

Laurent and g -vector counterexamples for quivers with homotopies

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29 October 2026

Abstract

We give negative answers to the general Laurent-phenomenon question and the g -vector sign-coherence question for quivers with homotopies posed by Li, Liu, Mou and Pan.

AI use and scholarly credit

The counterexamples and their proofs were obtained in one round of ChatGPT-6-Pro. Codex reviewed the arguments and polished the manuscript. The results passed review by Rethlas-OSV [13, 14].

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1 Introduction

Fomin and Zelevinsky introduced cluster algebras as an algebraic framework for dual canonical bases and total positivity. Their Laurent theorem states that every cluster variable is a Laurent polynomial in the variables of any chosen cluster, with coefficients in the coefficient group ring [1, Theorem 3.1]. Thus repeated division in the exchange relations produces only monomial denominators in the initial variables. Their theory of principal coefficients subsequently supplied a natural grading and the associated g -vectors [2]. The sign-coherence conjecture [2, Definition 6.12 and Conjecture 6.13] says that, within one cluster, each coordinate of these vectors has a common weak sign. Derksen, Weyman and Zelevinsky proved this for skew-symmetric exchange matrices using representations of quivers with potentials [3, Theorem 1.7]; Gross, Hacking, Keel and Kontsevich established it in the skew-symmetrizable setting by scattering-diagram methods [4, Theorem 5.11].

Extensions of cluster mutation have developed along several directions. Huang and Li constructed infinite-rank unfoldings of acyclic sign-skew-symmetric exchange matrices, obtaining positivity and the constant-term property for their F -polynomials [5]. Here an unfolding realizes matrix mutation by simultaneous mutations in suitable vertex orbits of a larger quiver. Li and Pan later used Newton polytopes to recover cluster-variable Laurent expressions from

2020 *Mathematics Subject Classification*. Primary 13F60; Secondary 05C20.

Keywords. Quiver mutation, homotopy, Laurent phenomenon, g -vector.

g -vectors in totally sign-skew-symmetric cluster algebras with principal coefficients [6, Theorem 4.1]. In another direction, Mou constructed scattering diagrams for Chekhov–Shapiro generalized cluster algebras with binomially factorized exchange polynomials, identifying their cluster variables with theta functions and obtaining positivity in the factor coefficients [7]. These results depend on the specified matrix and exchange-polynomial mutation rules.

Surfaces and group actions provide a second source of extensions. Fomin, Shapiro and Thurston described cluster complexes of marked bordered surfaces by tagged triangulations [8]. Paquette and Schiffler studied group actions on cluster algebras and cluster categories, constructing generalized orbit algebras and establishing Laurentness in their mutation-connected setting [9, Section 7]. Kaufman’s special foldings allow folded quivers beyond those encoded by skew-symmetrizable matrices and include examples from punctured surfaces with self-folded triangles [10]. The stability of the vertex orbits under repeated mutation is part of the folding condition.

Li, Liu, Mou and Pan introduced a mutation theory for loop-free quivers with oriented 2-cycles [11]. A homotopy is a normal subgroupoid of the fundamental groupoid; it specifies which 2-cycles are deleted. Their mutation is involutive and extends ordinary quiver mutation [11, Propositions 3.18 and 3.21]. For marked surfaces with colored punctures, their construction intertwines flips and homotopy mutations [11, Theorem 4.23]. We use their version 3, dated 13 October 2026. Theorem 5.8 establishes Laurentness and positivity when the associated cover remains weakly admissible under all orbit mutations. This includes homotopies containing the squares of all closed walks. Problem 6.1 asks for Laurentness for every algebra in the general theory; the authors report computational evidence for an affirmative answer. Proposition 5.9 defines g -vectors through homogeneous rational functions, and the following paragraph asks whether their sign coherence also persists.

Retaining opposite arrows changes the exchange numerators. A vertex joined to another in both directions contributes a common variable factor to the two exchange monomials. For comparison, the exchange polynomials in Lam–Pylyavskyy Laurent phenomenon algebras are irreducible and are not divisible by cluster variables [12, Section 2.1 and Theorem 5.1]. In the homotopy setting, both the retained multiplicities and the subsequent homotopy-dependent deletions enter the rational recurrence.

We give negative answers to both questions. A three-vertex quiver with six arrows produces a coefficient-free variable with an uncanceled factor $1 + x_1$ in its denominator after six mutations, where x_1 is the initial variable at vertex 1. A four-arrow quiver produces a mixed-sign row in its g -matrix after seven principal-coefficient mutations. The quotient vertex groups are infinite cyclic, so integer arrow labels determine every homotopy membership test. We give the presentations, all composite arrows and deletions, and the resulting exchange calculations. The six-arrow data also give a g -vector counterexample with nonzero initial skew-symmetric difference in Appendix A. In both examples the canonical cover acquires arrows within a deck orbit at the first mutation (Section 6).

2 Homotopies and exchange relations

Let Q be a finite loop-free quiver. Regard its underlying graph as a one-dimensional cell complex. We use vertices $1, \dots, n$, where n is a positive integer. Its fundamental groupoid $\pi(Q)$ has the vertices as objects and walks modulo cancellation of an edge with its inverse as morphisms. Path products are composed from right to left. Write $\pi_1(Q, i)$ for its group of closed walks at vertex i . A directed 2-cycle consists of two arrows $i \rightarrow j \rightarrow i$ with $i \neq j$. A homotopy is a normal subgroupoid containing all identity morphisms and only endomorphisms, with a normal subgroup $H(i, i)$ at each vertex and $pH(i, i)p^{-1} = H(j, j)$ for every walk

$p : i \rightarrow j$. It is reduced if it contains no directed 2-cycle of Q . The quotient groupoid $\pi(Q)/H$ has the same objects; walks $p, q : i \rightarrow j$ have the same image if $q^{-1}p \in H(i, i)$. Its group of endomorphisms at i is the quotient vertex group $\pi_1(Q, i)/H(i, i)$. For closed walks based at vertex 1, the notation $\langle\langle \cdots \rangle\rangle$ means the smallest normal subgroupoid containing them, including their conjugates at all vertices.

Put $p_{ij} = |\{\text{arrows } i \rightarrow j\}|$ and $P = (p_{ij})$. Parallel arrows are counted separately even when their images modulo H agree. Set $B = P - P^\top$, where the superscript denotes transpose. Mutation sequences are written in chronological order, from left to right. A subscript t records the number of mutations already performed, and P_t is the adjacency matrix after step t .

For the three-vertex examples, let $\mathcal{Z}$ be the groupoid with objects 1, 2, 3 and

$$\text{Hom}_{\mathcal{Z}}(i, j) = \{(i, j; r) : r \in \mathbb{Z}\}, \quad (j, k; s)(i, j; r) = (i, k; r + s).$$

An integer assignment to arrows extends uniquely to a functor $\lambda : \pi(Q) \rightarrow \mathcal{Z}$. With the endpoints understood, we also write $\lambda(p)$ for the integer component of the image of a walk p . Inverses have opposite labels. Its kernel consists of closed walks of label zero. In particular, a zero-label arrow between distinct vertices is not an identity morphism.

Lemma 2.1. *Let Q be loop-free on vertices 1, 2, 3, and let $H = \ker \lambda$ be reduced. Mutation at k is computed exactly as follows.*

- (1) *For each two-arrow path $i \rightarrow k \rightarrow j$ with $i \neq j$, add an arrow $i \rightarrow j$ whose label is the sum of the two labels.*
- (2) *Reverse each arrow incident with k and negate its label.*
- (3) *For $i, j \neq k$, delete opposite arrows with labels r and $-r$ in pairs, until no such pair remains.*

The resulting homotopy is the kernel of the label functor on the remaining quiver. For fixed endpoints and labels, the remaining multiplicities do not depend on the order of deletion.

Proof. Let Q' be the quiver after (1)–(2). Define a functor $\pi(Q') \rightarrow \pi(Q)/H$ by sending an unchanged arrow to its class, a reversed arrow to the class of its inverse, and the composite arrow for $i \xrightarrow{a} k \xrightarrow{b} j$ to the class of ba . These assignments extend to walks by composition and inversion because $\pi(Q')$ is a free groupoid. Its kernel is the premutated homotopy of [11, Definition 3.8].

The label functor induces an injection on each morphism set of $\pi(Q)/H$: two walks with the same endpoints and label differ by a closed walk in $\ker \lambda = H$. Composing with this injection gives the labels prescribed in (1)–(2), with the same kernel. Thus a directed 2-cycle belongs to the premutated homotopy precisely when its labels sum to zero. Deleting all such pairs is the reduction in [11, Definition 3.9]. By Definition 3.10 the homotopy on the remaining quiver is the kernel of the restricted functor, hence is again the label kernel.

Opposite arrows incident with k have their old nonzero cycle label negated, and thus remain reduced. After (3), every other opposite pair has nonzero label sum. For fixed i, j, r , only an $i \rightarrow j$ arrow of label r and a $j \rightarrow i$ arrow of label $-r$ can cancel each other. Their number of cancellations is the minimum of their two multiplicities, independently of their order. $\square$

A coefficient semifield $\mathbb{P}$ is a commutative multiplicative group with an auxiliary addition $\oplus$ that is associative, commutative and distributive over multiplication. Write $\mathbb{Z}\mathbb{P}$ for its integral group ring. The initial variables $x_1, \dots, x_n$ are algebraically independent over the fraction field of $\mathbb{Z}\mathbb{P}$. A seed consists of a free generating tuple $(X_1, \dots, X_n)$ of this rational function field, a coefficient tuple $(Y_1, \dots, Y_n) \in \mathbb{P}^n$, and a quiver with reduced homotopy. Write $X_{i,t}$ and $Y_{i,t}$ for the variable and coefficient at vertex i after step t , with $X_{i,0} = x_i$. At a current

seed we omit t ; a prime denotes the seed after one mutation. All products and sums with a vertex index range from 1 to n , and an empty product is 1. Mutation at k leaves the other variables fixed and gives

$$X'_k = \frac{Y_k \prod_j X_j^{p_{jk}} + \prod_j X_j^{p_{kj}}}{(1 \oplus Y_k) X_k}, \quad (2.1)$$

$$Y'_k = Y_k^{-1}, \quad Y'_j = Y_j Y_k^{p_{kj}} (1 \oplus Y_k)^{p_{jk} - p_{kj}} \quad (j \neq k). \quad (2.2)$$

These are [11, (5.1)–(5.2)]. For coefficient-free mutation we use the trivial semifield $\mathbb{P} = \{1\}$, so $1 \oplus 1 = 1$. For principal coefficients we use $\text{Trop}(y_1, \dots, y_n)$, the free abelian multiplicative group on $y_1, \dots, y_n$, with $\mathbf{y}^c = \prod_{j=1}^n y_j^{c_j}$ for $c = (c_1, \dots, c_n)^\top \in \mathbb{Z}^n$ and $\mathbf{y}^a \oplus \mathbf{y}^b = \mathbf{y}^{\min(a,b)}$. Here $\mathbf{y} = (y_1, \dots, y_n)$ and minima are taken coordinatewise. In particular,

$$1 \oplus \mathbf{y}^c = \mathbf{y}^{\min(c,0)}. \quad (2.3)$$

The initial coefficients are $Y_j = y_j$, and the coefficient group ring is $\mathbb{Z}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$. The ordinary addition in the numerator of (2.1) is distinct from $\oplus$.

The loop-free condition makes the numerator of (2.1) independent of X_k . Its two monomials have positive coefficients, so it is nonzero. Equation (2.1) expresses X_k as this same numerator divided by $(1 \oplus Y_k) X'_k$. Hence the old and new tuples generate the same rational function field; its transcendence degree n implies that the new n variables are algebraically independent. The algebra $\mathcal{A}(Q, H)$ is the $\mathbb{Z}\mathbb{P}$ -subalgebra generated by all variables reached by finite mutations from the initial seed. The Laurent property asks that all these variables belong to $\mathbb{Z}\mathbb{P}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

For principal coefficients, let e_i be the i th standard basis vector of $\mathbb{Z}^n$. Give the initial variables and coefficients degrees

$$\deg x_i = e_i, \quad \deg y_j = (p_{j1}^{(0)} - p_{1j}^{(0)}, \dots, p_{jn}^{(0)} - p_{nj}^{(0)})^\top, \quad (2.4)$$

where $p_{ij}^{(0)}$ is the (i, j) entry of P_0 . Degrees add under multiplication. A polynomial is homogeneous when all its nonzero monomials have the same degree; a quotient of nonzero homogeneous polynomials has degree equal to the numerator degree minus the denominator degree. This difference is independent of the representation: equality of two fractions gives an equality of nonzero homogeneous cross-products, whose degrees must agree. Proposition 5.9 of [11] gives homogeneity of the cluster variables as rational functions. Their degrees $g_{i,t} = \deg X_{i,t}$ are the g -vectors. Set $G_t = (g_{1,t} \ \cdots \ g_{n,t})$. Coordinatewise sign coherence means that each row of G_t is entirely nonnegative or entirely nonpositive; zero entries are allowed.

Theorem 2.2. *There exist loop-free quivers with reduced homotopies on three vertices, whose quotient vertex groups are isomorphic to $\mathbb{Z}$, with the following properties.*

- (i) *A six-arrow example has a coefficient-free cluster variable outside $\mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ after the sequence $(2, 1, 3, 2, 3, 2)$.*
- (ii) *A four-arrow example has, with principal coefficients, the cluster*

$$\left(\frac{x_3}{1 + y_3}, x_2, \frac{(1 + y_3)x_1}{x_3} \right)$$

after $(2, 3, 1, 2, 1, 2, 1)$. Its g -matrix, with g -vectors as columns, is

$$G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Part (i) is proved in Section 3 and part (ii) in Section 4.

3 A non-Laurent variable over $\mathbb{Q}$

3.1 Initial quiver and homotopy

Take the following six arrows.

Arrow	a	b	c	d	e	f
Direction	$1 \rightarrow 2$	$2 \rightarrow 1$	$2 \rightarrow 1$	$1 \rightarrow 3$	$1 \rightarrow 3$	$3 \rightarrow 1$
Label	1	1	0	0	0	1

Choose the unoriented spanning tree consisting of a, d . Collapsing this tree to vertex 1 leaves four independent circles, one for each remaining edge. Traversing the tree to an edge and back gives the free generators

$$u = ca, \quad v = ba, \quad w = d^{-1}e, \quad z = fd$$

of $\pi_1(Q, 1)$. Define, in the entire fundamental groupoid,

$$H = \langle\langle vu^{-2}, w, zu^{-1} \rangle\rangle. \quad (3.1)$$

Then

$$\pi_1(Q, 1)/H(1, 1) = \langle u, v, w, z \mid v = u^2, w = 1, z = u \rangle \cong \langle u \rangle \cong \mathbb{Z}. \quad (3.2)$$

Indeed, eliminating v, w, z leaves the free group on u . The labels give $(\lambda(u), \lambda(v), \lambda(w), \lambda(z)) = (1, 2, 0, 1)$. Consequently (3.2) identifies the quotient map at vertex 1 with λ , proving equality of kernels there. Conjugation by a connecting walk proves equality at the other vertices; the connecting walk and its inverse have cancelling labels. Hence (3.1) is exactly $\ker \lambda$ on the whole groupoid. Conjugating u along a walk from 1 to i gives a loop of label 1 at i . If $p : i \rightarrow j$ has label r , composing p with powers of this loop realizes every label $r + m$, $m \in \mathbb{Z}$, from i to j . Together with kernel equality, this proves $\pi(Q)/H \cong \mathcal{Z}$.

The directed 2-cycles based at vertex 1 are ca, ba, fd, fe , of labels 1, 2, 1, 1. Their versions at other basepoints have the same nonzero labels. Thus H is reduced. The relation $w = d^{-1}e \in H$ identifies the images of the parallel arrows d, e in the quotient groupoid. Both arrows remain in the quiver, since deletion removes opposite arrows.

3.2 Labelled mutations

Use the sequence $(2, 1, 3, 2, 3, 2)$. In Table 1, $L_{ij}^{(t)}$ is the multiset of labels on arrows $i \rightarrow j$ after step t . The column k specifies the vertex mutated at step t ; the initial row has $t = 0$. A repeated entry denotes a separate arrow, and $\emptyset$ denotes no arrows.

Table 1: The six-arrow example: all retained arrow labels.

t	k	$L_{12}^{(t)}$	$L_{21}^{(t)}$	$L_{13}^{(t)}$	$L_{31}^{(t)}$	$L_{23}^{(t)}$	$L_{32}^{(t)}$
0	–	1	1, 0	0, 0	1	$\emptyset$	$\emptyset$
1	2	–1, 0	–1	0, 0	1	$\emptyset$	$\emptyset$
2	1	1	1, 0	–1	0, 0	–1	0
3	3	$\emptyset$	0, –1	0, 0	1	0	1
4	2	0, 1	$\emptyset$	0	1, 1	–1	0
5	3	1	$\emptyset$	–1, –1	0	0	1
6	2	$\emptyset$	–1	–1, –1, 1	0	–1	0

Table 2 lists every composite arrow and every deletion. Here $ij : r$ means an arrow $i \rightarrow j$ with label r . Two labels separated by a comma on one arrow type specify two arrows. A pair in the last column specifies the two arrows removed. All remaining changes are the reversals prescribed in Lemma 2.1.

Table 2: Composite arrows and deletions in the six mutations.

Step	Composite arrows added	Pairs deleted
1	None	None
2	$23 : -1, -1; \quad 32 : 0, 1$	$(23 : -1, \quad 32 : 1)$
3	$12 : -1; \quad 21 : -1, -1$	$(12 : 1, \quad 21 : -1); (12 : -1, \quad 21 : 1)$
4	$31 : 1, 0$	$(13 : 0, \quad 31 : 0)$
5	$12 : 0; \quad 21 : 0, 0$	$(12 : 0, \quad 21 : 0)$, twice
6	$13 : 1$	None

At step 1 all two-arrow paths through 2 have equal endpoints and are omitted. At step 5 one deleted $12 : 0$ arrow was already present and one was newly added. Each deleted pair has label sum zero, and each retained opposite pair has nonzero label sum.

Counting Table 1 gives the six matrices used before the exchanges:

$$\begin{aligned}
P_0 &= \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \\
P_3 &= \begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{3.3}$$

3.3 Six exchanges and the uncanceled factor

Proof of Theorem 2.2(i). Work over the trivial semifield with initial cluster (x_1, x_2, x_3) . Equations (2.1) and (3.3) give the six new cluster variables U, V, W, Z, T, R , in order,

$$\begin{aligned}
U &= \frac{x_1(1+x_1)}{x_2}, \quad V = \frac{Ux_3(1+Ux_3)}{x_1}, \quad W = \frac{UV(1+V)}{x_3}, \\
Z &= \frac{W(1+V^2)}{U}, \quad T = \frac{VZ(1+V)}{W}, \quad R = \frac{T(1+V)}{Z}.
\end{aligned} \tag{3.4}$$

The clusters are successively (x_1, U, x_3) , (V, U, x_3) , (V, U, W) , (V, Z, W) , (V, Z, T) and (V, R, T) . In particular, the fourth and sixth exchange numerators are $W + V^2W$ and $VT + T$, respectively. Cancelling the intermediate variables in the last three equations yields

$$\begin{aligned}
R &= \frac{V(1+V)^2}{W} = \frac{x_3(1+V)}{U} = \frac{x_3}{U} + \frac{x_3^2}{x_1} + \frac{Ux_3^3}{x_1} \\
&= \frac{x_2x_3}{x_1(1+x_1)} + \frac{x_3^2}{x_1} + \frac{(1+x_1)x_3^3}{x_2}.
\end{aligned}$$

Thus the sixth mutation produces

$$\boxed{R = \frac{x_3[x_2^2 + (1+x_1)x_2x_3 + x_1(1+x_1)^2x_3^2]}{x_1(1+x_1)x_2}}. \tag{3.5}$$

In the Laurent ring $\mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ the element $1 + x_1$ is prime and not a unit: quotienting by it amounts to setting $x_1 = -1$, leaving $\mathbb{Q}[x_2^{\pm 1}, x_3^{\pm 1}]$. The numerator of (3.5) maps to $x_3 x_2^2 \neq 0$ in this quotient. Hence the factor $1 + x_1$ does not cancel and R is not in the Laurent ring. $\square$

4 A four-arrow counterexample to sign coherence

4.1 A homotopy with one relation

Now take

$$a : 1 \rightarrow 2, \quad b : 2 \rightarrow 1, \quad c : 1 \rightarrow 3, \quad d : 3 \rightarrow 1,$$

with labels $1, 0, 0, -2$, respectively. Collapsing the spanning tree consisting of a, c leaves two independent circles, represented by the closed walks $U = ba$ and $V = dc$ at vertex 1. These are free generators of the vertex group; the letters U, V in this section denote these walks. Define

$$H = \langle\langle U^2 V \rangle\rangle. \quad (4.1)$$

Then

$$\pi_1(Q, 1)/H(1, 1) = \langle U, V \mid U^2 V = 1 \rangle \cong \langle U \rangle \cong \mathbb{Z}, \quad U \mapsto 1, \quad V \mapsto -2.$$

Eliminating $V = U^{-2}$ proves that the label functor has exactly the kernel (4.1). As in the six-arrow example, conjugation extends the kernel equality to every vertex, and $\lambda(U) = 1$ makes the quotient groupoid isomorphic to $\mathcal{Z}$. Both directed 2-cycles have nonzero labels, 1 and -2 , so H is reduced.

Use the sequence $(2, 3, 1, 2, 1, 2, 1)$. Its complete retained arrow data are in Table 3.

Table 3: The four-arrow example: all retained arrow labels.

t	k	$L_{12}^{(t)}$	$L_{21}^{(t)}$	$L_{13}^{(t)}$	$L_{31}^{(t)}$	$L_{23}^{(t)}$	$L_{32}^{(t)}$
0	–	1	0	0	-2	$\emptyset$	$\emptyset$
1	2	0	-1	0	-2	$\emptyset$	$\emptyset$
2	3	0	-1	2	0	$\emptyset$	$\emptyset$
3	1	1	0	0	-2	1	0
4	2	0	-1	$\emptyset$	$\emptyset$	0	-1
5	1	1	0	$\emptyset$	$\emptyset$	0	-1
6	2	0	-1	$\emptyset$	$\emptyset$	1	0
7	1	1	0	$\emptyset$	$\emptyset$	1	0

The first two steps only reverse arrows. Step 3 adds $23 : 1$ and $32 : 0$, which do not cancel. Step 4 adds $13 : 2$ and $31 : 0$; these cancel the old arrows $31 : -2$ and $13 : 0$, respectively. Step 5 only reverses arrows. Step 6 adds $13 : 1$ and $31 : -1$, which cancel each other. Step 7 again only reverses arrows. This lists every composite and deletion, and every retained opposite pair has nonzero label sum.

These deletions can also be checked as words in the initial groupoid. At step 4 the new $1 \rightarrow 3$ and $3 \rightarrow 1$ arrows have images d^{-1} and c^{-1} , respectively, so their deletions with the old images d and c already hold by free cancellation. At step 6 the two new arrows have images cU and Ud . Their directed cycle at vertex 1 has image $(Ud)(cU) = UVU = U^{-1}(U^2V)U$, a conjugate of the defining relator. This justifies the deletion directly by (4.1).

Counting the arrows gives

$$\begin{aligned}
P_0 = P_1 = P_2 &= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & P_3 &= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \\
P_4 = P_5 = P_6 = P_7 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{4.2}$$

Every P_t in this sequence is symmetric.

4.2 Principal coefficients and degrees

Proof of Theorem 2.2(ii). Symmetry reduces (2.1)–(2.2) to

$$\begin{aligned}
Y'_k &= Y_k^{-1}, & Y'_j &= Y_j Y_k^{p_{kj}} \quad (j \neq k), \\
X'_k &= \frac{1 + Y_k \prod_j X_j^{p_{kj}}}{1 \oplus Y_k} X_k.
\end{aligned} \tag{4.3}$$

The coefficient tuples obtained from the initial tuple (y_1, y_2, y_3) are listed in Table 4.

Table 4: Coefficient tuples after each mutation.

t	k	$(Y_{1,t}, Y_{2,t}, Y_{3,t})$
0	–	(y_1, y_2, y_3)
1	2	$(y_1 y_2, y_2^{-1}, y_3)$
2	3	$(y_1 y_2 y_3, y_2^{-1}, y_3^{-1})$
3	1	$((y_1 y_2 y_3)^{-1}, y_1 y_3, y_1 y_2)$
4	2	$(y_2^{-1}, (y_1 y_3)^{-1}, y_1^2 y_2 y_3)$
5	1	$(y_2, (y_1 y_2 y_3)^{-1}, y_1^2 y_2 y_3)$
6	2	$((y_1 y_3)^{-1}, y_1 y_2 y_3, y_1)$
7	1	$(y_1 y_3, y_2, y_1)$

The factors $(1 + Y_k)/(1 \oplus Y_k)$ at the seven exchanges are, in order,

$$1 + y_2, \quad 1 + y_3, \quad 1 + y_1 y_2 y_3, \quad 1 + y_1 y_3, \quad 1 + y_2, \quad 1 + y_1 y_2 y_3, \quad 1 + y_1 y_3.$$

At the first four exchanges the current coefficient has nonnegative exponents, so its tropical sum with 1 is 1. At the last three it is the reciprocal of a monomial with nonnegative exponents. For $c \in \mathbb{Z}_{\geq 0}^n$, the identity

$$\frac{1 + \mathbf{y}^{-c}}{1 \oplus \mathbf{y}^{-c}} = \frac{1 + \mathbf{y}^{-c}}{\mathbf{y}^{-c}} = 1 + \mathbf{y}^c$$

gives the remaining factors. Using the adjacency matrices in (4.2), the seven exchanges are

$$X_{2,1} = \frac{(1+y_2)x_1}{x_2}, \quad X_{3,2} = \frac{(1+y_3)x_1}{x_3}, \quad (4.4)$$

$$\begin{aligned} X_{1,3} &= \frac{(1+y_1y_2y_3)X_{2,1}X_{3,2}}{x_1} \\ &= \frac{(1+y_2)(1+y_3)(1+y_1y_2y_3)x_1}{x_2x_3}, \end{aligned} \quad (4.5)$$

$$\begin{aligned} X_{2,4} &= \frac{(1+y_1y_3)X_{1,3}X_{3,2}}{X_{2,1}} \\ &= \frac{(1+y_3)^2(1+y_1y_2y_3)(1+y_1y_3)x_1}{x_3^2}, \end{aligned} \quad (4.6)$$

$$X_{1,5} = \frac{(1+y_2)X_{2,4}}{X_{1,3}} = \frac{(1+y_3)(1+y_1y_3)x_2}{x_3}, \quad (4.7)$$

$$X_{2,6} = \frac{(1+y_1y_2y_3)X_{1,5}X_{3,2}}{X_{2,4}} = x_2, \quad (4.8)$$

$$X_{1,7} = \frac{(1+y_1y_3)X_{2,6}}{X_{1,5}} = \frac{x_3}{1+y_3}. \quad (4.9)$$

The variables at the other vertices remain unchanged at each step. The resulting cluster is

$$\left(\frac{x_3}{1+y_3}, x_2, \frac{(1+y_3)x_1}{x_3} \right). \quad (4.10)$$

The initial adjacency matrix is symmetric, so (2.4) gives $\deg y_j = 0$. The final g -vectors, in the order of (4.10), are $e_3, e_2, e_1 - e_3$. Thus

$$G_7 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix},$$

whose third row has both a positive and a negative entry. $\square$

The first variable of (4.10) also fails the Laurent property over the principal coefficient group ring: $1+y_3$ is a nonunit prime in $\mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}][x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$, since setting $y_3 = -1$ identifies its quotient with the nonzero integral domain $\mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$. It therefore cannot divide the unit x_3 . Over the trivial semifield the final cluster is $(x_3/2, x_2, 2x_1/x_3)$, which is Laurent over $\mathbb{Q}$ but not over $\mathbb{Z}$.

5 The recurrence for symmetric adjacency matrices

For any n and $v = (v_1, \dots, v_n)^\top \in \mathbb{Z}^n$, write $\mathbf{x}^v = \prod_{i=1}^n x_i^{v_i}$, where $\mathbf{x} = (x_1, \dots, x_n)$. The symmetric case separates these variable exponents from the coefficient factors.

Proposition 5.1. *For a quiver on n vertices, if the initial adjacency matrix is symmetric, every mutated adjacency matrix is symmetric. With principal coefficients every cluster variable has the form*

$$X_{i,t} = R_{i,t}(\mathbf{y})\mathbf{x}^{g_{i,t}}, \quad R_{i,t} \in \mathbb{Q}(\mathbf{y}) \setminus \{0\}, \quad g_{i,t} \in \mathbb{Z}^n.$$

At the current seed, write R_i, g_i for its coefficient factors and exponent vectors. Write its coefficient at k as $Y_k = \mathbf{y}^{c_k}$, where $c_k \in \mathbb{Z}^n$, and put

$$h_k = \frac{1 + \mathbf{y}^{c_k}}{\mathbf{y}^{\min(c_k, 0)}}.$$

Then mutation at k satisfies

$$g'_k = -g_k + \sum_j p_{kj} g_j, \quad (5.1)$$

$$R'_k = h_k \frac{\prod_j R_j^{p_{kj}}}{R_k}, \quad (5.2)$$

while $g'_j = g_j$ and $R'_j = R_j$ for $j \neq k$.

Proof. For distinct $i, j \neq k$, premutation adds $p_{ik}p_{kj}$ arrows $i \rightarrow j$ and $p_{jk}p_{ki}$ arrows $j \rightarrow i$. These numbers agree if P is symmetric. Reversal of arrows incident with k preserves symmetry, and each deletion subtracts one arrow in each direction. The diagonal entries remain zero because premutation omits loops. This proves the first assertion inductively. Equation (4.3) then holds throughout. Initially $R_i = 1$, $g_i = e_i$, and $\deg y_j = 0$. Substituting $X_j = R_j \mathbf{x}^{g_j}$ into (4.3) gives

$$X'_k = \frac{1 + \mathbf{y}^{c_k}}{\mathbf{y}^{\min(c_k, 0)}} \frac{\prod_j R_j^{p_{kj}}}{R_k} \mathbf{x}^{-g_k + \sum_j p_{kj} g_j}.$$

The first two factors are nonzero rational functions of $\mathbf{y}$ and have degree zero. The last factor has the displayed exponent vector, proving (5.1)–(5.2) by induction. $\square$

For an irreducible coefficient factor ϕ in the Laurent polynomial ring $\mathbb{Q}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$, define $\nu_\phi(f)$ for a nonzero rational function f as the multiplicity of ϕ in its numerator minus its multiplicity in its denominator. Unique factorization makes this independent of the fraction representing f and gives $\nu_\phi(fg) = \nu_\phi(f) + \nu_\phi(g)$. Applying this identity to (5.2) gives

$$\nu_\phi(R'_k) = \nu_\phi(h_k) - \nu_\phi(R_k) + \sum_j p_{kj} \nu_\phi(R_j). \quad (5.3)$$

The normalizing factor is the polynomial

$$h_k = \mathbf{y}^{c_k - \min(c_k, 0)} + \mathbf{y}^{-\min(c_k, 0)},$$

since both exponent columns are nonnegative. In the four-arrow example, take $\phi = 1 + y_3$. Immediately before the last exchange, the valuations of (R_1, R_2, R_3) are $(1, 0, 1)$. The last exchange has $h_1 = 1 + y_1 y_3$ and $p_{12} = 1$, $p_{13} = 0$. Since $1 + y_1 y_3$ becomes $1 - y_1 \neq 0$ at $y_3 = -1$, its ϕ -valuation is zero. Thus (5.3) gives $\nu_\phi(R'_1) = 0 - 1 + 0 = -1$.

Over the trivial semifield, the same argument gives $X_{i,t} = 2^{r_{i,t}} \mathbf{x}^{g_{i,t}}$ with integer exponents $r_{i,t}$, and

$$r'_k = 1 - r_k + \sum_j p_{kj} r_j. \quad (5.4)$$

For the seven-step example the exponent triples are

$$(0, 0, 0), (0, 1, 0), (0, 1, 1), (3, 1, 1), (3, 4, 1), (2, 4, 1), (2, 0, 1), (-1, 0, 1).$$

6 The associated coverings

A quiver covering is a surjective map on vertices and arrows inducing bijections on the incoming and outgoing arrows at each vertex. A deck transformation is an automorphism of the covering quiver that commutes with the covering map. A connected covering is normal when its deck transformations act transitively on each vertex fiber. It is weakly admissible when its covering quiver has no oriented 2-cycles and no arrow joins two vertices in the same

deck-transformation orbit. An orbit mutation applies ordinary quiver mutation once at every vertex in one such orbit. Global weak admissibility requires weak admissibility after every finite sequence of orbit mutations; see [11, Section 2.3]. The mutations within one orbit commute because its vertices are pairwise nonadjacent. For the locally finite covers below, each arrow is affected by only finitely many of these mutations.

For either label assignment, the connected normal cover corresponding to $H = \ker \lambda$ has vertices (i, r) with $r \in \mathbb{Z}$. An arrow $i \rightarrow j$ of label ℓ lifts to

$$(i, r) \longrightarrow (j, r + \ell) \quad (r \in \mathbb{Z}).$$

The covering map forgets the second coordinate. Every base arrow has one lift with any prescribed initial or terminal vertex, giving the required local bijections. A walk of label m lifts from (i, r) to $(j, r + m)$, so a closed walk lifts to a closed walk exactly when its label is zero. Its subgroupoid of closed lifts is therefore H . The realization of every integer label between any two base vertices proves connectedness. Translations $(i, r) \mapsto (i, r + s)$, $s \in \mathbb{Z}$, are deck transformations and act transitively on each fiber, proving normality. Initially it is weakly admissible: base arrows have distinct endpoints, and a lifted directed 2-cycle would project to a directed 2-cycle of total label zero, which the reduced condition excludes.

In the four-arrow example, the first orbit mutation is at the orbit above vertex 2. For every r , the path

$$(1, r) \xrightarrow{a} (2, r + 1) \xrightarrow{b} (1, r + 1)$$

creates an arrow $(1, r) \rightarrow (1, r + 1)$ in the ordinary Fomin–Zelevinsky premutation of the cover. There was no arrow within this orbit, and all new within-orbit arrows point from level r to level $r + 1$. Consequently no opposite arrow can cancel it. The mutated cover has an arrow between two vertices in one deck orbit and is no longer weakly admissible.

The same obstruction occurs in the six-arrow example. The paths ca and ba through the first mutated orbit create arrows $(1, r) \rightarrow (1, r + 1)$ and $(1, r) \rightarrow (1, r + 2)$, respectively, with no opposite arrows. Thus its canonical covering also loses weak admissibility immediately. In the base quiver these paths have equal endpoints and are omitted by the homotopy premutation rule, allowing the subsequent base mutations to continue.

A Row sign incoherence with nonzero initial B

Proposition A.1. *For the six-arrow initial quiver and homotopy of Section 3, principal-coefficient mutation along $(2, 1, 3, 2, 3, 2, 3, 1)$ gives*

$$G_8 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 5 & 3 & 1 \end{pmatrix}.$$

Proof. Here

$$B_0 = P_0 - P_0^\top = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \neq 0,$$

and the columns of $-B_0$ are the degrees of the initial coefficients.

A.1 The last two quiver mutations

The additional label rows are given in Table 5. They continue Table 1.

Table 5: The last two label rows of the eight-step sequence.

t	k	$L_{12}^{(t)}$	$L_{21}^{(t)}$	$L_{13}^{(t)}$	$L_{31}^{(t)}$	$L_{23}^{(t)}$	$L_{32}^{(t)}$
7	3	-1, -1	-1	0	1, 1, -1	0	1
8	1	1	1, 1	-1, -1, 1	0	$\emptyset$	0, 0, 0, -2, -2

Step 7 adds $12 : -1, -1, 1$ and $21 : -1$, and deletes one pair $(12 : 1, 21 : -1)$. Step 8 adds $32 : 0, 0, 0, 0, -2, -2$ and $23 : -1$; it deletes $(23 : 0, 32 : 0)$ and $(23 : -1, 32 : 1)$. All other changes are reversals, and no further zero-sum pair remains. The matrices before these two exchanges are therefore

$$P_6 = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad P_7 = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}.$$

A.2 Coefficient exponents and degrees

For $j = 1, 2, 3$, let $c_j \in \mathbb{Z}^3$ be the exponent column in $Y_j = \mathbf{y}^{c_j}$, and let $g_j = \deg X_j$ at the current seed. Directly from (2.2), mutation at k gives

$$c'_k = -c_k, \quad c'_j = c_j + p_{kj}c_k + (p_{jk} - p_{kj}) \min(c_k, 0) \quad (j \neq k). \quad (\text{A.1})$$

The two numerator terms in (2.1) have the same degree precisely when

$$-B_0 c_k + \sum_i p_{ik} g_i = \sum_i p_{ki} g_i. \quad (\text{A.2})$$

Under this equality their common degree is the right-hand side, while the normalizing monomial has degree $-B_0 \min(c_k, 0)$. Consequently

$$g'_k = -g_k + \sum_i p_{ki} g_i + B_0 \min(c_k, 0), \quad g'_j = g_j \quad (j \neq k). \quad (\text{A.3})$$

Starting with $c_j = g_j = e_j$ for $j = 1, 2, 3$, the integer computations are recorded in Tables 6 and 7. Every tuple in these tables denotes a column vector. Table 8 gives the two numerator degrees and the normalizer degree at every exchange, using the seed immediately before that exchange. The sum of the third and fourth columns equals the fifth column, verifying (A.2). Subtracting the old g_k and the last column from the fifth column gives the new degree in Table 7.

Table 6: All coefficient exponent columns for the six-arrow data.

t	k	c_1	c_2	c_3
0	–	(1, 0, 0)	(0, 1, 0)	(0, 0, 1)
1	2	(1, 2, 0)	(0, –1, 0)	(0, 0, 1)
2	1	(–1, –2, 0)	(2, 3, 0)	(2, 4, 1)
3	3	(3, 6, 2)	(4, 7, 1)	(–2, –4, –1)
4	2	(11, 20, 4)	(–4, –7, –1)	(2, 3, 0)
5	3	(15, 26, 4)	(–2, –4, –1)	(–2, –3, 0)
6	2	(13, 22, 3)	(2, 4, 1)	(–4, –7, –1)
7	3	(1, 1, 0)	(–2, –3, 0)	(4, 7, 1)
8	1	(–1, –1, 0)	(0, –1, 0)	(5, 8, 1)

Table 7: All g -vector columns for the same principal-coefficient pattern.

t	k	g_1	g_2	g_3
0	–	(1, 0, 0)	(0, 1, 0)	(0, 0, 1)
1	2	(1, 0, 0)	(2, –1, 0)	(0, 0, 1)
2	1	(3, –2, 2)	(2, –1, 0)	(0, 0, 1)
3	3	(3, –2, 2)	(2, –1, 0)	(8, –5, 3)
4	2	(3, –2, 2)	(12, –8, 7)	(8, –5, 3)
5	3	(3, –2, 2)	(12, –8, 7)	(10, –7, 8)
6	2	(3, –2, 2)	(1, –1, 3)	(10, –7, 8)
7	3	(3, –2, 2)	(1, –1, 3)	(0, 0, 1)
8	1	(–1, 0, 5)	(1, –1, 3)	(0, 0, 1)

Table 8: Numerator and normalizer degrees before each exchange.

Step	k	$-B_0 c_k$	$\sum_i p_{ik} g_i$	$\sum_i p_{ki} g_i$	$-B_0 \min(c_k, 0)$
1	2	(1, 0, 0)	(1, 0, 0)	(2, 0, 0)	(0, 0, 0)
2	1	(2, –1, 1)	(2, –1, 1)	(4, –2, 2)	(0, 0, 0)
3	3	(3, –2, 2)	(5, –3, 2)	(8, –5, 4)	(0, 0, 0)
4	2	(6, –4, 4)	(8, –5, 3)	(14, –9, 7)	(0, 0, 0)
5	3	(3, –2, 2)	(15, –10, 9)	(18, –12, 11)	(0, 0, 0)
6	2	(–3, 2, –2)	(13, –9, 10)	(10, –7, 8)	(–3, 2, –2)
7	3	(–6, 4, –4)	(10, –7, 9)	(4, –3, 5)	(–6, 4, –4)
8	1	(1, –1, 1)	(1, –1, 6)	(2, –2, 7)	(0, 0, 0)

In the seventh exchange $c_3 = (–4, –7, –1)^\top = \min(c_3, 0)$ and $-B_0 \min(c_3, 0) = (–6, 4, –4)^\top$. Its new degree is

$$-(10, –7, 8)^\top + (3, –2, 2)^\top + (1, –1, 3)^\top - (–6, 4, –4)^\top = (0, 0, 1)^\top.$$

For the eighth exchange $c_1 = (1, 1, 0)^\top$, so $\min(c_1, 0) = 0$, and

$$g'_1 = -(3, –2, 2)^\top + 2(1, –1, 3)^\top + (0, 0, 1)^\top = (–1, 0, 5)^\top.$$

The resulting matrix has a mixed-sign first row:

$$G_8 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 5 & 3 & 1 \end{pmatrix}. \quad (\text{A.4})$$

□

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