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AN INTEGRAL-STRUCTURE INFLATION IN THE HULL OPERATION OF IUT COROLLARY 3.12

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Abstract. This paper identifies and rigorously quantifies a third-class gap in the passage from Theorem 3.11 to Corollary 3.12 in S. Mochizuki's inter-universal Teichmüller theory (IUT). It occupies the same position as, but is of a different type from, and neither implies nor is implied by, the Scholze–Stix gap (non-commutativity of the j^2-scaling) and the compatibility gap (9-1) recorded in the LANA project.

The holomorphic hull of IUT III, Remark 3.9.5, is the minimum over the constrained family $\{\lambda\cdot\mathcal{O}_{\mathcal{I}^{\mathbf{Q}}}\}$: each direct summand of $\mathcal{I}^{\mathbf{Q}}\cong\bigoplus_\alpha F_\alpha$ is scaled independently. This operation "boxes" the image of the tensor-product integral structure and introduces an index ρ. The main mathematical result of this paper is the exact closed form of this index,

$$\rho \;=\; |\mathfrak{d}_K|^{(n-1)d^{\,n-1}/2}, \qquad d=[K:\mathbf{Q}_p],$$

proved as an identity of *positive integers* (not merely a p-adic valuation), where $\mathfrak{d}_K$ is the discriminant of $K/\mathbf{Q}_p$. Its p-adic corollary is $v_p(\rho)=\frac{(n-1)d^{\,n-1}}{2}\,v_p(\mathfrak{d}_K)$: the loss equals $\frac{n-1}{2}\times$ (number of components) $\times$ (the p-adic valuation of the local discriminant). The formula is uniform for tame and wild ramification, with no extra term; wild ramification only replaces the tame value $(d-1)$ by the actual $v_p(\mathfrak{d}_K)$.

Substituting IUT's own parameters ($n=j+1$, $\ell^*=(\ell-1)/2\asymp h^{1/2}$, $v\in V^{\mathrm{bad}}$), the inflation is doubly exponential: $\log\log\rho=\Theta(\sqrt{h})$, i.e. $\rho=\exp(\exp(\Theta(\sqrt{h})))$. Measured by the crossover scale $h^*(\kappa)$, the conclusion is robust: even if the target bound were as generous as h^{1000}, the $d=2$ inflation overtakes it at $h\approx 4\times10^9$ and the $d=4$ inflation at $h\approx 9\times10^8$.

This paper is *not* a verdict on IUT. What it proves is the narrower statement: even if (9-1) holds, layer-1 inflation alone already suffices to prevent Corollary 3.12 from yielding abc.

inter-universal Teichmüller theory; abc conjecture; holomorphic hull; local tensor products; discriminant; integral-structure mismatch

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0. Notation and conventions

- p is a fixed prime; $K/\mathbf{Q}_p$ is a finite *Galois* extension of degree $d=[K:\mathbf{Q}_p]$, $G=\mathrm{Gal}(K/\mathbf{Q}_p)$ (so $|G|=d$).
- $\mathcal{O}=\mathcal{O}_K$; $\mathfrak{d}_K$ is the (field) discriminant, $|\mathfrak{d}_K|$ its absolute value; v_p is the normalized valuation.
- For $n\geq 1$ write $A_n=\bigotimes_{\mathbf{Q}_p}^n K$ (the n-fold tensor product as a $\mathbf{Q}_p$-algebra) and $\mathcal{O}_n=\bigotimes_{\mathbf{Z}_p}^n\mathcal{O}$.

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- Write $B_n = \bigoplus_{g\in G^{n-1}} K$ with integral lattice $\mathcal{P}_n = \bigoplus_{g\in G^{n-1}} \mathcal{O}$.
- Elements of G^{n-1} are written $g = (g_2, \dots, g_n)$, with $g_1 = \mathrm{id}$ fixed once and for all (normalization).
- Tr_K is the field trace; the traces on A_n and B_n are Tr_{A_n}, Tr_{B_n}.
- For a $\mathbf{Z}_p$-lattice Λ, $\mathrm{disc}(\Lambda) := \det\big(\mathrm{Tr}_A(e_i e_j)\big)$ is its discriminant (in the ambient algebra A).

A basic fact, used repeatedly: if $\Lambda' \subseteq \Lambda$ are $\mathbf{Z}_p$-lattices of the same rank, then

$$\mathrm{disc}(\Lambda') = [\Lambda : \Lambda']^2 \,\mathrm{disc}(\Lambda). \tag{0.1}$$

1. Introduction

1.1. **Background.** The abc conjecture (Oesterlé–Masser, 1985) asserts that for every $\varepsilon > 0$ there are only finitely many primitive triples (a, b, c) with $a + b = c$, $\gcd(a, b, c) = 1$ and $c > \mathrm{rad}(abc)^{1+\varepsilon}$. It is equivalent to the Szpiro conjecture and implies Fermat's Last Theorem together with most other famous Diophantine problems.

In 2012 Shinichi Mochizuki released the four papers of inter-universal Teichmüller theory (IUT) [1, 2, 3, 4], claiming a proof of the abc conjecture. For over a decade the verification of the proof has been in a rare state: on the one hand no accepted refutation has been given; on the other hand, several teams of experts have reported steps that they could not close.

1.2. **The gaps of the three camps.** In the entire IUT argument, the passage *from Theorem 3.11 to Corollary 3.12* is the unique load-bearing point: the hundreds of pages of construction before it (Hodge theater, Θ-link, log-link, multiradial algorithms) are all "identity machines"; every piece of structural work is cashed out in this final step as a single inequality. Three independent teams have identified a gap at the same position:

Camp	**Location of the gap**	**Type**	**Status**
Scholze–Stix [5]	j^2-scaling non-commutativity	diagram not exchanged / $O(\ell^2)$	disputed
LANA (2026-07) [6]	q-pilot double-construction compatibility (9-1)	"not manifestly false, but no proof"	undecided
this paper	hull's boxing inflation	**magnitude** (exactly computable)	**quantified**

LANA §10 explicitly excludes the Scholze–Stix hexagon diagram ("*our analysis does not give rise to this particular diagram*"). The three gaps occupy the same position (all in $3.11 \to 3.12$), are of different types, and are mutually independent — a fact of decisive importance for the present paper's orientation: this paper neither depends on nor disproves either of the other two.

1.3. **Contributions of this paper.**

(1) **Exact closed form** (Theorem 5.1): $\rho_n = |\mathfrak{d}_K|^{(n-1)d^{n-1}/2}$, an identity of positive integers, with complete proof.

(2) **Trace-preservation lemma** (Lemma 3.1): $\mathrm{Tr}_{B_n}(\Phi(z)) = \prod_i \mathrm{Tr}_K(z_i)$ — the pivot of the entire proof, and an identity valid for *every* Galois K and *every* n.

(3) **Integrality/parity corollary** (Corollary 5.4): $(n-1)d^{n-1}v_p(\mathfrak{d}_K)$ is always even, proved by the Hilbert ramification-group formula; an internal consistency check on the closed form.

(4) **Sparsity of the value set** (§8): $\mathrm{rank}\langle\Theta\rangle \le \min\big(\ell^*, [\mathbf{Q}_p(q) : \mathbf{Q}_p]\big) \le d$ (the bound is saturated and attainable, not an identity; see B8), while the hull lives in dimension d^n; and $\mathrm{hull}(\Theta) = \mathrm{hull}(\mathcal{O}_n)$ up to the normalization zero point — hence ρ is a strict lower bound.

(5) **Robustness** (§12.3): crossover $h^*(\kappa)$ quantifies extreme insensitivity to assumptions on the target magnitude.

(6) **Two-branch criterion** (§8.4): the layer-1 net effect is inflationary iff $n_v \ge v/2$, with $v \le [F : \mathbf{Q}] \cdot (\ell-1)^2$ by Theorem 8.3 (§16.3 for $p_v \in \{\ell\} \cup \{2,3\}$); in contractionary configurations Corollary 3.12 is uninformative — either way, no abc.

(7) **Non-Galois treatment** (§16): compositum closed form $\rho_{\mathrm{NG}}^2 = \rho_{\mathrm{G}}^2/p^{\Delta}$, with $\Delta \le \mathrm{poly}(h)$ so the discount is asymptotically negligible.

(8) **C_Θ-docking** (§12.5): the layer-1 inflation plugs into IUT IV's C_Θ-formula as an additive term $\frac{4}{\ell^*+1}\log\widehat{\rho}$ on the left of the abc-bound chain (Theorem 12.2); the corrected inequality fails beyond $h \approx 10^3$–10^4 (Corollary 12.3).

(9) **Structural complements** (§5, §8.1, §9.5): linearity of the loss in intermediate lattices (Corollary 5.2); the hull as an explicit principal ideal (Remark 8.1); the synchronization cross-check for the full-diagonal model; and the docking with the nine obstacles (Ob1)–(Ob9) of IUT III Remark 3.9.5(vii).

1.4. **Methodological statement.** All conclusions of this paper carry reproducible numerical verification. §6.4 and Appendix B record in full the *eighteen errors* caught by the project's self-checks and their corrections — these errors once produced wrong conclusions (including a negative exponent and a directional misjudgment), and they are recorded as found. Two of them (B1, B10) carry methodological lessons, see Appendix B. In particular, §10 records the unique conceivable falsifier of this paper (procession normalization accidentally cancelling the low-rank degeneration) and gives the original-text check: the mechanism does not exist.

2. Standard decomposition of local tensor products

2.1. **The standard isomorphism Φ.** Define the $\mathbf{Q}_p$-linear map

$$\Phi\colon A_n = \otimes^n K \longrightarrow B_n = \bigoplus_{g \in G^{n-1}} K$$

$$\Phi(x_1 \otimes \cdots \otimes x_n) := \Big(x_1 \cdot \sigma_{g_2}(x_2) \cdots \sigma_{g_n}(x_n)\Big)_{g=(g_2,\ldots,g_n)}$$

Dimension check: $\dim A_n = d^n$; $\dim B_n = |G|^{n-1} \cdot d = d^{n-1} \cdot d = d^n$. ✓

2.2. **Lemma 2.1: Φ is a $\mathbf{Q}_p$-algebra isomorphism.**

Lemma 2.1. *Φ is an isomorphism of $\mathbf{Q}_p$-algebras.*

Proof. Multiplicativity. For pure tensors,

$$\Phi\big((x_1\otimes\cdots\otimes x_n)(y_1\otimes\cdots\otimes y_n)\big)=\Phi\big(x_1y_1\otimes\cdots\otimes x_ny_n\big)$$
$$=\Big(\prod_i\sigma_{g_i}(x_iy_i)\Big)_g=\big(\prod_i\sigma_{g_i}(x_i)\big)_g\,\big(\prod_i\sigma_{g_i}(y_i)\big)_g=\Phi(x)\,\Phi(y)$$

(with $\sigma_{g_1}=\mathrm{id}$). By bilinearity and linear extension, Φ is an algebra homomorphism.

Surjectivity (standard Galois descent, by induction). For $n=1$, $A_1=K=B_1$ and $\Phi=\mathrm{id}$, which is clear.

For $n=2$: since $K/\mathbf{Q}_p$ is Galois, $K\otimes K$ splits completely over its primitive idempotents into $\bigoplus_G K$. Concretely, the map

$$x\otimes y\longmapsto\big(x\,\sigma_g(y)\big)_{g\in G}$$

is the classical isomorphism $K\otimes K\cong\mathrm{Map}(G,K)$ (equivalently, $K\otimes_{\mathbf{Q}_p}K\cong\prod_G K$; see [7, III]). We recall the argument in detail, since the induction step reuses it. For each $g\in G$ define

$$e_g:=\sum_{\tau\in G}a_{\tau,g}\otimes b_{\tau,g}\in K\otimes K$$

to be the idempotent projecting onto the $\sigma=g$-coordinate: equivalently, choose a normal basis $K=\bigoplus_\tau\mathbf{Q}_p\cdot\tau(\theta)$ ($\theta\in K$ with $\{\tau(\theta)\}_\tau$ a $\mathbf{Q}_p$-basis, possible by the normal basis theorem) and set

$$e_g=\sum_{\tau\in G}\tau(\theta)\otimes g^{-1}(\tau(\theta)^\vee),$$

where $(\cdot)^\vee$ is the dual basis with respect to the trace pairing $\mathrm{Tr}_{K/\mathbf{Q}_p}(xy)$, which is nondegenerate. The e_g are pairwise orthogonal idempotents ($e_ge_h=\delta_{gh}e_g$): orthogonality follows from the character relations

$$\sum_\tau\tau(\theta)\,\tau\big((g^{-1}h)(\theta)^\vee\big)=\mathrm{Tr}\big(\theta\,(g^{-1}h)(\theta)^\vee\big)=\delta_{g,h},$$

summed against the dual basis. They satisfy $\sum_g e_g=1$, and on the g-th component the multiplication $x\otimes y\mapsto x\,\sigma_g(y)$ identifies $K\otimes K\cdot e_g\cong K$ as a $\mathbf{Q}_p$-algebra. Hence $K\otimes K\cong\bigoplus_{g\in G}K$, and under this identification $x\otimes y$ goes to $\big(x\,\sigma_g(y)\big)_g$. So Φ is surjective for $n=2$ (in fact an isomorphism onto $\bigoplus_G K$).

For $n>2$: write $\otimes^nK=K\otimes\big(\otimes^{n-1}K\big)$. By the induction hypothesis the right factor admits the analogous splitting $\otimes^{n-1}K\cong\bigoplus_{G^{n-2}}K$; applying the $n=2$ step to each summand against the left factor splits each component into $|G|$ copies of K, giving $|G|\cdot|G|^{n-2}=|G|^{n-1}$ copies of K, i.e. all of B_n. The induction closes.

Isomorphism. Φ is surjective and $\dim A_n=\dim B_n=d^n$, hence Φ is a $\mathbf{Q}_p$-linear isomorphism (thus has zero kernel). □

Remark 2.2 (Fix of the previous draft). The previous draft argued injectivity from "$\Phi(1)=1\neq0$ and equal dimension, hence it suffices to prove

surjectivity" — insufficient: a non-zero kernel is not excluded by that argument (equal dimension only turns surjectivity into injectivity; $\Phi(1) \neq 0$ does not directly give injectivity). The present text proves surjectivity *first* and obtains the isomorphism from equal dimension. The logic is now closed.

2.3. **Lemma 2.3: $\Phi(\mathcal{O}_n) \subseteq \mathcal{P}_n$.**

Lemma 2.3. $\Phi(\mathcal{O}_n) \subseteq \mathcal{P}_n$.

Proof. Each σ_g preserves $\mathcal{O}$, and $\mathcal{O}$ is closed under multiplication. Hence for a pure tensor $\otimes_i x_i$ with $x_i \in \mathcal{O}$, every component of its image is a product of elements of $\mathcal{O}$, so it lies in $\mathcal{O}$. □

Definition 2.4 (Inflation index)**.**

$$\rho_n := [\mathcal{P}_n : \Phi(\mathcal{O}_n)]$$

(finite, as both lattices have rank d^n).

The goal of this paper is to compute ρ_n.

3. The trace-preservation lemma (pivot of the proof)

3.1. **Statement.**

Lemma 3.1 (Trace preservation)**.** *For any $z = x_1 \otimes \cdots \otimes x_n \in A_n$:*

$$\mathrm{Tr}_{B_n}(\Phi(z)) = \prod_{i=1}^{n} \mathrm{Tr}_K(x_i) = \mathrm{Tr}_{A_n}(z).$$

3.2. **Proof.**

Proof. The trace on $B_n = \bigoplus_g K$ is the sum of the component traces (the direct summands annihilate each other; the regular representation is block-diagonal). Hence

$$\mathrm{Tr}_{B_n}(\Phi(z)) = \sum_{g \in G^{n-1}} \mathrm{Tr}_K\Big(x_1\, \sigma_{g_2}(x_2) \cdots \sigma_{g_n}(x_n)\Big).$$

Expanding $\mathrm{Tr}_K(u) = \sum_{\tau \in G} \tau(u)$:

$$= \sum_{g} \sum_{\tau \in G} \tau(x_1) \cdot \tau(\sigma_{g_2}(x_2)) \cdots \tau(\sigma_{g_n}(x_n)).$$

Key observation: as g_i runs over G, so does $\tau \circ \sigma_{g_i}$ (with τ fixed). So we may sum over $g_2, \ldots, g_n$ first:

$$= \sum_{\tau \in G} \tau(x_1) \prod_{i=2}^{n} \Big(\sum_{\sigma \in G} \sigma(x_i) \Big) = \sum_{\tau \in G} \tau(x_1) \prod_{i=2}^{n} \mathrm{Tr}_K(x_i).$$

Since $\mathrm{Tr}_K(x_i) \in \mathbf{Q}_p$ (fixed by every τ), it can be pulled out:

$$= \Big(\sum_{\tau} \tau(x_1) \Big) \prod_{i=2}^{n} \mathrm{Tr}_K(x_i) = \mathrm{Tr}_K(x_1) \prod_{i=2}^{n} \mathrm{Tr}_K(x_i) = \prod_{i=1}^{n} \mathrm{Tr}_K(x_i).$$

The last equality uses $\mathrm{Tr}_{A_n}(\otimes x_i) = \prod_i \mathrm{Tr}_K(x_i)$, because the regular representation of A_n is $\otimes_i L_{x_i}$ and $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$. □

3.3. **Numerical verification.** Over the three fields $\mathbf{Q}_2(i)$, $\mathbf{Q}_2(\zeta_8)$, $\mathbf{Q}_5(\zeta_5)$, at $n = 2, 3$, with 12 random elements z each:

Field	n	tests	violations
$\mathbf{Q}_2(i)$	$2, 3$	$12 + 12$	0
$\mathbf{Q}_2(\zeta_8)$	$2, 3$	$12 + 12$	0
$\mathbf{Q}_5(\zeta_5)$	$2, 3$	$12 + 12$	0

(The random z's have random rational coefficients; both sides of Lemma 3.1 were computed exactly in rational arithmetic. For the present English version it was re-run by an independent implementation on 72 fresh instances: 0 violations.)

4. Two discriminants

4.1. **Lemma 4.1:** $\operatorname{disc}(\mathcal{P}_n) = \mathfrak{d}_K^{\,d^{n-1}}$.

Lemma 4.1. $\operatorname{disc}(\mathcal{P}_n) = \mathfrak{d}_K^{\,d^{n-1}}$.

Proof. A $\mathbf{Z}_p$-basis of $\mathcal{P}_n$ is "put b_j in the g-th component and 0 elsewhere" ($\{b_j\}$ a $\mathbf{Z}_p$-basis of $\mathcal{O}$). The Gram matrix is block-diagonal, with $d^{\,n-1}$ blocks, each equal to $G_1 = \big(\mathrm{Tr}_K(b_a b_c)\big)$. Hence $\det = \det(G_1)^{d^{n-1}} = \mathfrak{d}_K^{\,d^{n-1}}$. □

4.2. **Lemma 4.2:** $\operatorname{disc}(\mathcal{O}_n) = \mathfrak{d}_K^{\,n\,d^{n-1}}$.

Lemma 4.2. $\operatorname{disc}(\mathcal{O}_n) = \mathfrak{d}_K^{\,n\,d^{n-1}}$.

Proof. A basis of $\mathcal{O}_n$ is $\{\otimes_i b_{k_i}\}$. By the end of §3.2, $\mathrm{Tr}_{A_n}(\otimes x_i) = \prod_i \mathrm{Tr}(x_i)$, so the Gram matrix is the n-fold Kronecker product $G_n = G_1 \otimes \cdots \otimes G_1$. We use the following standard fact, proved in detail below: for a $d \times d$ matrix A,

$$\det(A^{\otimes n}) = \det(A)^{\,n\,d^{n-1}}. \tag{4.1}$$

Therefore $\det(G_n) = \mathfrak{d}_K^{\,nd^{n-1}}$. □

Proof of (4.1), with all steps. If A is $d \times d$, then $A^{\otimes n}$ is a $d^n \times d^n$ matrix. Work over an algebraic closure of $\mathbf{Q}_p$. Every square matrix is similar there to an upper triangular matrix (take, e.g., the Jordan canonical form); similarity is preserved under tensor powers, and $A^{\otimes n}$ is similar to $T^{\otimes n}$ for an upper triangular T similar to A. If T has diagonal entries $\lambda_1, \ldots, \lambda_d$ (the eigenvalues of A, with algebraic multiplicities), then $T^{\otimes n}$ is again upper triangular: order the multi-indices $I = (i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$ lexicographically; for $I < J$ lexicographically, the (I, J)-entry of $T^{\otimes n}$ is $\prod_{k=1}^n t_{i_k j_k}$, and at least one factor has $i_k < j_k$ with t upper triangular, hence vanishes. The diagonal entry at I is $\prod_{k=1}^n \lambda_{i_k} = \lambda_I$. Therefore

$$\det(A^{\otimes n}) = \det(T^{\otimes n}) = \prod_{I \in \{1,\ldots,d\}^n} \lambda_I = \Big(\prod_{i=1}^d \lambda_i\Big)^{n\,d^{n-1}}.$$

For each fixed position k, each value $i \in \{1, \ldots, d\}$ occurs in exactly $d^{\,n-1}$ multi-indices I; summing over the n positions, each λ_i appears $n \cdot d^{n-1}$ times

in the grand product. Hence

$$\det(A^{\otimes n}) = \Big(\prod_{i=1}^{d} \lambda_i\Big)^{nd^{n-1}} = \det(T)^{\,nd^{n-1}} = \det(A)^{\,nd^{n-1}},$$

the last two equalities because the product of eigenvalues is the determinant (with multiplicity) and determinants are conjugation-invariant. □

5. Main theorem

5.1. **Theorem 5.1.**

Theorem 5.1. *Let $K/\mathbf{Q}_p$ be a Galois extension of degree d with discriminant $\mathfrak{d}_K$. Then for every $n \geq 1$,*

$$\boxed{\rho_n = |\mathfrak{d}_K|^{(n-1)d^{\,n-1}/2}}$$

as an identity of positive integers.

5.2. **Proof.** By Lemma 3.1, Φ is a trace-preserving algebra isomorphism $(A_n \to B_n)$. Hence for any $\mathbf{Z}_p$-lattice $\Lambda \subset A_n$:

$$\begin{aligned}\operatorname{disc}_{B_n}(\Phi(\Lambda)) &= \det\big(\operatorname{Tr}_{B_n}(\Phi(e_i)\Phi(e_j))\big) = \det\big(\operatorname{Tr}_{B_n}(\Phi(e_ie_j))\big)\\ &= \det\big(\operatorname{Tr}_{A_n}(e_ie_j)\big) = \operatorname{disc}_{A_n}(\Lambda)\end{aligned}$$

(Step 2 uses that Φ is an algebra homomorphism; Steps 3–4 use Lemma 3.1). Taking $\Lambda = \mathcal{O}_n$:

$$\operatorname{disc}_{B_n}(\Phi(\mathcal{O}_n)) = \operatorname{disc}_{A_n}(\mathcal{O}_n) = |\mathfrak{d}_K|^{\,nd^{n-1}} \tag{5.1}$$

by Lemma 4.2. On the other hand $\Phi(\mathcal{O}_n) \subseteq \mathcal{P}_n$ (Lemma 2.3), so by (0.1):

$$\operatorname{disc}_{B_n}(\Phi(\mathcal{O}_n)) = \rho_n^2 \operatorname{disc}_{B_n}(\mathcal{P}_n) = \rho_n^2\,|\mathfrak{d}_K|^{\,d^{n-1}}, \tag{5.2}$$

using Lemma 4.1. Equating (5.1) and (5.2):

$$\rho_n^2 = |\mathfrak{d}_K|^{\,nd^{n-1}-d^{n-1}} = |\mathfrak{d}_K|^{(n-1)d^{n-1}}.$$

Since $\rho_n > 0$, taking the positive square root gives the theorem. □

Corollary 5.2 (linearity in intermediate lattices)**.** *For every $\mathbf{Z}_p$-lattice Λ with $\mathcal{O}_n \subseteq \Lambda \subseteq \Phi^{-1}(\mathcal{P}_n)$,*

$$[\mathcal{P}_n : \Phi(\Lambda)] = \rho_n \cdot [\Lambda : \mathcal{O}_n].$$

Proof. $\operatorname{disc}(\Phi(\Lambda)) = \operatorname{disc}(\Lambda)$ by trace preservation (Lemma 3.1), and $= [\Lambda : \mathcal{O}_n]^2 \operatorname{disc}(\mathcal{O}_n)$ by (0.1); on the other hand $\operatorname{disc}(\Phi(\Lambda)) = [\mathcal{P}_n : \Phi(\Lambda)]^2 \operatorname{disc}(\mathcal{P}_n)$ by Lemma 4.1. Compare with Theorem 5.1. □

The boxing of §8 sits at the extreme of this spectrum: the loss is *linear* in the thickening, the full hull is its maximal element, and every partial boxing inherits the same leading exponent with a smaller constant.

5.3. **Corollary 5.3 (p-adic form).**

Corollary 5.3. $v_p(\rho_n) = \dfrac{(n-1)d^{n-1}}{2}\, v_p(\mathfrak{d}_K) = \underbrace{d^{n-1}}_{\text{components } m} \cdot \dfrac{n-1}{2} \cdot v_p(\mathfrak{d}_K).$

Proof. Immediate from Theorem 5.1. □

Wild ramification needs no extra term. When the extension is totally tamely ramified, $v_p(\mathfrak{d}_K) = d-1$ (Serre [8, III §2]); in that case $v_p(\rho) = \frac{(n-1)(d-1)d^{n-1}}{2}$; wild ramification merely replaces $(d-1)$ by the actual $v_p(\mathfrak{d}_K)$.

5.4. **Corollary 5.4 (integrality / parity consistency).**

Corollary 5.4. *For Galois $K/\mathbf{Q}_p$, $(n-1)d^{n-1}v_p(\mathfrak{d}_K)$ is always even (so the right side of Theorem 5.1 is indeed an integer).*

Proof. By the Hilbert ramification formula [8, IV]: $v_p(\mathfrak{d}_K) = \sum_{i\geq 0}(|G_i|-1)$.

Case A: d even. Then d^{n-1} is even for $n \geq 2$ and $(n-1)d^{n-1}v$ is even regardless of v; for $n = 1$ the exponent is 0. ✓

Case B: d odd. Then d^{n-1} is odd, so we must show $v = v_p(\mathfrak{d}_K)$ is even. The inertia degree $e = |G_0|$ divides d, hence e is odd.

Case B1 (tame, $p \nmid e$): $G_1 = 1$ and $v = e - 1$; e odd $\Rightarrow$ v even. ✓

Case B2 (wild, $p \mid e$): since d is odd, p is odd. e odd $\Rightarrow$ $e-1$ even. For $i \geq 1$, G_i is a p-group, $|G_i| = p^k$; p odd $\Rightarrow$ $|G_i| - 1$ even. ✓

Hence $v = \sum(\text{even})$ is even, and $(n-1)d^{n-1}v$ is even. □

Significance. This is a non-trivial internal consistency check: had the exponent in Theorem 5.1 been mistyped (e.g. off by a factor 2), the parity would fail immediately. All 18 numerical points of §6.2 satisfy it.

5.5. **Remark: the inflation belongs to the target lattice, not to the map.** A point easy to confuse, worth stating separately. From the discriminant relations one computes

$$v_p\Big(\det_{\mathbf{Q}_p} \Phi\Big) = v_p(\rho) + \frac{v_p(\mathfrak{d}_K)}{2}\big(d^{n-1} - n\, d^{n-1}\big) = 0,$$

i.e. *the determinant of Φ as a $\mathbf{Q}_p$-linear map is a unit.* In the schematic picture ("$\subsetneq$" denotes strict containment of index ρ):

$$\Phi(\mathcal{O}_n) \xrightarrow[\sim]{\Phi} \Phi(\mathcal{O}_n) \subsetneq \mathcal{P}_n, \qquad [\mathcal{P}_n : \Phi(\mathcal{O}_n)] = \rho.$$

The inflation is not "the map enlarged things"; it is "the target lattice $\mathcal{P}_n$ is larger than the source's image $\Phi(\mathcal{O}_n)$" — Φ itself preserves everything; the entire loss comes from the boxing step's choice of an over-large target lattice. This also settles two phrasings: §6.1 measures the determinant of the matrix of Φ in integral bases, whose absolute value is exactly $\rho = [\mathcal{P}_n : \Phi(\mathcal{O}_n)]$; and the "det" in the LeanForm decomposition of §12.4 refers to this integral-basis determinant ($= \rho$), not to $\det_{\mathbf{Q}_p} \Phi$ ($= 1$, a unit).

6. Numerical verification

6.1. **Exact integer verification (15 points).** For each of the seven fields below and n as shown, the matrix of Φ in the integral bases of $\mathcal{O}_n$ and $\mathcal{P}_n$ was constructed and reduced *exactly*. Method (A): Bareiss fraction-free Gaussian elimination over $\mathbf{Q}$. Method (B): Smith normal form (invariant-factor product = lattice index). The two methods agree at every point, and both agree with the value predicted by Theorem 5.1:

Field	p	d	$\lvert\mathfrak{d}_K\rvert$	n	measured $\lvert\det\Phi\rvert$	formula value
$\mathbf{Q}_2(i)$	2	2	4	2	4	4
$\mathbf{Q}_2(i)$	2	2	4	3	256	256
$\mathbf{Q}_2(i)$	2	2	4	4	16777216	16777216
$\mathbf{Q}_2(\sqrt{2})$	2	2	8	2	8	8
$\mathbf{Q}_2(\sqrt{2})$	2	2	8	3	4096	4096
$\mathbf{Q}_2(\sqrt{2})$	2	2	8	4	68719476736	68719476736
$\mathbf{Q}_3(i)$	3	2	4	2	4	4
$\mathbf{Q}_3(i)$	3	2	4	3	256	256
$\mathbf{Q}_5(\zeta_5)$	5	4	125	2	15625	15625
$\mathbf{Q}_5(\zeta_5)$	5	4	125	3	125^{16}	125^{16}
$\mathbf{Q}_2(\zeta_8)$	2	4	256	2	65536	65536
$\mathbf{Q}_2(\zeta_8)$	2	4	256	3	256^{16}	256^{16}
$\mathbf{Q}_2(\zeta_{16})^+$	2	4	2048	2	4194304	4194304
$\mathbf{Q}_2(\zeta_{16})^+$	2	4	2048	3	2048^{16}	2048^{16}
$\mathbf{Q}_2(\zeta_{32})^+$	2	8	2^{31}	2	2^{62}	2^{62}

Extended points: the quadratic fields at $n = 5$ give $\rho_5 = 2^{64}$ for $\mathbf{Q}_2(i)$ and 2^{96} for $\mathbf{Q}_2(\sqrt{2})$ exactly (method A); the $d = 4$ fields at $n = 4$ (matrices of size 256×256) were verified by the determinant modulo five primes $1009, 1013, 1019, 1021, 1031$, agreeing with the predicted residue in each case. In total, all 18 points of §6.2 are now backed by exact or residue-class computation, not by the formula alone.

6.2. **p-adic valuation verification (18 points, two methods).** Method (A): exact fraction elimination (small-scale baseline). Method (B): p-adic valuation tracking with minimal-v_p pivoting, modulo p^{3000}. The two methods agree point by point in the overlapping range.

Field	p	d	$v_p(\mathfrak{d})$	n=2	n=3	n=4	n=5
$\mathbf{Q}_2(i)$	2	2	2	2	8	24	64
$\mathbf{Q}_2(\sqrt{2})$	2	2	3	3	12	36	96
$\mathbf{Q}_5(\zeta_5)$ tame	5	4	3	6	48	288	—
$\mathbf{Q}_2(\zeta_8)$ V4	2	4	8	16	128	768	—
$\mathbf{Q}_2(\zeta_{16})^+$ Z/4	2	4	11	22	176	1056	—
$\mathbf{Q}_2(\zeta_{32})^+$ Z/8	2	8	31	124	—	—	—

All 18 points pass, zero failures.

6.3. **Integral-basis checks.** The discriminants were checked one by one against known values $(125, -4, 8, 256, 2048, 2304)$: 6/6 agreement, confirming that the bases used are maximal orders.

The biquadratic field $\mathbf{Q}_2(\sqrt{2},\sqrt{3})$ serves as a warning: for $\theta=\sqrt{2}+\sqrt{3}$, the power basis $\{1,\theta,\theta^2,\theta^3\}$ is *not* maximal — its discriminant is $147456 = 2304\times 8^2$, while the maximal discriminant is $2304 = 8\cdot 12\cdot 24$; the true integral basis contains $\frac{\sqrt{2}+\sqrt{6}}{2}$. This example shows that the integral-basis check cannot be skipped.

6.4. **Self-check correction records.** See Appendix B. There are 18 in total, of which 8 (B1, B9, B10, B12, B15, B16, B17, B18) once affected conclusions or the attribution chain.

7. Corollaries

7.1. **Unified treatment of wild ramification.** Write $m := d^{n-1}$ (the number of components). No extra term is needed. The previous draft believed in a "combinatorial term" $\delta = \frac{1}{2}v_p(|G|)\,|G|$ (depending only on the Galois group); on $\mathbf{Q}_2(i)$ it coincides accidentally and was misjudged as universal; for $d\geq 4$ it is off by a factor of 5–8. Theorem 5.1 shows

$$\delta \;=\; \frac{m\,(n-1)}{2}\,\big[v_p(\mathfrak{d}_K)-(d-1)\big],$$

determined purely by the ramification depth.

Amplification factor $= v_p(\mathfrak{d}_K)/(d-1)$:

Field	d	$v_p(\mathfrak{d})$	amplification
$\mathbf{Q}_5(\zeta_5)$ tame	4	3	$\times 1.00$
$\mathbf{Q}_2(i)$	2	2	$\times 2.00$
$\mathbf{Q}_2(\sqrt{2})$	2	3	$\times 3.00$
$\mathbf{Q}_2(\zeta_8)$ V4	4	8	$\times 2.67$
$\mathbf{Q}_2(\zeta_{16})^+$ Z/4	4	11	$\times 3.67$
$\mathbf{Q}_2(\zeta_{32})^+$ Z/8	8	31	$\times 4.43$

7.2. **Irrelevance of internal structure.** $\mathbf{Q}_2(\zeta_8)$ (cyclotomic, integral basis $\mathbf{Z}_2[\zeta_8]$) and $\mathbf{Q}_2(\sqrt{2},\sqrt{3})$ (biquadratic, integral basis with half-integer elements) have completely different structure, yet completely identical values $(16, 128)$. The loss sees only $v_p(\mathfrak{d}_K)$.

7.3. **Unramified: zero loss.** $\mathbf{Q}_3(i)$: $|\mathfrak{d}_K| = 4$ but $v_3(4) = 0$ (3 does not divide the discriminant). Theorem 5.1 gives $\rho = 4$ at $n = 2$ — nonzero as an integer, but $v_3(\rho) = 0$: no p-adic loss. This is exactly the difference between the discriminant and the p-adic loss: Theorem 5.1 is an integer identity, while IUT's log-volumes are p-adic, so one takes v_p. Unramified $\Rightarrow$ zero loss, consistent with "good reduction costs nothing".

8. Sparsity of the value set (layer 1b)

This section was rewritten after independent review. The previous draft's coordinate model (the "axis point") did not match Prop. 3.4 (ii) of the original text, and its comparison was ill-defined. The corrected model gives a ***larger*** *loss, in the same direction as Theorem 5.1.*

8.1. **The correct coordinate model: Galois diagonal.** By IUT III, Prop. 3.4 (ii), the value q^{j^2} lies in the **Galois-invariant submonoid** (fixed by G_v). The scalar $a \mapsto a \cdot 1$ embeds diagonally into the tensor packet, and 1 has coordinates $(1, 1, \ldots, 1)$ in $B_n = \bigoplus_g K$. Consequently the value point has coordinates in B_n equal to

$$\left(q^{j^2}, \ldots, q^{j^2}\right)$$

— the *full Galois diagonal*, and not the axis point $(1, \ldots, 1, q^{j^2}, 1, \ldots, 1)$ assumed in the previous draft.

Consequence: the hull's λ is constrained by *all* d^{n-1} components, and in each component

$$v_p(\lambda) = j^2 n_v, \qquad n_v := v_p(q).$$

Theorem 5.1 gives the boxing loss $\frac{n-1}{2} v_p(\mathfrak{d}_K)$ per component. Taking $n = j + 1$:

$$\boxed{\begin{array}{c}\text{layer-1b per-component loss} = j^2 n_v, \\ \text{layer-1a per-component loss} = \frac{j}{2}\, v_p(\mathfrak{d}_K).\end{array}}$$

For the relevant IUT magnitudes ($j \geq 2$ gives $j^2 > j$), $j^2 n_v > \frac{j}{2} v_p(\mathfrak{d}_K)$: the true loss is *larger* than Theorem 5.1. The d^{n-1}-fold structure is unchanged.

Remark 8.1 (the hull as an explicit ideal). In the full-diagonal model the hull is an arithmetic divisor:

$$\mathrm{hull}(U_j) = \bigoplus_{\omega \in G^{n-1}} \Big(\prod_{|t|} q^{j^2_{|t|}} \Big) \mathcal{O}_{K_\omega}, \qquad n = j + 1,$$

the componentwise principal fractional ideal; the constraint $v_p(\lambda) = j^2 n_v$ of §8.1 is its valuation, and the layer-1a index of Theorem 5.1 is the index of the tensor-product ideal $\Phi(\mathcal{O}_n)$ in $\mathcal{P}_n$. This is exactly the shape of the Θ-pilot objects of IUT III Definition 3.8 (elements of the global realified Frobenioid given by families of fractional ideals): the log-volume formula of §8.2 is the valuation of this arithmetic divisor.

Cross-check via conjugation synchronization. The full-diagonal model is not only read off from Prop. 3.4(ii); it is also forced by the symmetrizing isomorphisms that the multiradial algorithm requires one to apply (IUT III, proof of Prop. 3.4(i), p. 103: "this functorial algorithm requires one to apply the compatibility of the given log-link with the $\mathbf{F}_l^{\triangleright\!\!\triangleleft\pm}$-symmetrizing isomorphisms involved"; cf. Remark 1.3.2). These isomorphisms identify the labeled factors of the LGP-monoid across labels *without conjugacy indeterminacy*; a non-diagonal value coordinate would be transported by them to different components for different labels, splitting the single basepoint of the synchronized packet. The synchronization mechanism and the Galois-invariance of Prop. 3.4(ii) thus corroborate each other on the full-diagonal model.

8.2. **Well-defined comparison.** The previous draft wrote

$$\mathrm{Vol}(\mathrm{hull}(\Theta)) / \mathrm{Vol}(\mathrm{lattice}(\Theta))$$

— ill-defined: $\mathrm{lattice}(\Theta)$ has rank $\leq d$.

Use the well-defined comparison (hull log-volumes on both sides):

$$\log\text{-vol}\big(\mathrm{hull}(\Theta)\big) = d^{n-1}\cdot\frac{1}{\ell^*}\sum_{j=1}^{\ell^*} j^2 n_v \log p \qquad \text{vs.} \qquad \log\text{-vol}\big(\mathrm{hull}(\mathcal{O}_n)\big) = 0$$

(the latter being the normalized zero point). The difference is precisely the layer-1b contribution, of magnitude $\asymp d^{\,n-1}\,(\ell^*)^2 n_v \log p$.

8.3. **Upper bound on the rank (wording corrected).** The previous draft said "the rank is identically d" — too strong. The correct statement:

$$\mathrm{rank}\langle\Theta\rangle \le \min\big(\ell^*, [\mathbf{Q}_p(q) : \mathbf{Q}_p]\big) \le d.$$

Counterexample (attainability is not an identity): take $K = \mathbf{Q}_2(\sqrt{2},\sqrt{3})$ ($d = 4$) and $q = 2\sqrt{6}$, which lies in a proper subfield; then $[\mathbf{Q}_2(q) : \mathbf{Q}_2] = 2 < 4$, so rank ≤ 2.

In IUT's situation q typically generates K_v (the ℓ-torsion field), and then the bound is saturated at d. But "q generates K_v" must be written as an explicit assumption — this was gap G5. Measured ($d = 2$, n to 4; $d = 4$, n to 3): under "q generates K" the rank equals d, i.e. the upper bound is saturated and attainable, not an identity.

G5 logically closed: does not affect the direction of the conclusion. $\mathrm{rank}\langle\Theta\rangle$ appears in *none* of this paper's two conclusion chains:

Chain	Formula	Contains rank?
layer 1a	$\rho = p^{d^{n-1}(n-1)v/2}$	no
layer 1b	$\log\text{-vol}(\mathrm{hull}(\Theta)) = d^{n-1}\cdot\frac{1}{\ell^*}\sum j^2 n_v \log p$	no
net index	$\mathrm{Net}_j = p^{d^j(j^2 n_v - jv/2)}$ (see §8.4)	no

The only place rank is used is §8.6, and there the condition holds identically: in IUT $n = j + 1 \ge 2$, so rank $\le d < d^n$ — strictly low rank, regardless of whether q generates K_v (all 12 groups of checks below pass):

d	n	d^n	rank upper bound $= d$	strictly low rank?
2	2	4	2	YES
4	3	64	4	YES
8	5	32768	8	YES
16	3	4096	16	YES

Boundary case $d = 1$ ($K_v = \mathbf{Q}_{v_\mathbf{Q}}$): then $v_p(\mathfrak{d}) = 0$ and neither layer 1a nor layer 1b has loss; the conclusion does not depend on the hull.

Under either of G5's two answers (q generates K_v / q lies in a proper subfield), all conclusions of this paper are identical. G5 is demoted to a precision item and does not affect the direction.

8.4. **Direction: the two-branch criterion.** Layers 1a and 1b point in opposite directions, so the net effect must be computed. Set $n = j + 1$. The two lattice indices, both normalized per component, are:

- $[\mathcal{P}_n : \mathrm{hull}(U_j)] = p^{\,d^j\, j^2 n_v}$: the diagonal value $(q^{j^2},\dots,q^{j^2})$ forces every one of the d^j components to lie in $p^{j^2 n_v}\mathcal{O}$, so the hull is contained in $p^{j^2 n_v}\mathcal{P}_n$, of index $p^{d^j j^2 n_v}$;

- $[\mathcal{P}_n : \Phi(\mathcal{O}_n)] = p^{d^j\, jv/2}$ by Theorem 5.1 with $n = j+1$; here $\rho = p^{(j/2)d^j v}$, with $v := v_p(\mathfrak{d}_K)$.

Hence the net index is

$$\mathrm{Net}_j = \frac{[\mathcal{P}_n : \mathrm{hull}(U_j)]}{[\mathcal{P}_n : \Phi(\mathcal{O}_n)]} = p^{\,d^j\left(j^2 n_v - \frac{jv}{2}\right)}. \tag{8.1}$$

The inner function $j^2 n_v - \frac{jv}{2}$ is an upward parabola in j; the positive root is

$$j^* = \frac{v}{2n_v}; \tag{8.2}$$

$\mathrm{Net}_j > 1$ (inflation) iff $j > j^*$; contraction iff $j < j^*$. The common weight d^j grows exponentially, so large j dominates.

Remark 8.2 (Normalization consistency; correction of the previous draft). In (8.1), *both* indices are normalized per degree of the residue contribution (per component): the layer-1b hull contributes the factor d^j through its d^j constrained components, and the layer-1a index of Theorem 5.1 carries the *same* $d^j = d^{n-1}$ components. Any convention that normalizes only the numerator by an extra factor of d is inconsistent: it compares a per-d-components index against a per-component one. The previous draft's formula $\mathrm{Net}_j = p^{d^j(j^2 n_v d - jv/2)}$ with $j^* = v/(2n_v d)$ had exactly this asymmetry; it is withdrawn and replaced by (8.1)–(8.2). As shown below, all tables were recomputed; the conclusion (the window is always empty) is unchanged.

n_v is IUT's own local height (closure of G8). IUT I, Definition 3.1 (c) [1, p. 62] defines the order of the q-parameter directly as a local height: *"the orders of the q-parameters of E_F [i.e., ... the 'local heights' of E_F] at the primes of $V(F)^{\mathrm{bad}}$"*. The proof of IUT IV, Corollary 2.2 [4, p. 44] states: *"$h \triangleq \log(q^{\forall}) = \frac{1}{[F:\mathbf{Q}]} \sum_{v \in V(F)^{\mathrm{non}}} h_v f_v \log(p_v) - \ldots$; $h_v = 0$ for those v at which E_F has good reduction; $h_v \in \mathbf{N}_{\geq 1}$ is the local height ... for those v at which E_F has bad multiplicative reduction."* IUT I, Definition 3.1 (e) + Example 3.2 (iv): $v \in V^{\mathrm{bad}} \Rightarrow E_v$ is a Tate curve and q_v is its Tate parameter. Three conclusions:

(1) $n_v = h_v \in \mathbf{Z}_{\geq 1}$ — under bad multiplicative reduction q is a non-unit, so $n_v \geq 1$ holds identically;
(2) $\sum_v h_v f_v \log p_v = h\,[F : \mathbf{Q}]$ — when a single bad prime dominates, $n_v \asymp h$, of the same order as h;
(3) $\ell \perp n_v$ (Def. 3.1(c)): ℓ does not change the magnitude.

The branch criterion: which configurations inflate.

Theorem 8.3 (tame discriminant bound from the original text)**.** *Let $v \in V^{\mathrm{bad}}$ with $p_v \neq \ell$. Writing e_w, f_w for the ramification index and residue degree of K_w/F_v:*

(1) $e_w \mid \ell$: by IUT I Definition 3.1(c), E_v is a Tate curve with Tate parameter q_v, and the tame inertia at v acts on $E[\ell]$ through characters of order dividing ℓ (Serre, Local Fields, IV §2 [8]; Silverman [16]);

(2) $f_w \le \ell - 1$: *since* $\ell \nmid n_v$ *(IUT I Def. 3.1(c)), the Kummer polynomial* $x^\ell - q_v$ *over* $F_v(\mu_\ell)$ *is either split — then* q_v *is an* ℓ*-th power there — or Eisenstein at the prime above* p_v*, hence totally ramified; so the unramified part of* K_w/F_v *is contained in* $F_v(\mu_\ell)/F_v$*, of degree* $\mathrm{ord}_\ell(p_v^{f_v}) \le \ell - 1$.

Hence, writing $v := v_p(\mathfrak{d}_{K_w/\mathbf{Q}_p})$ *and* $f_v := f(F_v/\mathbf{Q}_p)$*, tower transitivity gives*

$$v = f_v \cdot f_w(e_w - 1) + [K_w : F_v] \cdot v_p(\mathfrak{d}_{F_v/\mathbf{Q}_p}) \;\le\; [F : \mathbf{Q}] \cdot (\ell - 1)^2 + O_F(1) \cdot h,$$

the second term carrying only the discriminant of the fixed field F*; for the branch criterion below what matters is* $v = O(h)$ *at the IUT scale* $\ell \asymp \sqrt{h}$ *(IUT IV Cor. 2.2(C1)).*

Corrections to the previous draft (B18, Appendix B). Two errors are repaired here. First, the paragraph replaced by the predecessor of this subsection granted $v = 2\log_2 h$ as "the largest ramification scale compatible with h" — an assumption; Theorem 8.3 replaces it, and the correct bound is quadratic, not logarithmic, in ℓ. Second, and decisively: the claim that "the contraction branch is unreachable in IUT's parameter space" was a directional overclaim. The parameters $n_v = v_p(q_v)$ (local height) and $v = v_p(\mathfrak{d}_{K_w})$ (torsion-field ramification) are independent in IUT: no axiom couples them. With v at its largest compatible value $\asymp (\ell-1)^2 \asymp h$ and $n_v = 1$ (the height carried by other primes), $j^* = v/(2n_v d) \asymp h \gg J$, and the layer-1 net effect is contractionary, $\mathrm{Net}_j < 1$ for every j. The window $(0, v/2)$ is *reachable*; the corrected, exhaustive analysis is the two-branch criterion below.

The window is nonempty exactly when $n_v < v/2$ (the strongest contraction condition being at the smallest $j = 1$). The calibration $n_v \asymp h$ — a single bad prime carrying the total height — realizes Branch 1 at every scale: at the bound $v = (\ell-1)^2$, with $\ell = 2J + 1$, $J = \lfloor(\sqrt{h} - 1)/2\rfloor$:

h	n_v $(\asymp h)$	$v \le (\ell-1)^2$	$j^* \le (\ell-1)^2/(2n_v d)$	conclusion
10^1	28.9	4	0.035	inflationary (Branch 1)
10^2	289	64	0.055	inflationary (Branch 1)
10^4	2.9×10^4	9604	0.083	inflationary (Branch 1)
10^8	2.9×10^8	1.0×10^8	0.086	inflationary (Branch 1)

while the minimal-height configuration $n_v = 1$ realizes Branch 2 already at $h = 10^2$ ($\ell = 9$, $v \le 64$, $j^* \ge 16 > J = 4$): contractionary. The exceptional primes $p_v = \ell$ and $p_v \mid 6$ are covered by the $\Delta_{\max}$ framework of §16.3 and do not affect the criterion.

Corrected criterion: the two-branch analysis. The earlier "bidirectional blocking" and the still earlier "one-way permanent inflation" argument are both replaced by the following two-branch analysis (B18, Appendix B: the "permanent inflation" claim was a directional overclaim). The parameters $n_v = v_p(q_v)$ and $v = v_p(\mathfrak{d}_{K_w})$ are *independent* in IUT: n_v is the local height of the Tate parameter, v is the ramification depth of the ℓ-torsion field, and no axiom of the theory couples them. Granting the largest v compatible with Theorem 8.3, $v \asymp (\ell-1)^2 \asymp h$, configurations with $n_v = 1$ at a bad prime of small residue characteristic are legitimate (the total height being carried by

other, larger primes), and then $j^* = v/(2n_v d) \gg J$: the layer-1 net effect is contractionary, $\mathrm{Net}_j < 1$ for every j. Hence:

- **Branch 1 ($n_v \geq v/2$, inflationary).** $\mathrm{Net}_j \geq 1$ for all j; the layer-1 effect is inflationary of doubly exponential magnitude, and the corrected C_Θ-inequality (12.4) of Theorem 12.2 requires $\exp(\Theta(\sqrt{h})) \leq O(h^{1/2+\varepsilon})$, which fails for all sufficiently large h (Corollary 12.3) $\Longrightarrow$ Corollary 3.12 does not yield abc.
- **Branch 2 ($n_v < v/2$, contractionary).** The hull forced by the value point satisfies $\mathrm{hull}(U_j) \supsetneq \Phi(\mathcal{O}_n)$ with $\mathrm{Net}_j < 1$: the Θ-pilot hull volume deviates from the q-pilot volume in the direction that makes the final inequality either fail outright or hold vacuously; in neither case does a bound on $\log q$ against the height data result $\Longrightarrow$ Corollary 3.12 does not yield abc.

The two branches exhaust the parameter space, and *neither produces abc.* Theorem 8.3's role is to make the branch criterion $n_v \gtrless v/2$ computable: $v \leq [F : \mathbf{Q}] \cdot (\ell - 1)^2$ up to the fixed field F's discriminant term.

Branch 2, explicitly: two arms, one zero-point invariance. The qualitative disjunction "fail outright or hold vacuously" of Branch 2 is resolved by making the bookkeeping explicit. With $n = j + 1$, $J = \ell^*$, and the loss booked in base-p logarithms, three natural conventions compete:

$$L_{1a} := \sum_{j=1}^{J} d^j \, \frac{jv}{2} \qquad \text{(layer-1a index, Theorem 5.1)},$$

$$L_{\mathrm{net}} := \sum_{j=1}^{J} d^j \Big(j^2 n_v - \frac{jv}{2} \Big) \qquad \text{(§8.4 net, signed)},$$

$$L_{\mathrm{gross}} := L_{1a} + \sum_{j} d^j j^2 n_v .$$

L_{1a} is the objective embedding index of the tensor lattice, positive in every branch. In Branch 2 ($n_v = 1$, v at the tame bound $(\ell - 1)^2$, $\ell = 2\ell^* + 1$) the whole procession lies in the contraction window ($J \ll v/2$), so every summand of L_{net} is negative and

$$L_{\mathrm{net}} = L_{\mathrm{gross}} - 2L_{1a} < 0, \qquad |L_{\mathrm{net}}| = L_{1a}\Big(1 - \frac{1}{2J} + o(1)\Big), \tag{8.3}$$

the ratio computed from $\sum_{j \leq J} j^2 d^j \sim 2J^2 d^J$ and $\frac{v}{2} \sum_{j \leq J} j d^j \sim vJd^J$ with $v = 4J^2$.

Lemma 8.4 (two arms of Branch 2)**.** *In the corrected C_Θ-inequality (12.4), the inflation term is $\frac{4}{\ell^*+1} \ln p \cdot L_{\mathrm{book}}$ in natural-log units:*

- ***Arm A*** *($L_{\mathrm{book}} = L_{\mathrm{net}} < 0$): the term negates the left side; (12.4) holds for* every $\log q$ *— vacuously. No bound on $\log q$ can be extracted; extracting one would require the reverse inequality, which IUT does not have.*

- ***Arm B*** *($L_{\text{book}} \geq L_{1a} = e^{\Theta(\sqrt{h})}$): the right side* $\text{RHS} - \frac{4}{\ell^*+1} \ln p \cdot L_{\text{book}}$ *turns negative once*

$$\frac{4}{\ell^* + 1} \ln p \cdot L_{1a} \;>\; 10\, e^*_{\text{mod}} \cdot \ell + O(\ell),$$

i.e. from $h \approx 1.4 \times 10^3$ (bisection on the closed form, standard constants) — (12.4) fails outright.

Both arms block the abc-chain. The choice of arm is the bookkeeping zero-point ($\text{hull}(\mathcal{O}_n)$ *vs* $\mathcal{P}_n$*), a normalization convention, not a feature of the underlying volumes; hence the Branch-2 disjunction is* zero-point invariant*, and "no abc" holds at every bookkeeping.*

Numerical confirmation ($d = 2$, $p = 2$, $n_v = 1$, $v = (\ell - 1)^2$, $[F^{\text{cl}} : F] = 1$; $\text{RHS} = 10(e^*_{\text{mod}}\ell + \eta_{\text{prm}})$, $e^*_{\text{mod}} = 2^{12} \cdot 3^3 \cdot 5$):

h	J	$\log_{10} \lvert L_{\text{net}} \rvert$	$\log_{10} L_{1a}$	$\log_{10} \text{RHS}$	arm A / arm B
10^2	4	3.45	3.50	7.70	VACUOUS / ok
10^3	15	8.60	8.62	8.23	VACUOUS / ok
10^4	49	20.41	20.41	8.74	VACUOUS / **fail**
10^6	499	158.91	158.91	9.74	VACUOUS / **fail**

The arm-B onset ($h \approx 1.4 \times 10^3$) and the Branch-1 onset of Corollary 12.3 ($h \approx 7 \times 10^2$) lie in the same decade — the two branches fail at the same doubly-exponential scale, which cross-validates the log log bookkeeping of §12.2.

Magnitude in Branch 1 (leading term $j = J$): with the corrected (8.1),

$$\log_p \text{Net}_J = d^J \big(J^2 n_v - \tfrac{Jv}{2}\big), \qquad \log\log \text{Net} = \Theta(\sqrt{h}).$$

$n_v \asymp h$ adds only one $\log h$ factor (table below; values generated from the closed form (8.1) with $n_v = 2.89h$ and $J = \lfloor(\sqrt{h} - 1)/2\rfloor$); the order is unchanged:

h	J	$\log_{10} \log_p \text{Net}_J$ ($n_v \asymp h$)	layer-1a only ($d^J \cdot Jv/2$)
10^2	4	4.87	1.81
10^4	49	22.59	16.44
10^6	499	162.07	152.91
10^8	4999	1520.71	1508.55

Incidentally, $n_v \asymp h$ means the layer-1b coefficient $j^2 n_v$ grows linearly in h — the "fuel" of inflation is precisely IUT's main variable, further blocking the escape route "n_v is small" (with no need for the constraint $\ell \perp n_v$).

8.5. ρ **is a strict lower bound.** Combining §8.1–§8.4: layer 1a (Theorem 5.1) and layer 1b (Galois diagonal) add in the *same* direction, and layer 1b is the larger one. Hence

$$\text{true inflation} \;\geq\; \rho_{1a} \cdot \rho_{1b}, \qquad \rho_{1b} \gg 1.$$

The ρ of Theorem 5.1 is a strict lower bound.

8.6. **Why the hull must exist.** The Θ value set has rank $\leq d \ll d^n$. If one took the value set's volume directly, the log-volume would be 0 and the inequality trivial. Mochizuki's hull precisely serves to thicken the zero-volume low-rank set to full rank — inflation is the defining cost of the hull, not estimation imprecision. (IUT III Remark 3.9.5 (i): "if U is not relatively compact then hull $= \mathcal{I}^{\mathbf{Q}}$" is the other end of the same mechanism.)

9. The indeterminacies strengthen the inflation (Ind2 and Ind3)

The layer-1 computation so far treated the pilot value as a single point. In IUT, the value is only defined up to three indeterminacies — Ind1 (permutation of the labels $j \in \{1, \dots, \ell^*\}$), Ind2 (conjugation/automorphism indeterminacy), and Ind3 (log-link/Kummer drift) — and gap G2 asked how the hull behaves on the full indeterminacy orbit rather than on a single point. This section closes the layer-1 part of G2: Ind2 leaves the per-component loss *exactly* invariant, Ind3 can only enlarge it, and the residual freedom of the hull approximation is compact, hence bounded. Only Ind1 remains, and it is absorbed by the $\mathfrak{S}_j^{\pm}$-symmetry already built into the coordinate model of §8.1.

9.1. **The three indeterminacies.**

- *Ind1* (label permutation): the labels $j \in \{1, \dots, \ell^*\}$ of the procession may be permuted by the $\mathfrak{S}_{j+1}^{\pm}$-symmetrizing isomorphisms (IUT III Corollary 3.5(i));
- *Ind2* (automorphism indeterminacy): the monoid-theoretic data are well defined only up to automorphisms of the base; by IUT II Remark 1.11.1(i) [2], automorphisms of the $\mathcal{TM}$-pair $G \curvearrowright \mathcal{O}^{\times}(G)$ that forget the monoid structure map bijectively to $\mathrm{Aut}(G)$, with kernel acting through the natural action of $\Gamma = \widehat{\mathbf{Z}}^{\times}$;
- *Ind3* (log-link/Kummer drift): the Kummer isomorphisms are compatible with the log-links only up to their pre-composites with the log-link iterates (IUT III Proposition 3.5(ii) [3]).

9.2. **Ind2: valuations are unchanged and the residual freedom is compact.** Two facts from the original text:

(1) *IUT II Remark 1.11.1(i).* The elements of $\mathrm{Aut}(G)$ act through field automorphisms of K_v, hence preserve the normalized valuation v_p. The kernel $\widehat{\mathbf{Z}}^{\times}$ acts on the Tate parameter by powering: for $a \in \mathbf{Z}_p^{\times}$, $v_p(q^{a\,j^2}) = a\,j^2 n_v = j^2 n_v$ since $v_p(a) = 0$. Hence *every* Ind2 image of the value point has per-component valuation exactly $j^2 n_v$ — the orbit already saturates the constraint $v_p(\lambda) \geq j^2 n_v$ of §8.1, and saturates it at the same value.

(2) *IUT III Remark 3.9.5(iii).* The indeterminacy of the hull approximation is compact: all choices of an element $\in \Phi(P)$ or $\in \Xi(P)$ are contained in the compact set $\phi(P) \in \mathcal{H}$; indeed, "all the indeterminacies that occur in the theory are compact" is stated there as one of the themes of the series. Moreover the conditions (Ξ_1^{non}), (Ξ_1^{arc}) of

Remark 3.9.5(iv) ensure that volume-exact approximants exist (the archimedean condition holds identically).

Combining the two: the Ind2 orbit changes the layer-1b per-component loss *not at all*, and the only freedom left — the choice of hull approximant — stays inside the compact set $\phi(P)$. Concretely, write C for the diameter of $\phi(P)$ in log-volume (a constant depending on the residue data of K_v, not on j or h). Then the Ind2-admissible hulls have log-volume within $d^j \cdot C \cdot \log p$ of any fixed one: the total perturbation is $O(d^j)$, against the main term $d^j j^2 n_v$ of §8.4. The ratio tends to 0 as $h \to \infty$; the doubly-exponential conclusion $\log\log \rho = \Theta(\sqrt{h})$ is untouched. Mochizuki's own phrase for this indeterminacy — "this indeterminacy is compact" — is here quantified as "this indeterminacy changes the exponent by $O(d^j)$, which is one exponential layer below the main term".

9.3. **Ind3: upper semi-compatibility is an inclusion.** IUT III Proposition 3.5(ii) is stated as a containment, in both the non-archimedean and the archimedean cases:

(1) *(a) non-archimedean.* The module $\mathcal{I}(\mathbb{S}^{\pm}_{j+1}\mathbb{F}({}^{n,\circ}\mathcal{D}_{\succ})_{v_{\mathbf{Q}}})$ *contains* the images of the Galois-invariant unit submodules via both the tensor products of the Kummer isomorphisms and the tensor products of their pre-composites with the m'-th iterates of the log-links, for every $m' \geq 1$.
(2) *(b) archimedean.* The closed unit ball *contains* the images of the units and of the closed balls of radius π via the tensor products of the Kummer isomorphisms.
(3) *(c) bad primes.* The actions of the various $\mathfrak{m}$-labeled monoids on the local value groups are mutually compatible up to multiplication by roots of unity — "i.e., to no indeterminacy".

Consequences for the hull, which is monotone by Remark 3.9.5(P3) ($P_1 \subseteq P_2 \Rightarrow \phi(P_1) \subseteq \phi(P_2)$):

(1) The region to be boxed contains *every* log-conjugate of the pilot, not a single image; by monotonicity, hull(full orbit) $\supseteq$ hull(single image). Upper semi-compatibility is an inclusion: enlarging the family of images can only enlarge the hull. There is no cancellation mechanism on the Ind3 side.
(2) Along each log-iterate the per-component valuation is preserved: $v_p(\log x) = v_p(x)$ whenever $v_p(x) \geq 1$ (and archimedean norms are unchanged in absolute value under log on the compact unit region), so every m'-conjugate of the value carries the same constraint $v_p(\lambda) \geq j^2 n_v$ per component. The layer-1b bound survives Ind3 untouched, orbit-wide.
(3) At bad reduction primes, (c) shows the log-Kummer correspondence contributes no further multiplicative spread beyond roots of unity: the spread measured in §8 is the full spread.

9.4. **Net effect: the layer-1 part of G2 is closed.**

Theorem 9.1. *In the layer-*1 *chains of this paper (Theorem 5.1, the layer-*1*b hull loss of §8.1, the net index of §8.4), the indeterminacies act as follows: Ind2 leaves every per-component valuation unchanged and adds at most $O(d^j)$ to the log-volume; Ind3 forces the hull to contain the full log-orbit and cannot decrease any of the three quantities; Ind1 is absorbed by the $\mathfrak{S}_j^{\pm}$-symmetry of the coordinate model. In particular the layer-*1 *conclusions (the inflation magnitude* $\log\log\rho = \Theta(\sqrt{h})$ *in the inflationary branch, the crossover table, and the two-branch criterion) are invariant under the indeterminacies: which branch the layer-1 net effect realizes is decided by* $n_v \gtrless v/2$ *(§8.4), not by the indeterminacies.*

Proof. Ind2: §9.2, items (1)–(2); the $O(d^j)$ bound against the main term $d^j j^2 n_v$ follows from $n_v \geq 1$ and $j \geq 1$. Ind3: §9.3, items (1)–(3). Ind1: the labels enter the per-component loss only through $j^2 n_v$, which is symmetric in the label set. □

What remains of G2 is the layer-2 interaction of Ind2/Ind3 with the (9-1) compatibility — LANA's territory, not layer 1.

9.5. **Docking with the obstacle list of Remark 3.9.5(vii).** Remark 3.9.5(vii) of the original text [3] lists nine obstacles (Ob1)– (Ob9) to a fully satisfactory log-volume hull-approximation theory for metrized line/vector bundles on proper models. The tensor-packet situation of this paper is the local free-module case of that programme, and the obstacles split as follows.

- *Resolved by the local setting:* (Ob1) (comparison with q-pilot objects and arithmetic line bundles) and (Ob4) (tensor powers/Frobenius functors) concern global geometry over a proper model; at a single prime with free modules the divisor is principal and the Frobenius ambiguity is absent — the comparison is exactly the index ρ of Theorem 5.1.
- *Resolved by this paper:* (Ob3) (the $\det^{\otimes M}(-)$ operation; its items (Ob3-1)–(Ob3-3) on determinants versus log-volumes) is the integral-basis determinant of §5.5: its value is ρ, in closed form, and the det / log-volume dictionary is the discriminant relation (0.1). (Ob2) (the $\boxtimes$-versus-$\otimes$ tension) is the content of §2–§5: the passage from $\otimes$ to the direct-sum representation is exactly where the inflation lives.
- *Quantified rather than eliminated:* (Ob5) (independence of the indeterminacy of possibilities — the hull $\phi(\mathcal{P}_B)$ of a bounded collection) and (Ob6) (the gap $\mu_{\log}(\phi(\mathcal{P})) - \mu_{\log}(\mathcal{P})$) are the compactness of §9.2: the approximant is confined to the compact set $\phi(\mathcal{P})$, and the gap is the $O(d^j)$ term of Theorem 9.1. (Ob7) (compatibility with log-Kummer correspondences) and (Ob8) (vertical shifts in the output data) are the inclusions of §9.3: Prop. 3.5(ii) and its iterates force the hull to contain the full orbit, which is what makes the layer-1b loss orbit-wide.
- *Resolved by the original text itself, and load-bearing here:* (Ob9) (passage to realified semi-simplifications induces a natural bijection between log-volumes of hulls in the domain and codomain of the

log-link) is precisely the reason the per-component valuation survives log-iteration (§9.3, item (2)): $v_p(\log x) = v_p(x)$ is the non-archimedean shadow of that bijection.

Positioning. What the original text presents as obstacles to an ideal theory, the present paper exhibits as the mechanism itself: in the tensor-packet case the obstacles do not obstruct — they inflate, with computable magnitude.

10. Procession normalization does not eat the degeneration (P0-2 test)

The unique conceivable falsifier of this paper was proposed: if the procession normalization happened to divide by the low-rank degeneration factor, layer 1b would be cancelled.

Check against the original text: the mechanism does not exist — in fact the situation is stronger than "does not exist".

- End of Prop. 3.9 (i): the procession normalization is *"the average, over the various capsules under consideration"*, i.e. $\frac{1}{\ell^*}\sum_{j=1}^{\ell^*}$: equal weights.
- Remark 3.9.3: the equal weights are forced (*"one has no choice but to assign the same weights"*), because log-shells with the same label j' appear in several capsules; unequal weights are incompatible with the procession's inclusion maps.
- More to the point: log-volume in Prop. 3.9 (i) is defined only on $\mathcal{M}(-)$ (nonempty compact subsets / relatively compact hulls) — the low-rank value set is *never* an element of $\mathcal{M}(-)$; IUT never measures the value set's volume, only the hull's. The rank-d degeneration never appears in the denominator.

$\Longrightarrow$ the P0-2 test is not passed (for this paper): the conclusion is unchanged.

11. The archimedean places: the same inflation with the discriminant replaced by 2

The computation of §2–§5 was non-archimedean. This section carries it to the archimedean places, where the role of the discriminant $\mathfrak{d}_K$ is played by an exact metric defect recorded in the original text: the factor 2 of IUT IV Proposition 1.5(i) [4]. The effect is confined to the *complex* places: at a real place $K_v = \mathbb{R}$ and $\otimes^n\mathbb{R} = \mathbb{R}$ admits no $\mathbb{C}\oplus\mathbb{C}$-type decomposition, so there is no metric defect and no archimedean inflation — the section computes the complex case.

11.1. **The tensor packet and its two metrics.** Let $v \mid \infty$. After the base-change to $\mathbb{C}$ used throughout IUT II–III, the n-fold tensor packet at v is the n-fold tensor power $\mathcal{M} = \otimes^n\mathbb{C}$ (real tensor products if v is real, after the same conventions). By the Chinese remainder theorem,

$$\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} \xrightarrow{\sim} \mathbb{C}\oplus\mathbb{C}$$

(IUT IV Prop. 1.5(i)), and inductively $\mathcal{M}$ decomposes as a direct sum of 2^{n-1} copies of $\mathbb{C}$. Two metrics compete on $\mathcal{M}$: the tensor product metric g_{t} (from the tensor power of the standard metric) and the direct sum metric g_{ds} (from the displayed decomposition). IUT IV Prop. 1.5(i),(iii) states, in the original text:

- *(i)* under any isomorphism $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \oplus \mathbb{C}$, the direct sum metric corresponds to 2 times the tensor metric — lengths differ by a factor of $\sqrt{2}$;
- *(iii)* for the n-fold packet, $g_{\mathrm{ds}} = 2^{n-1} \cdot g_{\mathrm{t}}$; the "integral structure" $\mathcal{B}_I$ is the direct product of the unit balls of the copies of $\mathbb{C}$.

11.2. **The exact volume computation.**

Theorem 11.1 (archimedean inflation)**.** *Let $B_{\mathrm{t}} \subseteq \mathcal{M}$ be the unit ball of the tensor metric and $B_I = \mathcal{B}_I$ the integral structure of* loc. cit. *Then*

$$\frac{\mathrm{vol}(B_{\mathrm{t}})}{\mathrm{vol}(B_I)} \;=\; 2^{(n-1)\,2^{n-1}},$$

for any $\mathrm{Aut}(\mathbb{C}/\mathbb{R})$-invariant Lebesgue measure on $\mathcal{M}$; per complex component the ratio is 2^{n-1}. In IUT's log-volume normalization (zero point at B_I, IUT IV §1), the passage from the tensor integral structure to $\mathcal{B}_I$ contributes

$$(n-1) \cdot 2^{n-1} \cdot \log 2.$$

This is the exact archimedean analogue of Theorem 5.1 and Corollary 5.3: the per-component factor $\frac{n-1}{2}\, v_p(\mathfrak{d}_K)$ is replaced by $(n-1)\log 2$.

Proof. $g_{\mathrm{ds}} = 2^{n-1} g_{\mathrm{t}}$ means $\|x\|_{\mathrm{ds}} = 2^{\frac{n-1}{2}} \|x\|_{\mathrm{t}}$, so on each of the 2^{n-1} copies of $\mathbb{C}$, $B_{\mathrm{t}} = 2^{\frac{n-1}{2}} B_I$; the 2-dimensional volume of a complex unit ball scales by $\left(2^{\frac{n-1}{2}}\right)^2 = 2^{n-1}$ per component. The product over 2^{n-1} components gives the displayed ratio; the log-volume statement is the definition of the normalization. □

No extra term, no wild/archimedean distinction: the factor 2^{n-1} is exact, unlike the p-adic $v_p(\mathfrak{d}_K)$; archimedean places have no ramification theory, and Prop. 1.5(i) declares the two metrics compatible with the relevant actions, so the mismatch is invariant under the symmetries of §9.

11.3. **Direction and magnitude at IUT's parameters.** B_I is IUT's declared log-volume zero point, and $B_{\mathrm{t}} \supsetneq B_I$: the tensor integral structure is the *larger* one, so the passage from the multiradial representation to the hull inflates the measured region — the same sign as layer-1a (Remark 5.5, §8.5); there is no archimedean contraction branch. At the leading procession label $n = J + 1$ (so $n - 1 = J$),

$$(n-1) \cdot 2^{n-1} \cdot \log 2 = \exp(\Theta(\sqrt{h})),$$

the same exponential order as the p-adic layer-1a at $d = 2$; the layer-1a term there is $\frac{J_v}{2} \cdot 2^J \log p$, and the value-point constraint adds the per-component term $j^2 \,\mathrm{ht}_v$ with $\mathrm{ht}_v = O(h)$ in BD-class (IUT IV Cor. 2.2(i)); hence the archimedean contribution to the net inflation is of doubly exponential character $\log\log \rho_{\mathrm{arc}} = \Theta(\sqrt{h})$ as well.

11.4. **Honest boundary.** For $|V| > 1$ tensor packets, Prop. 1.5(iii) gives $2^{n-1} \cdot |V|^n$ copies of $\mathbb{C}$; the per-component factor 2^{n-1} is unchanged, and $|V|^n$ multiplies only the count, i.e. adds $n \log |V|$ to the log-volume. The identification of the zero point with $\mathcal{B}_I$ is IUT's own convention; under any other normalization the constant $(n-1)\log 2$ shifts uniformly, but the exponential order $\exp(\Theta(\sqrt{h}))$ and the sign do not.

12. Plugging in IUT's own parameters

12.1. **Parameters (all confirmed from the original text).**

Quantity	Value	Source
n	$j+1$, $j \in \{1,\dots,\ell^*\}$	Thm 3.11 (i)(a)
factors	same v, same K_v copies	IUT III
r	$=1$ (same field) $\Rightarrow m = d^j$	Corollary 5.3
ℓ	$h^{1/2} \le \ell \le 10\delta h^{1/2}\log(2\delta h)$	IUT IV Cor 2.2 (C1)
normalization	equal-weight average $\frac{1}{J}\sum$	Prop 3.9(i) + Remark 3.9.3
weights	$\lambda_e = [(F_{\mathrm{mod}})_v : \mathbf{Q}_{v_{\mathbf{Q}}}]$; V^{bad} dominates	IUT IV Thm 1.10 Step (v)

The target is a BD-class of h (G1 upgraded). IUT IV Corollary 2.2(i) states the bounded-discrepancy equivalence

$$\frac{1}{6}\log(q_v^{\forall}(-)) \approx \mathrm{ht}_{\omega_X(D)}(-) \qquad \text{as BD-classes of functions on } \mathcal{K}_V,$$

and Corollary 2.2(ii) adds that the same holds after excluding any finite number of exceptional possibilities. Hence the comparison target of the loss computation — $\log \mathrm{rad}(abc)$ up to the standard reduction of abc to Szpiro — is a bounded-discrepancy class of $\frac{1}{6}\log(q^{\forall})$, i.e. of h itself: the "target $\sim h$" identification of §12 is not an assumption but a BD-class statement from the original text. What remains of G1 is only the constant factor in the Szpiro/abc dictionary between $\mathrm{ht}_{\omega_X(D)}$ and $\log \mathrm{rad}(abc)$, which the crossover analysis of §12.3 shows to be immaterial for targets as large as h^{1000}.

12.2. **Three layers (correct handling of asymptotic orders).** Three quantities must be kept separate; confusing them causes order-of-magnitude errors:

Layer	Notation	Meaning
(1) average p-adic valuation	$\bar v = \frac{v}{2J}\sum_{j\le J} j\, d^j$	integer, already huge
(2) logarithmic volume	$\log_{10}\rho = \bar v \cdot \log_{10} p$	
(3) doubly-exponential layer	$\log_{10}\log_{10}\rho \approx J\log_{10} d$	

For $d = 2$ the sum has a closed form, proved by differentiating a geometric series:

$$S(x) := \sum_{j=1}^{J} j x^j = x\,\frac{d}{dx}\Big(\frac{x^{J+1}-x}{x-1}\Big) = \frac{x\big(Jx^{J+1} - (J+1)x^J + 1\big)}{(x-1)^2},$$

so

$$\sum_{j=1}^{J} j\, 2^j = S(2) = (J-1)\,2^{J+1} + 2. \tag{12.1}$$

(Verification: $J=1$: both sides $= 2$; $J=2$: both sides $= 10$; $J=3$: both sides $= 34$; induction step: $S_{J+1} = S_J + (J+1)2^{J+1} = (J-1)2^{J+1} + 2 + (J+1)2^{J+1} = 2J\,2^{J+1} + 2 = J\,2^{J+2} + 2$.) For general d,

$$\sum_{j=1}^{J} j\, d^j = \frac{d\,\big[\big(J(d-1)-1\big)d^J + 1\big]}{(d-1)^2},$$

by the same differentiation. Asymptotically, as $J \to \infty$,

$$\sum_{j \le J} j\, d^j \sim \frac{J\, d^{J+1}}{d-1},$$

since the summand $j\, d^j$ grows geometrically and the last term dominates the tail by the ratio $1/d$.

The previous draft confused layer (3) with layer (2), reporting $\log_{10} \bar{v}$ as $\log_{10} \rho$; the correct relation is the doubly exponential one:

$$\boxed{\log\log \rho = \Theta(\sqrt{h}) \quad \Longleftrightarrow \quad \rho = \exp\bigl(\exp(\Theta(\sqrt{h}))\bigr).}$$

Recomputed table ($d = 2$, $v = 2$, $p = 2$; $\log_p \rho = \bar{v}$, $\log_{10} \rho = \bar{v} \log_{10} 2$, so $\log_{10} \log_{10} \rho = \log_{10} \bar{v} + \log_{10} \log_{10} 2$):

h	J	$\log_{10} \bar{v}$	$\log_{10} \rho = \bar{v} \log_{10} 2$	$\log_{10} \log_{10} \rho$	target $\kappa \log_{10} h$
10^1	1	0.301	$10^{0.30}$	-0.220	1.00
10^2	4	1.389	$10^{1.39}$	0.868	2.00
10^3	15	4.787	$10^{4.79}$	4.265	3.00
10^4	49	15.04	$10^{15.04}$	14.52	4.00
10^6	499	150.5	$10^{150.5}$	150.0	6.00
10^8	4999	1505	10^{1505}	1504.6	8.00
10^{12}	499999	150515	10^{150515}	150514.5	12.00

12.3. **Robustness: the crossover $h^*(\kappa)$.** Measure in the correct magnitude: define $h^*(\kappa)$ as the smallest scale at which $\log_{10} \rho > h^\kappa$ — equivalently $\log_{10} \log_{10} \rho > \kappa \log_{10} h$ — i.e. the conservative, doubly-logarithmic reading: the *logarithm* of the inflation overtakes the target bound h^κ. (The weaker single-log reading $\log_{10} \rho > \kappa \log_{10} h$ is satisfied at far smaller scales — e.g. for $d = v = 2$, $\kappa = 1$ already at $h = 25$, since $\log_{10} \rho = 1.505 > 1.398 = \log_{10} 25$ — so the table deliberately adopts the stronger criterion.) Values below were recomputed by bisection from the closed form (12.1) (the preprint's corresponding table is superseded; see Remark 16.4).

κ	d=2, v=2	d=4, v=8	d=8, v=31
1	441	49	4
2	2401	289	81
5	2.16×10^4	3721	1225
10	1.15×10^5	2.10×10^4	7225
100	2.41×10^7	4.95×10^6	1.94×10^6
1000	4.08×10^9	8.83×10^8	3.59×10^8

Reading: even granting the abc target a magnitude as large as h^{1000} (beyond any plausible form), the $d = 2$ inflation still overtakes it at $h \approx 4 \times 10^9$ and the $d = 4$ inflation at $h \approx 9 \times 10^8$; $d = 8$ crosses even earlier. All three directions of parameter uncertainty (ℓ at its lower bound, d at its minimum, $v_p(\mathfrak{d})$ taken tame) *worsen* the picture: ℓ at the upper bound makes J larger; larger d amplifies through d^J; wild ramification makes v larger.

The qualitative conclusion is unchanged, and stronger than the previous draft's: under double exponentiality, any polynomial-magnitude target h^κ is

crossed at a moderate h. The previous draft's crossover values were too low by 10^{149} (at $h = 10^6$); those values are void and the present table governs.

12.4. **Mesh with the author-side decomposition.** The LeanForm report (2026-04) splits (xi-e) as

$$(\mathrm{dsc}) \longrightarrow (\mathbf{hull} + \mathbf{det}) \longrightarrow (\mathrm{SHE}),$$

where the **det** is exactly the quantity $\rho = |\det \Phi|$ computed in this paper. What this paper computes is not a peripheral estimate; it is the second arrow of the author's own decomposition.

12.5. **The inflation term in the C_Θ-inequality (G3 closed).** III Corollary 3.12 [3] is the inequality

$$-|\log(\Theta)| \;\geq\; -|\log(q)|,$$

i.e. $C_\Theta \geq -1$ for any C_Θ with $-|\log(\Theta)| \leq C_\Theta \cdot |\log(q)|$. Here $-|\log(\Theta)|$ is the procession-normalized mono-analytic log-volume of the *holomorphic hull of the union of the possible images of a Θ-pilot*, subject to (Ind1)(Ind2)(Ind3), and $-|\log(q)|$ is the same quantity for the q-pilot, which is not subject to the indeterminacies. IUT IV Theorem 1.10 [4] then estimates the Θ-pilot volume from height data:

(12.2)
$$C_\Theta = \frac{\ell^* + 1}{4|\log q|} \cdot \Big\{ (1 + 12 d_{\mathrm{mod}}/\ell^*)\big(\log \mathfrak{d}_{F_{\mathrm{tpd}}} + \log \mathfrak{f}_{F_{\mathrm{tpd}}}\big) + 10\big(e^*_{\mathrm{mod}} \cdot \ell^* + \eta_{\mathrm{prm}}\big) - \frac{1}{6}\Big(1 - \frac{12}{(\ell^*)^2}\Big)\log(q) \Big\} - 1,$$

with $d^*_{\mathrm{mod}} = 2^{12} \cdot 3^3 \cdot 5 \cdot d_{\mathrm{mod}}$, $e^*_{\mathrm{mod}} = 2^{12} \cdot 3^3 \cdot 5 \cdot e_{\mathrm{mod}}$, and applying $C_\Theta \geq -1$ yields the abc-bound chain

$$\frac{1}{6}\log(q) \;\leq\; \Big(1 + \frac{20\, d_{\mathrm{mod}}}{\ell^*}\Big)\big(\log \mathfrak{d}_{F_{\mathrm{tpd}}} + \log \mathfrak{f}_{F_{\mathrm{tpd}}}\big) + 20\big(e^*_{\mathrm{mod}} \cdot \ell^* + \eta_{\mathrm{prm}}\big). \tag{12.3}$$

Lemma 12.1 (docking of the layer-1 inflation)**.** *With the sign convention of §8.2 (zero point at* $\mathrm{hull}(\mathcal{O}_n)$*), the true hull volume exceeds the volume estimated in* loc. cit. *by the net index of §8.4: in IUT's convention,*

$$-|\log(\Theta)|_{\mathrm{true}} \;=\; -|\log(\Theta)|_{\mathrm{IUT}} - \log \widehat{\rho},$$

where $\log \widehat{\rho} = \log_p \mathrm{Net}_J \cdot \log p \asymp d^J J^2 n_v \log p$ *(§8.4), equivalently* $C_{\Theta,\mathrm{true}} = C_{\Theta,\mathrm{IUT}} - \log \widehat{\rho}/|\log q|$.

Proof. The inflation is the determinant of the integral-basis change of the hull's target lattice (§5.5, §12.4); log-volumes are additive under lattice indices by definition (IUT IV §1, cf. §8.2 of this paper). The magnitude $\log \widehat{\rho} = \log_p \mathrm{Net}_J \cdot \log p$ is the last display of §8.4. □

Theorem 12.2 (corrected C_Θ-inequality)**.** *With the true hull volume, the requirement* $C_{\Theta,\mathrm{true}} \geq -1$ *replaces* (12.3) *by*

$$\frac{1}{6}\Big(1 - \frac{12}{(\ell^*)^2}\Big)\log(q) + \frac{4}{\ell^* + 1} \cdot \log \widehat{\rho} \;\leq \Big(1 + \frac{12\, d_{\mathrm{mod}}}{\ell^*}\Big)\big(\log \mathfrak{d}_{F_{\mathrm{tpd}}} + \log \mathfrak{f}_{F_{\mathrm{tpd}}}\big) + 10\big(e^*_{\mathrm{mod}} \cdot \ell^* + \eta_{\mathrm{prm}}\big), \tag{12.4}$$

i.e. the inflation enters the abc-bound chain as an additive term $\frac{4}{\ell^*+1}\log\widehat{\rho}$ *on the left-hand side.*

Proof. Substitute $C_{\Theta,\mathrm{IUT}} = C_{\Theta,\mathrm{true}} + \log\widehat{\rho}/|\log q| \geq -1 + \log\widehat{\rho}/|\log q|$ into the C_Θ-formula. Since $|\log q| > 0$, the factor $\frac{\ell^*+1}{4|\log q|}$ is positive, and the condition $\{\cdots\} \geq \frac{4}{\ell^*+1}\log\widehat{\rho}$ is exactly (12.4). The computation is *sign-uniform*: in Branch 2 $\log\widehat{\rho} < 0$, the constraint weakens to a vacuity, and the same line of algebra exhibits both arms explicitly — see Lemma 8.4. □

Corollary 12.3 (quantitative failure of the chain)**.** *The right-hand side of* (12.4) *is* $O((\ell^*)^{1+\varepsilon})$ *at the IUT scale* $\ell^* \asymp \sqrt{h}$ *(the terms* $\log\mathfrak{d}_{F_{\mathrm{tpd}}}$, $e^*_{\mathrm{mod}}\cdot\ell^*$, η_{prm} *are bounded by* $(\ell^*)^{1+o(1)}$ *in the proof of IUT IV Corollary 2.2; see Step (viii) there and Proposition 1.6), while the inflation term satisfies* $\frac{4}{\ell^*+1}\log\widehat{\rho} = \exp(\Theta(\sqrt{h}))$ *(§12.2: removing the factor* $\ell^* \asymp \sqrt{h}$ *costs only* $\log\ell^*$ *on the doubly-logarithmic scale). Hence* (12.4) *demands* $\exp(\Theta(\sqrt{h})) \leq O(h^{1/2+\varepsilon})$ *and fails for all sufficiently large* h*; for the standard constants* ($e_{\mathrm{mod}} = 1$) *the onset lies between* $h = 10^2$ *and* $h = 10^3$ *— more precisely* $\approx 7\times10^2$ *by bisection on the closed form, comparing* $\frac{4}{\ell^*+1}\log\widehat{\rho}$ *(with* $\log\widehat{\rho} = \log\mathrm{Net}_J$ *of §8.4) against the full right-hand side of* (12.4)*, whose dominant term is IUT's own* $10\,e^*_{\mathrm{mod}}\cdot\ell^*$ *with* $e^*_{\mathrm{mod}} = 2^{12}\cdot3^3\cdot5 \approx 5.5\times10^5$*. Two neighbouring calibrations bracket the onset: with the layer-1a index in place of* Net_J *it moves to* $\approx 3\times10^3$*; with the explicit constant* e^*_{mod} *removed from the comparison it moves down to* $h \approx 10^1$*. The qualitative conclusion is calibration-independent.* Onset coincidence: *the Branch-2 arm-B onset* ($h \approx 1.4\times10^3$*, Lemma 8.4) lies in the same decade as this Branch-1 onset — the two branches fail at the same scale, as they must, since both are driven by the same* $e^{\Theta(\sqrt{h})}$ *object.*

This closes G3: the middle of the chain is now *computed* — with the true hull volume, Corollary 3.12 feeds an inequality whose left side outgrows any height-data right side. What breaks is the arrow (dsc) → (**hull + det**) of §12.4, exactly where this paper's $\widehat{\rho}$ lives; the remaining candidates for saving the chain are (9-1) and explicit-constant bookkeeping (§14), both outside layer 1.

Honest boundary. The docking uses the sign convention recorded in §8.2 and the placement of $\widehat{\rho}$ as the determinant of the second arrow (§12.4); the procession weights and λ_e-normalizations are those of §12.1, and no new normalization freedom is introduced. The onset range 10^3–10^4 depends on e_{mod} and η_{prm} and has the same status as the explicit constants of §14.

13. abc-side comparison (2185 primitive triples)

Quantity	Value
sample	2185 triples, $c \leq 4.4\times10^{11}$
max q	1.629912 ($2 + 3^{10}\cdot109 = 23^5$, i.e. a=2, b=6436341, c=6436343)
median q / median $\bar{v}$	1.0564 / 2.5137
median $\bar{v}/q$	2.3485
$q > 1.5$ / $q > 1.4$	5 / 24 triples
$\log(abc)$ range	[8.37, 60.94]

(Provenance: the sample, medians and ranges are quoted from the project's data files, scripts appended and reproducible; the record triple $2+3^{10}\cdot 109 = 23^5$ and $\log(abc) \in [8.37, 60.94]$ were spot-checked.)

Frontier saturation: scanning c from 10^7 to 10^{14} (seven orders of magnitude, +1123 new triples), $\max q$ did not move (1.629912).

△ *Logical correction:* "$\max q$ is not growing" is **not** evidence for abc — abc only requires that for each fixed $\lambda > 0$ there are finitely many triples with $q > 1 + \lambda$; it does not require $\max q \to 1$. The genuine evidence would be "no systematic construction".

14. Unverifiability of explicit constants

Measured $\ln(abc)$ range (natural log)	[8.37, 60.94]
Threshold $\ln\|abc\| \geq 700$ (explicit-estimates calibration)	requires $c \gtrsim 10^{152}$
Gap	$700 - 60.94 \approx 639$ in ln-units; equivalently ≈ 140 orders of magnitude in c (152 vs $\log_{10}(4.4 \times 10^{11}) \approx 11.6$)

Explicit Estimates in IUT [9] (RIMS Preprint 1933; *Kodai Math. J.* **45**(2) 175–236, 2022) Corollary 5.9 gives the explicit constant 2.453×10^{30}; Zhou [10] (arXiv:2503.14510) reduces it to 400.

△ *A falsifiable prediction:* if $\bar{v} \leq 3 + o(1)$ and $\bar{v} \approx 2.35q$, then asymptotically $q \lesssim 1.28$, while the measured $\max q = 1.6299$. The crossover must therefore occur at $\log\|abc\| \geq 700$. If, in the future, an example with $c > 10^{152}$ and $q > 1.3$ is found, this explicit form is refuted.

15. Other directions already closed

This section briefly records, in order to delimit the scope of the present work, several directions that were excluded during the earlier exploration and lie outside this paper's territory.

Direction	Conclusion
Archimedean side (periods/modular forms/heights)	cannot reach rad
L-values	$w = -1$ accounts for 49.8%; one half is identically zero
Baker / Matveev [11]	structural constant 30^{n+3}; pushed to $n = 1$ still short by 10^5
Stewart–Yu [12]	$\mathrm{rad}^{1/3}$ would need improvement 10^{44}
Pasten subexponent [13]	$\sqrt{v \log v}$ has the wrong shape; $\kappa' \to 0$ still short by e^{3703}
$\#I = 0$	$\kappa = 1 + \kappa'/2$ always > 1
Hilbert–Samuel multiplicity	over a DVR identically equal to 1; in dimension 2 can be non-integral [14]
Fargues–Fontaine slopes	X_E attached to a single p; does not unify all primes
Frey 2-descent	circular

Methodological significance: the reasons these directions were excluded differ, but they all reduce to the same structure — abc needs information of *"multiplicity-linear-in-position-logarithm"* type ($\sum \nu_p \log p$), while the listed mechanisms respectively produce information of *"log-of-multiplicity"* type ($\sum \log \nu_p$), of *"family-average"* type, or of *"$e^{\mathrm{poly}(\omega)}$ factor"* type. The hull inflation of this paper belongs to a fourth class: its magnitude can be computed exactly, and the computation shows that it runs away under IUT's own parameters.

16. The non-Galois case (boundary of applicability of Theorem 5.1)

This section was promoted to the main text on a reviewer's advice. Appendix A's one-liner was insufficient: non-Galois is the norm*, not the exception, in IUT.*

16.1. **Problem.** The proof of Theorem 5.1 depends on $K/\mathbf{Q}_p$ being Galois (the surjectivity in Lemma 2.1 uses $K \otimes K \cong \bigoplus_d K$). For $K/\mathbf{Q}_p$ non-Galois: $K \otimes K \cong \bigoplus_j K_j$, where K_j are the composita of K with its conjugates, of degrees > 1 in general, distinct from each other. Then:

- the target lattice is no longer $\bigoplus_d \mathcal{O}$ but $\bigoplus_j \mathcal{O}_{K_j}$;
- $\mathrm{disc}(\bigoplus_j \mathcal{O}_{K_j}) = \prod_j \mathfrak{d}_{K_j}$;
- the closed form must be rebuilt according to the compositum structure.

16.2. **Compositum theorem (withdrawing the previous "conservative direction" claim).** This section overturns the previous draft's claim (correction $\star$ of this paper). The previous draft said "non-Galois $\Rightarrow$ larger loss (conservative direction)". **This is wrong, and wrong on the side favorable to us; it must be corrected.**

Let $K/\mathbf{Q}_p$ be any separable extension of degree d, and $K \otimes K = \bigoplus_j K_j$ ($K_j \supseteq K$, $e_j = [K_j : K]$, $\sum_j e_j = d$).

- $\mathrm{disc}(\mathcal{O}\otimes\mathcal{O}) = \mathfrak{d}_K^{2d}$ (Kronecker discriminant formula for tensor products);
- $\mathrm{disc}(K_j) = \mathrm{N}_{K/\mathbf{Q}}(\mathfrak{d}_{K_j/K}) \cdot \mathfrak{d}_K^{e_j}$ (transitivity of the relative discriminant: for $K_j/K/\mathbf{Q}_p$, $\mathfrak{d}_{K_j/\mathbf{Q}_p} = \mathrm{N}_{K/\mathbf{Q}_p}(\mathfrak{d}_{K_j/K}) \cdot \mathfrak{d}_{K/\mathbf{Q}_p}^{[K_j:K]}$);
- $\rho^2 = \dfrac{\mathrm{disc}(\mathcal{O} \otimes \mathcal{O})}{\prod_j \mathrm{disc}(K_j)}$ (same argument as Theorem 5.1: Φ is still an isomorphism $A_2 \to B_2 = \bigoplus_j K_j$ preserving trace, $\Phi(\mathcal{O}_2)$ has discriminant $\mathfrak{d}_K^{2d}$, and $\bigoplus_j \mathcal{O}_{K_j}$ has discriminant $\prod_j \mathrm{disc}(K_j)$; the index is ρ).

Combining, the valuation of the index in the non-Galois case is

$$v_p(\rho^2) = 2d\,v - \sum_j \big(e_j v + v_p(\mathrm{N}\,\mathfrak{d}_{K_j/K})\big) = d\,v - \Delta, \qquad \Delta := \sum_j v_p\big(\mathrm{N}_{K/\mathbf{Q}_p}\,\mathfrak{d}_{K_j/K}\big) \geq 0. \tag{16.1}$$

Galois comparison (Theorem 5.1 at $n = 2$): $v_p(\rho^2) = d\,v$. Hence

$$\rho_{\mathrm{NG}}^2 = \frac{\rho_{\mathrm{G}}^2}{p^{\Delta}} \;\leq\; \rho_{\mathrm{G}}^2.$$

$\Longrightarrow$ in the non-Galois case ρ can only *decrease*. Theorem 5.1 gives an **upper bound**, not a lower bound. This is the opposite of the previous draft's claim, and it is *against* our thesis: if K_v is non-Galois, the inflation computed here is an overestimate.

Vanishing criterion for Δ. If the "extra part" of the Galois closure is unramified, i.e. $K_j = K \cdot L$ with $\mathfrak{d}_{L/K} = \mathcal{O}$, then $\Delta = 0$ and Theorem 5.1 holds *exactly*.

Tame Kummer type ($K = \mathbf{Q}_p(\pi^{1/d})$, $p \nmid d$): the extra part is generated by roots of unity, and $\mathbf{Q}_p(\zeta_d)/\mathbf{Q}_p$ is unramified (as $p \nmid d$) $\Longrightarrow$ $K(\zeta_d)/K$ unramified $\Longrightarrow$ $\Delta = 0$.

Verified example. $K = \mathbf{Q}_2(2^{1/3})$ ($d = 3$, S_3-type, non-Galois):

Quantity	Value
$\mathfrak{d}_K$	$-108 = 2^2 \cdot 27$, $v = 2$
components	$K \oplus \tilde{K}$, $\tilde{K} = K(\omega)$ (degree 6)
$\mathfrak{d}_{\tilde{K}}$	$-34992 = -2^4 \cdot 3^7$, $v_2 = 4$
Δ	$4 - 2v = 4 - 4 = \mathbf{0}$
$v_2(\rho^2)$	$dv - \Delta = 6 - 0 = \mathbf{6}$
Theorem 5.1 (Galois formula)	$dv = 6$ ✓

(Steps: $2dv = 12$; $\sum_j (e_j v + v_p(\mathrm{N}\,\mathfrak{d}_{K_j/K})) = 1 \cdot 2 + 2 \cdot 2 + 0 = 6$; $12 - 6 = 6$. The tower formula gives $v_2(\mathrm{N}\,\mathfrak{d}_{\tilde{K}/K}) = v_2(\mathfrak{d}_{\tilde{K}}) - 2v_2(\mathfrak{d}_K) = 4 - 4 = 0$, i.e. $\tilde{K}/K$ unramified, since $\mathbf{Q}_2(\zeta_3) = \mathbf{Q}_2(\sqrt{-3})$ is unramified over $\mathbf{Q}_2$.)

First empirical evidence for $\Delta > 0$. The example above gives $\Delta = 0$, which does not show whether Δ can be positive. A corrected review copy (Sage 10.9) ran five non-Galois samples, of which three give $\Delta > 0$:

Field	p	Group	d	v	$d\,v$	Δ	$v_p(\rho^2)$	$v_p(\rho)$	Galois
$\mathbf{Q}(\sqrt[3]{2})$	2	S_3	3	2	6	0	6	3	3
$\mathbf{Q}(\sqrt[3]{3})$	3	S_3	3	5	15	1	14	7	7.5
$x^4 - x - 1$	283	S_4	4	1	4	2	2	1	2
$x^4 - 8x^3 + 7x^2 + 5x - 3$	19	A_4	4	2	8	2	6	3	4
same	31	A_4	4	2	8	0	8	4	4

All five satisfy the closed form $v_p(\rho^2) = d\,v - \Delta$; $d\,v - \Delta$ is even in every row, and $(d\,v - \Delta)/2$ equals the measured $v_p(\rho)$ row by row. $\Longrightarrow$ $\Delta > 0$ does exist, and non-Galois ρ is strictly smaller than the Galois comparison value. These are the first nontrivial instances confirming the direction of §16.2.

Attribution and independence. The five examples were first produced by Part D of the review copy; a subsequent independent recheck on another temporary copy — which removed the single-prime gate according to the paper's own textual description of B13 and fixed the `Fraction`/numeric-type conversions, *without* using this package's corrected files — again gave 5/5. For $p = 283$ and $p = 19$ the diagnostic output explicitly shows "non-single-prime", yet the global tensor decomposition still yields $\Delta = 2$: direct empirical evidence that deleting the gate is correct, i.e. Δ depends only on the p-component of the *global* discriminant, not on the number of primes.

The package's corrected files themselves have not been run in that Sage environment.

The relative discount at a single prime can be large. Measuring against the Galois comparison value $d\,v/2$:

Sample	Galois comparison $v_p(\rho)$	Measured	Relative discount
$\mathbf{Q}(\sqrt[3]{2})$	3	3	0%
$\mathbf{Q}(\sqrt[3]{3})$	7.5	7	6.7%
$x^4 - x - 1$, $p = 283$	2	1	50%
A_4, $p = 19$	4	3	25%
A_4, $p = 31$	4	4	0%

From $v_p(\rho^2) = d\,v - \Delta \geq 0$ one gets $\Delta \leq d\,v$, so in the worst case the index can drop to 1 (a 100% discount). The statement "the discount is asymptotically negligible" in §16.3 is an $h \to \infty$ asymptotic statement (poly(h) versus $\exp(c\sqrt{h})$); it cannot be read off at a single prime, where Δ can consume the entire inflation of that prime.

Δ is not a function of (d, v): the E1QUICK scan. Running E1QUICK on a user-side temporary corrected copy (Sage 10.9) yields 31 records; grouping by (d, v), one of the 12 groups takes two values:

$$(d, v) = (4, 2): \qquad \Delta \in \{0, 2\}.$$

The two rows are the *same* A_4 quartic field at two different primes: $p = 31$ gives $\Delta = 0$, $p = 19$ gives $\Delta = 2$. All self-checks pass: parity violations 0, negative $v_p(\rho^2)$ 0, $\sum_j e_j \neq d$ 0, and among Galois-type samples none has $\Delta \neq 0$. $\Longrightarrow$ the hypothesis "Δ depends only on (d, v)" is refuted. Moreover, since the field is fixed, all global invariants (the field, the Galois group, d, $v_p(\operatorname{disc} K)$) are fixed, so the dependence is located precisely in the local arithmetic at p — whether the "extra part" of the Galois closure ramifies at p — which is exactly the defining quantity of Δ in (16.1). The data surfaced no new parameter; it confirmed that the old parameter is irreducible.

Two-term decomposition of Δ. Write

$$\Delta = \Delta_{\text{parity}} + \Delta_{\text{ram}}, \qquad \Delta_{\text{parity}} := (d\,v) \bmod 2, \quad \Delta_{\text{ram}} \geq 0,$$

where Δ_{parity} is the minimum forced by Corollary 16.1 below and Δ_{ram} is the ramification contribution of the closure's extra part. The 12 groups of the E1QUICK scan give:

Family	(d, v)	Δ	Δ_{parity} / Δ_{ram}
$d = 2$ (Galois comparison)	$(2,1), (2,2), (2,3)$	$0, 0, 0$	0 / 0
$d = 3$ (S_3 Kummer)	$v = 1, 2, 3, 4, 5$	$1, 0, 1, 0, 1$	v mod 2 / all 0
$d = 4$	$(4, 1)$	2	0 / 2
$d = 4$	$(4, 2)$	$0, 2$	0 / 0, 2
$d = 4$	$(4, 8), (4, 11)$	$0, 2$	0 / 0, 2

Two readings must be distinguished — which one is a restatement and which one is a fact: Δ_{ram} is always even (value set $\{0, 2\}$) is a trivial restatement of Corollary 16.1, not a new structure, and is not claimed as an independent discovery. *The nontrivial empirical fact is: all five $d = 3$ groups attain the parity lower bound*, i.e. $\Delta = v \bmod 2$ and $\Delta_{\text{ram}} = 0$. For S_3-type

$K = \mathbf{Q}(\sqrt[3]{m})$ the closure's extra part is $K(\zeta_3)/K$, and for $p \neq 3$ the extension $\mathbf{Q}_p(\zeta_3)/\mathbf{Q}_p$ is trivial or unramified quadratic, hence the extra part is unramified — consistent with the vanishing criterion of §16.2. Even the wild point $\mathbf{Q}(\sqrt[3]{3})$ ($v = 5$, $\Delta = 1$) attains the minimum.

Falsifiable prediction (P): for every S_3-type $K = \mathbf{Q}(\sqrt[3]{m})$ and every prime, $\Delta = v_p(\operatorname{disc} K) \bmod 2$, i.e. $\Delta_{\mathrm{ram}} = 0$.

Status of (P): proved completely on pure cubic fields (Corollary 16.2 below) — for $p \neq 3$, $\Delta = 0$ is a theorem; for $p = 3$ the per-prime relative-discriminant determination gives $\Delta = 1$. (P) is no longer a conjecture, and $\Delta \leq 1$ is an absolute constant bound.

Honesty note. 11 of the 12 groups carry a single Δ value, so this data can only refute "$\Delta = f(d, v)$"; it cannot distinguish among "depends on the residue class of p", "depends on the closure ramification group", and "depends on the decomposition type of p". Fitting a closed form requires more primes under the same (d, v, Gal).

Attribution. The 31 records come from the user's temporary corrected copy (gate deletion per the textual description of B13 and a `normalize()` compatibility fix), not from this package's files; the full E1 run was not executed.

Thirty-five-instance targeted scan and the analytic determination of Δ (S_3 pure cubics). A user-side Sage 10.9 scan of $K = \mathbf{Q}(\sqrt[3]{m})$, $m \in \{2, 3, 5, 6, 7, 10, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22\}$, at each ramified prime, gives 35 instances, 35/35 satisfying $\Delta = (3v) \bmod 2$, covering $p \bmod 3 \in \{0, 1, 2\}$. But "matching the parity" is itself a restatement of Corollary 16.1 and constitutes no new fact; the real increment is the analytic derivation below.

(i) *Δ is a valuation of relative discriminants (universally valid).* Let $K \otimes K \cong \bigoplus_j K_j$, $e_j = [K_j : K]$, $\sum_j e_j = d$. From $\operatorname{disc}(K_j) = \operatorname{disc}(K)^{e_j} \cdot \mathrm{N}_{K/\mathbf{Q}}(\mathfrak{d}_{K_j/K})$ and summation over j,

$$\Delta = \sum_j v_p\big(\mathrm{N}_{K/\mathbf{Q}}\, \mathfrak{d}_{K_j/K}\big).$$

This gives a constructive reason for $\Delta \geq 0$ (a relative discriminant is an integral ideal), more essential than the earlier inference from $v_p(\rho^2) \geq 0$.

(ii) *For S_3 type, the non-diagonal component is exactly the Galois closure.* $x^3 - m = (x - a)(x^2 + ax + a^2)$ with quotient discriminant $a^2 - 4a^2 = -3a^2$, and $a \in K \Longrightarrow M = K(\sqrt{-3}) = K(\zeta_3) = L$. Hence $\Delta = v_p(\mathrm{N}\,\mathfrak{d}_{L/K})$, a single term.

(iii) $\mathfrak{d}_{L/K} \mid (3)$. Taking ζ_3 as generator, the discriminant of $x^2 + x + 1$ is $(\zeta_3 - \zeta_3^2)^2 = -3$ (verified exactly in sympy), so $\mathrm{N}\,\mathfrak{d}_{L/K} \mid 3^{[K:\mathbf{Q}]} = 27$.

(iv) *Merge with Corollary 16.1:*

$$\Delta = 0 \quad (p \neq 3), \qquad \Delta \in \{1, 3\} \quad (p = 3),$$

the latter since $d\,v = 3v$ is odd for $p = 3$. The 35 instances split as:

p	$v_p(\operatorname{disc} K)$	Forced range	Value taken (35/35)
≥ 5	2 (even)	$\Delta = 0$ (theorem)	0
2	2 or 4 (even)	$\Delta = 0$ (theorem)	0
3	3 or 5 (odd)	$\Delta \in \{1, 3\}$	1

The last cell of (P) is now closed: the exact exponent of $\mathfrak{d}_{L/K}$ at 3 was determined by a user-side `relative_discriminant()` scan (192 examples) together with the local decomposition, and always gives $\Delta = 1$; the guess $\Delta = 3$ (B17) is withdrawn. The full determination is Corollary 16.2.

Calibration. sympy computes the power-basis $\mathbf{Z}[\sqrt[3]{m}]$ discriminant $-27m^2$; when $m \equiv \pm 1 \pmod 9$ the maximal-order discriminant loses 3^2, but v_3 remains odd and $v_p = 2$ for $p \geq 5$ is unchanged, so the parity conclusions are unaffected. The 35 instances use Sage's `K.discriminant()` (maximal order), the authoritative value. The A_4 sample $(4, 2) \Rightarrow \Delta \in \{0, 2\}$ shows that "Δ is decided by parity alone" is not universal — step (iii) depends on the specific generator.

Attribution. The 35 instances come from the user-side targeted scan; the analytic derivation was carried out in pure Python/sympy without Sage; the package files were not run in that environment.

Corollary 16.1 (parity of Δ is forced by $d\,v$)**.** *Since $v_p(\rho^2)$ must be even, the closed form $v_p(\rho^2) = d\,v - \Delta$ forces*

$$\Delta \equiv d\,v \pmod 2.$$

In particular, when $d\,v$ is odd, $\Delta \geq 1$ is forced, not accidental. The five examples match one by one: $d\,v = 6, 15, 4, 8, 8$ correspond to $\Delta = 0, 1, 2, 2, 0$.

This corollary also explains the second row's $\Delta = 1$: there $d = 3$, $v = 5$, $d\,v = 15$ is odd, while the Galois comparison value $\frac{(n-1)d^{n-1}}{2}v = \frac{3\cdot 5}{2} = 7.5$ is not an integer. The parity argument of Corollary 5.4 (Hilbert's ramification formula) relies on the Galois hypothesis; for non-Galois K, $d\,v$ can be odd, the Galois formula fails, and the deficit is absorbed by Δ. Hence Δ's lower bound is arithmetically forced, and its upper bound is given by §16.3 (poly(h)); neither harms the conclusion.

Corollary 16.2 (the pure-cubic family: Δ completely determined; (P) proved)**.** *Let $K = \mathbf{Q}(\sqrt[3]{m})$ (m not a cube) with Galois closure $L = K(\zeta_3)$. Since K is cubic and has no quadratic subfield, $\zeta_3 \notin K$ and $[L : K] = 2$; since $K \otimes K \cong K \oplus L$, the non-diagonal component is exactly L, so $\Delta = v_3\big(\mathrm{N}_{K/\mathbf{Q}}(\mathfrak{d}_{L/K})\big)$ (for $p = 3$; $\Delta = 0$ for $p \neq 3$ by (ii)–(iii) above). The minimal polynomial $x^2 + x + 1$ of ζ_3 has discriminant -3, and the relative-discriminant formula*

$$\mathrm{disc}(1, \zeta_3) = \mathfrak{d}_{L/K} \cdot [\mathcal{O}_L : \mathcal{O}_K[\zeta_3]]^2$$

gives, after taking $\mathrm{N}_{K/\mathbf{Q}}$, $27 = 3^3$ on the left, hence

$$\Delta = 3 - 2\,v_3\big(\mathrm{N}_{K/\mathbf{Q}}[\mathcal{O}_L : \mathcal{O}_K[\zeta_3]]\big), \tag{16.2}$$

consistent with Corollary 16.1 ($d = 3$, $v_3(\mathrm{disc}\,K)$ odd, Δ odd). $\Delta = 3$ would hold iff $\mathcal{O}_K[\zeta_3]$ were already maximal at 3. A user-side Sage 10.9 targeted scan ($-100 \leq m \leq 100$, 192 non-cube parameters, merging to 65 distinct discriminant values) gives $v_3\big(\mathrm{N}_{K/\mathbf{Q}}\,\mathfrak{d}_{L/K}\big) = 1$ in every case ($m = 2, 3, 5, 19$ all have relative discriminant norm 3), including the 40 cases $m \equiv \pm 1 \pmod 9$.

The per-prime determination at $p = 3$ is as follows. If $m \equiv \pm 1 \pmod 9$, then $x^3 - m$ has a root a in $\mathbf{Q}_3$, the quotient discriminant is $-3a^2$, and

$$K \otimes \mathbf{Q}_3 \cong \mathbf{Q}_3 \times \mathbf{Q}_3(\zeta_3),$$

with $L = K(\zeta_3)$ localizing over the two branches: over $\mathbf{Q}_3$ as $\mathbf{Q}_3(\zeta_3)/\mathbf{Q}_3$, and over $\mathbf{Q}_3(\zeta_3)$ trivially:

Branch	*Contains ζ_3?*	*Relative discriminant exponent*	f
$\mathbf{Q}_3$	*no*	1 *(tame quadratic, discriminant -3)*	1
$\mathbf{Q}_3(\zeta_3)$	*yes*	0 *(trivial extension)*	1

Both branches have $f = 1$, hence $v_3\big(\mathrm{N}_{K/\mathbf{Q}}\,\mathfrak{d}_{L/K}\big) = 1 \cdot 1 + 0 \cdot 1 = 1$. If $m \not\equiv \pm 1 \pmod 9$, then 3 is totally ramified in K ($f = 1$), K_3 does not contain ζ_3, and L_3/K_3 is a tame quadratic ($3 \nmid 2$) with relative discriminant exponent again 1. Merging the two cases with $\mathfrak{d}_{L/K} \mid (3)$:

$$\boxed{\Delta = \begin{cases} 0, & p \neq 3, \\ 1, & p = 3, \end{cases} \;=\; v_p(\operatorname{disc} K) \bmod 2.}$$

$\Longrightarrow$ (P) is proved on pure cubic fields (not merely supported), and $\Delta \leq 1$ is an absolute constant bound, far stronger than the poly(h) of §16.3. Hence "the discount is negligible" holds on S_3-type tensor packets at $n = 2$ *without* any asymptotic assumption.

Error mechanism of B17. The guess $\Delta = 3$ mistook "the decomposition type of 3" for "the number of times ζ_3 must be adjoined", accumulating one contribution per branch and ignoring that a branch already containing the root of unity contributes no relative discriminant. Δ is a valuation of a *relative* discriminant, not a sum over decomposition branches; the $(1, 2)$-type decomposition itself was correct, but the $\Delta = 3$ derived from it does not hold. This belongs to the same family as B10 ("directional claim without proof"), but the error is a misjudged additivity in the local–global correspondence, not a wrong direction.

Attribution and boundary. The 192 examples (including the 40 with $m \equiv \pm 1 \pmod 9$), the two branches' (e, f) diagnostics, and the outputs of the relative-discriminant computation are all from the user-side temporary Sage copy; the package files were not run in Sage. Corollary 16.2 is limited to $n = 2$ pure cubics: the A_4 sample $(4, 2) \Rightarrow \Delta \in \{0, 2\}$ shows $\Delta_{\mathrm{ram}} = 0$ does not extrapolate, and no closed form for Δ at $n \geq 3$ has been derived or verified.

Higher tensor packets ($n \geq 3$): the geometric discount (A4). The layer-boundary reservation of §16.3 — no closed form for Δ at $n \geq 3$ — is now removed for the S_3 pure-cubic family, by an exact discriminant computation that needs no matrices.

Theorem 16.3 (S_3 family at every n)**.** *Let $K = \mathbf{Q}(\sqrt[3]{m})$, $L = K(\zeta_3)$, and write $\Delta_1 := v_p\big(\mathrm{N}_{K/\mathbf{Q}}\,\mathfrak{d}_{L/K}\big) \in \{0, 1\}$ (Corollary 16.2). For every $n \geq 2$:*

(1) $K^{\otimes n} \cong K \oplus L^{a_n}$ as $\mathbf{Q}$-algebras, where $a_n = \frac{3^{n-1}-1}{2} = 1 + 3 + \cdots + 3^{n-2}$;

(2) *the standard isomorphism* $A_n \to B_n = K \oplus L^{a_n}$ *is trace-preserving — the proof of Lemma 3.1 does not use the Galois hypothesis — hence*

$$v_p(\rho_n^2) = (n-1)\,3^{n-1}\,v - a_n\,\Delta_1, \qquad \textit{i.e.} \quad \Delta_n = a_n \cdot \Delta_1;$$

(3) $\Delta_n \equiv (n-1)\,3^{n-1}v \pmod 2$ *is automatic, since* $a_n \equiv n-1 \pmod 2$ *and* $\Delta_1 \equiv v \pmod 2$ *(Corollary 16.1).*

Proof. (i) Induction: $K^{\otimes 2} \cong K \oplus L$ (§16.2), and since L/K is Galois of degree 2 and L splits K completely (L is normal), $L \otimes K \cong L^{[K:\mathbf{Q}]} = L^3$; hence $K^{\otimes(n+1)} \cong (K \oplus L^{a_n}) \otimes K \cong (K \oplus L) \oplus L^{3a_n} = K \oplus L^{a_{n+1}}$ with $a_{n+1} = 1 + 3a_n$.

(ii) The trace-preservation is Lemma 3.1 verbatim. The discriminant of the source lattice is $\mathrm{disc}(\mathcal{O}^{\otimes n}) = \mathfrak{d}_K^{n\,3^{n-1}}$ (Kronecker formula, Lemma 4.2); the target lattice B_n has $\mathrm{disc} = \mathrm{disc}(K) \cdot \mathrm{disc}(L)^{a_n}$ with $v_p\big(\mathrm{disc}(L)\big) = 2v + \Delta_1$ by transitivity of the relative discriminant. Equating through trace-preservation (Theorem 5.1, §5.2): $v_p(\rho_n^2) = n\,3^{n-1}v - v - a_n(2v + \Delta_1) = (n-1)3^{n-1}v - a_n\Delta_1$.

(iii) $a_n \bmod 2$: $n-1$ even gives $3^{n-1} \equiv 1 \pmod 8$, so a_n even; $n-1$ odd gives $3^{n-1} \equiv 3 \pmod 4$, so a_n odd. Then use $\Delta_1 \equiv 3v \equiv v \pmod 2$ from Corollary 16.1. □

Numerical anchor (exact integer arithmetic, no matrices): for $K = \mathbf{Q}(\sqrt[3]{2})$, $v_2(\mathfrak{d}_K) = 2$, $v_2(\mathrm{disc}\,L) = 4$, so $\Delta_1 = 0$ and $v_2(\rho_3^2) = 2 \cdot 9 \cdot 2 - 0 = 36$ — the Galois formula (Theorem 5.1) is *exact* at $n = 3$; at $p = 3$, $v_3(\mathfrak{d}_K) = 3$, $v_3(\mathrm{disc}\,L) = 7$, $\Delta_1 = 1$, and $v_3(\rho_3^2) = 54 - 4 = 50$. The same computation at $n = 4, 5$ gives $\Delta_4 = 13$, $\Delta_5 = 40$, all parities even.

Consequence for IUT. Since $\Delta_1 = 0$ for $p \neq 3$ (Corollary 16.2), Theorem 16.3 gives $\Delta_n = 0$ for *every* n: on S_3-type pure-cubic tensor packets at $p \neq 3$, Theorem 5.1 is exact at all tensor depths, with no asymptotic input. At $p = 3$ the discount $a_n\Delta_1 = a_n \leq \frac{3^{n-1}}{2}$ grows only like the component count d^{n-1}, the same weight the main term carries.

Corollary for IUT: $K = F(E[\ell])$ localized; for ℓ prime, $\ell \neq p$, $\mathbf{Q}_p(\zeta_\ell)/\mathbf{Q}_p$ is unramified $\Longrightarrow \Delta \to 0$ and Theorem 5.1 is exact.

16.3. **Explicit bound for Δ (G4 demoted to a precision item).**

Localization in the original text.

Location	Content
IUT I Remark 3.1.7(iii) [1, p. 68]	*"Write $F(\mu_\ell) \subseteq K$ for the subextension of K generated by the ℓ-th roots of unity"*
same page	$\mathrm{Gal}(K/F(\mu_\ell)) \cong SL_2(\mathbf{F}_\ell)$
IUT I Def 3.1(c)	$\ell \neq p_v$
IUT I Def 3.1(c) + Serre/Silverman	at bad multiplicative reduction E_v is a Tate curve, and the ℓ-torsion extension is tame of index dividing ℓ (Theorem 8.3)
IUT II §1 [2, p. 20]	locally contains $\mu_{4\ell}$ (compatible with $\sqrt{-1} \in F + \mu_\ell \subseteq K$)
IUT IV Cor 2.2 (C1)	$\ell \asymp h^{1/2}$
IUT I Def 3.1(a)	F contains $\sqrt{-1}$ but not μ_ℓ (hierarchy: F no, K yes)

Bound on Δ *(tightened by Theorem 8.3).* The crude bound $\Delta \le N(1+v_p(N))$ with $N \asymp \ell^4$ of the previous draft improves as follows. The conjugates of $K = F(E[\ell])$ over $\mathbf{Q}$ are contained in $F^{\mathrm{cl}}(E[\ell])$, so the extra part of the Galois closure satisfies $[K_v^{\mathrm{cl}} : K_v] \le [F^{\mathrm{cl}} : F]$ — $= 1$ when $F/\mathbf{Q}$ is Galois, in particular for the normalization $[F:\mathbf{Q}] = 2$ used throughout. At each completion Theorem 8.3 gives $e_w \mid \ell$ and $f_w \le \mathrm{ord}_\ell(p_v^{f_v}) \le \ell - 1$ (using $\ell \nmid n_v$, IUT I Def. 3.1(c)). Hence, for $p_v \notin \{2,3\}$ and $\ell \ne p_v$,

$$\Delta \le [F^{\mathrm{cl}} : F] \cdot (\ell-1)^2.$$

The primes $p_v \in \{2,3\}$ keep Serre's general bound $v_p(\mathfrak{d}_{L/\mathbf{Q}_p}) \le [L:\mathbf{Q}_p]\,(1+v_p([L:\mathbf{Q}_p]))$ [8, III §6]; there are finitely many of them, with $v_p(N) = O(\log \ell)$. Δ remains a polynomial in ℓ, hence in h ($\ell \asymp h^{1/2}$) — now of degree 2 with unit constant instead of degree 4 with a logarithmic factor.

⋆ *Core comparison.*

h	ℓ	$\Delta_{\max}$	$\log_{10}\Delta_{\max}$	J	$\log_{10}\log_p \rho$	share
10^2	10	81	1.91	4	1.39	1.37
10^4	100	9801	3.99	49	15.04	0.27
10^6	1000	1.0×10^6	6.00	499	150.51	0.040
10^8	10^4	1.0×10^8	8.00	4999	1505.15	0.0053
10^{12}	10^6	1.0×10^{12}	12.00	499999	150515.00	8.0×10^{-5}

(share $= \log_{10}\Delta_{\max}/\log_{10}\log_p\rho$; $\Delta_{\max} = (\ell-1)^2$; the previous draft's $\Delta_{\max}$ column — 5.3×10^4, 7.9×10^8, 1.0×10^{13}, 1.2×10^{17}, 1.6×10^{25} — is improved by two to thirteen orders of magnitude.) The table takes $[F^{\mathrm{cl}} : F] = 1$ ($F/\mathbf{Q}$ Galois); in general multiply $\Delta_{\max}$ by $[F^{\mathrm{cl}} : F] \le [F:\mathbf{Q}]$. At $h = 10^2$ the share exceeds 1: the discount is *not* negligible at small scales — asymptotic negligibility sets in from $h \gtrsim 10^4$ (share < 1), which is what the trend of the last column shows.

$$\frac{\Delta_{\max}}{\log_p \rho} = \frac{\mathrm{poly}(h)}{\exp(c\sqrt{h})} \longrightarrow 0.$$

Stress test: even blowing the bound up to $[F^{\mathrm{cl}} : F]^2(\ell-1)^2$ (far beyond any wild allowance), at $h = 10^6$ one gets 4.0×10^6 versus 10^{150} (more precisely $\log_p\rho = \bar v \approx 10^{150.51}$): still 144 orders of magnitude short.

Remark 16.4 (Recomputed values superseding the preprint's tables). The crossover table of §12.3 and the comparison table of this subsection were recomputed from the closed forms of this paper. They supersede the corresponding tables of the Chinese preprint. The $\Delta_{\max}$ column of the previous draft (last row 1.6×10^{25}, from the preprint's formula $N(1+v_2(N))$ with $N = 2.0\times10^{24}$, $v_2(N) = 7$; the preprint's displayed 6.2×10^{25} does not satisfy that formula and was corrected) is in turn superseded by the tame bound $\Delta_{\max} = (\ell-1)^2$ of Theorem 8.3. All conclusions are unaffected.

Verdict. **G4 is demoted to a precision item.** Three pieces assembled:

(1) original text: $K_v/\mathbf{Q}_{v_{\mathbf{Q}}}$ non-Galois is the norm;

(2) §16.2 closed form $\rho_{\mathrm{NG}}^2 = \rho_{\mathrm{G}}^2/p^{\Delta}$: Theorem 5.1 is always an upper bound;

(3) this round: $\Delta \le \mathrm{poly}(h) \ll \exp(\Theta(\sqrt{h}))$.

$\Longrightarrow$ the discount by Δ is asymptotically negligible; the crossover shifts to the right boundedly; the layer-1 conclusion is unchanged.

Honest reservations.

- $p_v \mid 6$ may have wild contribution ($E_{\mathrm{mod}}[2 \cdot 3 \cdot 5]$ can be present); the tame argument does not apply and cases must be checked individually;
- $\Delta = 0$ is sufficient but not necessary: IUT only guarantees the global picture, and localization does not preserve surjectivity. But this round's conclusion does not rely on $\Delta = 0$, only on $\Delta \leq \Delta_{\max}$;
- *hierarchy boundary:* the main theorem is applied to tensor packets with $n = j + 1 \geq 3$. For general non-Galois K no closed form of Δ at $n \geq 3$ has been derived or verified; the bound of this subsection uses the global degree N of the Galois closure and is valid for general n. For the S_3 pure-cubic family the closed form now exists at every n (Theorem 16.3), so in that case — which includes the $p \neq \ell$ situation relevant to IUT — the step *is* supported, indeed exactly;
- *asymptoticity of the discount:* as the table of §16.2 shows, the single-prime relative discount can reach 50%. "$\Delta \ll \log_p \rho$" holds only as $h \to \infty$; at small h the crossover position is given by §12.3, and one may not infer per-prime negligibility from it;
- Δ *has no general closed form:* E1QUICK refuted $\Delta = f(d, v)$, and §16.2 gives an analytic determination only on the S_3 pure-cubic family at $n = 2$. Tensor packets at $n \geq 3$ have a different component-set shape, and no closed form is available. This subsection uses only $\Delta \leq \mathrm{poly}(h)$, no closed form, so the conclusions are unaffected; "G4 closed" should be read as "Δ's upper bound is controlled", not "Δ is computed".

17. Gap list from the original text

Ordered by value. G1 is nearly closed (BD-class, §12.1), G3 is closed at layer 1 (§12.5), G4/G5/G8 are closed (see rows); the only substantial gap is layer 2, i.e. G2/G7 (9-1).

#	Gap	Why needed	Impact
G1	exact relation of $h = \log(q^{\forall})$ to log rad of abc	§12 uses "target $\sim h$" as the comparison	**nearly closed**: by IUT IV Cor. 2.2(i) the target is a BD-class of h (§12.1); only the constant factor in the Szpiro/abc dictionary remains, immaterial up to $\kappa \leq 1000$ by §12.3
G2	precise form of Ind1/Ind2 and how hull saturates on the Ind track	the quantifiable part of layer 2	**layer 1 closed** (§9, Theorem 9.1): Ind2 preserves every per-component valuation and adds only $O(d^j)$ to the log-volume; Ind3 is an inclusion (III Prop 3.5(ii)) and can only enlarge the hull; Ind1 is absorbed by the $\mathfrak{S}_j^{\pm}$-symmetry. Remainder: the layer-2 interaction with (9-1) (LANA)
G3	explicit formula for C_Θ in IUT IV Thm 1.10	the chain from qualitative inequality to quantitative one	**closed at layer 1** (§12.5): the inflation enters as $\frac{4}{\ell^*+1} \log \widehat{\rho}$ on the LHS of (12.4), and the chain breaks at the (dsc) $\rightarrow$ (**hull**+**det**) arrow; remainder: (9-1) and §14 bookkeeping
G4	whether K_v is Galois	—	**demoted to a precision item** (§16.3): closed form $\rho_{\mathrm{NG}}^2 = \rho_{\mathrm{G}}^2/p^{\Delta} + \Delta \leq \mathrm{poly}(h) \Rightarrow$ the discount is asymptotically negligible; on the S_3 pure-cubic family Δ is completely determined (Corollary 16.2) at every tensor depth (Theorem 16.3)
G5	whether q generates K_v	—	**logically closed**: rank enters none of the layer-1a/1b/net-index chains; the only use (§8.6) needs only rank $\leq d < d^n$, which holds since $n = j + 1 \geq 2$
G5′	full coordinates of Prop. 3.4(ii) (per-component exponent distribution)	item-by-item confirmation of the all-diagonal model	precision item only
G6	actual d and $v_p(\mathfrak{d})$ of V^{bad}	which value §12 plugs in	scanned with full coverage at $d = 2, 4, 8$; archimedean places covered exactly in §11
G7	exact statement of (9-1) + the λ_e weight distribution	layer 2	no literature-level solution; awaits LANA [6]
G8	order of magnitude of $n_v = v_p(q)$	—	**closed**: $n_v = h_v$ (IUT I Def 3.1(c) + IUT IV Cor 2.2); the branch is decided by the computable criterion $n_v \gtrless v/2$ (§8.4, Theorem 8.3)

18. Conclusion

Within IUT's own parameter range, the integral-structure inflation of the hull operation is doubly exponential:

$$\log\log\rho = \Theta(\sqrt{h}).$$

Measured by crossover: even granting the target a magnitude as large as h^{1000}, the $d = 4$ inflation still overtakes it at $h \approx 9 \times 10^8$. The conclusion is robust to parameter uncertainty in all three directions.

Accurate statement about the "lower bound": the ρ of Theorem 5.1 is the *exact* layer-1a value; layer 1b adds in the same direction (§8.5), making it a lower bound on the true inflation; while the compositum theorem of §16.2 gives $\rho_{\mathrm{NG}}^2 = \rho_{\mathrm{G}}^2/p^{\Delta}$ — i.e. in the non-Galois case ρ can only *decrease* — but §16.3 proves $\Delta \le \mathrm{poly}(h) \ll \exp(\Theta(\sqrt{h}))$, so the discount is asymptotically negligible. The three pieces combined leave the layer-1 conclusion unchanged.

Two branches, one outcome: the layer-1 net effect falls under one of the two branches of §8.4, decided by the computable criterion $n_v \gtrless v/2$ (Theorem 8.3): in the inflationary branch the corrected C_Θ-inequality fails for large h, in the contractionary branch the final inequality is uninformative — *in neither branch does Corollary 3.12 yield abc.* The contractionary side is not a disjunction: its two arms (vacuous and outright failing) are explicit and zero-point invariant (Lemma 8.4). Hence the conclusions above rely on no assumption of "branch luck". The same holds at the archimedean places (§11): the complex places contribute a fixed inflation, the real places nothing.

This paper does not falsify IUT. What it proves is the narrower statement: *even if (9-1) holds, layer-1 inflation alone already suffices to make Corollary 3.12 fail to yield abc.*

The three gaps (Scholze–Stix non-commutativity, LANA compatibility, and the inflation of this paper) occupy the same position, are of different types, and are mutually independent. The layer-1 evidence chain is now closed: the main formula is confirmed by exact computation, the non-Galois discount has a closed form and an explicit bound, and Δ is completely determined on the S_3 pure-cubic family (Corollary 16.2) at every tensor depth (Theorem 16.3). The only agent that can decide (9-1) is the follow-up work of LANA.

Appendix A. Applicability conditions of Theorem 5.1

The proof of Theorem 5.1 uses:

(1) $K/\mathbf{Q}_p$ Galois (surjectivity in Lemma 2.1 relies on $K \otimes K$ splitting completely into d copies of K);
(2) the trace-preservation Lemma 3.1 (valid for every Galois K and every n);
(3) the discriminant relation (0.1).

For the non-Galois case, see §16.

Appendix B. Self-check correction records (18 items)

#	Error	Consequence / correction
B1	Bareiss division silently truncated without an assertion	reported $n = 4, 5$ for $\mathbf{Q}_2(\zeta_{16})^+$ as $-3, -1$; the hard constraint $v_p(\det) \geq 0$ catches it immediately; switched to Fraction + p-adic tracking; true value 1056
B2	$v_2(\mathfrak{d}_{\zeta_{32}^+})$ misused as 65	inflation column wrong; actual discriminant 2^{31}, i.e. 31
B3	degree of $\mathbf{Q}_2(\zeta_{32})$ labeled 8	entire row void; $[\mathbf{Q}_2(\zeta_{32}) : \mathbf{Q}_2] = 16$
B4	automorphism of $\mathbf{Q}_5(\zeta_5)$ assumed to be the identity	"all" not full rank ($4 \neq 16$); corrected to $\sigma_a(\theta) = \theta^a$
B5	mixed tensor product component count taken as 2^{n-1}	all mixed-field values void; true count $m = 2^{n-r}$, r the $\mathbf{F}_2$-rank of the square classes
B6	§8 coordinate model error: value point assumed to be the axis point $(1, \ldots, 1, q^{j^2}, 1, \ldots, 1)$	contradicts Prop. 3.4(ii); per the original text, the value lies in the Galois-invariant submonoid and its coordinates are the full diagonal $(q^{j^2}, \ldots, q^{j^2})$; the corrected loss is **larger**
B7	§8.2 inequality ill-defined: Vol(lattice(Θ)) undefined for a low-rank lattice	comparison meaningless; changed to hull log-volumes on both sides
B8	§8.3 wording too strong: "rank identically d"	counterexample exists; corrected to rank $\leq \min(\ell^*, [\mathbf{Q}_p(q) : \mathbf{Q}_p]) \leq d$, counterexample $q = 2\sqrt{6}$
B9	§12 asymptotic-order error: treating $\log_{10} \log_{10} \rho$ as $\log_{10} \rho$	crossover table underestimated by 10^{149}; the truth is doubly exponential, $\log \log \rho = \Theta(\sqrt{h})$; everything recomputed
B10	§16.2 directional claim wrong: "non-Galois $\Rightarrow$ larger loss"	wrong on the side favorable to us; the compositum theorem proves $\rho_{\mathrm{NG}} \leq \rho_{\mathrm{G}}$: Theorem 5.1 is an **upper bound**

#	Error	Consequence / correction
B11	§12.3 definition/table mismatch (caught by external text-level review): the printed definition read $\log_{10}\rho > \kappa \log_{10} h$ while the table was generated from the stronger, conservative criterion $\log_{10}\rho > h^{\kappa}$ (equivalently $\log_{10}\log_{10}\rho > \kappa \log_{10} h$); the definition is corrected to match the table (all table values and headline scales unchanged). Related: the "gap" row of §14 mixed the $\ln(abc)$ scale with the c-scale; units are now stated explicitly (639 in ln-units; ≈ 140 orders in c)	
B12	three construction errors in the Sage scripts (found in review): **(1)** $\mathbf{Q}(\zeta_{16})^+$ was actually constructed as the whole cyclotomic field ($d = 8$) rather than the real subfield ($d = 4$); **(2)** `CyclotomicField(32).subfields(8)[0][0]` does not guarantee $\mathbf{Q}(\zeta_{32})^+$; **(3)** the integral-basis coordinate change missed a matrix transpose	(3) miscomputes integral-basis coordinates; (1)–(2) mismatch the field labels with the actually constructed fields (each internally consistent, but mislabeled). The review copy fixed these: explicit real-subfield construction + transpose. After the fix, Part C gives 17/17 matches, including $v_2(\mathfrak{d}_{\zeta_{32}^+}) = 31$ (once misrecorded as 65) and $v_2(\rho) = 124$
B13	the Δ script carried an extra "single-prime" gate, skipping the A_4, $p = 19$ and S_4, $p = 283$ samples, so REGRESS passed only 3/5; in addition `Fraction(SageInteger)` raised `TypeError`, and JSON wrote numbers as strings, crashing the analyzer	the gate mistook "local-field interpretation" for "the defining condition of Δ": Δ is a difference of *global* discriminants and is independent of the number of primes. Gate deleted (prediction 5/5; an independent copy reproduced 5/5), `int()` conversion, analyzer `normalize()`
B14	the analyzer once reported "strong support" for "Δ depends only on (d, v)" from the QUICK data (the $A_4/p = 19$ comparison point was then missing)	without the key comparison point, a single value within a group is misread as functional dependence; 11 of 12 groups being single-valued was a resolution problem. After the gate deletion the $(4, 2)$ group shows $\{0, 2\}$, refuting the claim; recorded: "data can refute but cannot distinguish alternative explanations"

#	Error	Consequence / correction
B15	REGRESS still failed on the new ZIP: the literal `2` in `Fraction(int(d)*int(v),2)` was pre-parsed by Sage into `Integer`	`.sage` pre-parsing applies to *all* integer literals, including denominators; B13 had only `int()`-converted the numerators. Fix: remove `fractions.Fraction` entirely and use "doubled quantities" all-integer arithmetic (`dv2`= $d \cdot v$ represents twice $(d\,v)/2$) to display half-integers; a user-side temporary copy ran 5/5 with `int(2)`; the package removes the root cause per the former
B16	the delivery package lacked version identification: two successive ZIPs were mounted as stale copies; when B15 claimed "0 remaining" the package comments still contained the word `Fraction`, so a user checking "contains `Fraction` / no `dv2`" misjudged it as the new package	the attribution chain was repeatedly broken and the same run was counted three times. The final package adds `VERSION.json` (per-file sha256) and `verify_package.py`, self-certifying before runs; all rational-class wording (including comments) purged, the in-package `Fraction` count being exactly 0
B17	guessed $\Delta = 3$ from the decomposition type: when $m \equiv \pm 1 \pmod 9$, 3 decomposes as $(1, 2)$ in K, and two primes were guessed each to contribute one adjoining of ζ_3, accumulating to $\Delta = 3$	left a nonexistent gap for (P) and misled the design of the "minimal experiment". The actual local decomposition is $\mathbf{Q}_3 \times \mathbf{Q}_3(\zeta_3)$: the branch already containing ζ_3 contributes 0, and only the other branch contributes the tame quadratic exponent 1; the 40 instances and the 192-example scan alike give 1. Corollary 16.2 gives $\Delta = v_p(\operatorname{disc} K) \bmod 2$
B18	§8.4 claimed "the layer-1 net effect is permanently inflationary; the contraction branch is unreachable in IUT's parameter space" (repeated in the contributions list and in §18) — a directional overclaim of exactly the B10 family: $n_v = v_p(q_v)$ and $v = v_p(\mathfrak{d}_{K_w})$ are independent parameters in IUT, and with v at its tame bound $(\ell - 1)^2$ (Theorem 8.3) and $n_v = 1$, the window $(0, v/2)$ is reachable and $\mathrm{Net}_j < 1$ for every j.	the contraction branch exists; repaired by the two-branch criterion of §8.4 (inflationary branch: (12.4) fails for large h; contractionary branch: the final inequality is uninformative; neither yields abc) — the conclusion survives in strengthened, exhaustive form. *Caught by external review.*

Three methodological lessons. Lesson from B1. -3 was once written into a report as a "new phenomenon" (albeit marked "not used as a conclusion"). The correct move is to stop it on the spot with a theoretical constraint: $v_p \geq 0$ is a hard constraint; a negative value is necessarily a bug.

Lesson from B9. When the value of a formula is itself a huge integer (2^{500}), "taking $\log_{10}$" and "taking $\log_{10}\log_{10}$" differ by one symbol, but the conclusions differ by an exponential layer. *Whenever large numbers are involved, the definition of every layer of quantity must be written out explicitly.*

Lesson from B10. (Same kind as B1 but more insidious.) B1 is a numerical violation of a hard constraint, decidable on the spot; B10 is a *directional* claim in the reasoning, with no hard constraint to catch it — and it was wrong *on the side favorable to us*. Such errors are the hardest to self-detect, because they make the conclusion "look stronger".

> **Rule.** Every directional claim favorable to our own conclusion must be given a proof or explicitly labeled as a conjecture; it may not be exempted from proof on the grounds of "conservative direction".

Lesson from B5. The automorphism of $\mathbf{Q}_2(\sqrt{2})$ is $\theta \mapsto -\theta$; it cannot be written in the form $\theta \mapsto \theta^a$ — the latter form is valid only for Kummer/cyclotomic types.

Reproducibility. All scripts, data, and excerpts of the original text are appended to the project files and reproduce every number in this paper.

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