

Algebraicity of rank-two scattering wall functions

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Abstract

We prove that every wall function in the rank-two scattering factorization determined by two polynomial initial walls over a field of characteristic zero is algebraic. This resolves the algebraicity question posed by Kontsevich and recorded by Gross and Pandharipande, and its cluster-scattering formulation in the work of Burcroff, Lee, Mou, Musiker, and Reineke.

AI use and scholarly credit

The main results and their proof were developed during a two-hour research conversation with ChatGPT-6-Pro. The research materials were subsequently subjected to an adversarial AI review in Codex. The manuscript also passed review by Rethlas-OSV [18], using OSVerify [17].

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1 Introduction

Scattering diagrams arose in the construction of spaces by gluing torus charts. The gluing transformations must compose to the identity around closed paths, and this requirement produces additional transformations supported on rays. Kontsevich and Soibelman developed the two-dimensional construction in their work on non-archimedean K3 surfaces [11]. Their later theory of stability structures and Donaldson–Thomas invariants [12] organized wall crossing through ordered products of transformations of a torus. The order records the phases, or slopes, of the contributing classes.

In the tropical vertex group, the transformations preserve the logarithmic symplectic form on a two-dimensional torus. Gross, Pandharipande, and Siebert [9] related their factorization coefficients to genus-zero relative Gromov–Witten invariants of toric surfaces. The relevant curves have prescribed contact with the toric boundary and full tangency at one unspecified point on the outgoing divisor. Scattering diagrams, tropical curve counts, degeneration formulas, and multiple-cover contributions enter this correspondence. On the representation-theoretic side, Reineke’s functional equations [20] relate generating series for stable quiver moduli to those for framed moduli and yield integrality statements for Donaldson–Thomas invariants.

Gross and Pandharipande [8] brought the quiver and curve-counting descriptions together and studied the rays and symmetries of the resulting diagrams. Their basic example starts

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with two initial functions $(1+tx)^{\ell_1}$ and $(1+ty)^{\ell_2}$, where ℓ_1, ℓ_2 are positive integers and t, x, y are independent formal variables. The commutator of the corresponding transformations has an ordered factorization indexed by positive rational slopes. Its factor in a primitive direction (d, e) , meaning $d, e > 0$ and $\gcd(d, e) = 1$, is determined by a series in the single monomial $t^{d+e}x^dy^e$. In Section 1.4 they record Kontsevich’s question whether all these series are algebraic over $\mathbb{Q}(t, x, y)$. This is the algebraicity question considered here.

The central slope supplied an early model. Gross and Pandharipande proposed an explicit formula there, noting that its equal-parameter specialization had been proved by Reineke. Reineke and Weist [21, Theorem 11.1 and Corollary 11.2] proved the formula for arbitrary ℓ_1, ℓ_2 through complete bipartite quivers and the refined Gromov–Witten/Kronecker correspondence. If $z = t^2xy$ and $C(z) \in 1 + z\mathbb{Q}[[z]]$ is the solution of

$$C(z) = 1 + zC(z)^{(\ell_1-1)(\ell_2-1)},$$

then the central wall function is $C(z)^{\ell_1\ell_2}$. Thus even diagrams with infinitely many rays have distinguished wall functions governed by finite algebraic equations. Their quiver-theoretic functional equations also describe other slopes, with the central case admitting this particular finite reduction.

Kontsevich announced a general factorization theorem in his Miami lecture of 30 January 2013. A two-variable formal logarithmic symplectomorphism is factored into the parts below, on, and above the diagonal; the theorem states that the original map is algebraic if and only if the three factors are algebraic. His handwritten notes [10, pp. 5–7] and Auroux’s notes [1, pp. 2–3] give a geometric proof outline using rational curves, marked points, tangent data, and a cross-ratio. A finite monomial covering sends any positive rational direction to the diagonal. Consequently this statement encompasses the full algebraicity question of Gross and Pandharipande; Section 7 proves the reduction.

Algebraic generating series also occur in spectral-network calculations. Galakhov, Longhi, Mainiero, Moore, and Neitzke [6, Section 3] obtained algebraic equations for families of BPS states and compared them with tropical vertex formulas. Mainiero [14, Section 1.1] reports an indirect algebraicity argument of Kontsevich and Soibelman, citing private communication. He studies explicit spectral-network equations, the relation between Donaldson–Thomas series and Euler-characteristic series, and the asymptotics obtained from algebraicity. These calculations concern specified families of dimension vectors and make the algebraic equations themselves accessible.

The subsequent work of Kontsevich and Soibelman [13] develops analytic stability data and proves preservation of analyticity under sector factorization. Section 6.3 also describes polynomial, rational, and algebraic stability data through non-archimedean geometry and states an algebraicity theorem for its algebraic class with rational central charge. The discussion records these facts with their detailed treatment assigned to future work. This gives a later account of the geometric approach alongside the 2013 lecture.

In cluster theory, Gross, Hacking, Keel, and Kontsevich [7] use scattering diagrams and broken lines to construct theta functions and prove positivity results. Mou [15] extends this construction to generalized cluster algebras with exchange polynomials factored into binomials. Burcroff, Lee, and Mou introduce tight gradings in [3] and prove a coefficient formula for arbitrary rank-two polynomial initial functions in [4, Theorem 1.2]. A tight grading is an integer decoration of a maximal Dyck path subject to compatibility and shadow conditions. Its weight is a monomial in the initial coefficients, so the formula gives both a finite coefficient calculation and coefficientwise positivity in the universal parameters.

Burcroff, Lee, Mou, Musiker, and Reineke [5] combine this combinatorics with quiver moduli to study rank-two cluster scattering coefficients, including their Weyl group symmetry and the structure of tight-grading frames. Their Section 9.3 asks again whether every wall

function is algebraic. Under the parameter correspondence $(\ell_1, \ell_2) = (c, b)$, its wall functions and the tropical vertex wall functions differ by a positive integral power; Proposition 6.1 gives the exact exponent. The same section treats coefficient growth as a separate question and formulates a slope-dependent limit in Conjecture 9.2.

We prove algebraicity for two arbitrary polynomial initial functions over any field of characteristic zero. The proof passes from the tight-grading formula to a finite system that treats all multiples of a fixed direction simultaneously. We identify a single rooted interval in each tight grading, describe its underlying word by a finite automaton, and record the matching constraints by a stack with one free-balance variable. This yields a finite system of discrete differential equations to which the theorem of Notarantonio and Yurkevich [16, Theorem 1.3] applies. Appendix B expands the coefficient formula in the precise case used here, following the triangular method of [4, Section 5.5]; Appendix A gives a second closure argument using nested algebraic approximation.

1.1 Wall functions and the main theorem

Let $\mathbb{F}$ be a field of characteristic zero, and let X, Y be independent formal variables. A vector $(d, e) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0, 0)\}$ is *primitive* if $\gcd(d, e) = 1$. For $G(z) \in 1 + z\mathbb{F}[[z]]$, define an $\mathbb{F}$ -algebra automorphism of $\mathbb{F}[[X, Y]]$ by

$$\theta_{(d,e),G}(X) = XG(X^dY^e)^{-e}, \quad \theta_{(d,e),G}(Y) = YG(X^dY^e)^d.$$

The inverse uses G^{-1} in place of G . These maps preserve the logarithmic two-form $dX/X \wedge dY/Y$, interpreted after inverting X and Y . Products of automorphisms below mean composition, with the rightmost map applied first, and are completed with respect to total degree in X, Y .

Fix polynomials

$$P_1(X) = 1 + \sum_{s=1}^a w_s X^s, \quad P_2(Y) = 1 + \sum_{h=1}^b v_h Y^h, \quad w_s, v_h \in \mathbb{F}, \quad (1)$$

where $a, b \geq 1$ are degree bounds. Coefficients may vanish, including all the nonconstant coefficients of either polynomial. Write $w_0 = v_0 = 1$, and extend these lists by zero beyond their respective bounds. Set

$$S = \theta_{(1,0),P_1}, \quad T = \theta_{(0,1),P_2}.$$

The ordered-factorization theorem for the tropical vertex group gives unique series $G_{d,e}(z) \in 1 + z\mathbb{F}[[z]]$, indexed by primitive $d, e > 0$, such that

$$T^{-1}STS^{-1} = \prod_{\substack{d,e>0 \\ \gcd(d,e)=1}}^{\rightarrow} \theta_{(d,e),G_{d,e}}. \quad (2)$$

The factors act in increasing order of e/d , so they are written from right to left in that order. Modulo any fixed total degree the product is finite. This is the convention of [8, Section 1.3] and [4, Section 1.1]. We call $G_{d,e}$ the wall function in direction (d, e) ; a missing wall has function 1.

Theorem 1.1. *For every choice of (1) and every primitive pair $d, e > 0$, the wall function $G_{d,e}(z)$ in (2) is algebraic over $\mathbb{F}(z)$.*

Here algebraicity means that there exists a nonzero polynomial $A(z, Z) \in \mathbb{F}[z, Z]$ with $A(z, G_{d,e}(z)) = 0$. The proof occupies Sections 2–5.

Taking $P_1 = (1 + X)^{\ell_1}$ and $P_2 = (1 + Y)^{\ell_2}$ gives the following answer to Kontsevich’s question.

Corollary 1.2. *For all positive integers ℓ_1, ℓ_2 and all primitive pairs $d, e > 0$, the wall function $G_{d,e}(t^{d+e}x^dy^e)$ associated to $(1 + tx)^{\ell_1}, (1 + ty)^{\ell_2}$ is algebraic over $\mathbb{Q}(t, x, y)$.*

Proof. Apply Theorem 1.1 over $\mathbb{Q}$ and substitute $z = t^{d+e}x^dy^e$ in a nonzero polynomial relation. The monomial substitution is injective on $\mathbb{Q}[z]$ and gives the required relation. $\square$

2 Tight gradings and protected matching

Fix a primitive pair $d, e > 0$ and write

$$G_{d,e}(z) = 1 + \sum_{k \geq 1} c_k z^k.$$

For a formal series J in named variables, brackets such as $[z^k]J$ denote the coefficient of the indicated monomial; absent monomials have coefficient zero. Thus $c_k = [z^k]G_{d,e}$.

For each $k \geq 1$, choose positive integers D_1, D_2 satisfying

$$eD_1 - dD_2 = 1, \quad D_1 > kd + 1, \quad D_2 > ke. \quad (3)$$

Bézout’s identity gives one integral solution, and adding a sufficiently large multiple of (d, e) gives these inequalities. In particular $\gcd(D_1, D_2) = 1$. All constructions involving D_1, D_2 in this section are for this fixed k .

2.1 Paths, shadows, and the coefficient formula

The maximal lower Dyck path $C(D_1, D_2)$ joins $(0, 0)$ to (D_1, D_2) by east and north unit edges and stays as close as possible to the segment from $(0, 0)$ to (D_1, D_2) while remaining below it. Its word is

$$\prod_{x=1}^{D_1} EN^{\lfloor xD_2/D_1 \rfloor - \lfloor (x-1)D_2/D_1 \rfloor},$$

where concatenation reads from left to right. We also read the path cyclically, or extend it periodically by translation by (D_1, D_2) . Its edges retain their labels under cyclic reading.

A grading ω assigns a nonnegative integer to each edge. An edge is selected if its grade is positive. For an oriented interval J of the path, let J_E, J_N be its horizontal and vertical edges, respectively, and write $\omega(J_E) = \sum_{h \in J_E} \omega(h)$ and $\omega(J_N) = \sum_{v \in J_N} \omega(v)$. We consider gradings of totals

$$p = \omega(C_N) = kd, \quad q = \omega(C_E) = ke.$$

For a selected horizontal edge h , its horizontal shadow interval starts at h and ends at the first vertical edge for which $|J_N| = \omega(J_E)$. Its shadow $\text{sh}(h)$ is the set of vertical edges in this interval. For a selected vertical edge v , read backwards from v to the first horizontal edge for which $|J_E| = \omega(J_N)$; the horizontal edges in that interval form $\text{sh}(v)$. A zero-grade edge has empty shadow. Since $q < D_2$ and $p < D_1$, these first returns exist within one period. Shadows of collections of edges mean unions of their shadows.

Compatibility means the following. For every selected horizontal edge h and selected vertical edge v , in the cyclic interval from h to v at least one of these conditions holds:

- a prefix ending at a vertical edge strictly before v has as many vertical edges as total horizontal grade;
- a suffix beginning at a horizontal edge strictly after h has as many horizontal edges as total vertical grade.

These are the compatible-grading conditions of [4, Definition 3.14], with the endpoints made explicit. A compatible grading is *shadowed* if all selected vertical edges lie in the horizontal shadows, or all selected horizontal edges lie in the vertical shadows. Under (3) and the prescribed totals, these are exactly the tight gradings of [4, Definition 5.4], since

$$pD_2 - qD_1 = -k = -\gcd(p, q).$$

Its weight is

$$\text{wt}(\omega) = \prod_{h \in C_E} v_{\omega(h)} \prod_{v \in C_N} w_{\omega(v)}. \quad (4)$$

Thus horizontal grades are bounded by b and vertical grades by a in every nonzero contribution.

We use the following coefficient formula, in precisely the determinant-one case of [4, Theorem 1.2]:

$$c_k = \sum_{\omega \text{ tight on } C(D_1, D_2)} \text{wt}(\omega). \quad (5)$$

The hypotheses follow from (3). Equivalently, [4, Corollary 5.14(2), Theorem 5.17] applies with exponent $|dD_2 - eD_1| = 1$. Appendix B derives the formula in this strict determinant-one case.

The formula is a polynomial identity in the initial coefficients and consequently holds over the given field with the given weights. One can implement specialization in the completed tropical vertex group by first substituting the universal coefficients as $p_{1,s} \mapsto w_s \tau^s$ and $p_{2,h} \mapsto v_h \tau^h$, with an auxiliary formal variable τ . These substitutions are continuous for the coefficient-ideal and τ -adic completions. Homogeneity identifies the result with $P_1(\tau x), P_2(\tau y)$, and identifies every coefficient of bidegree (p, q) after removing its common factor τ^{p+q} . This proves (5) for every specialization, including vanishing weights.

2.2 The matching rule

A positive horizontal grade h opens a packet of h blue tokens. A vertical edge removes one token from the most recently opened nonempty packet. The emptying of a packet is the first return defining its horizontal shadow interval: all packets opened inside it must empty before its last token is removed.

This description is valid also for the cyclic path. On the periodically extended path, give a horizontal edge increment equal to its grade and a vertical edge increment -1 . A selected horizontal edge starts a positive excursion; the total increment over one period is $q - D_2 < 0$, so its first return exists before one full period. If another excursion starts inside it, the later excursion returns before the earlier one: otherwise at the earlier return its increment from its own starting height would already be negative, and a downward step of size one must have crossed zero first. Thus the intervals are nested or disjoint, including across the cyclic endpoint. Every selected horizontal edge lies in a maximal interval. Summing the first-return equalities over disjoint maximal intervals shows that their total number of vertical edges is q . Since $q < D_2$, a vertical edge outside them exists; cutting there makes all maximal intervals ordinary linear intervals. Reading backwards with vertical increments equal to their grades and horizontal increments -1 proves the same facts for vertical shadows, whose total horizontal mass is p .

Independently, let each horizontal edge create one red token, and let a vertical edge of grade s remove the s most recent unmatched red tokens whenever possible. The first-return interpretation of this second matching will encode compatibility.

Lemma 2.1. *Suppose the red matching succeeds through a positive-grade vertical edge v . If h is the earliest horizontal edge whose token is removed at v , then the interval from h to v is the shortest backwards interval satisfying $|J_E| = \omega(J_N)$.*

Proof. Let the red height after a prefix be the number of its horizontal edges minus its total vertical grade. Write H for its value just after v . The token created at h was inserted from height H to height $H + 1$ and remained unmatched until v . Every horizontal edge strictly between h and v was therefore inserted from a height greater than H . The interval starting at h is balanced, and no interval starting at a later horizontal edge is balanced. $\square$

Consider a maximal blue interval. Its first horizontal edge is its root. A positive horizontal edge is called active until its blue packet empties. If it empties strictly before a given vertical edge, its closing edge supplies the horizontal compatibility witness. Otherwise its red token must be protected from the demand at that vertical edge. In particular the demand must stay above the most recent active root.

This gives a stack of records (r, z) . Here r is the remaining number of blue tokens of an active root, and z is the number of free red tokens above its protected token and below any more recent active root. Only the top record is changed by the next edge:

1. A zero-grade horizontal edge increases its z by one.
2. A horizontal edge of grade $h > 0$ pushes the record $(h, 0)$.
3. A vertical edge of grade $s \geq 0$ requires $z \geq s$ and replaces (r, z) by $(r - 1, z - s)$.
4. If r becomes zero, pop the record. If a parent remains, add $z + 1$ to its free balance; the extra token is the former root's token.

The demand in step 3 is tested before step 4. For example, adjacent edges of grades E_1N_1 fail this test and are incompatible, whereas $E_1E_0N_1$ is accepted with final free balance zero.

Lemma 2.2. *On a maximal blue interval, compatibility for all forward selected pairs is equivalent to legality of the protected matching. If the interval has $m + 1$ horizontal edges, its total vertical grade is at most m .*

Proof. Assume all earlier steps are legal, and let h be the most recent active root at a vertical edge v of positive grade s . If $s \leq z$, Lemma 2.1 gives a backwards witness strictly after h . It is also strictly after every older active root. Roots that have already closed have a strict horizontal witness.

If $s > z$, the relative red height after v , measured immediately after h , is negative. Before this step that relative height was nonnegative throughout the interval after h , since the token of h remained protected. In particular the height just before every intervening horizontal edge was nonnegative. None can start a balanced suffix ending at v . The horizontal witness for (h, v) also fails because h is active at v . Thus compatibility fails at the first illegal step. Zero-grade vertical edges impose no selected-pair condition and satisfy the demand automatically.

The root remains protected through the last vertical edge of the interval. Its token is therefore never consumed inside the interval. At most the other m red tokens can be consumed, proving the bound. $\square$

Cyclic pairs are covered by the same witnesses. If a selected vertical edge is earlier than a selected horizontal edge after our cut, the cyclic interval passes the latter's blue closing edge before reaching its target. This is a strict horizontal witness. The same argument treats pairs in different maximal blue intervals.

3 The single-root frame

We next determine the underlying words of all the tight gradings in (5). This will replace the path of k -dependent size by a fixed finite automaton.

Assign to an edge starting at (x, y) the rank

$$\rho(x, y) = D_2x - D_1y.$$

The horizontal starts have ranks in $[0, D_1)$ and the vertical starts have ranks in $[D_1, D_1 + D_2)$. Indeed, a horizontal edge starts at $(x, \lfloor D_2x/D_1 \rfloor)$, and a vertical edge in column x starts at a height between $\lfloor D_2(x-1)/D_1 \rfloor$ and $\lfloor D_2x/D_1 \rfloor - 1$. Along each step the rank increases by D_2 modulo $D_1 + D_2$. Coprimality implies that the ranks of all edge starts are exactly $0, 1, \dots, D_1 + D_2 - 1$.

Lemma 3.1. *Every tight grading of totals $(p, q) = (kd, ke)$ has all its selected vertical edges covered by horizontal shadows.*

Proof. We show that the other shadow inclusion would force $pD_2 - qD_1 > 0$. A maximal vertical shadow interval begins with a horizontal edge and ends with its root vertical edge. Write its numbers of edges as n_i horizontal and $m_i + 1$ vertical. First-return balance says that its total vertical grade is n_i .

For each selected horizontal edge in this interval, vertical compatibility with the root fails: the root's first backwards return reaches the beginning of the interval. Horizontal compatibility consequently supplies a first-return interval ending strictly before the root. All these horizontal shadows lie inside the maximal vertical interval and avoid its last edge. Restricting the horizontal grading to this interval preserves those shadows. Their union has size equal to its total horizontal grade, by the same blue matching as above. This grade is at most m_i .

Let r_i be the rank of the first horizontal edge. The start of the last vertical edge has rank $r_i + n_iD_2 - m_iD_1$. Since the former rank is below D_1 and the latter is at least D_1 , we have $n_iD_2 - m_iD_1 > 0$. All selected horizontal edges are in these maximal intervals under the assumed inclusion. Summing, and using the vertical shadow mass $\sum_i n_i = p$, gives

$$pD_2 - qD_1 = \sum_i (n_iD_2 - m_iD_1) + (\sum_i m_i - q)D_1 > 0.$$

But (3) gives $pD_2 - qD_1 = -k$. Thus only the stated shadow inclusion is possible. $\square$

Proposition 3.2. *A tight grading of totals (kd, ke) has exactly one maximal horizontal shadow interval. It has $kd + 1$ horizontal edges and ke vertical edges, begins at a horizontal edge of rank ℓ with $0 \leq \ell < k$, and ends with free red balance zero. All selected edges belong to this interval.*

Proof. By Lemma 3.1, all selected vertical edges belong to the maximal blue intervals. Write their edge counts as $(m_i + 1, n_i)$, their root ranks as r_i , and set

$$\delta_i = n_iD_1 - m_iD_2.$$

The last vertical edge has starting rank $r_i + D_1 + D_2 - \delta_i$, which is less than $D_1 + D_2$. Hence $0 \leq r_i < \delta_i$. Blue balance and Lemma 2.2 give $\sum_i n_i = q$ and $\sum_i m_i \geq p$. Therefore

$$k = qD_1 - pD_2 = \sum_i \delta_i + (\sum_i m_i - p)D_2. \quad (6)$$

All δ_i are positive, and $D_2 > ke \geq k$. The nonnegative integer $\sum_i m_i - p$ must be zero. Consequently $\sum_i m_i = p$ and $\sum_i \delta_i = k$.

Using $eD_1 - dD_2 = 1$ gives

$$d\delta_i - m_i = (dn_i - em_i)D_1.$$

Both $d\delta_i$ and m_i lie in $[0, kd]$, whose length is less than D_1 . Their difference is a multiple of D_1 , so

$$m_i = d\delta_i, \quad n_i = e\delta_i. \quad (7)$$

Starting at rank r_i , the horizontal edge reached after r_id horizontal steps has relative height

$$\left\lfloor \frac{r_i + r_idD_2}{D_1} \right\rfloor = r_ie,$$

and hence rank zero. It belongs to the interval because $r_i < \delta_i$ and $r_id \leq m_i$. Every maximal blue interval thus contains the same rank-zero horizontal edge. Since these intervals are disjoint, there is exactly one.

Its counts follow from (7) and (6), and its root rank satisfies $0 \leq \ell < k$. All positive horizontal grades belong to a blue interval by construction. The kd units of vertical grade consume all red tokens other than the root token, so its final free balance is zero. $\square$

Let A be the maximal lower Dyck word from $(0, 0)$ to (d, e) . It starts with E and ends with N ; write

$$A = EMN, \quad U = EM, \quad B = ENM, \quad (8)$$

where M is the intervening word, possibly empty. These words encode the two periodic phases of a frame. For example, $(d, e) = (2, 3)$ gives $A = ENENN$, $M = NEN$, $U = ENEN$, and $B = ENNEN$.

Lemma 3.3. *For every $0 \leq \ell < k$, the interval starting at the horizontal edge of rank ℓ and ending at local displacement $(kd + 1, ke)$ exists in $C(D_1, D_2)$ and has word*

$$W_{k,\ell} = A^\ell U B^{k-\ell-1} EN, \quad 0 \leq \ell < k. \quad (9)$$

In particular this describes the interval in Proposition 3.2.

Proof. After x horizontal steps from the root, the local height of the next horizontal edge is

$$\begin{aligned} y_\ell(x) &= \left\lfloor \frac{\ell + xD_2}{D_1} \right\rfloor \\ &= \left\lfloor \frac{ex}{d} \right\rfloor - \mathbf{1}_{\{d|x, x/d > \ell\}}, \quad 1 \leq x \leq kd. \end{aligned} \quad (10)$$

Here $\mathbf{1}$ is 1 if its subscripted condition holds and 0 otherwise. To verify the second equality, substitute $D_2 = (eD_1 - 1)/d$. If $d \nmid x$, the distance of ex/d to an integer is at least $1/d$, whereas the perturbation $(\ell - x/d)/D_1$ has absolute value less than $k/D_1 < 1/d$. For $x = jd$, the sign of $\ell - j$ gives the asserted correction.

At $x = kd$ the height is $ke - 1$. The horizontal edge there has rank $\ell + D_1 - k < D_1$; its successor has rank $\ell + D_1 + D_2 - k \geq D_1$, so the next edge is vertical. The frame stops at this edge. Formula (10) leaves the first ℓ copies of A unchanged and moves the last N of every later primitive block past the next E . This gives (9). $\square$

An *accepted decoration* of a word $W_{k,\ell}$ is a grading with horizontal grades at most b and vertical grades at most a , whose first horizontal grade is positive, for which the protected matching is legal, whose root closes exactly at the last vertical edge, and whose final free balance is zero. Let $\text{Accept}(W_{k,\ell}) \in \mathbb{F}$ be the sum of its weights as in (4).

Proposition 3.4. *For every $k \geq 1$,*

$$c_k = \sum_{\ell=0}^{k-1} \text{Accept}(W_{k,\ell}). \quad (11)$$

Proof. Proposition 3.2 and Lemmas 2.2 and 3.3 send each nonzero-weight tight grading to an accepted decoration with its root rank. Conversely, start an accepted decoration at the unique horizontal edge of rank ℓ in $C(D_1, D_2)$ and set all outside grades to zero. Lemma 3.3 identifies its word with the actual interval. The total length $k(d+e)+1$ is less than $D_1 + D_2$, so no edge is repeated. Protected matching and the cyclic-pair argument prove compatibility. The root shadow covers every selected vertical edge.

Blue and red balance respectively give total horizontal grade ke and total vertical grade kd , so this is a tight grading of the prescribed totals. The constructions are inverse and preserve weights. Distinct root ranks give distinct labelled gradings even when the words coincide. Equation (5) now gives (11). $\square$

4 A finite system with one catalytic variable

4.1 The frame automaton

For the fixed direction (d, e) , the family of words in (9), counted with its root-rank multiplicities, is the noncommutative series

$$\sum_{\ell, r \geq 0} A^\ell U B^r E N.$$

Construct a directed labelled graph with three distinguished vertices: the start, a second-stage vertex, and the finish. Attach an A -labelled cycle at the start, a U -labelled path from the start to the second stage, a B -labelled cycle at the second stage, and an EN -labelled path from the second stage to the finish. Replace each word-labelled path by its individual E, N edges, choosing pairwise distinct internal vertices for the four pieces. Write $|W|$ for the number of letters of a word W . There are no empty edges, since $|U| = d + e - 1 \geq 1$.

The graph has $m = 3(d + e)$ vertices: the four pieces contribute respectively $d + e - 1, d + e - 2, d + e - 1, 1$ internal vertices in addition to the three distinguished vertices. An accepting path is uniquely specified by the numbers ℓ, r of traversals of the two cycles, so it reads $W_{k,\ell}$ with $k = \ell + r + 1$.

Number the vertices from 1 to m . Let $\mathbf{E}, \mathbf{N}$ be the $m \times m$ matrices whose (i, j) entries count, respectively, the E -edges and N -edges from i to j . Let α be the row vector with a 1 at the start and zeros elsewhere, and let β be the corresponding column vector for the finish. For a word W , $W(\mathbf{E}, \mathbf{N})$ means the matrix product obtained by replacing each letter by its matrix in the same order; the empty word gives the identity matrix I . Then $\alpha W(\mathbf{E}, \mathbf{N}) \beta$ counts accepting paths reading W . This counts paths with their multiplicities, which is precisely the convention required by (11).

4.2 Subtrees and free red balance

Introduce two independent formal variables: t marks word length and u marks free red balance. All matrix series in this section have entries in $\mathbb{F}[u][[t]]$, the ring of formal t -series

with polynomial coefficients in u . For any fixed word length, the number of red tokens is bounded by that length, so each such coefficient is a polynomial.

A complete positive-root subtree starts at a positive horizontal edge, includes its nested packets, and stops when that root's blue packet empties. Its free balance at closure excludes the root's own token. Let F be the matrix series counting all such legal subtrees, weighted by (4), by $t^{|W|}$, by u to the free balance, and by $W(\mathbf{E}, \mathbf{N})$. Thus the (i, j) entry also counts all paths from vertex i to vertex j reading the decorated word. The balance at closure may be any nonnegative integer.

While a parent is active, the atoms between two successive vertical edges that consume its own blue tokens are zero-grade horizontal edges and complete child subtrees. The former contributes $tu\mathbf{E}$; a child counted by F contributes uF , because its formerly protected root token becomes free. If K counts arbitrary sequences of these atoms, including the empty sequence, then

$$K = I + (tu\mathbf{E} + uF)K. \quad (12)$$

The child cannot consume any of the parent's existing free tokens: its own root token separates them from its internal demand. Consequently its legality is independent of the balance with which the parent enters this atom. Every atom has positive length, so the geometric expansion $K = \sum_{j \geq 0} (tu\mathbf{E} + uF)^j$ is well defined coefficient by coefficient in t .

For a matrix series $J(t, u)$ over $\mathbb{F}[u][[t]]$, define the divided-difference operator and its weighted sum entrywise by

$$\Delta J(t, u) = \frac{J(t, u) - J(t, 0)}{u}, \quad \Phi(J) = \sum_{s=0}^a w_s \Delta^s J, \quad (13)$$

where Δ^0 is the identity and Δ^s is s successive applications. In particular $\Delta^s(u^j) = u^{j-s}$ for $j \geq s$ and is zero for $j < s$. Thus Φ implements a vertical demand with its weight and discards insufficient balances.

For $0 \leq j \leq b$, let P_j count prefixes after j vertical edges assigned directly to a root, with the root edge itself omitted and with no final atom sequence appended. The tree decomposition gives

$$P_0 = I, \quad P_j = t\Phi(P_{j-1}K)\mathbf{N} \quad (1 \leq j \leq b), \quad F = t\mathbf{E} \sum_{h=1}^b v_h P_h. \quad (14)$$

The product $P_{j-1}K$ combines the preceding free balance with that returned by the next sequence of atoms before the demand is applied. Blue matching uniquely determines the child intervals and the directly assigned vertical edges. Hence this decomposition counts each decoration once for each graph path reading it. Matrix multiplication sums over the intermediate graph vertex at each concatenation.

4.3 Polynomial fixed-point equations

To put (12)–(14) into the form required by a discrete-differential theorem, keep K, F as unknowns and introduce the matrices

$$R_0 = K, \quad R_j = P_j K \quad (1 \leq j < b).$$

The matrices R_j represent a prefix followed by the next atom sequence. The equations become

$$\begin{aligned} R_j &= t\Phi(R_{j-1})\mathbf{N}K & (1 \leq j < b), \\ K &= I + tu\mathbf{E}K + t^2u\mathbf{E}\left(\sum_{h=1}^b v_h\Phi(R_{h-1})\right)\mathbf{N}K, \\ F &= t^2\mathbf{E}\left(\sum_{h=1}^b v_h\Phi(R_{h-1})\right)\mathbf{N}. \end{aligned} \tag{15}$$

When $b = 1$, there are just the unknown matrices K, F . These equations are equivalent to the tree equations: from a solution of (15), define P_j by (14). Starting with $P_0K = R_0$, induction gives $P_jK = R_j$ for $j < b$; the formula for F and then (12) follow. The forward implication follows by substitution.

List all the matrix entries as scalar unknowns Z_i , $1 \leq i \leq (b+1)m^2$, in any fixed order. Let $f_i \in \{0, 1\}$ be their respective initial values. Expanding the products in (15) gives polynomials $\mathcal{Q}_i$ over $\mathbb{F}$ such that every equation has the form

$$Z_i = f_i + t\mathcal{Q}_i((\Delta^s Z_r)_{1 \leq r \leq (b+1)m^2, 0 \leq s \leq a}, t, u), \tag{16}$$

For example,

$$[\Phi(R_{j-1})\mathbf{N}K]_{i\ell} = \sum_{r=1}^m \sum_{s=1}^m \sum_{h=0}^a w_h \Delta^h (R_{j-1})_{ir} \mathbf{N}_{rs} K_{s\ell}.$$

Every other matrix product expands in the same way into a finite sum of products of scalar entries. Thus each $\mathcal{Q}_i$ is a polynomial in commuting scalar unknowns and their listed divided differences.

Lemma 4.1. *System (15) has a unique solution in matrices over $\mathbb{F}[u][[t]]$. It is the solution counted by the protected tree decomposition.*

Proof. Its constant terms are $K(0, u) = I$, $R_j(0, u) = 0$ for $j > 0$, and $F(0, u) = 0$. The operator Δ preserves divisibility by every power of t . Thus inputs agreeing modulo t^n give right-hand sides agreeing modulo t^{n+1} in (16). Successive substitution from the constant terms stabilizes each coefficient and gives a unique solution. Each stabilization involves finitely many polynomial operations in u and finitely many divided differences, so the coefficients remain polynomial in u . The tree decomposition satisfies the equations and the same constant terms, which identifies the solution. $\square$

5 Algebraicity and extraction

We use the following form of the system theorem of Notarantonio and Yurkevich [16, Theorem 1.3]. Over any field of characteristic zero, a finite system of the form (16), with polynomial initial terms, polynomial right-hand sides, one catalytic variable, and a fixed finite maximum order of divided differences, has a unique solution in $\mathbb{F}[u][[t]]$, and every component is algebraic over $\mathbb{F}(t, u)$. The difference operator in [16] is taken at an arbitrary field element; here that element is zero.

All these hypotheses hold for (15). In particular the state number $m = 3(d + e)$ and the difference order a are independent of k . It follows that every entry of F is algebraic over $\mathbb{F}(t, u)$.

Lemma 5.1. *If $J(t, u) \in \mathbb{F}[u][[t]]$ is algebraic over $\mathbb{F}(t, u)$, then $J(t, 0)$ is algebraic over $\mathbb{F}(t)$.*

Proof. Choose a nonzero polynomial $P(t, u, Z) \in \mathbb{F}[t, u, Z]$ annihilating J . Divide its coefficients by their greatest common power of u . Since multiplication by u is injective on $\mathbb{F}[u][[t]]$, the resulting polynomial still annihilates J and its specialization at $u = 0$ is nonzero. Evaluating the identity at $u = 0$ gives the desired relation. $\square$

Proof of Theorem 1.1. The graph selects the words $W_{k,\ell}$ with their root-rank multiplicities. Specializing $u = 0$ selects subtrees whose final free balance is zero. Proposition 3.4 therefore gives

$$\alpha F(t, 0)\beta = \sum_{k \geq 1} c_k t^{k(d+e)+1} = t(G_{d,e}(t^{d+e}) - 1). \quad (17)$$

The extra unit of length is the root horizontal edge. By the system theorem and Lemma 5.1, the left side is algebraic over $\mathbb{F}(t)$. Hence so is $G_{d,e}(t^{d+e})$. The extension $\mathbb{F}(t)/\mathbb{F}(t^{d+e})$ is finite algebraic, so transitivity makes this series algebraic over $\mathbb{F}(t^{d+e})$. Pulling back a nonzero relation through the injective substitution $z \mapsto t^{d+e}$ proves that $G_{d,e}(z)$ is algebraic over $\mathbb{F}(z)$. $\square$

Example 5.2. For $P_1(X) = 1 + w_1X$ and $P_2(Y) = 1 + v_1Y$, the direction $(1, 1)$ has wall function $G_{1,1}(z) = 1 + w_1v_1z$.

Proof. At $k = 1$ the only accepted decoration is $EEEN$ with grades $(1, 0, 1)$, of weight v_1w_1 . For $k > 1$ the frames are $(EN)^\ell E(EN)^{k-\ell-1}EN$. If $\ell > 0$, the root closes at the first N , too early. If $\ell = 0$, every EN before the final EN must be a child with grades $(1, 0)$: a zero horizontal grade would close the global root, and a positive vertical grade would consume the child's protected token. The last vertical edge can have grade at most one, whereas the required total is k . Thus no accepted decoration occurs for $k > 1$. $\square$

6 The cluster-scattering formulation

In this section b, c denote positive cluster parameters, replacing the polynomial degree bound b used in the preceding construction. Consider the rank-two diagram with initial functions $1 + x_1^b$ and $1 + x_2^c$, in the tropical vertex convention. For primitive $d, e > 0$, write its wall function on the ray $\mathbb{R}_{\leq 0}(db, ec)$ as

$$H_{d,e}(z) = 1 + \sum_{k \geq 1} \tau^{(b,c)}(kd, ke) z^k, \quad z = x_1^{db} x_2^{ec}.$$

This defines the coefficients $\tau^{(b,c)}(kd, ke)$ used here. The support in multiples of (db, ec) follows because the initial monomials lie in $b\mathbb{Z} \oplus c\mathbb{Z}$, a property preserved by the completed commutator and its homogeneous factorization. Put $g = \gcd(db, ec)$; the primitive lattice vector of this wall is $(db/g, ec/g)$.

Proposition 6.1. *Let $G_{d,e}$ correspond to $P_1(X) = (1 + X)^c$ and $P_2(Y) = (1 + Y)^b$. Then $g \mid bc$ and*

$$G_{d,e}(z) = H_{d,e}(z)^{bc/g}. \quad (18)$$

In particular the two series are algebraic simultaneously.

Proof. For every prime p , at least one of d, e is not divisible by p . Writing $\nu_p(n)$ for the exponent of p in a positive integer n , we obtain

$$\nu_p(g) = \min\{\nu_p(db), \nu_p(ec)\} \leq \nu_p(bc).$$

Thus bc/g is a positive integer.

Restrict the initial automorphisms to $X = x_1^b$, $Y = x_2^c$. They act as

$$(X, Y) \mapsto (X, Y(1 + X)^c), \quad (X, Y) \mapsto (X(1 + Y)^{-b}, Y).$$

The general wall with primitive vector $(db/g, ec/g)$ restricts to

$$X \mapsto XH_{d,e}(X^dY^e)^{-bce/g}, \quad Y \mapsto YH_{d,e}(X^dY^e)^{bcd/g}.$$

The change of lattice preserves slope order. Uniqueness of ordered factorization therefore gives (18). Algebraicity is preserved under integral powers. Conversely $H_{d,e}$ satisfies $Z^{bc/g} - G_{d,e} = 0$, so transitivity of algebraic extensions proves the other implication. $\square$

Lemma 6.2. *Let $J(z) \in \mathbb{F}[[z]]$ and let $q \in \mathbb{Z}_{\geq 0}^r \setminus \{0\}$. For independent formal variables $X_1, \dots, X_r$, write $X^q = \prod_i X_i^{q_i}$. Then $J(z)$ is algebraic over $\mathbb{F}(z)$ if and only if $J(X^q)$ is algebraic over $\mathbb{F}(X_1, \dots, X_r)$.*

Proof. The forward implication follows by substitution. For the converse, clear denominators in a nonzero polynomial relation to obtain

$$P(X, Z) \in \mathbb{F}[X_1^{\pm 1}, \dots, X_r^{\pm 1}, Z] \setminus \{0\}.$$

Group its finitely many X -exponents by their cosets in $\mathbb{Z}^r / \mathbb{Z}q$, choosing a representative r_C for each occurring coset. Then

$$P(X, Z) = \sum_C X^{r_C} P_C(X^q, Z), \quad P_C(z, Z) \in \mathbb{F}[z^{\pm 1}, Z].$$

After substituting $J(X^q)$, different cosets have disjoint monomial supports. Each component therefore vanishes separately. At least one P_C is nonzero, and clearing its negative powers of z gives a nonzero polynomial annihilating $J(z)$. $\square$

Theorem 1.1 and Proposition 6.1 prove algebraicity of every $H_{d,e}$. Lemma 6.2, applied to (db, ec) and $(d + e, d, e)$ respectively, identifies this assertion with the original multivariable statements for the cluster wall and for $G_{d,e}(t^{d+e}x^dy^e)$. The correspondence of initial parameters is $(\ell_1, \ell_2) = (c, b)$.

7 The diagonal factor in Kontsevich's statement

For a formal automorphism of $\mathbb{C}[[U, V]]$ tangent to the identity and preserving $dU/U \wedge dV/V$, group its positive-cone factorization into the directions below, on, and above the diagonal. Kontsevich's lecture states that the automorphism is algebraic if and only if all three grouped factors are algebraic [10, 1]. Here a map is algebraic when its coordinate series are algebraic over $\mathbb{C}(U, V)$. The notes give a geometric outline through rational curves in a surface obtained by two blowups of $\mathbb{P}^1 \times \mathbb{P}^1$; the diagonal factor is identified using marked points and tangent data.

Proposition 7.1. *Kontsevich's diagonal preservation statement implies algebraicity in every primitive positive direction for the two polynomial initial functions over $\mathbb{C}$, and implies Corollary 1.2 over $\mathbb{Q}$.*

Proof. Fix a primitive positive direction (d, e) in (2). Use the finite monomial covering

$$X = U^e, \quad Y = V^d.$$

An automorphism $X \mapsto XA(X, Y)$, $Y \mapsto YB(X, Y)$, where A, B have constant term 1, lifts uniquely to the branch tangent to the identity:

$$U \mapsto UA(U^e, V^d)^{1/e}, \quad V \mapsto VB(U^e, V^d)^{1/d}.$$

Fractional powers mean the unique formal roots with constant term 1. Uniqueness shows that lifting respects composition. The pullback of the logarithmic two-form is $de dU/U \wedge dV/V$, so the lift preserves the latter form.

The lifts of the polynomial initial maps are

$$(U, V) \mapsto (U, VP_1(U^e)^{1/d}), \quad (U, V) \mapsto (UP_2(V^d)^{-1/e}, V).$$

Their inverses have the same formulas with the exponents $1/d$ and $-1/e$ negated, since each initial map fixes the variable on which its multiplier depends. All four maps have algebraic coordinate series: the multipliers satisfy their defining root equations. Composition preserves this property. Indeed, substituting an algebraic formal automorphism into a polynomial relation gives a nonzero relation whose coefficients lie in a finite algebraic extension of $\mathbb{C}(U, V)$; injectivity of the substitution preserves nonzeroness, and transitivity gives algebraicity over $\mathbb{C}(U, V)$. The lifted commutator is therefore algebraic.

The monomial in a direction (r, s) becomes $U^{er}V^{ds}$; the slope is multiplied by d/e . Exactly the target direction (d, e) becomes diagonal. Its factor lifts to

$$U \mapsto UG_{d,e}((UV)^{de})^{-1}, \quad V \mapsto VG_{d,e}((UV)^{de}). \quad (19)$$

Ordered-factorization uniqueness identifies (19) with the diagonal factor of the lifted commutator. Thus the stated diagonal preservation theorem yields the algebraicity of $G_{d,e}((UV)^{de})$, and Lemma 6.2 yields that of $G_{d,e}(z)$ over $\mathbb{C}(z)$.

For the original binomial-power inputs, the coefficients of $G_{d,e}$ lie in $\mathbb{Q}$. A complex polynomial relation then descends to $\mathbb{Q}$: choose a $\mathbb{Q}$ -basis for the finite-dimensional space spanned by its coefficients and take the component along each basis vector. Since every coefficient of the series is rational, each component is a relation and at least one is nonzero. The monomial substitution of Corollary 1.2 gives the original formulation. $\square$

A Nested algebraic approximation

There is a second algebraic closure argument for the system (16). It uses nested algebraic approximation, as in Popescu [19, Theorem 1.4] and its combinatorial application in [2, Section 5.3]. We give the reduction because it explains the role of a single catalytic variable.

Write $\mathbb{F}\langle t, u \rangle$ for the subring of $\mathbb{F}[[t, u]]$ consisting of series algebraic over $\mathbb{F}(t, u)$, and similarly for $\mathbb{F}\langle t \rangle$. Nested approximation says that a finite polynomial system with a formal solution whose unknowns depend on prescribed initial segments of an ordered list of variables has an algebraic solution with the same dependence. We use the order t, u , so the allowed dependence sets are $\{t\}$ and $\{t, u\}$.

Alternative proof of algebraicity for (16). Let n be the number of scalar unknowns in (16). For $1 \leq i \leq n$, introduce unknowns $D_{i,j}(t, u)$, $0 \leq j \leq a$, and $B_{i,j}(t)$, $0 \leq j < a$, and impose

$$D_{i,0} = f_i + tQ_i((D_{r,j}), t, u), \quad (20)$$

$$uD_{i,j+1} = D_{i,j} - B_{i,j} \quad (0 \leq j < a). \quad (21)$$

The known formal solution supplies one by $D_{i,j} = \Delta^j Z_i$ and $B_{i,j} = (\Delta^j Z_i)(t, 0)$. Nested approximation gives an algebraic solution of (20)–(21) with $B_{i,j} \in \mathbb{F}\langle t \rangle$ and $D_{i,j} \in \mathbb{F}\langle t, u \rangle$.

To identify it, work in the larger complete ring $\mathbb{F}[[u]][[t]] = \mathbb{F}[[t, u]]$. The t -adic contraction in the proof of Lemma 4.1 applies unchanged in this ring. In any solution with the prescribed dependence, setting $u = 0$ in (21) gives $B_{i,j} = D_{i,j}(t, 0)$. Division by u consequently gives $D_{i,j+1} = \Delta D_{i,j}$. Thus $D_{i,0}$ solves the original system, and uniqueness identifies it with Z_i . All the original series are therefore algebraic. $\square$

B The tight-grading coefficient formula

We derive (5) in the strict determinant-one case used in the proof. The argument follows the triangular comparison of [4, Section 5.5]. The scattering input is the existence of the consistent completion and the chamber invariance of its formal theta functions, in the form of [4, Theorem 2.29], originating in [7, Theorem 3.5]. We state the latter input below and prove the required coefficient comparison.

Fix k, d, e, D_1, D_2 as in (3). In this appendix the letters p, q range over the entire rectangle

$$0 \leq p < D_1, \quad 0 \leq q < D_2. \quad (22)$$

All sums are weighted by (4). Write $\mathcal{C}(p, q)$ for the sum over compatible gradings on $C(D_1, D_2)$ of vertical total p and horizontal total q . Write $A_+(p, q)$ for the sum restricted to gradings whose selected vertical edges lie in horizontal shadows, and $A_-(p, q)$ for the sum restricted to gradings whose selected horizontal edges lie in vertical shadows. These sums equal 1 at $(0, 0)$; they are zero when either index is negative.

B.1 Two free-edge convolutions

Lemma B.1. *For every pair in (22),*

$$\mathcal{C}(p, q) = \sum_{s=0}^p [X^s] P_1(X)^{D_2-q} A_+(p-s, q), \quad (23)$$

$$\mathcal{C}(p, q) = \sum_{s=0}^q [Y^s] P_2(Y)^{D_1-p} A_-(p, q-s). \quad (24)$$

For $(p, q) \neq (0, 0)$ the two sums satisfy

$$A_+(p, q) = 0 \quad \text{if } qD_1 \leq pD_2, \quad A_-(p, q) = 0 \quad \text{if } qD_1 \geq pD_2. \quad (25)$$

Proof. Fix the horizontal grading of total q . Its horizontal shadows contain exactly q vertical edges by the first-return mass calculation in Section 2. The other $D_2 - q$ vertical edges may be graded freely. Indeed, if a selected vertical edge lies outside the shadow of a selected horizontal edge, the latter's first return is a strict horizontal witness. The only pairs that need a vertical witness have their vertical edge inside that horizontal shadow; every suffix under consideration then lies in the same shadow interval. Changing the grades outside all horizontal shadows therefore cannot affect these pairs. Setting all those grades to zero gives a grading counted by A_+ , and any assignment to them reverses this operation. Their generating polynomial for the total vertical grade is $P_1(X)^{D_2-q}$. Extracting total p proves (23). Reading backwards gives the analogous statement for the $D_1 - p$ horizontal edges outside vertical shadows, proving (24).

For a grading counted by A_+ with nonzero totals, use its maximal blue intervals. Write their edge counts as $(m_i + 1, n_i)$ and their initial ranks as r_i . The rank of the last vertical start is $r_i + D_1 + D_2 - (n_i D_1 - m_i D_2)$, so $n_i D_1 - m_i D_2 > r_i \geq 0$. Blue mass gives $\sum_i n_i = q$, and protected matching gives $\sum_i m_i \geq p$. Consequently

$$qD_1 - pD_2 = \sum_i (n_i D_1 - m_i D_2) + \left(\sum_i m_i - p \right) D_2 > 0.$$

If there is no selected horizontal edge, the shadow condition forces $p = q = 0$, already excluded. The vertical-shadow argument in the proof of Lemma 3.1 gives $pD_2 - qD_1 > 0$ for any nonzero grading counted by A_- . It uses only the strict bounds in (22) and the rank inequalities, and proves the other support assertion. $\square$

B.2 Broken lines and chamber invariance

Use the two-dimensional scattering diagram with the full horizontal axis carrying $P_1(X)$, the full vertical axis carrying $P_2(Y)$, and outgoing rays $\mathbb{R}_{\leq 0}(r, s)$ for positive primitive pairs (r, s) carrying $G_{r,s}(X^r Y^s)$. Products around closed paths are the identity; grouping equal rays gives the ordered factorization (2). This is the consistent completion of the two initial walls.

A broken line consists of a piecewise linear path $\gamma : (-\infty, 0] \rightarrow \mathbb{R}^2$ and a monomial $aX^{m_1}Y^{m_2}$ attached to each linear segment. Here $m = (m_1, m_2) \in \mathbb{Z}^2$ is its exponent and a is its coefficient. Its velocity on that segment is $-m$. We start with exponent $(-D_1, -D_2)$ and coefficient 1. At a wall with function f , choose a primitive integral normal n with $\langle n, m \rangle > 0$ for the incoming exponent. The outgoing monomial is one term of

$$aX^{m_1}Y^{m_2}f^{\langle n, m \rangle}.$$

This is the bending rule; the constant term permits crossing without a bend. A broken line avoids the origin and ends at a point off the walls. Its weight is the coefficient of its final monomial. All increments of exponents are nonnegative in each coordinate.

The formal theta function at an endpoint is the sum of these final monomials. The chamber-invariance theorem asserts that this sum is the same for endpoints in a common chamber of a consistent scattering diagram whose coordinates are linearly independent over $\mathbb{Q}$. The theorem is formal: it holds coefficient by coefficient in the completed monomial ring. Polynomial identity and the homogeneous specialization described after (5) then permit arbitrary values of the initial coefficients.

Choose an irrational real number $\varepsilon > 0$ with

$$\varepsilon < \min \left\{ \frac{e}{d}, \frac{1}{D_1}, \frac{1}{D_2} \right\}. \quad (26)$$

These bounds ensure the estimates below throughout the full rectangle (22). The endpoints $Q_+ = (1, \varepsilon)$ and $Q_- = (\varepsilon, 1)$ lie in the first-quadrant chamber. Let $\Theta(p, q)$ be the coefficient of $X^{-D_1+p}Y^{-D_2+q}$ in their common theta function.

Every contributing segment has positive velocity in both coordinates: its exponent increments are bounded by the final pair (p, q) in (22). Along the path the quantity

$$L = \gamma_2 m_1 - \gamma_1 m_2$$

is constant. It is constant on segments because $\gamma' = -m$, and at a bend because the exponent increment is parallel to the ray containing γ . Its final values are

$$L = (D_2 - q) - \varepsilon(D_1 - p) > 0 \quad \text{at } Q_+, \quad L = \varepsilon(D_2 - q) - (D_1 - p) < 0 \quad \text{at } Q_-.$$

At a positive horizontal crossing L is positive, and at a positive vertical crossing it is negative. Thus a path to Q_+ last enters the first quadrant through the positive horizontal axis, and one to Q_- through the positive vertical axis.

Let $B_+^s(p, q)$ be the weighted sum of paths to Q_+ for which this last crossing chooses the term of degree s in X from the horizontal-wall factor. Define $B_-^s(p, q)$ using the term of degree s in Y at the last positive vertical crossing. Write $B_+(p, q) = B_+^0(p, q)$ and $B_-(p, q) = B_-^0(p, q)$, and set these quantities to zero at negative indices.

Lemma B.2. *The broken-line sums satisfy*

$$\Theta(p, q) = \sum_{s=0}^p [X^s] P_1(X)^{D_2-q} B_+(p-s, q), \quad (27)$$

$$\Theta(p, q) = \sum_{s=0}^q [Y^s] P_2(Y)^{D_1-p} B_-(p, q-s), \quad (28)$$

and the support conditions (25) hold with $A_{\pm}$ replaced by $B_{\pm}$.

Proof. For Q_+ the wall power at the last crossing is $D_2 - q$, since its primitive inward normal is $(0, -1)$. Consider a path counted by $B_+(p-s, q)$. Its positive horizontal crossing is at

$$\left(1 - \frac{(D_1 - p + s)\varepsilon}{D_2 - q}, 0\right).$$

Scale its entire prefix before that crossing about the origin by the positive factor

$$\frac{1 - (D_1 - p)\varepsilon/(D_2 - q)}{1 - (D_1 - p + s)\varepsilon/(D_2 - q)}. \quad (29)$$

Both numerator and denominator are positive by (26). Radial scaling preserves all wall incidences, segment directions, and monomial choices; rescaling segment times preserves their prescribed velocities. At the new crossing choose the degree- s term of $P_1^{D_2-q}$ and append the final segment of velocity $(D_1 - p, D_2 - q)$ to Q_+ . The construction is invertible: remove that choice and scale the prefix by the reciprocal factor. Its weight is multiplied by $[X^s]P_1^{D_2-q}$. Summing over s proves (27). Interchanging coordinates proves (28) with $P_2^{D_1-p}$.

For a path counted by B_+ , all nonconstant bends occur before the last positive horizontal crossing. They are on the negative vertical axis or in the third quadrant. Write a bend point as $-\lambda(r, s)$ with $\lambda > 0$, $r \geq 0$, $s > 0$, and the incoming velocity as (v_x, v_y) . Choosing exponent increment $j(r, s)$, where $j > 0$, changes the velocity to $(v_x - jr, v_y - js)$. Since

$$L = \lambda(sv_x - rv_y) > 0,$$

the change in velocity slope is

$$\frac{v_y - js}{v_x - jr} - \frac{v_y}{v_x} = \frac{j(rv_y - sv_x)}{v_x(v_x - jr)} < 0.$$

The denominators are positive by (22). At nonzero total increment this gives

$$\frac{D_2 - q}{D_1 - p} < \frac{D_2}{D_1}, \quad \text{equivalently} \quad qD_1 > pD_2.$$

If there are no bends, the total increment is $(0, 0)$ and the unique straight path has weight 1. At Q_- the sign of L is reversed, so velocity slopes increase and $qD_1 < pD_2$. These are the required support statements and initial values. $\square$

Lemma B.3. *For all pairs in (22), $A_+(p, q) = B_+(p, q)$ and $A_-(p, q) = B_-(p, q)$.*

Proof. Equating the two convolutions for $\mathcal{C}$ gives

$$\begin{aligned} A_+(p, q) - A_-(p, q) &= \sum_{s=1}^q [Y^s] P_2^{D_1-p} A_-(p, q-s) \\ &\quad - \sum_{s=1}^p [X^s] P_1^{D_2-q} A_+(p-s, q). \end{aligned}$$

The same equation holds for $B_{\pm}$ by chamber invariance and Lemma B.2. The initial values are 1 at $(0,0)$. Every term on the right has smaller sum of indices. At a nonzero pair, the support conditions determine at least one of the two left-hand terms to be zero, so the equation determines the other. Induction on $p+q$ proves equality. In fact equality $qD_1 = pD_2$ occurs in the strict rectangle only at the origin, since D_1, D_2 are coprime. $\square$

B.3 The determinant-one coefficient

Proposition B.4. *With (3), the identity (5) holds.*

Proof. By Lemma 3.1, the tight gradings of totals (kd, ke) are precisely those counted by $A_+(kd, ke)$. By Lemma B.3, this sum is $B_+(kd, ke)$. We show that a path counted by the latter has exactly one nonconstant bend and that it reads the coefficient $[z^k]G_{d,e}(z)$.

Its exponent increments sum to (kd, ke) , and its last positive horizontal crossing has zero increment. While it lies in the third quadrant, the polar angle of its position increases strictly: its derivative is $L/|\gamma|^2 > 0$. Thus the ratios r/s of the negative ray directions at successive bends strictly decrease. The negative horizontal axis cannot carry a bend because there $L < 0$. At the first bend, positivity of L gives $r/s < D_1/D_2$, and therefore the same strict bound holds at every bend, including any negative vertical bend of ratio zero.

If there were more than one distinct direction, the weighted mean of their ratios would be d/e . Collect the increments with ratio strictly greater than d/e and let their sum be (A, B) . This group is nonempty, $0 < B \leq ke < D_2$, and

$$\frac{d}{e} < \frac{A}{B} < \frac{D_1}{D_2}.$$

Since $eD_1 - dD_2 = 1$, it follows that

$$0 < eA - dB < \frac{B}{D_2} < 1,$$

contradicting integrality. All increments therefore have direction (d, e) . Strict angular monotonicity allows only one encounter with that ray, so there is exactly one nonconstant bend.

There is a unique geometric path with that bend and those initial and final velocities. Its bend point and the time from the bend to Q_+ are respectively

$$-((D_2 - ke) - (D_1 - kd)\varepsilon)(d, e), \quad e - d\varepsilon.$$

The coefficient in parentheses and the time are positive by (26). Adding this time times the final velocity $(D_1 - kd, D_2 - ke)$ to the bend point gives $(1, \varepsilon)$, since $eD_1 - dD_2 = 1$. Extending backwards with velocity (D_1, D_2) gives the required initial segment; all other crossings choose constant terms.

The primitive normal at this bend is $(-e, d)$. Its pairing with the incoming exponent $(-D_1, -D_2)$ is 1. Thus the wall is used to the first power, and the term of increment (kd, ke) has coefficient $[z^k]G_{d,e}(z)$. Consequently $B_+(kd, ke) = c_k$, proving (5). $\square$

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