

THREE CUBIC TRINOMIAL SERIES FOR $1/\pi$

KHAI-HOAN NGUYEN-DANG

ABSTRACT. We prove the three identities in Sun's Conjecture VI, involving cubes of generalized central trinomial coefficients. A common double integral represents their generating functions. On two rational curves, this period reduces to quadratic expressions in Gauss hypergeometric functions associated with elliptic integrals. A first-order identity recovers the derivative transverse to each curve and supplies the linear weights in the three series. The constants are determined by an endpoint calculation and Bauer's identity. Classical modular equations of degrees three, five, and seven then complete the evaluations. We give explicit rational differential certificates for the period reductions and the transverse derivative identities.

1. INTRODUCTION

We prove the three cubic trinomial identities for $1/\pi$ in Sun's Conjecture VI [7, Section 5]. The main step is a pair of one-parameter transformations for a common two-variable period, together with identities for its transverse derivatives. These formulas connect the three series to classical elliptic periods and modular equations.

For $b, c \in \mathbb{C}$ and an integer $n \geq 0$, define

$$(1) \quad T_n(b, c) = [X^n](X^2 + bX + c)^n = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} \binom{2j}{j} b^{n-2j} c^j.$$

Sun recorded the discovery of the three formulas on 1 and 2 October 2011 [9, Section 4] and published them together as Conjecture VI in [7]. He still described all three type-VI series as open in 2020 [8, p. 1275]. Our main result proves the complete conjecture.

Theorem 1.1. *The following absolutely convergent identities hold:*

$$(2) \quad \sum_{n=0}^{\infty} (66n + 17) \frac{T_n(10, 121)^3}{55296^n} = \frac{540\sqrt{2}}{11\pi},$$

$$(3) \quad \sum_{n=0}^{\infty} (126n + 31) \frac{T_n(22, 441)^3}{(-512000)^n} = \frac{880\sqrt{5}}{21\pi},$$

$$(4) \quad \sum_{n=0}^{\infty} (3990n + 1147) \frac{T_n(62, 9025)^3}{(-23887872)^n} = \frac{432(195\sqrt{14} + 94\sqrt{2})}{95\pi}.$$

Ramanujan's 1914 paper [5] placed series for $1/\pi$ in the setting of elliptic integrals and modular equations. The subsequent theory includes transformations between different hypergeometric descriptions of elliptic periods; the work of Berndt, Bhargava, and Garvan [1] develops the alternative bases used below. Sun's formulas bring generalized central trinomial coefficients into this setting. Their relation to Legendre polynomials is

$$(5) \quad T_n(b, c) = (b^2 - 4c)^{n/2} P_n\left(\frac{b}{\sqrt{b^2 - 4c}}\right).$$

Date: 29 October 2026.

2020 Mathematics Subject Classification. Primary 33C20; Secondary 11F03, 33C45.

Key words and phrases. Ramanujan-type series, generalized central trinomial coefficients, Legendre polynomials, elliptic integrals, modular equations.

Here P_n is the Legendre polynomial normalized by $P_n(1) = 1$. For $b^2 \neq 4c$, the expression is independent of the square root; after expansion it is a polynomial, which also gives its value when $b^2 = 4c$. Thus the problem belongs to the study of generating functions of powers of Legendre polynomials.

Zudilin [11] obtained a transformation for a binomially weighted generating function of Legendre squares and used it to prove a family of Sun's series. Van Hoeij, van Straten, and Zudilin [4, Section 6] later gave a Hadamard-product representation for the generating function of Legendre cubes and studied its fourth-order differential equation. That work provides a general framework for the cubic problem. The reductions here concern two particular rational curves: on each, the relevant period satisfies a third-order equation that is a symmetric square of a second-order equation. For the second curve, the resulting equation is related to the hypergeometric transformation for Domb numbers in Rogers [6, Theorem 3.1, equation (3.4)].

There is a further derivative to determine. If $F(w) = \sum a_n w^n$, then a weight $An + B$ corresponds to applying $Aw d/dw + B$ to F . In our two-variable generating function, this derivative holds the trinomial parameter fixed, whereas restricting to a curve gives a tangent derivative. Differentiation and Legendre's relation have already been used systematically to obtain series for $1/\pi$ from products of elliptic periods; see Wan [10, Section 2]. Here we recover the missing direction through a first-order identity. Its constant is fixed analytically, by an endpoint estimate on the first curve and Bauer's identity on the second. The resulting formulas apply along the entire curves; the three series follow by evaluation at the required algebraic points.

The proof has three parts. Section 2 gives the double-integral representation. Sections 3–6 establish the first period transformation, its transverse derivative, and the first and third series. Section 7 proves the second series using a signature-three transformation and an explicit continuation path. Appendices A and B contain the rational differential certificates and the polynomial sign checks for that path. All square roots and continuation branches used in the proof are specified.

2. AN INTEGRAL AND A RATIONAL CURVE

Use the normalization

$$G(h, w) = \sum_{n=0}^{\infty} T_n(h, 1/4)^3 w^n, \quad \theta_w = w \frac{\partial}{\partial w}.$$

Since

$$T_n(h, 1/4) = \frac{1}{\pi} \int_0^\pi (h + \cos v)^n dv,$$

uniform convergence of the geometric series gives

$$(6) \quad G(h, w) = \frac{1}{\pi^3} \int_{[0, \pi]^3} \frac{dv_1 dv_2 dv_3}{1 - w \prod_{j=1}^3 (h + \cos v_j)}$$

whenever $|w|(|h| + 1)^3 < 1$. The integral also defines analytic continuation on real parameter intervals where its denominator remains positive.

Define

$$(7) \quad \Psi(s, t) = \frac{1}{\pi^2} \int_0^1 \int_0^1 \frac{du dv}{\sqrt{u(1-u)v(1-v)(1-suv)(1-t(1-u)(1-v))}}.$$

For real $s, t < 1$, all square roots in this formula are positive. Expanding the last two factors and evaluating beta integrals proves

$$(8) \quad \Psi(s, t) = \sum_{m, n \geq 0} \frac{(1/2)_m^3 (1/2)_n^3}{m! n! (m+n)!^2} s^m t^n \quad (|s|, |t| < 1).$$

Here $(a)_n = a(a+1) \cdots (a+n-1)$ and $(a)_0 = 1$. The substitution $(u, v) \mapsto (1-u, 1-v)$ gives $\Psi(s, t) = \Psi(t, s)$.

Lemma 2.1. *Let*

$$h = -\frac{a+b}{1+ab}, \quad w = \frac{a}{(h^2-1)(1+ah)},$$

and put

$$s = \frac{4a}{(1-b)(1+a)}, \quad t = -\frac{4a}{(1+b)(1-a)}.$$

Suppose $a, b \in (-1, 1)$, the displayed expressions are defined, and $s, t < 1$. Then, with G represented by (6),

$$(9) \quad G(h, w) = \frac{\sqrt{1-a^2}}{1+ab} \Psi(s, t).$$

In particular this identity holds on the curve

$$(10) \quad \begin{aligned} b &= -\frac{5a+1}{a+1}, & h &= \frac{a^2-4a-1}{5a^2-1}, \\ w &= -\frac{(5a^2-1)^3}{8(a-1)^2(a+1)^3(3a+1)}, & -\frac{1}{3} &< a < 0. \end{aligned}$$

There it becomes

$$(11) \quad g(a) := G(h(a), w(a)) = P(a) \Psi\left(\frac{2a}{3a+1}, \frac{a+1}{1-a}\right), \quad P(a) = \frac{(1+a)\sqrt{1-a^2}}{1-5a^2}.$$

Proof. Write $d\nu(X) = dX/(\pi\sqrt{1-X^2})$ on $(-1, 1)$. For $|r| < 1$, the increasing fractional linear map

$$M_r(X) = \frac{r+X}{1+rX} \quad \text{satisfies} \quad d\nu(M_r(X)) = \frac{\sqrt{1-r^2}}{1+rX} d\nu(X).$$

In the three cosine variables of (6), use respectively $M_a(X), M_a(Y), M_b(Z)$. The definitions of h, w give the polynomial identity

$$\begin{aligned} (1+aX)(1+aY)(1+bZ) - w(h+a+(1+ha)X)(h+a+(1+ha)Y)(h+b+(1+hb)Z) \\ = (1+ab)(1-ab+a(X+Y)+bZ+aXYZ). \end{aligned}$$

The Jacobian factors therefore cancel the three linear factors from the transformed denominator, leaving

$$G(h, w) = \frac{(1-a^2)\sqrt{1-b^2}}{1+ab} \int_{[-1,1]^3} \frac{d\nu(X)d\nu(Y)d\nu(Z)}{1-ab+a(X+Y)+(b+aXY)Z}.$$

Put $A = 1-ab+a(X+Y)$ and $B = b+aXY$. After $X = 1-2u, Y = 1-2v$, direct factorization gives

$$A-B = (1-b)(1+a)(1-suv), \quad A+B = (1+b)(1-a)(1-t(1-u)(1-v)).$$

Both factors are positive under the stated hypotheses, so $A > |B|$. Thus the integral in Z equals $(A^2 - B^2)^{-1/2}$. Moreover $d\nu(X)$ and $d\nu(Y)$ become the beta measures $du/(\pi\sqrt{u(1-u)})$ and $dv/(\pi\sqrt{v(1-v)})$. The constant extracted from $\sqrt{A^2 - B^2}$ is $\sqrt{(1-a^2)(1-b^2)}$, which proves precisely the prefactor in (9). This argument also proves positivity of the original denominator, since all the linear factors in the substitution are positive. For (10), substitution gives $s = 2a/(3a+1) < 0$ and $t = (a+1)/(1-a) \in (1/2, 1)$. Also $|a|, |b| < 1$ there. Hence the same direct integral argument proves (11) on the entire stated interval. $\square$

Set

$$(12) \quad U = \frac{1-a}{1+3a} > 1.$$

The curve in the (s, t) -plane is $s+t-2st=1$, parametrized by $s = (1-U)/2$ and $t = (1+U^{-1})/2$.

3. THE ELLIPTIC TRANSFORMATION

Write

$$f(x) = {}_2F_1(1/2, 1/2; 1; x), \quad H(x) = {}_3F_2(1/2, 1/2, 1/2; 1, 1; x).$$

Thus $f(x) = 2K(x)/\pi$, where $K(x) = \int_0^{\pi/2} (1 - x \sin^2 v)^{-1/2} dv$ uses the parameter convention. Our hypergeometric normalization is

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{n \geq 0} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!},$$

initially in its disk of convergence, and continued analytically where stated. We use Clausen's identity, Pfaff's transformation, and Legendre's relation in the forms recorded in [2, 16.12.2, 15.8.1, 19.7.1]; the quadratic forms used below also follow by substitution in the Gauss equation.

Theorem 3.1. *For $U > 1$, put*

$$m = \frac{1 - \sqrt{1 - U^{-2}}}{2}.$$

Then

$$(13) \quad \Psi\left(\frac{1-U}{2}, \frac{1+U^{-1}}{2}\right) = \frac{3f(m)^2 + f(1-m)^2}{4\sqrt{U}}.$$

The coefficient 3 in this formula is fixed by the local behavior at $U = 1$; it cannot be inferred from the differential equation alone.

Lemma 3.2. *The function on the left of (13) is annihilated by*

$$(14) \quad (2\vartheta - 1)^3 - U^2(2\vartheta + 1)^3, \quad \vartheta = U \frac{d}{dU}.$$

Proof. The equation follows by exact elimination from the Legendre recurrence. The derivation and explicit certificate are given in Appendix A. All identities used in that elimination are over a rational-function field. $\square$

Proof of Theorem 3.1. Let $f_0 = f(1/2)$ and $f_1 = f'(1/2)$. The Gauss equation and Legendre's relation give

$$(15) \quad f''(1/2) = f_0, \quad f_0 f_1 = 2/\pi.$$

Put $J(t) = {}_3F_2(1/2, 1/2, 1/2; 2, 2; t)$. Coefficient comparison in the defining series gives the contiguous identity

$$(16) \quad J(t) = 8H(t) + (32 - 8/t)\theta_t H(t) + 48(1 - 1/t)\theta_t^2 H(t).$$

The apparent singularity at $t = 0$ is removable. The quadratic Clausen identity is

$$(17) \quad H(t) = f\left(\frac{1 - \sqrt{1-t}}{2}\right)^2.$$

It follows from (15) that, with $r = \sqrt{1-t}$,

$$H(t) = f_0^2 - \frac{2}{\pi}r + \frac{f_1^2 + f_0^2}{4}r^2 + O(r^3).$$

Substitution in (16) cancels the terms in r^{-1} and gives

$$(18) \quad J(1) = 2f_0^2 - 6f_1^2.$$

From (8), uniformly for $0 \leq t \leq 1$ and small real s ,

$$\Psi(s, t) = H(t) + \frac{s}{8}J(t) + O(s^2).$$

For the uniform bound, the coefficient in (8) is at most the product of the corresponding coefficients of $H(s)H(t)$, and $H(1)$ is finite. Moreover $J'(1)$ is finite by absolute convergence of its differentiated series, so $J(t) = J(1) + O(1-t)$. Consequently, as $\delta \downarrow 0$,

$$(19) \quad \Psi \left(-\frac{\delta}{2}, \frac{1 + (1 + \delta)^{-1}}{2} \right) = f_0^2 - \frac{\sqrt{2}}{\pi} \sqrt{\delta} + \frac{f_1^2}{2} \delta + O(\delta^{3/2}).$$

To identify a local basis, put

$$u(U) = U^{-1/4} f(m), \quad v(U) = U^{-1/4} f(1-m).$$

The Gauss equation, followed by the change of variable $4m(1-m) = U^{-2}$, shows that both functions satisfy

$$Y'' + \frac{3U^2 - 1}{2U(U^2 - 1)} Y' + \frac{1}{16U^2} Y = 0.$$

Its symmetric square is exactly (14), after division by its leading coefficient. Hence u^2, uv, v^2 solve that equation. As $\delta = U - 1 \downarrow 0$, the three combinations

$$uv, \quad v^2 - u^2, \quad (v - u)^2$$

have leading terms

$$f_0^2, \quad 2\sqrt{2} f_0 f_1 \sqrt{\delta}, \quad 2f_1^2 \delta,$$

respectively, all with nonzero coefficients. They therefore form a local basis. The right side of (13) is $(3u^2 + v^2)/4$ and has the three coefficients in (19). Comparison in this basis proves the identity near $U = 1$. Since the leading coefficient of (14) does not vanish on $1 < U < \infty$, the identity continues along that interval. $\square$

4. THE TRANSVERSE DERIVATIVE

The weight $An + B$ requires a derivative in w with h fixed. The period transformation gives instead a derivative along the curve $a \mapsto (h(a), w(a))$. The next identity supplies the additional derivative needed to pass between these two directions.

On $-1/3 < a < 0$ define

$$\begin{aligned} A_0(a) &= \frac{3(3a+1)(3a^2+1)(5a^2+2a+1)}{8(a+1)(5a^2-1)^2}, \\ A_1(a) &= -\frac{(a-1)(3a+1)(a^2+6a+1)}{8(5a^2-1)}, \\ R(a) &= \frac{(1-a^2)(5a^2+2a+1)}{(1-5a^2)^2} \sqrt{-\frac{3a+1}{a}}. \end{aligned}$$

In particular $R(a) > 0$.

Proposition 4.1. *With g, h, w as in (10)–(11),*

$$(20) \quad (\theta_w G)(h(a), w(a)) + A_0(a)g(a) + A_1(a)g'(a) = \frac{3\sqrt{2}}{8\pi} R(a).$$

Proof. Set

$$N(a) = (\theta_w G)(h(a), w(a)), \quad E = N + A_0 g + A_1 g'.$$

The normal row identity in Appendix A gives $E' = \rho E$, where direct differentiation shows that $\rho = R'/R$. Hence E/R is constant.

An endpoint calculation determines the constant. As $a \rightarrow 0^-$,

$$h = 1 + 4a + O(a^2), \quad w = \frac{1}{8}(1 - 4a + O(a^2)), \quad q := w(h+1)^3 = 1 + 2a + O(a^2).$$

The elementary endpoint estimate in the single cosine integral is

$$T_n(h, 1/4) = (h+1)^n \sqrt{\frac{h+1}{2\pi n}} (1 + O(n^{-1})),$$

uniformly for h in a sufficiently small neighborhood of 1. The analogous differentiated estimate follows by inserting $n(h + \cos v)^{n-1}$ in the same integral. Expanding $\cos v = 1 - v^2/2 + O(v^4)$ near zero and bounding the remaining interval geometrically proves both estimates. Cubing and summing gives

$$N \sim \frac{1}{\pi\sqrt{1-q}}, \quad hG_h \sim \frac{3h}{h+1}N, \quad g = O(1).$$

Here the error in the differentiated sum is bounded by a convergent multiple of $\sum n^{-3/2}$, and $\sum_{n \geq 1} q^n n^{-1/2} \sim \sqrt{\pi}/\sqrt{1-q}$ follows by a rescaled Riemann sum. Since $h'/h \rightarrow 4$ and $w'/w \rightarrow -4$, we have $g' \sim 2N$. Finally,

$$A_0 \rightarrow \frac{3}{8}, \quad A_1 \rightarrow -\frac{1}{8}, \quad R \sim (-a)^{-1/2},$$

so $E/R \rightarrow 3\sqrt{2}/(8\pi)$. This proves (20) on a neighborhood of zero and hence throughout the interval by its first-order equation and analytic continuation of the positive integral. $\square$

5. CLASSICAL MODULAR EVALUATIONS

For $0 < \lambda < 1$, put

$$F_\lambda(u) = {}_2F_1(\lambda, 1-\lambda; 1; u), \quad H_\lambda(x) = {}_3F_2(1/2, \lambda, 1-\lambda; 1, 1; x), \quad \sigma_\lambda = \frac{\sin(\pi\lambda)}{\pi}.$$

The quadratic transformation and Clausen's identity give

$$(21) \quad H_\lambda(4u(1-u)) = F_\lambda(u)^2 \quad (0 < u < 1/2).$$

The modular calculation will be used for $\lambda = 1/2$ and $\lambda = 1/3$; its midpoint identity also treats $\lambda = 1/4$.

Lemma 5.1 (Modular differentiation). *For $0 < u < 1$, one has*

$$(22) \quad F_\lambda(u)F'_\lambda(1-u) + F'_\lambda(u)F_\lambda(1-u) = \frac{\sigma_\lambda}{u(1-u)}.$$

In particular,

$$(23) \quad F_\lambda(1/2)F'_\lambda(1/2) = 2\sigma_\lambda.$$

Let $N > 1$, and let $m \in (0, 1/2)$ be the unique point such that

$$\mathcal{R}_\lambda(m) = \sqrt{N}, \quad \mathcal{R}_\lambda(u) = \frac{F_\lambda(1-u)}{F_\lambda(u)}.$$

Write $p = m(1-m)$ and $d = 1-2m$. Let $\beta(\alpha)$ be the local branch through $(m, 1-m)$ determined by $\mathcal{R}_\lambda(\beta) = \mathcal{R}_\lambda(\alpha)/N$. Then $\beta'(m) = 1$ and

$$(24) \quad [\theta_x H_\lambda(x)]_{x=4p} + \left(1 + \frac{p\beta''(m)}{2d}\right) H_\lambda(4p) = \frac{\sigma_\lambda}{d\sqrt{N}}, \quad \theta_x = x \frac{d}{dx}.$$

Proof. Both $F_\lambda(u)$ and $F_\lambda(1-u)$ solve the Gauss equation

$$u(1-u)Y'' + (1-2u)Y' - \lambda(1-\lambda)Y = 0.$$

Abel's identity makes their Wronskian a constant multiple of $1/(u(1-u))$. Euler's integral, and the same integral differentiated in u , give

$$F_\lambda(1-u) = -\sigma_\lambda \log u + O(1), \quad \frac{d}{du} F_\lambda(1-u) \sim -\frac{\sigma_\lambda}{u} \quad (u \downarrow 0).$$

Since $F_\lambda(u) = 1 + O(u)$, the Wronskian constant is $-\sigma_\lambda$. This proves (22); its midpoint specialization gives (23). It also gives

$$(25) \quad \mathcal{R}'_\lambda(u) = -\frac{\sigma_\lambda}{u(1-u)F_\lambda(u)^2} < 0.$$

Together with $\mathcal{R}_\lambda(1/2) = 1$ and $\mathcal{R}_\lambda(u) \rightarrow \infty$ as $u \downarrow 0$, this proves existence and uniqueness of m .

At the complementary pair, $F_\lambda(1-m) = \sqrt{N}F_\lambda(m)$, so (25) gives $\beta'(m) = 1$. Set

$$A = \frac{F'_\lambda(m)}{F_\lambda(m)}, \quad B = \frac{F'_\lambda(1-m)}{F_\lambda(1-m)}.$$

Equation (22) and logarithmic differentiation of the defining equation for β give, respectively,

$$A + B = \frac{\sigma_\lambda}{p\sqrt{N}F_\lambda(m)^2}, \quad \beta''(m) = -\frac{2d}{p} + 2(B - A).$$

Finally, differentiation of (21) gives

$$[\theta_x H_\lambda(x)]_{x=4p} = \frac{2p}{d} A F_\lambda(m)^2.$$

Eliminating A, B proves (24). □

For $N = 3$ or 7 , put $\gamma = 1/(N+1)$. The two modular equations needed are

$$(26) \quad (\alpha\beta)^\gamma + ((1-\alpha)(1-\beta))^\gamma = 1.$$

Here $\alpha, \beta \in (0, 1)$ satisfy $\mathcal{R}_{1/2}(\beta) = N\mathcal{R}_{1/2}(\alpha)$, and all roots are positive. For $N = 3$ this is [3, (1.2), p. 31]; for $N = 7$ it is [3, Example 2, p. 39]. Their convention for degree is stated in their equation (1.1).

Lemma 5.2. *At*

$$m_3 = \frac{2 - \sqrt{3}}{4}, \quad m_7 = \frac{8 - 3\sqrt{7}}{16},$$

one has $f(1 - m_N) = \sqrt{N}f(m_N)$. Moreover,

$$(27) \quad (6\theta + 1)H(1/4) = \frac{4}{\pi}, \quad (42\theta + 5)H(1/64) = \frac{16}{\pi}, \quad \theta = x \frac{d}{dx}.$$

Operators in this notation are applied before evaluation.

Proof. Use Lemma 5.1 with $\lambda = 1/2$. The unique $m \in (0, 1/2)$ with $\mathcal{R}_{1/2}(m) = \sqrt{N}$ forms a complementary degree- N pair $(1-m, m)$. Equation (26) gives $2\{m(1-m)\}^\gamma = 1$, which yields the stated values m_N . For the inverse local branch through $(m, 1-m)$, put $p = m(1-m)$ and $d = 1 - 2m$. Twice differentiating the symmetric modular equation gives

$$\beta' = 1, \quad \beta'' = \frac{2(\gamma - 1) + 4p}{dp}.$$

Equation (24) consequently becomes

$$(28) \quad \theta H(4p) + \frac{\gamma - 2p}{1 - 4p} H(4p) = \frac{1}{\pi d \sqrt{N}}.$$

At $N = 3$, $p = 1/16$ and $d = \sqrt{3}/2$; at $N = 7$, $p = 1/256$ and $d = 3\sqrt{7}/8$. These substitutions give (27). □

6. EVALUATION OF THE FIRST AND THIRD SERIES

Proof of (2) and (4). On the curve put

$$W(a) = \frac{P(a)}{4\sqrt{U(a)}}, \quad m(a) = \frac{1 - \sqrt{1 - U(a)^{-2}}}{2}.$$

Theorem 3.1 says $g = W(3f(m)^2 + f(1 - m)^2)$. At the two points needed here, direct substitution gives the following exact data:

	$a = -1/7$	$a = -7/25$
U	2	8
P	$6\sqrt{3}/11$	$108/95$
A_0	$65/242$	$2509/10830$
A_1	$-8/539$	$1504/59375$
R	$240/121$	$7488\sqrt{7}/12635$

Let $V = f(m)^2$ at the relevant point. Differentiation, using Lemma 5.2 and the complementary derivative identities in the proof of Lemma 5.1, yields

$$\begin{aligned} g(-1/7) &= \frac{9\sqrt{6}}{22}V, & g'(-1/7) &= \frac{147\sqrt{6}}{484}V, \\ g(-7/25) &= \frac{27\sqrt{2}}{38}V, & g'(-7/25) &= \frac{1125\sqrt{2}}{161728} \left(225V - \frac{608}{\pi} \right). \end{aligned}$$

The derivatives are taken in the period identity before specializing the parameter.

For any constants A, B , Proposition 4.1 gives

$$(29) \quad (A\theta_w + B)G = \frac{3A\sqrt{2}}{8\pi}R + (B - AA_0)g - AA_1g'.$$

At $a = -1/7$, insert $(A, B) = (66, 17)$. The coefficient of V is zero, and the right side is $540\sqrt{2}/(11\pi)$. At $a = -7/25$, insert $(A, B) = (3990, 1147)$. Again the coefficient of V is zero, and the result is

$$\frac{40608\sqrt{2}}{95\pi} + \frac{16848\sqrt{14}}{19\pi} = \frac{432(94\sqrt{2} + 195\sqrt{14})}{95\pi}.$$

To identify the original series, use the scaling identity $T_n(b, c) = (2\sqrt{c})^n T_n(b/(2\sqrt{c}), 1/4)$. At the first point, $h = 5/11$ and $w = 1331/6912 = 22^3/55296$. At the second point, $h = -31/95$ and $w = 857375/2985984$; the parity identity $T_n(-h, 1/4) = (-1)^n T_n(h, 1/4)$ supplies the negative denominator in (4). Finally, $|T_n(b, c)| \leq (|b| + 2\sqrt{c})^n$ proves absolute convergence: the respective geometric bounds are $16/27$ and $343/512$. It also justifies termwise application of the linear weight. $\square$

7. THE SECOND IDENTITY

The second target lies on $s + t = -1$, rather than on the curve of Theorem 3.1. We determine its period and its transverse derivative separately. Write

$$f_3(u) = {}_2F_1(1/3, 2/3; 1; u), \quad H_3(v) = {}_3F_2(1/2, 1/3, 2/3; 1, 1; v).$$

Thus $f_3 = F_{1/3}$ and $H_3 = H_{1/3}$ in the notation of Section 5. The branch of each hypergeometric function is the one continued from its power series at zero to the plane cut along $[1, \infty)$.

Theorem 7.1. For $0 < s < 1$, put

$$x = s(s+1), \quad z = \frac{1-x+i\sqrt{x(2-x)}}{8}, \quad v = \frac{108z^2}{(1-4z)^3}, \quad A = \frac{\sqrt{z}}{1-4z}H_3(v),$$

where $\sqrt{z}$ is principal. Then

$$(30) \quad \Psi(s, -1-s) = -4 \operatorname{Re} A + 2\sqrt{6}|A|.$$

Proof. The rational certificate in Appendix A shows that $P(s) = \Psi(s, -1-s)$ satisfies the symmetric square of

$$(31) \quad Y'' + \frac{(2s+1)(s^2+s-1)}{s(s-1)(s+1)(s+2)}Y' + \frac{4s^2+4s-1}{16s(s-1)(s+1)(s+2)}Y = 0.$$

With $x = s(s+1)$, this equation becomes

$$Y_{xx} + \left(\frac{2}{4x+1} + \frac{x-1}{x(x-2)} \right) Y_x + \frac{4x-1}{16x(x-2)(4x+1)}Y = 0.$$

Substitute $x = 1-4z-1/(16z)$ and $Y = z^{1/4}V$. Direct differentiation gives

$$(32) \quad V'' + \frac{128z^2-30z+1}{z(4z-1)(16z-1)}V' + \frac{2(8z-1)}{z(4z-1)(16z-1)}V = 0.$$

Its symmetric square is the Domb equation:

$$\theta_z^3 D - 2z(2\theta_z + 1)(5\theta_z^2 + 5\theta_z + 2)D + 64z^2(\theta_z + 1)^3 D = 0.$$

Clausen's identity gives $H_3(v) = {}_2F_1(1/6, 1/3; 1; v)^2$. Substitution of $v = 108z^2/(1-4z)^3$ into the Gauss equation directly verifies that $V = (1-4z)^{-1/2}{}_2F_1(1/6, 1/3; 1; v)$ satisfies (32). In particular $D = (1-4z)^{-1}H_3(v)$ satisfies its symmetric square. Hence a local square root q of A satisfies (31). Its quadratic products with its conjugate satisfy the same third-order equation as P . The hypergeometric pullback above is the one in Rogers' generating function for Domb numbers [6, Theorem 3.1, equation (3.4)].

It remains to determine the linear combination, including its branches. The coefficient expansion (8), analytically continued in t to a neighborhood of -1 using the positive beta integral, and the contiguous identity (16) give

$$P(s) = H(-1) + s[H + 6\theta H + 12\theta^2 H]_{-1} + O(s^2).$$

Put

$$m_0 = \frac{1-\sqrt{2}}{2}, \quad f_0 = f(m_0), \quad f_1 = f'(m_0).$$

Clausen's identity and the differential equation for f therefore yield

$$(33) \quad P(s) = f_0^2 + \left(\frac{f_0^2}{4} - \frac{3\sqrt{2}}{4}f_0f_1 + \frac{3}{4}f_1^2 \right) s + O(s^2).$$

There is no term in $s^{1/2}$.

We use the signature-two/signature-three transformation

$$(34) \quad f_3 \left(\frac{27p^2(1+p)^2}{4(1+p+p^2)^3} \right) = \frac{1+p+p^2}{\sqrt{1+2p}} f \left(\frac{p^3(2+p)}{1+2p} \right),$$

which holds near $p = 0$ with the square root equal to one; see [1, Theorem 5.6, equation (5.17)]. At $p_0 = (\sqrt{2}-1+i)/2$ its two arguments are

$$u_0 = \frac{1+5i/\sqrt{2}}{2}, \quad m_0,$$

and its squared multiplier is $1/\sqrt{2} + i/2$. The path verification in Lemma B.1 identifies this continuation with the required upper boundary value. By the quadratic form of Clausen's identity, $H_3(4u(1-u)) = f_3(u)^2$, so

$$A(0+) = \kappa f_0^2, \quad \kappa = \frac{1}{2} + \frac{i}{2\sqrt{2}}.$$

Write $q = q_0 + q_1 s^{1/2} + q_2 s + O(s^{3/2})$, choosing $q^2 = A$. Differentiation of (34) gives

$$(35) \quad q_0^2 = \kappa f_0^2,$$

$$(36) \quad \frac{q_1}{q_0} = -\frac{1}{2} + \frac{i\sqrt{2}}{4} + \left(\frac{\sqrt{2}}{2} - \frac{i}{2} \right) \frac{f_1}{f_0}.$$

For an explicit check, the arguments in this differentiation satisfy

$$z = \frac{1}{8} + \frac{i}{4\sqrt{2}}\sqrt{s} + O(s), \quad u = u_0 - \frac{27}{4}\sqrt{s} + O(s), \quad p = p_0 - \frac{\sqrt{2} + i}{2}\sqrt{s} + O(s).$$

The coefficient of s^{-1} in (31) gives $q_2 = -q_0/16$. Using $|\kappa| = \sqrt{6}/4$, substitution into

$$-2(q^2 + \bar{q}^2) + 2\sqrt{6}q\bar{q}$$

gives exactly the coefficients in (33), including the zero coefficient of $s^{1/2}$. The second-order equation has exponents $0, 1/2$ at zero and two convergent Frobenius series without logarithms. Their quadratic products have exponents $0, 1/2, 1$, so these three coefficients uniquely determine a solution of the symmetric square. This proves (30) near zero.

For completeness, there is no hidden cut or zero in continuing the formula through $0 < s < 1$. Write $8z = e^{i\phi}$, $0 < \phi < \pi$. The sign of $\text{Im } v$ is the sign of $\sin \phi(16 \cos \phi - 11)$. Its only zero inside the interval is at $x = 5/16$, where $v = -4$. Thus v never meets $[1, \infty)$. Euler's integral for f gives $0 < f_1/f_0 < 1/2$, and consequently

$$\text{Im}(q_1/q_0) = \sqrt{2}/4 - f_1/(2f_0) > 0.$$

Thus q and $\bar{q}$ are independent. Their Wronskian never vanishes on this interval, so q cannot vanish there. In particular $|A| = q\bar{q}$ is analytic on the interval. Uniqueness for the ordinary differential equation completes the continuation. $\square$

7.1. Classical values needed for the second identity. Equation (21) with $\lambda = 1/3$ gives

$$(37) \quad H_3(4m(1-m)) = f_3(m)^2 \quad (0 < m < 1/2),$$

and the same identity holds by analytic continuation along negative real m . We first record a boundary value for the function H defined above.

Lemma 7.2. *One has*

$$(4\theta_x + 1)H(x)|_{x=-1} = \frac{2}{\pi}.$$

Proof. Put $J(x) = {}_2F_1(1/4, 1/4; 1; x)$ and $L(v) = {}_2F_1(1/4, 3/4; 1; v)$. Clausen's identity and Pfaff's transformation give

$$H(x) = J(x)^2, \quad J(x) = (1-x)^{-1/4} L\left(\frac{x}{x-1}\right).$$

At $x = -1$, differentiation yields

$$J(-1) = 2^{-1/4} L(1/2), \quad (\theta_x J)(-1) = 2^{-1/4} \left(-\frac{1}{8} L(1/2) + \frac{1}{4} L'(1/2) \right).$$

Hence $(4\theta_x + 1)H(-1) = \sqrt{2}L(1/2)L'(1/2)$. Equation (23) with $\lambda = 1/4$ gives $L(1/2)L'(1/2) = \sqrt{2}/\pi$, proving the identity. All evaluations above are of analytic functions near $x = -1$. $\square$

Lemma 7.3. *The classical signature-three evaluation is*

$$(38) \quad (33\theta_x + 4)H_3(x)|_{x=4/125} = \frac{15\sqrt{3}}{2\pi}.$$

Proof. We give the modular differentiation argument to specify the needed classical input. Set $R(m) = f_3(1-m)/f_3(m)$. The degree-five modular equation in signature three is

$$(39) \quad (\alpha\beta)^{1/3} + ((1-\alpha)(1-\beta))^{1/3} + 3\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1,$$

when $R(\beta) = 5R(\alpha)$; see [3, (1.5)]. The unique $m \in (0, 1/2)$ with $R(m) = \sqrt{5}$ forms a complementary degree-five pair with $1-m$. Substitution in (39) gives

$$p := m(1-m) = \frac{1}{125}, \quad c := 1-2m = \frac{11}{5\sqrt{5}}.$$

Consider the inverse local branch $\beta(\alpha)$ through $(m, 1-m)$, for which $R(\beta) = R(\alpha)/5$. The modular equation is symmetric, so it holds on this branch as well. Twice differentiating it gives, at this pair,

$$(40) \quad \beta' = 1, \quad \beta'' = \frac{-26 + 60p}{15cp}.$$

For an explicit check, its partial derivative in β is $-5cp^{-2/3}/6$, while its second derivative in direction $(1, 1)$ is $p^{-5/3}(-13/9 + 10p/3)$.

Apply Lemma 5.1 with $\lambda = 1/3$ and $N = 5$. Its formula gives

$$\theta_x H_3(4p) + \left(1 + \frac{p\beta''}{2c}\right) H_3(4p) = \frac{\sin(\pi/3)}{\pi c \sqrt{5}}.$$

By (40), the coefficient in parentheses is $(2/15 - 2p)/(1 - 4p) = 4/33$. The right side is $5\sqrt{3}/(22\pi)$, which proves (38). $\square$

Lemma 7.4. *At the negative argument needed below,*

$$(41) \quad (15\theta_x + 4)H_3(x)|_{x=-4} = \frac{3\sqrt{3}}{\pi}.$$

More precisely,

$$(42) \quad H_3(-4) = \frac{4}{5}H_3(4/125),$$

$$(43) \quad (\theta_x H_3)(-4) = -\frac{8}{75}H_3(4/125) + \frac{22}{25}(\theta_x H_3)(4/125).$$

Proof. The algebraic transformation of [1, Theorem 6.1] is

$$f_3(\alpha(v)) = (1+v)f_3(\beta(v)), \quad \alpha(v) = \frac{v(v+3)^2}{2(v+1)^3}, \quad \beta(v) = \frac{v^2(v+3)}{4}.$$

Although stated there for $0 \leq v < 1$, it extends from a neighborhood of zero along $v_0 \leq v \leq 0$, where $v_0 = 2/\sqrt{5} - 1$. On this interval $\alpha(v) \leq 0$, $0 \leq \beta(v) < 1/2$, and $1+v > 0$; thus no hypergeometric branch cut is crossed. With $A(v) = 4\alpha(v)(1-\alpha(v))$ and $B(v) = 4\beta(v)(1-\beta(v))$, squaring and using (37) gives

$$H_3(A(v)) = (1+v)^2 H_3(B(v)).$$

At $v = v_0$, exact differentiation gives

$$A = -4, \quad B = \frac{4}{125}, \quad A' = 30\sqrt{5}, \quad B' = -\frac{33\sqrt{5}}{125}, \quad (1+v_0)^2 = \frac{4}{5}.$$

These quantities give (42) and (43) by the chain rule. Consequently,

$$(15\theta_x + 4)H_3(-4) = \frac{2}{5}(33\theta_x + 4)H_3(4/125),$$

and Lemma 7.3 proves the result. $\square$

7.2. The transverse derivative and the weighted evaluation. On $0 \leq a < 1/3$, define

$$h_2 = -\frac{a^2 + 4a - 1}{3a^2 + 1}, \quad w_2 = \frac{(3a^2 + 1)^3}{8(a^2 - 1)^3}, \quad g_2(a) = G(h_2(a), w_2(a)),$$

$$U_0 = \frac{(a+1)(3a+1)(7a^2 - 4a + 1)}{4(3a^2 + 1)^2}, \quad U_1 = -\frac{a(a-1)^2(a+1)}{2(3a^2 + 1)}, \quad R_2 = \frac{(3a+1)(1-a^2)^{3/2}}{(3a^2 + 1)^2}.$$

Applying Lemma 2.1 with $b = (3a-1)/(a+1)$ gives, for $0 < a < 1/3$,

$$(44) \quad g_2(a) = k(a)P\left(\frac{2a}{1-a}\right), \quad k(a) = \frac{(1+a)\sqrt{1-a^2}}{1+3a^2}.$$

Lemma 7.5. For $0 < a < 1/3$,

$$(45) \quad (\theta_w G)(h_2, w_2) + U_0 g_2 + U_1 g'_2 = \frac{R_2}{2\pi}.$$

Proof. Set $E = (\theta_w G)(h_2, w_2) + U_0 g_2 + U_1 g'_2$. The normal row identity (50) gives $E' = (R'_2/R_2)E$, since

$$\frac{R'_2}{R_2} = -\frac{12a}{3a^2 + 1} + \frac{3}{3a+1} + \frac{3}{2(a+1)} + \frac{3}{2(a-1)}.$$

Thus E/R_2 is constant. At $a = 0$, one has $h_2 = 1$, $w_2 = -1/8$, $U_0 = 1/4$, $U_1 = 0$, and $R_2 = 1$. The triple integral is analytic near this point, because its denominator is $1 + \prod(1 + \cos v_j)/8 \geq 1$ there. In particular g'_2 stays bounded. Since $T_n(1, 1/4) = 2^{-n} \binom{2n}{n}$, Bauer's identity (Lemma 7.2) gives

$$E(0) = \frac{1}{4}[H(-1) + 4\theta H(-1)] = \frac{1}{2\pi}.$$

This proves (45). □

We can now prove the second formula in Conjecture VI. At $a = 1/9$,

$$h_2 = 11/21, \quad w_2 = -9261/64000 = 42^3/(-512000), \quad s = 1/4.$$

Let S_2 denote the left side of (3). Then $S_2 = 126\theta_w G + 31g_2$. At this point

$$U_0 = 65/294, \quad U_1 = -80/1701, \quad k = 10\sqrt{5}/21, \quad k'/k = 81/560, \quad s' = 81/32.$$

Therefore (45) and (44) yield

$$(46) \quad S_2 = \frac{240\sqrt{5}}{7\pi} + k(4P + 15P').$$

At $s = 1/4$ the quantities in Theorem 7.1 satisfy

$$z = \frac{11 + 3i\sqrt{15}}{128}, \quad v = -4, \quad \frac{\sqrt{z}}{1-4z} = \frac{\sqrt{3} + i\sqrt{5}}{6}.$$

Differentiation of their defining rational functions gives

$$\frac{d}{ds} \log \frac{\sqrt{z}}{1-4z} = -\frac{4}{3} + \frac{16i\sqrt{15}}{45}, \quad \frac{d}{ds} \log v = -4 + \frac{4i\sqrt{15}}{3}.$$

Since $H_3(-4)$ is real and positive, differentiating (30) now gives, with $\theta_v = v d/dv$,

$$P = \frac{2\sqrt{3}}{3}H_3(-4), \quad P' = \frac{8\sqrt{3}}{27}H_3(-4) + \frac{16\sqrt{3}}{9}\theta_v H_3(-4).$$

Substitution into (46) yields

$$S_2 = \frac{240\sqrt{5}}{7\pi} + \frac{160\sqrt{15}}{189}(4 + 15\theta_v)H_3(-4).$$

The classical value in Lemma 7.4 is $(4 + 15\theta_v)H_3(-4) = 3\sqrt{3}/\pi$. Consequently

$$S_2 = \frac{240\sqrt{5}}{7\pi} + \frac{160\sqrt{5}}{21\pi} = \boxed{\frac{880\sqrt{5}}{21\pi}}.$$

The coefficient bound $T_n(22, 441) \leq 64^n$ gives geometric ratio $64/125 < 1$ after cubing and dividing by 512000^n . This proves absolute convergence, including the linear weight, and completes the second identity.

APPENDIX A. EXACT DIFFERENTIAL CERTIFICATES

This appendix gives finite certificates for Lemma 3.2, Proposition 4.1, and the corresponding second-curve equations. All operations are differentiation and arithmetic of rational functions.

Put $\theta = z\partial_z$ and

$$F(y, z) = \sum_{n \geq 0} P_n(y)^3 z^n, \quad W = \theta \sum_{n \geq 0} P_n(y)^2 P_{n-1}(y) z^n.$$

The value assigned to P_{-1} does not affect W ; take $P_{-1} = 1$. For compactness define

$$\begin{aligned} A &= 1 + 4yz + z^2, & B &= 2yz + z^2, \\ C &= zy\{(y+z) + (4y+3z)\theta + 2(2y+z)\theta^2\}, \\ (d_0, d_1, d_2) &= (-3zy(y+3z, 4y+5z, 2(2y+z)), \\ L &= \theta^3 - zy^3(2\theta+1)^3 - 3z^2y^2(2\theta+3)(\theta+1)(2\theta+1) + z^2(\theta+1)^3. \end{aligned}$$

Multiplication by a coefficient in these expressions is always on the left. The Legendre recurrence gives the exact identities

$$(47) \quad A\theta W + BW = CF, \quad LF = (d_2\theta^2 + d_1\theta + d_0)W, \quad (y^2 - 1)F_y = 3(y\theta F - W).$$

For completeness, put $V_j = \sum_{n \geq 0} P_n(y)^{3-j} P_{n-1}(y)^j z^n$. Raising $(n+1)P_{n+1} = (2n+1)yP_n - nP_{n-1}$ to powers one, two, and three, respectively, gives

$$\begin{aligned} \theta V_2 &= z\{y(2\theta+1)F - \theta V_1\}, \\ \theta^2 V_1 &= z\{y^2(2\theta+1)^2 F - 2y(2\theta+1)\theta V_1 + \theta^2 V_2\}, \\ \theta^3 F &= z\{y^3(2\theta+1)^3 F - 3y^2(2\theta+1)^2 \theta V_1 + 3y(2\theta+1)\theta^2 V_2 - \theta^3 V_3\}. \end{aligned}$$

Since $V_3 = 1 + zF$, elimination proves the first two equations in (47). The third follows from $(y^2 - 1)P'_n = n(yP_n - P_{n-1})$.

Here is a short specification of every coefficient of the resulting four-dimensional connection. Define the operator

$$(48) \quad \mathcal{R} = \frac{A^2 L - d_2\{A\theta \circ C - (\theta A + B)C\} - d_1 AC}{Q} = \sum_{j=0}^3 r_j \theta^j,$$

where

$$Q = d_2\{B^2 + B\theta A - A\theta B\} - d_1 AB + d_0 A^2 = -3yz(z-1)(z+1)(8y^2 z - yz^2 - y - 3z).$$

Thus $W = \mathcal{R}F$. Write $C = \sum_{j=0}^2 c_j \theta^j$, set $r_{-1} = c_3 = 0$, and put

$$m_j = -\frac{A(\theta r_j + r_{j-1}) + B r_j - c_j}{A r_3} \quad (0 \leq j \leq 3).$$

Let M have its first three rows e_1, e_2, e_3 and its last row (m_0, m_1, m_2, m_3) , where e_j is the row vector with its j th coordinate equal to one, indexed from zero. Set

$$N_0 = \frac{3}{y^2 - 1}\{y e_1 - (r_0, r_1, r_2, r_3)\}, \quad N_{j+1} = \theta N_j + N_j M \quad (0 \leq j < 3),$$

and let N have rows $N_0, \dots, N_3$. For $V = (F, \theta F, \theta^2 F, \theta^3 F)^t$, equations (47) give

$$\theta V = MV, \quad \partial_y V = NV.$$

The differential identities are understood after clearing denominators at apparent poles; these poles need not be singularities of F .

We now give the substitution certificate. In either column of the following table, let $g(a) = G(h(a), w(a))$, where $G(h, w) = \sum T_n(h, 1/4)^3 w^n$.

	I	II
h	$\frac{a^2 - 4a - 1}{5a^2 - 1}$	$-\frac{a^2 + 4a - 1}{3a^2 + 1}$
w	$-\frac{(5a^2 - 1)^3}{8(a-1)^2(a+1)^3(3a+1)}$	$\frac{(3a^2 + 1)^3}{8(a-1)^3(a+1)^3}$
b	$-\frac{5a+1}{a+1}$	$\frac{3a-1}{a+1}$
s	$\frac{2a}{3a+1}$	$\frac{2a}{1-a}$
t	$\frac{a+1}{1-a}$	$-1-s$

In column I, $t = (1-s)/(1-2s)$. Put

$$y = \frac{h}{\sqrt{h^2 - 1}}, \quad z = w(h^2 - 1)^{3/2}, \quad T = y'N + \frac{z'}{z}M, \quad k = \frac{\sqrt{1-a^2}}{1+ab}.$$

These substitutions give $g = F(y, z) = k\Psi(s, t)$ and $V' = TV$. The square roots are taken consistently; every entry of T is rational in a after cancellation.

Put $\ell = k'/k$ and define rational rows

$$B_0 = e_0, \quad B_{j+1} = \frac{B'_j - \ell B_j + B_j T}{s'} \quad (0 \leq j < 3).$$

Then $d^j \Psi(s, t(s))/ds^j = k^{-1} B_j V$. The first certificate is the rational row identity

$$(49) \quad B_3 + 3pB_2 + (p_s + 4q + 2p^2)B_1 + (2q_s + 4pq)B_0 = 0,$$

with

	$p(s)$	$q(s)$
I	$\frac{6s^2 - 6s + 1}{2s(s-1)(2s-1)}$	$\frac{1}{4(2s-1)^2}$
II	$\frac{(2s+1)(s^2+s-1)}{s(s-1)(s+1)(s+2)}$	$\frac{4s^2+4s-1}{16s(s-1)(s+1)(s+2)}$

Consequently each restricted function $\Psi(s, t(s))$ satisfies the symmetric square of $Y'' + pY' + qY = 0$. For column I, putting $u = 1 - 2s$ gives, equivalently,

$$\{(2\theta_u - 1)^3 - u^2(2\theta_u + 1)^3\} \Psi\left(\frac{1-u}{2}, \frac{1+u^{-1}}{2}\right) = 0.$$

For the normal derivative certificate define

$$E = e_1 + U_0 e_0 + U_1 e_0 T.$$

The second row identity is

$$(50) \quad E' + ET = \rho E,$$

where

$$\begin{aligned}
U_0^I &= \frac{3(3a+1)(3a^2+1)(5a^2+2a+1)}{8(a+1)(5a^2-1)^2}, \\
U_1^I &= -\frac{(a-1)(3a+1)(a^2+6a+1)}{8(5a^2-1)}, \\
\rho^I &= -\frac{20a}{5a^2-1} + \frac{2(5a+1)}{5a^2+2a+1} + \frac{3}{2(3a+1)} + \frac{1}{a+1} + \frac{1}{a-1} - \frac{1}{2a}, \\
U_0^{II} &= \frac{(a+1)(3a+1)(7a^2-4a+1)}{4(3a^2+1)^2}, \\
U_1^{II} &= -\frac{a(a-1)^2(a+1)}{2(3a^2+1)}, \\
\rho^{II} &= -\frac{12a}{3a^2+1} + \frac{3}{3a+1} + \frac{3}{2(a+1)} + \frac{3}{2(a-1)}.
\end{aligned}$$

Both (49) and (50) are checked by inserting (48), differentiating, and clearing denominators: all four resulting numerators vanish identically. The displayed definitions specify these checks completely and avoid large expanded connection matrices. Since $EV = \theta G + U_0 g + U_1 g'$, the second identity proves the asserted first-order equation for the normal expression. Its multiplicative constant is then fixed by the analytic boundary arguments in the text.

APPENDIX B. THE CONTINUATION PATH

Recall that $p_0 = (\sqrt{2} - 1 + i)/2$ and $m_0 = (1 - \sqrt{2})/2$.

Lemma B.1. *Continuation of (34) along $p = rp_0$, $0 \leq r \leq 1$, gives the principal value of $f(m_0)$ and the upper boundary value $H_3(27/2 + i0)$.*

Proof. Denote the two arguments in (34) by $u(r)$ and $m(r)$. The denominators do not vanish: $\text{Re}(1 + 2p) > 0$ and

$$\text{Re}(1 + p + p^2) = 1 + \frac{\sqrt{2}-1}{2}r(1-r) > 0.$$

The following real polynomial checks specify the continuation explicitly. Put

$$D(r) = (-3 + 2\sqrt{2})r^4 + (4 - 3\sqrt{2})r^3 + (-4 + 3\sqrt{2})r^2 + (2 - 2\sqrt{2})r - 2.$$

Then $D(r) \leq -6 + 3\sqrt{2} < 0$ on $[0, 1]$, and

$$\text{Im } m(r) = \frac{r^3(r-1)((-7 + 5\sqrt{2})r + 4 - 3\sqrt{2})}{2((-4 + 2\sqrt{2})r^2 + (2 - 2\sqrt{2})r - 1)} < 0 \quad (0 < r < 1).$$

Also $\text{Im } u(r) = 27r^2 N(r)/(4D(r)^3)$, where

$$\begin{aligned}
N(r) &= (-41 + 29\sqrt{2})r^8 + (133 - 94\sqrt{2})r^7 + (-178 + 126\sqrt{2})r^6 + (122 - 86\sqrt{2})r^5 \\
&\quad + (-72 + 50\sqrt{2})r^3 + (60 - 44\sqrt{2})r^2 + (-28 + 18\sqrt{2})r + 4 - 4\sqrt{2}.
\end{aligned}$$

Its four coefficients of degrees five through eight are positive and all other nonzero coefficients are negative. Hence $N(r) \leq 40 - 29\sqrt{2} < 0$, proving $\text{Im } u(r) > 0$. Finally

$$\text{Re } u(r) - \frac{1}{2} = -\frac{(r-1)(r+2)J(r)}{4D(r)^3},$$

where

$$\begin{aligned} J(r) = & (-198 + 140\sqrt{2})r^{10} + (1038 - 734\sqrt{2})r^9 + (-2355 + 1665\sqrt{2})r^8 + (3620 - 2560\sqrt{2})r^7 \\ & + (-3398 + 2404\sqrt{2})r^6 + (2148 - 1512\sqrt{2})r^5 + (-768 + 560\sqrt{2})r^4 + (108 - 50\sqrt{2})r^3 \\ & + (60 - 18\sqrt{2})r^2 + (-20 + 24\sqrt{2})r + 8. \end{aligned}$$

Only its coefficients of degrees seven through ten are negative; therefore $J(r) \geq 2113 - 1489\sqrt{2} > 0$. Thus $\operatorname{Re} u(r) < 1/2$ for $r < 1$. It follows that $4u(r)(1 - u(r))$ remains in the upper half-plane and approaches $27/2$ from above. These signs prove all asserted branch choices. $\square$

AI ASSISTANCE

OpenAI ChatGPT/Codex (GPT-6-astra) assisted with symbolic calculations, literature checks. The main ideas were from the author. ChatGPT Astra pro assistance also included the developing of transverse derivative method. The October 2026 revisions used GPT-6 Astra Pro. Responsibility for the mathematical claims, references, and final text rests with the author.

REFERENCES

- [1] B. C. Berndt, S. Bhargava, and F. G. Garvan, *Ramanujan's theories of elliptic functions to alternative bases*, Trans. Amer. Math. Soc. **347** (1995), no. 11, 4163–4244. doi:10.1090/S0002-9947-1995-1311903-0.
- [2] NIST Digital Library of Mathematical Functions, <https://dlmf.nist.gov/>; accessed 29 October 2026.
- [3] H. H. Chan and W.-C. Liaw, *On Russell-type modular equations*, Canad. J. Math. **52** (2000), no. 1, 31–46. doi:10.4153/CJM-2000-002-0.
- [4] M. van Hoeij, D. van Straten, and W. Zudilin, *A hyperelliptic saga on a generating function of the squares of Legendre polynomials*, Journal of Experimental Mathematics **1** (2025), no. 2, 278–305. doi:10.56994/JXM.001.002.005.
- [5] S. Ramanujan, *Modular equations and approximations to π* , Quart. J. Math. **45** (1914), 350–372. <https://ramanujan.sirinudi.org/Volumes/published/ram06.pdf>.
- [6] M. D. Rogers, *New ${}_5F_4$ hypergeometric transformations, three-variable Mahler measures, and formulas for $1/\pi$* , Ramanujan J. **18** (2009), no. 3, 327–340. doi:10.1007/s11139-007-9040-x.
- [7] Z.-W. Sun, *On sums related to central binomial and trinomial coefficients*, in M. B. Nathanson (ed.), *Combinatorial and Additive Number Theory: CANT 2011 and 2012*, Springer Proceedings in Mathematics & Statistics **101**, Springer, New York, 2014, 257–312. doi:10.1007/978-1-4939-1601-6_18.
- [8] Z.-W. Sun, *New series for powers of π and related congruences*, Electron. Res. Arch. **28** (2020), no. 3, 1273–1342. doi:10.3934/era.2020070.
- [9] Z.-W. Sun, *List of conjectural series for powers of π and other constants*, arXiv:1102.5649v47, 29 December 2014.
- [10] J. G. Wan, *Series for $1/\pi$ using Legendre's relation*, Integral Transforms Spec. Funct. **25** (2014), no. 1, 1–14. doi:10.1080/10652469.2013.809072.
- [11] W. Zudilin, *A generating function of the squares of Legendre polynomials*, Bull. Aust. Math. Soc. **89** (2014), no. 1, 125–131. doi:10.1017/S0004972713000233.