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Associated Varieties of Finite-Group Orbifolds

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Abstract

Let V be a simple, strongly finitely generated vertex algebra with a connected nonnegative integer grading, and let G be a finite group of grading-preserving automorphisms. In this paper, we prove that $X_{V^G} \cong X_V/G$ as reduced Poisson varieties. More precisely, the canonical surjection $R_{V^G} \longrightarrow (R_V)^G$ has nilpotent kernel. The proof uses a uniform derivative bound in a relative C_2-quotient, which also implies that V^G is strongly finitely generated. Consequently, quasi-lisseness and C_2-cofiniteness descend to finite-group orbifolds, and hence, by existing results, so does strong rationality.

1 Introduction

A central theme in invariant theory is understanding which properties of an algebraic object are preserved under taking invariants. The setting of interest in this paper is that of a finite group G acting on a vertex algebra V, an algebraic structure arising naturally in two-dimensional conformal field theory. The invariant subalgebra V^G, also sometimes called an *orbifold* (although this differs from the meaning of the term in physics), gives rise to several fundamental questions. For example, if the representation theory of V is semisimple and has only finitely many irreducible modules, does the same hold for V^G? Similarly, if V is strongly finitely generated, must V^G also be strongly finitely generated? The latter is a vertex-algebra analogue of Hilbert's classical theorem on finite generation of invariants. These and related questions have been the subject of substantial work over the years [2, 4, 5, 11, 12, 21, 25, 30, 36, 37, 43]. In this paper, we continue this work by describing the associated variety of V^G in terms of the associated variety of V and the action of G, as well as showing strong finite generation descends.

We work throughout with ordinary vertex algebras over $\mathbb{C}$. By a connected nonnegatively graded vertex algebra, we mean a vertex algebra V equipped with a grading

$$V = \bigoplus_{n \geq 0} V_n$$

such that $V_0 = \mathbb{C}|0\rangle$ and $a_m V_k \subseteq V_{h+k-m-1}$ for $a \in V_h$, where $V_j = 0$ for $j < 0$. For a vertex algebra W, Zhu's C_2-algebra [44] is defined by

$$C_2(W) = \operatorname{span}_{\mathbb{C}}\{a_{-2}b \mid a, b \in W\}, \qquad R_W = W/C_2(W).$$

The algebra R_W carries a natural commutative Poisson algebra structure, with multiplication and Poisson bracket given by

$$\overline{a} \cdot \overline{b} = \overline{a_{-1}b}, \qquad \{\overline{a}, \overline{b}\} = \overline{a_0 b},$$

where $\overline{a}$ denotes the image of a in R_W [28, 1]. The associated variety of W is the reduced affine Poisson variety

$$X_W := \operatorname{Spec}\big((R_W)_{\mathrm{red}}\big).$$

For a finite group G acting on V, write $X_V/G = \mathrm{Spec}(((R_V)_{\mathrm{red}})^G)$ for the affine quotient of its associated variety. Our main result is the following.

Theorem 1.1 (Associated varieties of finite-group orbifolds)**.** *Let V be a simple, strongly finitely generated vertex algebra with a connected nonnegative integer grading, and let G be a finite group of grading-preserving automorphisms. The canonical Poisson homomorphism $p_G : R_{V^G} \twoheadrightarrow (R_V)^G$ has nilpotent kernel. In particular it induces an isomorphism of reduced Poisson varieties*

$$X_V/G \xrightarrow{\sim} X_{V^G}.$$

This proves Conjecture 13.2 of Lian and Linshaw [30]. The map p_G is induced by the inclusion $V^G \subseteq V$, and its surjectivity follows from the exactness of taking finite-group invariants. Its kernel is given by $\ker p_G = \frac{V^G \cap C_2(V)}{C_2(V^G)}$, in other words the kernel records the additional C_2-relations obtained by allowing fields from all of V, rather than only V^G. The proof of Theorem 1.1 uses a uniform bound on derivatives in a relative C_2-quotient (Proposition 4.1), namely $R_{V^G}(V)$ where V is considered as a V^G-module. The same uniform bound also gives descent of strong finite generation.

Descent of strong finite generation. A vertex algebra is strongly finitely generated if it is spanned by normally ordered products of derivatives of finitely many fields. As mentioned above, the descent of strong finite generation is a vertex-algebra analogue of Hilbert's theorem. Linshaw and collaborators have studied this problem extensively, establishing strong finite generation and related derivative-order bounds for several important families of orbifolds by arbitrary complex reductive groups, including free-field algebras, affine vertex (super)algebras, generic-level $\mathcal{W}$-(super)algebras, and their tensor products [31, 32, 33, 34, 6, 7]. The following theorem establishes strong finite generation uniformly for finite-group orbifolds under our hypotheses.

Theorem 1.2 (Finite strong generation)**.** *Let V be a simple, strongly finitely generated vertex algebra with a connected nonnegative integer grading, and let G be a finite group of grading-preserving automorphisms. Then V^G is strongly finitely generated.*

Theorems 1.1 and 1.2 together have several consequences for finite-group orbifolds. We begin with the quasi-lisse property.

Quasi-lisse orbifolds and symplectic leaves. Building on the work of Zhu and Li [44, 28, 29], Arakawa introduced the associated variety X_V of a vertex algebra and gave a geometric interpretation of C_2-cofiniteness [1]. Namely, for a strongly finitely generated vertex algebra with a connected nonnegative integer grading, C_2-cofiniteness is equivalent to the associated variety being a point. Generalizing this condition, Arakawa and Kawasetsu introduced the notion of a quasi-lisse vertex algebra: a strongly finitely generated vertex algebra whose associated variety, viewed as an affine Poisson variety, has finitely many symplectic leaves [3]. They showed that a simple quasi-lisse VOA of CFT type has only finitely many simple ordinary modules, and that their normalized characters satisfy a common modular linear differential equation [3]. Following a suggestion of Dražen Adamović, Arakawa, Dai, Fasquel, Li, and Moreau conjectured that the quasi-lisse property descends to orbifolds by finite solvable groups [2, Conjecture B].

Since finite Poisson quotients preserve the finiteness of symplectic leaves [15], Theorems 1.1 and 1.2 together establish descent of quasi-lisse in the finite group case.

Corollary 1.3 (Quasi-lisse descent)**.** *Let V be a simple quasi-lisse vertex algebra with a connected nonnegative integer grading, and let G be a finite group of grading-preserving automorphisms. Then V^G is quasi-lisse.*

Strong rationality and vertex tensor categories. A central question in the theory of vertex operator algebras has been whether strong rationality descends from a VOA V to its fixed-point subalgebra V^G. Recall that a VOA is called *strongly rational* if it is simple, of CFT type, self-contragredient, rational, and C_2-cofinite. By a theorem of Huang, the category of ordinary modules of a strongly rational VOA is a modular tensor category [23]. Thus, descent of strong rationality would in particular imply that the category of ordinary V^G-modules is modular.

For brevity, we refer to the general question of descent of strong rationality as the *orbifold problem*. Progress towards its resolution was made by Miyamoto, who established C_2-cofiniteness for cyclic groups, and consequently for solvable groups by iteration [37]. Carnahan and Miyamoto subsequently proved regularity for solvable-group orbifolds under suitable hypotheses [5]. For arbitrary finite groups, McRae showed that strong rationality of V^G follows from that of V, provided that V^G is C_2-cofinite [36, Corollary 4.23].

Our description of the associated variety supplies this remaining condition. Indeed, if V is C_2-cofinite, then X_V is a point, and Theorem 1.1 implies that X_{V^G} is also a point. Since V^G is strongly finitely generated by Theorem 1.2, Lemma 2.1 shows that R_{V^G} is finitely generated. As its reduced spectrum is a point, R_{V^G} is finite dimensional, so V^G is C_2-cofinite. Combined with McRae's result, we obtain the following theorem, resolving the orbifold problem.

Theorem 1.4 (Descent of Strong Rationality)**.** *Let V be a strongly rational VOA and G a finite group of VOA automorphisms. Then V^G is strongly rational.*

The representation theory of V^G is also closely connected to another important feature of orbifold constructions: the appearance of twisted sectors. In the language of vertex operator algebras, the corresponding objects are g-twisted V-modules whose monodromy is determined by the action of $g \in G$. The mathematical theory of twisted modules has been developed extensively, with foundational contributions including the work of Frenkel, Lepowsky, and Meurman [20] and Dong, Li, and Mason [10, 11]. In particular, every g-twisted V-module restricts naturally to a module for V^G, and it is natural to ask how the tensor products of the different twisted sectors are related to the representation theory of V^G.

Huang conjectured that, for a suitable VOA and finite group G, the categories of g-twisted modules should together form a G-crossed braided tensor category [25, Conjecture 6.2]. Du and Huang subsequently developed the theory of twisted intertwining operators and their tensor products [14]. Recently, Xu constructed a G-crossed braided fusion category of twisted modules for strongly rational VOAs [41], and established a further crossed modularity result [42].

McRae's equivariantization theorem provides another approach to the G-crossed tensor category structure, assuming that the fixed-point algebra admits a suitable vertex tensor category of modules [36]. By our C_2-cofiniteness result, together with Huang's logarithmic tensor product theory [24, 27], these hypotheses are satisfied under the assumptions below, without requiring rationality.

Write $\mathrm{Mod}(W)$ for the category of grading-restricted generalized modules of a VOA W, and $\mathrm{TwMod}_g(V)$ for the corresponding category of g-twisted V-modules.

Theorem 1.5 (Twisted sectors of a finite orbifold)**.** *Let V be a simple C_2-cofinite VOA of CFT type and $G \leq \mathrm{Aut}(V)$ a finite group. Then*

$$\mathrm{TwMod}_G(V) = \bigoplus_{g \in G} \mathrm{TwMod}_g(V)$$

admits a natural G-crossed braided tensor category structure. Its equivariantization is braided tensor equivalent to $\mathrm{Mod}(V^G)$.

In the strongly rational setting, this gives an alternative proof of the existence of the G-crossed braided tensor category structure, using McRae's categorical approach. More generally, Theorem 1.5

applies without assuming rationality or semisimplicity, and hence also applies in logarithmic conformal field theory.

A further consequence concerns the reconstruction of twisted group doubles by holomorphic orbifolds. Theorem 1.4 removes the orbifold-rationality hypothesis in the reconstruction theorem of Gannon and the author [21, Theorem 3.2], based on Evans–Gannon [16]. We record the resulting unconditional realization in Section 7.3.

Finally, there is also a consequence for genus-one correlation functions. The C_2-cofiniteness of V^G allows us to apply Huang's modular invariance theorem for logarithmic intertwining operators [26]. In particular, the genus-one correlation functions for V^G constructed from shifted pseudo-traces span spaces closed under modular transformations (Corollary 7.2). Here pseudo-traces generalize ordinary traces to the nonsemisimple setting; for strongly rational orbifolds the corresponding statement holds with ordinary traces [22].

Proof strategy and organization. The main technical ingredient in this paper is a uniform derivative bound in a relative C_2-quotient. Write $B = V^G$ and set

$$Q = R_B(V) = V/\operatorname{span}_{\mathbb{C}}\{b_{-2}v \mid b \in B,\ v \in V\}, \qquad q : V \longrightarrow Q.$$

Choose a finite-dimensional graded G-stable space U of strong generators of V, and write $r = |G|$. By Noether's degree bound, applied to an associated graded algebra, the construction of invariant generators reduces to normally ordered products of derivatives of at most r elements of U. In other words, the degree, or number of factors in each product, is at most $|G|$. The orders of differentiation, however, may be arbitrarily large. Translation covariance gives $(T^j u)_{-1} = j!\, u_{-1-j}$, so we seek a constant $H = H(U, r)$ such that, for $k \le r$, homogeneous $u_i \in U$, and $j_i \ge 0$,

$$q\big((u_1)_{-1-j_1} \cdots (u_k)_{-1-j_k}|0\rangle\big) = 0 \quad \text{whenever} \quad \sum_{i=1}^{k} j_i > H - \sum_{i=1}^{k} \operatorname{wt} u_i.$$

The class on the left is the constant term in $z_1, \dots, z_k$ of

$$\frac{1}{j_1! \cdots j_k!} \partial_{z_1}^{j_1} \cdots \partial_{z_k}^{j_k} q(Y(u_1, z_1) \cdots Y(u_k, z_k)|0\rangle)\,,$$

where q acts coefficientwise. Thus the desired bound amounts to showing that derivatives of the projected product have zero constant term in Q for sufficiently large total derivative order, uniformly in the u_i and $k \le r$. Proposition 4.1 gives such a bound for any fixed finite-dimensional graded G-stable subspace of V and any prescribed maximum number of insertions.

To that end, consider n-point functions $Y(a_1, z_1) \cdots Y(a_n, z_n)|0\rangle$, where $\mathbf{z} = (z_1, \dots, z_n)$ and the a_i lie in fixed homogeneous G-stable input spaces $E_i \subset V_{h_i}$. Multiplication by suitable locality factors gives a formal power series $H_V(\mathbf{z})$; write $H_Q = q \circ H_V$ for its image in Q. We first show that, after taking invariants in a chosen subset of the inputs, sufficiently high derivatives of H_Q vanish when those inputs are translated together. In particular, this gives a system of differential equations for H_Q. However, the required orders depend on the subset and the weights of the chosen states, so these equations do not yet provide the desired uniform bound. More precisely, for a nonempty subset $I \subseteq [n]$, we introduce collision coordinates $z_i = t + x_i$ for $i \in I$, with one $x_i = 0$. Locality and weak associativity regroup the corresponding fields into an iterate in the common coordinate t. Taking invariants in the tensor factors indexed by I makes the coefficients of this iterate lie in B. The relative C_2-relations then give the partial-translation equations

$$\partial_I^{d_I}(H_Q \circ P_I) = 0, \qquad \partial_I = \sum_{i \in I} \partial_{z_i}, \qquad d_I = 1 + \sum_{\substack{i \in I \\ j \notin I}} (h_i + h_j),$$

where P_I projects onto invariants in the indicated factors (Section 3).

We encode these equations algebraically by replacing the commuting differential operators ∂_{z_i} with polynomial variables p_i. Let $E = \bigotimes_i E_i$, $R = \mathbb{C}[p_1, \dots, p_n]$, and $p_I = \sum_{i \in I} p_i$. Define the universal module

$$M = \frac{R \otimes E}{\sum_{\varnothing \neq I \subseteq [n]} R\, p_I^{d_I} P_I E}.$$

The coefficient map

$$R \otimes E \longrightarrow Q, \qquad \mathbf{p}^{\boldsymbol{\alpha}} \otimes e \longmapsto (\partial_{\mathbf{z}}^{\boldsymbol{\alpha}} H_Q)(\mathbf{0})(e)$$

factors through M by the partial-translation equations.

We now append specially chosen auxiliary fields, obtaining an enlarged input space $\widetilde{E}$ and regularized field product $\widetilde{H}_Q$. The auxiliary fields have two properties. First, their representation-theoretic structure ensures that the submodule $\mathcal{N}$ of the corresponding enlarged universal module, generated by the prescribed tensors, satisfies

$$\operatorname{Supp} \mathcal{N} \subseteq \{\mathbf{0}\}.$$

Since $\mathcal{N}$ is finitely generated, some power of the maximal ideal at the origin annihilates it. This gives a uniform bound on the total derivative order of $\widetilde{H}_Q$ on the prescribed tensors. Second, the auxiliary fields occur in pairs whose leading operator product coefficient is the vacuum. Successively colliding these pairs and extracting their vacuum coefficients transfers the bound to the original H_Q, proving Proposition 4.1 (Section 4).

Finally, the uniform derivative bound and Noether's degree bound give finite generation of R_B, hence strong finite generation of B by Lemma 2.1 (Section 5). The fixed-weight comparison with a decreasing filtration then allows us to apply this bound to the kernel of $R_B \to (R_V)^G$. Together with a differential invariant-theoretic nilpotence argument, this proves that the kernel is nilpotent (Section 6). Section 7 develops the applications, and the appendix supplies the formal expansion and vacuum-coefficient arguments.

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2 The relative C_2 quotient

We begin by fixing notation and establishing several preliminary results that will be used throughout the paper.

Set $B = V^G$ and replace G by its image in $\operatorname{Aut}(V)$. Write $r = |G|$ and $P_G = r^{-1} \sum_{g \in G} g$ for the projection onto invariants. We use $Y(a, z) = \sum_{m \in \mathbb{Z}} a_m z^{-m-1}$, write T for translation and $\operatorname{wt} a$ for the degree of a homogeneous state, and associate normally ordered products to the right.

2.1 Relative relations and the comparison map

For a vertex algebra W and a W-module M, write $R_W(M) = M/\operatorname{span}\{b_{-2}v : b \in W,\ v \in M\}$. We apply this construction to V as a B-module, writing

$$D_B = \operatorname{span}\{b_{-2}v : b \in B,\ v \in V\}, \qquad Q = R_B(V) = V/D_B, \qquad q : V \to Q. \tag{1}$$

The quotient is a graded G-module. Two properties account for its use here. First, translation gives $(T^{m-2}b)_{-2} = (m-1)!\, b_{-m}$ for $m \geq 2$, so

$$q(b_{-m}v) = 0 \qquad (b \in B,\ v \in V,\ m \geq 2). \tag{2}$$

Second, if $\sum_i b^i_{-2}v_i \in D_B$ is invariant, applying P_G replaces each v_i by $P_G v_i$. It follows that

$$D_B \cap B = C_2(B), \qquad Q^G \simeq R_B. \tag{3}$$

Indeed, every invariant class has an invariant lift by exactness. Since $D_B \subseteq C_2(V)$, taking invariants in $Q \to R_V$ now gives

$$0 \longrightarrow K_G \longrightarrow R_B \xrightarrow{p_G} (R_V)^G \longrightarrow 0, \qquad K_G = \frac{B \cap C_2(V)}{C_2(B)}. \tag{4}$$

The product $[a_{-1}b]$ and bracket $[a_0 b]$ are preserved by inclusion, so p_G is Poisson. We use Q only as a vector space with grading and G-action. All fields act in V before applying q.

2.2 Weight and total derivative order

A subspace $U \subset V$ strongly generates V if the words

$$W = u^1_{-n_1} \cdots u^\ell_{-n_\ell}|0\rangle, \qquad u^i \in U, \quad n_i \geq 1, \tag{5}$$

span V, including the empty word. We say that V is strongly finitely generated if U can be chosen finite dimensional. For homogeneous u^i of positive weight,

$$\operatorname{wt} W = \sum_i \operatorname{wt} u^i + \sum_i (n_i - 1). \tag{6}$$

The identity $(T^{n-1}u)_{-1} = (n-1)!\, u_{-n}$ identifies $n_i - 1$ with the derivative order of the ith field. Their sum $\sum_i (n_i - 1)$ is the *total derivative order* of this word.

Lemma 2.1 (Strong-generation lifting criterion)**.** *Let W be a vertex algebra with a connected nonnegative integer grading, and let $S \subseteq W_{>0}$ consist of homogeneous states. Then S strongly generates W if and only if its images generate $R_W = W/C_2(W)$ as a unital commutative algebra. In particular, W is strongly finitely generated if and only if R_W is finitely generated as a commutative algebra.*

Proof. If S strongly generates W, the spanning words (5) (with W in place of V) map to products of the images of S in R_W, since modes $-n$ for $n \geq 2$ vanish in R_W.

Conversely, put

$$C_1(W) = TW + \operatorname{span}_{\mathbb{C}}\{a_{-1}b : a, b \in W_{>0}\}.$$

We have $C_2(W) \subseteq C_1(W)$: for positive-weight b, $a_{-2}b = (Ta)_{-1}b$, while $a_{-2}|0\rangle = Ta$. If the images of S generate R_W, every positive-weight state is congruent modulo $C_2(W)$ to a polynomial in S. Terms involving at least two positive-weight factors lie in $C_1(W)$, so S spans $W_{>0}/C_1(W)$. Li's strong-generation criterion [28, Theorem 4.10] now shows that S strongly generates W.

Finally, a finitely generated graded algebra R_W admits a finite homogeneous generating set, whose positive-weight elements have homogeneous lifts to W. This proves the last assertion. □

At fixed input weights, (6) turns an output-weight bound into a bound on total derivative order. To obtain it uniformly we keep the words as coefficients of field products and use (2) before extracting those coefficients.

3 Differential equations for partial translations

Before proving the uniform derivative bound, we establish a system of differential equations for regularized products of vertex operators modulo Q. More precisely, after taking invariants in selected input factors, sufficiently high derivatives in their common translation direction vanish. We prove this by using locality and weak associativity to regroup the corresponding fields, and then applying the defining relations of Q.

3.1 Regularized n-point functions

Fix finite-dimensional G-stable subspaces $E_i \subset V_{h_i}$, $1 \le i \le n$. Set $E = \bigotimes_i E_i$ and $\mathbf{z} = (z_1, \dots, z_n)$. For a pure tensor $e = a_1 \otimes \cdots \otimes a_n$, define

$$\Delta(\mathbf{z}) = \prod_{i<j} (z_i - z_j)^{h_i + h_j}, \tag{7}$$

$$H_V(\mathbf{z})(e) = \Delta(\mathbf{z}) Y(a_1, z_1) \cdots Y(a_n, z_n)|0\rangle, \qquad H_Q = q \circ H_V. \tag{8}$$

Before regularization the product is expanded in $V((z_1)) \cdots ((z_n))$ or equivalently $|z_1| > \cdots > |z_n|$. Locality clears its poles along the diagonals, and creation accounts for regularity in each remaining coordinate [19, 17]. The weight formula below converts a polynomial degree bound for the regularized product into the output-weight cutoff used in Proposition 4.1. The proof is in Appendix A.1.

Lemma 3.1 (Regularity)**.** *We have $H_V \in \mathrm{Hom}_{\mathbb{C}}(E, V[[z_1, \dots, z_n]])$. A coefficient of total degree d has weight*

$$\sum_i h_i + d - \deg \Delta. \tag{9}$$

In particular, projection to V_w gives a homogeneous polynomial in $z_1, \dots, z_n$.

Let $I = \{i_1 < \cdots < i_m\}$ be a nonempty subset of $[n] = \{1, \dots, n\}$, with complement I^c. Translate the coordinates in I together by writing

$$z_i = t + x_i \quad (i \in I), \qquad x_{i_m} = 0. \tag{10}$$

Here $t = z_{i_m}$ and the x_i record displacements from it. Factor $\Delta = \Delta_I \Delta_{I^c} \Delta_{\mathrm{cross}}$, retaining the orientation of each difference. The first two factors contain the differences internal to I and I^c, respectively. Only the cross factor depends on t, and its t-degree is

$$D_I = \sum_{i \in I,\, j \notin I} (h_i + h_j).$$

For $I^c = \{l_1 < \cdots < l_{n-m}\}$, write the two internal products as

$$C_I(\mathbf{x}) = \Delta_I(\mathbf{x}) Y(a_{i_1}, x_{i_1}) \cdots Y(a_{i_{m-1}}, x_{i_{m-1}}) a_{i_m} = \sum_{\boldsymbol{\alpha} \ge \mathbf{0}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}},$$

$$W_{I^c}(\mathbf{z}_{I^c}) = \Delta_{I^c}(\mathbf{z}_{I^c}) Y(a_{l_1}, z_{l_1}) \cdots Y(a_{l_{n-m}}, z_{l_{n-m}})|0\rangle = \sum_{\boldsymbol{\gamma} \ge \mathbf{0}} v_{\boldsymbol{\gamma}} \mathbf{z}_{I^c}^{\boldsymbol{\gamma}}.$$

The first is the regularized vacuum product for I at $x_{i_m} = 0$, so both are power series by Lemma 3.1. For a singleton I, $C_I = a_{i_1}$, and for $I^c = \varnothing$ we put $W_{I^c} = |0\rangle$. We use the usual multi-index conventions, in particular $|\boldsymbol{\alpha}| = \sum_i \alpha_i$ and $\boldsymbol{\alpha}! = \prod_i \alpha_i!$, and compare multi-indices componentwise. Brackets $[\mathbf{x}^{\boldsymbol{\alpha}}]$ extract the indicated coefficient and leave the remaining variables formal.

The following form of formal associativity keeps the cross factor explicit. In the two-field case the product and iterate are different expansions of one expression [17, Proposition 2.6]. Appendix A.2 verifies the corresponding reexpansion for the indicated subset of fields, coefficient by coefficient.

Lemma 3.2 (Reexpansion at a common coordinate)**.** *In $V((t))[[\mathbf{x}, \mathbf{z}_{I^c}]]$ one has*

$$H_V(\mathbf{z})(e)|_{z_i = t + x_i \ (i \in I)} = \Delta_{\mathrm{cross}}(t + \mathbf{x}, \mathbf{z}_{I^c}) Y(C_I(\mathbf{x}), t)\, W_{I^c}(\mathbf{z}_{I^c}). \tag{11}$$

The left side is substitution in a power series. Before multiplication by the cross factor, the coefficient of $\mathbf{x}^{\boldsymbol{\alpha}} \mathbf{z}_{I^c}^{\boldsymbol{\gamma}}$ on the right is $Y(c_{\boldsymbol{\alpha}}, t) v_{\boldsymbol{\gamma}}$. Its lower Laurent bound in t may depend on both multi-indices.

3.2 Translation after taking invariants

Let P_I take invariants for the diagonal action on the factors of E indexed by I, leaving the other factors fixed. When H_V is precomposed with P_I, equivariance replaces every $c_{\boldsymbol{\alpha}}$ in (11) by $P_G c_{\boldsymbol{\alpha}} \in B$. This is where the relative relations enter.

Lemma 3.3 (Partial translations)**.** *For each nonempty $I \subseteq [n]$, put $\partial_I = \sum_{i \in I} \partial_{z_i}$ and $d_I = D_I + 1$. Then*

$$\partial_I^{d_I} (H_Q \circ P_I) = 0. \tag{12}$$

Proof. For $b = P_G c_{\boldsymbol{\alpha}}$, relation (2) and lower truncation give

$$qY(b, t) v_{\boldsymbol{\gamma}} = \sum_{m \geq -1} q(b_m v_{\boldsymbol{\gamma}}) t^{-m-1} \in Q[t^{-1}].$$

Multiplication by Δ_{cross} can therefore produce no t-power above D_I, uniformly in the two multi-indices. On the other side of (11) regularity excludes negative powers of t. The result is polynomial in t of degree at most D_I. With the displacements held fixed, $\partial_t = \sum_{i \in I} \partial_{z_i}$, giving (12). □

Already for two fields there is a direction these equations can miss. Take $G = \{\pm 1\}$ and let both inputs lie in sign representations. Then $P_{\{1\}} = P_{\{2\}} = 0$, whereas $P_{\{1,2\}}$ is the identity and $d_{\{1,2\}} = 1$. The only equation is $(\partial_{z_1} + \partial_{z_2}) H_Q = 0$, which admits power series in $z_1 - z_2$ of arbitrarily high degree. To bound the coefficients of the actual field product, we obtain further equations from products with additional insertions.

4 Uniform bounds on derivatives

We now bound the weights of the full product, including the dependence on coordinate differences left open by Lemma 3.3. Here V is simple and the input space may be any finite-dimensional graded G-stable subspace.

Proposition 4.1 (Uniform derivative bound)**.** *Let $X \subset V$ be a finite-dimensional graded G-stable subspace and let $s \geq 1$. There is an integer $H = H(X, s) \geq 0$ such that, for every $1 \leq k \leq s$ and $a_1, \ldots, a_k \in X$, all coefficients of*

$$qY(a_1, z_1) \cdots Y(a_k, z_k)|0\rangle$$

belong to $\bigoplus_{w=0}^{H} Q_w$.

To read this as a derivative bound, take homogeneous $a_i \in X$ of weights h_i, with $1 \le k \le s$, and put $W = (a_1)_{-1-j_1} \cdots (a_k)_{-1-j_k}|0\rangle$ for $j_i \ge 0$. By (6), W has weight $\sum_i h_i + \sum_i j_i$. Hence

$$q(W) \neq 0 \quad \Longrightarrow \quad \sum_{i=1}^{k} j_i \le H(X,s) - \sum_{i=1}^{k} h_i.$$

This bounds the sum of the derivative orders across all k insertions and hence each individual order j_i. For a word in strong generators, this sum is the Li filtration index of its specified symbol in Section 6.

The proof enlarges the product so that the equations from partial translations apply to more subsets of positions. The additional fields must do two things. At the level of G-modules their prescribed tensor must lie in enough images of the projections P_I to force a degree bound. At the level of vertex operators it must be possible to remove the extra insertions while retaining that bound. Group differences supply the first property, and a vacuum coefficient supplies the second.

4.1 Choosing the auxiliary pairs

The two identities below give different ways to place a tensor in the images of subset projections. The first allows the subset to lie anywhere among r positions. In the second every subset contains a prescribed position, at the cost of inserting the vectors $(1-g)a_g$ in the other positions.

Lemma 4.2. *For r finite-dimensional G-modules $F_1, \dots, F_r$,*

$$\bigotimes_{i=1}^{r} F_i = \sum_{\varnothing \neq I \subseteq [r]} \operatorname{im} P_I. \tag{13}$$

For G-modules F_0 and F_g indexed by $g \neq 1$, $v \in F_0$, and $a_g \in F_g$,

$$v \otimes \bigotimes_{g \neq 1} (1-g)a_g \in \sum_{J \subseteq G \setminus \{1\}} \operatorname{im} P_{\{0\} \cup J}. \tag{14}$$

Both statements may be tensored with arbitrary additional factors.

Proof. Give $A = \operatorname{Sym}(\bigoplus_{i=1}^{r} F_i)$ its multigrading by the r summands, and let A_+ be the ideal of elements of positive total degree. For $x \in A_+$, the nonleading coefficients of the orbit polynomial $\prod_{g \in G}(t - gx)$ belong to $(A_+)^G$. Evaluating this polynomial at x shows that $x^r \in A(A_+)^G$. Polarizing over $\mathbb{C}$ gives $A_+^r \subseteq A(A_+)^G$. In particular, this ideal contains the whole multidegree-$(1,\dots,1)$ component $\bigotimes_i F_i$.

For a tensor $v_1 \otimes \cdots \otimes v_r$, this gives an expression as a sum of products ab, where $a \in A$ and $b \in (A_+)^G$. Since G preserves the multigrading, each multihomogeneous component of b is invariant. Taking the multidegree-$(1,\dots,1)$ component, we may therefore assume a and b are multihomogeneous with multidegrees summing to $(1,\dots,1)$. Their supports partition $[r]$. Let I be the set of indices where b has degree one; it is nonempty because $b \in A_+$. Then

$$b \in \left(\bigotimes_{i \in I} F_i \right)^G, \qquad a \in \bigotimes_{i \notin I} F_i,$$

so $ab = P_I(ab) \in \operatorname{im} P_I$. Summing over these products proves (13).

For (14), consider the identity

$$\sum_{h \in G} hv \otimes \bigotimes_{g \neq 1} (ha_g - ga_g) = v \otimes \bigotimes_{g \neq 1} (a_g - ga_g). \tag{15}$$

Indeed, when $h \neq 1$, the factor indexed by $g = h$ in the summand on the left is zero. Fix an order on $G \setminus \{1\}$. For $J \subseteq G \setminus \{1\}$, define

$$b_g^J = \begin{cases} a_g, & g \in J, \\ g a_g, & g \notin J. \end{cases}$$

Expanding the left side of (15) by choosing $h a_g$ in the factors indexed by J and $-g a_g$ in the remaining factors gives

$$v \otimes \bigotimes_{g \neq 1} (1-g) a_g = r \sum_{J \subseteq G \setminus \{1\}} (-1)^{r-1-|J|} P_{\{0\} \cup J} \left(v \otimes \bigotimes_{g \neq 1} b_g^J \right).$$

Here $P_{\{0\}\cup J}$ averages the diagonal G-action on the indicated tensor factors, leaving all other factors fixed. The factor r comes from the normalization of this projection. The sum includes $J = \varnothing$, whose term uses $P_{\{0\}}$; the tensor factors retain their original order. Each summand therefore belongs to the corresponding image, proving (14). □

To realize the group differences as auxiliary fields, choose a homogeneous a_g with $u_g = (1-g)a_g \neq 0$ for each $g \neq 1$. Faithfulness gives such an a_g, and its weight is positive since G fixes V_0. We need a partner for u_g whose leading product with it is the vacuum. This is the sole use of simplicity in the proof of the uniform derivative bound. Appendix A.3 proves the following statement using the vacuum projection.

Lemma 4.3 (Vacuum coefficients)**.** *For every nonzero $u \in V_h$, $h > 0$, there is a homogeneous $w \in V_k$, $k > 0$, such that*

$$u_{h+k-1} w \neq 0 \quad \text{in } \mathbb{C}|0\rangle.$$

Choose and rescale w_g so that

$$(u_g)_{h_g + k_g - 1} w_g = |0\rangle, \qquad h_g = \operatorname{wt} u_g, \quad k_g = \operatorname{wt} w_g. \tag{16}$$

All higher modes vanish by the grading. Put $U_g = \operatorname{span}(G a_g)$ and $W_g = \operatorname{span}(G w_g)$. Given k original input spaces $E_i \subset V_{h_i}$ as in Section 3, write $E = \bigotimes_i E_i$ and set

$$\widetilde{E} = E \otimes \bigotimes_{g \neq 1} (U_g \otimes W_g)^{\otimes (2r-1)}, \qquad \xi = \bigotimes_{g \neq 1} (u_g \otimes w_g)^{\otimes (2r-1)}.$$

There are $N = (r-1)(2r-1)$ pairs, written (u_t, w_t) in this order for $1 \leq t \leq N$. The multiplicity ensures that after any $2r-2$ positions have been specified, a copy of every u_g is still available elsewhere. When $r = 1$ there are no pairs and $\xi = 1$.

4.2 Support of the prescribed tensors

For the $n = k + 2N$ positions of $\widetilde{E}$, introduce a variable p_i for differentiation in the ith field coordinate. The differential equations (12) are encoded by the module

$$R = \mathbb{C}[p_1, \dots, p_n], \qquad p_I = \sum_{i \in I} p_i, \qquad M = \frac{R \otimes \widetilde{E}}{\sum_{\varnothing \neq I \subseteq [n]} R\, p_I^{d_I} P_I \widetilde{E}}. \tag{17}$$

Let $\mathcal{N} \subseteq M$ be the submodule generated by the classes of $e \otimes \xi$ for $e \in E$. It is finitely generated since E is finite dimensional. The two-field example at the end of Section 3 explains why the subset projections by themselves can leave support away from the origin. It suffices to exclude that support for $\mathcal{N}$, since these are the tensors from which the auxiliary pairs can be removed.

Lemma 4.4 (Support at the origin)**.** *For some integer $q_0 \geq 1$,*

$$(p_1, \ldots, p_n)^{q_0} \mathcal{N} = 0. \tag{18}$$

Proof. For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, write $\mathfrak{m}_{\boldsymbol{\lambda}} = (p_1 - \lambda_1, \ldots, p_n - \lambda_n)$. We first prove that $\mathcal{N}_{\mathfrak{m}_{\boldsymbol{\lambda}}} = 0$ for every $\boldsymbol{\lambda} \neq \mathbf{0}$, and then use finite generation to obtain the required power of $(p_1, \ldots, p_n)$.

Fix such a $\boldsymbol{\lambda}$ and let $S = \{i : \lambda_i \neq 0\}$. By (17), the relations in M include $p_I^{d_I} P_I v = 0$ for every $v \in \widetilde{E}$ and every nonempty $I \subseteq [n]$. If $p_I(\boldsymbol{\lambda}) \neq 0$, then p_I is a unit in the local ring $R_{\mathfrak{m}_{\boldsymbol{\lambda}}}$, so

$$P_I v = 0 \quad \text{in } M_{\mathfrak{m}_{\boldsymbol{\lambda}}} \qquad (v \in \widetilde{E}).$$

We apply this observation in two cases.

Suppose first that $|S| \geq 2r - 1$. Choose a real line through the origin in $\mathbb{C}$ containing none of the nonzero λ_i. One of the corresponding open half-planes contains at least r of these numbers; choose their indices $J \subseteq S$ with $|J| = r$. There exists a real-linear functional $\ell : \mathbb{C} \to \mathbb{R}$ with $\ell(\lambda_i) > 0$ for every $i \in J$. Hence, for every nonempty $I \subseteq J$,

$$\ell\big(p_I(\boldsymbol{\lambda})\big) = \sum_{i \in I} \ell(\lambda_i) > 0.$$

Thus $p_I(\boldsymbol{\lambda}) \neq 0$, and the image of $P_I \widetilde{E}$ vanishes in $M_{\mathfrak{m}_{\boldsymbol{\lambda}}}$ for each such I. Applying (13) to the r factors indexed by J, and tensoring with the remaining factors, gives

$$\widetilde{E} = \sum_{\varnothing \neq I \subseteq J} P_I \widetilde{E}.$$

Since these tensors generate M over R, we obtain $M_{\mathfrak{m}_{\boldsymbol{\lambda}}} = 0$ and therefore $\mathcal{N}_{\mathfrak{m}_{\boldsymbol{\lambda}}} = 0$.

Now suppose $1 \leq |S| \leq 2r - 2$ and choose $j \in S$. In our tensor order, the original factors occupy positions $1, \ldots, k$, and the tth auxiliary pair occupies positions $k + 2t - 1$ and $k + 2t$, with u_t first and w_t second. For each $g \neq 1$, let $A_g \subseteq [n]$ be the set of positions occupied by the prescribed copies of u_g in ξ. These sets are pairwise disjoint, and each has $2r - 1$ elements. As $|S| \leq 2r - 2$, we can choose

$$i_g \in A_g \setminus S \qquad (g \neq 1).$$

The selected positions are distinct, and none equals j.

By linearity it suffices to take a pure tensor $e = e_1 \otimes \cdots \otimes e_k \in E$. For each $J \subseteq G \setminus \{1\}$, set

$$I_J = \{j\} \cup \{i_g : g \in J\}.$$

Let $\theta_J \in \widetilde{E}$ be obtained from $e \otimes \xi$ by replacing the selected copy of $u_g = (1 - g)a_g$ at position i_g with a_g if $g \in J$ and with $g a_g$ otherwise, leaving all other factors unchanged. These replacements lie in the specified G-stable spaces U_g. The explicit identity (15), as expanded in the proof of Lemma 4.2, applies with distinguished position j and the positions i_g. It gives

$$e \otimes \xi = r \sum_{J \subseteq G \setminus \{1\}} (-1)^{r-1-|J|} P_{I_J} \theta_J.$$

The factor at j may be one of the original inputs or an auxiliary field, since the identity allows an arbitrary vector in that position. All other factors retain their original tensor order, and the case $J = \varnothing$ is included.

Since every i_g lies outside S, we have

$$p_{I_J}(\boldsymbol{\lambda}) = \lambda_j + \sum_{g \in J} \lambda_{i_g} = \lambda_j \neq 0.$$

Our initial localization observation therefore makes every $P_{I_J}\theta_J$ vanish in $M_{\mathfrak{m}_{\boldsymbol{\lambda}}}$. Thus every generator $e\otimes\xi$ of $\mathcal{N}$ vanishes after localization, proving $\mathcal{N}_{\mathfrak{m}_{\boldsymbol{\lambda}}}=0$ also in this case.

If $\mathcal{N}=0$, take $q_0=1$. Otherwise, as $\mathcal{N}$ is finitely generated over R, $\operatorname{Supp}\mathcal{N}=V(\operatorname{Ann}_R\mathcal{N})$. We have excluded all maximal ideals except $\mathfrak{m}_{\mathbf{0}}$ from this closed set. The Nullstellensatz gives

$$(p_1,\dots,p_n)\subseteq\sqrt{\operatorname{Ann}_R\mathcal{N}}.$$

Choose integers $\nu_i\geq 1$ such that $p_i^{\nu_i}\mathcal{N}=0$ and put

$$q_0=1+\sum_{i=1}^{n}(\nu_i-1).$$

Any monomial not divisible by one of the $p_i^{\nu_i}$ has total degree at most $\sum_i(\nu_i-1)$. Consequently, every monomial of degree at least q_0 annihilates $\mathcal{N}$, proving (18). □

4.3 Removing the auxiliary pairs

Lemma 4.4 bounds the total order of derivatives on the prescribed inputs. The next step transfers the resulting weight bound through a collision of each auxiliary pair.

Proof of Proposition 4.1. By multilinearity fix $k\leq s$ and homogeneous components $E_i=X\cap V_{h_i}$ for the original fields. Use the enlarged product (8) on $\widetilde{E}$, with variables in tensor order

$$\boldsymbol{\zeta}=(z_1,\dots,z_k,x_1,y_1,\dots,x_N,y_N).$$

Write $\widetilde{\Delta}$ for its regularizing polynomial (7), $\widetilde{H}_Q$ for its image in Q, and $H_t=\operatorname{wt}u_t$, $K_t=\operatorname{wt}w_t$ for the weights in pair t.

The weight bound with the pairs inserted. Taylor coefficients define a linear map

$$\Phi:R\otimes\widetilde{E}\longrightarrow Q,\qquad \mathbf{p}^{\boldsymbol{\alpha}}\otimes v\longmapsto(\partial_{\boldsymbol{\zeta}}^{\boldsymbol{\alpha}}\widetilde{H}_Q)(\mathbf{0})(v). \tag{19}$$

This factors through M. Indeed, differentiating (12) and taking its constant coefficient kills every multiple of $p_I^{d_I}\otimes P_Iv$. Put $F(\boldsymbol{\zeta})(e)=\widetilde{H}_Q(\boldsymbol{\zeta})(e\otimes\xi)$. Its $\boldsymbol{\alpha}$-coefficient is $\Phi(\mathbf{p}^{\boldsymbol{\alpha}}\otimes(e\otimes\xi))/\boldsymbol{\alpha}!$, hence vanishes for $|\boldsymbol{\alpha}|\geq q_0$ by (18). This bounds the total degree of F uniformly in e. If $\widetilde{h}=\sum_{i=1}^{k}h_i+\sum_{t=1}^{N}(H_t+K_t)$, the degree–weight identity (9) gives

$$F_w=0\quad\text{for } w>H,\qquad H=\max\{0,\widetilde{h}+q_0-1-\deg\widetilde{\Delta}\}, \tag{20}$$

where the subscript denotes projection to Q_w.

The effect of one collision. Let $H_V^{(j)}$ be the regularized product of the original fields and the remaining pairs $j+1,\dots,N$, in their original order. Put $H_Q^{(j)}=qH_V^{(j)}$, so $H_Q^{(0)}=F$ and $H_Q^{(N)}=H_Q$. For $t=j+1$ write $x_t=y_t+d$. The internal factor of this pair in the regularizer is $d^{H_t+K_t}$. Write $B_t^{(j)}(d)$ for the product of its cross factors with all remaining fields. By (16) and the grading,

$$d^{H_t+K_t}Y(u_t,d)w_t=|0\rangle+d\rho_t(d),\qquad \rho_t(d)\in V[[d]].$$

Reexpanding this pair by Lemma 3.2 gives

$$H_Q^{(j)}\Big|_{x_t=y_t+d}=B_t^{(j)}(d)\,q\Big(Y(|0\rangle+d\rho_t(d),y_t)H_V^{(j+1)}\Big).$$

The expression on the right has only nonnegative powers of d. Its constant term is $B_t^{(j)}(0)H_Q^{(j+1)}$, since the vacuum field acts as the identity. On the left regularity makes the same operation the specialization $x_t = y_t$.

Returning to the original product. Iterating the one-pair identity yields

$$F|_{x_1=y_1,\dots,x_N=y_N} = C(\mathbf{z},\mathbf{y})\, H_Q(\mathbf{z}), \tag{21}$$

where the accumulated cross factors are

$$C(\mathbf{z},\mathbf{y}) = \prod_{i=1}^{k}\prod_{t=1}^{N}(z_i - y_t)^{2h_i+H_t+K_t} \prod_{1\le t<v\le N}(y_t - y_v)^{2(H_t+K_t+H_v+K_v)}. \tag{22}$$

For an original field and a pair there are two cross factors, and between distinct pairs there are four. Their exponents add as in (22), while the factors among the original fields remain in H_Q. Only the coordinates *within* a pair were identified. The different y_t are independent, so C is nonzero, with $C = 1$ if $N = 0$.

Fix $w > H$. By (20) the left side of (21) vanishes in weight w. For each $e \in E$, $(H_Q)_w(e)$ is a Q_w-valued polynomial by Lemma 3.1, so multiplication by the nonzero scalar polynomial C is injective. Hence $(H_Q)_w(e) = 0$. Multiplication by Δ^{-1}, expanded in the original ordered Laurent space, commutes with the projection to Q_w. The same vanishing therefore holds before regularization. Finally X has only finitely many weights and $k \le s$. Taking the maximum over these finitely many choices gives $H(X,s)$. □

Remark 4.5. The choice in (16) used only one vacuum coefficient for each nonidentity element of the effective group. Thus in Proposition 4.1 and Theorems 1.1 and 1.2 simplicity may be replaced by the existence, for each such g, of homogeneous a_g, w_g such that $u_g = (1-g)a_g \neq 0$ and

$$(u_g)_{\mathrm{wt}\, u_g + \mathrm{wt}\, w_g - 1} w_g \in \mathbb{C}^\times|0\rangle.$$

5 Descent of Strong Finite Generation

Our goal in this section is to prove that strong finite generation descends to finite-group orbifolds. More precisely, if V is strongly finitely generated and satisfies the standing hypotheses of Section 1, then $B = V^G$ is also strongly finitely generated. By Lemma 2.1, it suffices to prove that

$$R_B = B/C_2(B)$$

is finitely generated as a commutative algebra. The main ingredient is the uniform derivative bound of Proposition 4.1. To apply it, however, we first need a uniform bound on the number of factors appearing in the normally ordered words required to generate R_B.

We obtain such a bound using the increasing generating-weight filtration $F_\bullet V$, defined by the sum of the weights of the strong generators appearing in a word. More precisely, for a finite-dimensional, graded, G-stable strong generating space U, we consider invariant parts of words of the form

$$\beta = P_G\left((T^{j_1}u_1)_{-1}\cdots(T^{j_\ell}u_\ell)_{-1}|0\rangle\right), \qquad u_i \in U, \quad j_i \ge 0.$$

Here ℓ is the number of factors, while the filtration degree p and conformal weight w are related by

$$p = \sum_{i=1}^{\ell} \mathrm{wt}\, u_i, \qquad w = p + \sum_{i=1}^{\ell} j_i.$$

Passing to the associated graded takes us to a commutative differential algebra, where we can apply classical invariant theory. In particular, Noether's bound states that the invariant ring of a polynomial algebra under a finite group G is generated by invariants of polynomial degree at most $|G|$. This allows us to choose generating words with

$$\ell \le |G|,$$

independently of their derivative orders. We can therefore apply Proposition 4.1 with $s = |G|$, obtaining a uniform bound $H = H(U, |G|)$ such that

$$q(\beta) \neq 0 \quad \Longrightarrow \quad w \le H, \qquad \sum_{i=1}^{\ell} j_i = w - p \le H - p.$$

Thus both the number of factors and their total derivative order are bounded. Since U is finite-dimensional, only finitely many such generating words remain. Their classes generate R_B, so Lemma 2.1 gives strong finite generation of V^G.

5.1 The generating-weight filtration

Take V simple and strongly finitely generated. Choose a finite-dimensional graded G-stable strong generating space $U \subset V_{>0}$ by taking the G-span of homogeneous generators. Write $U_h = U \cap V_h$, set $F_pV = 0$ for $p < 0$, and let

$$F_pV = \operatorname{span}\left\{ u^1_{-n_1} \cdots u^\ell_{-n_\ell}|0\rangle : n_i \ge 1,\ u^i \in U \text{ homogeneous},\ \sum_i \operatorname{wt} u^i \le p \right\} \qquad (p \ge 0). \tag{23}$$

The empty word gives $F_0V = \mathbb{C}|0\rangle$. In this filtration $T^j u$ has the same filtration bound as u and weight $\operatorname{wt} u + j$. Li's good-filtration theorem [28, Theorem 4.11] gives

$$(F_pV)_m F_qV \subseteq \begin{cases} F_{p+q}V, & m < 0, \\ F_{p+q-1}V, & m \ge 0. \end{cases}$$

Thus $A_F = \operatorname{gr}_F V$ is a commutative differential algebra. It is generated by $D^j\bar{u}$ for homogeneous $u \in U$ and $j \ge 0$, where $\bar{u}$ is the symbol in degree $\operatorname{wt} u$ and D is induced by T.

Restrict F to B and give R_B the quotient filtration. In weight w, $(F_pR_B)_w$ is the image of $F_pV \cap B_w$ in $(R_B)_w$. An invariant symbol in A_F has an invariant lift by applying P_G. Taking the quotient with the induced filtration on $C_2(B)$ then gives

$$\operatorname{gr}_F B = A_F^G, \qquad C_F := \operatorname{gr}_F R_B = A_F^G / \operatorname{gr}_F C_2(B). \tag{24}$$

We write $(C_F)_{p,w} = (F_pR_B)_w/(F_{p-1}R_B)_w$ for the component of filtration degree p and original weight w. For $u \in U_h$, the specified symbol $D^j\bar{u}$ in A_F has bidegree $(h, h+j)$ whenever it is nonzero.

5.2 Applying Noether's bound

The filtration in (24) serves two purposes. Generators of C_F lift to generators of R_B. Their bidegrees will also measure how many derivatives can survive in the comparison kernel.

Proposition 5.1. *Let $H_U = H(U, r)$ from Proposition 4.1. The algebra C_F is generated by the symbols of invariant parts of normally ordered products of derivatives of at most r elements of U, of total weight at most H_U. These symbols constitute a finite bihomogeneous generating set when the inputs range over a fixed homogeneous basis of U.*

Proof. For each $j \geq 0$, take a copy $U^{(j)}$ of the G-module U. There is a G-equivariant surjection

$$S = \mathrm{Sym}\left(\bigoplus_{j\geq 0} U^{(j)}\right) \longrightarrow A_F, \qquad u^{(j)} \longmapsto D^j \bar{u}.$$

It preserves both degrees and induces a surjection $S^G \to A_F^G$. Every polynomial belongs to some $S_{\leq J} = \mathrm{Sym}(\bigoplus_{j=0}^{J} U^{(j)})$. Noether's theorem [38] gives generators of $S_{\leq J}^G$ of polynomial degree at most r, independently of J. Taking the union over J shows that S^G is generated by $P_G\mathsf{m}$ for basis monomials $\mathsf{m} = u_1^{(j_1)} \cdots u_\ell^{(j_\ell)}$ with $1 \leq \ell \leq r$. For $h_i = \mathrm{wt}\, u_i$, write the number of factors, the sum of the generating weights, the total derivative order, and the original weight as

$$\deg_{\mathrm{pol}} \mathsf{m} = \ell, \qquad p = \sum_i h_i, \qquad j = \sum_i j_i, \qquad w = p + j. \tag{25}$$

Noether bounds ℓ. Even with ℓ and p fixed, the derivative orders can make w arbitrarily large.

Fix an order on the basis symbols. Lift $P_G\mathsf{m}$ to

$$\beta_{\mathsf{m}} = P_G\Big((T^{j_1}u_1)_{-1} \cdots (T^{j_\ell}u_\ell)_{-1}|0\rangle\Big) \in F_pB \cap B_w. \tag{26}$$

Its degree-p symbol is the image of $P_G\mathsf{m}$ by the good-filtration product rule. This statement allows that symbol to be zero. The specified symbols generate A_F^G and hence their images generate C_F.

The lift in (26) equals $\prod_i j_i!$ times the invariant part of

$$[z_1^{j_1} \cdots z_\ell^{j_\ell}] Y(u_1, z_1) \cdots Y(u_\ell, z_\ell)|0\rangle.$$

There are at most r insertions, whatever the j_i. Hence Proposition 4.1 and (3) give

$$w > H_U \quad \Longrightarrow \quad q(\beta_{\mathsf{m}}) = 0 \quad \Longrightarrow \quad \beta_{\mathsf{m}} \in C_2(B). \tag{27}$$

Its specified symbol then vanishes in C_F. For $w \leq H_U$, only finitely many basis words and derivative orders occur. Their images therefore generate the particular associated graded algebra in (24). □

We may now show finite strong generation descends to V^G.

Theorem 1.2 (Finite strong generation)**.** *Let V be a simple, strongly finitely generated vertex algebra with a connected nonnegative integer grading, and let G be a finite group of grading-preserving automorphisms. Then V^G is strongly finitely generated.*

Proof. Lift the finite generating symbols of Proposition 5.1 to R_B using (26). They generate R_B by induction on filtration degree. Indeed, express the symbol of a class in $(F_pR_B)_w$ as a bihomogeneous polynomial in these generators. Subtract the same polynomial in the lifts to obtain a class in $(F_{p-1}R_B)_w$. The induction starts with the constants in F_0R_B. Lemma 2.1 now gives finite strong generation of B. □

6 Associated varieties of finite-group orbifolds

We now prove Theorem 1.1. Write $B = V^G$ and $K_G = \ker(p_G : R_B \to (R_V)^G)$. Our goal is to show that K_G is nilpotent.

The proof uses two filtrations on V, which serve different purposes. The increasing generating-weight filtration $F_\bullet V$ from Section 5 was used to construct a finite set of generators of R_B. We now use the canonical decreasing filtration $E^\bullet V$, which records derivative order at fixed weight and

satisfies $E^1V = C_2(V)$. The two filtrations are related in each homogeneous weight by Arakawa's comparison [1, Proposition 2.6.1]:

$$E^jV \cap V_w = F_{w-j}V \cap V_w. \tag{28}$$

Thus the bounds on generating symbols obtained in Section 5 also give bounds in the decreasing filtration.

To detect K_G, we use the filtration on R_B induced by $B \cap E^jV$, rather than the intrinsic decreasing filtration of B. We first establish nilpotence in the associated graded algebra and then use the uniform derivative bound to lift this nilpotence to R_B.

6.1 The decreasing filtration and the fixed-weight comparison

The canonical decreasing filtration, introduced by Li [29], is defined by

$$E^jV = \operatorname{span}_{\mathbb{C}} \left\{ a^1_{-1-m_1} \cdots a^s_{-1-m_s} b : a^i, b \in V,\ m_i \geq 0,\ \sum_i m_i \geq j \right\} \qquad (j \geq 0). \tag{29}$$

Unlike the generating-weight filtration, this definition allows arbitrary states $a^i, b \in V$, not only the chosen strong generators. We have $E^0V = V$ and $E^1V = C_2(V)$. Translation takes E^jV into $E^{j+1}V$, and $(E^iV)_{-1}(E^jV) \subseteq E^{i+j}V$ [29, Theorem 2.12 and Corollary 3.6].

The fixed-weight comparison (28) can also be seen directly from the defining words. By (25), a defining U-word of weight w with generating-weight sum at most $w - j$ has at least j derivatives. Conversely, in a homogeneous word in (29) of weight w, the weights of $a^1, \dots, a^s, b$ sum to $w - \sum_i m_i \leq w - j$. Each input of weight h lies in F_hV by (6), so the negative-mode product rule puts the word in $F_{w-j}V$. In particular $E^jV \cap V_w = 0$ for $j > w$.

Set $A = \operatorname{gr}_E V$, with $A_{j,w} = (E^jV \cap V_w)/(E^{j+1}V \cap V_w)$. Write $A_j = \bigoplus_w A_{j,w}$ and $A_{>0} = \bigoplus_{j>0} A_j$. The differential generation theorem [29, Corollary 4.3] gives

$$A_0 = R_V, \qquad A_{>0} = A\,D(A), \tag{30}$$

where D is induced by T. Indeed, D raises E-degree by one, and A is differentially generated by A_0. Thus every monomial of positive E-degree contains a derivative.

6.2 Nilpotence of the comparison kernel

The following differential identity controls products of positive-degree symbols in the associated graded algebra. The uniform derivative bound controls the degree of each generating symbol.

Lemma 6.1. *Let A be a commutative $\mathbb{C}$-algebra with derivation D. If a group G of order r acts by algebra automorphisms commuting with D, then*

$$(A\,D(A))^{2r-1} \subseteq A\,D(A^G). \tag{31}$$

Proof. The ideal $I = A\,D(A^G)$ is D-stable since $Df \in A^G$ for $f \in A^G$. Every invariant in $\overline{A} = A/I$ lifts to A^G, so the induced derivation kills $\overline{A}^G$. For $a \in \overline{A}$, let $P(t) = \prod_{g\in G}(t - ga)$. Its coefficients are invariant. Differentiating $P(a) = 0$ gives $P'(a)Da = 0$. We claim that

$$P^{(j)}(a)(Da)^{2j-1} = 0 \qquad (1 \leq j \leq r),$$

where $P^{(j)}$ is the jth derivative in t. Applying D to the jth identity and multiplying by Da gives

$$P^{(j+1)}(a)(Da)^{2j+1} + (2j-1)P^{(j)}(a)(Da)^{2j-1}D^2a = 0.$$

The second term vanishes by induction. At $j = r$ we have $P^{(r)} = r!$, so $(Da)^{2r-1} = 0$ for every $a \in \overline{A}$. Polarizing over $\mathbb{C}$ shows that every product of $2r-1$ derivatives in $\overline{A}$ vanishes. This proves (31). $\square$

We are now ready to prove the main result of the paper.

Theorem 1.1 (Associated varieties of finite-group orbifolds)**.** *Let V be a simple, strongly finitely generated vertex algebra with a connected nonnegative integer grading, and let G be a finite group of grading-preserving automorphisms. The canonical Poisson homomorphism $p_G : R_{V^G} \twoheadrightarrow (R_V)^G$ has nilpotent kernel. In particular it induces an isomorphism of reduced Poisson varieties*

$$X_V/G \xrightarrow{\sim} X_{V^G}.$$

Proof. Identify the kernel in the inherited filtration. Give $B = V^G$ the filtration $B \cap E^j V$ inherited from V, and denote the resulting quotient filtration on R_B by $\mathcal{E}$. By $E^1 V = C_2(V)$ and (4),

$$\mathcal{E}^j R_B = \frac{(B \cap E^j V) + C_2(B)}{C_2(B)}, \qquad \mathcal{E}^1 R_B = \frac{B \cap C_2(V)}{C_2(B)} = K_G. \tag{32}$$

The first step of the intrinsic decreasing filtration of B is $C_2(B)$, whose image in R_B is zero. The (-1)-product rule for E gives $\mathcal{E}^i R_B \, \mathcal{E}^j R_B \subseteq \mathcal{E}^{i+j} R_B$.

Put $C = \operatorname{gr}_{\mathcal{E}} R_B$. Restricting (28) to B_w and taking images in $(R_B)_w$ gives $(\mathcal{E}^j R_B)_w = (F_{w-j} R_B)_w$. Passing to consecutive quotients therefore identifies the two gradings,

$$C_{j,w} = \frac{(\mathcal{E}^j R_B)_w}{(\mathcal{E}^{j+1} R_B)_w} = \frac{(F_{w-j} R_B)_w}{(F_{w-j-1} R_B)_w} = (C_F)_{w-j,w}. \tag{33}$$

Both products are induced by multiplication in R_B, so the generating symbols of Proposition 5.1 generate C after this change of indices. As for A, write $C_j = \bigoplus_w C_{j,w}$ and $C_{>0} = \bigoplus_{j>0} C_j$.

Nilpotence in the associated graded algebra. Exactness of invariants identifies $(A_j)^G$ with $(B \cap E^j V)/(B \cap E^{j+1} V)$ in each E-degree. By the definition of $\mathcal{E}$ this space maps onto C_j. Together these maps give a bigraded algebra surjection $\pi : A^G \to C$. For a bihomogeneous invariant symbol f, choose an invariant lift $b \in B$. In the next E-degree, Df is represented by $Tb = b_{-2}|0\rangle \in C_2(B)$, so $\pi(Df) = 0$. Taking invariant parts of the coefficients in a sum $\sum_i a_i D f_i$ with $f_i \in A^G$ gives

$$(A\, D(A^G))^G = A^G D(A^G) \subseteq \ker \pi.$$

Every class of positive $\mathcal{E}$-degree in C lifts in each bidegree to $A^G \cap A_{>0}$. By (30) and Lemma 6.1, the product of $2r-1$ such lifts belongs to $(A\, D(A^G))^G$. Hence

$$(C_{>0})^{2r-1} = 0. \tag{34}$$

Lift back to R_B. Use the finite set of generating symbols from Proposition 5.1, reindexed by (33). Each has $\mathcal{E}$-degree $j = w - p \le w \le H_U$, where $H_U = H(U, r)$. Let d be the largest positive $\mathcal{E}$-degree of the nonzero symbols in this set, with $d = 0$ if there is none. By (34) a nonzero monomial contains at most $2r-2$ positive-degree generators. Each contributes at most d, and generators of degree zero contribute nothing. Consequently

$$C_j = 0 \qquad (j > M), \qquad M = (2r-2)d. \tag{35}$$

Fix an original weight w. By (33) and (35), the spaces $(\mathcal{E}^j R_B)_w$ and $(\mathcal{E}^{j+1} R_B)_w$ agree for every $j > M$. Choose $J = \max\{M+1, w+1\}$, so $E^J V \cap V_w = 0$ by (28). These equalities give

$$(\mathcal{E}^{M+1} R_B)_w = (\mathcal{E}^J R_B)_w = 0.$$

This holds in every weight with the same M. By multiplicativity and (32), the kernel in the original algebra satisfies

$$K_G^{1+(2r-2)d} = (\mathcal{E}^1 R_B)^{M+1} \subseteq \mathcal{E}^{M+1} R_B = 0. \tag{36}$$

Since p_G is surjective with nilpotent kernel, it induces the first isomorphism below. The second follows by taking invariants in $R_V \to (R_V)_{\mathrm{red}}$, since the nilradical of $(R_V)^G$ is $(R_V)^G \cap \sqrt{(0)}$ in R_V. We obtain

$$(R_B)_{\mathrm{red}} \xrightarrow{\sim} ((R_V)^G)_{\mathrm{red}} \xrightarrow{\sim} ((R_V)_{\mathrm{red}})^G.$$

In characteristic zero the nilradical is stable under derivations, in particular under Hamiltonian derivations. Taking spectra therefore gives the canonical Poisson isomorphism $X_V/G \simeq X_B$. □

7 Consequences for finite-group orbifolds

We conclude by deriving several consequences of the description $X_{V^G} \simeq X_V/G$ and the descent of strong finite generation. The first concerns the Poisson geometry of associated varieties; the remaining applications concern the representation theory of vertex operator algebras.

7.1 Quasi-lisse orbifolds and ordinary characters

Theorem 1.1 identifies X_{V^G} with the finite Poisson quotient X_V/G. Since finite Poisson quotients preserve the finiteness of symplectic leaves [15], we obtain the following consequence.

Corollary 1.3 (Quasi-lisse descent)**.** *Let V be a simple quasi-lisse vertex algebra with a connected nonnegative integer grading, and let G be a finite group of grading-preserving automorphisms. Then V^G is quasi-lisse.*

Proof. By Theorem 1.2, the vertex algebra V^G is strongly finitely generated. Since G acts on X_V by Poisson automorphisms, the finite quotient X_V/G has finitely many symplectic leaves whenever X_V does [15]. Theorem 1.1 identifies this quotient with X_{V^G}. □

This has representation-theoretic consequences. If V is a simple quasi-lisse VOA of CFT type, then so is $B = V^G$. By the theorems of Arakawa and Kawasetsu [3], B has only finitely many simple ordinary modules, and the normalized characters of its ordinary modules satisfy a common modular linear differential equation. These conclusions do not require B to be rational or C_2-cofinite.

7.2 C_2-cofiniteness and strong rationality

The zero-dimensional case gives a stronger finiteness property. For a strongly finitely generated vertex algebra W with connected nonnegative grading, Lemma 2.1 shows that R_W is a finitely generated commutative algebra. Consequently, X_W is a point if and only if R_W is finite dimensional. Thus the condition that X_W is a point is equivalent to C_2-cofiniteness.

Corollary 7.1 (C_2-cofiniteness)**.** *Let V be a simple vertex algebra with a connected nonnegative integer grading, and let G be a finite group of grading-preserving automorphisms. If V is C_2-cofinite, then V^G is C_2-cofinite.*

Proof. Since R_V is finite dimensional, Lemma 2.1 shows that V is strongly finitely generated. Its associated variety is a point, so Theorem 1.1 gives that X_{V^G} is a point. By Theorem 1.2, the orbifold is strongly finitely generated. The preceding criterion therefore implies that R_{V^G} is finite dimensional. □

We can now complete the deduction of strong rationality.

Theorem 1.4 (Descent of Strong Rationality)**.** *Let V be a strongly rational VOA and G a finite group of VOA automorphisms. Then V^G is strongly rational.*

Proof. McRae proved that if V is strongly rational and G is finite, then V^G is strongly rational provided it is C_2-cofinite [36, Corollary 4.23]. The latter condition follows from Corollary 7.1. □

In particular, when V is strongly rational, the category of ordinary V^G-modules is a modular tensor category [23]. For the tensor-category structure of twisted sectors, the weaker hypothesis of C_2-cofiniteness already suffices.

7.3 Twisted sectors and holomorphic orbifolds

Let V now be a simple C_2-cofinite VOA of CFT type and $G \leq \mathrm{Aut}(V)$ finite. A grading-restricted generalized g-twisted V-module restricts naturally to a V^G-module, since every element of V^G has trivial g-monodromy. We restate the resulting categorical structure from the introduction.

Theorem 1.5 (Twisted sectors of a finite orbifold)**.** *Let V be a simple C_2-cofinite VOA of CFT type and $G \leq \mathrm{Aut}(V)$ a finite group. Then*

$$\mathrm{TwMod}_G(V) = \bigoplus_{g \in G} \mathrm{TwMod}_g(V)$$

admits a natural G-crossed braided tensor category structure. Its equivariantization is braided tensor equivalent to $\mathrm{Mod}(V^G)$*.*

Proof. Corollary 7.1 makes $B = V^G$ C_2-cofinite, and B inherits CFT type from V. Huang's cofiniteness theorem [24], together with logarithmic tensor-category theory [27], supplies the requisite braided vertex tensor category structure on $\mathrm{Mod}(B)$. McRae's Theorems 4.15 and 4.17 [36] then give the G-crossed braided category of twisted modules and identify its G-equivariantization with $\mathrm{Mod}(B)$. □

In the strongly rational case, related G-crossed tensor structures and crossed modularity are studied by Xu [41, 42].

Holomorphic orbifolds and reconstruction. Theorem 1.4 removes the orbifold-rationality assumption in the reconstruction theorem of Gannon and the author [21, Theorem 3.2], building on Evans–Gannon [16]. Consequently, for every finite group G and class $[\omega] \in H^3(G, \mathbb{C}^\times)$, there is a holomorphic VOA V with a G-action such that

$$\mathrm{TwMod}_G(V) \simeq \mathrm{Vec}_G^\omega, \qquad \mathrm{Mod}(V^G) \simeq \mathcal{Z}(\mathrm{Vec}_G^\omega),$$

as tensor and braided tensor categories, respectively. The reconstruction is that of [16, 21]; our result supplies the previously conditional orbifold-rationality input.

7.4 Genus-one modular invariance

A final consequence follows from C_2-cofiniteness without any semisimplicity assumption. Huang's modular invariance theorem for logarithmic intertwining operators [26] applies to the fixed-point VOA by Corollary 7.1.

Corollary 7.2 (Genus-one modular invariance)**.** *Let V be a simple C_2-cofinite VOA of CFT type, let $G \leq \mathrm{Aut}(V)$ be finite, and set $B = V^G$. Fix $n \geq 1$, grading-restricted generalized B-modules $W_1, \dots, W_n$, and vectors $w_i \in W_i$. For these insertions, the shifted pseudo-traces of products of geometrically modified logarithmic B-intertwining operators extend analytically to genus-one n-point functions. As the admissible pseudo-trace data vary, these functions span a space closed under modular transformations [26, Theorem 5.5]. If V is strongly rational, the same conclusion holds with ordinary traces.*

Proof. By Corollary 7.1, B is C_2-cofinite and has no negative conformal weights. The first assertion is Huang's modular invariance theorem [26, Theorem 5.5] applied to B. If V is strongly rational, so is B by Theorem 1.4, and the semisimple assertion follows from [22, Theorem 7.3]. □

A Formal products and vacuum coefficients

We verify the expansion conventions used for partial translations and collisions in Sections 3 and 4. All calculations of field products here take place in V. The last argument gives the vacuum partners used to remove the auxiliary pairs.

A.1 Regularity

Proof of Lemma 3.1. For homogeneous a_i, a_j, the grading gives $(a_i)_\ell a_j = 0$ for $\ell \geq h_i + h_j$. The commutator formula therefore makes $(z_i - z_j)^{h_i+h_j}$ a locality factor. In the regularized product, move $Y(a_i, z_i)$ next to the vacuum. Creation gives $e^{z_i T} a_i$, so no negative powers of z_i occur. This holds for every i, proving joint regularity. Before multiplication by Δ, each coefficient of total degree s has weight $\sum_i h_i + s$. Homogeneity of Δ gives (9). At fixed weight only one total degree is possible, so the projection of the power series is a polynomial. □

A.2 Reexpansion at a common coordinate

Proof of Lemma 3.2. We first establish the identity for an arbitrary fixed homogeneous vector $v \in V$. We then apply it coefficientwise to the expansion $W_{I^c}(\mathbf{z}_{I^c}) = \sum_{\boldsymbol{\gamma} \geq \mathbf{0}} v_{\boldsymbol{\gamma}} \mathbf{z}_{I^c}^{\boldsymbol{\gamma}}$.

A fixed complementary vector. For this calculation only, relabel the indices of I as $1, \dots, m$, preserving their order, and put $x_m = 0$. Write Δ_I for the corresponding internal polynomial and $C(\mathbf{x})$ for $C_I(\mathbf{x})$ in these labels. Fix $v \in V_d$ and put

$$F_v(\mathbf{z}) = \Delta_I(\mathbf{z}) Y(a_1, z_1) \cdots Y(a_m, z_m) v, \qquad K_i = h_i + d, \qquad \Pi(\mathbf{z}) = \prod_{i=1}^{m} z_i^{K_i}.$$

Insert $Y(v, s)$ immediately before the vacuum and put

$$\Psi(\mathbf{z}; s) = \Delta_I(\mathbf{z}) \prod_{i=1}^{m} (z_i - s)^{K_i} Y(a_1, z_1) \cdots Y(a_m, z_m) Y(v, s) |0\rangle.$$

By Lemma 3.1, $\Psi \in V[[\mathbf{z}, s]]$. Setting $s = 0$ and using creation gives $\Psi(\mathbf{z}; 0) = \Pi(\mathbf{z}) F_v(\mathbf{z})$, so

$$F_v \in \mathcal{A} := V[[z_1, \dots, z_m]][z_1^{-1}, \dots, z_m^{-1}].$$

There is a linear reexpansion map

$$\sigma : \mathcal{A} \longrightarrow V((t))[[x_1, \dots, x_{m-1}]]$$

defined by ordinary substitution $z_i = t + x_i$, $z_m = t$ on power-series numerators, and by

$$z_i^{-1} \longmapsto (t + x_i)^{-1} = \sum_{j \geq 0} (-1)^j t^{-j-1} x_i^j \quad (i < m), \qquad z_m^{-1} \longmapsto t^{-1}.$$

The numerator substitution preserves total degree, so each joint coefficient is a finite sum. If a denominator is $\prod_i z_i^{b_i}$, then the coefficient of $\mathbf{x}^{\boldsymbol{\alpha}}$ after substitution has no t-power below $-\sum_i b_i - |\boldsymbol{\alpha}|$. The transformed coordinates are units in $\mathbb{C}((t))[[\mathbf{x}]]$, so σ is independent of the denominator and commutes with multiplication by scalar polynomials. In particular $\Pi(t + \mathbf{x}, t)\sigma(F_v) = \Psi(t + \mathbf{x}, t; 0)$.

All scalar factors of Ψ are coordinate differences. Translation covariance and $T|0\rangle = 0$ therefore give

$$\left(\sum_{i=1}^{m} \partial_{z_i} + \partial_s \right) \Psi = T\Psi, \qquad \Psi(\mathbf{z} + u; s + u) = e^{uT} \Psi(\mathbf{z}; s).$$

The second equality is the formal Taylor identity for the first. Both sides are joint power series. Substituting $z_i = x_i$ for $i < m$, $z_m = 0$, $s = -t$, and $u = t$ consequently gives

$$\Psi(t + \mathbf{x}, t; 0) = e^{tT} \Psi(\mathbf{x}, 0; -t).$$

In Ψ, locality allows $Y(v, s)$ to be moved to the far left while retaining every factor $(z_i - s)^{K_i}$ in its original orientation. The field $Y(a_m, z_m)$ is then rightmost, so constant-coefficient extraction in z_m uses creation and yields

$$\Psi(\mathbf{x}, 0; -t) = \Pi(t + \mathbf{x}, t) Y(v, -t) C(\mathbf{x}).$$

For each coefficient $c_{\boldsymbol{\alpha}}$ of C, skew symmetry gives $e^{tT} Y(v, -t) c_{\boldsymbol{\alpha}} = Y(c_{\boldsymbol{\alpha}}, t)v$. Lower truncation makes the exponential action well defined at each coefficient. Combining the identities gives

$$\Psi(t + \mathbf{x}, t; 0) = \Pi(t + \mathbf{x}, t) Y(C(\mathbf{x}), t)v.$$

The constant coefficient of $\Pi(t + \mathbf{x}, t)$ in $\mathbf{x}$ is $t^{\sum_i K_i}$, so $\Pi(t + \mathbf{x}, t)$ is a unit in $\mathbb{C}((t))[[\mathbf{x}]]$. Cancellation gives

$$\sigma(F_v) = Y(C(\mathbf{x}), t)v.$$

For the two-field calculation with these expansion conventions, see [18, correction to Theorem 3.2.1].

Reassembling the complementary product. Restore the original labels and write $\mathbf{y} = \mathbf{z}_{I^c}$. Locality, with the original orientations of all factors retained, gives

$$H_V(\mathbf{z})(e) = \Delta_{\mathrm{cross}}(\mathbf{z}_I, \mathbf{y}) \sum_{\boldsymbol{\gamma} \geq \mathbf{0}} F_{v_{\boldsymbol{\gamma}}}(\mathbf{z}_I) \mathbf{y}^{\boldsymbol{\gamma}}.$$

Each $v_{\boldsymbol{\gamma}}$ is homogeneous when nonzero. Write the scalar cross factor as the finite sum

$$\Delta_{\mathrm{cross}}(\mathbf{z}_I, \mathbf{y}) = \sum_{\boldsymbol{\delta} \in D} p_{\boldsymbol{\delta}}(\mathbf{z}_I) \mathbf{y}^{\boldsymbol{\delta}}.$$

For any fixed $\boldsymbol{\beta}$, coefficient extraction gives

$$[\mathbf{y}^{\boldsymbol{\beta}}] H_V(\mathbf{z})(e) = \sum_{\substack{\boldsymbol{\delta} \in D \\ \boldsymbol{\delta} \leq \boldsymbol{\beta}}} p_{\boldsymbol{\delta}}(\mathbf{z}_I) F_{v_{\boldsymbol{\beta} - \boldsymbol{\delta}}}(\mathbf{z}_I).$$

Apply σ to this finite sum. On the left it is substitution in the power series H_V. On each term on the right it commutes with the scalar polynomial and gives the iterate already computed. This proves (11) at each $\mathbf{y}^{\boldsymbol{\beta}}$, allowing the coordinate-pole bounds to depend on the complementary coefficient. □

A.3 Vacuum coefficients

Proof of Lemma 4.3. Simplicity enters through the ideal of states invisible to all vacuum coefficients. Let $\pi_0 : V \to V_0 = \mathbb{C}|0\rangle$ be the graded projection, and put

$$J = \{v \in V : \pi_0 Y(a,z)v = 0 \text{ for all } a \in V\}.$$

For homogeneous a, the contributions of distinct homogeneous components of v to $\pi_0 Y(a,z)v$ occur at distinct powers of z. Thus J is graded. For $a, b \in V$ and $v \in J$, the definition of J gives

$$\pi_0 Y(Y(a,x)b,t)v = 0.$$

By formal associativity [17, Proposition 2.6], this iterate and $\pi_0 Y(a,z_1)Y(b,z_2)v$ are expansions of the same element of

$$V_0[[z_1,z_2]][z_1^{-1}, z_2^{-1}, (z_1-z_2)^{-1}].$$

For the iterate, set $z_1 = t + x$, $z_2 = t$ and expand $(t+x)^{-1}$ in nonnegative powers of x/t. Clearing the displayed denominators reduces injectivity of the expansion to the invertible linear change of variables on the power-series numerator. Thus $\pi_0 Y(a,z_1)Y(b,z_2)v = 0$. Taking coefficients shows that $b_m v \in J$ for all $b \in V$ and $m \in \mathbb{Z}$. Since $\pi_0 T = 0$, translation covariance gives

$$\pi_0 Y(a,z)Tv = -\partial_z \pi_0 Y(a,z)v = 0.$$

Thus J is stable under all left modes and T, hence is an ideal by skew symmetry. Since $|0\rangle \notin J$, simplicity gives $J = 0$.

Since $u \notin J$, some homogeneous $a \in V_k$ satisfies $a_{h+k-1}u \in \mathbb{C}^\times|0\rangle$. Necessarily $k > 0$, since $Y(|0\rangle, z)u = u$. Skew symmetry $Y(u,z)a = e^{zT}Y(a,-z)u$ gives

$$u_{h+k-1}a = (-1)^{h+k} a_{h+k-1}u.$$

The terms involving positive powers of T have zero vacuum component. Take $w = a$. □

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