

C_2 -Cofiniteness of Finite-Group Orbifolds

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Abstract

Let V be a simple vertex algebra with a connected nonnegative integer grading, and let G be a finite group of grading-preserving automorphisms. We prove that C_2 -cofiniteness of V implies C_2 -cofiniteness of V^G . The proof bounds the length and height of monomials in strong generators representing classes in a relative C_2 quotient. The length bound follows from a good filtration and finite-group invariant theory. The height bound follows from differential equations for products of fields, together with identities for invariants and vacuum coefficients. The argument also shows that finite strong generation passes to V^G , without assuming C_2 -cofiniteness. For strongly rational vertex operator algebras, the cofiniteness result and McRae's orbifold theorem imply strong rationality of every finite-group orbifold.

1 Introduction

Taking the fixed-point subalgebra of a vertex algebra under a finite group of automorphisms, or orbifolding, is a basic operation in vertex operator algebra theory and conformal field theory [2, 3, 7, 8]. A longstanding conjecture predicts that finite-group orbifolds of strongly rational vertex operator algebras are again strongly rational.

The C_2 condition, introduced by Zhu in his work on modular invariance [14], is central to this conjecture. For a vertex algebra A , write

$$C_2(A) = \text{span}_{\mathbb{C}}\{a_{-2}b : a, b \in A\}, \quad R_A = A/C_2(A).$$

The quotient R_A is a commutative algebra with product $[a][b] = [a_{-1}b]$, and A is C_2 -cofinite when R_A is finite dimensional. For vertex operator algebras, this condition implies that Zhu's algebra is finite dimensional [3, Proposition 3.6], and hence that there are only finitely many irreducible admissible modules up to isomorphism [3, Theorem 3.4(b)].

Miyamoto proved C_2 -cofiniteness for cyclic orbifolds of simple C_2 -cofinite vertex operator algebras of CFT type, and hence for finite solvable groups [12]. In subsequent work, Carnahan and Miyamoto established regularity of solvable fixed points for simple nonnegatively graded regular vertex operator algebras with a nondegenerate invariant bilinear form [1]. For arbitrary finite groups, McRae proved that a fixed-point algebra of a strongly rational vertex operator algebra is strongly rational as soon as it is C_2 -cofinite [11, Corollary 4.23]. This reduces the strong-rationality problem to C_2 -cofiniteness.

We prove that C_2 -cofiniteness passes to fixed points for every finite group. More precisely, let V be a simple vertex algebra over $\mathbb{C}$ with a *connected nonnegative integer grading*, meaning

$$V = \bigoplus_{n \geq 0} V_n, \quad V_0 = \mathbb{C}|0\rangle, \quad a_m V_k \subseteq V_{h+k-m-1} \quad (a \in V_h),$$

with $V_n = 0$ for $n < 0$. Let G be a finite group of grading-preserving automorphisms.

Theorem 1.1. *If V is C_2 -cofinite, then V^G is C_2 -cofinite.*

In particular, the theorem applies to simple C_2 -cofinite vertex operator algebras of CFT type. McRae's theorem then implies that every finite-group orbifold of a strongly rational vertex operator algebra is strongly rational (Corollary 5.1). We also prove that finite strong generation passes to fixed points, without assuming C_2 -cofiniteness.

Theorem 1.2. *Let V be a simple, strongly finitely generated vertex algebra with a connected nonnegative integer grading. For every finite group G of grading-preserving automorphisms, V^G is strongly finitely generated.*

Under the connected-grading assumption above, Theorem 1.2 resolves the finite-group case of the vertex algebra Hilbert problem formulated in [10, Conjecture 1.1].

Notation. We write tuples of formal variables and multi-indices in bold, and their individual components in italics. For $\mathbf{z} = (z_1, \dots, z_m)$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_{\geq 0}^m$, put

$$\mathbf{z}^{\boldsymbol{\alpha}} = \prod_{i=1}^m z_i^{\alpha_i}, \quad |\boldsymbol{\alpha}| = \sum_{i=1}^m \alpha_i, \quad \boldsymbol{\alpha}! = \prod_{i=1}^m \alpha_i!, \quad \partial_{\mathbf{z}}^{\boldsymbol{\alpha}} = \prod_{i=1}^m \partial_{z_i}^{\alpha_i}.$$

For a formal series with vector coefficients, $[\mathbf{z}^{\boldsymbol{\alpha}}]$ denotes coefficient extraction, leaving unextracted variables unchanged. Inequalities between multi-indices are componentwise. Write $[n] = \{1, \dots, n\}$, $I^c = [n] \setminus I$, and $\mathbf{z}_I = (z_i)_{i \in I}$, with indices in increasing order. All tensor products are over $\mathbb{C}$, and linear maps act coefficientwise on formal series.

Proof Strategy for Theorem 1.1. The argument has three steps.

1. *The Relative Quotient.* Set

$$Q = V / \text{span}_{\mathbb{C}}\{b_{-2}v : b \in V^G, v \in V\}.$$

Taking invariants gives $Q^G \simeq V^G / C_2(V^G)$, so it suffices to show that Q is finite dimensional.

2. *The Bounds.* Choose a finite-dimensional graded G -stable subspace $U \subset V_{>0}$ that strongly generates V . Then Q is spanned by classes of monomials

$$u_{-n_1}^1 \cdots u_{-n_\ell}^\ell |0\rangle, \quad u^i \in U, \quad n_i \geq 1.$$

We call ℓ the *length* and $\max_i n_i$ the *height* of a nonempty monomial. To prove that Q is finite dimensional, it suffices to bound both quantities for a spanning family.

The length bound follows from C_2 -cofiniteness and standard finite-group invariant theory, using a filtration by the sum of the generating states' weights (Section 2). This part of the argument is comparatively straightforward. The main difficulty is the height bound, which controls how large the mode indices can be. Its proof occupies Sections 3 and 4.

3. *The Height Bound.* We obtain a bound after adjoining auxiliary fields, then transfer it to the original n -point functions.

- A. *Differential Equations.* We derive differential equations that constrain the n -point functions after passing to Q .

- B. *Auxiliary Fields.* We extend these functions by adjoining suitable auxiliary fields. Their properties, together with the differential equations and commutative algebra, give a weight bound for the extended n -point functions.
- C. *Contractions.* Taking vacuum contractions removes the auxiliary fields and transfers the weight bound to the original n -point functions. This gives the required height bound.

Together, the length and height bounds give a finite spanning set for Q , proving Theorem 1.1.

The proof of the height bound does not use C_2 -cofiniteness. Under the same simplicity and grading assumptions, this allows us to show that finite strong generation also passes to V^G .

Proof Strategy for Theorem 1.2. The argument has four steps.

1. *The Associated Graded Algebra.* Choose a finite-dimensional graded G -stable strong generating space for V . A filtration lets us pass to a commutative associated graded algebra. We present this as a quotient of a symmetric algebra on infinitely many copies of the generating space, one for each derivative order.
2. *Noether's Bound.* Applying Noether's theorem to finitely many copies at a time gives a possibly infinite family of invariant generators of polynomial degree at most $|G|$. This bounds the number of factors in the corresponding lifts.
3. *Lifting and the Height Bound.* We lift these generators to invariant normally ordered products involving at most $|G|$ factors. Their classes generate the commutative algebra $V^G/C_2(V^G)$. The height bound makes all sufficiently high-weight lifts vanish in this quotient, leaving only finitely many generators.
4. *Strong Generation.* The images of this finite family then span $(V^G)_{>0}/C_1(V^G)$. Li's strong-generation criterion shows that the family strongly generates V^G , proving Theorem 1.2.

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2 The Length Bound

Throughout, V is a simple vertex algebra with a connected nonnegative integer grading, and G is a finite group of grading-preserving automorphisms. We may replace G by its image in $\text{Aut}(V)$ without changing V^G , and hence assume the action is faithful. Write $r = |G|$, $B = V^G$, and $P_G = r^{-1} \sum_{g \in G} g$ for the Reynolds projection on any G -module under consideration. We use the mode convention $Y(a, z) = \sum_{m \in \mathbb{Z}} a_m z^{-m-1}$ and write T for translation. For homogeneous $a \in V_h$, write $\text{wt } a = h$. Normally ordered products are associated to the right. In this section, assume that R_V is finite dimensional.

2.1 Strong Generators

A subset $S \subset V$ *strongly generates* V if V is spanned by the monomials

$$W = u_{-n_1}^1 \cdots u_{-n_\ell}^\ell |0\rangle, \quad \ell \geq 0, \quad u^i \in S, \quad n_i \geq 1. \quad (1)$$

The empty product is the vacuum. If S can be chosen finite, V is *strongly finitely generated*. A subspace strongly generates V if its elements do. For homogeneous positive-weight u^i , the monomial (1) has length ℓ and weight

$$\text{wt } W = \sum_i \text{wt } u^i + \sum_i (n_i - 1). \quad (2)$$

For a nonempty monomial, its height is $\max_i n_i$; in particular, $\text{ht}(W) \leq \text{wt } W$. The identity $(T^{n-1}u)_{-1} = (n-1)!u_{-n}$ expresses these monomials as right-associated normally ordered products of derivatives, up to nonzero scalar factors [10, Section 3].

Write $R_{V,+} = \bigoplus_{w>0} (R_V)_w$. Choose a graded linear section $s : R_{V,+} \rightarrow V_{>0}$ of the quotient map. Set

$$s_G(x) = \frac{1}{r} \sum_{g \in G} g s(g^{-1}x), \quad U = s_G(R_{V,+}) \subset V_{>0}.$$

The map s_G is again a section and is G -equivariant, so U is a finite-dimensional graded G -stable space of lifts. Let

$$C_1(V) = TV + \text{span}\{a_{-1}b : a, b \in V_{>0}\}.$$

The identities $a_{-2}b = (Ta)_{-1}b$ and $T|0\rangle = 0$ give $C_2(V) \subseteq C_1(V)$. Hence $V_{>0} = U + C_1(V)$, and induction on weight shows that U strongly generates V ; see [9, Theorem 4.10]. In particular, each V_w is finite dimensional: after choosing a finite homogeneous basis of U , both the length and the sum of the derivative orders of a monomial of weight w are bounded.

2.2 The Relative Quotient

For a vertex algebra W and a W -module M , write

$$R_W(M) = M / \text{span}\{w_{-2}m : w \in W, m \in M\}.$$

Thus $R_W(W) = R_W$. Regard V as a module over $B = V^G$ and abbreviate $R_B(V)$ to Q . Set

$$D_B = \text{span}\{b_{-2}v : b \in B, v \in V\}, \quad Q = V/D_B, \quad q : V \longrightarrow Q. \quad (3)$$

The subspace D_B is graded and G -stable, so Q inherits a grading and a G -action, and q is G -equivariant. Translation gives $(T^{m-2}b)_{-2} = (m-1)!b_{-m}$, so

$$q(b_{-m}v) = 0 \quad (b \in B, m \geq 2). \quad (4)$$

Applying P_G to $\sum_j b_{-2}^j v_j \in D_B \cap B$ replaces each v_j by $P_G v_j \in B$. Hence

$$D_B \cap B = C_2(B), \quad Q^G \simeq R_B. \quad (5)$$

For the second identification, an invariant class $q(v)$ has the invariant representative $P_G v$, since $q(P_G v) = P_G q(v) = q(v)$. The kernel of $q|_B$ is $D_B \cap B = C_2(B)$. It is therefore enough to show that Q is finite dimensional. We use Q as a graded vector space; all field operations below are performed in V before applying q .

2.3 The Length Estimate

Lemma 2.1. *Let A be a commutative G -algebra and J a G -stable ideal. Then*

$$J^r \subseteq AJ^G,$$

where AJ^G is the ideal generated by the invariant elements of J .

Proof. For $x \in J$, the nonleading coefficients of $\prod_{g \in G}(t - gx)$ lie in J^G . Evaluating at $t = x$ therefore gives $x^r \in AJ^G$. Apply the same argument in $A[t_1, \dots, t_r]$, with G fixing the variables, to $x = t_1 x_1 + \dots + t_r x_r$, where $x_i \in J$. The coefficient of $t_1 \dots t_r$ is $r! x_1 \dots x_r$; it belongs to AJ^G , since taking invariants commutes with adjoining the fixed variables. Since $r! \neq 0$ over $\mathbb{C}$, the claim follows. $\square$

The positive-weight ideal $R_{V,+}$ of the finite-dimensional, connected graded algebra R_V is nilpotent. Choose $\nu \geq 1$ with

$$R_{V,+}^\nu = 0. \quad (6)$$

For $p \geq 0$, define

$$F_p V = \text{span} \left\{ u_{-n_1}^1 \dots u_{-n_\ell}^\ell |0\rangle : \ell \geq 0, n_i \geq 1, u^i \in U \text{ homogeneous}, \sum_i \text{wt } u^i \leq p \right\}. \quad (7)$$

The empty monomial is $|0\rangle$; set $F_{-1}V = 0$, so $F_0V = \mathbb{C}|0\rangle$. This filtration uses the sum of the weights of the generating states, independently of the mode indices. Since $[T, u_{-n}] = nu_{-n-1}$ and $T|0\rangle = 0$, translation preserves the filtration while raising the weight by one. In particular, $T^j u \in V_{h+j} \cap F_h V$ for $u \in U \cap V_h$.

Since $U \subset V_{>0}$ is a graded strong generating subspace and $V_0 = \mathbb{C}|0\rangle$, Li's theorem [9, Theorem 4.11] gives the good-filtration identities

$$(F_p V)_m F_q V \subseteq \begin{cases} F_{p+q} V, & m < 0, \\ F_{p+q-1} V, & m \geq 0. \end{cases}$$

Thus $A = \text{gr}_F V$ is a commutative differential algebra. For homogeneous $u \in U$, let $\bar{u}$ be its class in $F_{\text{wt } u} V / F_{\text{wt } u-1} V$. The elements $T^j \bar{u}$, $j \geq 0$, generate A , where T denotes the induced derivation. Each $F_p V$ is graded and G -stable, and

$$V_w \subseteq F_w V, \quad C_2(V) \cap V_w \subseteq F_{w-1} V. \quad (8)$$

The first inclusion follows from (2). For $a \in V_h$ and $b \in V_k$, the filtration properties give $a_{-2}b = (Ta)_{-1}b \in F_{h+k} V$. This vector lies in V_{h+k+1} , proving the second inclusion.

Lemma 2.2 (Length Bound). *Every class in Q is represented by a linear combination of monomials (1) in U of length at most*

$$L = r\nu - 1.$$

Proof. Let $\mathfrak{a} \subset A$ be the ideal generated by the symbols of elements of U . We first show

$$\mathfrak{a}^\nu = 0. \quad (9)$$

Take ν homogeneous elements of U , of total weight w . Their normally ordered product lies in $C_2(V)$ by (6), so its symbol in filtration degree w vanishes by (8). That symbol is the product of the ν generating symbols. These products generate $\mathfrak{a}^\nu$ as an ideal, proving (9).

Consider the commutative algebra

$$\bar{A} = A/AT(A^G), \quad J = (\bar{A})_{>0},$$

where the subscript denotes positive filtration degree. The quotient is graded, with degree-zero part $\mathbb{C}|0\rangle$, and J is generated by the images of the $T^j \bar{u}$. We claim

$$(J^G)^\nu = 0. \quad (10)$$

Taking invariants gives

$$\bar{A}^G \simeq A^G / (A^G T(A^G)),$$

since applying P_G to the coefficients in $\sum_j a_j T(f_j)$, $f_j \in A^G$, gives $(AT(A^G))^G = A^G T(A^G)$. Every positive-degree invariant is a linear combination of the elements $P_G(m)$, where m is a monomial in the $T^j \bar{u}$. The identity

$$P_G((Ta)b) = TP_G(ab) - P_G(a(Tb))$$

gives, by iteration,

$$P_G((T^k \bar{u})b) \equiv (-1)^k P_G(\bar{u} T^k b) \pmod{AT(A^G)}.$$

Each positive-degree monomial has a factor $T^k \bar{u}$, so this identity applies to every $P_G(m)$. Moreover, $P_G(\bar{u} T^k b) = r^{-1} \sum_g (g\bar{u}) T^k (gb)$ belongs to $\mathfrak{a} \cap A^G$, because U is G -stable. Thus every element of J^G has a representative in $\mathfrak{a} \cap A^G$. Products of ν such representatives vanish by (9), proving (10).

Applying Lemma 2.1 in $\bar{A}$ and taking ν th powers gives

$$J^{r\nu} = (J^r)^\nu \subseteq (\bar{A} J^G)^\nu = \bar{A} (J^G)^\nu = 0.$$

Thus $\bar{A}$ is spanned by monomials in $T^j \bar{u}$ of length at most $r\nu - 1$.

Give D_B and Q the induced filtrations, so $F_p Q = (F_p V + D_B) / D_B$. Taking invariants in each filtration quotient gives $\text{gr}_F B = A^G$: an invariant symbol has an invariant lift obtained by applying P_G to any lift. If $b \in B \cap F_p V$, $v \in F_s V$, and σ_j denotes the class in $F_j V / F_{j-1} V$, then

$$\sigma_s(v) T \sigma_p(b) = \sigma_{p+s}(b_{-2} v) \in \text{gr}_F D_B.$$

Consequently

$$AT(A^G) \subseteq \text{gr}_F D_B, \quad \bar{A} \twoheadrightarrow \text{gr}_F Q \quad (11)$$

as graded vector spaces, since $\text{gr}_F Q = A / \text{gr}_F D_B$. For a class in $F_p Q$, express its degree- p symbol using the spanning monomials of $\bar{A}$ and lift those monomials to normally ordered products in V . Each lift has length at most L , and subtracting their linear combination leaves a class in $F_{p-1} Q$. Induction terminates at $F_{-1} Q = 0$. The filtration and all the maps used are compatible with weight, so the lifts can be chosen to preserve weight. $\square$

It remains to bound the heights of these monomials. We do this by bounding the weights in Q of coefficients in products of a fixed number of fields.

3 The Differential Equations

We retain $B = V^G$, D_B , and $q : V \rightarrow Q$, but no longer assume C_2 -cofiniteness. The arguments in this section use only the connected nonnegative grading and the finite group action.

Lemmas 3.1 and 3.2 are standard consequences of the vertex algebra axioms and the grading. We defer their proofs to Appendix A, where we give the details for completeness.

3.1 n -Point Functions

Fix $n \geq 1$ and finite-dimensional homogeneous G -stable subspaces $E_i \subset V_{h_i}$. Put $\mathbf{z} = (z_1, \dots, z_n)$, $E = \bigotimes_{i=1}^n E_i$, and

$$\Delta(\mathbf{z}) = \prod_{i < j} (z_i - z_j)^{h_i + h_j}. \quad (12)$$

For $e = a_1 \otimes \cdots \otimes a_n \in E$, define the *regularized n -point function*

$$H_V(\mathbf{z})(e) = \Delta(\mathbf{z})Y(a_1, z_1) \cdots Y(a_n, z_n)|0\rangle, \quad H_Q = q \circ H_V. \quad (13)$$

The ordered product initially belongs to $V((z_1)) \cdots ((z_n))$, and H_Q takes values in $Q((z_1)) \cdots ((z_n))$.

The following consequence of locality, creation, and the grading will also justify the substitutions used below; compare [6] and [4, Section 2.7].

Lemma 3.1 (Regularity). *We have $H_V \in \text{Hom}_{\mathbb{C}}(E, V[[z_1, \dots, z_n]])$. For every $e \in E$, the coefficient of a monomial of total degree d in $H_V(\mathbf{z})(e)$ has weight*

$$\sum_i h_i + d - \deg \Delta. \quad (14)$$

Consequently, the projection of H_V to each homogeneous component V_w is a homogeneous polynomial.

For each nonempty $I \subseteq [n]$, we will obtain a differential equation modulo D_B after taking invariants in the coordinates indexed by I . We write these coordinates as a common coordinate t , called the collision coordinate, plus displacements x_i . The differential equations will be in t .

To that end, fix a nonempty subset $I = \{i_1 < \cdots < i_m\} \subseteq [n]$ and set

$$z_i = t + x_i \quad (i \in I), \quad x_{i_m} = 0, \quad \mathbf{x} = (x_{i_1}, \dots, x_{i_{m-1}}). \quad (15)$$

In particular, we take $t = z_{i_m}$ as the collision coordinate. The remaining $n - m$ coordinates $\mathbf{z}_{I^c}$ are unchanged. In expressions such as $(t + \mathbf{x}, \mathbf{z}_{I^c})$, the coordinates indexed by I are $t + x_i$, including t at $i = i_m$.

We now factor the regularizing polynomial $\Delta = \Delta_I \Delta_{I^c} \Delta_{\text{cross}}$ by grouping its original factors according to whether both indices lie in I , both lie in I^c , or one lies in each. We call the last factor, Δ_{cross} , the cross factor. Its degree in t will determine the order of the differential equation. After the substitution (15), Δ_I depends only on $\mathbf{x}$, and the cross factor has t -degree

$$D_I := \deg_t \Delta_{\text{cross}}(t + \mathbf{x}, \mathbf{z}_{I^c}) = \sum_{i \in I, j \notin I} (h_i + h_j).$$

To obtain the differential equation, we split the regularized n -point function according to the partition induced by I . Namely, for $e = a_1 \otimes \cdots \otimes a_n$ and $I^c = \{l_1 < \cdots < l_{n-m}\}$, write

$$C_I(\mathbf{x}) = \Delta_I(\mathbf{x})Y(a_{i_1}, x_{i_1}) \cdots Y(a_{i_{m-1}}, x_{i_{m-1}})a_{i_m} = \sum_{\alpha \geq 0} c_{\alpha} \mathbf{x}^{\alpha},$$

$$W_{I^c}(\mathbf{z}_{I^c}) = \Delta_{I^c}(\mathbf{z}_{I^c})Y(a_{l_1}, z_{l_1}) \cdots Y(a_{l_{n-m}}, z_{l_{n-m}})|0\rangle = \sum_{\gamma \geq 0} v_{\gamma} \mathbf{z}_{I^c}^{\gamma}.$$

Here $\alpha \in \mathbb{Z}_{\geq 0}^{m-1}$ and $\gamma \in \mathbb{Z}_{\geq 0}^{n-m}$. Both expressions are power series by Lemma 3.1. To see this for C_I , first regularize the vacuum product for the indices in I , and then set its last coordinate x_{i_m} to zero. For $m = 1$, put $C_I = a_{i_1}$; for $I^c = \emptyset$, put $W_{I^c} = |0\rangle$. The following lemma allows us to isolate the cross factor.

Lemma 3.2 (Isolating the Cross Factor). *With this notation,*

$$\begin{aligned} H_V(\mathbf{z})(e)|_{z_i=t+x_i \ (i \in I)} &= \Delta_{\text{cross}}(t + \mathbf{x}, \mathbf{z}_{I^c})Y(C_I(\mathbf{x}), t)W_{I^c}(\mathbf{z}_{I^c}) \\ &= \Delta_{\text{cross}}(t + \mathbf{x}, \mathbf{z}_{I^c}) \sum_{\alpha, \gamma \geq 0} Y(c_{\alpha}, t)v_{\gamma} \mathbf{x}^{\alpha} \mathbf{z}_{I^c}^{\gamma}. \end{aligned} \quad (16)$$

The identity holds in $V((t))[[\mathbf{x}, \mathbf{z}_{I^c}]]$. The left side is ordinary substitution in H_V ; the field action on the right is interpreted coefficientwise, as in the second equality. The lower bound on powers of t may depend on (α, γ) .

3.2 The Differential Equations in Q

For each nonempty $I \subseteq [n]$, let P_I be the Reynolds projection for the action of G that is diagonal on the tensor factors indexed by I and trivial on the remaining factors.

Lemma 3.3 (The Differential Equations). *For $\emptyset \neq I \subseteq [n]$, put $\partial_I = \sum_{i \in I} \partial_{z_i}$ and $d_I = D_I + 1 = 1 + \sum_{i \in I, j \notin I} (h_i + h_j)$. Then*

$$\partial_I^{d_I} (H_Q \circ P_I) = 0. \quad (17)$$

Proof. Precompose (16) with P_I and apply q . Equivariance replaces c_α by $\bar{c}_\alpha = P_G c_\alpha \in B$ and leaves v_γ unchanged. By (4),

$$qY(\bar{c}_\alpha, t)v_\gamma = \sum_{m \geq -1} q((\bar{c}_\alpha)_m v_\gamma) t^{-m-1} \in Q[t^{-1}],$$

where lower truncation makes the sum finite. Each projected series $qY(\bar{c}_\alpha, t)v_\gamma$ has only nonpositive powers of t , and Δ_{cross} is a polynomial of degree D_I in t . Their product therefore has no powers of t above D_I . Regularity of the left side excludes negative powers of t . Together, these bounds give, for every $e \in E$,

$$(H_Q \circ P_I)(t + \mathbf{x}, \mathbf{z}_{I^c})(e) \in Q[[\mathbf{x}, \mathbf{z}_{I^c}]] [t]_{\leq D_I}.$$

Thus $\partial_t^{D_I+1}$ annihilates this expression. Under the invertible linear change of coordinates (15), whose inverse is $t = z_{i_m}$ and $x_i = z_i - z_{i_m}$, $\partial_t = \partial_I$. This proves (17). $\square$

4 The Height Bound

For a monomial in positive-weight states, a weight bound also bounds its height by (2). We will prove the following uniform weight bound in Q . As in Section 3, we do not assume C_2 -cofiniteness; simplicity is used below to supply the auxiliary pairs.

Proposition 4.1. *Let $X \subset V$ be a finite-dimensional graded G -stable subspace and let $s \geq 1$. There is an integer $H = H(X, s) \geq 0$ such that, for every $1 \leq k \leq s$ and $a_1, \dots, a_k \in X$, all coefficients of*

$$qY(a_1, z_1) \cdots Y(a_k, z_k) |0\rangle$$

belong to $\bigoplus_{w=0}^H Q_w$.

We adjoin pairs of auxiliary states whose leading OPE coefficient is the vacuum. The differential equations and two finite-group identities will give a weight bound after these auxiliary states have been fixed. We then remove the pairs one at a time by taking their vacuum coefficients. The point is that removing a pair leaves the regularized product of the remaining fields, multiplied by a scalar polynomial. This lets us pass the weight bound back to the original product.

4.1 The Auxiliary State

We begin with two finite-group identities needed for the auxiliary state. As before, P_I denotes the Reynolds projection for the diagonal action on the tensor factors indexed by I , with the remaining factors fixed.

Lemma 4.2. *For r finite-dimensional G -modules $F_1, \dots, F_r$,*

$$\bigotimes_{i=1}^r F_i = \sum_{\emptyset \neq I \subseteq [r]} \text{im } P_I. \quad (18)$$

For the second identity, let F_0 and F_g , $g \neq 1$, be G -modules. For $v \in F_0$ and $a_g \in F_g$,

$$v \otimes \bigotimes_{g \neq 1} (1 - g)a_g \in \sum_{J \subseteq G \setminus \{1\}} \text{im } P_{\{0\} \cup J}. \quad (19)$$

Both identities may be tensored with arbitrary additional G -modules.

Proof. Give $A = \text{Sym}(\bigoplus_{i=1}^r F_i)$ its multigrading by the r summands, and let A_+ be its ideal of elements of positive total degree. The whole multidegree- $(1, \dots, 1)$ component $\bigotimes_i F_i$ lies in A_+^r , hence in $A(A_+)^G$ by Lemma 2.1. For a tensor $v_1 \otimes \dots \otimes v_r$, this gives an expression as a sum of products ab , where $a \in A$ and $b \in (A_+)^G$. Since G preserves the multigrading, the multihomogeneous components of b are still invariant. Taking the multidegree- $(1, \dots, 1)$ component, we may therefore assume that a and b are multihomogeneous and that their multidegrees sum to $(1, \dots, 1)$. Each coordinate of these multidegrees is either zero or one, and their supports partition $[r]$. Let I be the set of indices where b has degree one; this set is nonempty because $b \in A_+$. Then $b \in (\bigotimes_{i \in I} F_i)^G$ and $a \in \bigotimes_{i \notin I} F_i$, so $ab = P_I(ab) \in \text{im } P_I$. Summing over these products proves (18).

For (19), use the identity

$$\sum_{h \in G} hv \otimes \bigotimes_{g \neq 1} (ha_g - ga_g) = v \otimes \bigotimes_{g \neq 1} (a_g - ga_g). \quad (20)$$

Every summand with $h \neq 1$ vanishes in its factor indexed by $g = h$. For $J \subseteq G \setminus \{1\}$, put

$$b_g^J = \begin{cases} a_g, & g \in J, \\ ga_g, & g \notin J. \end{cases}$$

Choosing ha_g in the factors indexed by J and $-ga_g$ in the remaining factors rewrites (20) as

$$v \otimes \bigotimes_{g \neq 1} (1 - g)a_g = r \sum_{J \subseteq G \setminus \{1\}} (-1)^{r-1-|J|} P_{\{0\} \cup J} \left(v \otimes \bigotimes_{g \neq 1} b_g^J \right).$$

The tensor factors remain in their original order, and the factor r comes from the normalization of the Reynolds projection. The sum includes $J = \emptyset$, whose term uses $P_{\{0\}}$. Each summand lies in the corresponding image, proving (19). $\square$

To remove an auxiliary pair later, we need its leading coefficient to be the vacuum. The next lemma supplies a partner for every nonzero homogeneous state of positive weight. It is a standard consequence of simplicity and the vertex algebra axioms; we give the proof in Appendix A.3 for completeness.

Lemma 4.3 (Vacuum Coefficients). *For every nonzero $u \in V_h$, $h > 0$, there is a homogeneous $w \in V_k$, $k > 0$, such that*

$$u_{h+k-1}w \neq 0 \quad \text{in } \mathbb{C}|0\rangle.$$

We can now construct the auxiliary state ξ . For each $g \neq 1$, faithfulness gives a homogeneous a_g with $ga_g \neq a_g$. Its weight is positive because G fixes $V_0 = \mathbb{C}|0\rangle$. Set $u_g = (1 - g)a_g$. By Lemma 4.3, we can choose and rescale a homogeneous w_g so that

$$(u_g)_{h_g+k_g-1}w_g = |0\rangle, \quad h_g = \text{wt } u_g, \quad k_g = \text{wt } w_g. \quad (21)$$

All higher modes vanish by the grading. Let

$$U_g = \text{span}\{ha_g : h \in G\}, \quad W_g = \text{span}\{hw_g : h \in G\}.$$

The auxiliary spaces U_g and W_g are finite dimensional and homogeneous, and $a_g, u_g \in U_g$.

Fix finite-dimensional homogeneous G -stable subspaces $E_i \subset V_{h_i}$, $1 \leq i \leq k$, and put $E = \bigotimes_i E_i$. Fix an order of $G \setminus \{1\}$, and use it in the tensor products below. Adjoin $2r - 1$ copies of each pair (u_g, w_g) by setting

$$\tilde{E} = E \otimes \bigotimes_{g \neq 1} (U_g \otimes W_g)^{\otimes (2r-1)}, \quad \xi = \bigotimes_{g \neq 1} (u_g \otimes w_g)^{\otimes (2r-1)}.$$

The tensor factors in this enlarged space retain the displayed order. We use $2r - 1$ copies so that any set of at most $2r - 2$ tensor positions leaves at least one prescribed copy of each u_g outside it. This will be used in the proof of Lemma 4.4. Write the pairs in ξ as (u_t, w_t) , $1 \leq t \leq N$, where

$$N = (r - 1)(2r - 1). \quad (22)$$

For $r = 1$, these tensor products are empty: $N = 0$, $\tilde{E} = E$, and $\xi = 1 \in \mathbb{C}$.

4.2 The Universal Module

We now encode the differential equations of Lemma 3.3 as relations over a polynomial ring. The enlarged space $\tilde{E}$ has $n = k + 2N$ factors; let d_I be the orders of the corresponding differential equations and set

$$R = \mathbb{C}[p_1, \dots, p_n], \quad p_I = \sum_{i \in I} p_i, \quad M = \frac{R \otimes \tilde{E}}{\sum_{\emptyset \neq I \subseteq [n]} R p_I^{d_I} P_I \tilde{E}}. \quad (23)$$

The variable p_i represents differentiation in the variable of the i th field. Let $\mathcal{N}$ be the R -submodule of M generated by the classes of $e \otimes \xi$, $e \in E$. This submodule is finitely generated because E is finite dimensional. We only need a bound for these prescribed tensors, not for all of M : we will show that every monomial of sufficiently large degree annihilates $\mathcal{N}$.

Lemma 4.4 (Support at the Origin). *For some integer $q_0 \geq 1$,*

$$(p_1, \dots, p_n)^{q_0} \mathcal{N} = 0. \quad (24)$$

Proof. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, write $\mathbf{m}_\lambda = (p_1 - \lambda_1, \dots, p_n - \lambda_n)$. We first show that $\mathcal{N}_{\mathbf{m}_\lambda} = 0$ whenever $\lambda \neq \mathbf{0}$, and then use finite generation to obtain the required power of $(p_1, \dots, p_n)$.

Fix such a λ and put $S = \{i : \lambda_i \neq 0\}$. The relations in (23) give $p_I^{d_I} P_I v = 0$ in M for every $v \in \tilde{E}$ and every nonempty $I \subseteq [n]$. If $p_I(\lambda) \neq 0$, then p_I is a unit in $R_{\mathbf{m}_\lambda}$, so

$$P_I v = 0 \quad \text{in } M_{\mathbf{m}_\lambda} \quad (v \in \tilde{E}).$$

We will apply this observation in two cases.

Suppose first that $|S| \geq 2r - 1$. Choose a line through the origin in $\mathbb{C}$ containing none of the nonzero λ_i . One of its open half-planes contains at least r of these numbers; choose a set $J \subseteq S$ of r corresponding indices. There is a real-linear functional $\ell : \mathbb{C} \rightarrow \mathbb{R}$ with $\ell(\lambda_i) > 0$ for all $i \in J$. Hence, for every nonempty $I \subseteq J$,

$$\ell(p_I(\lambda)) = \sum_{i \in I} \ell(\lambda_i) > 0.$$

Thus $p_I(\lambda) \neq 0$, and the image of $P_I \tilde{E}$ in $M_{\mathbf{m}_\lambda}$ is zero. Applying (18) to the r factors indexed by J , and tensoring with the remaining factors, gives

$$\tilde{E} = \sum_{\emptyset \neq I \subseteq J} P_I \tilde{E}.$$

Since these tensors generate M over R , we obtain $M_{\mathfrak{m}_\lambda} = 0$, and hence $\mathcal{N}_{\mathfrak{m}_\lambda} = 0$.

Now suppose $1 \leq |S| \leq 2r - 2$, and choose $j \in S$. In our tensor order, the original factors occupy positions $1, \dots, k$, and the t th auxiliary pair occupies positions $k + 2t - 1$ and $k + 2t$, with u_t first and w_t second. For each $g \neq 1$, let $A_g \subseteq [n]$ be the positions of the prescribed copies of u_g in ξ . The sets A_g are pairwise disjoint, each with $2r - 1$ elements. Since $|S| \leq 2r - 2$, we can choose

$$i_g \in A_g \setminus S \quad (g \neq 1).$$

The chosen positions are distinct, and none equals j .

It suffices by linearity to consider a pure tensor $e = e_1 \otimes \dots \otimes e_k \in E$. For $J \subseteq G \setminus \{1\}$, put

$$I_J = \{j\} \cup \{i_g : g \in J\}.$$

Let $\theta_J \in \tilde{E}$ be obtained from $e \otimes \xi$ by replacing the selected copy of u_g at position i_g by a_g if $g \in J$ and by ga_g otherwise, leaving all other factors unchanged. These replacements lie in U_g . The explicit identity in the proof of Lemma 4.2, with distinguished position j and positions i_g indexed by $g \neq 1$, gives

$$e \otimes \xi = r \sum_{J \subseteq G \setminus \{1\}} (-1)^{r-1-|J|} P_{I_J} \theta_J.$$

The factor at j may be original or auxiliary, since the identity allows an arbitrary vector there. All tensor factors remain in their fixed order. The sum includes $J = \emptyset$, for which $I_J = \{j\}$.

For every J , the selected positions i_g lie outside S , so

$$p_{I_J}(\lambda) = \lambda_j + \sum_{g \in J} \lambda_{i_g} = \lambda_j \neq 0.$$

Our initial observation therefore makes every $P_{I_J} \theta_J$ zero in $M_{\mathfrak{m}_\lambda}$. Thus every generator $e \otimes \xi$ of $\mathcal{N}$ vanishes after localization, proving $\mathcal{N}_{\mathfrak{m}_\lambda} = 0$ in this case as well.

If $\mathcal{N} = 0$, take $q_0 = 1$. Otherwise, finite generation gives $\text{Supp } \mathcal{N} = V(\text{Ann}_R \mathcal{N})$. We have excluded every maximal ideal except $\mathfrak{m}_0$ from this closed set. The Nullstellensatz therefore gives

$$(p_1, \dots, p_n) \subseteq \sqrt{\text{Ann}_R \mathcal{N}}.$$

Choose integers $\nu_i \geq 1$ such that $p_i^{\nu_i} \mathcal{N} = 0$, and put

$$q_0 = 1 + \sum_{i=1}^n (\nu_i - 1).$$

A monomial not divisible by any $p_i^{\nu_i}$ has total degree at most $\sum_i (\nu_i - 1)$. Hence every monomial of total degree at least q_0 annihilates $\mathcal{N}$, proving (24). $\square$

4.3 Proof of the Weight Bound

We recall the weight bound before proving it.

Proposition 4.1. *Let $X \subset V$ be a finite-dimensional graded G -stable subspace and let $s \geq 1$. There is an integer $H = H(X, s) \geq 0$ such that, for every $1 \leq k \leq s$ and $a_1, \dots, a_k \in X$, all coefficients of*

$$qY(a_1, z_1) \cdots Y(a_k, z_k)|0\rangle$$

belong to $\bigoplus_{w=0}^H Q_w$.

Proof of Proposition 4.1. By linearity, it suffices to take a_i in homogeneous components $E_i = X \cap V_{h_i}$. We fix k and $h_1, \dots, h_k$ and prove a bound independent of the a_i and the mode indices. Since $k \leq s$ and X has finitely many weights, the maximum of these bounds and 0 gives $H(X, s)$.

Form $\tilde{E}$ as above. Give the original fields variables $\mathbf{z} = (z_1, \dots, z_k)$ and the t th auxiliary pair variables x_t, y_t . Write $\mathbf{x} = (x_1, \dots, x_N)$ and $\mathbf{y} = (y_1, \dots, y_N)$, and put $H_t = \text{wt } u_t$ and $K_t = \text{wt } w_t$. For auxiliary states b_t, c_t in their respective tensor factors, write, in tensor order,

$$(A_1, \dots, A_n) = (a_1, \dots, a_k, b_1, c_1, \dots, b_N, c_N),$$

$$\zeta = (\zeta_1, \dots, \zeta_n) = (z_1, \dots, z_k, x_1, y_1, \dots, x_N, y_N).$$

Let η_i be the weight of the i th tensor factor and $\tilde{e} = A_1 \otimes \dots \otimes A_n$. Define

$$\tilde{\Delta}(\mathbf{z}, \mathbf{x}, \mathbf{y}) = \prod_{i < j} (\zeta_i - \zeta_j)^{\eta_i + \eta_j} \quad (25)$$

and

$$\tilde{H}_Q(\mathbf{z}, \mathbf{x}, \mathbf{y})(\tilde{e}) = \tilde{\Delta}(\mathbf{z}, \mathbf{x}, \mathbf{y}) q(Y(A_1, \zeta_1) \cdots Y(A_n, \zeta_n) | 0\rangle). \quad (26)$$

The polynomial $\tilde{\Delta}$ includes $\Delta(\mathbf{z})$, every $(x_t - y_t)^{H_t + K_t}$, and all factors between distinct pairs or between original and auxiliary fields. Lemma 3.1 gives $\tilde{H}_Q \in \text{Hom}_{\mathbb{C}}(\tilde{E}, Q[[\mathbf{z}, \mathbf{x}, \mathbf{y}]])$, and Lemma 3.3 gives its differential equations. Write $\mathbf{p} = (p_1, \dots, p_n)$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, put

$$\mathbf{p}^\alpha := p_1^{\alpha_1} \cdots p_n^{\alpha_n} \in R, \quad \partial^\alpha := \prod_{i=1}^n \partial_{\zeta_i}^{\alpha_i}.$$

For a formal power series, evaluation at $\mathbf{0}$ means taking its constant coefficient. Differentiation therefore defines a linear map

$$\Phi : R \otimes \tilde{E} \longrightarrow Q, \quad \mathbf{p}^\alpha \otimes v \longmapsto (\partial^\alpha \tilde{H}_Q)(\mathbf{0})(v). \quad (27)$$

By Lemma 3.3, this map factors through M . Indeed, the image of $\mathbf{p}^\alpha p_I^{d_I} \otimes P_I v$ is obtained by differentiating (17) and evaluating at zero, and is therefore zero. Set $b_t = u_t$ and $c_t = w_t$ for the auxiliary states, and put

$$F(\mathbf{z}, \mathbf{x}, \mathbf{y})(e) = \tilde{H}_Q(\mathbf{z}, \mathbf{x}, \mathbf{y})(e \otimes \xi). \quad (28)$$

For $|\alpha| \geq q_0$, Lemma 4.4 gives $\mathbf{p}^\alpha(e \otimes \xi) = 0$ in M , hence $(\partial^\alpha F)(\mathbf{0})(e) = 0$. Since Taylor coefficients differ from these derivatives by the nonzero factors $\alpha!$, F is a polynomial of total degree less than q_0 . Using ζ for the same variables in tensor order, we can write it explicitly as

$$F(\zeta)(e) = \sum_{|\alpha| < q_0} \frac{\zeta^\alpha}{\alpha!} \Phi(\mathbf{p}^\alpha \otimes (e \otimes \xi)).$$

The integer q_0 depends only on the spaces E_i , the chosen auxiliary pairs, and their weights, so the degree bound is uniform in e .

Put $\tilde{D} = \deg \tilde{\Delta}$ and let $\tilde{h}$ be the sum of the weights of the $k + 2N$ states. By (14), the projection F_w of F to Q_w is homogeneous of total degree $w - \tilde{h} + \tilde{D}$. Consequently

$$F_w = 0 \quad \text{if} \quad w > H := \max\{0, \tilde{h} + q_0 - 1 - \tilde{D}\}. \quad (29)$$

We now remove the auxiliary pairs one at a time. Let $H_V^{(j)}$ be the regularized product of the original fields and the auxiliary pairs $j + 1, \dots, N$, retaining their original order, and put $H_Q^{(j)} = qH_V^{(j)}$. Write

$\Delta^{(j)}$ for its regularizing polynomial, which contains only the factors among these remaining fields. Thus

$$H_Q^{(0)} = F, \quad H_Q^{(N)} = H_Q.$$

We suppress the dependence on e .

Suppose that the first j pairs have been removed, and consider the next pair, indexed by $t = j + 1$. Set $x_t = y_t + d$. The regularizing polynomial factors as

$$\Delta^{(j)} \Big|_{x_t=y_t+d} = d^{H_t+K_t} B_t^{(j)}(d) \Delta^{(j+1)}.$$

Here $d^{H_t+K_t}$ is the factor within the t th pair, $B_t^{(j)}(d)$ contains the factors between this pair and the other remaining fields, and $\Delta^{(j+1)}$ contains the factors among those other fields. We retain the original orientation of every factor.

By (21) and the grading,

$$d^{H_t+K_t} Y(u_t, d) w_t = |0\rangle + d\rho_t(d), \quad \rho_t(d) \in V[[d]].$$

Applying Lemma 3.2 to this pair, with common coordinate y_t , and then applying q , gives

$$H_Q^{(j)} \Big|_{x_t=y_t+d} = B_t^{(j)}(d) q \left(Y(|0\rangle + d\rho_t(d), y_t) H_V^{(j+1)} \right).$$

The field action is interpreted coefficientwise in the remaining variables. Every factor on the right has only nonnegative powers of d . Taking its constant coefficient and using $Y(|0\rangle, y_t) = \text{id}$, we obtain

$$H_Q^{(j)} \Big|_{x_t=y_t} = B_t^{(j)}(0) H_Q^{(j+1)}.$$

The specialization on the left is defined because $H_Q^{(j)}$ is a power series by Lemma 3.1.

To keep track of the scalar factors accumulated during these removals, put

$$F^{(j)} = F|_{x_1=y_1, \dots, x_j=y_j}.$$

We prove inductively that $F^{(j)} = C_j H_Q^{(j)}$, where C_j is a scalar polynomial. The assertion holds for $j = 0$, with $C_0 = 1$. Using the one-pair identity above, the induction step is

$$\begin{aligned} F^{(j+1)} &= (C_j H_Q^{(j)}) \Big|_{x_t=y_t} \\ &= C_j|_{x_t=y_t} B_t^{(j)}(0) H_Q^{(j+1)} \quad C_{j+1} := C_j|_{x_t=y_t} B_t^{(j)}(0). \\ &= C_{j+1} H_Q^{(j+1)}, \end{aligned}$$

The displayed definition makes C_{j+1} a scalar polynomial. This proves the induction. After removing all N pairs, we therefore have

$$F(\mathbf{z}, \mathbf{y}, \mathbf{y}) = C(\mathbf{z}, \mathbf{y}) H_Q(\mathbf{z}), \tag{30}$$

where $C = C_N$, H_Q is defined by $E_1, \dots, E_k$, and

$$\begin{aligned} C(\mathbf{z}, \mathbf{y}) &= \prod_{i=1}^k \prod_{t=1}^N (z_i - y_t)^{2h_i + H_t + K_t} \\ &\quad \cdot \prod_{1 \leq t < v \leq N} (y_t - y_v)^{2(H_t + K_t + H_v + K_v)}. \end{aligned} \tag{31}$$

The original factors $\Delta(\mathbf{z})$ remain in H_Q . The factors between original and auxiliary fields, and between distinct auxiliary pairs, give C : their exponents add, respectively, to $(h_i + H_t) + (h_i + K_t)$ and $(H_t + H_v) + (H_t + K_v) + (K_t + H_v) + (K_t + K_v)$. Thus (31) accounts for every remaining factor. The coordinates $y_1, \dots, y_N$ remain independent; only the two coordinates within each pair have been identified. Hence $C \neq 0$. Empty products are 1, including when $N = 0$.

For $w > H$, project (30) to Q_w . Fix $e \in E$. Equation (29) gives

$$C(\mathbf{z}, \mathbf{y})(H_Q)_w(\mathbf{z})(e) = 0.$$

By Lemma 3.1, $(H_Q)_w(\mathbf{z})(e)$ is a polynomial in $z_1, \dots, z_k$ with coefficients in Q_w . Choose a basis $v_1, \dots, v_m$ of the finite-dimensional subspace spanned by these coefficients, and write

$$(H_Q)_w(\mathbf{z})(e) = \sum_{i=1}^m f_i(\mathbf{z})v_i, \quad f_i(\mathbf{z}) \in \mathbb{C}[z_1, \dots, z_k].$$

The identity $C(\mathbf{z}, \mathbf{y})(H_Q)_w(\mathbf{z})(e) = 0$ therefore becomes

$$\sum_{i=1}^m C(\mathbf{z}, \mathbf{y})f_i(\mathbf{z})v_i = 0.$$

Comparing the coefficient of each monomial in $\mathbf{z}, \mathbf{y}$ and using the linear independence of the v_i , we obtain the scalar polynomial equalities

$$C(\mathbf{z}, \mathbf{y})f_i(\mathbf{z}) = 0 \quad (1 \leq i \leq m).$$

Since $C \neq 0$ and $\mathbb{C}[\mathbf{z}, \mathbf{y}]$ is an integral domain, every f_i is zero. Thus $(H_Q)_w(\mathbf{z})(e) = 0$. As e was arbitrary, $(H_Q)_w = 0$. Finally, multiplication by Δ is invertible on the ordered Laurent space $Q((z_1)) \cdots ((z_k))$. Its inverse is multiplication by Δ^{-1} , expanded according to the original order of the fields. This operation commutes with projection to Q_w . Thus the weight- w component of the original product is zero as well. $\square$

5 Cofiniteness and Its Consequences

We now combine the length and height bounds to prove C_2 -cofiniteness of V^G .

Theorem 1.1. *Let V be a simple vertex algebra with a connected nonnegative integer grading, and let G be a finite group of grading-preserving automorphisms. If V is C_2 -cofinite, then V^G is C_2 -cofinite.*

Proof of Theorem 1.1. Let U be the finite-dimensional graded G -stable strong generating space of Section 2.1, and choose ν with $R_{V,+}^\nu = 0$. By Lemma 2.2, Q is spanned by classes of monomials of length at most $L = r\nu - 1$. Apply Proposition 4.1 with $X = U$ and $s = \max\{1, L\}$. With $H = H(U, s)$, we obtain $Q_w = 0$ for $w > H$. Each V_w is finite dimensional, and hence so is $Q = \bigoplus_{w \leq H} Q_w$. Equation (5) now gives $\dim B/C_2(B) < \infty$.

More explicitly, for a homogeneous basis $u^1, \dots, u^d$ of U ,

$$Q = \text{span} \left\{ q(|0\rangle), q(u_{-n_1}^{i_1} \cdots u_{-n_\ell}^{i_\ell} |0\rangle) : 1 \leq \ell \leq L, 1 \leq i_j \leq d, 1 \leq n_j \leq H \right\}.$$

The height bound follows from (2). $\square$

Recall that a vertex operator algebra is *strongly rational* if it is simple, of CFT type, self-contragredient, rational, and C_2 -cofinite. Here rationality means complete reducibility of admissible modules.

Corollary 5.1. *Let V be a strongly rational vertex operator algebra and G a finite group of automorphisms. Then V^G is strongly rational.*

Proof. Theorem 1.1 gives C_2 -cofiniteness of V^G . McRae's theorem [11, Corollary 4.23] then gives strong rationality. $\square$

We next use the height bound to prove finite strong generation, without assuming C_2 -cofiniteness.

Theorem 1.2. *Let V be a simple, strongly finitely generated vertex algebra with a connected nonnegative integer grading. For every finite group G of grading-preserving automorphisms, V^G is strongly finitely generated.*

Proof of Theorem 1.2. Write $B = V^G$, $r = |G|$, and retain the relative quotient $q : V \rightarrow Q = V/D_B$ from (3). We first explain the structure of the proof.

Starting with a finite-dimensional graded G -stable strong generating space X for V , we use a good filtration to pass to a commutative associated graded algebra. Following the standard approach through commutative invariant theory [10, Sections 7–8], we present this algebra as a quotient of a symmetric algebra S on copies of X , one for each derivative order. Applying Noether's bound to the subalgebras involving finitely many copies gives generators for S^G of polynomial degree at most $|G|$. Their images in the associated graded algebra lift to invariant normally ordered products, giving a possibly infinite strong generating family $\mathcal{X}_G$ for V^G .

The degree bound ensures that each defining word has at most $|G|$ factors, so Proposition 4.1 applies with $s = |G|$. It gives a weight cutoff above which these invariant lifts lie in $C_2(B)$. The remaining family $\mathcal{X}_G^{\text{fin}}$ is finite, and its image generates $R_B = B/C_2(B)$. Its image consequently spans $B_{>0}/C_1(B)$, so Li's strong-generation criterion shows that $\mathcal{X}_G^{\text{fin}}$ strongly generates V^G .

A Finite Graded Generating Space and Its Filtration. Choose a finite-dimensional graded G -stable subspace $X \subset V_{>0}$ that strongly generates V . Such a space is obtained by taking the G -span of a finite homogeneous strong generating set of positive weight. Each V_w is then finite dimensional, since only finitely many normally ordered monomials in a homogeneous basis of X and its derivatives have weight w .

Define $F_p V$ by (7), with X in place of U . This definition and Li's good-filtration theorem [9, Theorem 4.11] do not require C_2 -cofiniteness. Set $A = \text{gr}_F V$ and give B the induced filtration $F_p B = B \cap F_p V$. The filtration is preserved by G and by translation, and A is a commutative differential algebra. In particular,

$$\text{gr}_F B = (\text{gr}_F V)^G = A^G. \quad (32)$$

To check the surjectivity implicit in (32), let $\alpha \in (F_p V / F_{p-1} V)^G$ and choose a lift $v \in F_p V$. Then $P_G v \in F_p B$ has symbol $P_G \alpha = \alpha$. This proves (32) in every filtration degree.

Noether's Uniform Degree Bound. Fix a finite homogeneous basis $\mathcal{X}$ of X . For each $j \geq 0$, let $X^{(j)}$ be a copy of the G -module X , and write $x^{(j)} \in X^{(j)}$ for the copy of $x \in X$. Set

$$S = \text{Sym} \left(\bigoplus_{j \geq 0} X^{(j)} \right), \quad S_{\leq J} = \text{Sym} \left(\bigoplus_{0 \leq j \leq J} X^{(j)} \right).$$

The map $x^{(j)} \mapsto T^j \bar{x}$ defines a surjective, G -equivariant homomorphism of commutative algebras

$$\rho : S \twoheadrightarrow A, \quad \rho(S^G) = A^G. \quad (33)$$

For the second assertion, take a preimage of an invariant element of A and apply P_G ; its image is unchanged. For a homogeneous $x \in X \cap V_h$, give $x^{(j)}$ the two degrees

$$\deg_F(x^{(j)}) = h, \quad \text{wt}(x^{(j)}) = h + j.$$

The map ρ preserves both gradings, although a generator may map to zero. We also write $\deg_{\text{pol}}$ for ordinary polynomial degree on S , defined by $\deg_{\text{pol}}(x^{(j)}) = 1$. Thus

$$\deg_{\text{pol}}(x_1^{(j_1)} \cdots x_d^{(j_d)}) = d.$$

We use this degree in S to bound the number of factors; we do not need it to descend to a grading of A .

Every polynomial in S belongs to some $S_{\leq J}$, and $S_{\leq J}$ is G -stable. Consequently

$$S^G = \bigcup_{J \geq 0} S_{\leq J}^G.$$

For each J , the classical Noether bound [13] states that $S_{\leq J}^G$ is generated as a commutative algebra by invariants of *ordinary polynomial degree at most $|G|$* . Since the invariant subspace of each homogeneous polynomial degree is the image of P_G , it is spanned by the elements $P_G(\mathbf{m})$, where $\mathbf{m}$ is a monomial of that degree in the variables $x^{(j)}$, with $0 \leq j \leq J$ and x ranging over our chosen homogeneous basis $\mathcal{X}$ of X . Thus the family

$$\{P_G(\mathbf{m}) : \mathbf{m} \text{ a monomial in the } x^{(j)}, \quad 1 \leq \deg_{\text{pol}} \mathbf{m} \leq |G|\} \quad (34)$$

generates S^G , and its image under ρ generates A^G . This is only a *bounded-degree* assertion: because j is unbounded, (34) is not yet a finite set.

Lifting Invariant Polynomials to V^G . Recall that $B = V^G$ and that $x^{(j)} \in X^{(j)}$ denotes the copy of the basis vector $x \in \mathcal{X}$ in the j th copy of X . Fix an ordering of these symbols and write each monomial in that order. For such a monomial

$$\mathbf{m} = x_1^{(j_1)} \cdots x_d^{(j_d)}, \quad 1 \leq d \leq |G|, \quad x_i \in \mathcal{X},$$

write $h_i = \text{wt } x_i$ and define the invariant state

$$\begin{aligned} \beta_{\mathbf{m}} &= P_G((T^{j_1} x_1)_{-1} \cdots (T^{j_d} x_d)_{-1} | 0 \rangle) \\ &= \left(\prod_{i=1}^d j_i! \right) P_G((x_1)_{-j_1-1} \cdots (x_d)_{-j_d-1} | 0 \rangle) \in B. \end{aligned} \quad (35)$$

Set

$$\mathcal{X}_G = \{\beta_{\mathbf{m}} : \mathbf{m} \text{ is as in (34)}\} \subseteq B.$$

The state $\beta_{\mathbf{m}}$ is homogeneous of weight

$$\text{wt } \beta_{\mathbf{m}} = \sum_{i=1}^d (h_i + j_i) \quad \text{if } \beta_{\mathbf{m}} \neq 0, \quad (36)$$

and lies in $F_p V$, where $p = \sum_i h_i$. Writing σ_p for the projection to $F_p V / F_{p-1} V$, the good-filtration product rule gives

$$\sigma_p(\beta_{\mathbf{m}}) = \rho(P_G(\mathbf{m})).$$

This equality also holds when $\beta_{\mathbf{m}}$ has lower filtration degree and both sides are zero.

We claim that $\mathcal{X}_G$ strongly generates B . Indeed, let $b \in B \cap F_p V$ be homogeneous. By (32), (33), and (34), its filtration- p symbol is a polynomial in the symbols of the elements of $\mathcal{X}_G$. Both filtration degree and weight are preserved, so this polynomial may be chosen homogeneous of weight $\text{wt } b$ and filtration degree p . Replace products in the polynomial by normally ordered products of the corresponding $\beta_{\mathbf{m}}$. The resulting state $b' \in B \cap F_p V$ has the same filtration- p symbol as b . Hence

$$b - b' \in B \cap F_{p-1} V.$$

Induction on p , starting with $F_0 B = \mathbb{C}|0\rangle$, proves the claim. In particular, the image of $\mathcal{X}_G$ generates the commutative algebra $R_B = B/C_2(B)$: normally ordered products become products of classes, and derivatives vanish in R_B .

The Weight Cutoff Makes the Family Finite. Recall that, using Noether's bound, we constructed $\mathcal{X}_G$ from monomials of polynomial degree at most $|G|$. Thus the corresponding words in (35) have at most $r = |G|$ factors. We therefore apply Proposition 4.1 to X with $s = r$, and denote the resulting weight bound by $H_X = H(X, r)$. Set

$$\mathcal{X}_G^{\text{fin}} = \{\beta \in \mathcal{X}_G \setminus \{0\} : \text{wt } \beta \leq H_X\}.$$

We will show that this set is finite and that its image generates R_B .

The word in the second line of (35) is the coefficient

$$(x_1)_{-j_1-1} \cdots (x_d)_{-j_d-1} |0\rangle = [z_1^{j_1} \cdots z_d^{j_d}] Y(x_1, z_1) \cdots Y(x_d, z_d) |0\rangle.$$

There are at most r fields, regardless of the derivative orders j_i . Proposition 4.1 therefore bounds the weight of its class in Q by H_X . Since the word is homogeneous of the weight in (36), this gives

$$\text{wt } \beta_{\mathbf{m}} > H_X \implies q(\beta_{\mathbf{m}}) = 0 \implies \beta_{\mathbf{m}} \in D_B \cap B = C_2(B). \quad (37)$$

The last implication uses (5); the equality $qP_G = P_G q$ follows from G -equivariance of q . Proposition 4.1 is used here only for the words of length at most r ; no global weight bound on Q is claimed. Only finitely many basis monomials $\mathbf{m}$ in (34) have $\sum_i (h_i + j_i) \leq H_X$: there are at most r factors, the x_i belong to a finite basis of positive-weight states, and the nonnegative integers j_i have bounded sum. Hence $\mathcal{X}_G^{\text{fin}}$ is finite. Every omitted element of $\mathcal{X}_G$ has zero image in R_B by (37), so the image of $\mathcal{X}_G^{\text{fin}}$ generates R_B as a commutative algebra.

From R_B to Strong Finite Generation. Recall

$$C_1(B) = TB + \text{span}\{a_{-1}b : a, b \in B_{>0}\}, \quad C_2(B) \subseteq C_1(B).$$

Because the elements of $\mathcal{X}_G^{\text{fin}}$ have positive weight and their images generate R_B , each homogeneous positive-weight element of B is, modulo $C_2(B)$, a sum of linear terms in elements of $\mathcal{X}_G^{\text{fin}}$ and normally ordered products of two or more such terms. Each such product lies in $C_1(B)$ by definition. Thus the images of $\mathcal{X}_G^{\text{fin}}$ span the *vector space* $B_{>0}/C_1(B)$; no algebra structure on this quotient is needed. Equivalently,

$$B_{>0} = \text{span } \mathcal{X}_G^{\text{fin}} + C_1(B).$$

The fixed-point algebra B inherits the connected nonnegative integer grading. Li's strong-generation criterion [9, Theorem 4.10] states that, in this setting, a homogeneous set strongly generates B if and only if its image spans $B_{>0}/C_1(B)$. The criterion now implies that the finite set $\mathcal{X}_G^{\text{fin}}$ strongly generates V^G . This proves Theorem 1.2. $\square$

Remark 5.2. In Theorems 1.1 and 1.2, simplicity can be replaced by the following condition: for each $g \in G$ acting nontrivially on V , there are homogeneous a_g, w_g such that $u_g = (1 - g)a_g \neq 0$ and

$$(u_g)_{\text{wt } u_g + \text{wt } w_g - 1} w_g \in \mathbb{C}^\times |0\rangle.$$

These pairs replace those supplied by Lemma 4.3.

A Technical VOA Appendix

We give the proofs deferred from Sections 3 and 4, retaining their notation.

A.1 Regularity

Proof of Lemma 3.1. For homogeneous a_i, a_j , the grading gives $(a_i)_\ell a_j = 0$ for $\ell \geq h_i + h_j$. The commutator formula therefore makes $(z_i - z_j)^{h_i + h_j}$ a locality factor. After multiplication by Δ , locality allows any one field $Y(a_i, z_i)$ to be moved to the rightmost position, without changing the scalar factors. Creation gives $Y(a_i, z_i)|0\rangle = e^{z_i T} a_i \in V[[z_i]]$. The other fields do not involve z_i , and multiplication by Δ introduces no negative powers. Thus the coefficient of any monomial with a negative z_i -power is zero. Since i was arbitrary, $H_V(\mathbf{z})(e) \in V[[z_1, \dots, z_n]]$.

The mode weight formula gives weight $\sum_i h_i + s$ for a coefficient of total degree s before multiplication by Δ . Since Δ is homogeneous, the coefficient of total degree d after multiplication has the weight in (14). For fixed w , its total degree must therefore be $w - \sum_i h_i + \deg \Delta$. There are finitely many monomials of that degree; if it is negative, the projection is zero. No finite-dimensionality assumption on V_w is needed here. $\square$

A.2 Isolating the Cross Factor

Proof of Lemma 3.2. We first prove the identity on a fixed homogeneous vector. We then apply it to the complementary coefficients, one at a time.

The Identity on a Fixed Vector. For this calculation only, relabel the indices of I as $1, \dots, m$, preserving their order, and put $x_m = 0$. Write Δ_I for the corresponding internal polynomial and $C(\mathbf{x})$ for $C_I(\mathbf{x})$ in these labels. Fix $v \in V_d$ and put

$$F_v(\mathbf{z}) = \Delta_I(\mathbf{z}) Y(a_1, z_1) \cdots Y(a_m, z_m) v, \quad K_i = h_i + d, \quad \Pi(\mathbf{z}) = \prod_{i=1}^m z_i^{K_i}.$$

Adjoin the field $Y(v, s)$ immediately before the vacuum:

$$\Psi(\mathbf{z}; s) = \Delta_I(\mathbf{z}) \prod_{i=1}^m (z_i - s)^{K_i} Y(a_1, z_1) \cdots Y(a_m, z_m) Y(v, s) |0\rangle.$$

Lemma 3.1 gives $\Psi \in V[[\mathbf{z}, s]]$. Creation and constant-coefficient extraction in s give

$$\Psi(\mathbf{z}; 0) = \Pi(\mathbf{z}) F_v(\mathbf{z}).$$

Thus all remaining coordinate poles of F_v are bounded:

$$F_v \in \mathcal{A} := V[[z_1, \dots, z_m]][z_1^{-1}, \dots, z_m^{-1}].$$

There is a linear reexpansion map

$$\sigma : \mathcal{A} \longrightarrow V((t))[[x_1, \dots, x_{m-1}]]$$

defined by ordinary substitution $z_i = t + x_i$, $z_m = t$ on power-series numerators, and by

$$z_i^{-1} \longmapsto (t + x_i)^{-1} = \sum_{j \geq 0} (-1)^j t^{-j-1} x_i^j \quad (i < m), \quad z_m^{-1} \longmapsto t^{-1}.$$

The numerator substitution preserves total degree, so each joint coefficient is a finite sum. If a denominator is $\prod_i z_i^{b_i}$, then the coefficient of $\mathbf{x}^\alpha$ after substitution has no t -power below $-\sum_i b_i - |\alpha|$. Thus the image lies in the stated space. The transformed coordinates are units in $\mathbb{C}((t))[[\mathbf{x}]]$, so this definition is independent of the monomial denominator. It is compatible with multiplication by scalar polynomials and agrees with ordinary substitution on $V[[\mathbf{z}]]$. In particular,

$$\sigma(F_v) = \Pi(t + \mathbf{x}, t)^{-1} \Psi(t + \mathbf{x}, t; 0), \quad \Pi(t + \mathbf{x}, t) = t^{K_m} \prod_{i < m} (t + x_i)^{K_i}.$$

All scalar factors of Ψ are coordinate differences. Translation covariance and $T|0\rangle = 0$ therefore give

$$\left(\sum_{i=1}^m \partial_{z_i} + \partial_s \right) \Psi = T\Psi, \quad \Psi(\mathbf{z} + u; s + u) = e^{uT} \Psi(\mathbf{z}; s).$$

The second equality is the formal Taylor identity for the first. Both sides are joint power series. Substituting $z_i = x_i$ for $i < m$, $z_m = 0$, $s = -t$, and $u = t$ consequently gives

$$\Psi(t + \mathbf{x}, t; 0) = e^{tT} \Psi(\mathbf{x}, 0; -t).$$

In Ψ , locality allows $Y(v, s)$ to be moved to the far left while retaining every factor $(z_i - s)^{K_i}$ in its original orientation. The field $Y(a_m, z_m)$ is then rightmost, so constant-coefficient extraction in z_m uses creation and yields

$$\Psi(\mathbf{x}, 0; -t) = \Pi(t + \mathbf{x}, t) Y(v, -t) C(\mathbf{x}).$$

These expressions are well defined in $V((t))[[\mathbf{x}]]$. For each vector coefficient c_α of C , skew symmetry gives

$$e^{tT} Y(v, -t) c_\alpha = Y(c_\alpha, t) v.$$

Lower truncation makes this identity and the exponential action well defined for each coefficient, without a common lower t -bound for all α . We obtain

$$\Psi(t + \mathbf{x}, t; 0) = \Pi(t + \mathbf{x}, t) Y(C(\mathbf{x}), t) v.$$

The scalar $\Pi(t + \mathbf{x}, t)$ is a unit in $\mathbb{C}((t))[[\mathbf{x}]]$: its constant coefficient in $\mathbf{x}$ is the invertible Laurent monomial $t^{\sum_i K_i}$. Cancellation therefore proves

$$\sigma(F_v) = Y(C(\mathbf{x}), t) v.$$

For $m = 1$, this is the identity $Y(a_1, t) v = Y(a_1, t) v$. This argument uses only regularity, translation, locality, creation, and skew symmetry; compare the two-field argument in [5, Section 3.2, Theorem 3.2.1].

The Complementary Coefficients. Restore the original labels and write $\mathbf{y} = \mathbf{z}_I^c$. Locality, with the original orientations of all factors retained, gives

$$H_V(\mathbf{z})(e) = \Delta_{\text{cross}}(\mathbf{z}_I, \mathbf{y}) \sum_{\gamma \geq \mathbf{0}} F_{v_\gamma}(\mathbf{z}_I) \mathbf{y}^\gamma.$$

Each v_γ is homogeneous when nonzero. Write the scalar cross factor as the finite sum

$$\Delta_{\text{cross}}(\mathbf{z}_I, \mathbf{y}) = \sum_{\delta \in D} p_\delta(\mathbf{z}_I) \mathbf{y}^\delta.$$

For any fixed β , coefficient extraction gives

$$[\mathbf{y}^\beta] H_V(\mathbf{z})(e) = \sum_{\substack{\delta \in D \\ \delta \leq \beta}} p_\delta(\mathbf{z}_I) F_{v_{\beta-\delta}}(\mathbf{z}_I).$$

Apply σ to both sides; the sum on the right is finite. On the left, regularity makes σ ordinary substitution in H_V . On the right, compatibility with multiplication by scalar polynomials and the identity just proved for a fixed vector give the $\mathbf{y}^\beta$ coefficient of (16). Since β was arbitrary, the lemma follows. The coordinate-pole bounds may depend on γ ; no bound uniform in all complementary coefficients is required. $\square$

A.3 Vacuum Coefficients

Proof of Lemma 4.3. Let $\pi_0 : V \rightarrow V_0 = \mathbb{C}|0\rangle$ be the graded projection, and put

$$J = \{v \in V : \pi_0 Y(a, z)v = 0 \text{ for all } a \in V\}.$$

For homogeneous a , the contributions of distinct homogeneous components of v to $\pi_0 Y(a, z)v$ occur at distinct powers of z . Thus J is graded. For $a, b \in V$ and $v \in J$, the definition of J gives

$$\pi_0 Y(Y(a, x)b, t)v = 0.$$

By formal associativity [4, Proposition 2.6], this iterate and $\pi_0 Y(a, z_1)Y(b, z_2)v$ are expansions of the same element of

$$V_0[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}].$$

For the iterate, set $z_1 = t + x$, $z_2 = t$ and expand $(t + x)^{-1}$ in nonnegative powers of x/t . This expansion is injective: after clearing the displayed denominators, it is an invertible linear change of variables on power-series numerators. Thus $\pi_0 Y(a, z_1)Y(b, z_2)v = 0$. Taking coefficients shows that $b_m v \in J$ for all $b \in V$ and $m \in \mathbb{Z}$. Since $\pi_0 T = 0$, translation covariance gives

$$\pi_0 Y(a, z)Tv = -\partial_z \pi_0 Y(a, z)v = 0.$$

Thus J is stable under all left modes and T , hence is an ideal by skew symmetry. It does not contain $|0\rangle$, and simplicity gives $J = 0$.

Since $u \notin J$, some homogeneous $a \in V_k$ satisfies $a_{h+k-1}u \in \mathbb{C}^\times |0\rangle$. Necessarily $k > 0$, since $Y(|0\rangle, z)u = u$. Skew symmetry $Y(u, z)a = e^{zT}Y(a, -z)u$ gives

$$u_{h+k-1}a = (-1)^{h+k} a_{h+k-1}u.$$

The terms involving positive powers of T have zero vacuum component. Take $w = a$. $\square$

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