

Asymptotic growth of rank-two scattering coefficients in the Badlands

Qiyue Tang

Abstract

We prove that the normalized logarithmic growth rate of ordinary wall-function coefficients in every wild rank-two cluster scattering diagram exists along all integer approaches to an interior Badlands slope and is continuous in that slope. This resolves Conjecture 9.2 of Burcroff, Lee, Mou, Musiker, and Reineke.

AI use and scholarly credit

The main results of this paper were obtained in a single conversation with ChatGPT-6-Pro within two hours. The arguments were reviewed and the manuscript was organized using Codex, and the proof underwent review with Rethlas-OSV [12, 16].

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1 Introduction

The mirror-symmetry constructions of Kontsevich and Soibelman [13] and of Gross and Siebert [11] led to scattering diagrams. These encode compatible changes of coordinates by attaching formal power series to walls. Gross, Hacking, Keel, and Kontsevich developed cluster scattering diagrams and used their broken lines to construct theta functions and prove Laurent positivity [9]. In rank two, the walls are lines and rays, so their attached coefficients form a two-dimensional array. In the wild case, where the product of the two positive exchange parameters exceeds four, an open sector between two limiting directions contains infinitely many nontrivial rays. Understanding the coefficients in this sector, usually called the Badlands, leads to a question about growth as both coordinates tend to infinity.

For an integer $m \geq 3$, the m -Kronecker quiver has two vertices and m parallel arrows from the first to the second. Its stable representations, with fixed dimensions at the vertices, form moduli spaces whose Euler characteristics occur in wall-crossing formulas. Gross and Pandharipande related the tropical vertex to these moduli spaces and to counts of curves on toric surfaces [10]. Reineke and Weist extended this correspondence to complete bipartite quivers and derived functional equations for the associated generating series [18]. Bridgeland subsequently constructed stability scattering diagrams from Hall algebras and identified them with cluster scattering diagrams for acyclic quivers [2, Theorem 1.5].

2020 Mathematics Subject Classification. Primary 13F60; Secondary 05A16.

Keywords. Rank-two cluster scattering diagrams, asymptotic growth, tight gradings, mechanical words.

Michael Douglas proposed that, with the number of Kronecker arrows fixed, the logarithm of the Euler characteristic divided by the first dimension should approach a continuous function of the ratio of dimensions. Weist records the original proposal as an oral communication from Douglas to Markus Reineke in August 2004 [19, Section 6.1 and reference 3]. A published formulation appears in Gross–Pandharipande [10, Section 6(iii)]; Weist gives a quantified version in [19, Conjecture 6.1]. The dimensions in these formulations are coprime.

Localization supplied early progress. Weist obtained an explicit formula for dimensions $(d, d+1)$, hence the growth rate at ratio one, and constructed families of stable trees giving exponential lower bounds [19, Theorem 6.6 and Section 6.3]. Combining localization with the Manschot–Pioline–Sen degeneration formula, he later proved an exponential upper bound [20, Theorem 3.11]. Okada studied a different asymptotic regime: for each fixed coprime dimension vector the number of arrows tends to infinity; the resulting logarithmic estimate also allows large dimensions when the arrow number is chosen sufficiently large in terms of them [17, Theorem 1 and Corollary 2]. Córdova and Shao calculated growth rates at integral ratios and studied their reflection and mutation symmetries [6, Sections 2–3]. Mainiero developed asymptotic formulas under algebraicity hypotheses on generating series, including detailed calculations for multiples of $(3, 2)$ in the three-arrow case [15, Sections 4 and 7].

On the combinatorial side, Lee, Li, and Zelevinsky introduced compatible pairs on Dyck paths in their construction of rank-two greedy bases [14]. Burcroff, Lee, and Mou introduced tight gradings and obtained positive counting formulas for rank-two wall-functions, together with generalized positivity results [4, 3]. Deformations and mutations had also yielded polynomiality and nonvanishing results in the dense region through the work of Gräfnitz and Luo [8]. Elgin, Reading, and Stella organized explicit computations into conjectures on scattering coefficients [7]; Akagi proved several of these using dilogarithm identities [1]. Burcroff, Lee, Mou, Musiker, and Reineke combined tight gradings with quiver moduli to establish further formulas and a combinatorial Weyl group symmetry [5]. Their Section 9.3 proposes the directional growth statement addressed here.

Fix positive integers b, c with $bc > 4$, and let x_1, x_2 be commuting formal variables. We use the plane and lattice conventions of [5, Section 2]: the initial walls are the coordinate lines, carrying $1+x_1^b$ and $1+x_2^c$, and all added walls have the form $\mathbb{R}_{\leq 0}v$ for $v \in \mathbb{Z}_{>0}^2$. Wall-functions have constant term one and belong to $\mathbb{Q}[[x_1, x_2]]$, with nonconstant exponents parallel to their supporting wall; for any fixed bound, only finitely many walls contain a nonconstant monomial whose total degree, the sum of its exponents, is at most that bound. At an oriented crossing of a wall with function F , let (n_1, n_2) be its primitive integral normal with negative pairing against the crossing direction. For $(u_1, u_2) \in \mathbb{Z}^2$, the crossing automorphism is

$$x_1^{u_1} x_2^{u_2} \mapsto x_1^{u_1} x_2^{u_2} F^{n_1 u_1 + n_2 u_2}.$$

Reversing the crossing gives the inverse. Consistency means that the ordered composition along every closed transverse path avoiding the origin is the identity, interpreted degree by degree. The consistent completion is unique up to equality of these path products [13, 9]; functions on coincident rays are multiplied. Assign function 1 to an absent ray. Let $\tau(i, j)$, for positive integers i, j , be the ordinary coefficients: for relatively prime positive integers d, e , the function on the outgoing ray $\mathbb{R}_{\leq 0}(bd, ce)$ is

$$1 + \sum_{k \geq 1} \tau(kd, ke) x_1^{bkd} x_2^{cke}. \quad (1)$$

Formula (1) fixes the normalization at nonprimitive indices as well. Write

$$Q(i, j) = bci j - bi^2 - cj^2, \quad I_{b,c} = \left(\frac{bc - \sqrt{bc(bc-4)}}{2c}, \frac{bc + \sqrt{bc(bc-4)}}{2c} \right). \quad (2)$$

Thus $Q(i, j) > 0$ for positive i, j precisely when $j/i \in I_{b,c}$. The interval $I_{b,c}$ records the Badlands in index coordinates; the slope of the corresponding ray in the plane is $cj/(bi)$. All logarithms below are natural logarithms.

In [5, Conjecture 9.2], Burcroff, Lee, Mou, Musiker, and Reineke ask whether $\log \tau(i, j) / \sqrt{Q(i, j)}$ has a continuous directional limit throughout $I_{b,c}$. Their exact central-wall formula [5, Proposition 3.6], obtained from [18, Section 11], already gives a ray growth rate at slope one when that slope is interior. We prove the conjecture.

Theorem 1.1. *For each $r \in I_{b,c}$, there is a finite nonnegative number $h(r)$ such that*

$$\lim_{\substack{i \rightarrow \infty \\ j/i \rightarrow r}} \frac{\log \tau(i, j)}{i} = h(r), \quad i, j \in \mathbb{Z}_{>0}. \quad (3)$$

The function h is continuous on $I_{b,c}$. Consequently,

$$f(r) = \lim_{\substack{i \rightarrow \infty \\ j/i \rightarrow r}} \frac{\log \tau(i, j)}{\sqrt{bcij - bi^2 - cj^2}} = \frac{h(r)}{\sqrt{bcr - b - cr^2}} \quad (4)$$

exists and is continuous on $I_{b,c}$.

For $b = c$, the identification at coprime indices in [5, Section 9.3] relates this theorem to Douglas's directional-growth question for Kronecker moduli on the open imaginary-root interval. The proof uses the tight-grading counting theorem [5, Theorem 4.10 and Remark 4.11] and comparisons of the finite sets it counts.

We express $\tau(i, j)$ as a sum of decoration counts on explicit path segments. On a fixed rational ray, a bounded extension produces a designated unselected east edge; substitution at that edge gives a supermultiplicative count and a ray limit. To compare nearby directions uniformly in their denominators, we find long common subsequences of the path words. Two last-in, first-out matchings allow each deleted letter to be repaired by removing at most one additional selection. The inequality $Q > 0$ supplies room to complete the edited decoration in a slightly larger target segment. This comparison transfers the rational-ray limits to arbitrary integer approaches.

2 Tight gradings and canonical frames

2.1 The counting formula

For positive integers D_1, D_2 , let $\mathcal{P}(D_1, D_2)$ be the maximal lattice path from $(0, 0)$ to (D_1, D_2) consisting of east steps **E** and north steps **N** and lying below the line joining its endpoints. Explicitly, after its x th east step it reaches height $\lfloor D_2 x / D_1 \rfloor$ before taking the next east step, for $1 \leq x \leq D_1$. We order its edges cyclically, identifying the endpoint with the initial vertex. A directed interval of edges includes both its first and its last edge and uses at most one circuit. For an interval T , let $E(T)$ and $N(T)$ be the numbers of its east and north edges. For any finite word W over $\{\mathbf{E}, \mathbf{N}\}$, the same notation counts its letters, and $|W| = E(W) + N(W)$ denotes its length. Concatenation of words is written by juxtaposition; W^m is the concatenation of m copies, with W^0 the empty word.

A grading assigns a nonnegative integer weight to each edge. In this paper an east edge has weight 0 or c , and a north edge has weight 0 or b . We call the positive-weight edges selected. A grading is compatible if, for each selected east edge u and selected north edge v , the cyclic interval from u to v contains either a north edge strictly before v at which the number of north edges equals the sum of the east weights since u , or an east edge strictly

after u at which the number of east edges up to v equals the sum of the north weights on that interval. These are the two strict-endpoint conditions of [5, Definition 4.5].

The horizontal shadow of a selected east edge is the set of north edges up to the first equality in the forward condition, now allowing the final north edge. The full interval from that east edge to this first return is its shadow interval, and the east edge is its root. The vertical shadow of a selected north edge is defined by the analogous backward equality and consists of east edges. If the equality is never reached in one circuit, the shadow is the entire set of edges of the opposite type. For a set of selected edges, its shadow is the union of these shadows.

Fix positive integers i, j , and put

$$p = bi, \quad q = cj, \quad g = \gcd(p, q), \quad P = p/g, \quad Q_0 = q/g. \quad (5)$$

The integers P, Q_0 are the dimensions of the primitive path period. Choose positive integers D_1, D_2 such that

$$qD_1 - pD_2 = g, \quad D_1 > p + 1, \quad D_2 > q. \quad (6)$$

Such choices can be made arbitrarily large: the equation is $Q_0D_1 - PD_2 = 1$, and adding (P, Q_0) to a solution preserves it. In particular D_1, D_2 are relatively prime. A compatible grading with j selected east edges and i selected north edges is tight in this sign convention when all selected north edges belong to the horizontal shadow of the selected east edges. The opposite containment corresponds to the opposite determinant sign [5, Definition 4.8 and Remark 4.9].

We use the following published counting formula:

$$\tau(i, j) = \#\{\text{tight gradings on } \mathcal{P}(D_1, D_2) \text{ with } j \text{ selected E and } i \text{ selected N}\}. \quad (7)$$

It is [5, Theorem 4.10 and Remark 4.11]; its value is independent of the choice in (6).

2.2 Frames and their words

The rank of an edge starting at (x, y) is $D_2x - D_1y$. On the maximal path, a vertex of rank less than D_1 is followed by an east step, and a vertex of rank at least D_1 by a north step. The new rank is respectively increased by D_2 or decreased by D_1 . Thus the ranks are successive additions of D_2 modulo $D_1 + D_2$, represented in $[0, D_1 + D_2)$. Since $\gcd(D_2, D_1 + D_2) = 1$, the edge starts have precisely all integer ranks in this interval, once each. In particular, the east edges have ranks $0, \dots, D_1 - 1$.

Lemma 2.1. *Every grading counted by (7) has a unique outer horizontal shadow. Its interval consists of $p + 1$ east edges and q north edges, starts at a selected east edge of rank $\ell \in \{0, \dots, g - 1\}$, and contains every selected edge. For sufficiently large D_1, D_2 , its word W_ℓ is determined by the heights*

$$y_\ell(x) = \left\lfloor \frac{qx}{p} \right\rfloor - \mathbf{1}_{\{P|x, x/P > \ell\}} \quad (1 \leq x \leq p), \quad y_\ell(p + 1) = q. \quad (8)$$

Here $y_\ell(x)$ is the height reached after the x th east edge and all following north edges before the next east edge; the final column stops at height q . The notation $\mathbf{1}$ is 1 when the indicated condition holds and 0 otherwise. Each of the g possible starting ranks occurs once on the ambient path.

Proof. Every horizontal first return exists in one circuit: the total east weight is $q < D_2$, whereas a full circuit has D_2 north edges, each decreasing the forward balance by one. First-return intervals are nested or disjoint, since a return begun inside an unfinished interval must

finish before that interval can return. Their maximal intervals contain all selected north edges by tightness. For such an interval indexed by a , write r_a for the rank of its first east edge and $m_a + 1, n_a$ for its east and north counts. Put $\Delta_a = n_a D_1 - m_a D_2$. The last north edge has rank $r_a + D_1 + D_2 - \Delta_a < D_1 + D_2$, so $0 \leq r_a < \Delta_a$. Its north count is the sum of the east weights in the interval. The maximal intervals partition the selected east edges, so $\sum_a n_a = q$.

Every selected north edge in an interval must satisfy the backward condition with its root, since that root's forward first return has not yet occurred. Its backward shadow therefore stays inside the interval and excludes the root. Backward last-in, first-out matching assigns b distinct east edges to each selected north edge. It follows that $\sum_a m_a \geq p$. Hence

$$0 < \sum_a \Delta_a = g - \left(\sum_a m_a - p \right) D_2 \leq g < D_2.$$

Integrality forces $\sum_a m_a = p$ and $\sum_a \Delta_a = g$. The Bezout identity gives

$$\Delta_a P - m_a = (n_a P - m_a Q_0) D_1.$$

Both terms on the left lie in $[0, p]$, and $p < D_1$, so $m_a = \Delta_a P$ and $n_a = \Delta_a Q_0$. At the east edge $r_a P$ horizontal steps after the root, the local height is

$$\left\lfloor \frac{r_a + r_a P D_2}{D_1} \right\rfloor = r_a Q_0,$$

and its rank is zero. Since $r_a < \Delta_a$, that edge lies inside the interval. Every maximal interval would therefore contain the unique rank-zero edge. There is only one interval, with the asserted counts and $0 \leq \ell < g$. This also recovers the frame structure of [5, Lemmas 4.12–4.13].

Starting at rank ℓ , the local height after x east steps is

$$\left\lfloor \frac{\ell + x D_2}{D_1} \right\rfloor = \left\lfloor \frac{qx}{p} + \frac{\ell - gx/p}{D_1} \right\rfloor.$$

For $1 \leq x \leq p$, the perturbation has absolute value at most g/D_1 . We may choose $D_1 > gP$, so that this is less than $1/P$. If $P \nmid x$, the fractional part of $qx/p = Q_0 x/P$ is a nonzero multiple of $1/P$, and the floor is unchanged. If $x = kP$, the perturbation has sign equal to that of $\ell - k$, which gives (8). Since $\ell \leq g - 1$, the height at p is $q - 1$; the final east step is followed by the north step reaching height q . Every rank in $[0, g)$ is the rank of exactly one east edge. Taking $D_1 > p + 1$ and $D_2 > q$ also ensures that a frame, read across the cyclic cut when necessary, contains no repeated ambient edge. $\square$

The word specified by (8) will be called a canonical frame, independently of the chosen ambient path. It starts with **E** and ends with **EN**. For example, if $p = q = 3$, its three possible words are

$$W_0 = \mathbf{EENENEN}, \quad W_1 = \mathbf{ENEENEN}, \quad W_2 = \mathbf{ENENEEN}.$$

In general let A be the lower lattice-path word with P east and Q_0 north letters, with column heights $\lfloor Q_0 x/P \rfloor$. Write $A = \mathbf{EMN}$, where M is the word obtained by removing its first and last letters and may be empty, and put $B = \mathbf{ENM}$. These two words describe the repeated primitive path blocks. Formula (8) gives

$$W_\ell = A^\ell \mathbf{EMB}^{g-\ell-1} \mathbf{EN} \quad (0 \leq \ell < g). \quad (9)$$

To verify the identity column by column, split the first p east steps into g groups of P . The first ℓ groups have word A . In the next group the last north step is removed, giving $\mathbf{EM}$. At every later group, a north step is inserted immediately after its first east step and its last north step is removed, giving $\mathbf{ENM} = B$. Thus only the columns $x = kP$ with $k > \ell$ end one unit below their usual height. The final $\mathbf{EN}$ restores height q , exactly as in (8).

2.3 Matchings on a frame

We describe the count on a frame by a stack, a list in which additions and removals occur at the last entry. Each selected east edge creates c blue tokens. A north edge consumes one token belonging to the most recent selected east edge whose tokens have not been exhausted. Such an east edge is called active. Each east edge also contributes one red token. A selected north edge consumes the latest b remaining red tokens, and these must all belong to inactive or unselected east edges. The red test occurs before the blue consumption at the same north edge.

The equivalent state description uses one entry (r, z) per active east edge, where r is its remaining blue count and z is the number of free red tokens above it and below the next active east edge, or above it altogether if it is the top entry. An unselected east edge increases the top z by one; a selected east edge pushes $(c, 0)$. An unselected north edge decreases the top r by one. A selected north edge first requires $z \geq b$, replaces z by $z - b$, and then decreases r . When an entry reaches $r = 0$, it is removed and adds $z + 1$ to its parent's red count. The additional token is the token of the east edge just closed.

A decoration is a choice of selected letters of each type. It is *accepted* if all the stated operations are legal, its first east edge is selected, the stack never becomes empty before the last letter, and the final removal leaves red remainder zero at the root. Let $F(W)$ be the number of accepted decorations of a word W .

Proposition 2.2. *For the canonical frames in Lemma 2.1,*

$$\tau(i, j) = \sum_{\ell=0}^{g-1} F(W_\ell). \quad (10)$$

*Each accepted decoration has j selected east edges and i selected north edges. Its penultimate letter **E** is unselected and its last letter **N** is selected.*

Proof. The blue exhaustion time of a selected east edge is exactly its forward first return. Likewise, red matching realizes backward first returns. For a selected pair $u < v$ in the linear frame, either u closes strictly before v , or compatibility requires the backward shadow of v to stop strictly after u . An active east token cannot have been consumed earlier: at its first such consumption, both strict compatibility conditions would fail. Consequently the latter requirement is precisely that the red operation consume only free tokens above the latest active east edge. This explains the order of the two operations at a north edge. For a pair whose east edge comes after its north edge in the linear frame, the east shadow finishes before the end of the frame, and therefore before the cyclic return to that north edge. Such pairs satisfy the forward condition.

The unique outer root in Lemma 2.1 stays active until the last north edge. Every red shadow remains inside it, since consuming or crossing its token would violate compatibility. The total blue supply is $cj = q$, and the total red demand is $bi = p$. Among the $p + 1$ east tokens, only the root token remains at the end. This yields an accepted decoration. Conversely, an accepted decoration is compatible by the preceding test, and its root shadow contains every selected north edge. Token conservation gives

$$c\#\{\text{selected E}\} = N(W), \quad b\#\{\text{selected N}\} = E(W) - 1. \quad (11)$$

It therefore gives a tight grading with the required support sizes. The distinct root ranks distinguish the summands in (10).

If the penultimate east edge were selected, its blue tokens would have to be exhausted by the single remaining north edge, while the original root was still active. This is impossible even for $c = 1$. That east edge is therefore unselected and supplies a free red token. The last north edge must be selected in order to leave remainder zero. $\square$

The interval from a selected east edge to its blue first return is its subtree. If its word is W and it closes with red remainder $s \geq 0$, conservation gives $b\#\{\text{selected N}\} = E(W) - s - 1$; it returns $s + 1$ red tokens to its parent. In particular, an accepted word consumes all its own blue supply and returns one red token, exactly the contribution of an unselected east edge. Replacing such an edge by an accepted word therefore preserves the external states. If the replacement position and length are known, both input decorations are recovered by restriction.

3 Editing forests and completing frames

3.1 Complete compatible forests

A decoration of a finite linear word is a *complete compatible forest* if every selected east edge has its forward first return inside the word, every selected north edge has its backward first return inside the word, and every selected pair $u < v$ satisfies one of the strict compatibility conditions. Blue matching may leave some north edges unused, and red matching may leave some east edges unused. Several roots are allowed. If H_X, V_X are the numbers of selected east and north edges in a forest on X , then

$$N(X) - cH_X \geq 0, \quad E(X) - bV_X \geq 0. \quad (12)$$

For a forest, keep a global red list of the east tokens not yet consumed. A selected north edge removes its last b tokens. At each prefix, the active east edges are those whose blue first returns have not yet occurred; their red tokens are protected. The test from Proposition 2.2 says that every red removal avoids these tokens. When there is no active east edge, all remaining red tokens are free. The *direct red owner* of an east token is the selected north edge that removes it, if any; the *direct blue owner* of a north edge is the selected east edge whose token it consumes, if any. These owners are unique. Reversing a word and interchanging E, N and b, c exchanges its two first-return conditions and preserves compatibility.

Lemma 3.1. *Delete one letter from a complete compatible forest. If the deleted letter is an east edge used by a selected north edge's direct red matching, deselect that north edge. If the deleted letter is a north edge used by a selected east edge's direct blue matching, deselect that east edge. The deleted letter's own selection disappears with it. The resulting decoration is a complete compatible forest.*

Proof. Consider deleting an east edge. If it has a red owner, all earlier red operations avoid its token and remain unchanged. At that owner's position, deselection releases the other $b - 1$ tokens in its demand group. They were free at this position in the original matching and can never become active later. The new red stack now contains the old red stack as an ordered subsequence, with only free extra tokens.

This property persists when both lists receive the same new token. It also persists when both remove their last b tokens. Indeed, any token surviving in the old list had at least b old tokens after it before the removal, hence at least b tokens after it in the new list, and survives there too. The extras are either former extras or tokens removed only from the old list; the latter are free because the old removal was legal. If a protected old token belonged to the last b tokens of the new list, fewer than b old tokens would follow it, contradicting the old red test. There are always at least as many new tokens as old tokens. Thus every retained demand is legal and has all its supplies inside the word. If the deleted token has no red owner, no old red operation reaches it, and all red removals keep their original owners.

If the deleted east edge was selected, its blue demand also disappears. For any other selected east edge, evaluate its forward balance at its original first-return endpoint. Deleting

a demand weakly decreases that balance; it starts positive and decreases only in unit north steps, so its new first zero occurs no later than the old one. All remaining blue returns therefore exist, and their active set at every retained position is a subset of the old active set. This preserves the red legality just established. If the deleted east edge was unselected, the blue matching is unchanged. Reversal and exchange of the two types proves the assertion for deletion of a north edge. $\square$

Lemma 3.2. *Inserting an unselected letter, or deselecting a selected letter without deleting it, preserves complete compatible forests.*

Proof. An inserted unselected east edge adds a free token to the red stack and leaves blue demands unchanged. The ordered-subsequence argument in Lemma 3.1 proves legality and completeness of subsequent red operations. An inserted north edge follows by reversal and exchange. Deselecting an east edge removes its blue demand and can only advance other blue closing times, while keeping all east supplies for the red matching. Deselecting a north edge follows by the same duality. $\square$

Fix a common subsequence of two words. Successively delete the other source letters using Lemma 3.1, then insert the missing target letters unselected. This gives a deterministic map on decorations after fixing the deletion order. If D source letters are deleted, at most D additional retained selections are erased. In particular each support size decreases by at most $2D$.

3.2 Completion in the interior cone

For this subsection let I, J be positive integers with $Q(I, J) > 0$, and use $p = bI$, $q = cJ$ for their frame dimensions. Put $\lambda = q/p$, the slope in path coordinates. The inequality $Q(I, J) > 0$ implies $0 < \lambda < c$ and

$$\frac{1}{\lambda} + \frac{1}{c - \lambda} < b. \quad (13)$$

Indeed, $Q(I, J) > 0$ first gives $J < bI$, hence $0 < \lambda < c$, and

$$b\lambda(c - \lambda) - c = \frac{cQ(I, J)}{bI^2} > 0.$$

Dividing by $\lambda(c - \lambda) > 0$ proves (13). For a fixed anchor ℓ , the height function (8) satisfies $y_\ell(x) \leq \lambda x$ for $1 \leq x \leq p$; set $y_\ell(0) = 0$.

Lemma 3.3. *In every canonical frame of dimensions $bI + 1, cJ$, selecting the first J east edges and the last I north edges gives an accepted decoration.*

Proof. We have

$$q - I - \lambda J = \frac{Q(I, J)}{bI} > 0.$$

Thus all selected east edges occur before the first selected north edge. During the initial selected east edges, the blue balance is positive since $cx - y_\ell(x) \geq (c - \lambda)x > 0$. After the last selected east edge it decreases to zero only at the last north edge, because the total supply is $cJ = q$.

Consider a selected north edge whose endpoint has height $q - t$, where $0 \leq t \leq I - 1$. Let x be the number of east edges before it and s the horizontal index of the current active top edge. The remaining blue supply below that top edge is

$$c(s - 1) - y_\ell(s - 1) \leq t,$$

where $y_\ell(0) = 0$. The inequality holds because the current top requires at least the present north edge and only t north edges follow. Hence

$$s \leq 1 + \frac{t}{c - \lambda}, \quad p - x \leq \frac{t}{\lambda}. \quad (14)$$

For the second bound, if $x \leq p$ use $q - t \leq y_\ell(x) \leq \lambda x$; if $x = p + 1$ it is immediate.

Exactly $I - t - 1$ selected north edges precede the present one. All their red removals lie above the current active edge, which has remained active since it appeared. Thus, assuming the earlier red tests have succeeded, its free red count is

$$z = x - s - b(I - t - 1).$$

The test $z \geq b$ is equivalent to $p - x + s \leq bt$. If $t \geq 1$, (14) and (13) give

$$p - x + s \leq 1 + t \left(\frac{1}{\lambda} + \frac{1}{c - \lambda} \right) < bt + 1.$$

The left side is an integer and hence at most bt . At the final north edge, $t = 0$, $x = p + 1$ and $s = 1$, giving equality. Induction proves all red tests. Conservation leaves only the root token at the last step, so the root remainder is zero. $\square$

Proposition 3.4. *Split a canonical frame for (I, J) into three consecutive words $W = LXR$, where R contains the final EN. Give X a complete compatible forest with support sizes H_X, V_X . If*

$$1 \leq J - H_X \leq E(L), \quad 1 \leq I - V_X \leq N(R), \quad (15)$$

then an accepted decoration of W is obtained by keeping the decoration on X , selecting the first $J - H_X$ east edges of W and the last $I - V_X$ north edges of W , and leaving the other edges of L, R unselected.

Proof. The new east selections all belong to L , and the new north selections all belong to R . During the initial consecutive east selections the blue balance is positive as in Lemma 3.3. At the start of X it is exactly

$$c(J - H_X) - N(L) = N(R) + N(X) - cH_X > 0.$$

After the last selected east edge of L , the blue balance decreases toward this positive value, so the root stays active through L . In the matching internal to X , precisely $N(X) - cH_X$ north edges are unused. During any prefix of X , the number of services taken from the external stack is at most this total. All selected east edges of X close within X , so the external root stays active throughout. Every selected north edge of X already has its entire backward matching inside X , and hence cannot consume an earlier protected token. Its internal legality is unchanged.

On entering R , every remaining active east edge belongs to the initial consecutive selections. The total remaining blue supply is $N(R)$, and no east edge of R is selected, so the root closes exactly at the final north edge. For a selected north edge of R , use t, x, s as in Lemma 3.3; here $0 \leq t < I - V_X$. The $s - 1$ earlier east edges were all selected, and their remaining blue supply is $c(s - 1) - y_\ell(s - 1)$. Every earlier selected north edge occurred after the current active east edge appeared. All its red removals are therefore above that edge. Since exactly $I - t - 1$ north selections precede the present one, the free red count is again $x - s - b(I - t - 1)$. The inequalities and the final-step equality from Lemma 3.3 apply unchanged. All red tests succeed. Total blue supply and red demand then give acceptance by (11). $\square$

Corollary 3.5. For $Q(i, j) > 0$,

$$1 \leq g \leq \tau(i, j) \leq g \binom{bi-1}{j-1} \binom{cj-1}{i-1} \leq g 2^{bi+cj}, \quad g = \gcd(bi, cj). \quad (16)$$

In particular $\log \tau(i, j)/i$ is bounded above uniformly when j/i stays in a compact subset of $I_{b,c}$.

Proof. Lemma 3.3 supplies at least one decoration for each of the g frames. For the upper bound, the first east edge and last north edge are selected, and the penultimate east edge is unselected by Proposition 2.2. Choosing the remaining supports and forgetting compatibility gives the two binomial factors. The final estimate and $g \leq bi + cj$ imply the local bound. $\square$

4 Limits on rational rays

Fix relatively prime positive integers d, e with $Q(d, e) > 0$, and put

$$g_0 = \gcd(bd, ce), \quad P = bd/g_0, \quad Q_0 = ce/g_0.$$

Along $(i, j) = (kd, ke)$, the words $A = \text{EMN}$ and $B = \text{ENM}$ in (9) are fixed and $g = kg_0$. To display the varying number of periods, write $W_\ell(g) = A^\ell \text{EMB}^{g-\ell-1} \text{EN}$. Its *threshold edge* is the east edge immediately following A^ℓ . Let $C_{k,\ell}$ count accepted decorations of $W_\ell(kg_0)$ in which this edge is unselected, and define

$$M_k = \max_{0 \leq \ell < kg_0} C_{k,\ell}.$$

If $\ell = 0$ the threshold is the root, so $C_{k,0} = 0$.

Lemma 4.1. For positive integers k, m , one has $M_{k+m} \geq M_k M_m$.

Proof. If either factor is zero there is nothing to prove. Otherwise choose maximizing anchors $\ell \geq 1$ and $\ell' \geq 1$. Replace the unselected threshold edge of the first decoration by the entire second accepted word. By the state-preserving substitution after Proposition 2.2, the result is accepted. The resulting word is

$$\begin{aligned} A^\ell W_{\ell'}(mg_0) M B^{kg_0-\ell-1} \text{EN} &= A^{\ell+\ell'} \text{EMB}^{mg_0-\ell'-1} \text{EN} M B^{kg_0-\ell-1} \text{EN} \\ &= W_{\ell+\ell'}((k+m)g_0). \end{aligned}$$

Its threshold is the inner threshold, which is unselected. The fixed substitution position and length recover both inputs, giving an injection into the corresponding target count. $\square$

A fixed extension compares M_k with the entire coefficient at a polynomial cost.

Lemma 4.2. There are positive integers h, K , depending only on b, c, d, e , such that for all $k \geq 2$,

$$M_k \leq \tau(kd, ke) \leq \binom{bkd}{h} (k+K)g_0 M_{k+K}. \quad (17)$$

Proof. Set $h = 2 \gcd(P, e)$. Choose a sufficiently large positive solution K of

$$Ke \equiv -h \pmod{P}, \quad KQ(d, e) \geq ceh.$$

The congruence is solvable since $\gcd(P, e)$ divides h , and it has arbitrarily large solutions. Define the numbers of prefix and suffix periods by

$$a = \frac{h + Ke}{P}, \quad t = Kg_0 - a.$$

They are integers and, for large K , satisfy $a \geq 1$, $t \geq 0$ and

$$aP = h + Ke, \quad tQ_0 - Kd = \frac{KQ(d, e) - ceh}{bd} \geq 0. \quad (18)$$

For the second identity, substitute $P = bd/g_0$ and $Q_0 = ce/g_0$ into $tQ_0 = (Kg_0 - (h + Ke)/P)Q_0$. This gives $tQ_0 = Kce - ce(h + Ke)/(bd)$, and subtracting Kd gives (18). Since $Q_0 > 0$, the same inequality gives $t \geq 0$.

Start with an accepted decoration of $W_\ell(kg_0)$. Delete its final east edge and deselect the last north edge, its direct red owner. Delete that last north edge and deselect the original root, its direct blue owner. Lemma 3.1 gives a complete forest on

$$X = A^\ell \text{EMB}^{kg_0 - \ell - 1}.$$

Deselect the original threshold if necessary, and then further east edges in a fixed order until exactly h original east selections have been erased. At most two distinct selections were required to be erased so far, and $2 \leq h \leq 2e \leq ke$, so this can be done. Lemma 3.2 preserves completeness. The resulting supports have sizes $ke - h$ and $kd - 1$.

The extended word is exactly

$$A^a X B^t \text{EN} = W_{\ell+a}((k + K)g_0).$$

Select all $aP = h + Ke$ east edges in the prefix and the last $Kd + 1$ north edges in the suffix. Its suffix has $tQ_0 + 1 \geq Kd + 1$ north edges by (18). Proposition 3.4 gives acceptance. The target threshold is the old threshold and remains unselected.

The target anchor determines the source anchor by subtracting a . Removing the fixed prefix and suffix recovers the edited core. Recording the positions of the h erased east selections recovers the source; its final north selection is restored deterministically. There are at most $\binom{bkd}{h}$ such records. Summing over at most $(k + K)g_0$ target anchors proves the second inequality of (17); the first follows from the definition of M_k . $\square$

Proposition 4.3. *For every relatively prime positive pair (d, e) with $Q(d, e) > 0$, the limit*

$$\lim_{k \rightarrow \infty} \frac{\log \tau(kd, ke)}{k}$$

exists and is finite.

Proof. Corollary 3.5 and Lemma 4.2 give $M_n \geq 1$ for every sufficiently large n , and $\log M_n = O(n)$. Lemma 4.1 makes $\log M_n$ superadditive wherever it is finite. Choose n_0 such that $M_n \geq 1$ for $n \geq n_0$. For a fixed $q \geq n_0$, write $n = mq + r$ with $n_0 \leq r < n_0 + q$. Then

$$\log M_n \geq m \log M_q + \log M_r \geq m \log M_q.$$

It follows that $\liminf_n n^{-1} \log M_n \geq q^{-1} \log M_q$ for every $q \geq n_0$. The supremum of these right sides also bounds $n^{-1} \log M_n$ for every $n \geq n_0$, so the liminf is at least the limsup. Thus $n^{-1} \log M_n$ converges. The logarithm of the factor $\binom{bkd}{h}(k + K)g_0$ in (17) is $O(\log k)$. Dividing that inequality by k gives

$$\frac{\log M_k}{k} \leq \frac{\log \tau(kd, ke)}{k} \leq \frac{k + K}{k} \frac{\log M_{k+K}}{k + K} + O\left(\frac{\log k}{k}\right).$$

Both bounds have the same limit. $\square$

5 Comparing nearby directions

5.1 Mechanical words

A mechanical word with frequency $\alpha \in [0, 1]$ and phase $x \in \mathbb{R}$ has letters given by

$$s_{\alpha, x}(n) = \lfloor x + (n+1)\alpha \rfloor - \lfloor x + n\alpha \rfloor, \quad n \in \mathbb{Z},$$

where 0 means E and 1 means N. A factor is a finite consecutive segment. A common subsequence of two words is obtained by deleting letters from each while preserving the order of those remaining.

Lemma 5.1. *Fix $\alpha_0 \in (0, 1)$. For each $\rho > 0$ there are $\delta > 0$ and L_0 such that any two mechanical factors of the same length $L \geq L_0$, with arbitrary phases and frequencies α, β satisfying*

$$|\alpha - \alpha_0| < \delta, \quad |\beta - \alpha_0| < \delta,$$

have a common subsequence of length at least $(1 - \rho)L$.

Proof. We may assume $0 < \rho < 1$. First suppose α_0 is irrational. Choose $0 < \epsilon < \rho/200$. There are coprime integers a, q with $q > 1/\epsilon$ and $|\alpha_0 - a/q| < 1/q^2$. Indeed, for any positive integer T , placing the $T+1$ fractional parts of $0, \alpha_0, \dots, T\alpha_0$ in T equal intervals gives integers $1 \leq q \leq T$ and a with $|q\alpha_0 - a| < 1/T$. Reducing the fraction preserves the bound $|\alpha_0 - a/q| < 1/q^2$. The reduced denominators are unbounded as $T \rightarrow \infty$, since otherwise the errors could not tend to zero for irrational α_0 .

The first q points of a rotation by a/q form an equally spaced grid, and the corresponding points for α_0 are within $1/q$ of them on $\mathbb{R}/\mathbb{Z}$. Consequently the first q points for α_0 are 2ϵ -dense. Also, in any block of q consecutive points, an arc of length u is visited at most $qu + 4$ times, uniformly in the initial phase: enlarge the arc by $1/q$ at each end and count the grid points.

Put $K = q - 1$ and take a multiple B of q with $B > K/\epsilon$. Every length- B orbit segment visits the two radius- 10ϵ neighborhoods of 0 and $1 - \alpha_0$ at most

$$(40\epsilon q + 8)B/q \leq 48\epsilon B$$

times. Choose $\delta > 0$ so that $(B + K + 1)\delta < \epsilon/10$. Divide the two given factors into aligned blocks of length B . In each pair, the density just proved gives $0 \leq k \leq K$ such that shifting the first phase by $k\alpha$ brings it within 3ϵ of the second. Delete its first k letters and the other block's last k letters. The compared phases stay within 4ϵ for all remaining positions. Each also stays within ϵ of the corresponding α_0 orbit with the same initial phase. A mechanical letter is the indicator of the arc $[1 - \alpha, 1)$ evaluated at its phase, so mismatches can occur only in the two neighborhoods just counted. Deleting the two letters at every mismatch gives a common subsequence with loss at most

$$K + 48\epsilon B < 49\epsilon B$$

per block and at most B for the final incomplete block. Since $49\epsilon < \rho/2$, taking $L_0 > 2B/\rho$ proves the claim.

Now let $\alpha_0 = p_0/q_0$ in lowest terms. Choose an integer $B > 2(5q_0 + 4)/\rho$, then choose $\delta > 0$ with $(B + 1)\delta < 1/q_0$. On a block write

$$\lfloor x + n\alpha \rfloor = \lfloor x + n(\alpha - p_0/q_0) + np_0/q_0 \rfloor.$$

The varying phase $x + n(\alpha - p_0/q_0)$ is monotone and travels less than $1/q_0$. All discontinuities relevant to the q_0 possible residues of np_0/q_0 lie on the grid $q_0^{-1}\mathbb{Z}$. Thus it crosses at most

one grid point. While two consecutive phases stay in the same half-open grid interval, their floors agree with those obtained using any one fixed phase in that interval. Each increment then agrees with a mechanical word of frequency p_0/q_0 . At most two letters adjacent to the crossing need be deleted; this includes an equality at the grid point. If the varying phase is constant, the whole block is already periodic.

Every mechanical word of this rational frequency is periodic with period q_0 , and its phase choices give cyclic shifts of the same word. Indeed, adding p_0/q_0 to the phase shifts the sequence by one letter, and these additions permute the q_0 phase intervals. Remove from both blocks all positions exceptional for either block, at most four in total. At most five paired intervals remain. On each, two cyclic shifts can be aligned by deleting fewer than q_0 initial letters from one and the same number of final letters from the other. If an interval is shorter than the required shift, delete it entirely. The loss from each block is at most $5q_0 + 4 < \rho B/2$. Discarding the final incomplete block costs at most B , so $L_0 > 2B/\rho$ gives the asserted bound. $\square$

Lemma 5.2. *The conclusion of Lemma 5.1 holds for any two equal-length factors of canonical frames whose index slopes lie sufficiently close to a fixed positive slope. The constants are independent of the anchors and the denominators of the slopes. Moreover, for a factor T of a frame with dimensions $p+1, q$,*

$$\left| N(T) - \frac{q}{p+q}|T| \right| \leq 4, \quad \left| E(T) - \frac{p}{p+q}|T| \right| \leq 4. \quad (19)$$

Proof. The lower word $A = \text{EMN}$ in (9) is mechanical with frequency $Q_0/(P+Q_0) = q/(p+q)$. Indeed, its north count after n letters is $\lfloor nQ_0/(P+Q_0) \rfloor$: a vertex (x, y) with $x+y = n$ lies below the diagonal precisely when $(P+Q_0)y \leq Q_0n$, and the maximal path takes the largest allowed integer height. The upper path has cumulative north count $\lceil nQ_0/(P+Q_0) \rceil$. For $0 < n < P+Q_0$ these two counts differ by one, because $\gcd(Q_0, P+Q_0) = 1$. Their successive differences therefore agree except at the first and last letters, proving that the upper word is NME . It too is a mechanical word: the ceiling values are obtained by floors with phase $1 - 1/(P+Q_0)$. Thus $B = \text{ENM}$ is a cyclic shift of the upper mechanical word. Consequently $A^\ell \text{EM}$ and $B^{g-\ell-1}$ are each mechanical factors of that frequency, and the final EN contributes two boundary letters.

A mechanical factor has north count differing from its frequency times its length by less than one, by telescoping the defining floors. The two mechanical pieces and the two boundary letters prove (19). For the subsequence assertion, fix $\rho > 0$ and apply Lemma 5.1 with loss fraction $\rho/2$ and threshold L_0 . Each frame factor splits into at most two mechanical pieces and two exceptional single letters. Taking the union of the two sets of cuts gives at most seven paired intervals. Delete every position exceptional in either factor, at most four, and every remaining interval shorter than L_0 , at total cost at most $7L_0$. On the other intervals apply the mechanical lemma. For total length L , the loss is at most $\rho L/2 + 7L_0 + 4 \leq \rho L$ when $L \geq 2(7L_0 + 4)/\rho$. The frequency associated to index slope $s = j/i$ is $cs/(b+cs)$, so continuity of this function supplies the required slope neighborhood. $\square$

5.2 A comparison after a small dilation

Proposition 5.3. *Fix $r \in I_{b,c}$. For every sufficiently small $\eta > 0$ and every $\varepsilon > 0$, there exist $\delta > 0$ and N_0 such that positive integer pairs $(i, j), (I, J)$ satisfying*

$$\left| \frac{j}{i} - r \right| < \delta, \quad \left| \frac{J}{I} - r \right| < \delta, \quad 1 + \eta \leq \frac{I}{i} \leq 1 + 2\eta, \quad i \geq N_0$$

obey

$$\log \tau(I, J) \geq \log \tau(i, j) - \varepsilon i. \quad (20)$$

Proof. Choose a real number θ with

$$\frac{r}{b} < \theta < 1 - \frac{1}{cr}. \quad (21)$$

This interval is nonempty because $bcr - b - cr^2 > 0$. Choose a source anchor whose count is at least $\tau(i, j)/\gcd(bi, cj)$, and fix any target anchor. In the target word let L consist of the first $\lfloor \theta(b + cJ/I)(I - i) \rfloor$ letters and let X be the next $bi + cj + 1$ letters. Denote the remaining suffix by R .

We take $0 < \eta < 1$ and $0 < \delta \leq 1$, with the slope neighborhood inside $I_{b,c}$. The two slope bounds and $I/i \leq 1 + 2\eta < 3$ give

$$|J - j - r(I - i)| \leq \delta(I + i) \leq 4\delta i.$$

Since $|X| = bi + cj + 1$, the suffix length is

$$\begin{aligned} |R| &= b(I - i) + c(J - j) - \lfloor \theta(b + cJ/I)(I - i) \rfloor \\ &= (1 - \theta)(b + cr)(I - i) + O(\delta i + 1). \end{aligned}$$

Here and below the implicit constants depend only on b, c, r, θ , with $0 < \eta < 1$. Since $I - i \geq \eta i$ and $\theta < 1$, choosing δ sufficiently small relative to η and then increasing i makes $|R| \geq 2$. Thus X exists and R contains the final EN.

The east and north frequencies of the target frame are $b/(b + cJ/I)$ and $(cJ/I)/(b + cJ/I)$. Applying (19) to L and R gives, uniformly in the anchors,

$$E(L) = \frac{b}{b + cJ/I} \lfloor \theta(b + cJ/I)(I - i) \rfloor + O(1) = \theta b(I - i) + O(1), \quad (22)$$

$$N(R) = \frac{cJ/I}{b + cJ/I} |R| + O(1) = cr(1 - \theta)(I - i) + O(\delta i + 1). \quad (23)$$

Given a small $\rho > 0$, Lemma 5.2 supplies a fixed common subsequence of the source word and X with at most

$$D \leq \rho n, \quad n = bi + cj + 1,$$

deleted letters on each side, once the slope neighborhood is small enough and i is large enough. Fix this editing scheme for all source decorations. Lemmas 3.1 and 3.2 turn each one into a complete forest on X with

$$j - 2D \leq H_X \leq j, \quad i - 2D \leq V_X \leq i.$$

Using (22)–(23), we obtain

$$\begin{aligned} E(L) - (J - H_X) &\geq (\theta b - r)(I - i) - O(\delta i + D + 1), \\ N(R) - (I - V_X) &\geq (cr(1 - \theta) - 1)(I - i) - O(\delta i + D + 1). \end{aligned}$$

Both leading coefficients are positive by (21). First choose ρ small enough relative to η and these margins, then shrink δ and increase N_0 . The upper capacity bounds in (15) follow. The lower bounds follow from

$$I - V_X \geq I - i \geq \eta i, \quad J - H_X \geq J - j = r(I - i) + O(\delta i) > 0.$$

Proposition 3.4 now maps every source decoration into an accepted decoration of the fixed target frame.

To bound a fiber of this map, restrict a target decoration to its known middle word X . The fixed subsequence positions recover all retained flags except those additionally deselected

during repair. The deleted source positions have at most 2^D possible flag assignments; at most D retained flags were erased. The number of preimages is at most

$$2^{2D} \sum_{a=0}^{2D} \binom{n}{a} \leq 2^{2D} \left(\frac{n}{2D} \right)^{2D} \exp(2D) = \exp(2D(1 + \log(n/D))) \quad (24)$$

for $1 \leq D \leq \rho n$ and fixed $\rho < 1/4$. Indeed, $0 < 2D/n \leq 1$ and the binomial theorem give

$$\left(\frac{2D}{n} \right)^{2D} \sum_{a=0}^{2D} \binom{n}{a} \leq \sum_{a=0}^n \binom{n}{a} \left(\frac{2D}{n} \right)^a = \left(1 + \frac{2D}{n} \right)^n \leq \exp(2D).$$

If $D = 0$, the fiber bound is 1. Since $\delta \leq 1$, we have $n \leq (b + c(r + 1) + 1)i$. The function $x(1 + \log(1/x))$ is increasing on $(0, 1)$, so the exponent in (24) is at most

$$2(b + c(r + 1) + 1)\rho(1 + \log(1/\rho))i.$$

Choose ρ to make this less than $\varepsilon i/2$ as well as satisfying the capacity margins. The source-anchor loss is $\log \gcd(bi, cj) = O(\log i)$, smaller than $\varepsilon i/2$ for large i . These two losses give (20). $\square$

6 Directional limits

Proof of Theorem 1.1. Fix $r \in I_{b,c}$. Corollary 3.5 gives a finite constant C bounding $\log \tau(i, j)/i$ for all sufficiently large indices with slopes in a fixed compact neighborhood of r . It also bounds the rational-ray limits there.

Given small $\eta, \varepsilon > 0$, take the neighborhood supplied by Proposition 5.3 and choose a rational slope $s = e/d$ in it, with d, e relatively prime and positive. Proposition 4.3 gives

$$h(s) = \lim_{k \rightarrow \infty} \frac{\log \tau(kd, ke)}{kd}.$$

Consider any positive integer sequence (i_n, j_n) with $i_n \rightarrow \infty$ and $j_n/i_n \rightarrow r$. Define positive integers, for all sufficiently large n , by

$$k_n^+ = \left\lceil \frac{(1 + \eta)i_n}{d} \right\rceil, \quad k_n^- = \left\lfloor \frac{i_n}{(1 + \eta)d} \right\rfloor.$$

The ceiling adds less than d to the first coordinate, and the floor subtracts less than d . Since d is fixed and $i_n \rightarrow \infty$, these choices satisfy

$$1 + \eta \leq \frac{k_n^+ d}{i_n} \leq 1 + 2\eta, \quad 1 + \eta \leq \frac{i_n}{k_n^- d} \leq 1 + 2\eta$$

for large n . Apply Proposition 5.3 first to $(i_n, j_n), (k_n^+ d, k_n^+ e)$ and then to $(k_n^- d, k_n^- e), (i_n, j_n)$. After division by i_n , it gives

$$\begin{aligned} \frac{\log \tau(i_n, j_n)}{i_n} &\leq \frac{k_n^+ d}{i_n} \frac{\log \tau(k_n^+ d, k_n^+ e)}{k_n^+ d} + \varepsilon, \\ \frac{\log \tau(i_n, j_n)}{i_n} &\geq \frac{k_n^- d}{i_n} \frac{\log \tau(k_n^- d, k_n^- e)}{k_n^- d} - \varepsilon \frac{k_n^- d}{i_n}. \end{aligned}$$

Both ray quotients tend to $h(s) \geq 0$, so

$$\begin{aligned} \limsup_n \frac{\log \tau(i_n, j_n)}{i_n} &\leq (1 + 2\eta)h(s) + \varepsilon, \\ \liminf_n \frac{\log \tau(i_n, j_n)}{i_n} &\geq \frac{h(s)}{1 + 2\eta} - \varepsilon. \end{aligned}$$

The same s and the same two bounds work for every sequence approaching r . Their difference is at most

$$\left(1 + 2\eta - \frac{1}{1 + 2\eta}\right) C + 2\varepsilon.$$

Letting η, ε tend to zero proves that all such sequences have the same limit $h(r)$. It is finite and nonnegative by Corollary 3.5.

To prove continuity, let $r_n \rightarrow r$ with $r_n \in I_{b,c}$. The directional limit at each r_n permits a choice of integers $i_n \geq n$, $j_n > 0$ with

$$|j_n/i_n - r_n| < 1/n, \quad \left| \frac{\log \tau(i_n, j_n)}{i_n} - h(r_n) \right| < 1/n.$$

Then $j_n/i_n \rightarrow r$, so the already established limit gives $h(r_n) \rightarrow h(r)$. Finally,

$$\frac{\sqrt{Q(i, j)}}{i} = \sqrt{bc(j/i) - b - c(j/i)^2} \rightarrow \sqrt{bcr - b - cr^2} > 0.$$

This proves (4) and its continuity. □

References

- [1] R. Akagi, *Some coefficients of rank 2 cluster scattering diagrams*, arXiv:2507.21916 (2025), <https://arxiv.org/abs/2507.21916>.
- [2] T. Bridgeland, *Scattering diagrams, Hall algebras and stability conditions*, *Algebr. Geom.* **4** (2017), no. 5, 523–561, [doi:10.14231/AG-2017-027](https://doi.org/10.14231/AG-2017-027).
- [3] A. Burcroff, K. Lee, and L. Mou, *Positivity of generalized cluster scattering diagrams*, arXiv:2503.03719 (2025), <https://arxiv.org/abs/2503.03719>.
- [4] A. Burcroff, K. Lee, and L. Mou, *Scattering diagrams, tight gradings, and generalized positivity*, *Proc. Natl. Acad. Sci. USA* **122** (2025), no. 18, e2422893122, [doi:10.1073/pnas.2422893122](https://doi.org/10.1073/pnas.2422893122).
- [5] A. Burcroff, K. Lee, L. Mou, G. Musiker, and M. Reineke, *Cluster scattering diagrams via quiver moduli and tight gradings*, *J. London Math. Soc.* (2) **114** (2026), no. 2, e70642, [doi:10.1112/jlms.70642](https://doi.org/10.1112/jlms.70642). Numbered statements are as in [arXiv:2511.14672v1](https://arxiv.org/abs/2511.14672v1).
- [6] C. Córdova and S.-H. Shao, *Asymptotics of ground state degeneracies in quiver quantum mechanics*, *Commun. Number Theory Phys.* **10** (2016), no. 2, 339–371, <https://arxiv.org/abs/1503.03178>.
- [7] T. Elgin, N. Reading, and S. Stella, *Cluster scattering coefficients in rank 2*, arXiv:2404.11381 (2024), <https://arxiv.org/abs/2404.11381>.
- [8] T. Gräfnitz and P. Luo, *Scattering diagrams: polynomiality and the dense region*, arXiv:2312.13990 (2023), <https://arxiv.org/abs/2312.13990>.
- [9] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras*, *J. Amer. Math. Soc.* **31** (2018), no. 2, 497–608, [doi:10.1090/jams/890](https://doi.org/10.1090/jams/890).
- [10] M. Gross and R. Pandharipande, *Quivers, curves, and the tropical vertex*, *Port. Math.* **67** (2010), no. 2, 211–259, [doi:10.4171/PM/1865](https://doi.org/10.4171/PM/1865).
- [11] M. Gross and B. Siebert, *From real affine geometry to complex geometry*, *Ann. of Math.* (2) **174** (2011), no. 3, 1301–1428, <https://arxiv.org/abs/math/0703822>.

- [12] H. Ju, G. Gao, J. Jiang, B. Wu, Z. Sun, S. Liu, L. Chen, Y. Wang, Y. Wang, Z. Wang, W. He, P. Wu, L. Xiao, R. Liu, B. Dai, and B. Dong, *Automated conjecture resolution with formal verification*, arXiv:2604.03789 (2026), <https://arxiv.org/abs/2604.03789>.
- [13] M. Kontsevich and Y. Soibelman, *Affine structures and non-Archimedean analytic spaces*, in *The Unity of Mathematics*, Progr. Math. **244**, Birkhäuser, Boston, 2006, 321–385, <https://arxiv.org/abs/math/0406564>.
- [14] K. Lee, L. Li, and A. Zelevinsky, *Greedy elements in rank 2 cluster algebras*, Selecta Math. (N.S.) **20** (2014), no. 1, 57–82, doi:10.1007/s00029-012-0115-1.
- [15] T. Mainiero, *Algebraicity and asymptotics: an explosion of BPS indices from algebraic generating series*, arXiv:1606.02693 (2016), <https://arxiv.org/abs/1606.02693>.
- [16] F. Nie, H. Ye, S. Liu, Z. Wang, S. Ermon, and J. Zou, *One-sided verification under limited expert review*, arXiv:2610.18089 (2026), <https://arxiv.org/abs/2610.18089>.
- [17] S. Okada, *On Euler characteristics for large Kronecker quivers*, C. R. Math. Acad. Sci. Paris **350** (2012), nos. 5–6, 273–276, doi:10.1016/j.crma.2012.02.008.
- [18] M. Reineke and T. Weist, *Refined GW/Kronecker correspondence*, Math. Ann. **355** (2013), no. 1, 17–56, doi:10.1007/s00208-012-0778-0.
- [19] T. Weist, *Localization in quiver moduli spaces*, Represent. Theory **17** (2013), 382–425, <https://arxiv.org/abs/0903.5442>.
- [20] T. Weist, *On the Euler characteristic of Kronecker moduli spaces*, J. Algebraic Combin. **38** (2013), no. 3, 567–583, <https://arxiv.org/abs/1203.2740>.

ZHILI COLLEGE, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA
 Email address: tangqy24@mails.tsinghua.edu.cn