

Near-Null Limits and Contact Distributions for Maxwell Composite Operators

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Abstract

We investigate the simultaneous near-null, longitudinal-continuum, and ultraviolet limits of four-point correlation functions of quadratic composite operators in Maxwell theory. Taking the external momenta to be collinear and spacelike, we weight each loop integral by a smooth test function of the routed longitudinal momentum. All three connected Wick contraction cycles are generated from a common Gaussian source functional and regulated by a cutoff on the total proper time. For the resulting integrals restricted to a fixed ultraviolet proper-time interval, we establish the uniform bound

$$\left| \mathcal{I}_{\delta, M}^{\epsilon}[\phi] - \mathcal{I}_{0, \infty}^{\epsilon}[\phi] \right| \leq C \|\phi\|_{C_b^8} \left(\delta \epsilon^{-3} + M^{-2} \delta^{-3} \epsilon^{-1} \right),$$

where δ denotes the dimensionless near-null tilt parameter and M specifies the longitudinal lattice resolution. This estimate determines the joint scaling regimes under which a universal counterterm subtraction yields an identical renormalized finite part. We further show that the longitudinal logarithmic divergence of the alternating chiral cycle is governed by the measure $8\mathbf{1}_{(0,1)}d\xi + 3\delta_0 + 3\delta_1$, whereas each of the other two cycles contributes $8\mathbf{1}_{(0,2)}d\xi$. These distributional coefficients emerge from the complete covariant numerators, including the instantaneous longitudinal terms. Finally, for the alternating cycle and any smooth compactly supported test function coinciding with ξ^2 in a neighborhood of $[0, 1]$, the critical trajectory $\delta = c\epsilon^3$ with $M^2\delta^3\epsilon \rightarrow \infty$ leaves a finite discrepancy of $30c$, demonstrating the sharpness of the tilt scaling. The proof combines exact Gaussian moment reductions, whole-line quadrature error bounds, and an explicit evaluation of boundary jets.

Keywords. Maxwell theory; composite operators; light-front quantization; contact terms; ultraviolet limits; proper-time regularization.

1 Introduction

Deforming a quantization surface from spacelike to null alters the analytic structure of constrained gauge-field components. While transverse propagation smoothly approaches a light-front kernel, longitudinal degrees of freedom tend to concentrate at equal light-front time. For composite operators, taking these limits also entails evaluating propagator derivatives and distributional products along partial coincidence diagonals. A consistent comparison between the spacelike and light-front formulations therefore requires both a common distributional extension of the field-strength covariance and a uniform treatment of the ultraviolet regime.

The sensitivity of near-null limits to compactification and longitudinal zero modes is already evident in perturbative scalar field theories. Hellerman and Polchinski analyzed the singular behavior arising in the lightlike compactification limit [1], while Harindranath, Martinovič, and Vary systematically

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compared near-null spacelike quantization, infinite-momentum frame calculations, and quantization formulated directly on the light front [2]. For gauge theories, eliminating constrained components in the presence of external sources yields the bridge identities established by Bassetto and De Pol [3], and the renormalization of composite operators in light-cone gauge was investigated by Acerbi and Bassetto [4]. Collectively, these studies underscore the critical role played by constrained fields and boundary data when relating distinct gauge-field representations.

In four-dimensional gauge theories, position-dependent gauge and theta couplings naturally organize scalar and pseudoscalar composite insertions within a single quadratic source functional. Osborn computed the resulting one-loop effective action and local Weyl anomaly response [5], and the corresponding short-distance expansions can be systematically expressed in terms of heat-kernel coefficients [6]. In the present work, we refine this probe by inserting a smooth weight $\phi(k/P)$ prior to integrating or summing over the longitudinal loop momentum k , where $P > 0$ denotes the external longitudinal momentum scale and the routing cut defining k is fixed for each contraction cycle. The resulting linear functional resolves longitudinal endpoint contributions that would otherwise be merged in the unweighted case $\phi = 1$.

Specifically, we consider $d_G = N^2 - 1$ independent Maxwell fields and the quadratic composite operators constructed from their self-dual and anti-self-dual field strengths. Because all insertions enter quadratically via external sources, the four-point correlation function is given by an exact Gaussian cumulant. We choose collinear external momenta $p, -p, p, -p$ with $p^2 > 0$. The three connected Wick cycles are regulated by a common lower cutoff ϵ on the total proper time, with the upper cutoff normalized to unity in units of p^2 . The remaining deformation parameters are the dimensionless tilt δ of the quantization surface and the longitudinal lattice resolution M .

Our first main result is the uniform error estimate

$$\left| \mathcal{I}_{\delta, M}^\epsilon[\phi] - \mathcal{I}_{0, \infty}^\epsilon[\phi] \right| \leq C \|\phi\|_{C_b^8} \left(\delta \epsilon^{-3} + M^{-2} \delta^{-3} \epsilon^{-1} \right), \quad (1.1)$$

where the normalized ultraviolet integrals $\mathcal{I}$ are defined in definition 4.1. Theorem 5.1 establishes this bound for a general class of polynomial numerators encompassing all three Maxwell Wick cycles. In particular, choosing the scaling trajectory $\delta = \epsilon^4$ and $M \geq \epsilon^{-7}$ yields an $O(\epsilon)$ discrepancy. Consequently, applying a common power-law and logarithmic counterterm subtraction produces the exact same renormalized ultraviolet finite part in both formulations. Along such trajectories, the physical spatial circumference diverges, corresponding to a decompactifying near-null limit.

Our second main result characterizes the longitudinal logarithmic divergence as an explicit distribution. For the alternating chiral cycle, we find

$$\mathfrak{R}_a = 8\mathbf{1}_{(0,1)} d\xi + 3\delta_0 + 3\delta_1, \quad (1.2)$$

whereas each of the two contact cycles yields $8\mathbf{1}_{(0,2)} d\xi$. In the alternating cycle, the two endpoint Dirac masses originate from the third derivatives of a fourth-order differential coefficient possessing cubic zeros at $\xi = 0$ and $\xi = 1$. In the contact cycles, by contrast, integration by parts over the two adjacent Feynman-parameter subdomains generates equal and opposite first-derivative boundary terms at their interface $\xi = 1$. Thus, all three logarithmic coefficients collapse to positive Radon measures, even though the raw Gaussian reduction involves test-function derivatives up to fourth order.

Our third result pinpoints the critical scaling regime at which the renormalized finite part of the alternating cycle becomes path-dependent. An exact evaluation of its second longitudinal moment reveals a leading discrepancy proportional to $30\delta\epsilon^{-3}$. Using a Gaussian tail localization estimate, we extend this result to any test function in $C_c^\infty(\mathbb{R})$ that coincides with ξ^2 on a neighborhood of $[0, 1]$. Along the critical trajectory $\delta = c\epsilon^3$ with $M^2\delta^3\epsilon \rightarrow \infty$, the limiting finite part of this cycle shifts by $30c$. This explicit counterexample confirms that the $\delta\epsilon^{-3}$ tilt scaling in equation (1.1) is sharp within the stated test-function space.

Throughout our analysis, the finite-tilt covariance is defined via Feynman analytic continuation from spacelike slices, thereby retaining longitudinal zero modes with nonvanishing transverse momentum.

The comparison of weighted loop integrals is performed on a fixed ultraviolet proper-time interval, where the Gaussian integrands remain regular at vanishing loop momentum. Prescribing characteristic initial data for null *gauge potentials*—such as those underlying the Mandelstam–Leibbrandt prescription [7]—constitutes a distinct boundary-value problem not addressed here. Rather, our results provide rigorous distributional matching conditions for light-front and worldsheet ribbon-graph representations of Maxwell composite insertions, including frameworks motivated by planar gauge theory [8].

The remainder of the paper is organized as follows. Section 2 establishes the source functional, normal-ordering conventions, and contact propagators. Section 3 constructs the finite-tilt covariance and analyzes its near-null limiting kernels. Section 4 derives the longitudinally weighted four-point integrals via Gaussian differential operators. The uniform convergence bound and universal subtraction are proved in section 5, while the distributional logarithmic coefficients and path dependence are established in sections 6 and 7, respectively. Explicit polynomial coefficients, general Euclidean finite parts, and determinant identities are collected in the appendices.

2 Maxwell sources and longitudinal reduction

2.1 Source normalization and contact propagators

In four-dimensional Euclidean space, we consider the free Maxwell action for d_G decoupled gauge fields,

$$S_0 = \frac{1}{4} \sum_{A=1}^{d_G} \int_{\mathbb{R}^4} F_{\mu\nu}^A F_{\mu\nu}^A d^4x, \quad F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A. \quad (2.1)$$

Endowing the space of two-forms with the pointwise inner product $\langle F, G \rangle = \frac{1}{2} F_{\mu\nu} G_{\mu\nu}$, we introduce the chiral Hodge projectors $P_\pm = (1 \pm *)/2$ and choose orthonormal bases within the corresponding three-dimensional self-dual and anti-self-dual eigenspaces. Denoting the projected field-strength components in these bases by $f_{\pm,a}^A$, we define the normalized chiral composite operators

$$O_\pm(x) = \frac{1}{\sqrt{6d_G}} : \sum_{A=1}^{d_G} \sum_{a=1}^3 (f_{\pm,a}^A(x))^2 :. \quad (2.2)$$

Here normal ordering subtracts the vacuum self-contraction within each individual composite insertion while leaving Wick contractions between distinct insertions—including their distributional extensions to coincident points—unchanged.

Given external sources $J_\pm$, we set $j_\pm = J_\pm/\sqrt{6d_G}$ and consider the connected generating functional

$$W[J_+, J_-] = \log \left\langle \exp \int (J_+ O_+ + J_- O_-) d^4x \right\rangle_0. \quad (2.3)$$

Combining the free action with the composite insertions yields the source-dependent quadratic action

$$S[f, \vartheta] = \frac{1}{4} \sum_A \int \left(f F_{\mu\nu}^A F_{\mu\nu}^A + \vartheta F_{\mu\nu}^A (*F^A)_{\mu\nu} \right) d^4x, \quad f = 1 - j_+ - j_-, \quad \vartheta = j_- - j_+. \quad (2.4)$$

Correlation functions are obtained by differentiating with respect to the sources at $f = 1$ and $\vartheta = 0$. Subtracting a term linear in the sources implements the normal ordering in (2.2) and has a vanishing fourth functional variation. For sufficiently small real sources satisfying $f > |\vartheta|$, the quadratic form in (2.4) remains positive definite upon gauge fixing. Here ϑ acts as a real Euclidean source; analytic continuation to Lorentzian signature introduces the standard factor of i in front of the topological density.

A unified distributional extension of the field-strength two-point function is obtained by applying two exterior derivatives to the Feynman-gauge potential propagator

$$D(x) = \frac{1}{4\pi^2 |x|^2} = \frac{1}{4\pi^2} \int_0^\infty e^{-\sigma |x|^2} d\sigma. \quad (2.5)$$

Acting on the six-dimensional space of two-forms, let $X_x = x \wedge \iota_x$. The resulting field-strength covariance density with respect to inverse proper time σ takes the form

$$C_\sigma(x) = \frac{e^{-\sigma|x|^2}}{\pi^2} \left(\sigma I_6 - \sigma^2 X_x \right). \quad (2.6)$$

Invoking the algebraic two-form identity $P_\pm X_x P_\pm = |x|^2 P_\pm / 2$, we find that the same-chirality projections are exact σ -derivatives:

$$P_\pm C_\sigma P_\pm = \frac{1}{2} \partial_\sigma \delta_\sigma(x) P_\pm, \quad \delta_\sigma(x) = \frac{\sigma^2}{\pi^2} e^{-\sigma|x|^2}. \quad (2.7)$$

Because $\delta_\sigma \rightarrow \delta^{(4)}$ as $\sigma \rightarrow \infty$ and $\delta_\sigma \rightarrow 0$ as $\sigma \rightarrow 0$ in the sense of distributions, integrating over $\sigma \in (0, \infty)$ yields the pure contact relation

$$P_\pm \langle FF \rangle P_\pm = \frac{1}{2} \delta^{(4)}(x) P_\pm. \quad (2.8)$$

Notably, although the same-chirality field-strength correlator vanishes identically at separated points, it acquires a nonvanishing local contact term under proper-time regularization.

For the connected four-point function of $O_+(x_1)$, $O_-(x_2)$, $O_+(x_3)$, and $O_-(x_4)$, the Wick contractions organize into three unoriented four-cycles, each comprising sixteen equivalent slot pairings. Two of these cycles contain one same-chirality contraction of each sign; we refer to them as the *contact cycles*, and to the remaining cycle with strictly alternating chiralities along its edges as the *alternating cycle*. Away from full coincidence ($x_1 \neq x_2$), the sum of the two contact cycles reduces in position space to

$$\frac{2}{3\pi^4 d_G} \delta^{(4)}(x_1 - x_3) \delta^{(4)}(x_2 - x_4) \frac{1}{|x_1 - x_2|^8}. \quad (2.9)$$

Extending this expression across the remaining coincidence diagonal $x_1 = x_2$ requires an ultraviolet renormalization prescription. For instance, the differential renormalization extension

$$[|x|^{-8}]_\mu = -\frac{1}{768} \square^3 \left(\frac{\log(\mu^2 |x|^2)}{|x|^2} \right) \quad (2.10)$$

exhibits the scale variation $\mu \partial_\mu [|x|^{-8}]_\mu = \pi^2 \square^2 \delta^{(4)}(x) / 96$. Such distributional extensions and the associated local counterterm ambiguities are standard features of composite-operator renormalization in position space [9]. In our momentum-space analysis below, imposing a single cutoff on the total proper time automatically supplies a consistent common extension across all reduced polarization channels.

2.2 Longitudinal elimination

In strict light-front coordinates with metric $ds^2 = -2dx^+ dx^- + dx_\perp^2$, we restrict attention to the nonzero longitudinal momentum sector where the light-cone gauge condition $A_- = 0$ is admissible. In this sector, the constrained component $a \equiv A_+$ enters the source-dependent action quadratically:

$$S = S_\perp + \frac{1}{2} (a, K_f a) + (a, \mathcal{J}), \quad K_f = -\partial_- f \partial_- . \quad (2.11)$$

Performing the Gaussian functional integral over a generates both the nonlocal Schur-complement interaction $-\frac{1}{2} (\mathcal{J}, K_f^{-1} \mathcal{J})$ and the functional determinant factor $(\det' K_f)^{-d_G/2}$. Crucially, the external sources embedded in f and $\mathcal{J}$ are retained prior to integrating out a ; this is the structural mechanism underlying source-retaining bridge identities [3].

When the longitudinal coordinate x^- is compactified on a circle, the zero-mean Green's operator $G_f = K_f^{-1}$ can be written down explicitly. Let $\mathbf{P}$ denote the orthogonal projection onto zero-mean

functions on the longitudinal circle, and let $\mathbb{T} = \partial_-^{-1}$ denote the unique zero-mean antiderivative on that subspace. For a strictly positive profile $f > 0$, we set $w = f^{-1}$. The periodic inverse on the zero-mean subspace is then given by

$$G_f = -\mathbb{T} \left(w - \frac{|w\rangle\langle w|}{\int w} \right) \mathbb{T}, \quad K_f G_f = G_f K_f = \mathbb{P}. \quad (2.12)$$

To verify (2.12), take any zero-mean source J and set $u = \mathbb{T}J$. Integrating $-\partial_-(fa') = J$ once yields $fa' = -u + C$, where periodicity of a fixes the integration constant to be $C = (\int wu)/(\int w)$. Pairing with J then gives the quadratic form

$$(J, G_f J) = \int wu^2 - \frac{(\int wu)^2}{\int w}. \quad (2.13)$$

The spatially constant A_+ mode imposes a separate global Gauss constraint, and periodic gauge transformations likewise leave the longitudinal zero mode of A_- invariant. Equation (2.12) characterizes the inverse within the nonzero longitudinal sector and does not eliminate those global degrees of freedom.

The fourth-order source expansion of the full block determinant—including the purely longitudinal contraction cycle—is detailed in section C. All resulting terms inherit the common proper-time measure introduced below.

2.3 Polarization completeness and proper-time regularization

For any four-momentum p with nonvanishing longitudinal component p^+ , we define two off-shell transverse polarization vectors together with the null gauge vector n :

$$E_i(p) = \left(0, \frac{p_i}{p^+}, \delta_{i1}, \delta_{i2} \right)^t, \quad n = (0, 1, 0, 0)^t, \quad \mathbb{B}(p) = (E_1(p), E_2(p), n/p^+). \quad (2.14)$$

Setting $\widehat{D}(p) = \text{diag}((p^2)^{-1}, (p^2)^{-1}, 1)$, a direct algebraic computation yields the completeness relation

$$\mathbb{B}(p)\widehat{D}(p)\mathbb{B}(-p)^t = \frac{g^{-1}}{p^2} - \frac{pn^t + np^t}{p^2(p \cdot n)}. \quad (2.15)$$

Here the first two diagonal entries of $\widehat{D}(p)$ correspond to the two propagating transverse polarizations, while the third entry represents the instantaneous longitudinal Coulomb flux, with its $(p^+)^{-2}$ weight absorbed into $\mathbb{B}(p)$ and $\mathbb{B}(-p)^t$. Note that the third column of $\mathbb{B}(p)$ reverses sign under $p \mapsto -p$.

Up to the overall normalization already absorbed into equation (2.2), the scalar source vertex tensor reads

$$V_f(p, r)_{\mu\nu} = (p \cdot r)g_{\mu\nu} - r_\mu p_\nu, \quad (2.16)$$

while for the pseudoscalar vertex tensor we adopt the bilinear convention

$$u^\mu V_\theta(p, r)_{\mu\nu} v^\nu = \frac{1}{4} \varepsilon^{\alpha\beta\gamma\lambda} F(p, u)_{\alpha\beta} F(r, v)_{\gamma\lambda}, \quad F(p, u)_{\alpha\beta} = p_\alpha u_\beta - p_\beta u_\alpha, \quad (2.17)$$

with $\varepsilon^{+-12} = 1$. Because both vertex tensors are manifestly transverse to the incoming and outgoing leg momenta ($p^\mu V_{\mu\nu} = 0$ and $V_{\mu\nu} r^\nu = 0$), the longitudinal gauge remainder proportional to $p_\mu n_\nu + n_\mu p_\nu$ in equation (2.15) drops out identically upon contracting successive vertices around a loop. Consequently, tracing over the two transverse polarization channels and the third instantaneous flux channel reproduces the full covariant numerator trace whenever $p^+ \neq 0$ on every internal edge.

Upon Wick rotation to Euclidean proper time $\ell > 0$, the common three-channel edge density takes the form

$$\widehat{D}(\ell, p) = e^{-\ell p^2} \text{diag}(1, 1, p^2), \quad \ell > 0, \quad (2.18)$$

and ultraviolet regularization is implemented by restricting the total proper time $T = \sum_e \ell_e$. Although the identity $\int_0^\infty p^2 e^{-\ell p^2} d\ell = 1$ recovers the unregularized instantaneous contact factor in the formal limit, at any finite cutoff on the total proper time the third channel must be kept in the exponential form of equation (2.18). Replacing the third entry by a bare constant prior to regularization while cutting off only the transverse propagators would spoil covariance and alter the regulated amplitude. Equivalently, in the Schur-complement expansion equation (C.4), every effective vertex retains the proper-time parameters of the underlying instantaneous edges rather than assigning a reduced cutoff after edge contraction.

Finally, the extension of the integrand across $p^+ = 0$ is uniquely fixed by the finite-tilt field-strength covariance constructed in section 3.

3 Near-null covariance and propagation

3.1 Spacelike slices approaching a null surface

To interpolate smoothly between spacelike and light-front quantization, we introduce the tilted coordinates

$$t = x^+ + \frac{\eta^2}{2} x^-, \quad s = x^-, \quad ds_{\text{metric}}^2 = -2dt ds + \eta^2 ds^2 + dx_\perp^2, \quad \eta > 0. \quad (3.1)$$

When the longitudinal coordinate s is identified with period L_- , the constant- t hypersurfaces are spacelike cylinders of physical circumference ηL_- . The linear coordinate transformation

$$T_{\text{cart}} = t/\eta, \quad X_{\text{cart}} = \eta s - t/\eta \quad (3.2)$$

brings the metric into standard Minkowski form. Crucially, the Feynman $i0^+$ continuation is defined with respect to the timelike coordinate T_{cart} at finite $\eta > 0$ prior to taking the near-null limit $\eta \downarrow 0$. This ordering of limits is essential when analyzing compactified longitudinal modes [1, 2].

Rotating the transverse plane so that the momentum lies along x^1 , we adopt the Fourier convention $\exp(-i\omega t - i\kappa s + i\kappa x^1)$, for which the quadratic Lorentz invariant reads

$$p^2 = \kappa^2 - 2k\omega - \eta^2 \omega^2. \quad (3.3)$$

In Lorentzian signature, we write the source action as $-\frac{1}{4} \int (fF^2 + \vartheta_L F\tilde{F})$, where ϑ_L is the analytic continuation of the Euclidean pseudoscalar source in equation (2.4), including its Wick-rotation factor. The covariance relations below are evaluated at zero sources; overall factors of i associated with the Feynman measure are suppressed.

3.2 Nonzero and zero longitudinal modes

For modes with nonzero longitudinal momentum $k \neq 0$, we impose the gauge condition $A_s = 0$. Restricted to the remaining components (A_t, A_1, A_2) , the kinetic Hessian matrix is given by

$$\mathbb{K}_{\eta,k} = \begin{pmatrix} k^2 + \eta^2 \kappa^2 & \kappa(k + \eta^2 \omega) & 0 \\ \kappa(k + \eta^2 \omega) & 2k\omega + \eta^2 \omega^2 & 0 \\ 0 & 0 & -p^2 \end{pmatrix}, \quad \det \mathbb{K}_{\eta,k} = k^2 (p^2)^2. \quad (3.4)$$

For the longitudinal zero mode $k = 0$ at nonvanishing transverse momentum $\kappa \neq 0$, the gauge $A_s = 0$ becomes degenerate; we instead impose the transverse Coulomb gauge $A_1 = 0$ and retain the components (A_t, A_s, A_2) . The resulting kinetic matrix and its inverse are

$$\mathbb{K}_{\eta,0} = \begin{pmatrix} \eta^2 \kappa^2 & \kappa^2 & 0 \\ \kappa^2 & \omega^2 & 0 \\ 0 & 0 & \eta^2 \omega^2 - \kappa^2 \end{pmatrix}, \quad (3.5)$$

$$\mathbb{K}_{\eta,0}^{-1} = \begin{pmatrix} -\omega^2/(\kappa^2 Q_\eta) & 1/Q_\eta & 0 \\ 1/Q_\eta & -\eta^2/Q_\eta & 0 \\ 0 & 0 & -1/Q_\eta \end{pmatrix}, \quad Q_\eta = \kappa^2 - \eta^2 \omega^2, \quad (3.6)$$

with determinant $\det \mathbb{K}_{\eta,0} = \kappa^2 Q_\eta^2$.

Proposition 3.1 (Common field-strength covariance). *Let $\mathbb{C}(p)$ denote the 6×4 exterior derivative matrix in momentum space, with entries $\mathbb{C}(p)_{(\mu\nu),a} = p_\mu \delta_{\nu a} - p_\nu \delta_{\mu a}$. Let D_{red} denote the 4×4 potential propagator obtained by embedding the 3×3 inverse of either equation (3.4) (for $k \neq 0$) or equation (3.5) (for $k = 0, \kappa \neq 0$) into the four-component potential space. Then, on each respective gauge sector, the induced field-strength covariance coincides with the covariant Feynman expression:*

$$\mathbb{C}(p) D_{\text{red}} \mathbb{C}(p)^t = \mathbb{C}(p) \left(-\frac{g_\eta}{p^2} \right) \mathbb{C}(p)^t. \quad (3.7)$$

Proof. Let $p = (-\omega, -k, \kappa, 0)^t$ denote the covariant four-momentum and set $D_{\text{cov}} = -g_\eta/p^2$. For any linear gauge condition $n^t A = 0$ with $n^t p \neq 0$, define the projection matrix

$$R_n = I - \frac{p n^t}{n^t p}.$$

In the sector $k \neq 0$ we choose $n = (0, 1, 0, 0)^t$, whereas for $k = 0, \kappa \neq 0$ we choose $n = (0, 0, 1, 0)^t$; in either case $n^t p \neq 0$. Because the unconstrained Maxwell kinetic operator is $p^\sharp (p^\sharp)^t - p^2 g_\eta^{-1}$, its restriction to the subspace $\ker n^t$ has the embedded inverse

$$D_{\text{red}} = R_n D_{\text{cov}} R_n^t, \quad (3.8)$$

whose nonvanishing 3×3 blocks reproduce the inverses of (3.4) and (3.5). Since the exterior derivative annihilates pure-gauge modes, $\mathbb{C}(p)p = 0$, it follows that $\mathbb{C}(p)R_n = \mathbb{C}(p)$, which immediately yields (3.7) as a rational identity away from the mass-shell poles. Applying the same Feynman $i0^+$ prescription to both sides extends the identity to the corresponding distributional covariances. $\square$

At fixed frequency ω and nonzero transverse momentum $\kappa \neq 0$, the zero-mode propagator (3.6) admits the pointwise limit as $\eta \downarrow 0$:

$$\begin{pmatrix} -\omega^2/\kappa^4 & 1/\kappa^2 & 0 \\ 1/\kappa^2 & 0 & 0 \\ 0 & 0 & -1/\kappa^2 \end{pmatrix}. \quad (3.9)$$

However, the poles at $\omega = \pm\kappa/\eta$ recede to infinity as $\eta \downarrow 0$, and our Feynman prescription retains their residues when integrating over $\omega \in \mathbb{R}$. Although the explicit gauge choice in (3.5) assumes $\kappa \neq 0$, passing to the gauge-invariant field-strength representation (3.7) and introducing a positive proper time $\ell > 0$ yields a Gaussian integrand that is completely regular at $\kappa = 0$.

3.3 Drift–diffusion kernels and the zero-mode limit

After Wick rotation $t = -i\tau$ and $\omega = i\nu$, we Fourier transform in the Euclidean frequency ν at fixed spatial momenta (k, κ) . For proper time $\ell > 0$ and Euclidean time separation τ , the resulting drift–diffusion heat kernel is

$$K_\eta(\ell, \tau; k, \kappa) = \frac{\exp[-(\tau - 2k\ell)^2/(4\eta^2\ell) - \ell\kappa^2]}{\sqrt{4\pi\eta^2\ell}}. \quad (3.10)$$

By standard Gaussian convolution, these kernels satisfy the semigroup property

$$\int_{\mathbb{R}} K_\eta(\ell_2, \tau_2 - \tau; k, \kappa) K_\eta(\ell_1, \tau - \tau_0; k, \kappa) d\tau = K_\eta(\ell_1 + \ell_2, \tau_2 - \tau_0; k, \kappa). \quad (3.11)$$

For $(k, \kappa) \neq (0, 0)$, integrating over all proper times $\ell \in (0, \infty)$ yields the Euclidean Green’s function

$$G_\eta(\tau; k, \kappa) = \int_0^\infty K_\eta(\ell, \tau; k, \kappa) d\ell = \frac{\exp[(k\tau - \sqrt{k^2 + \eta^2\kappa^2}|\tau|)/\eta^2]}{2\sqrt{k^2 + \eta^2\kappa^2}}, \quad (3.12)$$

which satisfies the differential equation $(-\eta^2 \partial_\tau^2 + 2k \partial_\tau + \kappa^2)G_\eta = \delta_0(\tau)$, with its overall normalization fixed by the derivative jump at $\tau = 0$. In the near-null limit $\eta \downarrow 0$, $G_\eta(\tau; k, \kappa)$ converges in the sense of distributions to the Euclidean light-front kernel

$$\frac{\mathbf{1}_{\tau>0}}{2k} e^{-\tau \kappa^2 / (2k)} \quad (3.13)$$

for fixed $k > 0$, and to the branch supported on $\tau < 0$ for fixed $k < 0$. At $k = 0$ and fixed $\kappa > 0$, the kernel compresses into an instantaneous contact interaction:

$$G_\eta(\tau; 0, \kappa) = \frac{e^{-\kappa|\tau|/\eta}}{2\eta\kappa} \longrightarrow \frac{\delta_0(\tau)}{\kappa^2}. \quad (3.14)$$

More quantitatively, expanding any test function $\psi \in C_b^2(\mathbb{R})$ to second order around $\tau = 0$ against this even exponential density yields the error bound

$$\left| \langle G_\eta(\cdot; 0, \kappa), \psi \rangle - \frac{\psi(0)}{\kappa^2} \right| \leq \frac{\eta^2}{\kappa^4} \|\psi''\|_\infty. \quad (3.15)$$

Thus, at fixed nonzero spatial momentum, both the light-front modes ($k \neq 0$) and the zero-mode contact interaction ($k = 0, \kappa > 0$) arise from the same finite-tilt kernel. The displayed zero-mode error bound is uniform when κ is bounded away from zero.

For a four-cycle loop with edge proper times ℓ_e and longitudinal momenta k_e , integrating over the relative vertex times at fixed overall translation produces the cyclic constraint factor

$$\frac{1}{\sqrt{4\pi\eta^2 T}} \exp \left[-\frac{(\sum_e \ell_e k_e)^2}{\eta^2 T} \right], \quad T = \sum_e \ell_e. \quad (3.16)$$

In particular, for the momentum routing $k_e = (k, k - P, k, k - P)$, the exponent simplifies to $-T(k - \nu P)^2/\eta^2$, where $\nu = (\ell_2 + \ell_4)/T$. Equivalently, each proper-time step of length h admits the first-order phase-space representation

$$\int \frac{d\nu}{2\pi} \exp[-h\eta^2 \nu^2 + i\nu(\Delta\tau - 2hk)], \quad (3.17)$$

which reproduces $K_\eta(h, \Delta\tau; k, \kappa)$ upon multiplication by $e^{-h\kappa^2}$. In correlation functions of composite operators, vertex energy factors act as τ -derivatives prior to setting the intermediate vertex times equal; maintaining this order of operations automatically preserves all inverse-propagator contact terms and provides the continuum drift-diffusion counterpart of discretized light-front worldsheet graphs [8].

4 Longitudinally weighted four-point integrals

4.1 External momenta and loop routing

We now specialize to collinear spacelike external momenta with covariant components in the coordinate system of equation (3.1) given by

$$p_\mu = (0, -P, Q, 0), \quad P > 0, \quad Q > 0, \quad p^2 = Q^2. \quad (4.1)$$

We consider the four-point correlator with alternating external insertions $O_+(p), O_-(-p), O_+(p), O_-(-p)$, stripping off the overall momentum-conserving delta function.

In the alternating chiral cycle, the four internal edge momenta are routed as $(\ell, \ell - p, \ell, \ell - p)$. In each of the two contact cycles, the internal momenta are routed as $(\ell, \ell - p, \ell - 2p, \ell - p)$, with the reference cut placed on the first edge ℓ immediately following the initial self-dual insertion. We weight the loop integrand by $\phi(\xi)$, where $\xi = k/P$ is the dimensionless longitudinal momentum carried by the

cut edge ℓ . Shifting the cut to another edge simply acts on the resulting longitudinal distribution by a coordinate translation or reflection.

To cast the loop integrals into dimensionless form, we introduce the parameters

$$\delta = \frac{\eta^2 Q^2}{P^2}, \quad M = \frac{PL_-}{2\pi} \in \mathbb{N}, \quad T = Q^2 \sum_e \ell_e, \quad \epsilon = Q^2 \ell_{UV}, \quad (4.2)$$

where ℓ_{UV} is the lower cutoff on the sum of the four edge proper times. On the compactified longitudinal lattice, the quantized loop momentum takes the values $\xi = j/M$ with $j \in \mathbb{Z}$. Rescaling transverse and longitudinal momenta by Q and P , respectively, sets $p^2 = P = 1$ in the dimensionless loop formulas below. In these units, the ultraviolet proper-time window is $\epsilon \leq T \leq 1$, and the physical spacelike circumference reads

$$\eta L_- = \frac{2\pi M \sqrt{\delta}}{Q}. \quad (4.3)$$

Both the source action and the smooth test function ϕ are held fixed while the dimensionless parameters (δ, M, ϵ) in equation (4.2) are varied.

4.2 Chiral numerators and Feynman parameters

For any four-vector $r = (r_0, \mathbf{r})$ in Euclidean signature, we define the 3×3 matrix

$$\mathbb{N}(r) = (r_0^2 - |\mathbf{r}|^2)I_3 + 2\mathbf{r}\mathbf{r}^t + 2r_0[\mathbf{r}]_\times, \quad [\mathbf{r}]_\times u = \mathbf{r} \times u. \quad (4.4)$$

With respect to compatible orthonormal bases of the self-dual and anti-self-dual two-form spaces, the opposite-chirality block of the field-strength propagator is proportional to $\mathbb{N}(r)/(2r^2)$ (up to a fixed orthogonal basis transformation), whereas the same-chirality block reduces to the contact identity $I_3/2$. The matrix $\mathbb{N}(r)$ satisfies the trace identities

$$\text{tr}[\mathbb{N}(r)\mathbb{N}(s)^t] = 4(r \cdot s)^2 - r^2 s^2, \quad (4.5)$$

$$\text{tr}[(\mathbb{N}(r)\mathbb{N}(s)^t)^2] = 16(r \cdot s)^4 - 16r^2 s^2 (r \cdot s)^2 + 3(r^2 s^2)^2. \quad (4.6)$$

The first relation follows by direct multiplication, while the second follows from the property $\mathbb{N}(r)\mathbb{N}(r)^t = (r^2)^2 I_3$: for any nonzero real vectors r, s , the normalized product $R = \mathbb{N}(r)\mathbb{N}(s)^t/(r^2 s^2)$ belongs to $SO(3)$ and therefore has eigenvalues $\{1, e^{i\theta}, e^{-i\theta}\}$, implying the Cayley–Hamilton trace relation $\text{tr}(R^2) = (\text{tr } R)^2 - 2 \text{tr } R$. Because both sides of (4.6) are polynomial in the components of r and s , the identities extend by analyticity to the complexified shifted momenta encountered below.

Writing the edge proper times as $\ell_e = T\alpha_e$ on the standard simplex $\sum_{e=1}^4 \alpha_e = 1$, we parameterize the alternating cycle by setting

$$\begin{aligned} v &= \alpha_2 + \alpha_4, & w &= v(1 - v), & Z &= x^2, & Y &= x \cdot p, & x &= \ell - vp, \\ A &= Z + 2vY + v^2, & B &= Z + 2(v - 1)Y + (v - 1)^2, \\ U &= Z + (2v - 1)Y - w, \\ \mathfrak{n}_a(Z, Y; v) &= 16U^4 - 16ABU^2 + 3A^2B^2. \end{aligned} \quad (4.7)$$

Integrating out the remaining two simplex coordinates at fixed $v \in [0, 1]$ produces the Jacobian weight $v(1 - v) = w$, and completing the square in the Gaussian exponent yields the Feynman-parameter variance $\mathcal{F} = w$.

Similarly, for each contact cycle we introduce the linear combinations

$$v = \alpha_2 + 2\alpha_3 + \alpha_4, \quad g = \alpha_3, \quad 0 \leq v \leq 2, \quad \max(0, v - 1) \leq g \leq v/2. \quad (4.8)$$

Integrating over the remaining simplex coordinate at fixed (v, g) yields the measure weight $\rho(v, g)$ and completed-square variance $\mathcal{F}(v, g)$:

$$\rho(v, g) = v - 2g, \quad \mathcal{F}(v, g) = v + 2g - v^2 \geq 0. \quad (4.9)$$

Along with A and B from equation (4.7), we define the shifted quadratic invariants

$$C = Z + 2(v - 2)Y + (v - 2)^2, \quad U_{02} = Z + 2(v - 1)Y + v(v - 2), \quad (4.10)$$

in terms of which the contact-cycle numerator polynomial reads

$$\mathfrak{n}_c(Z, Y; v) = B^2(4U_{02}^2 - AC). \quad (4.11)$$

Here the factor B^2 originates from the two inverse-propagator factors $(\ell - p)^2$ carried by the same-chirality contact edges, whose proper-time parameters α_2, α_4 remain active in the simplex domain equation (4.8).

Crucially, both $\mathfrak{n}_a$ in equation (4.7) and $\mathfrak{n}_c$ in equation (4.11) are polynomials of total degree at most four in the pair of invariants (Z, Y) . Because each application of the four-dimensional momentum-space Laplacian reduces the total degree in (Z, Y) by at least one, this degree bound controls all inverse powers of T and δ in our subsequent error estimates.

4.3 The Gaussian differential operator

Acting on polynomials in the invariants $Z = x^2$ and $Y = x \cdot p$ (with $p^2 = 1$), the four-dimensional Laplacian $\square_x$ takes the reduced form

$$\mathcal{L} = 4Z\partial_Z^2 + 8\partial_Z + 4Y\partial_Z\partial_Y + \partial_Y^2. \quad (4.12)$$

For each cycle $\Gamma \in \{a, c\}$, we define the associated Gaussian differential symbol

$$\mathcal{P}_{\delta, \Gamma}(T, h; v) = \sum_{j=0}^4 \frac{1}{4^j j! T^j} (\mathcal{L}^j \mathfrak{n}_\Gamma) \left(\frac{\delta h^2}{4T^2}, \frac{h}{2T}; v \right), \quad (4.13)$$

where the formal variable h acts as a derivative operator with respect to the longitudinal test-function argument. Because $\deg_{(Z, Y)} \mathfrak{n}_\Gamma \leq 4$, the polynomial $\mathcal{P}_{\delta, \Gamma}(T, h; v)$ has degree at most eight in h .

To incorporate the longitudinal momentum sum or integral, we introduce the normalized one-dimensional Gaussian kernel and its lattice sampling operator:

$$g_s(x) = \frac{e^{-x^2/s^2}}{\sqrt{\pi}s}, \quad s^2 = \frac{\delta}{T}, \quad \mathcal{B}_{\delta, M}(T, \zeta; \phi) = \frac{1}{M} \sum_{j \in \mathbb{Z}} g_s(j/M - \zeta) \phi(j/M). \quad (4.14)$$

In the longitudinal continuum limit $M = \infty$, the Riemann sum is replaced by the whole-line convolution integral

$$\mathcal{B}_{\delta, \infty}(T, \zeta; \phi) = \int_{\mathbb{R}} g_s(\xi - \zeta) \phi(\xi) d\xi = e^{\delta \partial_\zeta^2 / (4T)} \phi(\zeta). \quad (4.15)$$

For any $\delta > 0$ and $T > 0$, the series in (4.14) and all of its ζ -derivatives converge absolutely and locally uniformly in ζ for bounded ϕ , permitting termwise differentiation to arbitrary order.

Definition 4.1 (Weighted ultraviolet integrals). For $\phi \in C_b^8(\mathbb{R})$, the normalized ultraviolet amplitude for the alternating cycle is defined by

$$\mathcal{I}_{\delta, M, a}^\epsilon[\phi] = \int_\epsilon^1 T dT \int_0^1 w e^{-Tw} [\mathcal{P}_{\delta, a}(T, \partial_\zeta; v) \mathcal{B}_{\delta, M}(T, \zeta; \phi)]_{\zeta=v} dv, \quad (4.16)$$

and for each of the two contact cycles by

$$\mathcal{I}_{\delta,M,c}^\epsilon[\phi] = \int_\epsilon^1 T dT \int_0^2 dv \int_{\max(0,v-1)}^{v/2} dg (v-2g) e^{-T\mathcal{F}(v,g)} [\mathcal{P}_{\delta,c}(T, \partial_\zeta; v) \mathcal{B}_{\delta,M}(T, \zeta; \phi)]_{\zeta=v}. \quad (4.17)$$

Here the auxiliary variable ζ is treated as independent of the Feynman parameters (v, g) ; all ∂_ζ -derivatives are evaluated prior to setting $\zeta = v$. The corresponding null-continuum integrals $\mathcal{I}_{0,\infty,\Gamma}^\epsilon[\phi]$ are defined by replacing $\mathcal{P}_{\delta,\Gamma}$ with $\mathcal{P}_{0,\Gamma}$ and $\mathcal{B}_{\delta,M}(T, \zeta; \phi)$ with $\phi(\zeta)$.

Restoring physical dimensions, the ultraviolet contribution of an individual cycle Γ to the connected four-point function is related to the normalized integral $\mathcal{I}_{\delta,M,\Gamma}^\epsilon$ by

$$\tilde{G}_{\Gamma,\text{UV}} = \frac{Q^4}{576\pi^2 d_G} \mathcal{I}_{\delta,M,\Gamma}^\epsilon. \quad (4.18)$$

To verify the numerical prefactor in equation (4.18), note that the sixteen Wick contractions per cycle combined with the four normalization factors $(6d_G)^{-1/2}$ from equation (2.2) yield $16d_G/(6d_G)^2 = 4/(9d_G)$. The four chiral propagator projections contribute 2^{-4} , the four-dimensional Gaussian loop integral contributes $(16\pi^2 T^2)^{-1}$, and the simplex change of variables $\ell_e = T\alpha_e$ supplies the Jacobian $T^3 dT$, leaving the net measure $T dT$ and the overall factor $(576\pi^2 d_G)^{-1}$.

4.4 Derivation for real test functions

We now derive (4.13) directly as an exact identity for real Gaussian densities, thereby justifying its action on general non-analytic test functions $\phi \in C_b^8(\mathbb{R})$.

Let $u = \xi - v$ denote the centered longitudinal loop variable. After shifting the finite-tilt frequency integration contour to its Gaussian saddle point, the shifted quadratic invariants take the form

$$Z = x_1^2 + x_2^2 + \delta v^2 + u^2/\delta, \quad Y = x_1 + iv + u/\delta, \quad (4.19)$$

where (x_1, x_2, v) are independent centered real Gaussian variables with variances $(2T)^{-1}$, $(2T)^{-1}$, and $(2\delta T)^{-1}$, respectively, while u is distributed with the one-dimensional Gaussian density $g_{\sqrt{\delta/T}}(u)$ and weights the test function $\phi(v + u)$ along the real axis.

To compute the conditional expectation over (x_1, x_2, v) at fixed u , we assemble all four variables into the isotropic four-dimensional Gaussian vector $X = (x_1, x_2, \sqrt{\delta}v, u/\sqrt{\delta}) \in \mathbb{R}^4$ with expectation

$$\mathbb{E}_T[F] = \left(\frac{T}{\pi}\right)^2 \int_{\mathbb{R}^4} e^{-T|X|^2} F(X) d^4 X,$$

and introduce the constant complex four-vectors

$$p_E = (1, 0, i/\sqrt{\delta}, 1/\sqrt{\delta}), \quad q_E = (0, 0, 0, \sqrt{\delta}). \quad (4.20)$$

These vectors satisfy the Euclidean dot-product relations $p_E^2 = 1$, $q_E^2 = \delta$, and $p_E \cdot q_E = 1$, so that $Z = X^2$, $Y = X \cdot p_E$, and $u = X \cdot q_E$. Introducing a dual generating parameter $h \in \mathbb{C}$ and completing the square in the Gaussian integral over X yields the exact moment-generating identity

$$\mathbb{E}_T[\mathbf{n}(X^2, X \cdot p_E; v) e^{hu}] = e^{\delta h^2/(4T)} \mathcal{P}_\delta(T, h; v). \quad (4.21)$$

Because $\mathbf{n}$ is polynomial, applying the heat operator $e^{\square_X/(4T)}$ terminates at fourth order and produces (4.13) via the chain rule on (Z, Y) .

Expanding the polynomial symbol as $\mathcal{P}_\delta(T, h; v) = \sum_{r=0}^8 c_r(\delta, T, v) h^r$ and inverting the Laplace transform in h via integration by parts shows that the marginal density of $u = \xi - v$ satisfies the pointwise identity

$$g_s(\xi - v) \mathbb{E}_T[\mathbf{n} \mid \xi - v] = \sum_{r=0}^8 c_r(\delta, T, v) \partial_\zeta^r g_s(\xi - \zeta) \Big|_{\zeta=v}, \quad (4.22)$$

where the coefficients $c_r(\delta, T, \nu)$ are independent of ξ . Multiplying equation (4.22) by any smooth test function $\phi(\xi) \in C_b^8(\mathbb{R})$ and either integrating over $\xi \in \mathbb{R}$ or summing over the lattice points $\xi = j/M$ immediately yields the representations equations (4.16) and (4.17). Notably, while the analytic continuation acts on the polynomial numerator and the frequency contour, the longitudinal test function ϕ is evaluated exclusively along the real axis.

5 Uniform convergence and the ultraviolet finite part

5.1 Polynomial numerators and the uniform estimate

Both the alternating cycle and the two contact cycles can be treated within a single unified framework. Let Ω be a compact parameter domain of finite Lebesgue measure, let $\rho \in L^1(\Omega)$ be a real weight, and let $\nu : \Omega \rightarrow \mathbb{R}$ be a bounded measurable function. Suppose further that the measurable variance function satisfies $0 \leq \mathcal{F}(\omega) \leq F_*$ on Ω , and consider a family of numerator polynomials of total degree at most four,

$$\mathbf{n}_\omega(Z, Y) = \sum_{m+\ell \leq 4} c_{m\ell}(\omega) Z^m Y^\ell, \quad c_{m\ell} \in L^\infty(\Omega). \quad (5.1)$$

For each integer $k \geq 0$, we equip the space of bounded C^k test functions with the norm

$$\|\phi\|_{C_b^k} = \max_{0 \leq r \leq k} \|\phi^{(r)}\|_\infty, \quad C_b^k(\mathbb{R}) = \{\phi \in C^k(\mathbb{R}) : \|\phi\|_{C_b^k} < \infty\}.$$

Associating the differential symbol $\mathcal{P}_{\delta, \omega}$ to $\mathbf{n}_\omega$ via (4.13), we consider the general weighted ultraviolet integral

$$\mathcal{I}_{\delta, M}^\epsilon[\phi] = \int_\epsilon^1 T \, dT \int_\Omega \rho(\omega) e^{-T\mathcal{F}(\omega)} [\mathcal{P}_{\delta, \omega}(T, \partial_\zeta) \mathcal{B}_{\delta, M}(T, \zeta; \phi)]_{\zeta=\nu(\omega)} d\omega, \quad (5.2)$$

along with its null-continuum counterpart $\mathcal{I}_{0, \infty}^\epsilon[\phi]$, obtained by replacing $\mathcal{P}_{\delta, \omega}$ with $\mathcal{P}_{0, \omega}$ and $\mathcal{B}_{\delta, M}(T, \zeta; \phi)$ with $\phi(\zeta)$. The alternating cycle corresponds to $\Omega = [0, 1]$ with $\rho(\nu) = \mathcal{F}(\nu) = \nu(1 - \nu)$, while the contact cycles correspond to the two-dimensional domain and weights specified in (4.8)–(4.9).

Because every term in the reduced Laplacian equation (4.12) lowers the total polynomial degree in (Z, Y) by at least one, each monomial generated in the expansion of $\mathcal{L}^j \mathbf{n}_\omega(Z, Y)$ has the form $c_{j m \ell}(\omega) Z^m Y^\ell$ with exponents obeying the fundamental degree constraint

$$j + m + \ell \leq 4. \quad (5.3)$$

Substituting $Z = \delta h^2 / (4T^2)$ and $Y = h / (2T)$ into (4.13), we see that each such monomial contributes to $\mathcal{P}_{\delta, \omega}(T, \partial_\nu)$ a constant numerical multiple $4^{-j-m-2\ell} / j!$ of the differential monomial

$$c_{j m \ell}(\omega) \delta^m T^{-j-2m-\ell} \partial_\nu^{2m+\ell}. \quad (5.4)$$

Collecting the $L^1(\Omega)$ norms of all resulting coefficients into the finite norm

$$C_{\mathbf{n}, \rho} = \sum_{j, m, \ell} \frac{1}{4^{j+m+2\ell} j!} \int_\Omega |\rho(\omega) c_{j m \ell}(\omega)| d\omega, \quad (5.5)$$

the constants in the two error estimates proving theorem 5.1 are bounded by universal numerical multiples of $C_{\mathbf{n}, \rho}$.

Theorem 5.1 (Joint near-null, lattice, and ultraviolet estimate). *For any integral of the form (5.2), there exists a universal numerical constant $C_0 > 0$ such that, with $C = C_0 C_{\mathbf{n}, \rho}$, one has for all $\phi \in C_b^8(\mathbb{R})$, $M \in \mathbb{N}$, and $0 < \delta \leq \epsilon < 1$ the uniform bound*

$$|\mathcal{I}_{\delta, M}^\epsilon[\phi] - \mathcal{I}_{0, \infty}^\epsilon[\phi]| \leq C \|\phi\|_{C_b^8} \left(\frac{\delta}{\epsilon^3} + \frac{1}{M^2 \delta^3 \epsilon} \right). \quad (5.6)$$

Here C is completely independent of δ, ϵ , and M . In particular, equation (5.6) holds for the alternating amplitude equation (4.16), for each contact amplitude equation (4.17), and for their total sum.

5.2 The continuum tilt error

We first bound the continuum tilt discrepancy $\mathcal{I}_{\delta,\infty}^\epsilon[\phi] - \mathcal{I}_{0,\infty}^\epsilon[\phi]$. Let $H_a = e^{a\partial_v^2}$ denote the one-dimensional heat semigroup for $a > 0$, which obeys the standard L^∞ bounds

$$\|H_a\psi\|_\infty \leq \|\psi\|_\infty, \quad \|(H_a - 1)\psi\|_\infty \leq a\|\psi''\|_\infty \quad (5.7)$$

for every $\psi \in C_b^2(\mathbb{R})$, where the second inequality follows by integrating $\partial_s H_s \psi = H_s \psi''$ over $s \in [0, a]$.

Consider a generic monomial term equation (5.4). When $m \geq 1$, the corresponding term is absent in the null operator $\mathcal{P}_{0,\omega}$. Since $\delta \leq \epsilon \leq T \leq 1$ and $j + m + \ell \leq 4$, we can factor out one power of $\delta/T \leq 1$ to obtain

$$\delta^m T^{-j-2m-\ell} \leq \delta T^{-j-m-\ell-1} \leq \delta T^{-5}. \quad (5.8)$$

Because the derivative order satisfies $2m + \ell \leq 8$, the first bound in equation (5.7) controls the action of this term on $H_{\delta/(4T)}\phi$ by $\|\phi\|_{C_b^8}$.

When $m = 0$, the differential prefactor $c_{j0\ell}(\omega)T^{-j-\ell}\partial_v^\ell$ is identical in $\mathcal{P}_{\delta,\omega}$ and $\mathcal{P}_{0,\omega}$, so the discrepancy arises solely from the heat-smoothing difference $(H_{\delta/(4T)} - 1)\phi^{(\ell)}$. Applying the second estimate in equation (5.7) and using $j + \ell \leq 4$, we bound this contribution by

$$T^{-j-\ell} \frac{\delta}{4T} \|\phi^{(\ell+2)}\|_\infty \leq C\delta T^{-5} \|\phi\|_{C_b^8}, \quad (5.9)$$

which in fact requires only six derivatives of ϕ since $\ell \leq 4$. Multiplying by the proper-time measure TdT , using $e^{-T\mathcal{F}} \leq 1$, and integrating over Ω and $T \in [\epsilon, 1]$, we arrive at the continuum tilt estimate

$$|\mathcal{I}_{\delta,\infty}^\epsilon[\phi] - \mathcal{I}_{0,\infty}^\epsilon[\phi]| \leq C\|\phi\|_{C_b^8} \delta \int_\epsilon^1 T^{-4} dT \leq C\|\phi\|_{C_b^8} \frac{\delta}{\epsilon^3}. \quad (5.10)$$

5.3 The longitudinal lattice error

To bound the discretization error $\mathcal{I}_{\delta,M}^\epsilon[\phi] - \mathcal{I}_{\delta,\infty}^\epsilon[\phi]$, we employ a whole-line trapezoidal quadrature bound under explicit $W^{2,1}(\mathbb{R})$ regularity.

Lemma 5.2 (Whole-line sampling). *For every $f \in W^{2,1}(\mathbb{R})$ and lattice spacing $h > 0$, the series $\sum_{j \in \mathbb{Z}} f(jh)$ converges absolutely and satisfies*

$$\left| h \sum_{j \in \mathbb{Z}} f(jh) - \int_{\mathbb{R}} f(x) dx \right| \leq \frac{h^2}{8} \|f''\|_{L^1}. \quad (5.11)$$

Proof. Identify f with its continuously differentiable representative. On each lattice cell $I_j = [jh, (j+1)h]$, the fundamental theorem of calculus yields

$$h|f(jh)| \leq \int_{I_j} |f(x)| dx + h \int_{I_j} |f'(x)| dx,$$

and summing over $j \in \mathbb{Z}$ establishes the absolute convergence of the series. Let $L_j f$ denote the affine interpolant of f matching its values at the endpoints of I_j . Integrating by parts twice against the Peano kernel on I_j gives the exact local quadrature identity

$$\int_{I_j} (L_j f - f) dx = \frac{1}{2} \int_{I_j} (x - jh)((j+1)h - x) f''(x) dx.$$

Bounding the quadratic kernel by $h^2/8$, summing over $j \in \mathbb{Z}$, and noting that $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$ proves (5.11). By absolute continuity of f and f' , the identity holds across $W^{2,1}(\mathbb{R})$. $\square$

Fix a monomial term equation (5.4) of derivative order $r = 2m + \ell$, and consider the sampled integrand as a function of $x \in \mathbb{R}$:

$$f_{r,s,v}(x) = \partial_v^r g_s(x - v)\phi(x). \quad (5.12)$$

By scale invariance of the Gaussian kernel, the L^1 norms of its derivatives scale as $\|g_s^{(a)}\|_1 = C_a s^{-a}$. Because $s^2 = \delta/T \leq 1$, expanding $f_{r,s,v}''(x)$ via the Leibniz rule yields the uniform L^1 bound

$$\|f_{r,s,v}''\|_1 \leq C_r \|\phi\|_{C_b^2} s^{-r-2}, \quad (5.13)$$

independently of the center point v . Applying lemma 5.2 with step size $h = M^{-1}$ and multiplying by the prefactor $\delta^m T^{-j-2m-\ell}$ from equation (5.4), we find that each monomial contributes to the lattice discrepancy at most

$$\begin{aligned} \frac{C}{M^2} \delta^m T^{-j-2m-\ell} s^{-(2m+\ell+2)} &= \frac{C}{M^2} \delta^{-1-\ell/2} T^{1-j-m-\ell/2} \\ &\leq \frac{C}{M^2} \delta^{-3} T^{-3}, \end{aligned} \quad (5.14)$$

where the second line uses $\ell \leq 4$, $\delta \leq 1$, and the inequality $4 - j - m - \ell/2 \geq 0$ implied by equation (5.3). Multiplying by the proper-time measure TdT and integrating over $T \in [\epsilon, 1]$ yields the discretization bound

$$|\mathcal{I}_{\delta,M}^\epsilon[\phi] - \mathcal{I}_{\delta,\infty}^\epsilon[\phi]| \leq \frac{C\|\phi\|_{C_b^2}}{M^2\delta^3} \int_\epsilon^1 T^{-2} dT \leq \frac{C\|\phi\|_{C_b^2}}{M^2\delta^3\epsilon}. \quad (5.15)$$

Combining equation (5.10) and equation (5.15) via the triangle inequality completes the proof of theorem 5.1.

5.4 Joint scaling and decompactification

Corollary 5.3. *If the parameters (δ, M, ϵ) approach the joint limit along any trajectory satisfying*

$$\frac{\delta}{\epsilon^3} \longrightarrow 0, \quad M^2\delta^3\epsilon \longrightarrow \infty, \quad (5.16)$$

then the total discrepancy in equation (5.6) vanishes for every $\phi \in C_b^8(\mathbb{R})$. In particular, along power-law trajectories $\delta = \epsilon^a$ and $M \geq \epsilon^{-b}$ with $a > 3$ and $2b > 3a + 1$, the error decays as $O(\epsilon^{a-3} + \epsilon^{2b-3a-1})$.

For instance, the benchmark choice $a = 4$ and $b = 7$ guarantees an $O(\epsilon)$ convergence rate while sending the physical circumference $\eta L_- \rightarrow \infty$. More generally, the joint scaling conditions equation (5.16) automatically enforce $M^2\delta \rightarrow \infty$ via the identity $M^2\delta = (M^2\delta^3\epsilon)/(\delta^2\epsilon)$. In view of equation (4.3), any trajectory satisfying (5.16) necessarily decompactifies the spacelike longitudinal circle in physical units.

To illustrate the role of lattice effects, note that for constant $\phi = 1$ the sampling kernel admits the exact Poisson summation representation

$$\frac{1}{M} \sum_{j \in \mathbb{Z}} g_{\sqrt{\delta/T}}(j/M - v) = \sum_{r \in \mathbb{Z}} e^{-\pi^2 M^2 \delta r^2 / T} e^{2\pi i r M v}. \quad (5.17)$$

If M were held fixed while taking $\delta \downarrow 0$, the non-zero winding modes ($r \neq 0$) in (5.17) would fail to decay. The second error term in (5.6) bounds these oscillatory wrapping contributions even after up to eight longitudinal derivatives have been applied.

5.5 The ultraviolet finite part in the fixed window

To isolate the ultraviolet divergences as $\epsilon \downarrow 0$, we expand the null differential symbol $\mathcal{P}_{0,\Gamma}(T, h; \nu)$ in inverse powers of proper time:

$$\mathcal{P}_{0,\Gamma}(T, h; \nu) = \sum_{k=0}^4 T^{-k} p_{k,\Gamma}(\nu, h), \quad \deg_h p_{k,\Gamma} \leq k, \quad (5.18)$$

where for a general parameter space Ω the polynomials $p_{k,\Gamma}$ may also depend on the remaining Feynman parameters, while the differential argument $h = \partial_\zeta$ acts solely on the test function ϕ . We define the quadratic and linear power-divergence functionals by

$$\begin{aligned} A_\Gamma[\phi] &= \int_\Omega \rho [p_{4,\Gamma}(\partial_\zeta)\phi(\zeta)]_{\zeta=\nu} d\omega, \\ B_\Gamma[\phi] &= \int_\Omega \rho [(p_{3,\Gamma} - \mathcal{F} p_{4,\Gamma})(\partial_\zeta)\phi(\zeta)]_{\zeta=\nu} d\omega, \end{aligned} \quad (5.19)$$

and the logarithmic divergence functional by

$$\mathfrak{R}_\Gamma[\phi] = \int_\Omega \rho [p_{2,\Gamma} - \mathcal{F} p_{3,\Gamma} + \tfrac{1}{2} \mathcal{F}^2 p_{4,\Gamma}] (\partial_\zeta)\phi(\zeta)|_{\zeta=\nu} d\omega. \quad (5.20)$$

Because $\mathcal{F}$ and all polynomial coefficients are bounded on Ω , subtracting the three singular Laurent terms in T leaves a proper-time integrand that is uniformly bounded as $T \downarrow 0$. Integrating over $T \in [\epsilon, 1]$ therefore yields the small- ϵ asymptotic expansion

$$\mathcal{I}_{0,\infty,\Gamma}^\epsilon[\phi] = \frac{A_\Gamma[\phi]}{2\epsilon^2} + \frac{B_\Gamma[\phi]}{\epsilon} + \mathfrak{R}_\Gamma[\phi] \log(1/\epsilon) + \mathcal{F}_{\text{UV},\Gamma}[\phi] + O(\epsilon \|\phi\|_{C_b^4}). \quad (5.21)$$

Specifically, letting

$$\mathcal{J}_\Gamma(T; \phi) = T \int_\Omega \rho e^{-T\mathcal{F}} [\mathcal{P}_{0,\Gamma}(T, \partial_\zeta)\phi(\zeta)]_{\zeta=\nu} d\omega,$$

the renormalized ultraviolet finite part on the interval $T \in (0, 1]$ takes the explicit form

$$\begin{aligned} \mathcal{F}_{\text{UV},\Gamma}[\phi] &= -\tfrac{1}{2} A_\Gamma[\phi] - B_\Gamma[\phi] \\ &\quad + \int_0^1 \left[\mathcal{J}_\Gamma(T; \phi) - \frac{A_\Gamma[\phi]}{T^3} - \frac{B_\Gamma[\phi]}{T^2} - \frac{\mathfrak{R}_\Gamma[\phi]}{T} \right] dT. \end{aligned} \quad (5.22)$$

The bracketed integrand in (5.22) is $O(1)$ as $T \downarrow 0$, ensuring both the absolute convergence of $\mathcal{F}_{\text{UV},\Gamma}[\phi]$ and the $O(\epsilon \|\phi\|_{C_b^4})$ remainder bound in (5.21). Evaluating these coefficients for the unweighted alternating cycle ($\phi = 1$) gives

$$\mathcal{I}_{0,\infty,a}^\epsilon[1] = \frac{30}{\epsilon^2} - \frac{48}{\epsilon} + 14 \log(1/\epsilon) + \mathcal{F}_{\text{UV},a}[1] + O(\epsilon). \quad (5.23)$$

Introducing the dimensionless renormalization scale $\widehat{\mu}^2 = \mu^2/Q^2$, we define the universal counterterm functional

$$\mathcal{S}_\Gamma^{\epsilon,\mu}[\phi] = \frac{A_\Gamma[\phi]}{2\epsilon^2} + \frac{B_\Gamma[\phi]}{\epsilon} + \mathfrak{R}_\Gamma[\phi] \log \frac{1}{\epsilon \widehat{\mu}^2}. \quad (5.24)$$

Corollary 5.4 (Agreement after a common subtraction). *Along any scaling trajectory satisfying equation (5.16), subtracting the common counterterm equation (5.24) from the finite-tilt lattice integral yields the exact null-continuum renormalized limit:*

$$\lim_{\epsilon \downarrow 0} \left(\mathcal{I}_{\delta,M,\Gamma}^\epsilon - \mathcal{S}_\Gamma^{\epsilon,\mu} \right) [\phi] = \mathcal{F}_{\text{UV},\Gamma}[\phi] + \mathfrak{R}_\Gamma[\phi] \log \widehat{\mu}^2, \quad (5.25)$$

with uniform convergence on bounded subsets of $C_b^8(\mathbb{R})$.

Proof. Subtract equation (5.24) from both terms on the left-hand side of equation (5.6) and combine theorem 5.1 with the asymptotic expansion equation (5.21). $\square$

Equation (5.24) uniquely specifies the minimal subtraction scheme as a functional of the longitudinal weight ϕ on the fixed proper-time window $T \in (0, 1]$. Any additional finite renormalization applied identically to both representations shifts their limits by the same amount. Whereas finite renormalizations of the unweighted generating functional correspond to local source counterterms [5, 6], modifying the subtraction for weighted integrals requires specifying how the counterterm acts on ϕ .

6 Longitudinal contact distributions

6.1 The three logarithmic coefficients

Although the logarithmic divergence functional (5.20) formally involves test-function derivatives up to fourth order, for the covariant Maxwell numerators we can integrate by parts to transfer all derivatives onto the Feynman-parameter coefficient polynomials. A careful evaluation of the resulting endpoint jets reveals that all higher-derivative boundary terms cancel, leaving pure order-zero measures.

Theorem 6.1 (Longitudinal logarithmic coefficients). *Under the normalization and loop-momentum routings specified in section 4, the logarithmic divergence functionals of the alternating and contact cycles are given by*

$$\mathfrak{R}_a[\phi] = 8 \int_0^1 \phi(v) dv + 3\phi(0) + 3\phi(1), \quad (6.1)$$

$$\mathfrak{R}_{c,1}[\phi] = \mathfrak{R}_{c,2}[\phi] = 8 \int_0^2 \phi(v) dv. \quad (6.2)$$

These identities hold for every test function ϕ that is of class C^4 on an open neighborhood of the respective closed intervals $[0, 1]$ and $[0, 2]$.

6.2 Endpoint terms in the alternating cycle

Evaluating equations (4.12) and (4.13) on the alternating-cycle numerator equation (4.7) and substituting the result into equation (5.20), we obtain the fourth-order representation

$$\mathfrak{R}_a[\phi] = \int_0^1 \sum_{r=0}^4 a_r(v) \phi^{(r)}(v) dv, \quad w = v(1-v), \quad (6.3)$$

where the coefficient polynomials $a_r(v)$ read explicitly

$$\begin{aligned} a_0 &= 6w(1 + 20w + 210w^2), \\ a_1 &= 30w^2(2v - 1)(1 + 21w), \\ a_2 &= w^2(4 + 141w - 378w^2), \\ a_3 &= 7w^3(2v - 1)(2 - 3w), \\ a_4 &= \frac{1}{2}w^3(1 - w)(1 - 3w). \end{aligned} \quad (6.4)$$

We emphasize that all five polynomials in (6.4) arise solely from the smooth bulk numerator equation (4.7) without any a priori endpoint insertions. In the interior of $(0, 1)$, their formal Lagrange adjoint collapses to a constant:

$$\sum_{r=0}^4 (-\partial_v)^r a_r(v) = 8. \quad (6.5)$$

Integrating (6.3) by parts repeatedly over $[0, 1]$ generates the boundary sum

$$\sum_{r=1}^4 \sum_{k=0}^{r-1} (-1)^k \left[a_r^{(k)}(v) \phi^{(r-1-k)}(v) \right]_0^1. \quad (6.6)$$

Because $w = v(1-v)$ vanishes to first order at both endpoints, the polynomials a_1, a_2, a_3 vanish at $v \in \{0, 1\}$ to orders at least 2, 2, 3, respectively, and therefore contribute no boundary terms at all to (6.6). For the fourth-order coefficient a_4 , the endpoint expansions

$$a_4(v) = \frac{1}{2}v^3 + O(v^4), \quad a_4(v) = \frac{1}{2}(1-v)^3 + O((1-v)^4) \quad (6.7)$$

show that a_4, a'_4 , and a''_4 vanish at both $v = 0$ and $v = 1$, eliminating all boundary terms involving ϕ', ϕ'' , and ϕ''' . Only the $k = 3$ term survives, with $a_4'''(0) = 3$ and $a_4'''(1) = -3$. Combining (6.5) with (6.6) establishes equation (6.1). Thus, the logarithmic divergence of the alternating cycle defines a positive Radon measure with endpoint Dirac masses, even though its Gaussian derivation proceeds through fourth-order derivatives of ϕ .

6.3 Boundary cancellation in the contact cycles

For each contact cycle, we first extract the coefficients of h^r in the logarithmic combination prior to integrating over the auxiliary simplex parameter g :

$$R_r(v, g) = [h^r] \left(p_{2,c}(v, h) - \mathcal{F}(v, g) p_{3,c}(v, h) + \frac{1}{2} \mathcal{F}(v, g)^2 p_{4,c}(v, h) \right). \quad (6.8)$$

Because the integration limits for g in equation (4.8) change piecewise at $v = 1$, integrating over g defines two sets of polynomial coefficients on the subintervals $[0, 1]$ and $[1, 2]$:

$$\begin{aligned} b_r^<(v) &= \int_0^{v/2} (v-2g) R_r(v, g) dg, & 0 \leq v \leq 1, \\ b_r^>(v) &= \int_{v-1}^{v/2} (v-2g) R_r(v, g) dg, & 1 \leq v \leq 2. \end{aligned} \quad (6.9)$$

The explicit polynomials $b_r^\lessgtr(v)$ are recorded in section A. On both subintervals, their formal adjoints reduce to the same constant:

$$\sum_{r=0}^4 (-\partial_v)^r b_r^<(v) = \sum_{r=0}^4 (-\partial_v)^r b_r^>(v) = 8. \quad (6.10)$$

At the exterior boundaries $v = 0$ and $v = 2$, all boundary jets in (6.6) vanish identically. At the internal interface $v = 1$, integrating by parts on $[0, 1]$ yields the nonvanishing boundary term $+\frac{1}{4}\phi'(1)$, whereas integrating by parts on $[1, 2]$ yields the opposite boundary term $-\frac{1}{4}\phi'(1)$, with all other derivative orders vanishing at $v = 1$. Adding the two subintervals cancels the $\phi'(1)$ interface contribution exactly and establishes equation (6.2). In particular, the piecewise kink in the simplex domain at $v = 1$ leaves neither a Dirac mass nor a dipole distribution in the final logarithmic coefficient.

6.4 Physical normalization and momentum routing

Multiplying the normalized residue equation (6.1) by the physical prefactor equation (4.18) and by the factor of 2 from $\mu \partial_\mu \log \mu^2$, we find that the renormalization-scale derivative of the alternating cycle under the subtraction equation (5.24) is

$$\frac{Q^4}{288\pi^2 d_G} \left(8 \int_0^1 \phi(v) dv + 3\phi(0) + 3\phi(1) \right). \quad (6.11)$$

Weight	Alternating cycle	Each contact cycle	Sum of three
1	14	16	46
ξ	7	16	39
ξ^2	17/3	64/3	145/3

Table 1: Moments of the normalized longitudinal logarithmic residues. Polynomial weights can be evaluated either directly from the finite Gaussian formulas or by choosing compactly supported smooth test functions that coincide with ξ^m on an open neighborhood of the support of the residue measure.

Summing over all three cycles for the unweighted correlator ($\phi = 1$) yields the total logarithmic scale coefficient $23Q^4/(144\pi^2 d_G)$, in agreement with Table 1.

We note that the support and endpoint locations in equations (6.1) and (6.2) are tied to our choice of momentum-routing cut. If the new longitudinal loop variable is $\xi' = \xi - a$, the pushed-forward residue satisfies $\Re'[\phi] = \Re[\phi(\cdot - a)]$, and its support, including any endpoint Dirac masses, translates by $-a$. In the original routing, choosing a test function ϕ that vanishes near 0 and 1 filters out the alternating-cycle endpoint masses altogether. Thus, both the momentum-routing convention and the behavior of the weight at the endpoints form an integral part of the definition of the weighted observable.

Finally, our analysis highlights a sharp conceptual distinction between two types of contact contributions. First, equation (2.8) represents a *spacetime* contact distribution arising when two composite insertions are joined by a same-chirality propagator edge; this mechanism generates the two contact cycles. Second, the Dirac masses $3\delta_0 + 3\delta_1$ in equation (6.1) are *longitudinal momentum-fraction* endpoint distributions that emerge in the alternating cycle after integrating out the transverse and frequency loop momenta and extracting the ultraviolet logarithm. Distinguishing these two mechanisms avoids any ambiguity or double counting when comparing covariant and light-front loop expansions.

7 Path dependence of the joint limit

7.1 An exact quadratic-moment difference

To test the sharpness of the $\delta\epsilon^{-3}$ continuum tilt bound in theorem 5.1, we first evaluate the alternating cycle exactly on the quadratic weight $\phi(\xi) = \xi^2$, for which the heat convolution reads

$$\mathcal{B}_{\delta,\infty}(T, v; \xi^2) = v^2 + \frac{\delta}{2T}. \quad (7.1)$$

Because $\partial_v^r(v^2 + \delta/(2T))$ vanishes for $r \geq 3$, only the terms of degree $r \in \{0, 1, 2\}$ in equation (4.13) contribute. Substituting these terms gives a continuum tilt discrepancy that is strictly linear in δ :

$$\mathcal{I}_{\delta,\infty,a}^\epsilon[\xi^2] - \mathcal{I}_{0,\infty,a}^\epsilon[\xi^2] = \delta \int_\epsilon^1 dT \int_0^1 dv T w e^{-Tw} \mathcal{D}_2(T, w), \quad (7.2)$$

where the rational kernel $\mathcal{D}_2(T, w)$ takes the explicit form

$$\begin{aligned} \mathcal{D}_2(T, w) = \frac{1}{2T^5} [& 3T^4 w^4 - T^3(72w^3 + 16w^2) \\ & + T^2(540w^2 + 160w + 16) \\ & - T(1440w + 320) + 1080]. \end{aligned} \quad (7.3)$$

Expanding the exponential e^{-Tw} in powers of Tw and integrating termwise over $v \in [0, 1]$ yields the small- T Laurent expansion of the proper-time integrand:

$$\int_0^1 T w e^{-Tw} \mathcal{D}_2(T, w) dv = \frac{90}{T^4} - \frac{206}{3T^3} + \frac{55}{3T^2} - \frac{8}{3T} + O(1).$$

Integrating over $T \in [\epsilon, 1]$ then produces the exact small- ϵ asymptotics

$$\begin{aligned} \mathcal{I}_{\delta, \infty, a}^\epsilon[\xi^2] - \mathcal{I}_{0, \infty, a}^\epsilon[\xi^2] &= \delta \left[\frac{30}{\epsilon^3} - \frac{103}{3\epsilon^2} + \frac{55}{3\epsilon} \right. \\ &\quad \left. - \frac{8}{3} \log(1/\epsilon) + O(1) \right], \end{aligned} \quad (7.4)$$

where the $O(1)$ remainder is independent of δ .

7.2 Compactly supported test functions

Although the polynomial ξ^2 is unbounded and thus lies outside $C_b^\infty(\mathbb{R})$, the rapid decay of the Gaussian kernel $g_s(\xi - v)$ allows us to localize (7.4) to compactly supported smooth test functions with an exponentially small error.

Lemma 7.1 (Localization of polynomial test functions). *Fix $a > 0$ and let $\phi \in C_c^\infty(\mathbb{R})$ be any smooth compactly supported test function that coincides with ξ^2 on the expanded interval $[-a, 1+a]$. Then there exists a constant $C_{a, \phi} > 0$ such that, for all $0 < \delta \leq \epsilon < 1$,*

$$|\mathcal{I}_{\delta, \infty, a}^\epsilon[\phi - \xi^2]| \leq C_{a, \phi} \delta^{-2} \epsilon^{-2} \exp\left(-\frac{a^2 \epsilon}{4\delta}\right), \quad (7.5)$$

while for the null-continuum integral one has identically $\mathcal{I}_{0, \infty, a}^\epsilon[\phi - \xi^2] = 0$.

Proof. Consider a generic differential monomial equation (5.4) of derivative order $r = 2m + \ell$ and set $s^2 = \delta/T$. For every $v \in [0, 1]$, the difference $\phi(\xi) - \xi^2$ vanishes identically whenever $|\xi - v| < a$ and satisfies the quadratic growth bound $|\phi(\xi) - \xi^2| \leq C(1 + |\xi|^2)$ on the exterior region $|\xi - v| \geq a$. Rescaling the integration variable by s and absorbing the Hermite polynomial and quadratic growth factors into half of the Gaussian exponent yields the tail bound

$$\int_{|\xi - v| \geq a} |\partial_v^r g_s(\xi - v)| |\phi(\xi) - \xi^2| d\xi \leq C_{a, r, \phi} s^{-r} e^{-a^2/(4s^2)}. \quad (7.6)$$

Multiplying by the coefficient $\delta^m T^{-j-2m-\ell}$ from equation (5.4) and using $s^2 = \delta/T$, $\ell \leq 4$, and $j + m + \ell \leq 4$, we bound each monomial contribution at fixed T by

$$C \delta^{-\ell/2} T^{-j-m-\ell/2} e^{-a^2 T/(4\delta)} \leq C \delta^{-2} T^{-4} e^{-a^2 T/(4\delta)}. \quad (7.7)$$

Multiplying by $T dT$, bounding $e^{-a^2 T/(4\delta)} \leq e^{-a^2 \epsilon/(4\delta)}$ for $T \geq \epsilon$, and integrating over $T \in [\epsilon, 1]$ establishes equation (7.5). Finally, because $\mathcal{I}_{0, \infty, a}^\epsilon$ depends only on the fourth-order jet of the test function on $[0, 1]$, where $\phi(\xi) = \xi^2$, the null-continuum difference vanishes identically. $\square$

Proposition 7.2 (A nonzero joint-limit discrepancy). *Fix $c > 0$ and let $\phi \in C_c^\infty(\mathbb{R})$ be chosen as in lemma 7.1. Along any joint scaling trajectory with $\delta = c\epsilon^3$ and $M^2 \delta^3 \epsilon \rightarrow \infty$ (for instance, $M \geq \epsilon^{-6}$), one has*

$$\lim_{\epsilon \downarrow 0} \left(\mathcal{I}_{\delta, M, a}^\epsilon[\phi] - \mathcal{I}_{0, \infty, a}^\epsilon[\phi] \right) = 30c. \quad (7.8)$$

Proof. Because $\phi \in C_c^\infty(\mathbb{R}) \subset C_b^\infty(\mathbb{R})$, the lattice discretization error $\mathcal{I}_{\delta, M, a}^\epsilon[\phi] - \mathcal{I}_{\delta, \infty, a}^\epsilon[\phi]$ vanishes as $\epsilon \downarrow 0$ by equation (5.15). Furthermore, along $\delta = c\epsilon^3$ the ratio $\epsilon/\delta = (c\epsilon^2)^{-1}$ diverges, so the localization error in lemma 7.1 decays faster than any power of ϵ . The remaining continuum difference is governed by equation (7.4), whose leading term $30\delta\epsilon^{-3}$ converges to $30c$ while all subleading terms vanish as $\epsilon \downarrow 0$. $\square$

Proposition 7.2 demonstrates that a single fixed test function in $C_c^\infty(\mathbb{R})$ saturates the continuum error scale $\delta\epsilon^{-3}$ in theorem 5.1. In particular, although the trajectory $\delta = c\epsilon^3$ satisfies $\delta/\epsilon \rightarrow 0$, subtracting the common counterterm equation (5.24) still leaves a finite, path-dependent shift of $30c$ in the renormalized alternating-cycle amplitude under the lattice scaling hypothesis of the proposition.

ϵ	$(\mathcal{I}_{\epsilon^4, \infty, a}^\epsilon[\xi^2] - \mathcal{I}_{0, \infty, a}^\epsilon[\xi^2])/\epsilon$
0.1	26.7300853675
0.05	28.3264479092
0.025	29.1527563758
0.0125	29.5736482349
0.00625	29.7861261514

Table 2: Numerical quadrature of the exact difference (7.2) along the trajectory $\delta = \epsilon^4$. The rescaled discrepancy converges linearly to the analytical coefficient 30.

8 Conclusions

By regulating all polarization channels with a common lower cutoff on the total proper time, we have established a rigorous quantitative comparison between near-null spacelike Maxwell loop integrals and their null light-front counterparts as functionals of a longitudinal momentum weight. Retaining external sources prior to eliminating the constrained longitudinal gauge field uniquely fixes both the induced nonlocal Schur interactions and the accompanying functional determinant. The full covariant chiral numerators then reduce each connected four-cycle to an explicit eighth-order Gaussian differential operator acting on the longitudinal test function.

Our main convergence estimate, theorem 5.1, simultaneously controls the continuum near-null tilt error and the longitudinal lattice discretization error uniformly on bounded subsets of $C_b^8(\mathbb{R})$. In doing so, it delineates the joint scaling regime equation (5.16) within which a single universal counterterm subtraction yields identical renormalized finite parts. At the same time, our explicit integration-by-parts analysis (theorem 6.1) reveals that the longitudinal logarithmic divergence of the alternating cycle acquires two endpoint Dirac masses $3\delta_0 + 3\delta_1$ from third-order boundary jets, whereas the internal interface terms in the contact cycles cancel completely. Finally, the localized counterexample in proposition 7.2 proves that the continuum tilt scaling $\delta\epsilon^{-3}$ is sharp, producing an explicit finite discrepancy of $30c$ in the alternating cycle along $\delta = c\epsilon^3$ with $M^2\delta^3\epsilon \rightarrow \infty$.

Taken together, these results specify the precise distributional counterterms and endpoint matching conditions that any light-front or worldsheet discretization must satisfy to reproduce the covariant renormalized four-point correlator of Maxwell composite operators. While our weighted analysis was carried out for collinear spacelike momenta on a fixed ultraviolet proper-time window, the unweighted parameter representation in section B provides a complementary formula for the full proper-time integral at general nonexceptional Euclidean momenta. Extending these distributional matching techniques to non-Abelian gauge theories and interacting light-front Hamiltonians remains an important direction for future investigation.

A Explicit contact-cycle coefficients

Evaluating the parameter integrals in equation (6.9) over the left subinterval $0 \leq v \leq 1$ yields the five polynomial coefficients

$$\begin{aligned} b_0^< &= \frac{3}{2}v^2(210v^4 - 720v^3 + 775v^2 - 264v + 19), \\ b_1^< &= \frac{1}{2}v^2(v-1)(630v^4 - 2040v^3 + 1980v^2 - 508v + 21), \\ b_2^< &= \frac{1}{12}v^2(1134v^6 - 5724v^5 + 11115v^4 - 10176v^3 \\ &\quad + 4230v^2 - 586v + 12), \\ b_3^< &= \frac{1}{24}v^3(v-1)(252v^5 - 1224v^4 + 2292v^3 - 2004v^2 + 753v - 64), \\ b_4^< &= \frac{1}{48}v^4(v-1)^2(3v^2 - 6v + 4)(6v^2 - 16v + 11). \end{aligned} \quad (\text{A.1})$$

On the right subinterval $1 \leq v \leq 2$, the corresponding polynomials are related to $b_r^<$ by reflection symmetry across $v = 1$:

$$b_r^>(v) = (-1)^r b_r^<(2-v). \quad (\text{A.2})$$

Applying the integration-by-parts boundary formula equation (6.6) to (A.1) and (A.2) on their respective subintervals gives

$$\int_0^1 \sum_{r=0}^4 b_r^<(v) \phi^{(r)}(v) dv = 8 \int_0^1 \phi(v) dv + \frac{1}{4} \phi'(1), \quad (\text{A.3})$$

$$\int_1^2 \sum_{r=0}^4 b_r^>(v) \phi^{(r)}(v) dv = 8 \int_1^2 \phi(v) dv - \frac{1}{4} \phi'(1). \quad (\text{A.4})$$

Adding the two expressions cancels the interface derivative $\pm \frac{1}{4} \phi'(1)$ and implies that for every integer moment $m \geq 0$,

$$\Re_a[\xi^m] = \frac{8}{m+1} + 3 + 3\mathbf{1}_{m=0}, \quad \Re_c[\xi^m] = \frac{8 \cdot 2^{m+1}}{m+1}. \quad (\text{A.5})$$

B Finite parts at general Euclidean momenta

At general nonexceptional Euclidean external momenta, the unweighted correlator regulated by a common cutoff on the total proper time admits a closed Feynman-parameter representation over the full interval $T \in [\varepsilon, \infty)$. Writing the edge momenta as $r_e = \ell + q_e$ and edge proper times as $\ell_e = T\alpha_e$ on the three-simplex Δ_3 , we define

$$\bar{q} = \sum_e \alpha_e q_e, \quad v_e = q_e - \bar{q}, \quad \mathcal{F} = \sum_e \alpha_e q_e^2 - \bar{q}^2. \quad (\text{B.1})$$

In terms of the shifted loop momentum $x = \ell + \bar{q}$, the alternating-cycle numerator reads

$$\mathbf{n}(x) = \text{tr} \left[\mathbb{N}(x + v_1) \mathbb{N}(x + v_2)^t \mathbb{N}(x + v_3) \mathbb{N}(x + v_4)^t \right], \quad (\text{B.2})$$

while for a contact cycle it takes the form

$$\mathbf{n}(x) = |x + v_2|^2 |x + v_4|^2 \text{tr} \left[\mathbb{N}(x + v_1) \mathbb{N}(x + v_3)^t \right]. \quad (\text{B.3})$$

Setting

$$M_j = \frac{\square_x^j \mathbf{n}(0)}{4^j j!}, \quad 0 \leq j \leq 4, \quad R = M_2 - \mathcal{F} M_3 + \frac{1}{2} \mathcal{F}^2 M_4, \quad (\text{B.4})$$

the unweighted amplitude integrated over the semi-infinite proper-time interval $T \in [\varepsilon, \infty)$ evaluates to

$$\frac{1}{576\pi^2 d_G} \int_{\Delta_3} d^3\alpha \sum_{j=0}^4 M_j \mathcal{F}^{j-2} \Gamma(2-j, \varepsilon \mathcal{F}), \quad (\text{B.5})$$

where $\varepsilon = \ell_{UV}$ denotes the dimensionful proper-time cutoff (related to the dimensionless cutoff in the main text by $\epsilon = Q^2 \varepsilon$), and the overall normalization matches equation (4.18).

Subtracting the divergent counterterm integrand

$$\frac{M_4}{2\varepsilon^2} + \frac{M_3 - \mathcal{F}M_4}{\varepsilon} - R [\log(\varepsilon\mu^2) + \gamma_E] \quad (\text{B.6})$$

under the simplex integral and taking $\varepsilon \downarrow 0$ yields the renormalized Euclidean amplitude

$$\frac{1}{576\pi^2 d_G} \int_{\Delta_3} d^3\alpha \left[\frac{M_0}{\mathcal{F}^2} + \frac{M_1}{\mathcal{F}} - \mathcal{F}M_3 + \frac{3}{4}\mathcal{F}^2 M_4 - R \log \frac{\mathcal{F}}{\mu^2} \right]. \quad (\text{B.7})$$

Pointwise for $\mathcal{F} > 0$, equation (B.7) follows immediately from the small-argument expansion of the incomplete gamma function $\Gamma(2-j, \varepsilon\mathcal{F})$. To justify interchanging the limit $\varepsilon \downarrow 0$ with the simplex integration, write

$$\mathcal{F} = \sum_{e < f} \alpha_e \alpha_f |q_e - q_f|^2.$$

When the external shifts q_e vary over a compact Euclidean domain with pairwise separations $|q_e - q_f|$ bounded away from zero, $\mathcal{F}$ is strictly positive away from the four vertices of Δ_3 and vanishes linearly with the distance to each vertex. Because the simplex has dimension three, $\mathcal{F}^{-2}$ and $\log \mathcal{F}$ are integrable over Δ_3 . Furthermore, the Taylor remainder of the subtracted proper-time integrand on $0 < T < 1$ is uniformly bounded across Δ_3 , while the infrared tail $T \geq 1$ is dominated by $C(1 + \mathcal{F}^{-2} + |\log \mathcal{F}|)$. Dominated convergence therefore justifies taking $\varepsilon \downarrow 0$ inside the integral and guarantees uniform convergence on compact nonexceptional momentum sets.

Note that the full-line subtraction in equation (B.7) differs from the fixed-window subtraction equation (5.24) by explicit finite local terms (such as the Euler–Mascheroni constant γ_E) and by the infrared proper-time integral over $T \in (1, \infty)$. While standard local heat-kernel expansions [5] organize such unweighted source-local finite shifts, our fixed-window formulation in the main text retains the longitudinal test function ϕ throughout the joint limiting comparison.

C The fourth-order source determinant

Within any common finite-dimensional Gaussian regularization, we can decompose the source-dependent kinetic Hessian into transverse and longitudinal blocks:

$$\mathbb{H} = \begin{pmatrix} T_0 + L & C^t \\ C & K_0 + V \end{pmatrix}, \quad (\text{C.1})$$

where L, C, V are homogeneous of degree one in the external sources and $G = K_0^{-1}$ denotes the free longitudinal inverse on the nonzero longitudinal subspace. Factoring out the longitudinal block via the Schur complement gives

$$\det \mathbb{H} = \det(K_0 + V) \det(T_0 + L - C^t(K_0 + V)^{-1}C).$$

Expanding $(K_0 + V)^{-1} = G(1 + VG)^{-1}$ in powers of the source insertion V , we define the effective transverse vertices of source orders one through four:

$$U_1 = L, \quad U_2 = -C^t G C, \quad (\text{C.2})$$

$$U_3 = C^t G V G C, \quad U_4 = -C^t G V G V G C, \quad (\text{C.3})$$

and set $A_r = T_0^{-1} U_r$. Extracting the fourth-order homogeneous source term of $\text{Tr} \log(1 + \sum_{r=1}^4 A_r)$ yields

$$\text{Tr} \left(A_4 - A_1 A_3 - \frac{1}{2} A_2^2 + A_1^2 A_2 - \frac{1}{4} A_1^4 \right),$$

where cyclicity of the operator trace has been used to combine distinct orderings of noncommuting source matrices. Combining this Schur-complement expansion with the fourth-order expansion of $\log \det(1 + GV)$, we find that the complete fourth-order coefficient of the effective action $-\frac{1}{2}d_G \log \det \mathbb{H}$ is given by

$$d_G \text{Tr} \left[\frac{1}{8}A_1^4 - \frac{1}{2}A_1^2A_2 + \frac{1}{4}A_2^2 + \frac{1}{2}A_1A_3 - \frac{1}{2}A_4 + \frac{1}{8}(GV)^4 \right]. \quad (\text{C.4})$$

Here the final term $\frac{1}{8}d_G \text{Tr}[(GV)^4]$ represents the purely longitudinal contraction cycle generated by $\det(K_0 + V)$. In the unified three-channel propagator formalism of equations (2.15) and (2.18), this contribution is automatically generated by the product of four third-channel instantaneous entries. Thus, the block-determinant decomposition (C.4) and the three-channel trace provide two algebraically equivalent ways of accounting for the purely longitudinal loop exactly once. Under the total proper-time regularization equation (2.18), every term in (C.4) retains all four edge proper-time parameters prior to integration.

Data and code availability

The supplementary material contains the symbolic polynomial coefficients and high-precision numerical quadrature data reported in this paper. With Python 3.10 or later, SymPy 1.14.0, and mpmath 1.3.0, running `python code/reproduce.py` from the supplementary directory regenerates `data/coefficients.json` and `data/limit_values.csv`. The scripts verify the Gaussian differential reductions and covariance identities, and test the fourth-order block-determinant expansion on exact matrix examples.

Declaration of AI assistance

Large language models were used for proofreading, language editing, and assistance with checking calculations and proofs during preparation of this manuscript.

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