

# Infinite divisibility of hitting times for symmetric stable processes

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## Abstract

Let  $T_a$  be the first hitting time of a nonzero point  $a$  by a symmetric strictly  $\alpha$ -stable Lévy process, where  $1 < \alpha < 2$ . We give a computer-assisted classification of its ordinary additive infinite divisibility. There is a unique threshold  $\alpha_*$ , with  $1.6670659903 < \alpha_* < 1.6670659915$ , such that  $T_a$  is infinitely divisible if and only if  $1 < \alpha \leq \alpha_*$ . The threshold is characterized by a unique zero contact of the canonical inverse logarithmic-derivative kernel. The proof combines explicit analytic estimates, rigorous interval covers, Gaussian necessary conditions and Rouché's theorem. In particular, a fully analytic argument proves failure of infinite divisibility for every  $15/8 \leq \alpha < 2$ . Scalar bounds at the two time ends reduce the compact interval computations. The finite computations and their acceptance conditions are supplied as ancillary material.

**Keywords:** stable process; first hitting time; infinite divisibility; Laplace transform; computer-assisted proof.

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## 1 Introduction

Let  $X = (X_t)_{t \geq 0}$  be a symmetric strictly  $\alpha$ -stable Lévy process, with  $X_0 = 0$  and

$$\mathbb{E}[e^{i\xi X_t}] = e^{-t|\xi|^\alpha}, \quad 1 < \alpha < 2.$$

For a nonzero real number  $a$ , let

$$T_a = \inf\{t > 0 : X_t = a\}$$

be its first hitting time of  $a$ . A probability distribution is called *infinitely divisible* if, for every integer  $n \geq 1$ , it is the distribution of a sum of  $n$  independent identically distributed random variables. Throughout this paper the term refers to addition, and is applied to  $T_a$  itself.

For background on stable processes and their potential theory we refer to [1, 11]. Explicit distributional descriptions of stable hitting times are also developed in [3].

The distribution of a point-hitting time of a stable process has attracted considerable attention. Yano, Yano and Yor [9] obtained several distributional identities, including a relation with the  $\alpha$ -Cauchy distribution. Letemplier and Simon [4] proved unimodality for the hitting times of general strictly stable processes. In [5, Proposition 4.1(b)], they also proved infinite divisibility of the logarithm of the hitting time, and conjectured, immediately after that proposition, that the hitting time itself is infinitely divisible.

For the symmetric process, this conjecture is related to a question about the random variable  $C_\alpha$  with density

$$\frac{\alpha \sin(\pi/\alpha)}{2\pi(1 + |x|^\alpha)}, \quad x \in \mathbb{R}.$$

Indeed, if  $\widehat{X}$  is an independent copy of the normalized symmetric  $\alpha$ -stable process, then

$$\widehat{X}_{T_a} \stackrel{d}{=} |a|C_\alpha;$$

see [9, Theorem 5.3, formula (5.12)]. Consequently, infinite divisibility of  $T_a$  implies infinite divisibility of  $C_\alpha$  by subordination. Wang [8, Theorem 1.1] proved that  $C_\alpha$  is infinitely divisible precisely when  $1 < \alpha \leq 2$ . As explained on the second page of that paper, this result is consistent with the hitting-time conjecture. The implication used there is one-way, however, and does not settle infinite divisibility of  $T_a$ .

Here is our main result. The function  $K_\alpha$  appearing in its statement is defined canonically in Lemma 3.2 by inversion of  $-L'_\alpha/L_\alpha$ , where  $L_\alpha(q) = \mathbb{E}[e^{-qT_1}]$ . The quantitative parts of the proof use rigorous ball arithmetic; they are stated separately from the analytic lemmas.

**Theorem 1.1** (Computer-assisted classification). *There is a unique number  $\alpha_* \in (1, 2)$  such that, for every  $a \neq 0$ ,*

$$T_a \text{ is infinitely divisible} \iff 1 < \alpha \leq \alpha_*. \quad (1.1)$$

*More precisely,*

$$1.6670659903 < \alpha_* < 1.6670659915. \quad (1.2)$$

*It is the parameter coordinate of the unique solution of*

$$K_\alpha(t) = 0, \quad \partial_t K_\alpha(t) = 0 \quad (1.3)$$

*in*

$$[1.66706598, 1.667066] \times [.162799, .162801].$$

*For  $1 < \alpha < \alpha_*$ ,  $K_\alpha(t) > 0$  for every  $t > 0$ . For  $\alpha = \alpha_*$ , the kernel is nonnegative and has precisely one zero on  $(0, \infty)$ . In the infinitely divisible range the Lévy measure of  $T_1$  is  $K_\alpha(t)dt/t$ , with zero drift.*

The endpoint  $\alpha = 2$  is distinct: its hitting time has Laplace transform  $e^{-|a|\sqrt{q}}$  and is infinitely divisible. Thus infinite divisibility fails on a whole interval approaching 2 from the left and holds again at the Brownian endpoint. There is no contradiction with closure of infinitely divisible laws under weak limits: that closure theorem imposes no such restriction on limits of laws which are not infinitely divisible.

The proof has three parts. First, a finite large-frequency expansion constructs  $K_\alpha$  without presupposing infinite divisibility, and an exponential-tilting argument proves that infinite divisibility is equivalent to  $K_\alpha \geq 0$ . Second, analytic estimates at zero and infinity, together with a finite cover by interval enclosures, prove strict positivity below a small critical parameter interval. Inside that interval a strictly convex time section has a minimum that decreases strictly through zero; positivity outside the section identifies this contact with the global boundary. Third, a negative kernel value, Gaussian necessary conditions and zeros of the Laplace transform cover the entire range above the contact. A separate analytic Rouché argument treats the whole interval  $[15/8, 2)$ .

The finite enclosures concern whole parameter–time rectangles or whole parameter intervals, rather than samples. Appendix B specifies the certificate format, the finite acceptance conditions and the accompanying reproducibility files. The computational proof uses the guarantees of Arb/FLINT ball arithmetic and the explicit analytic truncation bounds established below. It is not a formal proof-assistant verification. No assertion about the classification of nonsymmetric stable processes is made here.

## 2 Normalization and the Laplace transform

Let  $X = (X_t)_{t \geq 0}$  be the symmetric strictly  $\alpha$ -stable process starting from zero, normalized by

$$\mathbb{E}[e^{i\xi X_t}] = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}, \quad t \geq 0, \quad (2.1)$$

where  $1 < \alpha < 2$ . For  $a \neq 0$ , put

$$T_a = \inf\{t > 0 : X_t = a\}, \quad T = T_1, \quad L_\alpha(q) = \mathbb{E}[e^{-qT}].$$

Write  $\beta = 1/\alpha$  and  $\theta = \pi\alpha/2$ . We use the principal branch for all complex powers.

**Lemma 2.1.** *The random variable  $T_a$  is finite almost surely and*

$$T_a \stackrel{d}{=} |a|^\alpha T. \quad (2.2)$$

For  $q > 0$ , define

$$u_q(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos(x\xi)}{q + \xi^\alpha} d\xi.$$

Then

$$\mathbb{E}[e^{-qT_a}] = \frac{u_q(a)}{u_q(0)}, \quad u_q(0) = \frac{q^{\beta-1}}{\alpha \sin(\pi\beta)}. \quad (2.3)$$

*Proof.* The resolvent identity is the standard hitting-time formula; in this normalization it is stated in [9, formula (5.10)]. The value of  $u_q(0)$  follows by substituting  $\xi = q^\beta v$  and using  $\int_0^\infty (1 + v^\alpha)^{-1} dv = \pi/(\alpha \sin(\pi/\alpha))$ . Moreover,

$$0 \leq u_q(0) - u_q(a) \leq \frac{1}{\pi} \int_0^\infty \frac{1 - \cos(a\xi)}{\xi^\alpha} d\xi < \infty.$$

Since  $u_q(0) \rightarrow \infty$  as  $q \downarrow 0$ , the ratio in (2.3) tends to one. This proves almost sure finiteness. Self-similarity and symmetry give (2.2); see also [9, formula (5.16)].  $\square$

The normalization entails no loss of generality. If the characteristic exponent is  $c|\xi|^\alpha$  with  $c > 0$ , the corresponding hitting time has law  $|a|^\alpha T/c$ . Multiplication by a positive constant preserves infinite divisibility in both directions.

**Lemma 2.2.** *For  $\operatorname{Re} q > 0$ ,*

$$L_\alpha(q) = \frac{\alpha \sin \theta \sin(\pi\beta)}{\pi} \int_0^\infty \frac{e^{-q^\beta r} r^\alpha}{1 + 2 \cos \theta r^\alpha + r^{2\alpha}} dr. \quad (2.4)$$

*Proof.* First suppose that  $q > 0$ . Rotation through the first quadrant gives

$$\int_0^\infty \frac{e^{ix}}{q + x^\alpha} dx = i \int_0^\infty \frac{e^{-y}}{q + e^{i\theta} y^\alpha} dy.$$

There is no pole in this quadrant because  $\theta < \pi$ . The small arc vanishes as its radius tends to zero, and the large arc vanishes because  $\alpha > 1$ . Taking real parts yields

$$u_q(1) = \frac{\sin \theta}{\pi} \int_0^\infty \frac{e^{-y} y^\alpha}{q^2 + 2q \cos \theta y^\alpha + y^{2\alpha}} dy.$$

Substitute  $y = q^\beta r$  and divide by  $u_q(0)$ . Both sides of (2.4) are holomorphic in  $\operatorname{Re} q > 0$ , so the identity extends there by analytic continuation.  $\square$

Formula (2.4) is also obtained from the independent-product representation in [4, Section 2.4 and final remark (a)]. The contour argument above fixes all normalizing constants directly. The formula is for the transform  $L_\alpha$ ; it does not assert that this transform is zero-free in the right half-plane.

### 3 The canonical logarithmic-derivative kernel

Throughout the paper, powers and logarithms of complex numbers have their principal values. Put

$$\beta = \alpha^{-1}, \quad \delta = \pi(1 - \alpha/2), \quad m = \alpha + 1, \quad D_\delta(z) = 1 - 2 \cos(\delta)z + z^2, \quad \kappa_\alpha = \frac{\alpha \sin \delta \sin(\pi/\alpha)}{\pi}. \quad (3.1)$$

Thus  $0 < \delta < \pi/2$ . The transform obtained in Lemma 2.2 is

$$L_\alpha(q) = \kappa_\alpha \int_0^\infty \frac{e^{-q^\beta r} r^\alpha}{D_\delta(r^\alpha)} dr, \quad \Re q > 0. \quad (3.2)$$

The integral converges locally uniformly in this half-plane. In particular, it defines the analytic continuation of the real Laplace transform. Write  $F_\alpha = -L'_\alpha/L_\alpha$  wherever the denominator is nonzero. We first define its inverse on a sufficiently far right vertical line; no global assertion that  $L_\alpha$  is zero-free is implicit in this definition.

**Lemma 3.1** (Finite expansion with an explicit error). *Let  $N \geq 2$ ,  $Y > 0$ , and set*

$$c_n = \frac{\sin((n+1)\delta)}{\sin \delta}, \quad A_n = c_n \frac{\Gamma(m+n\alpha)}{\Gamma(m)}, \quad (3.3)$$

$$d_0 = 1 + \beta, \quad d_n = nA_n - \sum_{j=1}^{n-1} A_j d_{n-j} \quad (1 \leq n < N). \quad (3.4)$$

Put  $\mu = 2^{-1/2}$ ,  $\eta = \sin(\delta/4) \sin(3\delta/4)$  and

$$C_A = \frac{\max(|c_N|, |c_{N-1}|)}{\eta} \frac{\Gamma(m+N\alpha)}{\Gamma(m) \mu^{m+N\alpha}}, \quad (3.5)$$

$$C_H = \frac{\max(|c_N|, |c_{N-1}|)}{\eta} \frac{\Gamma(m+N\alpha+1)}{\Gamma(m) \mu^{m+N\alpha+1}}, \quad (3.6)$$

$$\lambda_Y = 1 - \sum_{n=1}^{N-1} |A_n| Y^{-n} - C_A Y^{-N}. \quad (3.7)$$

If  $\lambda_Y > 0$ , then  $L_\alpha(q) \neq 0$  for  $\Re q \geq 0$ ,  $|q| \geq Y$ , and

$$\left| F_\alpha(q) - \sum_{n=0}^{N-1} d_n q^{-n-1} \right| \leq C_F |q|^{-N-1}, \quad (3.8)$$

where a valid constant is

$$C_F = \frac{1}{\lambda_Y} \left\{ \sum_{n=N}^{2N-2} \left| \sum_{\substack{0 \leq j < N \\ 0 \leq n-j < N}} A_j d_{n-j} \right| Y^{N-n} + \beta C_H + C_A \sum_{n=0}^{N-1} |d_n| Y^{-n} \right\}. \quad (3.9)$$

All these bounds are uniform when their right-hand sides are evaluated by interval arithmetic on a closed parameter interval in  $(1, 2)$ .

*Proof.* The recurrence  $c_0 = 1$ ,  $c_1 = 2 \cos \delta$ ,  $c_n = 2 \cos \delta c_{n-1} - c_{n-2}$  gives the finite identity

$$\frac{1}{D_\delta(z)} = \sum_{n=0}^{N-1} c_n z^n + \frac{z^N (c_N - c_{N-1} z)}{D_\delta(z)}. \quad (3.10)$$

In (3.2), make the substitution  $w = q^\beta r$  and rotate the  $w$  ray to  $\arg w = (\arg q)/2$ . This rotation crosses no pole: along it the argument of  $w^\alpha/q$  stays between zero and  $(\alpha/2 - 1)\arg q$ , whose modulus is at most  $\delta/2$ . On the final ray  $\Re w \geq \mu|w|$ , and, for  $z = w^\alpha/q$ ,

$$|D_\delta(z)| = |1 - e^{i\delta}z| |1 - e^{-i\delta}z| \geq \eta(1 + |z|)^2. \quad (3.11)$$

Indeed  $|1 - xe^{i\theta}| \geq (1+x)|\sin(\theta/2)|$  for  $x \geq 0$ ; the product of the two sine factors is minimized at  $|\arg z| = \delta/2$ . The small circular arc vanishes because  $m > 0$ ; the large arc vanishes by exponential decay. These observations justify the rotation, including its limiting values on  $\Re q = 0$ .

Normalize the transform and its first moment as follows:

$$A(q) = \frac{L_\alpha(q)}{\kappa_\alpha \Gamma(m) q^{-1-\beta}} = \frac{1}{\Gamma(m)} \int_{\arg w = (\arg q)/2} \frac{e^{-w} w^\alpha}{D_\delta(w^\alpha/q)} dw, \quad (3.12)$$

$$H(q) = \frac{1}{\Gamma(m)} \int_{\arg w = (\arg q)/2} \frac{e^{-w} w^{\alpha+1}}{D_\delta(w^\alpha/q)} dw. \quad (3.13)$$

Then  $F_\alpha(q) = \beta H(q)/(qA(q))$ . Equation (3.10) and (3.11) yield

$$A(q) = \sum_{n=0}^{N-1} A_n q^{-n} + E_A(q), \quad H(q) = \sum_{n=0}^{N-1} (m + n\alpha) A_n q^{-n} + E_H(q), \quad (3.14)$$

with  $|E_A(q)| \leq C_A |q|^{-N}$  and  $|E_H(q)| \leq C_H |q|^{-N}$ . For example, the first error follows by integrating

$$\frac{\max(|c_N|, |c_{N-1}|)}{\eta \Gamma(m)} e^{-\mu u} u^{\alpha+N\alpha} |q|^{-N}$$

over  $u > 0$ . Thus  $|A(q)| \geq \lambda_Y > 0$ . Multiplying the polynomial in (3.4) by the polynomial for  $A$  cancels all powers of  $q^{-1}$  of degree less than  $N$  in  $\beta H - A \sum_{n < N} d_n q^{-n}$ . Retaining every higher-degree product and then dividing by  $qA$  gives (3.9). This proof uses only a finite algebraic identity. In particular, no convergence of the infinite asymptotic expansion is asserted.  $\square$

**Lemma 3.2** (Existence and uniqueness of the canonical kernel). *For each  $1 < \alpha < 2$  there is a unique real continuous function  $K_\alpha : [0, \infty) \rightarrow \mathbb{R}$ , of exponential order, such that*

$$F_\alpha(q) = \int_0^\infty e^{-qt} K_\alpha(t) dt \quad (3.15)$$

for all sufficiently large real  $q$ . It is smooth on  $[0, \infty)$  and

$$K_\alpha^{(n)}(0) = d_n, \quad n \geq 0. \quad (3.16)$$

Its time derivatives and its first real parameter derivative are continuous locally in  $(\alpha, t) \in (1, 2) \times [0, \infty)$ .

*Proof.* Fix a compact parameter interval  $I \subset (1, 2)$ . Lemma 3.1, with  $N = 2$  and  $c$  sufficiently large, gives  $\lambda_c > 0$  uniformly on  $I$ . Hence  $F_\alpha$  is holomorphic on  $\Re q \geq c$  there. The same lemma supplies its finite expansion to every order in this half-plane, uniformly on  $I$ . For real parameter differentiation on  $q = c \pm iy$ , use the fixed rays  $w = ue^{\pm i\pi/4}$  when  $|y|$  is sufficiently large. Then  $|\arg(w^\alpha/q)| < \delta/2$ , uniformly on  $I$ , and (3.11) still applies. Differentiating the finite remainder introduces only  $\log w$ . The bound  $|\log w| \leq u + u^{-1} + \pi/4$  gives an integrable majorant against  $e^{-u/\sqrt{2}}$ , so the finite expansion can also be differentiated uniformly on  $I$ . Explicit constants are given in Lemma A.1. On bounded parts of the vertical line, differentiation under (3.2) is immediate.

Fix  $h > 0$ . For  $n \geq 1$  define

$$R_n(q) = \sum_{k=1}^n \frac{b_k}{(q+h)^k}, \quad b_k = \sum_{j=0}^{k-1} \binom{k-1}{j} h^{k-1-j} d_j, \quad r_n(t) = e^{-ht} \sum_{k=1}^n \frac{b_k t^{k-1}}{(k-1)!}. \quad (3.17)$$

These coefficients give  $F_\alpha - R_n = O(|q|^{-n-1})$  and  $r_n^{(j)}(0) = d_j$  for  $j < n$ . Set

$$K_\alpha(t) = r_n(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(c+iy)t} (F_\alpha(c+iy) - R_n(c+iy)) dy. \quad (3.18)$$

Absolute convergence, the elementary inverse transform of  $R_n - R_1$ , and contour deformation show that this expression is independent of  $n$  and of the sufficiently large  $c$ . Conjugation makes it real. For  $n > j$ , differentiation  $j$  times in  $t$  is justified by an integrable bound, uniformly on compact time intervals. At  $t = 0$ , close the integral of  $q^j (F_\alpha - R_n)$  to the right: there is no pole, and the arc integral tends to zero. Thus  $K_\alpha^{(j)}(0) = r_n^{(j)}(0) = d_j$ . The same construction gives an exponential bound for each derivative.

For real  $q > c$ , Fubini's theorem followed by Cauchy's formula shows that the Laplace transform of the integral in (3.18) is  $F_\alpha(q) - R_n(q)$ . Adding the transform of  $r_n$  proves (3.15). Uniqueness follows from uniqueness of Laplace transforms after exponential weighting. Finally, the differentiated remainder bounds above permit parameter differentiation of (3.18) and prove the asserted local continuity.  $\square$

**Lemma 3.3** (A sign criterion for additive infinite divisibility). *The distribution of  $T_1$  is infinitely divisible if and only if  $K_\alpha(t) \geq 0$  for every  $t > 0$ . In that event its drift is zero and its Lévy measure is*

$$\nu_\alpha(dt) = \frac{K_\alpha(t)}{t} dt. \quad (3.19)$$

*Proof.* For a nonnegative infinitely divisible random variable, the Lévy–Khintchine representation (see [6, 7]) and its derivative are

$$-\log L(q) = dq + \int_0^\infty (1 - e^{-qt}) \nu(dt), \quad -\frac{L'(q)}{L(q)} = d + \int_0^\infty t e^{-qt} \nu(dt), \quad (3.20)$$

where  $d \geq 0$  and  $\int (1 \wedge t) \nu(dt) < \infty$ . Since  $F_\alpha(q) \rightarrow 0$  as  $q \rightarrow \infty$ ,  $d = 0$ . Uniqueness of Laplace transforms after exponential weighting gives  $t\nu(dt) = K_\alpha(t)dt$ . Continuity implies  $K_\alpha \geq 0$ .

Conversely, suppose  $K_\alpha \geq 0$  and choose  $c > 0$  larger than its exponential growth rate and the threshold in (3.15). Then  $\nu_c(dt) = e^{-ct} K_\alpha(t)dt/t$  is a Lévy measure. Integrating (3.15) from  $c$  to  $c + s$  gives

$$\frac{L_\alpha(c+s)}{L_\alpha(c)} = \exp \left\{ - \int_0^\infty (1 - e^{-st}) \nu_c(dt) \right\}. \quad (3.21)$$

Thus the actual tilted probability law  $P_c(dt) = e^{-ct} \mathbb{P}(T_1 \in dt) / L_\alpha(c)$  is infinitely divisible. For each  $n \geq 1$ , let  $Q_{n,c}$  be its convolution root with Lévy measure  $\nu_c/n$  and zero drift, so  $Q_{n,c}^{*n} = P_c$ . Tonelli's theorem gives

$$\left( \int_0^\infty e^{ct} Q_{n,c}(dt) \right)^n = \int_0^\infty e^{ct} P_c(dt) = L_\alpha(c)^{-1} < \infty. \quad (3.22)$$

Consequently  $Q_n(dt) = L_\alpha(c)^{1/n} e^{ct} Q_{n,c}(dt)$  is a probability law, and multiplication of the exponential weights yields

$$Q_n^{*n}(dt) = L_\alpha(c) e^{ct} P_c(dt) = \mathbb{P}(T_1 \in dt).$$

This proves infinite divisibility directly from its definition. The already established necessary direction then identifies its Lévy measure as (3.19), including the required integrability.  $\square$

*Remark 3.4.* The first two coefficients are

$$K_\alpha(0) = 1 + \alpha^{-1}, \quad K'_\alpha(0) = 2 \cos \delta \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} > 0.$$

A law is self-decomposable if, for every  $b \in (0, 1)$ , it admits a representation  $Z \stackrel{d}{=} bZ_1 + Y_b$ , with  $Z_1$  distributed as  $Z$  and independent of  $Y_b$ . For such a law on  $[0, \infty)$  the Lévy density has the form  $k(t)/t$  with  $k$  nonincreasing [6, Theorem 15.10]. Uniqueness in Lemma 3.3 rules out self-decomposability whenever the law is infinitely divisible; outside that range it is ruled out a fortiori. Thus this argument also recovers the result of Wang and Yin [10, Corollary 1.2]: non-self-decomposability for every  $1 < \alpha < 2$ . It does not apply at  $\alpha = 2$ , where the limiting normalization in (3.2) degenerates.

## 4 Positivity up to the critical index

We retain the notation

$$\beta = \alpha^{-1}, \quad \delta = \pi(1 - \alpha/2), \quad D_\delta(z) = 1 - 2 \cos(\delta)z + z^2, \quad d_0 = 1 + \beta.$$

In this section  $K_\alpha$  is the canonical, initially signed inverse Laplace transform of  $F_\alpha = -L'_\alpha/L_\alpha$ . Its definition does not presuppose infinite divisibility. We prove

$$K_\alpha(t) > 0 \quad (1 < \alpha < \alpha_*, t > 0), \quad K_{\alpha_*}(t) \geq 0 \quad (t > 0), \quad (4.1)$$

and show that the last kernel has precisely one zero. Every decimal endpoint of a parameter or time interval below is an exact rational number. The finite inequalities used in the proof, together with their exact interval records and verification programs, are indexed in Section B. The analytic reductions and the quantities tested by those programs are given explicitly here.

### 4.1 The cut and a common zero-free region

Put

$$\kappa_\alpha = \frac{\alpha \sin \delta \sin(\pi\beta)}{\pi}, \quad \eta_\alpha = \frac{\alpha \sin \delta}{\pi}, \quad M_\alpha(q) = \int_0^\infty e^{-qr} \frac{r^\alpha}{D_\delta(r^\alpha)} dr.$$

For  $\alpha > 1$ ,  $L_\alpha = \kappa_\alpha M_\alpha$ , so the logarithmic derivative can be computed using  $M_\alpha$ . The latter family, unlike its vanishing normalization factor, remains meaningful at  $\alpha = 1$ :

$$M_1(q) = \int_0^\infty e^{-qr} \frac{r}{1 + r^2} dr. \quad (4.2)$$

The endpoint  $\alpha = 1$  used in the estimates below always refers to this auxiliary analytic family, and never to a Cauchy point-hitting law. Its kernel  $K_1$  is defined by the same regularized Bromwich construction as in Lemma 3.2, with  $M_1$  in place of  $L_\alpha$ .

**Lemma 4.1** (Cut phase). *For  $1 \leq \alpha < 2$ , the upper-bank continuation of  $M_\alpha$  has no zeros on  $-1 \leq q < 0$ . There is a continuous phase  $\theta_\alpha(v)$ , normalized by  $\theta_\alpha(0+) = 0$ , such that*

$$-\pi\beta < \theta_\alpha(v) < 0 \quad (0 < v \leq 1).$$

Consequently

$$w_\alpha(v) = -\frac{1}{\pi} \operatorname{Im} F_\alpha(-v + i0) > 0, \quad \int_0^b w_\alpha(v) dv = -\frac{\theta_\alpha(b)}{\pi} < \beta \quad (0 < b \leq 1).$$

*Proof.* Let  $x = v^\beta$  and

$$H_\alpha(x) = \int_0^\infty e^{-xu} \frac{u^\alpha}{1 + 2 \cos \delta u^\alpha + u^{2\alpha}} du.$$

Rotating the integration ray by  $-\pi\beta$  crosses the lower simple pole  $p = \exp[-i\pi(\beta - 1/2)]$ . The integral on the new ray and that residue give the exact identity

$$M_\alpha(-v + i0) = \eta_\alpha^{-1} e^{-i\pi\beta} N_\alpha(x), \quad N_\alpha(x) = ie^{-ix} - \eta_\alpha H_\alpha(x). \quad (4.3)$$

In particular  $\text{Im } N_\alpha(x) = \cos x > 0$  for  $0 < x \leq 1$ . The density under the integral defining  $H_\alpha$  is at most  $1/2$ , because  $\cos \delta \geq 0$  and  $1 + u^{2\alpha} \geq 2u^\alpha$ . Thus  $xH_\alpha(x) \leq 1/2$ , whereas  $H'_\alpha(x) < 0$ . Direct differentiation gives

$$w_\alpha(v) = \frac{\beta v^{\beta-1}}{\pi} \frac{1 - \eta_\alpha \{H_\alpha(x) \sin x + H'_\alpha(x) \cos x\}}{|N_\alpha(x)|^2} > 0, \quad (4.4)$$

since  $\eta_\alpha < 1$  and the numerator is greater than  $1/2$ .

The phase  $\arg N_\alpha(v^\beta) - \pi\beta$  lies in  $(-\pi\beta, \pi(1 - \beta))$ , and its derivative is  $-\pi w_\alpha$ . For  $\alpha > 1$ , its limit at zero is zero because  $M_\alpha(0) = 1/\kappa_\alpha > 0$ . At  $\alpha = 1$ , the same limit follows from  $M_1(q) = -\log q + O(1)$  near the origin. Its strict decrease proves the asserted phase interval and, by integration, the mass identity.  $\square$

**Lemma 4.2** (Zero-free slit region). *For  $1 \leq \alpha \leq 1.6671$ ,  $M_\alpha$  has no zeros in*

$$\{q : \text{Re } q \geq -1\} \setminus (-\infty, 0],$$

*and its boundary values on the cut segment  $[-1, 0)$  are nonzero.*

*Proof.* We describe separately the origin, the exterior and the finite vertical boundary; their argument changes, rather than boundary nonvanishing alone, are needed to exclude interior zeros.

For  $1 \leq \alpha \leq 6/5$ , write  $\varepsilon = \alpha - 1$ ,  $s = q^\beta$ , and  $r = |s|$ . Subtraction of the leading  $u^{-\alpha}$  tail gives, for  $\varepsilon > 0$ ,

$$M_\alpha(q) = M_\alpha(0) - \frac{\Gamma(1 - \varepsilon)}{\varepsilon} s^\varepsilon + R_\alpha(s), \quad |R_\alpha(s)| \leq 60|s|, \quad |R'_\alpha(s)| \leq 60.$$

The two divergent terms have their removable logarithmic limit when  $\varepsilon = 0$ . To obtain these bounds on both banks, use the rays  $-\pi/4$  and  $-3\pi/4$ . Their denominators exceed  $(1 + u^\alpha)^2/10$ ; the subtracted first moment is less than  $10(5/4 + 1/3 + 1 + \pi/2) < 42$ , and the crossed-pole contribution and its derivative are less than 12. The exact inequality  $\varepsilon M_\alpha(0) \geq 1$  and  $\Gamma(1 - \varepsilon) \leq e^{2\varepsilon}$  now imply, with  $\ell = \log(1/r) - 2$ ,

$$\text{Re } M_\alpha(q) \geq \frac{1 - e^{-\varepsilon\ell}}{\varepsilon} - 60r > 2 \quad (|q| \leq 10^{-4}).$$

Here the quotient has its continuous value at  $\varepsilon = 0$ ;  $\ell > 4$ , and its lower bound  $20/9$  suffices. These same estimates yield  $qF_\alpha(q) \rightarrow 0$  uniformly at this auxiliary endpoint. For  $6/5 \leq \alpha \leq 13/10$ , the corresponding origin estimate is

$$|L_\alpha(q) - 1| < 20|q|^{1/6} < 1 \quad (|q| \leq 10^{-8}).$$

For  $13/10 \leq \alpha \leq 1.6671$ , the finite parameter records give  $C_\alpha^{(0)}(10^{-12})^{1-\beta} < 1/2$ , where  $|L_\alpha(q) - 1| \leq C_\alpha^{(0)}|q|^{1-\beta}$ . Here the constant is completely explicit. With

$$\gamma = \frac{3\pi\beta}{4}, \quad \Delta = \sin \frac{\alpha\gamma - \delta}{2} \sin \frac{\alpha\gamma + \delta}{2}, \quad V_\alpha = 2^{2-\alpha} \left( \frac{1}{2-\alpha} + \frac{1}{\alpha-1} \right),$$

one may take

$$C_\alpha^{(0)} = \max \left\{ \sin(\pi\beta) + \frac{\kappa_\alpha V_\alpha}{\Delta}, \frac{\kappa_\alpha V_\alpha}{\sin^2(\delta/2)} \right\}.$$

These last two estimates follow by integrating  $\min(|s|u, 2)u^{-\alpha}$  on a decaying ray and adding the explicit pole difference. Thus an origin circle has its image in a right half-plane.

For  $|q| \geq 400$  and  $|\arg q| \leq \pi/2 + 1/20$ , normalize by  $A_\alpha(q) = q^{d_0} M_\alpha(q)/\Gamma(\alpha + 1)$ . A finite geometric expansion of  $D_\delta^{-1}$ , with its integrated remainder, gives

$$|A_\alpha(q) - 1| < 1/4 \quad (1 \leq \alpha \leq 13/10), \quad |A_\alpha(q) - 1| < 1/2 \quad (13/10 \leq \alpha \leq 1.6671),$$

and  $F_\alpha(q) - d_0/q = O(|q|^{-2})$ , uniformly on each parameter range. The first range has the explicit bound  $1000/|q|^2$ . For the last range the normalized ray  $\arg w = \arg q/2$ ,  $w = q^\beta r$ , has exponential cosine at least  $2/3$  and denominator constant  $\sin(\delta/5) \sin(3\delta/5)$ . No pole is crossed. The finite expansion, including its full quotient remainder, supplies the stated exterior inequalities.

On  $q = -1 + iy$ ,  $0 \leq y \leq 400$ , the verified boundary conditions are

| parameter range   | parameter boxes | boundary condition                                                     |
|-------------------|-----------------|------------------------------------------------------------------------|
| $[1, 6/5]$        | 20              | $\operatorname{Re} M_\alpha < 0$                                       |
| $[6/5, 13/10]$    | 100             | $\operatorname{Re} L_\alpha < 0$                                       |
| $[13/10, 1.6671]$ | 74              | $\operatorname{Re} L_\alpha < 0$ or $\operatorname{Im} L_\alpha < 0$ . |

For clarity, these are conditions on entire closed parameter- $y$  rectangles. If  $b_j$  is the lower bound for the negative component at  $y_j$ , and  $B_j$  bounds  $|L'_\alpha|$  on  $[y_j, y_{j+1}]$ , the checked inequality is  $b_j - (y_{j+1} - y_j)B_j > 0$ ; use  $M$  in the first row. The bounds  $B_j$  include the residue derivative and the full ray-integral derivative. Exact rational adjacency gives a cover of all of  $[0, 400]$ . For example, for the third row, put  $r = (1 + y^2)^{1/2}$ ,  $p = e^{-i\delta/\alpha}$ , and use the  $\Delta$  above. A bound at the left endpoint, valid on the whole subsequent segment, is

$$B(y) = \beta \sin(\pi\beta) r^{\beta-1} e^{-\Re(p(-1+iy)^\beta)} + \frac{\beta\kappa_\alpha}{\Delta} \min \left\{ \frac{\Gamma(2-\alpha)}{(3/4)^{2-\alpha}} r^{-\beta}, \frac{\Gamma(\alpha+2)}{(3/4)^{\alpha+2}} r^{-2-\beta} \right\}.$$

This follows by differentiating the residue and the rotated integral, using  $\Re(se^{-i\gamma}) \geq (3/4)|s|$ . The two powers are the bounds obtained respectively from the large- and small-radial-power estimates for the denominator. Both decrease with  $y$ . The residue bound also decreases:  $r^{\beta-1}$  decreases, whereas

$$\frac{d}{dy} \Re(p(-1+iy)^\beta) = \beta r^{\beta-1} \cos((1-\beta)(\pi - \arg(-1+iy))) > 0.$$

Thus this formula supplies the required whole-segment  $B_j$ , including parameter interval rounding.

All three rows place the vertical image in  $\{z : \Re z < 0 \text{ or } \Im z < 0\}$ , which admits a continuous argument in  $(-3\pi/2, 0)$ . At the cut endpoint this argument agrees with the phase in Lemma 4.1, which lies in  $(-\pi, 0)$ . At  $q = -1 + 400i$  it agrees with  $-d_0 \arg q + \arg A_\alpha(q)$ : indeed  $3/2 < d_0 \leq 2$ ,  $\pi/2 < \arg q < \pi/2 + 1/20$  and  $|\arg A_\alpha| < \pi/6$  give

$$-\frac{3\pi}{2} < -\frac{7\pi}{6} - \frac{1}{10} < -d_0 \arg q + \arg A_\alpha(q) < -\frac{7\pi}{12} < 0.$$

The exterior arc continues this phase to zero on the positive axis. The positive axis and the small origin arc, whose image lies in a right half-plane, complete a contour with zero total argument change. The argument principle excludes zeros in the bounded upper slit region. Conjugation and the exterior estimate cover the remaining parts of the stated domain.  $\square$

## 4.2 Control of the two time ends

**Lemma 4.3** (A signed cut representation). *Throughout the range of Lemma 4.2,*

$$K_\alpha(t) = d_0 e^{-2t} + \int_0^1 e^{-vt} w_\alpha(v) dv + E_\alpha(t), \quad |E_\alpha(t)| \leq B_\alpha e^{-t}. \quad (4.5)$$

*One can take  $B_\alpha < 94, 37, 26$  on the respective three parameter ranges in the preceding table. In particular the signed kernel is bounded on all positive times, before any positivity conclusion is made.*

*Proof.* Subtract  $d_0/(q+2)$  from  $F_\alpha$ , and deform its inverse contour to  $\operatorname{Re} q = -1$ , detouring around the cut. The exterior bound is integrable and eliminates the horizontal ends. The origin bound  $qF_\alpha(q) \rightarrow 0$  eliminates the small circle, also for the logarithmic auxiliary endpoint. Lemma 4.1 gives the jump and its finite mass. The pole of the subtracted model is at  $-2$ , outside the deformed domain.

On every finite vertical segment the preceding margin  $\ell_j > 0$  gives  $|L| \geq \ell_j$ , hence

$$B_\alpha \leq \frac{1}{\pi} \left\{ \sum_j h_j \left( \frac{B_j}{\ell_j} + \frac{d_0}{\sqrt{1+y_j^2}} \right) + \text{the complete exterior tail} \right\}.$$

For the first two ranges the last tail is at most  $1004/400$ . Evaluating this finite inequality on every parameter box gives the constants stated above; the first sharper bound is  $93.0745928287506103515625$ . The integral on the left line has absolute value at most  $B_\alpha e^{-t}$ , which proves the representation. Its cut mass is less than  $\beta$ , so it also proves  $|K_\alpha(t)| \leq 1 + 2\beta + B_\alpha$ .  $\square$

**Lemma 4.4** (Time-end positivity). *The following inequalities hold uniformly on the indicated parameter sets:*

| $\alpha$ -range | small time                                 | large time                          |
|-----------------|--------------------------------------------|-------------------------------------|
| $[1, 6/5]$      | $K_\alpha(t) > 1/2$ ( $0 < t \leq 1/4$ )   | $K_\alpha(t) > 0$ ( $t \geq 33/5$ ) |
| $[6/5, 13/10]$  | $K_\alpha(t) > 1/2$ ( $0 < t \leq 1/5$ )   | $K_\alpha(t) > 0$ ( $t \geq 6$ )    |
| $[13/10, 3/2]$  | $K_\alpha(t) > 1/2$ ( $0 < t \leq 9/100$ ) | $K_\alpha(t) > 0$ ( $t \geq 6$ )    |
| $[3/2, 1.6671]$ | $K_\alpha(t) > 1/2$ ( $0 < t \leq 9/200$ ) | $K_\alpha(t) > 0$ ( $t \geq 6$ ).   |

*In particular  $K_\alpha(t) > 0$  for  $0 < t \leq 9/200$  on the parameter ranges used in the contact and bridge arguments below.*

*Proof.* For small time we sharpen the finite expansion by splitting its inversion line. The proof of Lemma 3.1 permits any lower bound  $\mu$  for the exponential cosine; use  $\max(|c_N|, |c_{N-1}|) \leq 1/\sin \delta$  in its remainder constants. On  $q = c + iy$ ,  $c = N/t$ , the part  $|\arg q| \leq \pi/3$  permits  $\mu = \sqrt{3}/2$ , while the remaining part has  $|q| \geq 2c$  and permits  $\mu = 1/\sqrt{2}$ . For  $0 < t \leq T$ , let  $C_F^{\text{in}}$  and  $C_F^{\text{out}}$  be the corresponding constants computed at  $Y = N/T$  and  $Y = 2N/T$ . Their checked positive denominator bounds also exclude zeros on  $\Re q \geq N/T$ , so the canonical inversion line may be moved there. The inner integral is bounded by the whole-line integral; on the outer part,

$$\int_{|y| \geq \sqrt{3}c} |c + iy|^{-N-1} dy \leq (2c)^{1-N} \int_{|y| \geq \sqrt{3}c} \frac{dy}{c^2 + y^2} = c^{-N} \frac{\pi}{3} 2^{1-N}.$$

Thus the entire inversion error is at most

$$E_N(T) = e^N (T/N)^N \left\{ \frac{C_F^{\text{in}} \Gamma(N/2)}{2\sqrt{\pi} \Gamma((N+1)/2)} + \frac{C_F^{\text{out}} 2^{1-N}}{6} \right\}.$$

To retain the positive polynomial terms, let  $\underline{d}_n$  be lower bounds for  $d_n$  on the parameter box and use the Bernstein coefficients

$$\mathcal{B}_k = \sum_{n=0}^k \frac{\underline{d}_n T^n}{n!} \frac{\binom{k}{n}}{\binom{N-1}{n}}, \quad 0 \leq k < N.$$

The Bernstein basis is nonnegative and sums to one on  $[0, 1]$ , giving  $\sum_{n < N} \underline{d}_n t^n / n! \geq \min_k \mathcal{B}_k$  for  $0 \leq t \leq T$ . The finite checks are therefore

$$\min_{k < N} \mathcal{B}_k - E_N(T) > \frac{1}{2}. \quad (4.6)$$

They use  $N = 40, 32, 24, 16$  on the successive parameter ranges  $[1, 6/5]$ ,  $[6/5, 13/10]$ ,  $[13/10, 3/2]$  and  $[3/2, 1.6671]$ , with four adjacent parameter boxes in each range, of maximum widths  $1/20, 3/100, 1/20, 1/20$ , respectively; the last box is shortened to the range endpoint. This is a finite expansion bound on the entire time interval, with no convergence assertion about an infinite series.

For large time we use one cut-mass estimate on all three parameter ranges. Put  $b = 1/20$ ,  $\chi = \cos \delta$  and  $\lambda = (1 - 2\chi^2)/(2(1 + \chi))$ . The elementary inequality

$$\frac{1}{1 + 2\chi v + v^2} \leq 1 - \chi v - \lambda v^2, \quad 0 \leq v \leq 1,$$

follows by multiplying through: the difference is  $v(1 - v)\{\chi + (\chi + \lambda + 2\chi\lambda)v + \lambda v^2\}$ . For  $\lambda \geq 0$  all coefficients in braces are nonnegative; for  $\lambda < 0$  the quadratic is concave with positive endpoint values  $\chi$  and  $1 + 2\chi - 2\chi^2$ . Applying this inequality with  $v = \min(u^\alpha, u^{-\alpha})$  gives

$$\begin{aligned} H_\alpha(x) &\leq \widehat{H}_\alpha(x) := I_\alpha(x) - \chi I_{2\alpha}(x) - \lambda I_{3\alpha}(x), \\ I_r(x) &:= \int_0^1 e^{-xu} u^r du + \int_1^\infty e^{-xu} u^{-r} du \\ &= x^{-r-1} \gamma(r+1, x) + x^{r-1} \Gamma(1-r, x). \end{aligned}$$

The lower and upper incomplete Gamma functions here represent convergent real integrals, including when  $1 - r \leq 0$ .

The function  $H_\alpha(x)$  decreases with  $x$  and with  $\alpha$ : its radial denominator is  $2 \cosh(\alpha \log u) + 2 \cos \delta$ . For a parameter box  $A = [a_0, a_1]$  set

$$x_0 = b^{1/a_0}, \quad x_1 = b^{1/a_1}, \quad \bar{\eta}_A = \frac{a_1 \sin \delta(a_0)}{\pi}, \quad n_A = \bar{\eta}_A \widehat{H}_{a_0}(x_0) - \sin x_0.$$

On the boxes below  $n_A > 0$ . Since  $x_0 \leq x \leq x_1$  and  $\eta_\alpha \leq \bar{\eta}_A$ , the exact cut phase yields

$$W_\alpha(b) := \int_0^b w_\alpha(v) dv \geq \frac{1}{a_1} - \frac{1}{2} - \frac{1}{\pi} \arctan\left(\frac{n_A}{\cos x_1}\right).$$

Only the following scalar inequalities are required:

| $\alpha$ -range   | boxes | maximum width | $W_\alpha(1/20) >$ | $T$  |
|-------------------|-------|---------------|--------------------|------|
| $[1, 6/5]$        | 5     | 1/10          | 899/5000           | 33/5 |
| $[6/5, 13/10]$    | 2     | 1/20          | 1351/10000         | 6    |
| $[13/10, 1.6671]$ | 10    | 1/10          | 441/5000           | 6    |

The exact endpoints of these nonuniform partitions are listed in the scalar certificates. These whole-box bounds use real incomplete Gamma and elementary functions, with no transform quadrature. The final time comparisons follow from  $e^{627/100} > 523$ ,  $e^{57/10} > 298$  and

$$\frac{899}{5000} 523 > 94, \quad \frac{1351}{10000} 298 > 37, \quad \frac{441}{5000} 298 > 26.$$

The two exponential bounds already follow from their degree-16 Taylor sums. Lemma 4.3 now gives  $K_\alpha(t) \geq W_\alpha(b)e^{-bt} - B_\alpha e^{-t} > 0$  for all  $t \geq T$ , since  $W_\alpha(b)e^{(1-b)t}$  increases with  $t$ .  $\square$

### 4.3 Validated inversion on compact rectangles

We use the rational model  $R_p$  and its inverse  $r_p$  from (3.17), and write  $R = R_p$ ,  $r = r_p$ ,  $G = F - R$  and  $g = K - r$ . Thus  $G = O(q^{-p-1})$ . Choose  $P > t$ ,  $\omega = 2\pi/P$  and  $J\omega \geq Y > h$ . The following bound controls both omitted frequencies and the error caused by periodization.

**Lemma 4.5** (Complete Fourier error). *If  $|g(t)| \leq M$  for all  $t > 0$ , the finite expression*

$$\mathcal{H}_J(t) = r(t) + \frac{e^{\sigma t}}{P} \left\{ G(\sigma) + 2 \operatorname{Re} \sum_{j=1}^J e^{ij\omega t} G(\sigma + ij\omega) \right\}$$

satisfies, for  $0 < t < P$ ,

$$|K(t) - \mathcal{H}_J(t)| \leq \frac{Me^{-\sigma P}}{1 - e^{-\sigma P}} + \frac{e^{\sigma t}}{\pi} \left\{ \sum_{n=p}^{N-1} \frac{|d_n - r_n|}{nY^n} + \frac{C_F + C_R}{NY^N} \right\},$$

where  $r_n$  are the finite inverse-power coefficients of  $R$ , and

$$C_R = \sum_{k=1}^p |b_k| \binom{N}{k-1} \frac{h^{N+1-k}}{(1 - h/Y)^k}.$$

The same assertion holds for  $|g(t)| \leq Me^{\lambda t}$ ,  $\sigma > \lambda$ , if the alias term is replaced by  $Me^{\lambda t} e^{-(\sigma-\lambda)P} / (1 - e^{-(\sigma-\lambda)P})$ .

*Proof.* Extend  $g$  by zero to negative times and periodize  $e^{-\sigma t} g(t)$ . Its Fourier coefficients are  $G(\sigma + ij\omega)/P$ . The matched coefficients give an absolutely summable Fourier series, hence, at  $0 < t < P$ ,

$$g(t) = \frac{e^{\sigma t}}{P} \sum_{j \in \mathbb{Z}} e^{ij\omega t} G(\sigma + ij\omega) - \sum_{\ell \geq 1} e^{-\sigma \ell P} g(t + \ell P).$$

The last sum has exactly the stated alias bound. Expand the rational model to the same finite order as  $F$ ; its convergent binomial remainder is bounded by  $C_R |q|^{-N-1}$ . For each remaining power use  $\sum_{j>J} j^{-n-1} \leq J^{-n}/n$  and  $J\omega \geq Y$ . Including both frequency signs and the factor  $1/P$  gives the stated  $1/\pi$  bound. This accounts for every omitted frequency.  $\square$

The bound in Lemma 4.3 supplies  $M$ , after adding

$$\sup_{t \geq 0} |r(t)| \leq |b_1| + \sum_{k=2}^p \frac{|b_k|}{(k-1)!} \left( \frac{k-1}{eh} \right)^{k-1}.$$

A real parameter mean-value enclosure is applied to the finite function  $\mathcal{H}_J$ , and the uniform inversion error is added afterwards. No derivative of an unknown alias error is used. On a time interval centered at  $t_m$ , finite Taylor enclosures through degree four use the whole interval for the fifth derivative. A complex coefficient rectangle can instead be split into its exact binary center and its radius: the uncertainty is then added once using  $|e^{ij\omega t}| = 1$ , rather than being differentiated repeatedly with the Taylor polynomial.

For the earlier ranges one uses directly the absolutely convergent Bromwich integral after subtracting  $d_0/q$ , or a two-term shifted model. The finite inverse-power tail at a time midpoint is evaluated exactly using the upper incomplete gamma function. For its time variation, the  $q^{-2}$  term has the bound  $2e^{ct}/(\pi tY)$  obtained by one integration by parts; all subsequent terms have absolutely

integrable derivatives. The remaining expansion error is  $e^{ct} C_F / (\pi N Y^N)$ . Parameter derivatives are obtained on fixed rays before integration, with separate bounds on the omitted origins and tails. For example, putting  $w = e^{i\phi} u$ ,  $z = w^\alpha / q$ ,

$$A = \frac{e^{i(\alpha+1)\phi}}{\Gamma(\alpha+1)} \int_0^\infty \frac{e^{-e^{i\phi} u} u^\alpha}{D_\delta(z)} du, \quad F = \frac{\beta N}{q A},$$

where  $N$  has one extra factor  $w$ . The fixed-ray integrand derivative is its value times

$$\log u + i\phi - \psi(\alpha+1) - \frac{\partial_\alpha D_\delta(z)}{D_\delta(z)}, \quad \partial_\alpha D_\delta(z) = -\pi \sin \delta z - 2 \cos \delta z \log w + 2z^2 \log w.$$

Thus  $\partial_\alpha F = -\beta F + (\beta/q)(N_\alpha/A - N A_\alpha/A^2)$ . The ray is fixed on the entire parameter box. Its angular distance from both poles and its exponential decay are checked before it is used; a representation crossing a pole includes its residue explicitly.

**Lemma 4.6** (The lower parameter ranges). *For  $1 < \alpha \leq 33/20$ ,  $K_\alpha(t) > 0$  for every  $t > 0$ .*

*Proof.* The compact parts not covered by Lemma 4.4 have the following finite, closed-rectangle covers. The bounds in the last column are conservative rational lower bounds, restricted to the displayed compact time domains.

| $\alpha$ -range | $t$ -range  | $\alpha$ -boxes | rectangles | $K$ lower bound |
|-----------------|-------------|-----------------|------------|-----------------|
| [1, 6/5]        | [1/4, 33/5] | 20              | 6360       | 1/50000         |
| [6/5, 13/10]    | [1/5, 6]    | 20              | 5800       | 7/100000        |
| [13/10, 3/2]    | [9/100, 6]  | 40              | 11840      | 63/1000         |
| [3/2, 33/20]    | [9/200, 6]  | 30              | 8941       | 2/125           |

The first two rows use  $c = 1/8, Y = 40, N = 12$ . The third uses  $c = 1/32, Y = 120, N = 8$ , a two-term shifted model with  $h = 1$ , and the whole-kernel enclosure

$$K_\alpha(t) \in K_{\alpha_m}(t_m) + (t - t_m) \partial_t K_{\alpha_m}(T) + (\alpha - \alpha_m) \partial_\alpha K_A(T).$$

The last row uses Lemma 4.5 with  $\sigma = 1/8, P = 64, J = 4096, Y = 400, N = 18, p = 4, h = 20$ . Every accepted output is an enclosure of the full parameter and time rectangle. Exact rational adjacency joins the rectangles; a failed rectangle is subdivided, never skipped. The saved enclosure lower endpoints are strictly above the displayed bounds. In the first two rows, input containment is justified by the exact rational construction in the frozen producer; later rows also retain their binary input balls for a separate containment check. Every retained saved output has a strictly positive lower endpoint. The finite subset is selected by exact interval endpoints: a rectangle crossing a new time cutoff is retained and restricted to the required subinterval. The parameter intervals and clipped time intervals meet exactly. Combining these compact covers with both analytic time ends proves the proposition. The auxiliary row endpoint  $\alpha = 1$  makes no assertion about a point-hitting distribution at that index.  $\square$

#### 4.4 The contact and completion of the positive range

Put

$$\alpha_0 = 1.66706599, \quad u = 10^{-8}, \quad t_0 = .1628, \quad v = 10^{-6},$$

$$A_* = [\alpha_0 - u, \alpha_0 + u], \quad I_* = [t_0 - v, t_0 + v].$$

**Lemma 4.7** (One local contact). *There is exactly one pair  $(\alpha_*, t_*) \in A_* \times I_*$  satisfying  $K_{\alpha_*}(t_*) = \partial_t K_{\alpha_*}(t_*) = 0$ , and*

$$1.6670659903 < \alpha_* < 1.6670659915, \quad .162799 < t_* < .162801.$$

*Each  $K_\alpha$ ,  $\alpha \in A_*$ , has a unique interior minimum on  $I_*$ , and its minimum value decreases strictly with  $\alpha$ .*

*Proof.* The interval inversion gives the following rational inequalities:

$$\begin{aligned} 3 \cdot 10^{-9} < K_{\alpha_0}(t_0) < 13 \cdot 10^{-9}, \quad |\partial_t K_{\alpha_0}(t_0)| < 57 \cdot 10^{-6}, \\ 1402 < \partial_t^2 K_\alpha(t) < 1421, \quad -9.068 < \partial_\alpha K_\alpha(t) < -9.059 \quad ((\alpha, t) \in A_* \times I_*). \end{aligned}$$

For these local derivative evaluations, the line is  $\text{Re } q = 20$ , with  $Y = 1000$ ,  $N = 18$ ; the regularized inverse of  $q^j F$  subtracts its polynomial terms and its  $d_j/q$  term and adds back  $d_j$ . Appendix A gives the complete finite remainder and its parameter derivative. Lemma 4.2 identifies the inverse with the canonical kernel, independently of positivity.

At  $\alpha_0$ , strong convexity raises either time endpoint above the center by at least  $-57 \cdot 10^{-6}v + 701v^2$ . The variation of this difference when  $|\alpha - \alpha_0| \leq u$  is at most  $u(9.068 - 9.059)$ . Thus both endpoints exceed the center by more than

$$-57 \cdot 10^{-12} + 701 \cdot 10^{-12} - 90 \cdot 10^{-12} = 554 \cdot 10^{-12} > 0.$$

Strict convexity therefore gives a unique interior minimizer for every  $\alpha \in A_*$ . Write  $m(\alpha) = \min_{t \in I_*} K_\alpha(t)$ . For  $a < b$  in  $A_*$ , integrate the parameter bounds and then take minima to obtain

$$-9.068(b - a) \leq m(b) - m(a) \leq -9.059(b - a). \quad (4.7)$$

In particular  $m$  is continuous and strictly decreasing; no differentiation of its minimizer is required. Strong convexity also gives

$$m(\alpha_0) > 3 \cdot 10^{-9} - \frac{(57 \cdot 10^{-6})^2}{2 \cdot 1402} > 0, \quad m(\alpha_0 + u) < 13 \cdot 10^{-9} - 9.059 \cdot 10^{-8} < 0.$$

Consequently  $m$  has exactly one zero, and (4.7) implies

$$\frac{3 \cdot 10^{-9} - (57 \cdot 10^{-6})^2 / (2 \cdot 1402)}{9.068} < \alpha_* - \alpha_0 < \frac{13 \cdot 10^{-9}}{9.059}.$$

These bounds give the stated enclosure. The unique interior minimizer at  $\alpha_*$  is  $t_*$ ; strict convexity proves uniqueness of the pair in  $A_* \times I_*$ .  $\square$

**Lemma 4.8** (Complete positive side). *For every  $1 < \alpha < \alpha_*$ ,  $K_\alpha(t) > 0$  for all  $t > 0$ . The kernel at  $\alpha_*$  is nonnegative and has exactly one zero, at the time  $t_*$  of Lemma 4.7. The corresponding hitting-time laws are infinitely divisible under ordinary addition.*

*Proof.* By Lemmas 4.6 and 4.4, it remains to consider  $33/20 \leq \alpha \leq \alpha_*$  and  $.045 \leq t \leq 6$ . Put  $a_- = \alpha_0 - u = 1.66706598$ .

First let  $a_- \leq \alpha \leq \alpha_*$ . On  $A_* \times [.145, .175]$ , thirty adjacent time intervals give  $\partial_t^2 K_\alpha > 242$ . Hence the minimum on  $I_*$  is also the unique minimum on this larger interval. Its value is nonnegative by Lemma 4.7, and is zero only when  $(\alpha, t) = (\alpha_*, t_*)$ . On the complement

$$A_* \times ( [.045, .145] \cup [.175, 6] ),$$

891 adjacent time rectangles give  $K_\alpha > 10^{-5}$ . These use  $\sigma = 1/2, P = 32, J = 2048, Y = 400, N = 18, p = 4, h = 20$  in Lemma 4.5. The convexity calculation uses the filtered second derivative with its full inversion remainder. The finite sum enclosures and error constants are verified independently at higher precision while preserving the exact original parameter centers.

Now let  $33/20 \leq \alpha \leq a_-$ . On  $[33/20, a_-] \times [.15, .17]$ , ninety-eight parameter boxes and 980 time rectangles give  $\partial_\alpha K_\alpha < -1$ . Its derivative alias bound is independent of infinite divisibility. Specifically, put  $Z(q) = L_\alpha(q)(q+2)^{d_0}/(\kappa_\alpha \Gamma(\alpha+1))$ . The finite vertical boundary margins, the cut bound  $|L_\alpha(-v + i0)| \geq \sin(\pi\beta) \cos 1$ , and the exterior estimate give  $|Z| > 0.1476$  by the maximum principle applied to  $1/Z$ . For  $\operatorname{Re} q \geq 1$ , this implies  $|A_\alpha| > 0.0252$ . Bounding the fixed-ray moment derivatives by  $|\log u| \leq u + u^{-1}$  yields  $|\partial_\alpha F_\alpha(q)| < 4.178 \cdot 10^9/|q|$ . An integrable inversion after subtraction of  $-\beta^2/(q+2)$ , with the finite derivative remainder on  $|y| \geq 400$ , then gives

$$|\partial_\alpha K_\alpha(t)| < 5.319 \cdot 10^{11} e^t.$$

The exponential-growth version of Lemma 4.5, with  $\sigma = 5, P = 8, J = 512$ , supplies the complete alias and omitted-frequency errors in the derivative cover. Integration in  $\alpha$  now gives the direct comparison

$$K_\alpha(t) \geq K_{a_-}(t) + (a_- - \alpha) > 0, \quad .15 \leq t \leq .17,$$

where positivity at  $a_-$  was proved in the first case. On the remaining compact times  $[.045, .15] \cup [.17, 6]$ , eighteen parameter boxes and 3096 rectangles give  $K_\alpha > 2 \cdot 10^{-4}$ . All parameter and time endpoints meet exactly, so the two cases and the time-end bounds prove the kernel assertions. Lemma 3.3 gives infinite divisibility and the Lévy representation.  $\square$

The negative-side argument is needed separately to establish that this locally unique contact is the unique classification threshold. No claim that other contacts cannot occur at larger indices is used or implied.

## 5 The non-infinitely-divisible range

The local minimum constructed below the critical index already gives a negative value of  $K_\alpha$  immediately to the right of that index. For larger parameters two necessary conditions for infinite divisibility are more efficient than a global inversion of  $F_\alpha$ . We give both conditions, their validated evaluation, and an analytic argument that closes the interval at 2.

**Lemma 5.1** (Gaussian necessary condition). *Suppose a nonnegative random variable has an infinitely divisible law with Laplace transform  $L$ . Put  $F = -L'/L$ . For  $\sigma, h > 0, t_0 \in \mathbb{R}$  and  $\Omega > 0$ , define*

$$J(\sigma, t_0, h, \Omega) = \sqrt{\frac{2}{\pi}} h \Re \int_0^\Omega e^{-h^2 y^2/2 - i y t_0} F(\sigma - i y) dy. \quad (5.1)$$

Then

$$J(\sigma, t_0, h, \Omega) + \sqrt{\frac{2}{\pi}} F(\sigma) \frac{e^{-h^2 \Omega^2/2}}{h \Omega} \geq 0. \quad (5.2)$$

*Proof.* By (3.20),  $F(q) = \int_{[0, \infty)} e^{-qt} \mu(dt)$ , where  $\mu = d\delta_0 + t\nu(dt)$  is nonnegative and  $\int e^{-\sigma t} \mu(dt) = F(\sigma) < \infty$ . Consequently  $|F(\sigma - i y)| \leq F(\sigma)$ . The Fourier formula for a Gaussian and Fubini's theorem give

$$0 \leq \int_{[0, \infty)} e^{-\sigma t - (t-t_0)^2/(2h^2)} \mu(dt) = \sqrt{\frac{2}{\pi}} h \Re \int_0^\infty e^{-h^2 y^2/2 - i y t_0} F(\sigma - i y) dy.$$

The absolute value of the omitted part is at most

$$\sqrt{\frac{2}{\pi}} h F(\sigma) \int_{\Omega}^{\infty} e^{-h^2 y^2 / 2} dy \leq \sqrt{\frac{2}{\pi}} F(\sigma) \frac{e^{-h^2 \Omega^2 / 2}}{h \Omega}.$$

This proves the assertion. Notice that the estimate is used under the infinite-divisibility hypothesis; no signed inverse kernel is needed.  $\square$

**Lemma 5.2** (Rotated moments and their error bounds). *Let  $k$  be a nonnegative integer and let  $s = q^\beta$ , with  $\Re q > 0$ . Set*

$$J_k(s) = \int_0^{\infty} \frac{v^{\alpha+k} e^{-sv}}{D_\delta(v^\alpha)} dv.$$

*On a ray  $v = e^{i\gamma}u$ , chosen so that  $\Re(se^{i\gamma}) = \lambda > 0$ , the same integral equals*

$$J_k(s) = P_k(s) + e^{i(\alpha+k+1)\gamma} \int_0^{\infty} \frac{u^{\alpha+k} e^{-se^{i\gamma}u}}{D_\delta(e^{i\alpha\gamma}u^\alpha)} du. \quad (5.3)$$

*Here  $P_k = 0$  if  $0 \leq \alpha\gamma < \delta$ , whereas, if  $\delta < \alpha\gamma$  and  $\alpha\gamma + \delta < \pi$ ,*

$$P_k(s) = \frac{\pi p^{k+1} e^{-ps}}{\alpha \sin \delta}, \quad p = e^{i\delta/\alpha}. \quad (5.4)$$

*In either case a denominator lower bound is*

$$|D_\delta(e^{i\alpha\gamma}u^\alpha)| \geq D_* := \sin |\alpha\gamma - \delta| \sin(\alpha\gamma + \delta) > 0. \quad (5.5)$$

*For  $0 < u_0 < U$ , the errors from deleting  $(0, u_0)$  and  $(U, \infty)$  from (5.3) are bounded respectively by*

$$\frac{u_0^{\alpha+k+1}}{(\alpha+k+1)D_*}, \quad \frac{U^{\alpha+k} e^{-\lambda U}}{D_*[\lambda - (\alpha+k)/U]}, \quad (5.6)$$

*provided the denominator in the latter bound is positive. Moreover  $F_\alpha(q) = \beta s J_1(s)/(q J_0(s))$ .*

*Proof.* Rotation of the original contour proves (5.3). In the second case only  $v = p$  is crossed. At that pole the residue is  $p^{k+1} e^{-ps}/(2i\alpha \sin \delta)$ , giving (5.4). The factorization of  $D_\delta$  and  $|1 - x e^{i\theta}| \geq |\sin \theta|$  for  $x \geq 0$  give (5.5). The first estimate in (5.6) follows by dropping the exponential. For the second, write  $u = U + v$  and use  $u^p \leq U^p e^{pv/U}$ ,  $p = \alpha + k$ . Differentiating (3.2) under the integral gives the stated formula for  $F$ . The rotations are valid by exponential decay on the outer arcs and the positive power at zero. Conjugation gives the corresponding formulas in the opposite half-plane.  $\square$

For completeness, the parameter differentiation used in the Gaussian cover is specified here. The ray is held fixed while  $\alpha$  varies. Writing  $\theta = \pi\alpha/2$ ,  $c = \cos \theta$ ,  $z = v^\alpha$ , one has

$$s_\alpha = -s \log q / \alpha^2, \quad c' = -(\pi/2) \sin \theta, \quad (5.7)$$

$$(D_\delta(z))_\alpha = 2z[(z+c) \log v + c'], \quad (5.8)$$

$$\partial_\alpha \left( \frac{v^{\alpha+k} e^{-sv}}{D_\delta(z)} \right) = \frac{v^{\alpha+k} e^{-sv}}{D_\delta(z)} \left( \log v - s_\alpha v - \frac{(D_\delta(z))_\alpha}{D_\delta(z)} \right). \quad (5.9)$$

If  $Q_k$  denotes this differentiated moment, including the derivative of the pole term when present, then

$$\partial_\alpha F_\alpha(q) = F_\alpha(q) \left( -\frac{1}{\alpha} - \frac{\log q}{\alpha^2} \right) + \frac{\beta s}{q} \left( \frac{Q_1}{J_0} - \frac{J_1 Q_0}{J_0^2} \right). \quad (5.10)$$

For the pole term,  $p_\alpha/p = -i\pi/\alpha^2$ , and

$$\partial_\alpha P_k = P_k \left\{ -\alpha^{-1} + \frac{\pi}{2} \cot \delta + (k+1) \frac{p_\alpha}{p} - p_\alpha s - p s_\alpha \right\}.$$

These identities include the dependence of  $q^{1/\alpha}$  on  $\alpha$ . Endpoint errors for (5.9) are obtained from

$$\int_0^{u_0} u^p (|\log u| + \gamma) du = u_0^{p+1} \left( \frac{-\log u_0 + \gamma}{p+1} + \frac{1}{(p+1)^2} \right), \quad u_0 \leq 1, \quad (5.11)$$

$$|\log(ue^{i\gamma})| \leq 2u \quad (u \geq 100, 0 \leq \gamma < 1), \quad (5.12)$$

together with (5.6), applied also to the powers  $3\alpha + k$ ,  $2\alpha + k$  and their successors. Thus these derivatives are enclosed on whole parameter intervals.

**Lemma 5.3** (The finite Gaussian cover; computer-assisted). *For every  $\alpha \in [1.6671, 1.7386]$ , the distribution of  $T_1$  is not infinitely divisible.*

*Proof.* All decimals in this proof are exact rationals. For the first part take  $\sigma = 1$ ,  $t_0 = .163$ ,  $h = 1/3000$  and  $\Omega = 7/h = 21000$ . The evaluated test at  $\alpha = 1.6671$ , including the conditional tail, is enclosed by

$$[-1.403446562329232 \cdot 10^{-7} \pm 3.90 \cdot 10^{-10}].$$

For the same finite function (5.1), the derivative intervals on the four successive parameter intervals of radius .0001 and centers 1.6672, 1.6674, 1.6676, 1.6678 are contained respectively in

$$[-.00963, -.00304], \quad [-.00962, -.00301], \quad [-.00960, -.00299], \quad [-.00956, -.00299].$$

Their exact unions cover  $[1.6671, 1.6679]$ . Each corresponding uniform tail bound, when added to the upper bound for the finite test at the leftmost endpoint, gives a value less than  $-1.39 \cdot 10^{-7}$ . Monotonicity and Lemma 5.1 prove non-infinite-divisibility throughout that interval.

For the rest of the cover, 33 overlapping parameter intervals cover  $[1.6679, 1.7386]$ . Each uses either direct interval evaluation or the following mean-value enclosure. If  $\alpha \in [a - r, a + r]$ ,  $J(a)$  is enclosed by  $B$ , and  $|\partial_\alpha J| \leq M$  on the interval, then

$$\sup B + rM + R < 0 \quad (5.13)$$

certifies the entire interval, where  $R$  is the uniform conditional Gaussian tail in (5.2). The parameters, input balls, integral enclosures, derivative balls and tails of every interval are supplied with the ancillary certificate. The exact coverage test is to sort the rational endpoints, check that the first left endpoint is at most 1.6679, each subsequent left endpoint is at most the current right endpoint, and the final right endpoint is at least 1.7386. All 33 inequalities (5.13) are strict. Two of the intervals use more of the parameter range already covered by their derivative enclosures:

| $a$         | $r$    | available derivative radius |
|-------------|--------|-----------------------------|
| 1.685422064 | .0011  | .0012842363                 |
| 1.687466206 | .00095 | .0013556707                 |

Recombining the same point, derivative and tail balls on these radii still gives strict negativity. Their overlap makes one former bridging interval unnecessary; two further intervals are redundant. The active list and its exact endpoint checks are supplied in the ancillary material. For example, the last interval  $[1.7356, 1.7386]$  has its entire test enclosed by  $[-.00515256635094 \pm .00126000000001]$ .

Here are the evaluation details needed to reproduce the inequalities. The radial integral is regularized by  $u = x^4$ , truncated at  $u_0 = 2^{-64}$  and  $U = 100$ , and evaluated by complex ball integration. The deleted parts are bounded by (5.6) and its differentiated versions. The noncrossing ray is

$\gamma = \pi(1/\alpha_0 - 1/2)/2$ , fixed at the interval center; where used, the crossing ray is  $\gamma = \pi/4$  with the residue (5.4). Every ray inequality is checked on the whole parameter interval. High frequencies in the point tests use the finite identity (3.10), with  $N = 18$ , retaining the integrated remainder for each moment. Direct differentiated moments are used for the derivative intervals. The last Fourier tail is always included with  $h\Omega = 7$ . No sign claim is inferred from a floating-point inverse transform.  $\square$

**Lemma 5.4** (A uniform zero certificate). *Let  $I \subset (1, 2)$  be a parameter interval. Set  $G_\alpha(s) = \sin \delta J_0(s)$ , using analytic continuation by (5.3). Suppose the disk  $|s - z_0| \leq r$  lies in  $\Re s > 0$  and in its domain of analyticity for every  $\alpha \in I$ . If a nonzero complex number  $D_0$  satisfies*

$$\sup_{\alpha \in I} |G_\alpha(z_0)| + r \sup_{\alpha \in I} |G'_\alpha(z_0) - D_0| + \frac{r^2}{2} \sup_{\substack{\alpha \in I \\ |s - z_0| \leq r}} |G''_\alpha(s)| < r|D_0|, \quad (5.14)$$

and  $\Re(s^\alpha) > 0$  on the whole disk for every  $\alpha \in I$ , then  $T_1$  is not infinitely divisible for any  $\alpha \in I$ .

*Proof.* Taylor's formula with integral remainder gives

$$G_\alpha(z_0 + z) = G_\alpha(z_0) + zG'_\alpha(z_0) + z^2 \int_0^1 (1-u)G''_\alpha(z_0 + uz)du.$$

On  $|z| = r$ , (5.14) compares this expression with  $D_0 z$ . Rouché's theorem gives exactly one zero in the disk, counting multiplicity. Since  $|\arg s| < \pi/2$  and  $\alpha < 2$ , the condition  $\Re(s^\alpha) > 0$  forces  $|\alpha \arg s| < \pi/2$ . The principal mapping  $q = s^\alpha$  therefore sends that zero into  $\Re q > 0$  and satisfies  $q^{1/\alpha} = s$  there. It gives a zero of  $L_\alpha$ . The representation (3.20) shows that a nonnegative infinitely divisible law has a zero-free Laplace transform throughout that half-plane, a contradiction.  $\square$

**Lemma 5.5** (The finite zero cover; computer-assisted). *The distribution of  $T_1$  is not infinitely divisible for  $1.7386 \leq \alpha \leq 15/8$ .*

*Proof.* The active certificate contains 102 parameter intervals and disks to which Lemma 5.4 applies. Their rational endpoints cover the stated closed interval with overlaps. The moment enclosures in Lemma 5.2, on the ray  $\gamma = \pi/4$ , give

$$G_\alpha = \sin \delta J_0, \quad G'_\alpha = -\sin \delta J_1, \quad G''_\alpha = \sin \delta J_2.$$

For the last expression the complex rectangle  $[\Re z_0 - r, \Re z_0 + r] + i[\Im z_0 - r, \Im z_0 + r]$  contains the whole disk. Exact rational comparisons also give  $\Re z_0 - r > 5.63$  for every disk. The same rectangle is used to enclose  $\exp(\alpha \log s)$ , and its real part has a strictly positive lower bound in every record. All 102 inequalities (5.14) hold strictly. Recombining the saved balls gives a maximum left/right ratio below .999668 and a minimum lower bound for  $\Re(s^\alpha)$  above .0266059. These margins concern entire parameter intervals and entire disks. Approximate root finding is used only to choose the centers  $z_0$ ; it is not an existence argument.  $\square$

**Lemma 5.6** (An analytic zero for every  $15/8 \leq \alpha < 2$ ). *For  $15/8 \leq \alpha < 2$ , the transform  $L_\alpha$  has a zero in  $\Re q > 0$ . Consequently the law of  $T_1$  is not infinitely divisible throughout this interval.*

*Proof.* Put  $\varepsilon = 2 - \alpha$ , so  $0 < \varepsilon \leq 1/8$  and  $\delta = \pi\varepsilon/2 \leq \pi/16$ . Write

$$B = \kappa_\alpha \Gamma(m), \quad p = e^{-i\delta/\alpha}, \quad d = \frac{\pi \kappa_\alpha}{\alpha \sin \delta}, \quad A = dp, \quad \ell = \log(d/B), \quad \nu = \pi + \delta.$$

We shall use the elementary digamma bound

$$\psi(t) > \log(t - 1/2), \quad t > 1/2. \quad (5.15)$$

Indeed, subtracting the integral representations of its two sides gives

$$\psi(t) - \log(t - 1/2) = \int_0^\infty e^{-(t-1/2)u} \left( \frac{1}{u} - \frac{1}{2 \sinh(u/2)} \right) du > 0.$$

Since  $\sin \delta \leq \pi \varepsilon / 2$ , integration of (5.15) gives

$$\ell \geq -\log(\varepsilon(2 - \varepsilon)) + \int_{3-\varepsilon}^3 \log(t - 1/2) dt.$$

This lower bound decreases with  $\varepsilon \in (0, 1/8]$ ; its derivative is  $-1/\varepsilon + 1/(2 - \varepsilon) + \log(5/2 - \varepsilon) < 0$ . Consequently

$$\ell \geq \log \frac{64}{15} + \frac{5}{2} \log \frac{5}{2} - \frac{19}{8} \log \frac{19}{8} - \frac{1}{8} > \frac{31}{20}. \quad (5.16)$$

The duplication formula and (5.15) also give

$$\frac{d}{d\alpha} \log \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} = \log 4 + \psi(\alpha + 1/2) > \log(4\alpha) \geq \log \frac{15}{2} > 2.$$

Here  $e < 87/32$  and  $(87/32)^2 < 15/2$ . Thus

$$\frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} \leq 12e^{-2\varepsilon}. \quad (5.17)$$

We first locate an exact zero of the exponential model. For  $x > m$  let  $y(x) > 0$  be the unique solution of

$$y - m \arctan(y/x) = \nu.$$

Existence and uniqueness follow because the left side increases from zero to infinity, with derivative at least  $1 - m/x > 0$ . Implicit differentiation gives

$$y'(x) = -\frac{my}{x^2 + y^2 - mx}, \quad \frac{d}{dx} \left( x - \frac{m}{2} \log(x^2 + y^2) \right) = \frac{(x - m)^2 + y^2}{x^2 + y^2 - mx} > 0.$$

The expression differentiated on the left is negative as  $x \downarrow m$  and tends to infinity as  $x \rightarrow \infty$ . By (5.16), there is therefore exactly one  $x > m$  for which

$$w_0 = x + iy(x), \quad w_0 - m \log w_0 = \ell + i\nu. \quad (5.18)$$

For useful uniform bounds, put  $m_0 = 23/8$  and let  $h(x)$  solve  $h - m_0 \arctan(h/x) = \pi$ ,  $x > m_0$ . Then  $y(x) \geq h(x)$ . At  $x_0 = 79/10$  the elementary comparisons

$$\frac{14}{3} - \frac{23}{8} \arctan \frac{140}{237} < \pi, \quad \frac{79}{10} - \frac{23}{16} \log \frac{75769}{900} < \frac{31}{20} \quad (5.19)$$

show that  $h(x_0) > 14/3$  and that the real equation in (5.18) cannot hold for  $x \leq x_0$ . Indeed its left side is at most  $x - m_0 \log \sqrt{x^2 + h(x)^2}$ , an increasing function which is below  $31/20$  at  $x_0$ . Also  $\sqrt{x^2 + h(x)^2}$  increases with  $x$ . Finally,  $\arctan(51/79) < 7/12$  and  $\nu \leq 17\pi/16 < 67/20 = 51/10 - 3(7/12)$  imply  $y(x) < 51/10$ . We have obtained

$$x > \frac{79}{10}, \quad |w_0| > \frac{55}{6}, \quad \pi < y < \frac{51}{10}. \quad (5.20)$$

Put  $r_0 = 13/10$  and consider the closed disk  $|w - w_0| \leq r_0$ . Write  $U = \Re w$ ,  $V = \Im w$  and  $\lambda = 7/8$ . The center satisfies  $\lambda x - y > 29/16$ , while

$$r_0 \sqrt{1 + \lambda^2} < \frac{29}{16}.$$

Cauchy–Schwarz therefore gives  $\lambda U - V > 0$  throughout the disk. Also  $V > \pi - r_0 > 0$ . Set

$$\theta = \frac{\pi}{2\alpha}, \quad \gamma = \theta - \frac{\delta}{\alpha} = \frac{\pi(\alpha - 1)}{2\alpha}.$$

Since

$$0 < \arg w < \arctan \frac{7}{8} < \frac{73}{100} < \frac{7\pi}{30} \leq \gamma,$$

the point  $s = w/p$  satisfies  $0 < \arg s < \theta$ , and  $q = s^\alpha$  belongs to  $\Re q > 0$  with  $q^{1/\alpha} = s$ . All powers and logarithms hence have their principal values. Moreover, if  $a = \pi/4 - \gamma = \delta/(2\alpha) \leq \pi/60$ , then

$$\frac{U - V}{U + V} > \frac{1}{15} > \frac{1}{30} > \tan(a/2).$$

Expanding  $\cos(\pi/4 - a)$  and  $\sin(\pi/4 - a)$  proves

$$\Re(se^{-i\theta}) = \Re(we^{-i\gamma}) \geq \frac{U + V}{\sqrt{2}}. \quad (5.21)$$

Rotate the  $r$  contour in (3.2) downward through angle  $\theta$ . The sole crossed pole is  $p$ , and the clockwise residue contribution is  $Ae^{-ps}$ . The small arc vanishes; the large arc decays because  $0 < \arg s < \theta < \pi/2$ . Consequently

$$\mathcal{G}(s) := L_\alpha(s^\alpha) = Ae^{-ps} + J(s), \quad (5.22)$$

$$J(s) = \kappa_\alpha e^{-im\theta} \int_0^\infty \frac{e^{-se^{-i\theta}u} u^\alpha}{D_\delta(-iu^\alpha)} du. \quad (5.23)$$

For  $z = -ib$ ,  $b \geq 0$ , direct squaring gives

$$4|D_\delta(-ib)|^2 - |2\cos\delta + ib|^2 = 4\sin^2\delta + (16\cos^2\delta - 9)b^2 + 4b^4 \geq 0.$$

Here  $\cos\delta > 3/4$ . Applying  $D_\delta(z)^{-1} = 1 + z(2\cos\delta - z)/D_\delta(z)$  therefore gives

$$J(s) = Bs^{-m}(1 + R_0(s)), \quad |R_0(w/p)| \leq 2 \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} \frac{|w|^m}{\Re(we^{-i\gamma})^{2\alpha+1}}. \quad (5.24)$$

The remainder is analytic on a neighborhood of the disk.

To bound (5.24), define

$$Q_\alpha(U, V) = \frac{(U^2 + V^2)^{(\alpha+1)/2}}{(U + V)^{2\alpha+1}}.$$

This function decreases in each coordinate on  $(0, \infty)^2$ . It also decreases along  $(U, V) = (x - r_0, h(x) - r_0)$  for  $x \geq 79/10$ . For completeness,

$$|h'(x)| \leq \frac{m_0}{2\sqrt{x(x - m_0)}} < \frac{1}{4}, \quad |\partial_U \log Q_\alpha| > \frac{1}{4} |\partial_V \log Q_\alpha|.$$

The second inequality follows from positivity of  $(2\alpha - 1)U^2 - 3(\alpha + 1)UV + (7\alpha + 4)V^2$ ; its discriminant is  $-47\alpha^2 + 14\alpha + 25 < 0$ . Both coordinate derivatives are negative, as follows similarly from  $1 - 2\alpha - 7\alpha^2 < 0$ . Hence, with  $u_0 = 33/5$  and  $v_0 = 101/30$ ,

$$Q_\alpha(U, V) \leq Q_\alpha(x - r_0, y - r_0) \leq Q_\alpha(x - r_0, h(x) - r_0) \leq Q_\alpha(u_0, v_0).$$

Put  $H = (u_0 + v_0)^2 / \sqrt{u_0^2 + v_0^2}$ . Direct rational comparisons, using  $e > 163/60$ , give

$$H < \frac{59}{4} < 2e^2, \quad 24\sqrt{32} Q_2(u_0, v_0) < \frac{9}{16}.$$

Combining (5.17), (5.21) and (5.24), and writing  $R(w) = R_0(w/p)$ , we conclude throughout the disk that

$$|R(w)| \leq 24\sqrt{32} Q_2(u_0, v_0) \left( \frac{H}{2e^2} \right)^\varepsilon < \frac{9}{16}. \quad (5.25)$$

Now let  $\Psi(w) = w - m \log w - \ell - i\nu$ . Equation (5.18) implies, on  $|w - w_0| = r_0$ ,

$$|\Psi(w) - (w - w_0)| \leq -m \log(1 - r_0|w_0|^{-1}) < 3 \log \frac{275}{236} < \frac{47}{100}. \quad (5.26)$$

Since  $|R| < 9/16$ , the function  $\Xi(w) = \Psi(w) + \log(1 + R(w))$  is analytic on a neighborhood of the disk. Furthermore,

$$|\log(1 + R(w))| < \log \frac{16}{7} < \frac{83}{100}.$$

Thus  $|\Xi(w) - (w - w_0)| < 13/10 = r_0$  on the boundary. A single application of Rouché's theorem gives a zero  $w_*$  of  $\Xi$  inside it. Finally,  $Bp^m/A = e^{-\ell-i\delta}$  yields the exact identity

$$\frac{\mathcal{G}(w/p)}{Bp^m w^{-m}} = 1 + R(w) - e^{-\Psi(w)}.$$

At  $w_*$  this is zero, so  $L_\alpha$  has a zero at  $q = (w_*/p)^\alpha \in \{\Re q > 0\}$ . Equation (3.20) excludes infinite divisibility.

All fixed scalar comparisons in this proof reduce to rational arithmetic. One may use  $157/50 < \pi < 22/7$ ,  $163/60 < e < 87/32$ , and the alternating series for  $\arctan z$ . The logarithmic comparisons in (5.16) and (5.19) follow respectively from

$$\left( \frac{64}{15} \right)^{40} \left( \frac{5}{2} \right)^{100} \left( \frac{8}{19} \right)^{95} > \left( \frac{87}{32} \right)^{67}, \quad \left( \frac{75769}{900} \right)^{115} > \left( \frac{87}{32} \right)^{508}.$$

The remainder bound is rational after squaring. The two last logarithmic bounds follow respectively from  $-\log(1 - t) \leq t + t^2/(2(1 - t))$  at  $t = 39/275$  and the first five terms of  $e^{83/100}$ . No interval enclosure of a transform or parameter subdivision is used in this lemma.  $\square$

**Lemma 5.7** (Completion of the negative side). *For every  $\alpha_* < \alpha < 2$ , the distribution of  $T_1$  is not infinitely divisible.*

*Proof.* The contact argument gives a negative minimum for  $\alpha_* < \alpha \leq 1.667066$ . At  $t_0 = .1628$  its point and derivative bounds imply

$$K_{1.667066}(t_0) < 13 \cdot 10^{-9} - 9.059 \cdot 10^{-8} < 0.$$

On  $\alpha \in [1.667066, 1.6671]$  the separate uniform derivative enclosure is

$$\partial_\alpha K_\alpha(t_0) \in [-9.06079735696894 \pm 2.78000000000001] \subset (-\infty, -6).$$

It is evaluated by the same regularized inversion and differentiated finite-tail estimates as the contact enclosure. Canonical identification is supplied by the zero-free strip for  $1.3 \leq \alpha \leq 1.6671$ . Thus a negative value persists through 1.6671, and Lemma 3.3 applies. Lemmas 5.3 and 5.5 then cover  $[1.6671, 15/8]$ . Lemma 5.6 covers  $[15/8, 2)$  analytically, completing the range without an extrapolation from finitely many zero disks.  $\square$

## 6 Proof of the main theorem

*Proof of Theorem 1.1.* Lemma 2.1 reduces the assertion to  $a = 1$ . Lemma 4.7 constructs the unique contact in the stated rectangle and gives (1.2). Lemma 4.8 gives  $K_\alpha > 0$  on  $(0, \infty)$  for  $1 < \alpha < \alpha_*$ , and  $K_{\alpha_*} \geq 0$  with exactly one zero. By Lemma 3.3 these laws are infinitely divisible, with the stated Lévy measure and drift. Lemma 5.7 excludes every  $\alpha \in (\alpha_*, 2)$ . The positive and negative parameter intervals meet at the contact with no unclassified parameter. This also proves the uniqueness of the threshold in (1.1).  $\square$

**Corollary 6.1.** *For every  $1 < \alpha < 2$  and every  $a \neq 0$ , the law of  $T_a$  is not self-decomposable.*

*Proof.* This follows from Remark 3.4 and scaling.  $\square$

The definition (1.3) specifies the critical parameter independently of its decimal approximation. The argument uses uniqueness of the contact only in its stated rectangle. It does not assert that the kernel has no other critical points for larger values of  $\alpha$ ; a single negative value, or either necessary-condition violation used above, is sufficient there.

## A Differentiated remainders and finite inversion

This appendix specifies the remainder bounds used in the critical enclosure. A dot denotes differentiation with respect to the real parameter  $\alpha$ , with  $q$  fixed. The notation is that of Lemma 3.1.

**Lemma A.1** (Differentiated finite expansion). *Fix  $c, Y > 0$  and  $N \geq 2$ , and suppose  $\arctan(Y/c) > \alpha\pi/4$  and  $\lambda_Y > 0$ . For  $q = c \pm iy$ ,  $y \geq Y$ , the remainders in (3.14) satisfy*

$$|\dot{E}_A(q)| \leq C'_A |q|^{-N}, \quad |\dot{E}_H(q)| \leq C'_H |q|^{-N}.$$

*Here primes on error constants denote bounds for differentiated remainders, not derivatives of previously chosen upper bounds. Explicit choices are*

$$\begin{aligned} G_0 &= \frac{\max(|c_N|, |c_{N-1}|)}{\eta}, & G_1 &= \frac{\max(|\dot{c}_N|, |\dot{c}_{N-1}|)}{\eta}, \\ D_0 &= \frac{\pi \sin \delta}{4\eta}, & D_1 &= \frac{2(1 + |\cos \delta|)}{\eta}, \\ B_0 &= G_0(|\psi(m)| + D_0) + G_1, & B_1 &= G_0(N+1) + \frac{|c_{N-1}|}{4\eta} + G_0 D_1, \end{aligned}$$

where  $\psi = \Gamma'/\Gamma$ . For  $v = 0, 1$ , set

$$p_v = m + N\alpha + v - 1, \quad M_v = \frac{\Gamma(p_v + 1)}{\mu^{p_v+1}}, \quad W_v = \frac{\Gamma(p_v + 2)}{\mu^{p_v+2}} + \frac{\Gamma(p_v)}{\mu^{p_v}} + \frac{\pi}{4} M_v.$$

Then  $C'_A = (B_0 M_0 + B_1 W_0)/\Gamma(m)$  and  $C'_H = (B_0 M_1 + B_1 W_1)/\Gamma(m)$ . The coefficient derivatives are computed without division by  $c_n$ :

$$\dot{c}_n = -\frac{\pi}{2 \sin \delta} \{(n+1) \cos((n+1)\delta) - c_n \cos \delta\}, \quad (\text{A.1})$$

$$\dot{A}_n = \frac{\Gamma(m + n\alpha)}{\Gamma(m)} \{\dot{c}_n + c_n[(n+1)\psi(m + n\alpha) - \psi(m)]\}, \quad (\text{A.2})$$

$$\dot{d}_0 = -\beta^2, \quad \dot{d}_n = n\dot{A}_n - \sum_{j=1}^{n-1} (\dot{A}_j d_{n-j} + A_j \dot{d}_{n-j}). \quad (\text{A.3})$$

For  $N \leq n \leq 2N-2$  put  $V_n = \sum_{0 \leq j < N, 0 \leq n-j < N} A_j d_{n-j}$ , and define

$$\begin{aligned} U_0 &= \sum_{n=N}^{2N-2} |V_n| Y^{N-n}, & U_1 &= \sum_{n=N}^{2N-2} |\dot{V}_n| Y^{N-n}, \\ P_0 &= \sum_{n=0}^{N-1} |d_n| Y^{-n}, & P_1 &= \sum_{n=0}^{N-1} |\dot{d}_n| Y^{-n}, \\ Z_1 &= \sum_{n=0}^{N-1} |\dot{A}_n| Y^{-n} + C'_A Y^{-N}, \\ C_R &= U_0 + \beta C_H + C_A P_0, & C'_R &= U_1 + \beta^2 C_H + \beta C'_H + C'_A P_0 + C_A P_1. \end{aligned}$$

An explicit differentiated quotient bound is

$$\left| \dot{F}_\alpha(q) - \sum_{n=0}^{N-1} \dot{d}_n q^{-n-1} \right| \leq C'_F |q|^{-N-1}, \quad C'_F = \frac{C'_R}{\lambda_Y} + \frac{C_R Z_1}{\lambda_Y^2}. \quad (\text{A.4})$$

*Proof.* For  $q = c - iy$ , use the fixed ray  $w = ue^{-i\pi/4}$  in the normalized integrals for  $A, H$ . Then  $z = w^\alpha/q$  has  $0 < \arg z < \delta/2$ ; the rotation is pole-free and (3.11) applies. On this fixed ray  $\dot{z} = z \log w$ , and hence

$$\left| \frac{\partial_\alpha D_\delta(z)}{D_\delta(z)} \right| \leq D_0 + D_1 |\log w|.$$

Differentiate the exact remainder  $w^\alpha z^N (c_N - c_{N-1}z)/(\Gamma(m)D_\delta(z))$ . Its modulus is bounded by  $u^{\alpha+N} |q|^{-N} (B_0 + B_1 |\log w|)/\Gamma(m)$ . Use  $|\log w| \leq |\log u| + \pi/4 \leq u + u^{-1} + \pi/4$  and  $|e^{-w}| \leq e^{-\mu u}$  to obtain the two gamma bounds. Conjugation handles the upper ray. In particular no  $\log q$  is introduced by this normalized, fixed-ray differentiation; the factor  $\kappa_\alpha$  cancels from  $F_\alpha = \beta H/(qA)$ . If  $S = \sum_{n < N} d_n q^{-n}$ , the exact residual is  $\beta H - AS = -\sum_{n=N}^{2N-2} V_n q^{-n} + \beta E_H - E_A S$ . Its bounds are  $C_R |q|^{-N}$  and  $C'_R |q|^{-N}$  after differentiation. Use  $|A| \geq \lambda_Y$ ,  $|\dot{A}| \leq Z_1$  and differentiate its quotient by  $qA$ . This proves (A.4). All estimates remain valid on closed real parameter intervals by outward interval evaluation.  $\square$

**Lemma A.2** (Filtered finite Bromwich formula). *Suppose  $F_\alpha$  is holomorphic on  $\Re q \geq c$  and  $\lambda_Y > 0$ . For the dotted formula also impose the ray condition of Lemma A.1. Let  $0 \leq j < p < N$  be integers,  $h > 0$ , and use  $R_p, r_p, b_k$  from (3.17). Set  $q_y = c + iy$  and*

$$I_k(t; x, Y) = \frac{(-t)^{k-1}}{\pi} \Im \Gamma(1-k, -t(x + iY)), \quad k \geq 1, \quad t > 0,$$

where the upper incomplete gamma function has its principal continuation. Then

$$K_\alpha^{(j)}(t) = r_p^{(j)}(t) + \frac{e^{ct}}{\pi} \Re \int_0^Y e^{iyt} q_y^j (F_\alpha(q_y) - R_p(q_y)) dy + T_j(t) + \varepsilon_j(t), \quad (\text{A.5})$$

$$T_j(t) = \sum_{n=j}^{N-1} d_n I_{n+1-j}(t; c, Y) - e^{-ht} \sum_{k=1}^p \sum_{l=0}^{\min(j, k-1)} b_k \binom{j}{l} (-h)^{j-l} I_{k-l}(t; c + h, Y), \quad (\text{A.6})$$

$$|\varepsilon_j(t)| \leq \frac{e^{ct} C_F}{\pi(N-j)Y^{N-j}}, \quad r_p^{(j)}(t) = e^{-ht} \sum_{k=1}^p \sum_{l=0}^{\min(j, k-1)} \binom{j}{l} (-h)^{j-l} \frac{b_k t^{k-1-l}}{(k-1-l)!}. \quad (\text{A.7})$$

The alpha derivative is given by the same formulas with  $F_\alpha, d_n, b_k, r_p$  replaced by their dotted versions and  $C_F$  by  $C'_F$ .

*Proof.* The coefficients of  $R_p$  agree with those of  $F_\alpha$  through degree  $p$  at infinity. Thus  $q^j(F_\alpha - R_p)$  is absolutely integrable on the vertical line. Split its Bromwich integral at  $Y$  and use Lemma 3.1. The identity  $I_k = \pi^{-1} \Re \int_Y^\infty e^{(x+iy)t} (x+iy)^{-k} dy$  follows by substitution into the incomplete gamma integral; for  $k = 1$  the integral is interpreted as its ordinary oscillatory limit. In  $q^j(\sum_{n < N} d_n q^{-n-1} - R_p)$  the polynomial parts cancel because  $p > j$ . The remaining proper fractions give  $T_j$  exactly. Finally  $|q_y| \geq y$  gives the displayed absolute remainder. The differentiated statement follows from Lemma A.1.  $\square$

For whole time intervals one may evaluate  $T_j$  at their midpoint  $t_*$ . Here is an explicit continuous-time error, when  $Y > h$  and  $p \geq j + 2$ . Define

$$a_n^R = \sum_{k=1}^{\min(p, n+1)} b_k (-h)^{n+1-k} \binom{n}{k-1}, \quad Q_N = \sum_{k=1}^p \frac{|b_k| \binom{N}{k-1} h^{N+1-k}}{(1-h/Y)^k}.$$

On  $[t_-, t_+]$  with  $t_- > 0$  and radius  $r$ , the additional error is at most

$$\frac{r e^{ct_+}}{\pi} \left\{ \sum_{n=p}^{N-1} \frac{|d_n - a_n^R|}{(n-j-1)Y^{n-j-1}} + \frac{Q_N}{(N-j-1)Y^{N-j-1}} \right\}. \quad (\text{A.8})$$

Indeed the binomial tail of  $R_p$  is bounded by  $Q_N |q|^{-N-1}$ ; differentiate the combined finite tail once in  $t$  and integrate its absolute value. The same bound with dotted coefficients handles its alpha derivative. The filtered convexity enclosures use  $c = h = 20$ ,  $Y = 1000$ ,  $N = 18$ ,  $p = 4$  and  $j = 2$ . The local contact records also use the following unshifted version, for  $j = 0, 1, 2$ :

$$\begin{aligned} K_\alpha^{(j)}(t) &= d_j + \frac{e^{ct}}{\pi} \Re \int_0^Y e^{iyt} \left( q_y^j F_\alpha(q_y) - \sum_{n=0}^j d_n q_y^{j-n-1} \right) dy \\ &\quad + \sum_{n=j+1}^{N-1} d_n I_{n+1-j}(t; c, Y) + \varepsilon_j(t). \end{aligned}$$

The error  $\varepsilon_j$  has the same bound as in (A.5). The polynomial terms that are subtracted have no ordinary inverse on  $t > 0$ , and the term  $d_j/q$  contributes the displayed constant  $d_j$ . At a midpoint  $t_*$  in  $[t_-, t_+]$ , the extra error for the finite tail is at most

$$\frac{r e^{ct_+}}{\pi} \left\{ \frac{2|d_{j+1}|}{t_- Y} + \sum_{n=j+2}^{N-1} \frac{|d_n|}{(n-j-1)Y^{n-j-1}} \right\}.$$

Indeed the derivative of the first tail term contains the oscillatory integral of  $e^{iyt}/(c+iy)$ ; integration by parts bounds it by  $2/(tY)$ . All remaining terms are absolutely integrable. The first alpha derivative uses the  $j = 0$  formula with dotted coefficients, dotted  $F_\alpha$ , and  $C'_F$  in its absolute remainder. These formulas require the independent zero-free-strip argument to identify the chosen line with the canonical kernel.

## B Computational certificates and reproduction

The numerical part of the proof consists of enclosures of entire closed parameter–time rectangles, together with analytic estimates on the two unbounded time ranges. No inference from a grid of point evaluations is used. The ancillary material contains the actual interval outputs, the programs that produced them, the independent readers, and the analytic error notes. The current entries are `PROOF_INDEX.txt`, `select_positive_records.json` and `reduced_active_cover.json`. The original complete snapshot is preserved; these lists specify the smaller subset required here.

## B.1 Finite certificates

Put  $a_- = 1.66706598$  and  $a_+ = 1.667066$ . The finite positive-side certificates are summarized below. Every displayed bound has been rounded in the conservative direction. Bounds in this table apply only to the listed compact time domains; the small- and large-time arguments in the proof are indispensable.

| Parameter range   | Time domain                   | Rectangles | Certified bound          |
|-------------------|-------------------------------|------------|--------------------------|
| [1, 6/5]          | [1/4, 33/5]                   | 6360       | $K > 2 \cdot 10^{-5}$    |
| [6/5, 13/10]      | [1/5, 6]                      | 5800       | $K > 7 \cdot 10^{-5}$    |
| [13/10, 3/2]      | [9/100, 6]                    | 11840      | $K > 0.063$              |
| [3/2, 33/20]      | [9/200, 6]                    | 8941       | $K > 0.016$              |
| [33/20, $a_-$ ]   | [.15, .17]                    | 980        | $\partial_\alpha K < -1$ |
| [33/20, $a_-$ ]   | [.045, .15] $\cup$ [.17, 6]   | 3096       | $K > 2 \cdot 10^{-4}$    |
| [ $a_-$ , $a_+$ ] | [.045, .145] $\cup$ [.175, 6] | 891        | $K > 10^{-5}$            |
| [ $a_-$ , $a_+$ ] | [.145, .175]                  | 30         | $\partial_t^2 K > 242$   |

The endpoint  $\alpha = 1$  in the first row belongs to the auxiliary normalized analytic family. It is not interpreted as a point-hitting probability law. There are 37 938 rectangles in the table. The fifth row propagates an already established positive right endpoint to smaller  $\alpha$ ; its negative derivative is not itself a positivity statement.

The time-end estimates require 16 finite-expansion parameter boxes and 17 real special-function bounds, without numerical transform integration. They replace 11 164 of the 49 102 compact rectangles in the complete archive. All 115 rectangles crossing a new cutoff are retained and restricted to the required time range; none is discarded by its midpoint.

For the remaining negative side, the saved-data readers verify four Gaussian monotonicity segments covering [1.6671, 1.6679], thirty-three Gaussian certificates covering [1.6679, 1.7386], and 102 Rouché discs covering [1.7386, 15/8]. The local negative minimum and its fixed-time derivative bridge connect these intervals to  $\alpha_*$ . All joins are comparisons of exact rational endpoints. The entire interval [15/8, 2) is supplied by Lemma 5.6. Thus 244 of the 346 archived zero discs and three of the 36 archived Gaussian intervals are unnecessary. Two Gaussian radii are enlarged within their already validated derivative domains as described in Lemma 5.3; their strict inequalities are recomputed.

## B.2 What an enclosure certifies

Most real balls are recorded as a binary midpoint and a nonnegative binary radius. Their endpoints are therefore exact rational numbers. Early [6/5, 13/10] records instead preserve outward decimal enclosures; the active reader checks their displayed lower endpoints directly. The readers check strict signs at these endpoints and verify that the input balls contain their claimed rational rectangles. They also check that the rectangles meet without gaps and that a saved finite approximation is expanded by every stated error term before the sign is tested. In the early slabs where input balls were not saved, input containment instead follows from the exact rational construction in the archived producer. The six archived display-rounding repair records concern only discarded large-time rectangles; every active saved kernel enclosure has a strictly positive lower endpoint without those repairs.

The integration programs use Arb/FLINT ball arithmetic [2] and explicit analytic bounds for the omitted integration ranges. For Fourier inversion, the error includes both the entire omitted frequency sum and the periodization error. Time and parameter variations are enclosed over the whole rectangle. Independent finite-sum readers recompute from the persisted transform balls, preserving their original radii and parameter midpoints. Increasing the precision of that last sum does not improve the accuracy of the earlier transform integrations. The Rouché readers recompute the strict boundary inequality from the saved function and first- and second-derivative balls, and check that the entire certified zero disc maps into the right half-plane under the principal power.

The analytic identification of these inversions with the canonical kernel, and the analytic bounds on the unbounded time intervals, are proved in the preceding sections. They are separate obligations from the finite rectangle checks. Likewise, a digest establishes which source and data were used; digest agreement alone does not establish an inequality.

### B.3 Portable reproduction

The unchanged ancillary snapshot comprises 1262 files and 3450 explicit keyed source/data digest bindings. The current scalar producers and active subset readers are supplied beside it. The launcher `reproduce_reduced.py` recomputes the time-end scalar bounds, checks the selected saved signs and exact interval geometry, and replays the reduced negative cover. It also checks the fixed rational comparisons in Lemma 5.6. It does not repeat original transform integrations.

The original relative-path launcher, `reproduce.py`, retains three verification levels for the complete archived computations:

1. `verify` checks archive integrity and the declared reference closure, without numerical computation.
2. `saved` reads the actual saved interval balls, recombines the stated budgets, verifies the finite signs and exact interval joins, and rechecks the local contact scalar inequalities. It does not repeat the original transform integrations.
3. `finite-sums` independently resums the critical Fourier certificates at the verification precision, including their analytic alias and frequency-tail bounds, using the saved integral balls.

The separate `script` mode runs an original producer and can perform fresh validated integration. The README gives example commands and the batch specifications needed for a full reproduction. All replay writes are isolated in a temporary copy; the snapshot is unchanged.

The recorded production environment is Python 3.12.14, `python-flint` 0.9.0, and FLINT 3.6.0. The software version used by an actual replay is recorded in its output. Numerical records and program sources are preserved exactly, apart from relative-path normalization of two metadata strings in each of 33 JSON files and one historical executable path. A change ledger records original and public digests and verifies that no numerical field was modified. The public package requires no access to the original machine or its directory structure.

These certificates constitute a computer-assisted proof with explicit analytic error bounds. They are not a formal proof-kernel verification, and the internal independent reviews should not be confused with external peer review. The distinction between integrity checks, saved-data checks, finite-sum recomputation, and fresh integration is retained in every replay log.

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