and

$$\nu_n(\eta) = \rho^n (1 - \eta^n)^{s(\eta)} (1 - \tau \eta^n)^{s(\eta)}.$$

Hence $\nu_n(\eta) \rightarrow \rho^n$ as $\eta \rightarrow 0^+$ for any n . Moreover, since $0 < s(\eta) \leq 1$ and the last two factors belong to $(0, 1]$, we have $\nu_n(\eta) \geq c_* \rho^n$ for any $n \geq 1$ and any $0 < \eta \leq Q^{-\alpha}$, where $c_* \stackrel{\text{def}}{=} (1 - Q^{-\alpha})(1 - \tau)$.

Let Π_n^η and Π_n^0 be the orthogonal projections onto $\mathbb{C}\tilde{e}_n^\eta$ and $\mathbb{C}e_n^0$. By Lemma 9.1, $\Pi_n^\eta \rightarrow \Pi_n^0$ in norm for each n . Let f be either $\lambda \mapsto (1 + \lambda)^{-1}$ or $\lambda \mapsto e^{-t\lambda}$ with $t > 0$, which is a decreasing function on $[0, \infty)$ with limit zero at infinity. For any integer $N \geq 1$,

$$\|f(L_\eta^{s(\eta)}) - f(\mathcal{M}_{\text{rad}}^\alpha)\| \leq \sum_{n \leq N} \|f(\nu_n(\eta))\Pi_n^\eta - f(\nu_n)\Pi_n^0\| + f(c_* \rho^{N+1}) + f(\rho^{N+1}),$$

since the parts of the two operators on the orthogonal complements of the first $N+1$ eigenvectors have norms at most $\sup_{n>N} f(\nu_n(\eta)) \leq f(c_*\rho^{N+1})$ and $f(\rho^{N+1})$. Given $\varepsilon > 0$, we first choose N such that the last two terms are smaller than ε , and then η small enough for the finite sum to be smaller than ε . This proves part 2. $\blacksquare$

Remark 9.3 The family $(L_\eta)_{0<\eta<1}$ contains the zonal Casimir operator L_τ of Theorem 7.7, and, for $\eta = Q^{-\alpha}$, the operator compared with R_{rad}^α in Theorem 4.1; for $\eta = Q^{-\alpha}$ we have $s(\eta) = 1$. The subordination exponent $s(\eta)$ tends to zero, and it cannot be omitted: without it, $\Lambda_n(\eta) \rightarrow \infty$ for every $n \geq 1$ as $\eta \rightarrow 0^+$, and the resolvents of L_η converge in norm to the projection onto the constants. In the parametrization $p_n^{a,b;q}(x) = p_n(q^x; q^{-1}a, q^{-1}b; q)$ of [KO07, (2.2)], the eigenfunctions of L_η are $\varphi_n^\eta(\eta^k) = p_n^{\tau,\eta;\eta}(k)$. Thus Lemma 9.1 is a variant of the limit theorem of Haran [Har01] and Koornwinder–Onn [KO07, Theorem 2.1], in which the parameter $b = \eta$ tends to zero together with $q = \eta$. The limit functions are the little 0-Laguerre functions $p_n^{\tau,0;0}$, which are the spherical functions of $\text{GL}_d(\mathcal{O})$ acting on $\mathcal{O}^d$ [KO07, Section 5.2], in accordance with the remark following Theorem 2.3. See also the ladder structures of [Har08, Chapter 4].

10 Quantum projective spaces and the orders $\frac{d}{n-1}$

In this section, $n \geq 2$ is an integer and $0 < q < 1$. We show that the operator L_η of Section 3, with

$$(10.1) \quad \eta = q^2 \quad \text{and} \quad \tau = \tau_n \stackrel{\text{def}}{=} q^{2(n-1)},$$

is the zonal part of the Casimir operator of the standard quantum complex projective space $\mathbb{CP}_q^{n-1}$. For $n = 2$, this is the Podleś sphere of Section 5, with $q = r$ and $\eta = \tau$. Together with Theorem 4.1, this shows that the Taibleson operator of order $\alpha = \frac{d}{n-1}$ on $\mathcal{O}^d$ corresponds to $\mathbb{CP}_q^{n-1}$ with $q^2 = Q^{-\frac{d}{n-1}}$. We use the computation of the zonal spherical functions of $\mathbb{CP}_q^{n-1}$ by Noumi, Yamada and Mimachi [NYM93]. Throughout this section, the parameter τ of Section 3 is τ_n , so that $w_k = (1 - \tau_n)\tau_n^k$, $\varphi_l^\eta(x) = p_l(x; \tau_n\eta^{-1}, 1; \eta) = p_l(x; q^{2(n-2)}, 1; q^2)$ and $Y_\eta = \{q^{2k} : k \geq 0\} \cup \{0\}$.

10.1 Zonal spherical functions

Let $U_q(n)$ be the compact quantum unitary group, whose $*$ -algebra $\text{Pol}(U_q(n))$ is generated by the entries of the fundamental corepresentation and the inverse of the quantum determinant [NYM93]. We denote by h_n its Haar state, by ε its counit, by Δ its coproduct and by $C(U_q(n))$ the universal C^* -completion of $\text{Pol}(U_q(n))$. Since the fusion rules of $U_q(n)$, together with the classical dimension function, are those of the compact group $U(n)$ [NYM93], the argument of [NT13, Theorem 2.7.14] shows that $U_q(n)$ is coamenable, see also [Ban99]. Hence the universal and the reduced C^* -algebras of $U_q(n)$ coincide [NT13, Theorem 2.7.10], and h_n is faithful on $C(U_q(n))$; recall that coamenability is equivalent to the faithfulness of the Haar state on the universal C^* -algebra [BMT01]. Let K_n be the quantum subgroup $U_q(n-1) \times \mathbb{T}$ of block diagonal matrices, which corresponds to the case $\sigma = \infty$ of [DN98], and let $\pi_K : C(U_q(n)) \rightarrow C(K_n)$ be the restriction homomorphism and h_K the Haar state of K_n . The algebra of the standard quantum complex projective space is

$$C(\mathbb{CP}_q^{n-1}) \stackrel{\text{def}}{=} \{x \in C(U_q(n)) : (\text{Id} \otimes \pi_K)\Delta(x) = x \otimes 1\},$$

and we put $\text{Pol}(\mathbb{CP}_q^{n-1}) \stackrel{\text{def}}{=} \text{Pol}(U_q(n)) \cap C(\mathbb{CP}_q^{n-1})$. The map

$$(10.2) \quad \mathbb{E}_K \stackrel{\text{def}}{=} (h_K \pi_K \otimes \text{Id} \otimes h_K \pi_K) \circ (\Delta \otimes \text{Id}) \circ \Delta$$

is a unital completely positive idempotent onto the K_n -biinvariant elements of $C(U_q(n))$, and it preserves h_n .

The irreducible unitary corepresentations of $U_q(n)$ are labeled by the dominant integral weights, as for $U(n)$. Those which admit a nonzero K_n -invariant vector are the corepresentations π_l of highest weight $l(\varepsilon_1 - \varepsilon_n)$, $l \in \mathbb{N}_0$, and the space of K_n -invariant vectors of π_l is one-dimensional [NYM93], see also [DN98, Section 5]. For any $l \in \mathbb{N}_0$, we choose an orthonormal basis (e_i) of the space of π_l which contains a K_n -invariant unit vector e_0 , and we denote by $u_{ij}^{(l)}$ the corresponding matrix coefficients. Since π_l is unitary, we have $\|u_{ij}^{(l)}\| \leq 1$. For any irreducible corepresentation π with matrix coefficients u_{ij}^π in an orthonormal basis, the matrix $(h_K \pi_K(u_{ij}^\pi))_{i,j}$ is the orthogonal projection onto the K_n -invariant vectors of π . Consequently,

$$(10.3) \quad \mathbb{E}_K(u_{ij}^{(l)}) = \delta_{i0} \delta_{j0} u_{00}^{(l)}, \quad \mathbb{E}_K(u_{ij}^\pi) = 0 \text{ if } \pi \text{ is not equivalent to some } \pi_l.$$

The zonal spherical function of π_l is $\varphi_l^{(n)} \stackrel{\text{def}}{=} u_{00}^{(l)}$. It is K_n -biinvariant and satisfies $\varepsilon(\varphi_l^{(n)}) = 1$. By the Peter–Weyl decomposition [NT13, Theorem 1.5.3] and (10.3), the K_n -biinvariant elements of $\text{Pol}(U_q(n))$ form the space $\text{span}\{\varphi_l^{(n)} : l \in \mathbb{N}_0\}$, and $\text{Pol}(\mathbb{CP}_q^{n-1}) = \text{span}\{u_{i0}^{(l)} : l \in \mathbb{N}_0, i\}$.

Let $(x_{ij})_{1 \leq i,j \leq n}$ be the fundamental corepresentation of $U_q(n)$ in the conventions of [NYM93], and put

$$x_n \stackrel{\text{def}}{=} \sum_{i=1}^{n-1} x_{ni} x_{ni}^* = 1 - x_{nn} x_{nn}^*.$$

This is the element ζ_{n-1} of [NYM93, (4.26)], written with the elements $z_i = x_{ni}$ and $w_i = z_i^*$ of [NYM93, (4.6)], and the second equality is [NYM93, (4.9.d)]. The element x_n is self-adjoint, and $\varepsilon(x_n) = 0$ since $\varepsilon(x_{ni}) = \delta_{ni}$. The following identification is due to Noumi, Yamada and Mimachi. By [NYM93, Proposition 4.2], the irreducible corepresentation of highest weight $\ell \varepsilon_1 - m \varepsilon_n$, $\ell, m \in \mathbb{N}_0$, contains a $U_q(n-1)$ -invariant vector, unique up to a scalar, which is of weight $(\ell - m)\varepsilon_n$ for the diagonal torus; it is invariant under the factor $\mathbb{T}$ of K_n if and only if $\ell = m$. By [NYM93, Theorem 4.7], the corresponding matrix coefficient $\varphi^{\ell,m}$ normalized by $\varepsilon(\varphi^{\ell,m}) = 1$ is given by [NYM93, (4.34.a)]. For $\ell = m = l$, this formula and the definition [NYM93, (4.33)] of the little q -Jacobi polynomials give

$$(10.4) \quad \varphi_l^{(n)} = P_l^{(n-2,0)}(x_n; q^2) = p_l(x_n; q^{2(n-2)}, 1; q^2) = \varphi_l^\eta(x_n), \quad l \in \mathbb{N}_0,$$

where $p_l(\cdot; a, b; q^2)$ is the little q^2 -Jacobi polynomial of [KLS10, Section 14.12]. Indeed, both polynomials are equal to ${}_2\phi_1(q^{-2l}, q^{2(l+n-1)}; q^{2(n-1)}; q^2, q^2 x)$. The paper [NYM93] uses left invariant elements of the quotient by $U_q(n-1)$; since $\varphi_l^{(n)}$ is K_n -biinvariant, (10.4) does not depend on this convention. See also [Flo97, Theorem 3.3.1], where x_n is denoted Q_{n-1} and the zonal elements are expressed by q -disk polynomials, and [DN98, Section 5]. By Lemma 4.4, we have $x_n = \frac{1-\tau_n}{1-\tau_n \eta}(1 - \varphi_1^{(n)})$, so that x_n is K_n -biinvariant. For $n = 2$, (10.4) reduces to Proposition 5.3, with $x_2 = A$ and $q = r$.

Lemma 10.1 *The spectrum of x_n is Y_η , and the distribution of x_n under the Haar state is given by*

$$h_n(F(x_n)) = (1 - \tau_n) \sum_{k \geq 0} \tau_n^k F(q^{2k}) = \sum_{k \geq 0} w_k F(\eta^k), \quad F \in C(Y_\eta).$$

Moreover, we have $\sup_{l \geq 0} \sup_{Y_\eta} |\varphi_l^\eta| \leq 1$.

Proof : Let μ be the distribution of the self-adjoint element x_n under h_n . It is a probability measure with compact support contained in the spectrum of x_n . By the orthogonality relations [NT13, Theorem 1.4.3], $h_n(u_{00}^{(l)}) = 0$ for any $l \geq 1$, since π_l is not trivial. By (10.4), we deduce that $\int \varphi_l^\eta d\mu = 0$ for any $l \geq 1$. The probability measure $\nu \stackrel{\text{def}}{=} \sum_k w_k \delta_{\eta^k}$ has the same property, since the polynomials φ_l^η are orthogonal in $\ell_{\mathbb{N}_0}^2(w)$ and $\varphi_0^\eta = 1$, by part 4 of Proposition 3.1. Since $(\varphi_l^\eta)_{l \geq 0}$ is a basis of the space of polynomials, all the moments of μ and ν coincide. Both measures have compact support in $\mathbb{R}$, hence $\mu = \nu$ by the Weierstrass theorem. Since h_n is faithful on $C(U_q(n))$, the spectrum of x_n is the support of μ , that is Y_η . Finally, $\|\varphi_l^\eta(x_n)\| = \sup_{Y_\eta} |\varphi_l^\eta|$, and $\|\varphi_l^{(n)}\| = \|u_{00}^{(l)}\| \leq 1$ by (10.4). ■

Remark 10.2 Lemma 10.1 agrees with the computations of [NYM93]. By [NYM93, Theorem 4.6], the Haar state on the polynomials in $\zeta_1, \dots, \zeta_{n-1}$ is the iterated Jackson integral [NYM93, (4.27)], see [DLMF, (17.2.46)] for the Jackson integral. For a polynomial F in ζ_{n-1} alone, the $n-2$ inner integrals are equal to $(1-q^2)^{n-2} (q^2; q^2)_{n-2}^{-1} \zeta_{n-1}^{n-2}$, see [NYM93, p. 78], so that

$$h_n(F(x_n)) = \frac{1-q^{2(n-1)}}{1-q^2} \int_0^1 F(\zeta) \zeta^{n-2} d_{q^2} \zeta = (1-\tau_n) \sum_{k \geq 0} \tau_n^k F(q^{2k}),$$

by [NYM93, (4.24)]. Moreover, [NYM93, (4.48)] with $\ell = m = \ell' = m' = l$ gives $h_n((\varphi_l^{(n)})^* \varphi_l^{(n)}) = d_l^{-1}$, where

$$d_l = q^{-2l(n-1)} \frac{1-q^{2(2l+n-1)}}{1-q^{2(n-1)}} \left(\frac{(q^{2(n-1)}; q^2)_l}{(q^2; q^2)_l} \right)^2$$

is the quantum dimension of π_l . By (4.6) with $\eta = q^2$ and $\tau = \tau_n$, this is $\|\varphi_l^\eta\|_{\ell_{\mathbb{N}_0}^2(w)}^{-2}$, in accordance with Lemma 10.1. The orthogonality relation of the polynomials (10.4) is [NYM93, (4.49)].

10.2 The Casimir operator and the energy bridge

Let C be the central element $\sum_{i,j} q^{2(n-i)} L_{ij}^+ S(L_{ji}^-)$ of $U_q \mathfrak{gl}_n$ of [DN98, (5.5)]. It acts on the irreducible module of highest weight λ by the scalar $\chi_\lambda(C) = \sum_{k=1}^n q^{2(\lambda_k + n - k)}$ [DN98, Section 5], and in particular on the module of π_l by

$$(10.5) \quad \chi_l \stackrel{\text{def}}{=} q^{2(l+n-1)} + q^{-2l} + \frac{q^2 - q^{2n-2}}{1-q^2},$$

see [DN98, (5.7)]. A direct computation gives, for any $l \in \mathbb{N}_0$,

$$(10.6) \quad \chi_l - \chi_0 = q^{-2l} + q^{2(l+n-1)} - 1 - q^{2(n-1)} = q^{-2l} (1 - q^{2l}) (1 - q^{2(l+n-1)}) \stackrel{(1.4)}{=} \Lambda_l(\eta).$$

Through the duality between $U_q \mathfrak{gl}_n$ and $\text{Pol}(U_q(n))$, the operator $\Omega^{(n)}: x \mapsto (\text{Id} \otimes C) \Delta(x)$ acts on the matrix coefficients of π_l by the scalar χ_l . For $n = 2$, (10.6) is (5.11) with $r = q$.

Theorem 10.3 *Let $n \geq 2$ and $0 < q < 1$, and put $\eta = q^2$ and $\tau_n = q^{2(n-1)}$.*

1. For any polynomial F , we have $(\Omega^{(n)} - \chi_0)(F(x_n)) = (L_\eta F)(x_n)$, where L_η is the operator (1.3) with $\tau = \tau_n$ acting on polynomials in η^k . In other words, L_η is the zonal part of the Casimir operator of $\mathbb{CP}_q^{n-1}$, normalized by (10.6).
2. Let (T_t^η) be the Feller semigroup e^{-tL_η} of Proposition 3.3, and μ_t^η its law at the origin. The maps $\mathcal{S}_t^{(n)} \stackrel{\text{def}}{=} (\text{Id} \otimes \mu_t^\eta \circ \mathbb{E}_K) \circ \Delta$ restrict to a Haar-preserving strongly continuous semigroup of unital completely positive maps on $C(\mathbb{CP}_q^{n-1})$, with $\mathcal{S}_t^{(n)}(u_{i0}^{(l)}) = e^{-t\Lambda_l(\eta)} u_{i0}^{(l)}$, and $\mathcal{S}_t^{(n)}(F(x_n)) = (T_t^\eta F)(x_n)$ for any $F \in C(Y_\eta)$.
3. Suppose that $\tau_n = Q^{-d}$, that is $q^2 = Q^{-\alpha}$ with $\alpha = \frac{d}{n-1}$. Then the map

$$J_n : L^\infty(\mathcal{O}^d)_{\text{rad}} \rightarrow W^*(x_n), \quad J_n(f(-\log_Q \|\cdot\|_K)) = \sum_{k \geq 0} f(k) \mathbf{1}_{\{\eta^k\}}(x_n),$$

is a normal unital state-preserving $*$ -isomorphism onto the von Neumann algebra generated by x_n in the GNS representation of h_n . The closed form $\mathcal{E}_\eta$ of the radial negative generator L_η of $(\mathcal{S}_t^{(n)})$ is equivalent to the closed form $\mathcal{E}_R$ of R_{rad}^α , with the constants c_1 and c_2 of Theorem 4.1. In particular, for $n = d + 1$, the Taibleson operator of order 1 on $\mathcal{O}^d$ corresponds in this way to $\mathbb{CP}_q^d$ with $q = Q^{-\frac{1}{2}}$.

Proof : 1. By (10.4), we have $\Omega^{(n)}(\varphi_l^\eta(x_n)) = \Omega^{(n)}(u_{00}^{(l)}) = \chi_l \varphi_l^\eta(x_n)$. On the other hand, $L_\eta \varphi_l^\eta = \Lambda_l(\eta) \varphi_l^\eta$ by part 3 of Proposition 3.1, and $\Lambda_l(\eta) = \chi_l - \chi_0$ by (10.6). Since $(\varphi_l^\eta)_{l \geq 0}$ is a basis of the space of polynomials, the assertion follows by linearity.

2. By Lemma 10.1, we have $C^*(x_n) = C(Y_\eta)$, so that the probability measure μ_t^η on Y_η defines a state on $C^*(x_n)$. The map $\mathbb{E}_K$ is unital completely positive and maps $\text{Pol}(U_q(n))$ onto the polynomials in x_n , by (10.3) and (10.4), hence $C(U_q(n))$ into $C^*(x_n)$ by continuity. Thus $\omega_t \stackrel{\text{def}}{=} \mu_t^\eta \circ \mathbb{E}_K$ is a state on $C(U_q(n))$, and $\mathcal{S}_t^{(n)} = (\text{Id} \otimes \omega_t) \circ \Delta$ is unital completely positive. By invariance of the Haar state, $h_n \circ \mathcal{S}_t^{(n)} = (h_n \otimes \omega_t) \Delta = h_n$. By (10.3), (10.4) and (3.7),

$$\omega_t(u_{ij}^{(l)}) = \delta_{i0} \delta_{j0} \int \varphi_l^\eta d\mu_t^\eta = \delta_{i0} \delta_{j0} e^{-t\Lambda_l(\eta)},$$

and $\omega_t(u_{ij}^\pi) = 0$ if π is not equivalent to some π_l . Hence $\mathcal{S}_t^{(n)}(u_{i0}^{(l)}) = \sum_j u_{ij}^{(l)} \omega_t(u_{j0}^{(l)}) = e^{-t\Lambda_l(\eta)} u_{i0}^{(l)}$. By invariance of h_K , we have $(\pi_K \otimes h_K \pi_K) \Delta = h_K \pi_K(\cdot) 1$, hence $(\pi_K \otimes \mathbb{E}_K) \Delta = 1 \otimes \mathbb{E}_K$ by coassociativity, and $(\pi_K \otimes \omega_t) \Delta = \omega_t(\cdot) 1$. Using coassociativity again, we obtain $(\text{Id} \otimes \pi_K) \Delta \circ \mathcal{S}_t^{(n)} = (\text{Id} \otimes (\pi_K \otimes \omega_t) \Delta) \circ \Delta = \mathcal{S}_t^{(n)}(\cdot) \otimes 1$. Hence $\mathcal{S}_t^{(n)}$ maps $C(U_q(n))$, and in particular $C(\mathbb{CP}_q^{n-1})$, into $C(\mathbb{CP}_q^{n-1})$. The map $(\text{Id} \otimes h_K \pi_K) \circ \Delta$ is a contractive projection of $C(U_q(n))$ onto $C(\mathbb{CP}_q^{n-1})$ which maps $\text{Pol}(U_q(n))$ onto $\text{Pol}(\mathbb{CP}_q^{n-1})$, so that $\text{Pol}(\mathbb{CP}_q^{n-1})$ is dense in $C(\mathbb{CP}_q^{n-1})$. On $\text{Pol}(\mathbb{CP}_q^{n-1})$, the maps $\mathcal{S}_t^{(n)}$ are diagonal in the vectors $u_{i0}^{(l)}$, satisfy the semigroup law and are strongly continuous, the sums being finite. Since the maps $\mathcal{S}_t^{(n)}$ are contractions, the semigroup law and the strong continuity extend to $C(\mathbb{CP}_q^{n-1})$. Finally, $\mathcal{S}_t^{(n)}(\varphi_l^\eta(x_n)) = e^{-t\Lambda_l(\eta)} \varphi_l^\eta(x_n) = (T_t^\eta \varphi_l^\eta)(x_n)$ by part 2 of Proposition 3.3. Both sides of the last assertion are continuous in $F \in C(Y_\eta)$ for the uniform norm, and they agree on polynomials, which are dense by the Weierstrass theorem.

3. By Lemma 10.1, the spectral measure of x_n in the GNS representation of h_n is supported by Y_η and charges each point η^k with mass $w_k = (1 - \tau_n) \tau_n^k$, whereas $\{0\}$ is a null set. By (2.2), the shell S_k of $\mathcal{O}^d$ has the Haar measure $(1 - Q^{-d}) Q^{-dk} = w_k$. Hence J_n is a normal

unital state-preserving $*$ -isomorphism, exactly as in Theorem 5.6. Under J_n , the radial negative generator of $(\mathcal{S}_t^{(n)})$ is the operator L_η on $\ell_{\mathbb{N}_0}^2(w)$, by part 2. Since $\eta = Q^{-\alpha}$ and $\tau_n = Q^{-d}$, Theorem 4.1 applies and gives the equivalence of the forms, with its constants. For $n = d + 1$, we have $\alpha = 1$ and $q^2 = Q^{-1}$. $\blacksquare$

Remark 10.4 Theorem 10.3 shows that the quantum group $\mathrm{SU}_r(2)$ with $r^2 = Q^{-d}$ of the previous sections is the case $n = 2$ of a family. For $n = 2, \dots, d + 1$, the quantum projective space $\mathbb{CP}_q^{n-1}$ with $q^2 = Q^{-\frac{d}{n-1}}$ carries a Casimir dynamics whose radial energy is equivalent to the Taibleson energy of order $\frac{d}{n-1}$, and whose radial Haar law is the radial Haar law of $\mathcal{O}^d$. In the extreme case $n = d + 1$, the quantum parameter $q = Q^{-\frac{1}{2}}$ does not depend on d , the rank of the quantum group matches the dimension of $\mathcal{O}^d$, and the order is $\alpha = 1$; this is the regime $t = Q^{-1}$ of Freund and Zabrodin [FZ92, Section 3]. In the classical limit, $\Lambda_l(q^2) \sim (2 \log \frac{1}{q})^2 l(l + n - 1)$, which is proportional to the eigenvalues of the Laplace–Beltrami operator of $\mathbb{CP}^{n-1}$. The analogue of Theorem 7.10, together with the corresponding case of Theorem 4.6, holds for any $n \geq 2$, see Theorem 10.5 below.

10.3 Powers of the Casimir operator and covariant semigroups

We now extend Theorem 7.10 to $\mathbb{CP}_q^{n-1}$. By the proof of Theorem 10.3, the map $\mathbb{E}_R \stackrel{\text{def}}{=} (\mathrm{Id} \otimes h_K \pi_K) \circ \Delta$ is a contractive projection of $C(U_q(n))$ onto $C(\mathbb{CP}_q^{n-1})$. It is unital and completely positive. For $x \in C(\mathbb{CP}_q^{n-1})$, coassociativity gives

$$(\mathrm{Id} \otimes \mathbb{E}_R)\Delta(x) = (\mathrm{Id} \otimes \mathrm{Id} \otimes h_K \pi_K)(\Delta \otimes \mathrm{Id})\Delta(x) = \Delta(\mathbb{E}_R(x)) = \Delta(x).$$

Hence Δ maps $C(\mathbb{CP}_q^{n-1})$ into the range $C(U_q(n)) \otimes_{\min} C(\mathbb{CP}_q^{n-1})$ of $\mathrm{Id} \otimes \mathbb{E}_R$, and for any bounded linear map $S: C(\mathbb{CP}_q^{n-1}) \rightarrow C(\mathbb{CP}_q^{n-1})$ which is completely bounded, the map $(\mathrm{Id} \otimes S) \circ \Delta$ is defined on $C(\mathbb{CP}_q^{n-1})$.

Theorem 10.5 *Let $n \geq 2$ and $0 < q < 1$, and put $\eta = q^2$ and $\tau_n = q^{2(n-1)}$.*

1. *Suppose that $\tau_n = Q^{-d}$ and let $\alpha > 0$. The closed forms of R_{rad}^α and $L_\eta^{\frac{(n-1)\alpha}{d}}$ are equivalent if and only if $\alpha < \frac{(n+1)d}{n-1}$. If $\alpha \geq \frac{(n+1)d}{n-1}$, then no operator of the form $F(L_\eta)$ has the same form domain as R_{rad}^α .*
2. *Let $(S_t)_{t \geq 0}$ be a strongly continuous semigroup of unital completely positive maps on $C(\mathbb{CP}_q^{n-1})$ which is covariant, in the sense that*

$$\Delta \circ S_t = (\mathrm{Id} \otimes S_t) \circ \Delta \quad \text{on } C(\mathbb{CP}_q^{n-1}), \quad t \geq 0.$$

There exists a function $\psi: \mathbb{N}_0 \rightarrow [0, \infty)$ such that $S_t(u_{i0}^{(l)}) = e^{-t\psi(l)} u_{i0}^{(l)}$ for any $l \in \mathbb{N}_0$, any i and any $t \geq 0$. In particular, S_t leaves $C^(x_n)$ invariant, its restriction T_t to $C^*(x_n) = C(Y_\eta)$ satisfies $T_t \varphi_l^\eta = e^{-t\psi(l)} \varphi_l^\eta$, and $\psi(l) \leq K_{\tau_n, \eta} \psi(1) \Lambda_l(\eta)$ for any $l \geq 1$.*

3. *Suppose that $\tau_n = Q^{-d}$ and let $\alpha > 0$. There exists a semigroup $(S_t)_{t \geq 0}$ as in part 2 such that the closed form $f \mapsto \sum_{l \geq 1} \psi(l) |\langle f, e_l^\eta \rangle|^2$ of the negative generator of the radial semigroup (T_t) on $\ell_{\mathbb{N}_0}^2(w)$ is equivalent to the closed form of R_{rad}^α if and only if $\alpha \leq \frac{d}{n-1}$. For $\alpha \leq \frac{d}{n-1}$, one can take $S_t = \int_{[0, \infty)} \mathcal{S}_u^{(n)} d\nu_t^{(s)}(u)$ with $s = \frac{(n-1)\alpha}{d}$, where $\nu_t^{(s)}$ is the measure of Corollary 7.8; its radial negative generator is L_η^s .*

4. For any $s > 1$, the semigroup $(e^{-tL_\eta^s})_{t \geq 0}$ on $\ell_{\mathbb{N}_0}^2(w)$ is not positivity preserving.

Proof : 1. This is Theorem 4.6 and Proposition 4.7 with $\vartheta = \frac{d}{n-1}$, since $\eta = Q^{-\frac{d}{n-1}}$, $\frac{\alpha}{\vartheta} = \frac{(n-1)\alpha}{d}$ and $d + 2\vartheta = \frac{(n+1)d}{n-1}$.

2. Let $\omega_t \stackrel{\text{def}}{=} \varepsilon \circ S_t$, which is a state on $C(\mathbb{CP}_q^{n-1})$. Applying $\text{Id} \otimes \varepsilon$ to the covariance identity and using $(\text{Id} \otimes \varepsilon) \circ \Delta = \text{Id}$, we obtain $S_t = (\text{Id} \otimes \omega_t) \circ \Delta$ on $C(\mathbb{CP}_q^{n-1})$. Let $l \in \mathbb{N}_0$. For any j , the element $u_{j0}^{(l)}$ belongs to $\text{Pol}(\mathbb{CP}_q^{n-1})$, and $\Delta(u_{i0}^{(l)}) = \sum_j u_{ij}^{(l)} \otimes u_{j0}^{(l)}$, so that

$$S_t(u_{i0}^{(l)}) = \sum_j u_{ij}^{(l)} \omega_t(u_{j0}^{(l)}).$$

Since $S_t(u_{i0}^{(l)})$ belongs to $C(\mathbb{CP}_q^{n-1})$, we have

$$\sum_k u_{ik}^{(l)} \omega_t(u_{k0}^{(l)}) \otimes 1 = (\text{Id} \otimes \pi_K) \Delta(S_t(u_{i0}^{(l)})) = \sum_k u_{ik}^{(l)} \otimes \sum_j \pi_K(u_{kj}^{(l)}) \omega_t(u_{j0}^{(l)}).$$

By the linear independence of the matrix coefficients [NT13, Theorem 1.4.3], we deduce that $\sum_j \pi_K(u_{kj}^{(l)}) \omega_t(u_{j0}^{(l)}) = \omega_t(u_{k0}^{(l)}) 1$ for any k . In other words, the vector $v \stackrel{\text{def}}{=} \sum_j \omega_t(u_{j0}^{(l)}) e_j$ satisfies $(\text{Id} \otimes \pi_K)(u^{(l)})(v \otimes 1) = v \otimes 1$, that is v is K_n -invariant. Since the space of K_n -invariant vectors of π_l is $\mathbb{C}e_0$, we obtain $\omega_t(u_{j0}^{(l)}) = 0$ for $j \neq 0$, and

$$S_t(u_{i0}^{(l)}) = \omega_t(\varphi_l^{(n)}) u_{i0}^{(l)}.$$

The number $\omega_t(\varphi_l^{(n)})$ is real and belongs to $[-1, 1]$, since $\varphi_l^{(n)} = \varphi_l^\eta(x_n)$ is self-adjoint with norm at most 1 by Lemma 10.1. It is multiplicative in t by the semigroup law, and it depends continuously on t with value 1 at $t = 0$ by strong continuity. Hence it is equal to $e^{-t\psi(l)}$ for a unique $\psi(l) \in [0, \infty)$. By (10.4), the polynomials in x_n are the linear combinations of the elements $\varphi_l^{(n)}$, which are dense in $C^*(x_n)$. Thus S_t leaves $C^*(x_n)$ invariant, and $T_t \varphi_l^\eta = e^{-t\psi(l)} \varphi_l^\eta$ under the identification $C^*(x_n) = C(Y_\eta)$ of Lemma 10.1. The maps T_t are positive and unital, and $|\varphi_l^\eta| \leq 1$ on Y_η by Lemma 10.1. Hence Proposition 7.9 with $\tau = \tau_n$ gives the last assertion.

3. Put $\vartheta = \frac{d}{n-1}$ and $s = \frac{\alpha}{\vartheta}$. Suppose that the forms are equivalent. Then $e_l^\eta \in \text{dom } \mathcal{E}_R$ for any $l \geq 1$. By the assumed equivalence and part 1 of Theorem 4.6, there exist constants $c, c' > 0$ such that $\psi(l) \geq c \mathcal{E}_R(e_l^\eta) \geq c' \Lambda_l(\eta)^s$ for any $l \geq 1$. By part 2, $\Lambda_l(\eta)^{s-1}$ is bounded in l . Since $\Lambda_l(\eta) \rightarrow \infty$, we conclude that $s \leq 1$, that is $\alpha \leq \frac{d}{n-1}$.

Conversely, suppose that $s \leq 1$. By part 2 of Theorem 10.3, the map $u \mapsto \mathcal{S}_u^{(n)}(x)$ is continuous and bounded for any $x \in C(\mathbb{CP}_q^{n-1})$, so that $S_t(x) \stackrel{\text{def}}{=} \int \mathcal{S}_u^{(n)}(x) d\nu_t^{(s)}(u)$ is a Bochner integral. As in the proof of Corollary 7.8, the maps S_t form a strongly continuous semigroup of unital completely positive maps on $C(\mathbb{CP}_q^{n-1})$, with $S_t(u_{i0}^{(l)}) = e^{-t\Lambda_l(\eta)^s} u_{i0}^{(l)}$, whose restriction to $C^*(x_n) = C(Y_\eta)$ is $\int T_u^\eta d\nu_t^{(s)}(u) = e^{-tL_\eta^s}$. Each map $\mathcal{S}_u^{(n)} = (\text{Id} \otimes \omega_u) \circ \Delta$ is covariant by the coassociativity computation of the proof of Theorem 7.10, and integrating with respect to $\nu_t^{(s)}$ shows that S_t is covariant. Here $\psi(l) = \Lambda_l(\eta)^s$, and the closed form of part 3 is the closed form of L_η^s . Since $\alpha \leq \vartheta < d + 2\vartheta$, part 2 of Theorem 4.6 gives the equivalence of the forms.

4. Suppose that the operators $e^{-tL_\eta^s}$ are positivity preserving. As in the proof of part 3 of Theorem 7.10, they restrict to positive unital maps on $C(Y_\eta)$ which map each φ_l^η to $e^{-t\Lambda_l(\eta)^s} \varphi_l^\eta$. By Lemma 10.1, Proposition 7.9 with $\tau = \tau_n$ applies with $\psi(l) = \Lambda_l(\eta)^s$, and shows that $\Lambda_l(\eta)^{s-1}$ is bounded in l , which contradicts $s > 1$. ■

Remark 10.6 Suppose that $\tau_n = Q^{-d}$. Theorem 10.5 describes the orders α of the Taibleson energy seen by $\mathbb{CP}_q^{n-1}$. For $\alpha \leq \frac{d}{n-1}$, the energy is realized by covariant quantum Markov semigroups on $C(\mathbb{CP}_q^{n-1})$. For $\frac{d}{n-1} < \alpha < \frac{(n+1)d}{n-1}$, it is the energy of a power of the zonal Casimir operator, but no covariant quantum Markov semigroup realizes it. For $\alpha \geq \frac{(n+1)d}{n-1}$, no function of the zonal Casimir operator has the right form domain. In terms of the exponent $s = \frac{(n-1)\alpha}{d}$ of the power, these three ranges are $s \leq 1$, $1 < s < n+1$ and $s \geq n+1$, independently of d . For $n = 2$, this is the description of Theorems 4.6 and 7.10. For $n = d+1$, we have $q^2 = Q^{-1}$ and $s = \alpha$: the Taibleson energies of the orders $0 < \alpha < d+2$ are the energies of the powers $L_{Q^{-1}}^\alpha$ of the zonal Casimir operator of $\mathbb{CP}_q^d$, and those of the orders $0 < \alpha \leq 1$ are realized by covariant quantum Markov semigroups on $C(\mathbb{CP}_q^d)$.

10.4 The frontier between $\mathbb{CP}_q^{n-1}$ and S_q^{2n-1}

We now show that the phenomenon of Section 8 occurs in the whole family. Recall that K_n is the quantum subgroup of block diagonal matrices, so that $\pi_K(x_{ij})$ is the (i, j) entry of the fundamental corepresentation of $U_q(n-1)$ for $i, j \leq n-1$, $\pi_K(x_{nn}) = 1 \otimes z$, where z is the coordinate function of $\mathbb{T}$, and $\pi_K(x_{in}) = \pi_K(x_{ni}) = 0$ for $i \leq n-1$. Let $\pi_{K'} \stackrel{\text{def}}{=} (\text{Id} \otimes \varepsilon_1) \circ \pi_K: C(U_q(n)) \rightarrow C(U_q(n-1))$, where ε_1 is the evaluation at $1 \in \mathbb{T}$. It is the restriction homomorphism to the quantum subgroup $U_q(n-1)$ of K_n . The algebra of the quantum sphere is

$$C(S_q^{2n-1}) \stackrel{\text{def}}{=} \{x \in C(U_q(n)) : (\text{Id} \otimes \pi_{K'})\Delta(x) = x \otimes 1\}.$$

It contains $C(\mathbb{CP}_q^{n-1})$. Since $\Delta(x_{nn}) = \sum_j x_{nj} \otimes x_{jn}$, we have $(\text{Id} \otimes \pi_{K'})\Delta(x_{nn}) = x_{nn} \otimes 1$ and $(\text{Id} \otimes \pi_K)\Delta(x_{nn}) = x_{nn} \otimes (1 \otimes z)$. Thus x_{nn} belongs to $C(S_q^{2n-1})$ but not to $C(\mathbb{CP}_q^{n-1})$.

Lemma 10.7 *We have*

$$x_{nn}x_{nn}^* = 1 - x_n, \quad x_{nn}^*x_{nn} = 1 - \eta x_n, \quad x_n x_{nn} = \eta x_{nn} x_n.$$

Consequently, $\|x_{nn}\| = 1$ and, for any $k \in \mathbb{N}_0$, the spectral projection $p_k \stackrel{\text{def}}{=} \mathbf{1}_{\{\eta^k\}}(x_n)$ belongs to $C^*(x_n)$ and satisfies $x_{nn}p_0 = 0$, $x_{nn}^*p_k = p_{k-1}x_{nn}^*$ for $k \geq 1$, and $(x_{nn}^*p_k)^*(x_{nn}p_k) = (1 - \eta^k)p_k$.

Proof : With the notation of [NYM93, (4.6) and (4.26)], we have $z_n = x_{nn}$, $w_n = x_{nn}^*$, $\zeta_{n-1} = x_n$ and $\zeta_n = 1$. By [NYM93, (4.28.a)] with $k = n$, we have $z_n w_n = 1 - \zeta_{n-1}$ and $w_n z_n = 1 - q^2 \zeta_{n-1}$, and by [NYM93, (4.28.b)] with $k = n$ and $i = n-1$, we have $z_n \zeta_{n-1} = q^{-2} \zeta_{n-1} z_n$. This gives the three identities. By Lemma 10.1, the spectrum of x_n is $Y_\eta \subset [0, 1]$ and contains 0, so that $\|x_{nn}\|^2 = \|1 - x_n\| = 1$. Taking adjoints in the third identity, we obtain $x_{nn}^* x_n = \eta x_n x_{nn}^*$, hence $x_{nn}^* P(x_n) = P(\eta x_n) x_{nn}^*$ for any polynomial P . Since $\eta Y_\eta \subset Y_\eta$, uniform approximation on Y_η gives $x_{nn}^* F(x_n) = F(\eta x_n) x_{nn}^*$ for any $F \in C(Y_\eta)$, where $F(\eta x_n)$ is the functional calculus of $x \mapsto F(\eta x)$. The function $F = \mathbf{1}_{\{\eta^k\}}$ is continuous on Y_η , since η^k is an isolated point. For $x \in Y_\eta$, we have $F(\eta x) = \mathbf{1}_{\{\eta^{k-1}\}}(x)$ if $k \geq 1$, and $F(\eta x) = 0$ if $k = 0$. Finally, $(x_{nn}^* p_k)^*(x_{nn} p_k) = p_k x_{nn} x_{nn}^* p_k = p_k (1 - x_n) p_k = (1 - \eta^k) p_k$. ■

For $n = 2$, Lemma 10.7 shows that x_{22}^* and x_2 satisfy the relations (5.2) of b and A with $r^2 = q^2$. In general, x_{nn}^* acts as a weighted shift on the spectral subspaces of x_n , whose points are η^k whereas the Haar weights are $(1 - \tau_n)\tau_n^k$.

Theorem 10.8 *Let $n \geq 2$ and $0 < q < 1$, and put $\eta = q^2$ and $\tau_n = q^{2(n-1)}$. Let $\mathcal{A}$ be a unital C^* -subalgebra of $C(U_q(n))$ which contains x_n and x_{nn} , for instance $C(S_q^{2n-1})$ or $C(U_q(n))$.*

1. Let $S: \mathcal{A} \rightarrow \mathcal{A}$ be a unital completely positive map. Suppose that there exists a stochastic matrix K on $\mathbb{N}_0$ whose rows $K(m, \cdot)$ converge in $\ell^1_{\mathbb{N}_0}$ to a probability measure μ as $m \rightarrow \infty$, such that, for any $F \in C(Y_\eta)$, we have $S(F(x_n)) = G(x_n)$ for a function $G \in C(Y_\eta)$ satisfying $G(\eta^m) = \sum_k K(m, k)F(\eta^k)$ for any $m \in \mathbb{N}_0$. Then

$$\|S(x_{nn}) - x_{nn}\| \geq 1 - F(\mu) \geq 1 - \sqrt{1 - \left(\sup_k \mu(k)\right)^2},$$

where $F(\mu)$ is defined in (8.2).

2. Suppose that $\tau_n = Q^{-d}$, let $\alpha > 0$, and let J_n be the isomorphism of part 3 of Theorem 10.3. Any family $(S_t)_{t>0}$ of unital completely positive maps on $\mathcal{A}$ whose restrictions to $C^*(x_n)$ are $J_n e^{-tR_{\text{rad}}^\alpha} J_n^{-1}$ satisfies

$$\|S_{\rho^{-j}}(x_{nn}) - x_{nn}\| \geq 1 - \sqrt{1 - \delta_T^2} > 0, \quad j \geq 1,$$

where δ_T is the constant of Lemma 2.6.

3. For any $0 < t \leq \Lambda_1(\eta)^{-1}$, any unital completely positive map $S: \mathcal{A} \rightarrow \mathcal{A}$ such that $S(F(x_n)) = (T_t^\eta F)(x_n)$ for any $F \in C(Y_\eta)$ satisfies

$$\|S(x_{nn}) - x_{nn}\| \geq 1 - \sqrt{1 - \delta_C(\tau_n, \eta)^2},$$

where $\delta_C(\tau_n, \eta)$ is the constant of Proposition 8.5.

4. Let $(\omega_t)_{t>0}$ be a family of states on $C(U_q(n))$ such that $\omega_t(x_{jn}) \rightarrow \delta_{jn}$ as $t \rightarrow 0^+$ for any $1 \leq j \leq n$. There exists $t_1 > 0$ such that, for any $0 < t \leq t_1$, the map $(\text{Id} \otimes \omega_t) \circ \Delta$ does not coincide with $F(x_n) \mapsto (T_t^\eta F)(x_n)$ on $C^*(x_n)$.

Proof : 1. We realize $C(U_q(n))$ faithfully on a Hilbert space H . By Lemma 10.1, η^m belongs to the spectrum of x_n , hence $p_m \neq 0$ for any $m \in \mathbb{N}_0$. Let $m \geq 1$ and let $\xi \in p_m H$ be a unit vector. By Lemma 10.7, the vector $\xi' \stackrel{\text{def}}{=} (1 - \eta^m)^{-\frac{1}{2}} x_{nn}^* \xi$ is a unit vector of $p_{m-1} H$, and $\langle x_{nn}^* \xi, \xi' \rangle = (1 - \eta^m)^{\frac{1}{2}}$. By Stinespring's theorem, there exist a Hilbert space $\mathcal{K}$, a representation Π of $\mathcal{A}$ on $\mathcal{K}$ and an isometry $V: H \rightarrow \mathcal{K}$ such that $S(x) = V^* \Pi(x) V$ for any $x \in \mathcal{A}$. Put $\zeta \stackrel{\text{def}}{=} V\xi$ and $\zeta' \stackrel{\text{def}}{=} V\xi'$. For any $F \in C(Y_\eta)$ and any $i \in \mathbb{N}_0$, the operator $F(x_n)$ acts on $p_i H$ by the scalar $F(\eta^i)$. By the hypothesis on S , applied to $F = \mathbf{1}_{\{\eta^k\}}$, we deduce that

$$\|\Pi(p_k)\zeta\|^2 = \langle S(p_k)\xi, \xi \rangle = K(m, k), \quad \|\Pi(p_k)\zeta'\|^2 = K(m-1, k), \quad k \in \mathbb{N}_0.$$

Since $\sum_k K(m, k) = 1 = \|\zeta\|^2$ and the projections $\Pi(p_k)$ are pairwise orthogonal, we have $\zeta = \sum_k \Pi(p_k)\zeta$, and similarly $\zeta' = \sum_k \Pi(p_k)\zeta'$. By Lemma 10.7, we have $x_{nn}^* p_0 = 0$ and $x_{nn}^* p_k = p_{k-1} x_{nn}^* p_k$ for $k \geq 1$. Consequently,

$$\langle S(x_{nn}^*)\xi, \xi' \rangle = \langle \Pi(x_{nn}^*)\zeta, \zeta' \rangle = \sum_{k \geq 1} \langle \Pi(x_{nn}^* p_k) \Pi(p_k)\zeta, \Pi(p_{k-1})\zeta' \rangle,$$

and, since $\|x_{nn}^* p_k\| \leq 1$, we obtain $|\langle S(x_{nn}^*)\xi, \xi' \rangle| \leq F_m \stackrel{\text{def}}{=} \sum_{k \geq 1} \sqrt{K(m, k)K(m-1, k-1)}$. As in the proof of Lemma 8.1, $F_m \rightarrow F(\mu)$ as $m \rightarrow \infty$. Since S is positive, we have $S(x_{nn}^*) = S(x_{nn})^*$, hence

$$\|S(x_{nn}) - x_{nn}\| = \|S(x_{nn}^*) - x_{nn}^*\| \geq |\langle (S(x_{nn}^*) - x_{nn}^*)\xi, \xi' \rangle| \geq (1 - \eta^m)^{\frac{1}{2}} - F_m.$$

Letting $m \rightarrow \infty$ gives the first inequality, and Lemma 8.2 gives the second one.

2. By Lemma 2.6, the rows of the transition matrix P_t of $e^{-tR_{\text{rad}}^\alpha}$ converge in $\ell_{\mathbb{N}_0}^1$ to the entrance law μ_t . Under J_n , the function $k \mapsto f(k)$ corresponds to $F(x_n)$ with $F(\eta^k) = f(k)$. Hence S_t satisfies the hypothesis of part 1 with $K = P_t$ and $\mu = \mu_t$, and $\sup_k \mu_{\rho^{-j}}(k) \geq \delta_T$ by Lemma 2.6.

3. By Lemma 10.1, we have $|\varphi_l^\eta| \leq 1$ on Y_η for any l . By Proposition 3.3, the operator T_t^η is given by a stochastic matrix whose rows converge in $\ell_{\mathbb{N}_0}^1$ to μ_t^η . Proposition 8.5, which holds for any $0 < \tau < 1$, applied with $\tau = \tau_n$, and part 1 give the estimate.

4. The map $S_t \stackrel{\text{def}}{=} (\text{Id} \otimes \omega_t) \circ \Delta$ is unital completely positive on $C(U_q(n))$, and $S_t(x_{nn}) = \sum_j x_{nj} \omega_t(x_{jn})$. Since the fundamental corepresentation is unitary, we have $\|x_{nj}\| \leq 1$, hence $\|S_t(x_{nn}) - x_{nn}\| \leq \sum_j |\omega_t(x_{jn}) - \delta_{jn}| \rightarrow 0$ as $t \rightarrow 0^+$. Choose $0 < t_1 \leq \Lambda_1(\eta)^{-1}$ such that this sum is less than $1 - \sqrt{1 - \delta_C(\tau_n, \eta)^2}$ for any $0 < t \leq t_1$. By part 3 with $\mathcal{A} = C(U_q(n))$, S_t does not coincide with $F(x_n) \mapsto (T_t^\eta F)(x_n)$ on $C^*(x_n)$ for these t . ■

Remark 10.9 Theorem 10.8 shows that the natural domain of the radial Taibleson and Casimir dynamics is $\mathbb{CP}_q^{n-1}$, for any $n \geq 2$. On $C(\mathbb{CP}_q^{n-1})$, the Casimir semigroup $(\mathcal{S}_t^{(n)})$ of Theorem 10.3 is strongly continuous. It is of the form $(\text{Id} \otimes \omega_t) \circ \Delta$ with $\omega_t = \mu_t^\eta \circ \mathbb{E}_K$, and $\omega_t(x_{nn}) = 0$ for $t > 0$ by (10.3), since x_{nn} is a coefficient of the fundamental corepresentation, which is not equivalent to some π_l . By part 3, no unital completely positive map on $C(S_q^{2n-1})$ with the same radial part is close to the identity on x_{nn} . By part 4, no weak* continuous convolution semigroup of states on $C(U_q(n))$, that is no Lévy process on $U_q(n)$ in the sense of Schürmann, see [CFK14], has the Casimir dynamics of $\mathbb{CP}_q^{n-1}$ as radial part. As in Section 8, the mechanism is the persistent atom of the entrance laws at the origin, which is incompatible with the shift x_{nn}^* .

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Bibliography

- [Arh26b] C. Arhancet. A bridge between analysis on local fields and quantum groups: the noncompact case. Preprint (2026). 8
- [AV91] I. Ya. Aref'eva and I. V. Volovich. Quantum group particles and non-archimedean geometry. Phys. Lett. B 268 (1991), no. 2, 179–187. doi:10.1016/0370-2693(91)90801-V. Section and equation numbering refers to CERN-TH.6137/91 (June 1991). 2
- [AK26] C. Arhancet and C. Kriegler. Functional calculus and semilinear evolution equations for the Taibleson operator on non-Archimedean local fields. J. Evol. Equ. 26 (2026), no. 2, Paper No. 63. doi:10.1007/s00028-026-01216-8. Equation numbering refers to arXiv:2407.10508v5. 9
- [AI84] R. Askey and M. E. H. Ismail. Recurrence relations, continued fractions, and orthogonal polynomials. Mem. Amer. Math. Soc. 49 (1984), no. 300. 42, 45, 53
- [Ass21] T. Assiotis. Infinite p -adic random matrices and ergodic decomposition of p -adic Hua measures. Preprint, arXiv:2009.04762v2 (2021).

- [AK04] N. M. Atakishiyev and A. U. Klimyk. On q -orthogonal polynomials, dual to little and big q -Jacobi polynomials. *J. Math. Anal. Appl.* 294 (2004), no. 1, 246–257. Preprint [arXiv:math/0307250](#). 45, 53
- [Ban99] T. Banica. Representations of compact quantum groups and subfactors. *J. Reine Angew. Math.* 509 (1999), 167–198. 73
- [BMT01] E. Bédos, G. J. Murphy and L. Tuset. Co-amenability of compact quantum groups. *J. Geom. Phys.* 40 (2001), no. 2, 130–153. [doi:10.1016/S0393-0440\(01\)00024-9](#). 73
- [BGP+14] A. Bendikov, A. Grigor’yan, C. Pittet and W. Woess. Isotropic Markov semi-groups on ultra-metric spaces. *Russian Math. Surveys* 69 (2014), no. 4, 589–680. [doi:10.1070/RM2014v069n04ABEH004907](#). 12
- [Bia90] P. Biane. Marches de Bernoulli quantiques. Séminaire de Probabilités XXIV, 1988/89, Lecture Notes in Math. 1426, Springer, Berlin, 1990, 329–344. 30
- [Bia91] P. Biane. Quantum random walk on the dual of $SU(n)$. *Probab. Theory Related Fields* 89 (1991), no. 1, 117–129. 36, 53
- [Bia92] P. Biane. Équation de Choquet–Deny sur le dual d’un groupe compact. *Probab. Theory Related Fields* 94 (1992), no. 1, 39–51. 28
- [Bia94] P. Biane. Théorème de Ney–Spitzer sur le dual de $SU(2)$. *Trans. Amer. Math. Soc.* 345 (1994), no. 1, 179–194. 28, 53
- [Bia03] P. Biane. La fonction zêta de Riemann et les probabilités. In: *La fonction zêta*, Éd. Éc. Polytech., Palaiseau, 2003, 165–193. 51
- [Bia06] P. Biane. Le théorème de Pitman, le groupe quantique $SU_q(2)$, et une question de P. A. Meyer. In *memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX*, Lecture Notes in Math. 1874, Springer, Berlin, 2006, 61–75. 5, 30, 35, 36, 53
- [Bia08] P. Biane. Introduction to random walks on noncommutative spaces. In: *Quantum potential theory*, Lecture Notes in Math. 1954, Springer, Berlin, 2008, 61–116. 28, 30
- [BBO05] P. Biane, P. Bougerol and N. O’Connell. Littelmann paths and Brownian paths. *Duke Math. J.* 130 (2005), no. 1, 127–167.
- [BPY01] P. Biane, J. Pitman and M. Yor. Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. *Bull. Amer. Math. Soc. (N.S.)* 38 (2001), no. 4, 435–465. 50, 51
- [BF93] L. Brekke and P. G. O. Freund. p -adic numbers in physics. *Phys. Rep.* 233 (1993), no. 1, 1–66. [doi:10.1016/0370-1573\(93\)90043-D](#). 3
- [CV22] M. Caspers and G. Vos. BMO spaces of σ -finite von Neumann algebras and Fourier–Schur multipliers on $SU_q(2)$. *Studia Math.* 262 (2022), no. 1, 45–91. [doi:10.4064/sm201202-18-6](#). Formula numbering refers to [arXiv:2011.10333v4](#). 21, 22
- [CFK14] F. Cipriani, U. Franz and A. Kula. Symmetries of Lévy processes on compact quantum groups, their Markov semigroups and potential theory. *J. Funct. Anal.* 266 (2014), no. 5, 2789–2844. [doi:10.1016/j.jfa.2013.11.026](#). Formula numbering refers to [arXiv:1210.6768v2](#). 6, 7, 21, 23, 65, 71, 81
- [CCSW24] H. S. Cohl, R. S. Costas-Santos and X.-S. Wang. Asymptotics, orthogonality relations and duality for the q and q^{-1} -symmetric polynomials in the q -Askey scheme. Preprint, [arXiv:2411.09040v1](#) (2024). 42, 45, 53
- [Col04] B. Collins. Martin boundary theory of some quantum random walks. *Ann. Inst. H. Poincaré Probab. Statist.* 40 (2004), no. 3, 367–384. [doi:10.1016/j.anihpb.2003.10.004](#). 36, 53
- [DS03] L. Dąbrowski and A. Sitarz. Dirac operator on the standard Podleś quantum sphere. In: *Noncommutative geometry and quantum groups* (Warsaw, 2001), Banach Center Publ. 61, Polish Acad. Sci. Inst. Math., Warsaw, 2003, 49–58. [doi:10.4064/bc61-0-4](#). Numbering refers to [arXiv:math/0209048v3](#).
- [DN98] M. S. Dijkhuizen and M. Noumi. A family of quantum projective spaces and related q -hypergeometric orthogonal polynomials. *Trans. Amer. Math. Soc.* 350 (1998), no. 8, 3269–3296. [doi:10.1090/S0002-9947-98-01971-0](#). 73, 74, 75
- [DR75] C. F. Dunkl and D. E. Ramirez. A family of countable compact P_* -hypergroups. *Trans. Amer. Math. Soc.* 202 (1975), 339–356. 10

- [DLMF] NIST Digital Library of Mathematical Functions. <https://dlmf.nist.gov/>, Release 1.2.8 of 2026-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl and M. A. McClain, eds. [13](#), [37](#), [40](#), [41](#), [42](#), [45](#), [46](#), [47](#), [48](#), [50](#), [75](#)
- [Fil09] J. A. Fill. The passage time distribution for a birth-and-death chain: strong stationary duality gives a first stochastic proof. *J. Theoret. Probab.* 22 (2009), no. 3, 543–557. [doi:10.1007/s10959-009-0235-5](#). [53](#)
- [Flo97] P. G. A. Floris. Addition formula for q -disk polynomials. *Compositio Math.* 108 (1997), no. 2, 123–149. Theorem numbering refers to [arXiv:math/9411231](#). [74](#)
- [FZ92] P. G. O. Freund and A. V. Zabrodin. Macdonald polynomials from Sklyanin algebras: a conceptual basis for the p -adics-quantum group connection. *Comm. Math. Phys.* 147 (1992), no. 2, 277–294. [doi:10.1007/BF02096588](#). [3](#), [8](#), [77](#)
- [FOT11] M. Fukushima, Y. Oshima and M. Takeda. Dirichlet forms and symmetric Markov processes. Second revised and extended edition. *De Gruyter Studies in Mathematics* 19, Walter de Gruyter, Berlin, 2011. [15](#)
- [GR04] G. Gasper and M. Rahman. Basic hypergeometric series. Second edition, *Encyclopedia of Mathematics and its Applications* 96, Cambridge University Press, Cambridge, 2004. [doi:10.1017/CBO9780511526251](#). [13](#), [37](#), [40](#)
- [GMZ12] Y. Gong, Y.-H. Mao and C. Zhang. Hitting time distributions for denumerable birth and death processes. *J. Theoret. Probab.* 25 (2012), no. 4, 950–980. [doi:10.1007/s10959-012-0436-1](#). Numbering refers to [arXiv:1007.1039v1](#). [49](#), [53](#)
- [Haa79] U. Haagerup. L^p -spaces associated with an arbitrary von Neumann algebra. In: *Algèbres d’opérateurs et leurs applications en physique mathématique* (Marseille, 1977), *Colloq. Internat. CNRS* 274, CNRS, Paris, 1979, 175–184. [62](#)
- [HJX10] U. Haagerup, M. Junge and Q. Xu. A reduction method for noncommutative L_p -spaces and applications. *Trans. Amer. Math. Soc.* 362 (2010), no. 4, 2125–2165. [doi:10.1090/S0002-9947-09-04935-6](#). [62](#), [63](#)
- [Haa06] M. Haase. The functional calculus for sectorial operators. *Operator Theory: Advances and Applications* 169, Birkhäuser, Basel, 2006. [17](#), [63](#)
- [Har93] S. Haran. Analytic potential theory over the p -adics. *Ann. Inst. Fourier (Grenoble)* 43 (1993), no. 4, 905–944. [3](#)
- [Har01] M. J. S. Haran. The mysteries of the real prime. *London Mathematical Society Monographs, New Series* 25, Oxford University Press, 2001. [3](#), [7](#), [10](#), [73](#)
- [Har08] S. M. J. Haran. Arithmetical investigations. Representation theory, orthogonal polynomials, and quantum interpolations. *Lecture Notes in Mathematics* 1941, Springer, Berlin, 2008. [doi:10.1007/978-3-540-78379-4](#). [3](#), [7](#), [73](#)
- [Izu02] M. Izumi. Non-commutative Poisson boundaries and compact quantum group actions. *Adv. Math.* 169 (2002), no. 1, 1–57. [doi:10.1006/aima.2001.2053](#). [4](#), [28](#), [30](#)
- [INT06] M. Izumi, S. Neshveyev and L. Tuset. Poisson boundary of the dual of $SU_q(n)$. *Comm. Math. Phys.* 262 (2006), no. 2, 505–531. [doi:10.1007/s00220-005-1439-x](#). [4](#), [28](#), [53](#)
- [Kai92] V. A. Kaimanovich. Measure-theoretic boundaries of Markov chains, 0–2 laws and entropy. In: *Harmonic Analysis and Discrete Potential Theory* (Frascati, 1991), *Plenum*, New York, 1992, 145–180. [57](#)
- [Kig10] J. Kigami. Dirichlet forms and associated heat kernels on the Cantor set induced by random walks on trees. *Adv. Math.* 225 (2010), no. 5, 2674–2730. [27](#)
- [KLS10] R. Koekoek, P. A. Lesky and R. F. Swarttouw. Hypergeometric orthogonal polynomials and their q -analogues. *Springer Monographs in Mathematics*, Springer, Berlin, 2010. [3](#), [13](#), [14](#), [18](#), [36](#), [70](#), [74](#)
- [KS96] R. Koekoek and R. F. Swarttouw. The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue. Preprint, [arXiv:math/9602214v1](#) (1996). [36](#), [40](#), [41](#), [42](#), [45](#), [52](#), [53](#)
- [Koe96] H. T. Koelink. Askey–Wilson polynomials and the quantum $SU(2)$ group: survey and applications. *Acta Appl. Math.* 44 (1996), 295–352. [24](#)

- [Koo89] T. H. Koornwinder. Representations of the twisted $SU(2)$ quantum group and some q -hypergeometric orthogonal polynomials. *Indag. Math.* 51 (1989), 97–117. [3](#), [24](#)
- [KO07] T. H. Koornwinder and U. Onn. LU factorizations, $q = 0$ limits, and p -adic interpretations of some q -hypergeometric orthogonal polynomials. *Ramanujan J.* 13 (2007), 365–387. [doi:10.1007/s11139-006-0258-9](#). Numbering refers to [arXiv:math/0405309v4](#). [3](#), [7](#), [10](#), [71](#), [73](#)
- [Let04] G. Letzter. Quantum zonal spherical functions and Macdonald polynomials. *Adv. Math.* 189 (2004), no. 1, 88–147. [doi:10.1016/S0001-8708\(03\)00353-0](#).
- [Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Second edition. Oxford Mathematical Monographs, Oxford University Press, New York, 1995. [2](#)
- [MMNNU91] T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi and K. Ueno. Representations of the quantum group $SU_q(2)$ and the little q -Jacobi polynomials. *J. Funct. Anal.* 99 (1991), 357–386. [3](#), [24](#)
- [Ner13] Yu. A. Neretin. Hua measures on the space of p -adic matrices and inverse limits of Grassmannians. *Izv. Math.* 77 (2013), no. 5, 941–953.
- [NT04] S. Neshveyev and L. Tuset. The Martin boundary of a discrete quantum group. *J. Reine Angew. Math.* 568 (2004), 23–70. [doi:10.1515/crll.2004.018](#). [28](#), [53](#)
- [NT13] S. Neshveyev and L. Tuset. *Compact Quantum Groups and Their Representation Categories*. *Cours Spécialisés*, vol. 20, Société Mathématique de France, Paris, 2013. [6](#), [21](#), [22](#), [23](#), [24](#), [57](#), [64](#), [73](#), [74](#), [75](#), [78](#)
- [Nor97] J. R. Norris. *Markov chains*. Cambridge Series in Statistical and Probabilistic Mathematics 2, Cambridge University Press, Cambridge, 1997. [48](#), [50](#)
- [Nou96] M. Noumi. Macdonald’s symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces. *Adv. Math.* 123 (1996), 16–77.
- [NYM93] M. Noumi, H. Yamada and K. Mimachi. Finite-dimensional representations of the quantum group $GL_q(n; \mathbb{C})$ and the zonal spherical functions on $U_q(n-1) \backslash U_q(n)$. *Japan. J. Math. (N.S.)* 19 (1993), no. 1, 31–80. [doi:10.4099/math1924.19.31](#). [7](#), [73](#), [74](#), [75](#), [79](#)
- [Onn06] U. Onn. From p -adic to real Grassmannians via the quantum. *Adv. Math.* 204 (2006), no. 1, 152–175. [doi:10.1016/j.aim.2005.05.012](#). [3](#)
- [PB09] J. Pearson and J. Bellissard. Noncommutative Riemannian geometry and diffusion on ultrametric Cantor sets. *J. Noncommut. Geom.* 3 (2009), no. 3, 447–480.
- [PX97] G. Pisier and Q. Xu. Non-commutative martingale inequalities. *Comm. Math. Phys.* 189 (1997), no. 3, 667–698. [doi:10.1007/s002200050224](#). [62](#)
- [Pod87] P. Podleś. Quantum spheres. *Lett. Math. Phys.* 14 (1987), no. 3, 193–202. [5](#), [25](#)
- [RS78] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press, New York, 1978. [44](#)
- [RP81] L. C. G. Rogers and J. W. Pitman. Markov functions. *Ann. Probab.* 9 (1981), no. 4, 573–582. [36](#)
- [SSV10] R. L. Schilling, R. Song and Z. Vondraček. *Bernstein functions. Theory and applications*. De Gruyter Studies in Mathematics 37, Walter de Gruyter, Berlin, 2010. [65](#)
- [SS98] M. Schürmann and M. Skeide. Infinitesimal generators on the quantum group $SU_q(2)$. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 1 (1998), no. 4, 573–598. [7](#), [71](#)
- [Ser80] J.-P. Serre. *Trees*. Springer, Berlin, 1980. [30](#)
- [Sim05] B. Simon. *Trace ideals and their applications*. Second edition. Mathematical Surveys and Monographs 120, American Mathematical Society, Providence, RI, 2005. [43](#)
- [Sze75] G. Szegő. *Orthogonal polynomials*. Fourth edition, American Mathematical Society Colloquium Publications 23, Providence, RI, 1975. [14](#), [44](#)
- [Tak03] M. Takesaki. *Theory of operator algebras. II*. *Encyclopaedia of Mathematical Sciences* 125, Springer, Berlin, 2003. [60](#)
- [Ter81] M. Terp. L^p spaces associated with von Neumann algebras. Notes, Mathematical Institute, Copenhagen University, 1981. [62](#)
- [Tes00] G. Teschl. *Jacobi operators and completely integrable nonlinear lattices*. Mathematical Surveys and Monographs 72, American Mathematical Society, Providence, RI, 2000. [52](#)

- [VS88] L. L. Vaksman and Ya. S. Soibelman. An algebra of functions on the quantum group $SU(2)$. *Funct. Anal. Appl.* 22 (1988), no. 3, 170–181. [3](#), [24](#)
- [Ven15] V. Venkateswaran. Symmetric and nonsymmetric Hall–Littlewood polynomials of type BC. *J. Algebraic Combin.* 42 (2015), no. 2, 331–364. [doi:10.1007/s10801-015-0583-4](#).
- [Wei74] A. Weil. *Basic number theory*. Third edition, Grundlehren der Mathematischen Wissenschaften 144, Springer, Berlin, 1974.
- [Woe12] W. Woess. On the duality between jump processes on ultrametric spaces and random walks on trees. Preprint, [arXiv:1211.7216](#) (2012). [27](#)
- [XZ21] Z. Xu and H. Zhang. From the Littlewood-Paley-Stein inequality to the Burkholder-Gundy inequality. Preprint, [arXiv:2111.05164v2](#) (2021). [6](#)
- [Zag08] D. Zagier. Elliptic modular forms and their applications. In: *The 1-2-3 of modular forms*, Universitext, Springer, Berlin, 2008, 1–103. [45](#), [47](#)

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